Notices from the ISMS

September 2007

CONTENTS

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INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

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RIEMANN-HILBERT TYPE PROBLEMS IN LEIBNIZ ALGEBRAS WITH LOGARITHMS.

ABSTRACT. We give a survey of Riemann-Hilbert type problems in Leibniz algebras with logarithms solved by means of Algebraic Analysis in PR[8] (cf. Chapter 14), PR[13] and PR[14]. These problems correspond to such classical problems when the Cauchy transformation is a involution. It is shown that this involution is not multiplicative. On the other hand, in the same book equations with multiplicative involutions were considered. These results can be applied to equations with an involutive transformation of argument, in particular, to equations with transformed argument by means of a function of Carleman type. Riemann-Hilbert type problems with an additional multiplicative involution in commutative Leibniz algebras with logarithms are examined in PR[13]. Results obtained there can be applied not only to problems with a transformation of argument but also to problems with the conjugation (in the complex sense). In PR[14] there are considered similar problems in several variables with Riemann- Hilbert condition posed on each variable separately. For instance, these problems correspond in the classical case to problems for polyanalytic functions on polydiscs (cf. BD[1], Ms[1]).

KEY WORDS: Algebraic analysis, Riemann-Hilbert problem, analytic function, polyanalytic function, Cauchy singular integral, algebra with unit, Leibniz condition, Carleman function, transformed argument, conjugation, involution, multiplicative involution logarithmic mapping, antilogarithmic mapping

2000 AMS SUBJECT CLASSIFICATION: 47C05, 30E25, 30H05, 32A70, 39B32

0. Introduction.

The classical Riemann-Hilbert problem can be stated as follows.

Example 0.1. Let $n \in \mathbb{N}$. Let $j = 1, ..., n$. Let $\mathbf{\Omega_j} \subset \mathbb{C}$ be domains with the boundaries $\partial\Omega_j = \Gamma_j$ which are pairwise disjoint (for $n > 1$) closed regular arcs, i.e $\Gamma_j = \{z = z_j(t) : \alpha_j \leq t \leq \beta_j,\ z_j(\alpha_j) = z_j(\beta_j)\}$, where the functions $z_j \in C^1(\alpha_j, \beta_j)$, are one-to-one, $z_j'(t) \neq 0$ for $t \in (\alpha_j, \beta_j)$ and $\lim_{t\to\alpha_j+0} z_j'(t) = \lim_{t\to\beta_j-0} z_j'(t) \neq 0$.

We assume that the system $\Gamma = (\Gamma_1, ..., \Gamma_2)$ is *oriented* (cf. for instance, PR[5]). It means that the plane is divided into components $\mathbf{\Omega_0}$, $\mathbf{\Omega_1}, ..., \mathbf{\Omega_n}$ and we associate the sign "-" with the component $\mathbf{\Omega_0}$ containing the point ∞, and the sign "+" with the components having a common boundary with $\mathbf{\Omega_0}$. Next, we associate the sign "-" with the components having a common boundary with components having the sign "+", but not with $\mathbf{\Omega_0}$, and so on. Hence on the left of each of these arcs lies a domain with the sign "+", and on the right, a domain with the sign "-". If $n = 1$ then $\mathbf{\Omega_0} = \mathbb{C} \setminus \mathbf{\Omega_1}$ (cf. PR[8], Chapter 14; PR[13]).

We have to find a vector function $\Phi = (\Phi_1, ..., \Phi_n)$ with Φ_j piecewise analytic in the domains $\Omega_j^+ = \Omega_j$ and $\Omega_j^- = \mathbb{C} \setminus \overline{\Omega_j}$, bounded at infinity $^{*)}$ and such that their boundary values Φ^+ and Φ^- satisfy the following condition on the oriented system Γ:

$$\Phi_j^+(t) = G_j(t)\Phi_j^-(t) + g_j(t) \quad \text{for } t \in \Gamma_j, \quad (j = 1, ..., n) \tag{2.8}$$

where functions g_j, G_j are given. The solution of the problem is well-known (for $n = 1$ cf. for instance, MICHLIN Mi[1], POGORZELSKI P[1], MEISTER Me[1], WEGERT Wg[1],[2], WENDLAND Wn[1]; for $n > 1$, cf. for instance, BD[1], Ms[1]). In order to solve (2.8), we have to use properties of logarithmic and exponential functions (cf. ANOSOV AND BOLIBRUCH AB[1]) and of singular integral operators S_j $(j = 1, ..., n)$ defined by the *Cauchy principal value* of an integral, namely,

$$(S_j\varphi)(t_j) = \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{\Gamma_j \setminus \{z \in \mathbb{C} \,:\, |z - t_j| < \varepsilon\}} \frac{\varphi(\tau)}{\tau - t_j} d\tau \overset{\text{def}}{=} \tag{2.9}$$

$$\overset{\text{def}}{=} \frac{1}{\pi i} \int_{\Gamma_j} \frac{\varphi(\tau)}{\tau - t_j} d\tau \quad (t_j \in \Gamma_j) \quad (j = 1, ..., n).$$

In the case, when g_j, G_j belong to the space $H^\mu(\Gamma_j)$ of functions satisfying the Hölder condition on Γ_j with an exponent μ, $0 < \mu < 1$, $(j = 1, ..., n)$, the singular integral operators S_j defined by (2.9) are involutions in the space $X_j = H^\mu(\Gamma_j)$: $S_j^2 = I$ on X_j. Thus there are disjoint projectors P_j^+ and P_j^- giving the partition of unit and such that $\Phi_j^+ = P_j^+ x_j$, $\Phi_j^- = P_j^- x_j$ for an $x_j \in X_j$ $(j = 1, ..., n)$. Clearly, X_j are also commutative algebras over $\mathbb{C}$ with the pointwise multiplication. It not difficult to verify that the operators S_j defined by (2.9), the operators $D_j = \frac{\mathrm{d}}{\mathrm{d}t_j}$ and the usual logarithmic functions satisfy Condition $_n$ with $S = (S_1, ..., S_n)$ and $D = (D_1, ..., D_n)$ (cf. Section 2). Thus Φ is to be found if we apply logarithms to the homogeneous problem (2.8) (i.e. with $g_j = 0$ for $j = 1, ..., n$). Having found Φ_j $(j = 1, ..., n)$, we get $x = \Phi = (\Phi_1, ..., \Phi_n)$. Then the jth non-homogeneous problem $(j = 1, ..., n)$ is solved by a use of the already found solution to the jth homogeneouus problem (cf. Formula (2.7) for $n = 1$.).

This problem can be also formulated and solved in the same manner if some of Γ_j are oriented systems of finite sets of closed regular arcs. □

Riemann-Hilbert type problems in Leibniz algebras with logarithms have been studied and solved by means of ALgebraic Analysis in PR[8] (cf. Chapter 14). These problems correspond to such classical problems when the Cauchy transformation is an involution. It was shown that this involution is **not** multiplicative. On the other hand, in the same book (cf. Chapter 16; also PR[9], PR[12]) equations with multiplicative involutions were considered. These results can be applied

$^{*)}$ cf. Example 2.3.

to equations with an involutive transformation of argument, in particular, to equations with transformed argument by means of a function of Carleman type.

Riemann-Hilbert type problems with an additional multiplicative involution in commutative Leibniz algebras with logarithms are examined in PR[13]. Results obtained there can be applied not only to problems with a transformation of argument but also to problems with the conjugation (in the complex sense).

There are considered also similar problems in several variables with Riemann-Hilbert condition posed on each variable separately. For instance, these problems correspond in the classical case to problems for polyanalytic functions on polydiscs (cf. HD[1], Ms[1], PR[14]).

In order to find solutions to a generalized Riemann-Hilbert problem, we consider algebras with logarithms induced by a linear operator D and with an involution S. In commutative algebras solutions will be obtained in a similar manner as in the classical case (cf. Example 0.1). In noncommutative algebras we shall need some additional assumptions and some modifications of considerations used in the commutative case. We should point out that logarithmic and antilogarithmic mappings are **non**-linear.

The next step is to consider such problems with an additional involution which, by assumption, is multiplicative. This involution may correspond in the classical case to an involutive transformation of argument and/or to the conjugation in the complex sense: $x \to \overline{x}$ (cf. Examples 2.1. 2.2).

In order to consider multidimensional problems. we shall generalize problems mentioned above to a Cartesian product of a (finite) number of algebras with logarithms and involutions.

1. Preliminaries.

Let X be a linear space over a field $\mathbb{F}$ of scalars of the characteristic zero. Recall that $L(X)$ is the set of all linear operators with domains and ranges in X and $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$.

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we shall write $D \in \mathbf{A}(X)$. The set of all *commutative* algebras belonging to $\mathbf{A}(X)$ will be denoted by $\quad (X)$. If $D \in \mathbf{A}(X)$ then

$$f_D(x, y) = D(xy) - c_D[xDy + (Dx)y] \qquad \mathit{for} \quad x, y \in \text{dom } D,$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \quad (X)$. This form is called a *non-Leibniz component*. Non-Leibniz components have been introduced for right invertible operators $D \in \quad (X)$ (cf. PR[6], Section 6.1). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x, y) \qquad \mathit{for} \quad x, y \in \text{dom } D.$$

If $D \in \quad (X)$ is right invertible then the algebra X is said to be a *D-algebra*.

We shall consider in $\mathbf{A}(X)$ the following sets:

- the set of all *multiplicative* mappings (not necessarily linear) with domains and ranges in X: $M(X) = \{A : \text{dom } A \subset X,\ A(xy) = A(x)A(y)\ for\ x, y \in \text{dom } A\}$;
- the set $I(X)$ of all invertible elements belonging to X;
- the set $R(X)$ of all right invertible operators belonging to $L(X)$;
- the set $\mathcal{R}_D = \{R \in L_0(X)\ :\ DR = I\}$ of all right inverses to a $D \in R(X)$;
- the set $\mathcal{F}_D = \{F \in L_0(X)\ :\ F^2 = F,\quad FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_\mathcal{D}}\ FR = 0\}$ of all *initial* operators for a $D \in R(X)$;
- the set $\mathcal{I}(X)$ of all invertible operators belonging to $L(X)$.

Clearly, if $\ker D \neq \{0\}$ then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. Elements of the kernel of a $D \in R(X)$ are said to be *constants*. If $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$. We also have dom $D = RX \oplus \ker D$ independently of the choice of an $\mathcal{R}_D$. It is well-known that F is an initial operator for a $D \in R(X)$ if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on dom D. Moreover, if F' is any projection onto $\ker D$ then F' is an initial operator for D corresponding to the right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$ The already mentioned properties of initial operators are fundamental for Algebraic Analysis (cf. PR[6]).

Suppose that $D \in \mathbf{A}(X)$. Let $\Omega_r, \Omega_l : \text{dom } D \longrightarrow 2^{\text{dom } D}$ be multifunctions defined as follows:

$$(1.1) \quad \Omega_r u = \{x \in \text{dom } D : Du = uDx\}, \quad \Omega_l u = \{x \in \text{dom } D : Du = (Dx)u\}$$

for $u \in$ dom D . The equations

$$(1.2)\ Du = uDx \quad for\ (u, x) \in \text{graph } \Omega_r\ , \quad Du = (Dx)u \quad for\ (u, x) \in \text{graph } \Omega_l$$

are said to be the *right* and *left basic equations*, respectively. Clearly,

$$\Omega_r^{-1}x = \{u \in \text{dom } D : Du = uDx\}, \quad \Omega_l^{-1}x = \{u \in \text{dom } D : Du = (Dx)u\}$$

for $x \in$ dom D. The multifunctions Ω_r, Ω_l are well-defined and dom $\Omega_r \cap$dom $\Omega_l \supset$ $\ker D$.

Suppose that $(u_r, x_r) \in$ graph Ω_l, $(u_l, x_l) \in$ graph Ω_r, L_r, L_l are selectors of Ω_r, Ω_l, respectively, and E_r, E_l are selectors of Ω_r^{-1}, Ω_l^{-1}, respectively. By definitions, $L_r u_r \in$ dom Ω_r^{-1}, $E_r x_r \in$ dom Ω_r, $L_l u_l \in$ dom Ω_l^{-1}, $E_l x_l \in$ dom Ω_l and the following equations are satisfied:

$$Du_r = u_r D L_r u_r, \qquad D E_r x_r = (E_r x_r) D x_r;$$

$$Du_l = (DL_l u_l)u_l, \quad DE_l x_l = (Dx_l)(E_l x_l).$$

Any invertible selector L_r of Ω_r is said to be a *right logarithmic mapping* and its inverse $E_r = L_r^{-1}$ is said to be a *right antilogarithmic mapping*. If $(u_r, x_r) \in$ graph Ω_r and L_r is an invertible selector of Ω_r then the element $L_r u_r$ is said to be a *right logarithm* of u_r and $E_r x_r$ is said to be a *right antilogarithm* of x_r. By $G[\Omega_r]$ we denote the set of all pairs (L_r, E_r), where L_r is an invertible selector of Ω_r and $E_r = L_r^{-1}$. Respectively, any invertible selector L_l of Ω_l is said to be a *left logarithmic mapping* and its inverse $E_l = L_l^{-1}$ is said to be a *left antilogarithmic mapping*. If $(u_l, x_l) \in$ graph Ω_l and L_l is an invertible selector of Ω_l then the element $L_l u$ is said to be a *left logarithm* of u_l and $E_l x_l$ is said to be a *left antilogarithm* of x_l. By $G[\Omega_l]$ we denote the set of all pairs (L_l, E_l), where L_l is an invertible selector of Ω_l and $E_l = L_l^{-1}$.

If $D \in \quad (X)$ then $\Omega_r = \Omega_l$ and we write $\Omega_r = \Omega$ and $L_r = L_l = L$, $E_r = E_l = E$, $(L, E) \in G[\Omega]$. Selectors L, E of Ω and Ω^{-1} are said to be *logarithmic* and *antilogarithmic* mappings, respectively. For any $(u, x) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively (cf. PR[8]).

Clearly, by definition, for all $(L_r, E_r) \in G[\Omega_r]$, $(u_r, x_r) \subset$ graph Ω_r, $(L_l, E_l) \in G[\Omega_l]$, $(u_l, x_l) \in$ graph Ω_l we have

$$\text{(1.3)} \qquad E_r L_r u_r = u_r, \ L_r E_r x_r = x_r; \quad E_l L_l u_l = u_l, \ L_l E_l x_l = x_l;$$

$$\text{(1.4)} \qquad DE_r x_r = (E_r x_r)Dx_r, \quad Du_r = u_r DL_r u_r;$$

$$DE_l x_l = (Dx_l)(E_l x_l), \quad Du_l = (DL_l u_l)u_l.$$

A right (left) logarithm of zero is not defined. If $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$ then $L_r(\ker D \setminus \{0\}) \subset \ker D$, $E_r(\ker D) \subset \ker D$, $L_l(\ker D \setminus \{0\}) \subset \ker D$, $E_l(\ker D) \subset \ker D$. In particular, $E_r(0)$, $E_l(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant.

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*: $D(xy) = xDy + (Dx)y$ for $x, y \in$ dom D then X is said to be a *Leibniz algebra*.

Let $D \in \quad (X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x+y) = (Ex)(Ey)$ for $x, y \in$ dom Ω. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *Leibniz commutative algebra* (cf. PR[8])). In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for $u \in$ dom Ω is that $u \in I(X)$ (cf. PR[8]).

By $\mathbf{Lg_r}(D)$, $\mathbf{Lg_l}(D)$, $\mathbf{Lg}(D)$ we denote the classes of these algebras with $D \in R(X)$ and with unit $e \in$ dom Ω for which there exist invertible selectors of Ω_r, Ω_l, Ω, respectively, i.e. there exist $(L_l, E_l) \in G[\Omega_l]$, $(L_r, E_r) \in G[\Omega_r]$, $(L, E) \in G[\Omega]$, respectively.

In the sequel we shall consider multidimensional Leibniz algebras, i.e. a Cartesian product of finite number of Leibniz algebras with logarithms.

Suppose then that we are given n commutative algebras X_j (over the field $\mathbb{F}$) with $D_j \in R(X_j)$ and with units $e_j \in X_j$ and multifunctions Ω_j $(j = 1, ..., n)$. We assume that $X_j \in \mathbf{Lg}(D_j)$, $(L_j, E_j) \in G[\Omega_j]$ and $e_j \in \text{dom}\, \Omega_j$ $(j = 1, ..., n)$. Consider the Cartesian product

$$X = X_1 \times ... \times X_n$$

with the coordinatewise addition and multiplications of elements, multiplication of elements by scalars and coordinatewise operations of all mappings acting on X_j, i.e. for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $t = (t_1, ..., t_n) \in \mathbb{F}^n$, $T_j \in L(X_j)$ $(j = 1, ..., n)$ we have

$$(1.5) \qquad x + y = (x_1 + y_1, ..., x_n + y_n), \quad xy = (x_1 y_1, ..., x_n y_n),$$

$$(1.6) \qquad tx = (t_1 x_1, ..., t_n x_n),$$

$$Tx = (T_1 x_1, ..., T_n x_n) \quad \textit{whenever } x_j \in \text{dom}\, T_j.$$

Clearly, X has the unit e. Namely, $e = (e_1, ..., e_n)$.

Consequently, we shall write

$$(1.7) \qquad I(X) = \big(I(X_1), ..., I(X_n)\big), \quad \Omega = (\Omega_1, ..., \Omega_n),$$

$$(L, E) = \big((L_1, E_1, ..., (L_n, E_n)\big), \quad G[\Omega] = \big(G[\Omega_1], ..., G[\Omega_n]\big),$$

$$\mathbf{Lg}(D) = \big(\mathbf{Lg}(D_1), ..., \mathbf{Lg}(D_n)\big), \qquad (X) = \big(\ (X_1), ..., \ (X_n)\big).$$

Similar denotations are admitted in non-commutative cases. Namely,

$$(1.8) \qquad (L_r, E_r) = \big((L_{r1}, E_{r1}, ..., (L_{rn}, E_{rn})\big), \quad G[\Omega_r] = \big(G[\Omega_{r1}], ..., G[\Omega_{rn}]\big),$$

$$\mathbf{Lg_r}(D) = \big(\mathbf{Lg_r}(D_1), ..., \mathbf{Lg_r}(D_n)\big), \quad \mathsf{A}(X) = \big(\mathsf{A}(X_1), ..., \mathsf{A}(X_n)\big).$$

$$(L_l, E_l) = \big((L_{l1}, E_{l1}, ..., (L_{ln}, E_{ln})\big), \quad G[\Omega_l] = \big(G[\Omega_{l1}], ..., G[\Omega_{ln}]\big),$$

$$\mathbf{Lg_l}(D) = \big(\mathbf{Lg_l}(D_1), ..., \mathbf{Lg_l}(D_n)\big).$$

If $D_j \in \mathsf{A}(X_j)$ $(j = 1, ..., n)$ satisfy the Leibniz condition then $D \in \mathsf{A}(X)$ also satisfy that condition and X is said to be a *Leibniz algebra* (of dimension n).

If in order to prove some statement it is enough to prove it for an arbitrarily fixed j ($j = 1, ..., n$), we shall omit that proof and we refer either to PR[8] (Chapter 14) or to PR[13], respectively, i.e. to one-dimensional case: $n = 1$.

We shall denote the identity operators in all spaces $X_1, ..., X_n$ by the same letter I, since it does not lead here to any misunderstanding.

2. Commutative case.

Here and in the sequel we shall consider an n-dimensional commutative Leibniz algebra $X \in \mathbf{L}(D)$ (cf. (1.5), (1.6), (1.7)).

Theorem 2.1. (cf. PR[3] for $n = 1$.). *Let $n \in N$ and let $j = 1, ..., n$. Let X_j be linear spaces over an algebraically closed field $\mathbb{F}$ with involutions $S_j \in L_0(X_j)$ (i.e. $S_j^2 = I$, $S_j \neq I$). Then for an arbitrarily fixed $1 \leq j \leq n$ $P_j^+ = \frac{1}{2}(I + S_j)$ and $P_j^- = \frac{1}{2}(I - S_j)$ are disjoint projectors giving partition of unity: $(P^{\pm})_j^2 = P_j^{\pm}$; $P_j^+ P_j^- = P_j^- P_j^+ = 0$ and $P_j^+ + P_j^- = I$. Moreover, $P_j^+ - P_j^- = S_j$ and $S_j P_j^{\pm} = P_j^{\pm} S_j = \pm P_j^{\pm}$. So that, if we write $X_j^{\pm} = P_j^{\pm} X_j$, $x_j^{\pm} = P_j^{\pm} x_j$ for $x_j \in X_j$, then we have*

$$X_j = X_j^+ \oplus X_j^-; \quad x_j^{\pm} \in X_j^{\pm}; \quad S_j x_j^+ = x_j^+; \; S_j x_j^- = -x_j^- \quad \textit{for } x_j \in X_j.$$

Corollary 2.1. *Let all assumptions of Theorem 2.1 be satisfied. Let $1 \leq j \leq n$ be arbitrarily fixed. If $u_j \in X_j$, $u_j^{\pm} = P_j^{\pm} u_j$ and*

$$u_j^+ - u_j^- = v_j, \qquad \textit{where } v_j \in X_j, \tag{2.1}$$

then $u_j = S v_j$. In particular, if $v_j = 0$ then $u_j = 0$.

Lemma 2.1. (cf. PR[8] for $n = 1$) *Let $n \in \mathbb{N}$. Let $D \in \quad (X)$, where $X \in \mathbf{Lg}(D)$ is an n-dimensional Leibniz algebra with unit e and with an involution $S = (S_1, ..., S_n) \in L_0(X)$. Let $(L, E) \in G[\Omega]$. Then the following conditions are equivalent:*

(i) $Lu^{\pm} \in X^{\pm}$ *for all* $u \in \operatorname{dom} \Omega$;

(ii) $Ev^{\pm} \in X^{\pm}$ *for all* $v \in \operatorname{dom} \Omega^{-1}$;

(iii) $P^{\pm} L u^{\pm} = \pm L u^{\pm}$ *for all* $u \in \operatorname{dom} \Omega$;

(iv) $P^{\pm} E v^{\pm} = \pm E v^{\pm}$ *for all* $v \in \operatorname{dom} \Omega^{-1}$.

Let $n \in \mathbb{N}$ be arbitrarily fixed. We admit the following condition:

$_n$ Let $\mathbb{F}$ be algebraically closed. Let $D \in \quad (X)$, where $X \in \mathbf{Lg}(D)$ is an n-dimensional Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $(L, E) \in G[\Omega]$ and

$$L(X^+ \cap \operatorname{dom} \Omega) \subset X^+, \quad L(X^- \cap \operatorname{dom} \Omega) \subset X^-. \tag{2.2}$$

In particular, (2.2) implies that $X^{\pm} \cap \operatorname{dom} \Omega \subset \operatorname{dom} \Omega^{-1}$.

Lemma 2.2. (cf. PR[8] for $n = 1$) *Suppose that Condition* $\quad_n$ *is satisfied. Then*

(i) $uv \in X^{\pm}$ *whenever* $u, v \in X^{\pm} \cap \operatorname{dom} \Omega$;

(ii) $(u^{\pm})^{-1} \in I(X^{\pm})$ *whenever* $u^{\pm} \in I(X^{\pm})$.

On the other hand, we have

Proposition 2.1. (cf. PR[8] for $n = 1$) *Suppose that* X *is a commutative algebra over a field* $\mathbb{F}$ *with a multiplicative involution* $S \in L_0(X$, *i.e.* $S(xy) = (Sx)(Sy)$ *for* $x, y \in X$. *Then* $x^{\pm}y^{\pm} \in X^{+}$, $x^{+}y^{-}, x^{-}y^{+} \in X^{-}$ *for all* $x, y \in X$.

Proposition 2.2. (cf. PR[8] for $n = 1$) *Suppose that Condition* $\quad_n$ *is satisfied. Then the involution* S *is not multiplicative.*

Proposition 2.2 is important, since there are several examples of functional-differential equations with a multiplicative involution which can be also solved by means of algebraic methods (cf. PR[4], PR[5], PR[6]). Namely, any transformation of argument is a multiplicative operation (cf. also PR[8], PR[9], PR[12], PR[13]). However, by Proposition 2.2, any involution under consideration cannot be multiplicative if Condition $\quad_n$ is assumed.

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$\quad_n$ is satisfied. Find an $x_0 = (x_{01}, ..., x_{0n}) \in \operatorname{dom} \Omega$ such that

$$x_0^{+} = a x_0^{-}, \quad \text{where } a \in \operatorname{dom} \Omega \text{ is given.} \tag{2.3}$$

Theorem 2.2. (cf. PR[8] for $n = 1$) *Suppose that Condition* $\quad_n$ *is satisfied. If* $a \neq \pm e$ *then the homogeneous Riemann-Hilbert problem (2.3) has a solution*

$$x_0 = E(P^{+}La) + E(-P^{-}La) \quad \text{and} \quad x_0^{\pm} = E(\pm P^{\pm}La) \in I(X). \tag{2.4}$$

If $a = \pm e$ *then the only solution of (2.3) is* $x_0 = 0$.

Clearly, the solution (2.4) is dependent on the choice of selectors L.

Corollary 2.2. (cf. PR[8] for $n = 1$) *If all assumptions of Theorem 2.2 are satisfied then the solution to the problem (2.3) can be written in the form:* $x_0 = (a + e)E(-P^{-}La)$.

Consider now the set of all elements from $Y \subset X$ having *k-th roots*:

$$I_k(Y) = \{x \in Y : \exists_{y \in I(Y)} \quad y^k = x\} \qquad (k \in \mathbb{N}).$$

Here $n \in \mathbb{N}$ is fixed (for $n = 1$ cf. PR[8]). If $x \in I_k(Y)$ and $y^k = x$ then we write $y = x^{1/k}$ $(k \in \mathbb{N})$. By definition, $x \in I(Y)$.

Corollary 2.3. (cf. PR[8] for $n = 1$) *Suppose that all assumptions of Theorem 2.2 are satisfied and $a \in I_2(\text{dom}\ \Omega)$. Then the solution to the problem (2.3) can be written in the form:* $x_0 = (a^{\frac{1}{2}} + a^{-\frac{1}{2}})[E(SLa)]^{\frac{1}{2}}$.

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dition $\quad {}_n$ is satisfied. Find an $x \in \text{dom}\ \Omega$ such that

$$x^+ = ax^- + b, \quad \textit{where } a \in \text{dom}\ \Omega,\ b \in X \textit{ are given.} \tag{2.5}$$

Theorem 2.3. (cf. PR[8] for $n = 1$) *Suppose that Condition* $\quad {}_n$ *is satisfied. If $a \neq \pm e$ and x_0 is a solution of the homogeneous Riemann-Hilbert problem (2.3) then the nonhomogeneous Riemann-Hilbert problem (2.5) has a solution*

$$x = x_0 + \frac{1}{2}(x_0 S y_0 + y_0 S x_0), \quad \textit{where } y_0 = a^{-1}bE(P^- La) = a^{-1}b(P^- x_0)^{-1}. \tag{2.6}$$

If $a = e$ then a solution to (2.5) is $x = Sb$. If $a = -e$ then a solution to (2.5) is $x = b$.

Corollary 2.4. (cf. PR[8] for $n = 1$) *Suppose that Condition* $\quad {}_n$ *is satisfied. If $a \neq \pm e$ and x_1, x_2 are two solutions to the nonhomogeneous Riemann-Hilbert problem (2.5) then their difference $\tilde{x} = x_1 - x_2$ is a solution to the homogeneous problem (2.3). If $a = e$ then the problem (2.5) has a unique solution $x = Sb$. If $a = -e$ then (2.5) has a unique solution $x = b$.*

Corollary 2.5. (cf. PR[13] for $n = 1$) *Suppose that Condition* $\quad {}_n$ *is satisfied and $a \neq \pm e$, x_0 is a solution of the homogeneous Riemann-Hilbert problem (2.3) then the nonhomogeneous Riemann-Hilbert problem (2.5) has solutions of the form*

$$x = x_0 + S(x_0 y_0), \quad \textit{where } y_0 = a^{-1}bE(P^- La) = a^{-1}b(P^- x_0)^{-1}. \tag{2.7}$$

If $a = e$ then the problem (2.5) has a unique solution $x = Sb$. If $a = -e$ then (2.5) has a unique solution $x = b$.

Note 2.1. If X is a commutative Leibniz algebra for a $D \in L(X)$ then X is a Leibniz algebra for a $D' = dD$, where $d \in X \setminus \{0\}$. Indeed, for all $x, y \in \text{dom}\ D' = \text{dom}\ D$ we have

$$D'(xy) = dD(xy) = d(xDy + yDx) = xdDy + ydDx = xD'y + yD'x.$$

In particular, if $D \in R(X)$, $R \in \mathcal{R}_D$ and $g = Re \in I(X)$, then X is a Leibniz algebra for $D_n = g^n D$ $(n \in \mathbb{N})$. If $n = 1$, then antilogarithms induced by $D' = gD$ are $E'(\lambda g) = g^\lambda = E(\lambda Lg)$. Observe that in the classical case of the operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and $R = \int_0^t$ in $C[0, T]$ we have $D_n = t^n \frac{\mathrm{d}}{\mathrm{d}t}$. The corresponding logarithms are (up to an additive constant) $Lx = \ln x$, as in the case of the operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ and antilogarithms are ct^λ $(c \in \mathbb{R})$. $\square$

pose that Condition $_n$ is satisfied, $T_j \in L_0(X)$ are multiplicative involutions with projectors $Q_j^{\pm}$ giving the partition of unity $(j = 1, ..., n)$, $T = (T_1, ..., T_n)$ and $a(t) = a_0 + a_1 t$, where $a_0, a_1, b \in \operatorname{dom} \Omega$ are given. By definition, T is also a multiplicative involution. Find an $x \in \operatorname{dom} \Omega$ such that

$$x^+ = a(T)x + b. \tag{2.10}$$

We shall use the following theorem, different than results obtained before for equations with involutions (cf. PR[3], PR[5], PR[8], PR[12]).

Theorem 2.4. (cf. PR[13] for $n = 1$) *Suppose that Condition $_n$ is satisfied and X is a commutative algebra over $\mathbb{F}$ with unit e, $T \in L_0(X)$ is a multiplicative involution with projectors $Q^{\pm}$ giving the partition of unity, $a(t) = a_0 + a_1 t$, where $a_0, a_1 \in X$ are given and either $Ta_k = a_k$ or $Ta_k = a_k T$, $k = (0, 1)$.*

(i) If $a(\pm 1) = a_0 \pm a_1 \in I(X)$ then the equation

$$a(T)x = y, \qquad y \in X \tag{2.11}$$

has a unique solution

$$x = (a_0^2 - a_1^2)^{-1} a(-T)y. \tag{2.12}$$

(ii) If $a_0 = \pm a_1$ and $a(\pm 1) = a_0 \pm a_1 \in I(X)$, then $a(T) = a(\pm 1)Q^{\pm}$ and all solutions of Equation (2.12) are of the form

$$x = [a(\pm 1)]^{-1} y + x^{\mp}, \quad \textit{where } x^{\mp} \in X^{\mp} \textit{ are arbitrary.} \tag{2.13}$$

Corollary 2.6. (cf. PR[13] for $n = 1$) *Theorem 2.4 holds without the assumption that T is multiplicative if, in particular, a_0, a_1 are scalar multiples of unit e.*

Note 2.2. Suppose that the involution T appearing in Theorem 2.4 satisfies the condition: $Ta_k - a_k T \in J$ $(k = 0, 1)$, where J is a proper two-sided ideal in the algebra X. Then we may apply Theorem 2.4 to the quotient algebra X/J and we obtain similar results up to an additive component belonging to J (cf. PR[5], PR[13] for $n = 1$). □

Theorem 2.5. (cf. PR[13] for $n = 1$) *Suppose that Condition $_n$ is satisfied and $a \neq \pm e$, $T_j \in L_0(X)$ are multiplicative involutions with projectors $Q_j^{\pm}$ giving the partition of unity $(j = 1, ..., n)$, $T = (T_1, ..., T_n)$, $a(t) = a_0 + a_1 t$, where $a_0, a_1 \in \operatorname{dom} \Omega$ are given and either $Ta_k = a_k$ or $Ta_k = a_k T$ $(k = 0, 1)$. Let*

$$x_0 = E(P^+ L a_0) + E(-P^- L a_0), \tag{2.14}$$

$$a_0' = e,\ a_1' = a_1(e + a_0)^{-1}),\ a'(t) = a(t) - eI - a_1' t, \quad b' = (e + a_0^{-1})b. \tag{2.15}$$

If $a'(t)$ satisfies all assumptions of Theorem 2.4, then the problem (2.10) has solutions of the form:

$$x = a'(T)x^- + b' + [(a_0 - e)^2 - (a_1 - a_1')^2]^{-2} a'(-T)(b' - b). \tag{2.16}$$

Example 2.1. (cf. PR[13] for $n = 1$) Let $\Gamma_j = \{z : |z| = r_j\}$, $(r_j > 0)$ $(j = 1, .., n)$. Suppose that X and S are defined as in Example 0.1. Consider the following problem: Find a vector function $\Phi = (\Phi_1, ..., \Phi_n)$ with Φ_j piecewise analytic in the domains $\Omega_j^+ = \Omega_j$ and $\Omega_j^- = \mathbb{C} \setminus \overline{\Omega_j}$, bounded at infinity and such that their boundary values Φ_j^+ and Φ_j^- satisfy the following condition:

$$\Phi_j^+(t) = G_{j_0}(t)\Phi_j^-(t) + G_{j_1}(t)\Phi_j^+(h_j(t)) + G_{j_2}(t)\Phi_j^-(h_j(t)) + g_j(t) \tag{2.17}$$

$$for\ t \in \Gamma_j, \quad (j = 1, ..., n),$$

where functions g_j, h_j, G_{kj} $(k = 0, 1, 2; j = 1, ..., n)$ are given, $h_j(\Gamma_j) \subset \Gamma_j$, $h_j(h_j(t)) \equiv t$, $h_j'(t) \neq 0$ for $t \in \Gamma_j$, $G_k(h_j(t)) = G_k(t)$ $(k = 0, 1, 2; j = 1, ..., n)$. Write: $(Tx)(t) = (x_1(h_1(t), ..., x_n(h_n(t)))$ for $x \in X$, $t \in \Gamma_j$ $(j = 1, ..., n$. Clearly, T is a multiplicative involution and $TG_k = G_k$ $(k = 0, 1, 2)$. Thus we can apply Theorem 2.5 in order to solve the problem (2.17).

In particular, if for a j we have $\Gamma_j = \mathbb{R}$ (i.e. $r_j = \infty$), $h_j(t) = -t$ for $t \in \Gamma_j$, then the functions G_{j_k} are even. Indeed, for $k = 0, 1, 2$ we have $G_{j_k}(t) = (TG_{j_k})(t) = G_{j_k}(-t)$. □

Example 2.2. (cf. PR[13] for $n = 1$) Let $j = 1, ..., n$. Let $\overline{\Gamma_j} = \Gamma_j$, i.e. $\bar{z} \in \Gamma_j$ whenever $z \in \Gamma_j$. Suppose that X and S are defined as in Example 2.1. Consider the following problem: Find a vector function $\Phi = (\Phi_i, ..., \Phi_n)$ with $\Phi_1, ..., \Phi_n$ piecewise analytic in the domains $\Omega_j{}^+ = \Omega_j$ and $\Omega_j^- = \mathbb{C} \setminus \overline{\Omega_j}$, bounded at infinity and such that their boundary values Φ_j^+ and Φ_j^- satisfy the following condition:

$$\Phi_j^+(t) = G_{j_0}(t)\Phi_j^-(t) + G_{j_1}(t)\overline{\Phi_j^+(t)} + G_{j_2}(t)\overline{\Phi_j^-(t)} + g_j(t) \tag{2.18}$$

$$for\ t \in \Gamma_j, \quad (j = 1, ..., n),$$

where functions g_j, G_{j_k} $(k = 0, 1, 2)$ are given, $\overline{G_{j_k}} = G_{j_k}$ $(k = 0, 1, 2)$. Clearly, the functions G_{j_k} are real. Write: $(Tx) = \overline{x} = (\overline{x_1}, ..., \overline{x_n})$ for $x \in X$. Clearly, T is a multiplicative involution and $TG_{j_k} = \overline{G_{j_k}} = G_{j_k}$ $(k = 0, 1, 2)$. Thus we can apply Theorem 2.5 in order to solve the problem (2.18). □

Example 2.3. In Examples 0.1, 2.1 and 2.3 the condition that functions, we are looking for, are bounded at infinity can be replaced by the condition of finite order of their growth at infinity (cf. Hizhyel H[1]). □

3. Noncommutative case.

In this section instead of Condition $\quad_n$ we shall admit either the condition

$(\mathsf{A})_n^l$, where $n \in \mathbb{N}$ (cf. PR[8], PR[13] for $n = 1$). Let $\mathbb{F}$ be algebraically closed. Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_l}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $uv \in X^\pm$ whenever $u, v \in X^\pm$ and $DS = SD$ on dom D, $(L_l, E_l) \in G[\Omega_l]$ and

$$(3.1) \qquad L_l(X^+ \in \text{dom } \Omega_l) \subset X^+, \quad L_l(X^- \cap \text{dom } \Omega_l) \subset X^-.$$

or the condition

$(\mathsf{A})_n^r$, where $n \in N$ (cf. PR[8], PR[13] for $n = 1$). Let $\mathbb{F}$ be algebraically closed. Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_r}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$, $uv \in X^\pm$ whenever $u, v \in X^\pm$ and $DS = SD$ on dom D, $(L_r, E_r) \in G[\Omega_r]$ and

$$(3.2) \qquad L_r(X^+ \cap \text{dom } \Omega_r) \subset X^+, \quad L_r(X^- \cap \text{dom } \Omega_r) \subset X^-.$$

In particular, (3.1) implies that $X^\pm \cap \text{dom } \Omega_l \subset \text{dom } \Omega_l^{-1}$, (3.2) implies that $X^\pm \cap \text{dom } \Omega_r \subset \text{dom } \Omega_r^{-1}$.

Note 3.1. (cf. PR[13] for $n = 1$) Let either Condition $(\mathsf{A})_n^l$ or Condition $(\mathsf{A})_n^r$ be satisfied. The assumed condition that $uv \in X^\pm$ whenever $u, v \in X^\pm$ in the commutative case is proved by Lemma 2.2. It is so, because in that case logarithms are of the exponential type. Without this property, we cannot prove a lemma corresponding to Lemma 2.2.

We should point out also that here, in the noncommutative algebras, we shall need essentially the property that D and S commute each with another. Thus Condition $\quad_n$ is not a particular case of Conditions $(\mathsf{A})_n^l$ and $(\mathsf{A})_n^r$. □

Lemma 3.1. (cf. PR[8], PR[13] for $n = 1$) *Let $D \in \mathbf{A}(X)$. Suppose that $X \in \mathbf{Lg_l}(D)$ is a Leibniz algebra with unit e and with an involution $S \in L_0(X)$ and $(L_l, E_l) \in G[\Omega_l]$ ($X \in \mathbf{Lg_r}(D)$ and $(L_r, E_r) \in G[\Omega_r]$, respectively). Then the following conditions are equivalent for $u_l \in \text{dom } \Omega_l$, $u_r \in \text{dom } \Omega_r$, $v_l \in \text{dom } \Omega_l^{-1}$, $v_r \in \text{dom } \Omega_r^{-1}$:*

(i) $L_l u_l^\pm \subset X^\pm$ *($L_r u_r^\pm \subset X^\pm$, respectively);*

(ii) $E_l v_l^\pm \subset X^\pm$ *($E_r v_r^\pm \subset X^\pm$, respectively.);*

(iii) $P^\pm L_l u_l^\pm = \pm L_l u_l^\pm$ *($P^\pm L_r u_r^\pm = \pm L_r u_r^\pm$, respectively);*

(iv) $P^\pm E_l v_l^\pm = \pm E_l v_l^\pm$ *($P^\pm E_r v_r^\pm = \pm E_r v_r^\pm$, respectively).*

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$(\mathsf{A})_n^l$ or Condition $(\mathsf{A})_n^r$ be satisfied. Find an $x_0 = (x_{01}, \dots, x_{0n}) \in X$ such that

$$(3.3) \qquad x_{0j}^+ = a x_{0j}^-, \quad (j = 1, \dots, n),$$

where $a \in I(X) \cap \operatorname{dom} \Omega_l$. ($a \in I(X) \cap \operatorname{dom} \Omega_r$, *respectively*).

Theorem 3.1. (cf. PR[8], PR[13] for $n = 1$) *Suppose that Condition* $(\mathbf{A})_n^l$ *(Condition* $(\mathbf{A})_n^r$*, respectively) holds. Then the Riemann-Hilbert problem (3.3) has a solution if and only if there a* u *such that*

$$u^+a - au^- = Da \qquad (au^+ - u^-a = Da, \quad \textit{respectively}). \tag{3.4}$$

If it is the case and $D \in R(X)$, *then the solution, we are looking for, is*

$$x_0 = E_lP^+(Ru+z) + E_lP^-(Ru+z), \quad \textit{where } z \in \ker D \tag{3.5}$$

$$(x_0 = E_rP^+(Ru+z) + E_rP^-(Ru+z), \quad \textit{respectively}).$$

Moreover, $x_0^\pm = E_lP^\pm(Ru+z) \in I(X)$, $x_0^\pm - E_lP^\pm(Ru+z) \in I(X)$, respectively.

Theorem 3.2. (cf. PR[8], PR[13] for $n = 1$) *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_l^r$ *holds. If*

$$aX^-a^{-1} \subset X^- \quad (a^{-1}X^+a \subset X^+, \textit{ respectively}), \tag{3.6}$$

then the Riemann-Hilbert problem (3.3) has a solution of the form (3.5), where

$$u = DL_lP^+a - a^{-1}(DL_lP^-a)a; \tag{3.7}$$

$$(u = a(DL_rP^+a)a^{-1} - DL_rP^{-1}a, \textit{ respectively})$$

Corollary 3.1. (cf. PR[8], PR[13] for $n = 1$) *Suppose that all assumptions of Theorem 3.2 are satisfied. Then* $P^-a \in I(X) \cap \operatorname{dom} \Omega_r$ ($P^-a \in I(X) \cap \operatorname{dom} \Omega_l$, *respectively) and*

$$u = DL_lP^+a + a^{-1}[DL_r(P^-a)^{-1}]a$$

$$(u = a(DL_lP^+a)a^{-1} + DL_r(P^-a)^{-1}, \textit{ respectively}).$$

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$(\mathbf{A})_n^l$ or Condition $(\mathbf{A})_n^r$ be satisfied. Let $a \in I(X) \cap \operatorname{dom} \Omega_l$, ($a \in I(X) \cap \operatorname{dom} \Omega_r$, respectively). Find an $x = (x_1, ..., x_n) \in X$ such that

$$x_j^+ = a_jx^- + b_j, \qquad \textit{where } b_j \in X, \ (j = 1, ..., n). \tag{3.8}$$

Theorem 3.3. (cf. PR[8], PR[13] for $n = 1$) *Suppose that all assumptions of Theorem 3.2 are satisfied,* $a \neq \pm e$ *and* x_0 *is a solution of the homogeneous problem (3.3). Then the nonhomogeneous* n*-dimensional Riemann-Hilbert problem (3.8) has a solution of the form*

$$x = x_0 + \frac{1}{2}[x_0Sy_0 + (Sx_0)y_0], \quad y_0 = (x_0^+)^{-1}a^{-1}b. \tag{3.9}$$

If either $a = e$ or $a = -e$ then (3.8) has a unique solution, namely $x = Sb$ or $x = b$, respectively.

Corollary 3.2. (cf. PR[8], PR[13] for $n = 1$ *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_n^r$ *is satisfied. If $a \neq \pm e$ and x', x'' are two solutions to the non-homogeneous n-dimensional Riemann-Hilbert problem (3.8) then their difference $x = x' - x''$ is a solution to the homogeneous problem (3.3). If $a = e$ then the problem (3.8) has a unique solution $x = Sb$ (cf. Corollary 2.4). If $a = -e$ then (3.8) has a unique solution $x = b$.*

Corollary 3.3. (cf. PR[8], PR[13] for $n = 1$) *Suppose that either Condition* $(\mathbf{A})_n^l$ *or Condition* $(\mathbf{A})_n^r$ *is satisfied. Then the involution S is not multiplicative* (cf. Proposition 2.2).

The already obtained solutions to the Riemann-Hilbert problem can be used in order to solve linear equations with involutions in Leibniz algebras with logarithms. Namely, we have

Example 3.1. (cf. PR[13] for $n = 1$) Suppose that either Condition $_n$ or Condition $(\mathbf{A})_n^r$ or Condition $(\mathbf{A})_n^l$ is satisfied. Consider the equation:

$$(aI + bS)x = y, \quad \textit{where } a, b, y \in X. \tag{3.10}$$

Since $aI + bS = a(P^+ + P^-) + b(P^+ - P^-) = (a + b)P^+ + (a - b)P^-$, Equation (3.10) can be rewritten as follows

$$(a + b)x^+ + (a - b)x^- = y. \tag{3.11}$$

If $a = \pm b$ and $a \pm b \in I(X)$ then solutions to this equations exist if and only if $(a \pm b)^{-1}y \in X^{\pm}$. If it is the case, then $x^{\pm} = (a \pm b)^{-1}y + x^{\mp}$, where $x^{\mp} \in X^{\mp}$ is arbitrary.

Suppose now that $(a + b)^{-1}(a - b) \in I(X) \in \text{dom } \Omega$ (dom Ω_r, dom Ω_l, respectively). Then Equation (3.11) can be written as

$$x^+ = (a + b)^{-1}(a - b)x^{-1} + (a + b)^{-1}y,$$

i.e. we have a Riemann-Hilbert problem $x^+ = \tilde{a}x^- + \tilde{y}$ with $\tilde{a} = (a + b)^{-1}(a - b)$ and $\tilde{b} = (a + b)^{-1}y$.

Solutions to Equation (3.10) have been obtained earlier in another way under other assumptions, for instance, that a, b commute (anticommute) with the involution S (cf. PR[3]-PR[5], PR[12], PR[13] for $n = 1$).

Having already solved Equation (3.10), we can solve the equation

$$(aI + bS + cT)x = y. \quad \textit{where } a, b, c, y \in X, \tag{3.12}$$

where $T \in L_0(X)$ is a multiplicative involution satisfying all assumptions of Theorem 2.5. □

This method without any essential change can be applied to problems with the Hilbert transform, a singular integral with the Cauchy kernel on a curve closed at infinity (cf. PR[2]), also with the cotangent Hilbert transform in appropriate spaces of functions.

The dependence of solutions on the choice of selectors has been examined in PR[8] for $n = 1$.

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Communications:

(Ⅰ) BIOCOMP 2007

Communicated by L.M. Ricciardi *

Stimulated by some friends, and on the grounds of the successful experience of *BIOCOMP2002 and "BIOCOMP2005*: Conferences, Prof. L.M. Ricciardi has now been induced to plan another Conference to be held in the same location (Vietri sul Mare, Italy), September 24-28, 2007.
The title is ***BIOCOMP2007 - Collective Dynamics: Topics on Competition and Cooperation in the Biosciences.***

The title is motivated by the nature of our sponsors and supporting grants, but the main purpose of this Conference is to bring together a limited number of well-known specialists in the fields of applied mathematics, physics and theoretical biology for an in-depth discussion of model building and computational strategies in some selected areas of the life sciences with special emphasis on theoretical neurobiology, molecular motors and quantitative problems in ecology and population dynamics.
This will be implemented through a program of plenary talks, parallel sessions and a poster session.
The interdisciplinary nature of the conference will allow cross-fertilization of recent advances in applied nonlinear mathematics and computational approaches. Several invited lectures on different topics of biomathematical interest will also be given, especially tailored on the needs of graduate students and young researches.

(Ⅱ) The 7th International Conference on Optimization (ICOTA 7)

Communicates by Wuyi Yue**

The 7th International Conference on Optimization(ICOTA 7) : Techniques and applications
December 12-December 15, 2007, International Conference Center, Kobe, Japan
ICOTA webpage is: http://www.iict.konan-u.ac.jp/ICOTA7/
See also pages 8 of Notices from the ISMS, November 2006.

(Ⅲ) Conferences for Young Algebraists

Klaus Denecke***

The following list gives some information on the future conferences of young algebraists:

1. CYA (75. AAA), Darmstadt (Germany), November 1-4, 2007
2. AAA Linz (Austria), May 22-25, 2008
3. CYA (77. AAA), Potsdam, Februar 2009
   (dedicated to K. Denecke)
4. AAA Bern (Switzerland), June 11-14, 2009
5. CYA (79. AAA), Olomouc (Czech Republic), February 2010

See also pages 10 and 11 of Notices from the ISMS, January 2007.

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* L.M.Ricciardi is a professor of Dipartimento di Matematica e Applicazioni, Universita di Napoli Federico II, and an International Advisor of SCMJ.

** Wuyi Yue is a professor of Konan University, an editor of SCMJ, and an editor of Notice from the ISMS.
***Klaus Denecke is a professor of University of Potsdam, Institute of Mathematics, an Editor of SCMJ, and an Editor of Notices from the ISMS.

(IV) Announcement of Meetings in Topology

Communicated by Gerhard Preuss*

1) December 3-7, 2007:
International Conference on Topology and its Applications 2007
(Jointly with 4th Japan Mexico Topology Conference)
Department of Mathematics, Kyoto University, Kitashirakawa-Oiwakecho,
Sakyoku, Kyoto, Japan

Organizing Committee:
Chair:
Akira Kono (Kyoto University)
Salvador Garcia-Ferreira (UNAM)
Algebraic Topology:
Norio Iwase (Kyushu University)
Miguel A. Xicote'ncatl (CINVESTAV) xico at math.cinvestav.mx
Knot Theory:
Akio Kawauchi (Osaka City University)
Mario Eudave (Instituto de Matematicas-UNAM)
Set Theory, Set-theoretic Topology:
Tsugunori Nogura (Ehime University)
Angel Tamariz-Mascarua (Facultad de Ciencias-UNAM)
Diego Rebolledo-Rojas (Instituto de Matematicas-UNAM)
Geometric Topology, Continuum Theory:
Hisao Kato (Tsukuba University)
Sergey Antonyan (Facultad de Ciencias-UNAM)
Alejandro Illanes (Instituto de Matematicas-UNAM)
Dynamical System:
Hiroshi Kokubu (Kyoto University)

See: http://www.math.sci.ehime-u.ac.jp/jamex/

2) June 9-19, 2008
Advances in Set-Theoretic Topology
in Honour of Tsugunori Nogura on his 60th Birthday
Centre for Scientific Culture "Ettore Majorana"
International School of Mathematics "G. Stampacchia"
Erice, Sicily, Italy

Organizers:
Szymon Dolecki (Burgundy University, France)
Yasunao Hattori (Shimane University, Japan)
Dmitri Shakhmatov (Ehime University, Japan)
Gino Tironi (University of Trieste, Italy)

Topics:
Convergence properties and convergence structures;
Dimension theory and related fields;
General topology and its applications in other areas of mathematics;
Hyperspaces, set-valued mapping and their selections;
Set theoretic methods in mathematics;
Topological algebra (topological groups, functions spaces, etc...).

See: http://www.math.sci.ehime-u.ac.jp/erice/

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*Gerhard Preuss is a professor of Freie Universität Berlin, FB Mathematik, an Editor of SCMJ, and an Editor of Notices from the ISMS.

The ISMS
(I) International Society for Mathematical Sciences
----- Contributions

Dear Colleagues and Friends,

In September 2007, we establish the following two funds.

(1) **International ISMS Prizes Fund**
in order to award the prizes for the original papers or survey works published in Scientiae Mathematicae Japonicae or Notices from the ISMS.

(2) **International Research Promoting Fund**
in order to promote and support international joint meetings by IVMS.

The contributions are classified into the following five categories.

(A) ¥ 500,000 (or $5,000) and above
(B) ¥ 100,000 (or $1,000) and above
(C) ¥ 50,000 (or $500) and above
(D) ¥ 10,000 (or $100) and above
(E) Less than ¥10,000 (or $100)

We deeply appreciate your generous contributions to support the above activities of our society.

Your remittance to the following accounts of ours will be much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai, Osaka 590-0075, Japan

(2) Through a bank, make remittance to the following account of JAMS.
A/C 94103518
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

Kiyoshi Iseki
Tadashige Ishihara

(Ⅱ) Election

(1) ELECTION of OFFICERS **(Term of office: Jan. 1, 2008～Dec. 31, 2010)**

With the deadline of Aug.20, 2007, we have accepted four candidates for Treasure, Secretary in charge of Publishing (SCMJ), Secretary in charge of Meetings (IVMS), and Secretary in charge of Prizes. As there is only one candidate for one seat, the confidence vote shall be done with the deadline of Oct 10, 2007.

(2) PRESIDENT ELECT **(Term of office: Jan. 1, 2008 ～June 30, 2009)**

There has been no candidate for President Elect. So, the seat is vacant until June 30 2009.

(3) ELECTION OF COUNCIL MEMBERS

After the 2008 Officers Election, which will be conducted from August 2007 through October 2007, the Board of Officers will nominate the new members and the confidence vote by the membership shall be conducted.

Confidence Vote

On August 20, 2007, we closed registration of candidates for Secretaries (Officers) who are to be elected based on Articles Ⅰ, Ⅳof newly enacted Bylaws 2007. The ISMS members are requested to vote, confidence (Yes) or non-confidence (No), to the following candidates. The vote can be made from the web: (http://www.jams.or.jp/hp/tohyo/Officers_tohyo_E.html), by e-mail to pgp7j@jams.jp, by fax to: 81-72-222-7987, or by airmail to: International Society for Mathematical Sciences, Minami Hanadaguchi, Sakai, Sakai-ku, Osaka 590-0075, Japan. The vote should be made between October 1, 2007 and October 10, 2007. When you vote by fax or airmail, please use the following as a ballot paper.

At the same time, the confidence vote of some council members for the vacant seats are to be done. These candidates are nominated by the Board of Officers according to the Article IV of Bylaws 2007.

Your Name and Membership No.			
E-mail Address			
(Officers)			
Treasurer	**Toshio Nishida**	**Yes**	**No**
Publishing	**Tadashige Ishihara**	**Yes**	**No**
Meetings (IVMS)	**Atsushi Yagi**	**Yes**	**No**
Prizes	**Masako Sato**	**Yes**	**No**
(Council Members)			
Jair Minoro Abe (Brazil)		**Yes**	**No**
Anthony To-Ming Lau (Canada)		**Yes**	**No**
Milan Vlach (Czech Republic)		**Yes**	**No**
Klaus Denecke (Germany)		**Yes**	**No**
Pal Dömösi (Hungar)		**Yes**	**No**
Vladimir V. Mazalov, (Russia)		**Yes**	**No**
Luis M. Sanchez Ruiz (Spain)		**Yes**	**No**
Steven J. Brams (U.S.A.)		**Yes**	**No**
Henryk Hudzik (Poland)		**Yes**	**No**
Congxin Wu (P.R. China)		**Yes**	**No**
Yoshisada Murotsu (Japan)		**Yes**	**No**
Yoshifumi Usami (Japan)		**Yes**	**No**
Hisashi Choda (Japan)		**Yes**	**No**
Koyu Uematsu (Japan)		**Yes**	**No**

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutioanl Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$300. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members do including the discount of the page charge. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$300)	☐Print ＋ Online (US$300)
University (Institution)		
Department		
Postal Address where SCMJ should be sent		
E-mail address		
Person in charge	Name: Signature:	
Payment Check one of the two.	☐Bank transfer	☐Credit Card (Visa, Master)
Name of Associate Membership	1.	
	2.	

Call for regular Members
ISMS Membership Dues from 2007

A new category "life member" has been established and can be applied for from 2005. An eligible member may become a life member by making a one-time payment of dues. A member who has been an ISMS member for ten years or more is eligible for a life member. The amounts of dues are : ¥70,000 for the domestic members, US$ 600 (€480)for the foreign members, and US$ 500 (€400)for the members in developing countries.

We have reduced the ISMS memership dues since 2001 and copies of the printed journal have not been distributed to the members, free of charge. Instead, we give User Name and Password to each member so that he/she can view or print out the full text of the papers published in SCMJ except papers in the international plaza from our Web site (http://www.jams.or.jp).

The Membership Dues for each category is as follows. Applications for the 3-year members can be made only in 2005 and in every three years.

Membership Dues for 2007

Membership	JAPAN	S-JAPAN	Foreign	S-Foreign	Developing
1-year	A1 ¥7,000	SA1 ¥3,500	F1 US$50 €40	SF1 US$30 €24	D1 US$30 €24
3-year	A3 ¥18,000	SA3 ¥9,000	F3 US$120 €96	SF3 US$60 €48	D3 US$70 €56
Life Member	Life ¥70,000	Life ¥70,000	FL US$600 €480	FL US$600 €480	DL US$500 €400

Category D is for those who reside in the countries of Eastern Europe, CIS or developing countries. Category S is for students and for the aged (older than 70). The figure 1 and 3 means a year and 3 years respectively.

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Payment Instructions

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.
(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance
(1) to our Post Office Transfer Account - 00930-3-73982 or
(2) to our account No.1565679 at SUMITOMO BANK, Sakai, Osaka, Japan.
All the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Membership Application Form (from 2007 September)

To determine what membership category you are eligible for, read "Join ISMS" on the inside of the back cover.

1. Name: Family Name, First Name, Middle Name (in this order)
2. Home Address
3. Name of Firm or Institution affiliation
4. Postal address to which correspondence should be sent
5. e-mail address
6. Telephone Number, Fax Number
7. Membership Category
8. Panel (Please choose one out of the following 14panels in the page 26 and write the panel number. You could choose one or more.)
9. Would you like to buy the printed copies of SCMJ, whose prices a year are US$60(6,000yen) for 1-year-members(A1, D1, S-A1, S-D1)and US$55(5,500yen) for 4-year-members(A4, D4, S-A4, S-D4) ? Type YES or NO.
10. If you apply for an aged member (70 years old or over), please type the year of your birth.
11. If you wish to be a student member, please verify.
12. Is your university (institution) an Academic or Institutional Member of the ISMS? Yes or No.
13. If the answer of 12 is Yes, please answer the following. Are you designated associate member by your university (institution)?
14. Date
15. Signature

For Japanese Applicants, please send two application forms, one in English and the <u>other in Japanese.</u>

I wish to enroll as a member of ISMS and will pay to International Society for Mathematical Sciences the annual dues upon presentation of an invoice. Copies of Mathematica Japonica, Scientiae Mathematicae and Scientiae Mathematicae Japonicae received as an ISMS member will be for my personal use and shall not be placed in institutional, university or other libraries or organizations, nor can membership subscriptions be used for library purposes.

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Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)**, which enjoyed an international reputation, for about sixty years in print and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The number of the annual total pages of the print version has been from 900 to 1,200 pages in six issues since January 1978. The whole number of SCMJ exceeds 240, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 90 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The submitted papers are received directly by the editors and are refereed quickly. The accepted papers are published online with no lead time after compiling or proofreading. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) The original papers and surveys of distinguished mathematical scientist appear in every issue of SCMJ. The section called "International Plaza" of SCMJ has very interesting expository papers written by the eminent mathematical scientist of the world. Presentations of recent research frontier including award lectures by the winners of the ISMS Prize or Shimizu Prize are made.
4) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.
5) The ISMS holds inter-regional videoconferences called **International Videoconference of Mathematical Sciences (IVMS)** via internet. There is no need for the participants to travel abroad.

Privileges to ISMS Members: (1) Free access (**including printing out**) to the online version of SCMJ, (2)Discounted price for the printed version of SCMJ (See **Table 1**), (3) Discounted page charges (See **Table 2**).

Privileges to Institutional Members: (1) Two associate members can be registered, free of charge, from an institution. (2) The discounted page charges (Table 2) are applied to the associate members.

Table 1: Subscription Price (from 2007)

	Individual 1-year mem.	Individual 3-year mem.	Institutional member	List Price
Print / year	￥6,000 US$60, €48	￥5,500 * US$55, €44	￥33,000 US$300, €240	￥45,000 US$400, €320
Online/year	Free	Free	—	—
Online+Print / year	￥6,000 US$60, €48	￥5,500 * US$55, €44	￥33,000 US$300, €240	￥45,000 US$400, €320

Postal charge is US$2 (€1.6) per issue. *In case three-year members make the payment at a time in advance, the price for 3 years is ￥15,000 (US$150, €120). The authors can buy a copy of the print version at a price of ￥1,200 (US$12) per issue including postage.

Table 2: Page Charge per printed page

	Individual/Associate Member	Non Member
Paper : P	￥3,850 (US$35, €28)	￥4,450 (US$43、€35)
TeX: T	￥2,200 (US$18, €14)	￥2,800 (US$26, €21)
ISMS style: Js	￥1,100 (US$8, €7)	￥1,700 (US$16, €14)

The above page charges include 20 offprints.

Table 3: Membership Dues for this year

Categories	Domestic	Overseas	Developing countries
1-year member (1A)	A1: ￥7,000	F1: US$50 , €40	D1: US$30, €24
3-year member (3A)	A3: ￥18,000	F3: US$120, €96	D3: US$70, €56
1-year students or aged (1S)	SA1: ￥3,500	SF1: US$30, €24	SD1: US$20, €16
3-year students or aged (3S)	SA3: ￥9,000	SF3: US$70, €56	SD3: US$50, €40
Life member* (L)	AL: ￥70,000	FL: US$600, €480	DL: US$500, €400

*The members who have been the ISMS members for more than 10 years are eligible for this category.The categories 1S and 3S are for students or persons over 70 years old.