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CONTENTS

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A Survey on Clones

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Abstract

Clones are sets of operations defined on the same set A which are closed under composition and contain all projection operations. Nowadays clones are understood in a more general sense. It turns out that clones of operations form many-sorted algebras. They belong to a variety of many-sorted algebras whose members are called abstract clones. In this sense not only operations, but also partial operations, sets of operations, terms, sets of terms, cooperations and relations form clones. The notion of a clone was first used in P. M. Cohn's book "Universal Algebra", but probably it goes back to Ph. Hall. The concept of a clone is one of the basic concepts of General Algebra and has applications in the theory of data bases and in other fields of Theoretical Computer Science.

Contents

1 Composition of Functions, Clones

First we consider clones of operations. Let A be a set and let $f : A^n \to A$ be an n-ary operation defined on A. By $O^n(A)$ we denote the set of all n-ary operations defined on A. The *n-ary projections* e_i^n on A are defined by $e_i^n(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A, n \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and $1 \leq i \leq n$. Let $O(A) := \bigcup\limits_{n \geq 1} O^n(A)$ be the set of all operations defined on A. Up to now we excluded nullary operations $f : A^0 = \{\emptyset\} \to A$ which exist if $A \neq \emptyset$. In many cases one can consider nullary operations as constant unary ones. If $f \in O^n(A)$ and $g_1, \ldots, g_n \in O^m(A)$, then we define a new m-ary operation $f(g_1, \ldots, g_n)$ by

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in A^m$. The operation $f(g_1, \ldots, g_n)$ is said to be the *composition* of f with $g_1, \ldots, g_n$. Since $O(A)$ is the collection of all operations defined on A, it is closed under composition and contains all projection operations. Arbitrary subsets $C \subseteq O(A)$ which have both properties, to be closed under composition and containing all projections, are called *clones*. Therefore, $O(A)$ is the greatest clone of operations defined on A. It is easy to see that $J(A)$, the set of all projection operations, is also a clone. It is the least clone of operations defined on A. If A is a finite set, then $O(A)$ is also called set of all functions of many-valued logic. If especially A is two-element, one may assume that $A = \{0, 1\}$. Operations defined on $\{0, 1\}$ are also called *Boolean operations*. Important examples of Boolean operations are *conjunction* ($\wedge$), *disjunction* ($\vee$), *negation* ($\neg$), *implication* ($\Rightarrow$) and *equivalence* ($\Leftrightarrow$) which are given by the tables

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\Rightarrow$	0	1
0	1	1
1	0	1

$\Leftrightarrow$	0	1
0	1	0
1	0	1

	$\neg x$
0	1
1	0

.

It is well-known that any Boolean operation can be generated by conjunction and negation and the projections or by disjunction and negation and the projections using composition.

2 Clones as Algebraic Structures

There are several ways to regard clones as algebraic structures. We want to begin with the so-called *Mal'cev operations* defined on sets of operations ([24]). To define them we need the concept of a *fictitious (fictive)* variable in an operation. Let $f \in O^n(A)$. Then we say that the i-th variable (better: the i-th place) of f is *essential* if there are n-tuples

$$\underline{a} = (a_1, \ldots, a_{i-1}, d, a_{i+1}, \ldots, a_n)$$

and

$$\underline{b} = (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n)$$

such that $d \neq c$ and $f(\underline{a}) \neq f(\underline{b})$. In the opposite case, the i-th variable (i-th place) of f is said to be *fictitious* (or *fictive* or *non-essential*). If the i-th variable of f is not fictitious, we say that f depends on the i-th variable.

A. I. Mal'cev considered the following four operations on $O(A)$: $*$, ξ, τ and Δ:

$$* : \quad O^n(A) \times O^m(A) \to O^{m+n-1}(A) \text{ defined by } (f, g) \mapsto f * g$$

with

$$(f * g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) := f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}),$$

for all $x_1, \ldots, x_{m+n-1} \in A$;

$$\xi : \quad O^n(A) \to O^n(A) \text{ defined by } f \mapsto \xi(f)$$

with

$$\xi(f)(x_1, \ldots, x_n) := f(x_2, \ldots, x_n, x_1);$$

$$\tau : \quad O^n(A) \to O^n(A) \text{ defined by } f \mapsto \tau(f)$$

with

$$\tau(f)(x_1, \ldots, x_n) := f(x_2, x_1, x_3, \ldots, x_n);$$

$$\Delta : \quad O^n(A) \to O^{n-1}(A) \text{ defined by } f \mapsto \Delta(f)$$

with

$$\Delta(f)(x_1, \ldots, x_{n-1}) \ := \ f(x_1, x_1, x_2, \ldots, x_{n-1})$$

for every $n > 1$ and for all $x_1, \ldots, x_n \in A$; and

$$\xi(f) \ = \ \tau(f) \ = \ \Delta(f) \ = \ f, \ \text{ if } \ n \ = \ 1.$$

We also use the nullary operation e_1^2 which picks out the binary projection on the first coordinate; as defined above, $e_1^2(a_1, a_2) = a_1$, for all $a_1, a_2 \in A$. With these five operations, we obtain an algebra $(O(A);\ *, \xi, \tau, \Delta, e_1^2)$ of type $(2, 1, 1, 1, 0)$. This algebra is called the *full iterative algebra* on A. Clearly, the operation $*$ is associative. Therefore the full iterative algebra is a semigroup with additional unary and an nullary operation. It is not difficult to see that the universe of any subalgebra of the algebra $(O(A);\ *, \xi, \tau, \Delta, e_1^2)$ is a clone and that conversely each clone is closed under the operations $*, \xi, \tau, \Delta, e_1^2$.

Another algebraic approach to clones is via *many-sorted algebras*. To define many-sorted algebras one needs to consider *S-sorted sets* $A = (A_s)_{s\in S}$, i.e. S-indexed families of sets where the set S is called set of sorts. For any S-sorted sets $A = (A_s)_{s\in S}, B = (B_s)_{s\in S}$ the inclusion $A \subseteq B$ means $A_s \subseteq B_s$ for every $s \in S$. Union and intersection are defined by $A \cup B := (A_s \cup B_s)_{s\in S}$ and $A \cap B := (A_s \cap B_s)_{s\in S}$. A *sorted relation* $\theta := (\theta_s)_{s\in S}$ on an S-sorted set $A = (A_s)_{s\in S}$ is an S-sorted family of relations such that for each $s \in S$, θ_s is a relation on A_s. If each θ_s is an equivalence relation on A_s one speaks of a *sorted equivalence*. Let $(\theta_s)_{s\in S}$ be a sorted equivalence on $A = (A_s)_{s\in S}$. The *sorted quotient set* is $A/\theta := (A_s/\theta_s)_{s\in S}$ where $A_s/\theta_s := \{[a]_s \mid a \in A_s\}$, $s \in S$. A *sorted mapping* $\varphi : A \to B$ from an S-sorted set $A = (A_s)_{s\in S}$ to an S-sorted set $B = (B_s)_{s\in S}$ is an S-sorted family $\varphi : (\varphi_s)_{s\in S}$ of mappings $\varphi_s : A_s \to B_s, (s \in S)$. Now let $\mathbb{N}^+$ be the set of sorts and let $(O^n(A))_{n\in\mathbb{N}^+}$ be a sorted set, where the different sorts are the sets of n-ary operations on A. As operations we choose the composition operations $S^n_m, m, n \in \mathbb{N}^+$, defined by

$$S^n_m : O^n(A) \times (O^m(A))^n \to O^m(A)$$

with $(f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$ where $f(g_1, \ldots, g_n)$ was already defined in section 1. As nullary operations we choose the projections $e_i^n, i \leq n, n \in \mathbb{N}^+$. Then we obtain the many-sorted algebra

$$\mathcal{C}lone A := ((O^n(A))_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (e_i^n)_{1\leq i\leq n, n\in\mathbb{N}^+}).$$

Clearly $O(A) = \bigcup\limits_{n\in\mathbb{N}^+} O^n(A)$ is a clone and for each sorted subalgebra $\mathcal{S}$ of $\mathcal{C}lone A$ the union $\bigcup\limits_{n\in\mathbb{N}^+} S^n$ is a clone and conversely, each clone of operations on A can be obtained as union of the universes of some many-sorted subalgebra of $\mathcal{C}lone A$.

3 The Galois Connection $Pol - Inv$

There is a completely different way to define clones: by relations. We start again with a fixed base set A, and with as one of our sets of objects the set $O(A)$ of all operations on A. Then we consider an interconnection between operations from $O(A)$ and relations $\theta \subseteq A^h$, namely that an operation could be compatible with, or preserve, a binary relation. We denote by $R^h(A)$ the set of all h-ary relations defined on A, and by $R(A) \ = \ \bigcup\limits_{h\geq 1} R^h(A)$ the set of all finitary relations defined on A.

We say $f \in O^n(A)$ *preserves* the $h-$ary relation $\rho \in R(A)$, if whenever

$$(a^1_1, \ldots, a^1_h) \in \rho, \ \ \ldots, \ (a^n_1, \ldots, a^n_h) \in \rho,$$

it follows that also

$$(f(a^1_1, \ldots, a^n_1), \ \ \ldots, \ f(a^1_h, \ldots, a^n_h)) \in \rho.$$

This connection between operations and relations determines a mapping which associates to any relation a set of operations. For any relation $\rho \in R(A)$, we can consider the set of all operations from $O(A)$ which preserve ρ. This set will be denoted by $Pol_A\rho$, i.e.,

$$Pol_A\rho \ = \ \{f \mid f \in O(A) \text{ and } f \text{ preserves } \rho\}.$$

We remark that in fact $Pol_A\rho$ is a clone on A.

Example 3.1 Let A be the set $\{0, 1\}$.

1. Let ρ be the unary relation $\{0\}$. Notice that unary relations on a set are simply subsets of this set. Then $Pol_A\{0\}$ is the set of all Boolean functions which preserve $\{0\}$, so

$$Pol_A\{0\} \ = \ \{f \in O(A) \mid f(0, \ldots, 0) \ = \ 0\}.$$

2. Let

$$\alpha \ = \ \{(a, b, c, d) \in A^4 \mid a + b \ = \ c + d\},$$

where $+$ is the addition modulo 2. An n-ary operation f on A is called *linear*, if there are elements $a_1, \ \ldots, \ a_n, c \in \{0, 1\}$ such that

$$f(x_1, \ldots, x_n) \ = \ a_1x_1 + \ \cdots \ + a_nx_n + c$$

for all $x_1, \ldots, x_n \in \{0, 1\}$, where again $+$ is the addition modulo 2. It can be shown that a Boolean function f is linear iff it preserves α. Thus $Pol_A\alpha$ is the set of all linear Boolean functions.

These examples show how, for any given relation ρ on A, we can look for the set of all operations which preserve ρ. In the other direction, for a given operation $f \in O(A)$ one can look for the set of all relations from $R(A)$ which are preserved by f. Such relations are called *invariants* of f, and the set of all such invariants is denoted by Inv_Af, that is,

$$Inv_Af \ = \ \{\rho \in R(A) \mid f \text{ preserves } \rho\}.$$

We can also extend the maps Pol_A and Inv_A to sets of relations or functions, respectively. If $F \subseteq O(A)$ is a set of operations on A, we define Inv_AF to be the set of all relations which are invariant for all $f \in F$, and similarly for a set $Q \subseteq R(A)$ of relations on A, we denote by Pol_AQ the set of all operations which preserve every relation $\rho \in Q$.

These maps Pol_A and Inv_A, between sets of relations and sets of operations on A, show the basic idea of a Galois-connection, between two sets of objects.
The following theorem of Pöschel and Kalužnin ([31]) shows that in this example, we have the two important properties that will be used shortly to define a Galois-connection.
These properties are that our maps between the two sets of objects map larger sets to smaller images, and that mapping a set twice returns us to a set which contains the starting set.

Theorem 3.2 ([31]) *The following interconnections hold between sets of the form Pol_AQ and Inv_AF, for $Q \subseteq R(A)$ and $F \subseteq O(A)$:*

$$\begin{array}{llll} (i) & F_1 \subseteq F_2 \subseteq O(A) & \Rightarrow & Inv_AF_1 \supseteq Inv_AF_2, \\ & Q_1 \subseteq Q_2 \subseteq R(A) & \Rightarrow & Pol_AQ_1 \supseteq Pol_AQ_2; \\ (ii) & F \subseteq Pol_AInv_AF, & & \\ & Q \subseteq Inv_APol_AQ. & & \end{array}$$

This theorem forms a model for the definition of a Galois-connection. In our case we consider the Galois-connection (Pol, Inv). From (i) and (ii) one easily obtains that the products $PolInv$ and $InvPol$ are closure operators. The closed sets under $PolInv$ are precisely the sets of operations of the form $PolR$ for a set R of relations defined on A. We mentioned already that for a single relation ϱ, the set $Pol\varrho$ is a clone. Therefore also sets of the form $PolR$ for sets R of relations are clones. The set of all fixed points with respect to the closure operator $PolInv$ forms a complete lattice $\mathcal{L}_A$. The greatest element of this lattice is the clone $O(A)$ of all operations defined on A. The clone $O(A)$ can be written as $Pol\Delta_A$ for the diagonal $\Delta_A := \{(a, a) \mid a \in A\}$ since every operation preserves the diagonal relation. The least element of this lattice is the set J_A of all projections. The fixed points with respect to the closure operator $InvPol$ are called relational clones. They form another complete lattice which is dually isomorphic to $\mathcal{L}_A$. The duality between clones of operations and clones of relations defined on the same set A is an important tool for the study of $\mathcal{L}_A$.

4 An Axiomatic Approach to Clones

It is routine work to check that the many-sorted algebra $\mathcal{C}loneA$ satisfies the following three identities:

(C1) $\tilde{S}^p_m(\tilde{Z}, \tilde{S}^n_m(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n_m(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n))$
$\approx \tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n),\ (m, n, p = 1, 2, \ldots),$

(C2) $\tilde{S}^n_m(e^{\tilde{n}}_i, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_i,\ (m = 1, 2, \ldots, 1 \leq i \leq n, n \in \mathbb{N}^+),$

(C3) $\tilde{S}^n_n(\tilde{Y}, e^{\tilde{n}}_1, \ldots, e^{\tilde{n}}_n) \approx \tilde{Y},\ (n = 1, 2, \ldots).$

Here $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n$, are variables for terms, $\tilde{S}^n_m, \tilde{S}^p_m, \tilde{S}^p_n, \tilde{S}^n_n$ are operation symbols and $e^{\tilde{n}}_i$ are symbols for variables.
The axiom (C1) is called the *superassociative law*. It generalizes the associative law which we obtain from (C1) if we set for m, n and p the integer 1. The identities (C1), (C2), (C3) define a variety K_0 of $\mathbb{N}^+$-sorted algebras of clone type which is called the *variety of all clones*. The $\mathbb{N}^+$-sorted algebra *cloneA* belongs to K_0. To make a difference between clones of operations and the members of K_0 we call the elements of the variety K_0 *abstract clones* and clones of operations *concrete* ones. We mentioned already that every concrete clone is an abstract one. Conversely it was proved that every abstract clone is isomorphic to a concrete one. This result generalizes the theorem of Cayley about the representation of groups and semigroups by permutation groups and transformation semigroups, resp. Our many-sorted clones correspond to algebraic theories, particular categories, in the sense of F. W. Lawvere ([22]).
The set $O^n(A)$ of all n-ary operations defined on A is closed under the composition operation $S^n := S^n_n : (O^n(A))^{n+1} \to O^n(A)$. Then for any $n \geq 1$ we obtain an algebra $(O^n(A); S^n)$ of type $\tau = (n + 1)$ which satisfies axiom (C1). Algebras with one n-ary fundamental operation satisfying (C1) were considered by K. Menger ([26]) and are called *Menger algebras of rank n*. The algebra $(O^n(A); S^n, e^n_1, \ldots, e^n_n)$ of type $\tau = (n + 1, 0, \ldots, 0)$ is an example of a unitary Menger algebra of rank n. It satisfies (C3) and axioms corresponding to (C2) and (C3). For a survey on Menger algebras see [13].

5 Clones of Terms and Clones of Polynomials

In this section we want to give another important class of clones. We define *terms of type* τ using an indexed sequence $(f_i)_{i\in I}$ of operation symbols and individual variables from an alphabet X. Let $X_n := \{x_1, \ldots, x_n\}$ be a finite alphabet and let $X := \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. To every operation symbol f_i there belongs an integer n_i as its *arity*. The type of the formal language which we want to define is the indexed set $(n_i)_{i\in I}$ of the arities. We define n-ary terms of type τ as follows:

Definition 5.1 *Let $n \geq 1$. The n-ary terms of type τ are defined in the following inductive way:*

(i) *Every variable $x_i \in X_n$ is an n-ary term.*

(ii) *If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.*

(iii) The set $W_\tau(X_n) = W_\tau(x_1, \ldots, x_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

Remark 5.2 *1. It follows immediately from the definition that every n-ary term is also k-ary, for $k > n$.*

2. Our definition does not allow nullary terms. This could be changed by adding a fourth condition to the inductive definition, stipulating that every nullary operation symbol of our type is an n-ary term.

We denote by $W_\tau(X)$ the set of all terms of type τ over the countably infinite alphabet X:

$$W_\tau(X) \quad = \quad \bigcup_{n\geq 1} W_\tau(X_n).$$

To define *polynomials* we need one more set C, a set of nullary operation symbols. Then to the definition of terms we add a requirement saying that every nullary operation symbol is a term of type τ. Let $P_\tau(X)$ be the set of all polynomials of type τ. On the $\mathbb{N}^+$-sorted sets $(W_\tau(X_n))_{n\in\mathbb{N}^+}$ and $(P(X_n))_{n\in\mathbb{N}^+}$ we define composition operations $\bar{S}^n_m$ as follows:

Definition 5.3 *Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ. Then the composition operations $\bar{S}^n_m$ (for terms) are inductively defined by the following steps:*

(i) *If $x_j \in X_n$ is a variable and $t_1, \ldots, t_n \in W_\tau(X_m)$, then*
$\bar{S}^n_m(x_j, t_1, \ldots, t_n) := t_j$, for $1 \leq j \leq n$;

(ii) *If* $f_i(s_1,\ldots,s_{n_i})$ *is a composite term, then* $\bar{S}^n_m(f_i(s_1,\ldots,s_{n_i}),t_1,\ldots,t_n) := f_i(\bar{S}^n_m(s_1,t_1,\ldots,t_n),\ldots,\bar{S}^n_m(s_{n_i},t_1,\ldots,t_n))$.

The composition for polynomials can be defined in a similar way. In both cases, for terms and for polynomials we obtain $\mathbb{N}^+$-sorted algebras $\mathcal{C}lone\tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}; (\bar{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\le i\le n, n\in\mathbb{N}^+})$ and $\mathcal{P}clone\tau := ((P_\tau(X_n))_{n\in\mathbb{N}^+}; (\bar{S}^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{1\le i\le n, n\in\mathbb{N}^+})$, the term clone and the polynomial clone of type τ. Both algebras satisfy (C1), (C2), (C3). These algebras are free with respect to the variety of many-sorted algebras defined by (C1), (C2), (C3), freely generated by the set $\{f_i(x_1,\ldots,x_{n_i}) \mid i\in I\}$ in the first case and by $\{f_i(x_1,\ldots,x_{n_i}) \mid i\in I\}\cup C$ in the second one.
In [23] the following generalization of the composition operation for terms was defined:

Definition 5.4 (i) If $t = x_i, 1\le i\le n,$ then $S^n_g(x_i,t_1,\ldots,t_n) := t_i$ for $t_1,\ldots,t_n \in W_{\tau_n}(X)$.

(ii) If $t = x_i,\ n < i$, then $S^n_g(x_i,t_1,\ldots,t_n) := x_i$.

(iii) If $t = f_i(s_1,\ldots,s_n)$, then

$$S^n_g(t,t_1,\ldots,t_n) := f_i(S^n_g(s_1,t_1,\ldots,t_n),\ldots,S^n_g(s_n,t_1,\ldots,t_n)).$$

It turns out that this generalized composition of terms satisfies (C1) and (C3), but (C2) is not satisfied ([9]).
Now we consider one more important example of a clone, the clone of *term operations* of an algebra.
Terms are formal expressions on our formal language of type τ. In order to formulate statements using terms which are true or false in a given algebra $\mathcal{A}$, we have to evaluate the variables in the terms by elements of the concrete set A, and we have to interpret the operation symbols by concrete operations on this set. It is this process which produces term operations from terms. We shall continue to denote by X the countably infinite set $\{x_1,x_2,x_3,\ldots\}$ of variables.

Definition 5.5 *Let* $\mathcal{A} := (A; f_i^{\mathcal{A}})_{i\in I})$ *be an algebra of type* τ *and let* t *be an* n*-ary term of type* τ *over* X*. Then* t *induces an* n*-ary operation* $t^{\mathcal{A}}$ *on* $\mathcal{A}$*, called the term operation induced by the term* t *on the algebra* $\mathcal{A}$*, via the following steps:*

1. *If* $t = x_i \in X_n$*, then* $t^{\mathcal{A}} = x_j^{\mathcal{A}} = e_j^{n,A}$*; here* $e_j^{n,A}$ *is the* n*-ary projection on* A *defined by* $e_j^{n,A}(a_1,\ldots,a_n) = a_j$ *for all* $a_1,\ldots,a_n\in A$.

2. *If* $t = f_i(t_1,\ldots,t_{n_i})$ *is an* n*-ary term of type* τ*, and* $t_1^{\mathcal{A}},\ldots,t_{n_i}^{\mathcal{A}}$ *are the term operations which are induced by* $t_1,\ldots,t_{n_i}$*, then* $t^{\mathcal{A}} = f_i^A(t_1^{\mathcal{A}},\ldots,t_{n_i}^{\mathcal{A}})$.

In part (ii) of this definition, the right hand side of the equation refers to the composition of operations, so that

$$t^{\mathcal{A}}(a_1,\ldots,a_n) := f_i^A(t_1^{\mathcal{A}}(a_1,\ldots,a_n),\ldots,t_{n_i}^{\mathcal{A}}(a_1,\ldots,a_n)),$$

for all $a_1,\ldots,a_n\in A$.

Since any concrete operation on a set has an arity attached to it, if we want to induce such an operation from a term, we must also have an arity attached to the term. It is for this reason that our definition of terms begins in the definition with terms of each fixed arity n.
Roughly speaking, if t is an n-ary term and $\mathcal{A}$ is an algebra of type τ, then to obtain the operation $t^{\mathcal{A}}$ we substitute elements from the set A for the variables occurring in t, and interpret the operation symbols f_i for $i\in I$ as the corresponding fundamental operations f_i^A. More precisely, we map the variables x_j occurring in t to elements $a_1,\ldots,a_n$ of A by a function $f : X\to A$, and then our term operation $t^{\mathcal{A}}$ is the unique extension $\hat{f} : \mathcal{F}_\tau(X)\to\mathcal{A}$. We will denote by $W_\tau(X_n)^{\mathcal{A}}$ the set of all n-ary term operations of the algebra $\mathcal{A}$, and by $W_\tau(X)^{\mathcal{A}}$ the set of all (finitary) term operations on $\mathcal{A}$.

There is another way to obtain the set $W_\tau(X)^{\mathcal{A}}$ of all term operations on $\mathcal{A}$, using clone operations.

Theorem 5.6 *Let* $\mathcal{A} = (A; (f_i^A)_{i\in I})$ *be an algebra of type* τ*, and let* $W_\tau(X)$ *be the set of all terms of type* τ *over* X*. Then* $W_\tau(X)^{\mathcal{A}}$ *is a clone on* $\mathcal{A}$*, called the term clone of* $\mathcal{A}$*. Moreover, the clone* $W_\tau(X)^{\mathcal{A}}$ *is generated by the set of all fundamental operations of the algebra* $\mathcal{A}$*. That is,* $W_\tau(X)^{\mathcal{A}} = \langle\{f_i^A | i\in I\}\rangle$.

The clone of term operations of the algebra $\mathcal{A}$ will be also denoted by $T(\mathcal{A})$.
In a similar way to each polynomial and to each algebra of the corresponding type there belongs the polynomial operation induced on the algebra by that polynomial. All polynomial operations of a given algebra form a clone which is generated by the fundamental operations together with the interpretation of the constant symbols as constant operations of the algebra.

It turns out that the mapping which maps each term of type τ to the term operation which is induced on an algebra $\mathcal{A}$ of type τ is an $\mathbb{N}^+$-sorted homomorphism from $\mathcal{C}lone\tau$ to the clone of all term operations of $\mathcal{A}$ regarded as $\mathbb{N}^+$-sorted algebra. In fact, one has

$$(\bar{S}^n_m(t, t_1, \ldots, t_n))^{\mathcal{A}} = S^n_m(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$$

for all $t \in W_\tau(X_n), t_1, \ldots, t_m \in W_\tau(X_m)$.
This mapping is not injective. If two terms s, t are mapped to the same operation $s^{\mathcal{A}} = t^{\mathcal{A}}$, then $s \approx t$ is an identity in $\mathcal{A}$.

6 Primal Algebras

One aim of algebraic research is to classify all algebras. Many group theoretists have worked hard to classify all simple finite groups (see [17]). If the classification principle is the location of $T(\mathcal{A})$ in the lattice $\mathcal{L}_A$ of all clones of operations defined on A, one has to begin with algebras having the property $T(\mathcal{A}) = O(A)$. For finite universes such algebras are called primal. In a primal algebra any operation defined on A is expressible as term operation of the algebra. A finite algebra is called functionally complete if the clone of all polynomial operations of $\mathcal{A}$ is equal to $O(A)$.
Primality of $\mathcal{A}$ thus means that for every $n \geq 1$ and every operation $f : A^n \to A$, there exists a term operation $t^{\mathcal{A}}$ of $\mathcal{A}$ such that $f = t^{\mathcal{A}}$; so for all n-tuples $(a_1, \ldots, a_n) \in A^n$ the equation

$$f(a_1, \ldots, a_n) = t^{\mathcal{A}}(a_1, \ldots, a_n)$$

is satisfied. Functional completeness of an algebra $\mathcal{A}$ means that for every $f \in O(A)$ there exists a polynomial operation $p^{\mathcal{A}}$ on $\mathcal{A}$ with $f = p^{\mathcal{A}}$. This property is a generalization of the *interpolation* property in ring theory, where an arbitrary unary operation is interpolable by a polynomial over a ring.
It follows from the definition that any primal algebra is functionally complete.
It is very easy to see in this way that the two-element Boolean algebra is primal. We list here some more examples of primal and functionally complete algebras.

Example 6.1 1. For $k \geq 2$, let $\mathcal{A}_k = (\{0, \ldots, k-1\}; min, g)$ be the type $(2, 1)$ algebra with min the minimum with respect to the usual order of the natural numbers $0, 1, \ldots, k-1$ and g defined by

$$g(x) = \begin{cases} x+1 & \text{if} \quad x \neq k-1 \\ 0 & \text{if} \quad x = k-1. \end{cases}$$

E. L. Post showed in [29] that $\mathcal{A}_k$ is primal. (Note that for $k = 2$, this is the algebra $(\{0, 1\}; \wedge, \neg)$.)

2. For $k \geq 2$, let $\mathcal{B}_k = (\{0, \ldots, k-1\}; /)$ be the type (2) algebra with

$$x/y = \begin{cases} 0 & \text{if} \quad x = y = k-1 \\ min(x, y) + 1 & \text{otherwise.} \end{cases}$$

The algebras $\mathcal{B}_k$ were shown to be primal by D. L. Webb in [39].

3. Let $\mathcal{A} = (A; \cdot, g)$ be an algebra of type $(2, 1)$ in which there is an element $0 \in A$ such that

(i) $(A \setminus \{0\}; \cdot)$ is a group,

(ii) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in A$, and

(iii) g is a cyclic permutation on A.

This algebra was shown to be primal by A. L. Foster in [14].

Primal algebras have a nice characterization by majority terms. Under a majority term we understand a ternary term in the language of the algebra $\mathcal{A}$ which satisfies the identities

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Then as an application of a famous theorem by Baker and Pixley ([1]) one has:

Theorem 6.2 *A finite algebra $\mathcal{A}$ is primal iff there is a majority term which induces a term operation on $\mathcal{A}$ and $\mathcal{A}^2$ has only itself and the diagonal Δ_A as subalgebras.* ■

Varieties of algebras are classes of algebras of the same type which are closed under formation of arbitrary direct products, subalgebras and homomorphic images, or, equivalently, which are model classes of sets of equations. A variety is called *congruence distributive* if the congruence lattices of all its algebras are distributive and *congruence permutable* if any two congruences of any algebra of the variety are permutable. Congruence distributive and congruence permutable varieties are said to be *arithmetical*. An algebra $\mathcal{A}$ is called *simple* if it has only the trivial congruences Δ_A and $A \times A$.
Primal and functionally complete algebras have the following properties:

Proposition 6.3 (Prop. 10.5.7 in [8])

(i) *Let $\mathcal{A}$ be a primal algebra. Then*

1. *$\mathcal{A}$ has no proper subalgebras,*
2. *$\mathcal{A}$ has no non-identical automorphisms,*
3. *$\mathcal{A}$ is simple, and*
4. *$\mathcal{A}$ generates an arithmetical variety.*

(ii) *Every functionally complete algebra is simple.*

It turns out that the four properties of primal algebras given in the previous Proposition are sufficient to characterize primal algebras. An additional characterization was given by H. Werner in [40] using the so-called ternary discriminator term:

$$t(x,y,z) = \begin{cases} z & \text{if } x = y \\ x & \text{otherwise.} \end{cases}$$

Theorem 6.4 *For a finite algebra $\mathcal{A}$ the following propositions are equivalent:*

(i) *$\mathcal{A}$ is primal.*

(ii) *$\mathcal{A}$ generates an arithmetical variety, has no non-identical automorphisms, has no proper subalgebras and is simple.*

(iii) *There is a ternary discriminator term which induces a term operation on $\mathcal{A}$, and the algebra $\mathcal{A}$ has no proper subalgebras and no non-identical automorphisms.*

Functionally complete algebras can be characterized also by the ternary discriminator:

Theorem 6.5 (Corr. 10.5.9 in [8]) *A finite algebra $\mathcal{A}$ is functionally complete if and only if the ternary discrimibnator t induces a polynomial operation on $\mathcal{A}$.*

7 Clones of Partial Operations

An n-ary operation on set A is called *partial* if it is defined on its domain $domf \subseteq A^n$, i.e. if $f(a_1, \ldots, a_n)$ exists for all $(a_1, \ldots, a_n) \in domf$. Let $P^n(A) := \{f : A^n \multimap A\}$ be the set of all n-ary partial operations on set A and let $P(A) := \bigcup\limits_{n \geq 1} P^n(A)$ be the set of all partial oeations on A.

Composition of partial operations can also be described by operations $\bar{S}^n_m$ as follows:

$$\bar{S}^n_m(f, g_1, \ldots, g_n) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m)$ for which $g_1, \ldots, g_n$ are defined and for which the values $b_1 = g_1(a_1, \ldots, a_m), \ldots, b_n = g_n(a_1, \ldots, a_m)$ form an n-tuple $(b_1, \ldots, b_n)$ belonging to the domain of f. Then one can consider the many-sorted algebra $((P^n(A))_{n \geq 1}; (\bar{S}^n_m)_{m,n \geq 1})$.
If we add a family of nullary operation symbols, corresponding to the total projections, then we obtain a many-sorted algebra

$$P - cloneA := ((P^n(A))_{n \geq 1}; (\bar{S}^n_m)_{m,n \geq 1}, (e^n_i)_{n \geq 1, 1 \leq i \leq n}).$$

This algebra satisfies (C1) and (C3). The equation (C2) is not satisfied. One can check that the following identities are satisfied:

(R) $\bar{S}^n_n(f, \bar{S}^2_n(e^2_1, e^n_1, g), \ldots, \bar{S}^2_n(e^2_1, e^n_n, g)) \approx \bar{S}^2_n(e^2_1, f, g)$,

(L1) $\bar{S}^1_n(e^1_1, f) = f$,

(L2) $\bar{S}^2_n(e^2_1, f, e^n_i) = f, 1 \leq i \leq n$,

(L3) $\bar{S}^2_n(e^2_1, \bar{S}^2_n(e^2_1, f, g), h) \approx \bar{S}^2_n(e^2_1, f, \bar{S}^2_n(e^2_1, g, h))$,

(L4) $\bar{S}^n_m(e^n_{i+1}, g_n, g_0, \ldots, g_{n-1}) \approx \bar{S}^n_m(e^n_i, g_1, \ldots, g_n), 1 \leq i \leq n$,

(L5) $\bar{S}^2_m(e^2_1, g_1, \bar{S}^n_m(e^n_1, g_1, \ldots, g_n))$.

In [2] a many-sorted algebra of the corresponding type was called an abstract P-clone algebra if (C1), (R), (L1), (L2), (L3), (L4), (L5) are satisfied. The algebra $P - cloneA$ is an abstract P-clone algebra. P-clone algebras consisting of partial operations are called concrete P-clone algebras. Answering to a question of H. J. Hoehnke (posed on occasion of the 4-th Conference for Young Algebraists, Potsdam 1988), F. Börner proved in [2] that any abstract P-clone algebra is isomorphic to a subdirect product of concrete ones.

8 Clones of Cooperations

Let A be a non-empty set. For each $n \geq 1$ we denote the n-th copower of A, that is the union of n disjoint copies of A by $A^{\sqcup n}$. Specifically, $A^{\sqcup n} := \{1, \ldots, n\} \times A$, and an element (i, a) corresponds to the element a in the i-th copy of A. An n-ary cooperation on A is then a mapping $f : A \to A^{\sqcup n}$. Each n-ary cooperation is uniquely determined by a pair (f_1, f_2) of mappings $f_1 : A \to \{1, \ldots, n\}$ and $f_2 : A \to A$.

Let $cO^{(n)}_A(A)$ be the set of all n-ary cooperations defined on A and let $cO_A := \bigcup_{n\geq 1} cO^{(n)}_A$ be the set of all cooperations defined on A. If $f \in cO^{(n)}_A$ and $g_1, \ldots, g_n \in cO^{(k)}_A$, then we define a k-ary cooperation $f[g_1, \ldots, g_n] : A \to A^{\sqcup k}$ by

$$a \mapsto ((g_{(f_1)(a)})_1(f_2(a)), (g_{(f_1)(a)})_2(f_2(a)))$$

for all $a \in A$. The cooperation $f[g_1, \ldots, g_n]$ is called the composition of f and $g_1, \ldots, g_n$. Instead of $f[g_1, \ldots, g_n]$ we will also write $comp^n_k(f, g_1, \ldots, g_n)$. The injections $\imath^{n,A}_i$ are special cooperations which are defined by $\imath^{n,A}_i : A \to A^{\sqcup n}$ with $a \mapsto (i, a)$ for $1 \leq i \leq n$. Then we get a many-sorted algebra

$$((cO^{(n)}_A)_{n\geq 1}; (comp^n_k)_{k,n\geq 1}, (\imath^{n,A}_i)_{1\leq i\leq n}).$$

In [5] it is mentioned that this many-sorted algebra is an abstract clone, i.e. satisfies the clone axioms (C1),(C2),(C3):

All 16 binary Boolean cooperations are given by the following table:

	$\imath^2_1$	$\imath^2_2$	c^1_0	c^2_0	c^3_0	c^4_0	c^1_1	c^2_1
0	(1,0)	(2,0)	(1,0)	(2,0)	(1,0)	(2,0)	(1,1)	(2,1)
1	(1,1)	(2,1)	(1,0)	(2,0)	(2,0)	(1,0)	(1,1)	(2,1)
	c^3_1	c^4_1	f	d	g_2	h_{10}	h_8	h_7
0	(1,1)	(2,1)	(1,1)	(1,0)	(1,1)	(2,1)	(2,0)	(2,1)
1	(2,1)	(1,1)	(1,0)	(2,1)	(2,0)	(1,0)	(1,1)	(2,0)

All clones of Boolean cooperations were determined in [5] (See also [10], [11]).
Let $I_{\{0,1\}}$ be the set of all injections, i.e. $I_{\{0,1\}} = \bigcup_{n\geq 1} I^{(n)}_{\{0,1\}}$. We denote the negation by $\neg$.

$\mathbf{C_{0c}} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} \mid \exists i \in \{1, \ldots, n\}(f(0) = (i, 0))\}$,

$\mathbf{C_{1c}} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} \mid \exists i \in \{1, \ldots, n\}(f(1) = (i, 1))\}$,

$\mathbf{D_c} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ (f_1(a) = f_1(\neg a) \textit{ and } f_2(a) = \neg f_2(\neg a), a \in \{0,1\}\}$,

$\mathbf{M_c} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ (a_1 \leq a_2 \Rightarrow f_1(a_1) = f_1(a_2), f_2(a_1) \leq f_2(a_2), \ a_1, a_2 \in \{0,1\}\}$,

$\mathbf{L_c} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i \in \{1,\ldots,n\}, \ \exists a_0, a_1 \in \{0,1\}, \ \forall x \in \{0,1\}, \ (f(x) = (i, a_0 +_2 a_1 x))\}$,

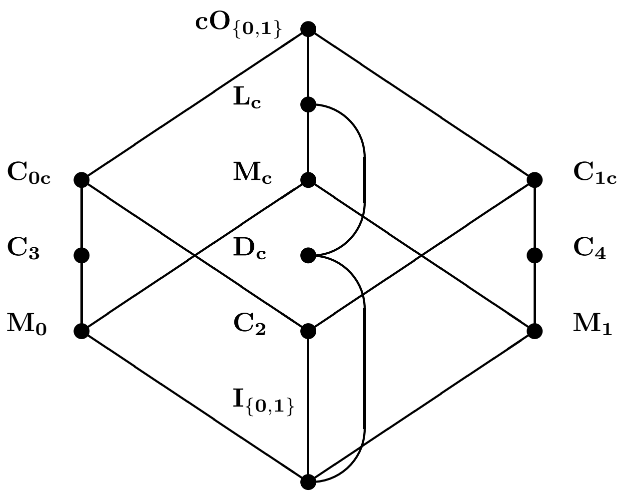
$\mathbf{C_2} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i,j \in \{1,\ldots,n\}, f(0) = (i,0), \ f(1) = (j,1)\} = \mathbf{C_{0c}} \cap \mathbf{C_{1c}}$,

$\mathbf{C_3} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i,j \in \{1,\ldots,n\}, f(0) = (i,0), \ f(1) = (j,0)\} \cup I_{\{0,1\}}$,

$\mathbf{C_4} = \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i,j \in \{1,\ldots,n\}, f(0) = (i,1), \ f(1) = (j,1)\} \cup I_{\{0,1\}}$,

$\mathbf{M_0} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i \in \{1,\ldots,n\}, f(0) = (i,0), \ f(1) = (i,0)\} \cup I_{\{0,1\}} = \mathbf{C_3} \cap \mathbf{M_c}$ and

$\mathbf{M_1} := \bigcup_{n\geq 1} \{f \in cO^{(n)}_{\{0,1\}} | \ \exists i \in \{1,\ldots,n\}, f(0) = (i,1), \ f(1) = (i,1)\} \cup I_{\{0,1\}} = \mathbf{C_4} \cap \mathbf{M_c}$.



9 Clones of Sets of Operations, Sets of Terms and of Relations

Any subset of the set $W_\tau(X_n)$ of all n-ary terms of type τ , i.e. any element of the power set $\mathcal{P}(W_\tau(X_n))$ is called a *tree language*. Tree languages may be accepted or not by *tree recognizers*. For tree languages one may define the following composition operations

$$\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times \mathcal{P}(W_\tau(X_m))^n \to \mathcal{P}(W_\tau(X_m))$$

inductively by the following steps:

Definition 9.1 *Let $m,n \in \mathbb{N}^+ (:= \mathbb{N} \setminus \{0\})$ and let $B \in \mathcal{P}(W_\tau(X_n))$ for $B_1,\ldots,B_n$ $\in \mathcal{P}(W_\tau(X_m))$*

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}^n_m(\{x_j\}, B_1,\ldots,B_n) := B_j$.

(ii) If $B = \{f_i(t_1,\ldots,t_{n_i})\}$, and if we assume that $\hat{S}^n_m(\{t_j\}, B_1,\ldots,B_n)$ for $1 \leq j \leq n$ are already defined, then $\hat{S}^n_m(\{f_i(t_1,\ldots,t_{n_i})\}, B_1,\ldots,B_n) := \{f_i(r_1,\ldots,r_{n_i}) \mid r_j \in \hat{S}^n_m(\{t_j\}, B_1,\ldots,B_n)$ for $1 \leq j \leq n_i\}$.

(iii) If B is an arbitrary subset of $W_\tau(X_n)$, we define

$$\hat{S}^n_m(B, B_1,\ldots,B_n) := \bigcup_{b\in B} \hat{S}^n_m(\{b\}, B_1,\ldots,B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}^n_m(B, B_1, \ldots, B_n) = \emptyset$. Then we may consider the many-sorted algebra

$$\mathcal{P}ow - clone\ \tau := ((\mathcal{P}(W_\tau(X_n)))_{n\in\mathbb{N}^+}; (\hat{S}^n_m)_{m,n\in\mathbb{N}^+}, (\{x_i\})_{i\leq n, n\in\mathcal{N}^+})$$

which is called the power clone of τ ([12]). We mention that $\mathcal{P} - clone\ \tau$ satisfies the well-known clone axioms (C1), (C2), (C3) (see e.g. [38], [12]).
A similar structure can be obtained if one defines a superposition for sets of operations. Let $O^{(n)}(A)$ be the set of all n-ary operations ($n \geq 1$) defined on the set A and let $O(A) := \bigcup\limits_{n\geq 1} O^{(n)}(A)$ be the set of all operations defined on A. Let $e^{n,A}_i$ be the n-ary projection defined on A and let $\mathcal{P}(O^{(n)}(A))$ be the power set of $O^{(n)}(A)$

Definition 9.2 *Let $m, n \in \mathbb{N}^+$ and $B \in \mathcal{P}(O^{(n)}(A)), B_1, \ldots, B_n \in \mathcal{P}(O^{(m)}(A))$*

(i) *If $B = \{e^{n,A}_j\}$ for $1 \leq j \leq n$, then $\hat{S}^{n,A}_m(\{e^{n,A}_j\}, B_1, \ldots, B_n) := B_j$.*

(ii) *If $B = \{f^A_i(t^A_1, \ldots, t^A_{n_i})\}$ with $f^A_i \in O^{(n_i)}(A), t^A_j \in O^{(n)}(A)$ and assume that $\hat{S}^{n,A}_m(\{t^A_j\}, B_1, \ldots, B_n)$ for $1 \leq j \leq n_i$ are already defined, then*
$\hat{S}^{n,A}_m(\{f^A_i(t^A_1, \ldots, t^A_{n_i})\}, B_1, \ldots, B_n) :=$
$\{f^A_i(r^A_1, \ldots, r^A_{n_i}) \mid r^A_j \in \hat{S}^{n,A}_m(\{t^A_j\}, B_1, \ldots, B_n), 1 \leq j \leq n_i\}$

(iii) *If $B \in \mathcal{P}(O^{(n)}(A))$ is arbitrary, then we define*

$$\hat{S}^{n,A}_m(B, B_1, \ldots, B_n) := \bigcup_{b\in B} \hat{S}^{n,A}_m(\{b\}, B_1, \ldots, B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, then we define $\hat{S}^{n,A}_m(B, B_1, \ldots, B_n) := \emptyset$
In this case we consider the heterogeneous algebra

$$\mathcal{P}ow_A - clone := (\mathcal{P}(O^{(n)}(A))_{n\in\mathbb{N}^+}, (\hat{S}^{n,A}_m)_{m,n\in\mathbb{N}^+}, (e^{n,A}_i)_{i\leq n, n\in\mathbb{N}^+}).$$

Clearly, (C1), (C2), (C3) are satisfied also in this case.
Let $\mathcal{A} = (A; (f^A_i)_{i\in I})$ be an algebra of type τ. Then we may consider the subclone which is defined as follows.

Definition 9.3 *Let $n \in \mathbb{N}^+$ and $B \in \mathcal{P}(W_\tau(X_n))$. Then we define the set B^A of term operations induced on the algebra $\mathcal{A} = (A; (f^A_i)_{i\in I})$ as follows:*

(i) *If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^{\mathcal{A}} := \{e^{n,A}_j\}$.*

(ii) *If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ then $B^{\mathcal{A}} = \{f^{\mathcal{A}}_i(t^{\mathcal{A}}_1, \ldots, t^{\mathcal{A}}_{n_i})\}$ where $f^{A_i}_i$ is the fundamental operation of $\mathcal{A}$ corespondending to the operation symbol f_i and where t^A_j are term operations on $\mathcal{A}$ which are induced in the usual way by the t_j's.*

(iii) *If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then we define $B^{\mathcal{A}} := \bigcup\limits_{b\in B} \{b\}^{\mathcal{A}}$. If the set B is empty, then we define $B^{\mathcal{A}} := \emptyset$.*

Let $\mathcal{P}(W_\tau(X_n))^{\mathcal{A}}$ be the collection of all sets of n-ary term operations induced by sets of n-ary terms of type τ on the algebra $\mathcal{A} = (A; (f^A_i)_{i\in I})$.
¿From these definitions we obtain the following

Proposition 9.4

$$\mathcal{P}ow_A - clone\mathcal{A} = ((\mathcal{P}(W_\tau(X_n))^{\mathcal{A}})_{n\in\mathbb{N}^+}, (\hat{S}^{n,A}_m)_{m,n\in\mathbb{N}^+}, (e^{n,A}_i)_{i\leq n, n\in\mathbb{N}^+})$$

is a subalgebra of $\mathcal{P}ow_A - clone$.

Finally we mention that the composition of relations can be defined in the following way:
Let $\sigma_1, \ldots, \sigma_n, \varrho$ be $(n+1)$-ary relations, where $\sigma_i \subseteq A_1 \times \cdots \times A_n \times B, i = 1, \cdots, n, \varrho \subseteq B_1 \times \cdots \times B_n \times C$. Then we define an $(n+1)$-ary relation

$$\varrho[\sigma_1, \ldots, \sigma_n] \subseteq A_1 \times \cdots \times A_n \times C$$

by

$$\varrho[\sigma_1, \ldots, \sigma_n] := \{(\overline{a}, c) \mid \exists \overline{b}((\overline{a}, b_1) \in \sigma_1 \wedge \cdots \wedge (\overline{a}, b_n) \in \sigma_n \wedge (\overline{b}, c) \in \varrho)\},$$

where $\overline{b} = (b_1, \ldots, b_n) \in B_1 \times \cdots \times B_n$, $\overline{a} = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. Then for $\chi_i \subseteq A_1 \times \cdots \times A_n \times B_i, \sigma_i \subseteq B_1 \times \cdots \times B_n \times C_i, i = 1, \ldots, n$ and $\varrho \subseteq C_1 \times \cdots \times C_n \times D$ the inclusion

$$\varrho[\sigma_1, \ldots, \sigma_n][\chi_1, \ldots, \chi_n] \subseteq \varrho[\sigma_1[\chi_1, \ldots, \chi_n], \ldots, \sigma_n[\chi_1, \ldots, \chi_n]]$$

is satisfied (see [13]).

10 Many-Sorted Clones

To define *many-sorted* Σ-*terms* for an index set I we need an I-sorted alphabet $X := (X_i)_{i \in I}$ where $X_i := \{x_{i_1}, \ldots, x_{i_n}\}, i \in I$, are pairwise disjoint sets, we need pairwise disjoint sets $\Sigma_n \subseteq \{(w, i) \mid w \in I^n, n \in \mathbb{N}^+\}, \Sigma_0 := \{(\varepsilon, i) \mid i \in I\}$, where ε is the empty word, $\Sigma := \bigcup_{n \in \mathbb{N}} \Sigma_n, \Omega_n := \{f_{(w,i)} \mid (w, i) \in \Sigma_n, n \in \mathbb{N}\}, \Omega := \bigcup_{n \in \mathbb{N}^+} \Omega_n$.
The sets Ω_n are also pairwise disjoint and pairwise disjoint with the sets $X_i, i \in I$. The set Ω is said to be set of many-sorted operation symbols. The symbols $f_{(\varepsilon,i)}, i \in I$ are called *nullary* operation symbols. Then for every $i \in I$ we define inductively:

(i) $W_0(i) := X_i \cup \Omega_0$

(ii) $W_{l+1}(i) := W_l(i) := W_l(i) \cup \{f_{(w,i)}(t_1, \ldots, t_n) \mid f_{(w,i)} \in \Omega_n, t_1 \in W_l(i_1), \ldots, t_n \in W_l(i_n)\}$

$W(i) := \bigcup_{l \geq 1} W_l(i), W := W_\Sigma(X) := (W(i))_{i \in I}$.

Definition 10.1 *Let $W_\Sigma(X) = (W(i))_{i \in I}$ be a family of sets of Σ-terms. Then an operation $S_\alpha : W(i) \times W(k_1) \times \cdots \times W(k_n) \to W(i)$, where $\alpha = (k_1, \ldots, k_n, i), k_1, \ldots, k_n, i \in I, n \in \mathbb{N}^+$ is inductively defined as follows:*

1. *If $t = x_{ij} \in X_i, i \in I, j \geq 1$, then*
   1.1 $S_\alpha(x_{ij}, t_1, \ldots, t_n) := x_{ij}$ *if $i \neq k_q$ for all $q \in \{1, \ldots, n\}$,*
   1.2 $S_\alpha(x_{ij}, t_1, \ldots, t_n) := x_{ij}$ *if there exists an element $q \in \{1, \ldots, n\}$ such that $i = k_q$ and $j \neq q$,*
   1.3 $S_\alpha(x_{ij}, t_1, \ldots, t_n) := t_j$ *if there exists an element $q \in \{1, \ldots, n\}$ such that $i = k_q$ and $j = q$,*
   1.4 $S_\alpha(x_{ij}, t_1, \ldots, t_n) := x_{ij}$ *if $i = k_q$ for all $q \in \{1, \ldots, n\}$ and $j \neq q$,*
   1.5 $S_\alpha(x_{ij}, t_1, \ldots, t_n) := t_j$ *$i = k_q$ for all $q \in \{1, \ldots, n\}$ and $j = q$.*

2. *If $t = f_\gamma(s_1, \ldots, s_m)$, where $\gamma = (i_1, \ldots, i_m, i)$ and $m \in \mathbb{N}^+$, then*
   $S_\alpha(f_\gamma(s_1, \ldots, s_m), t_1, \ldots, t_n) := f_\gamma(S_{\alpha_1}(s_1, t_1, \ldots, t_n), \ldots,$
   $S_{\alpha_m}(s_m(t_1, \ldots, t_n))$
   where $\alpha_1 = (k_1, \ldots, k_n, i_1), \ldots, \alpha_m = (k_1, \ldots, k_n, i_m)$ if we assume that $S_{\alpha_1}(s_1, t_1, \ldots, t_n), \ldots, S_{\alpha_m}(s_m, t_1, \ldots, t_n)$ are already defined.

For a many-sorted type Σ we define $\Lambda_n := \{(w, k) \mid w \in I^n$ and there is an $m \in \mathbb{N}^+$, an $\gamma \in \Sigma_m$ and an j with $1 \leq j \leq m$ such that $\gamma(j) = k \in I\}$. Here $\gamma(j)$ denotes the j-th component of γ. Finally, let $\Lambda := \bigcup_{n \geq 0} \Lambda_n$.

Now we may consider the many-sorted algebra

$$m - clone\Sigma := ((W(i))_{i \in I}; (S_\alpha)_{\alpha \in \Lambda}, (x_{ij})_{i \in I, 1 \leq i \leq n_i, n_i \in \mathbb{N}})$$

and obtain:

Theorem 10.2 *The many-sorted algebra $m - clone\Sigma$ satisfies for $\alpha = (k_1, \ldots, k_n, i) \in \Sigma, \alpha_1 = (i_1, \ldots, i_n, k_1), \ldots, \alpha_n = (i_1, \ldots, i_n, k_n), \gamma = (i_1, \ldots, i_m, i), \alpha_i, \gamma \in \Lambda$ the identity*

$$S_\alpha(z, S_{\alpha_1}(y_1, x_1, \ldots, x_n), \ldots, S_{\alpha_n}(y_n, x_1, \ldots, x_n)) \approx S_\gamma(S_\alpha(z, y_1, \ldots, y_n), x_1, \ldots, x_n)).$$

We notice that (C3) is also satisfied. Instead of (C2) we have two different identities.

11 The Lattice of all Clones of Operations Defined on Finite Sets

Now we will come back to the lattice $\mathcal{L}_A$ of all clones of operations defined on a finite set A with $|A| \geq 2$. We noticed already that these lattices $\mathcal{L}_A$ are complete and algebraic, with the least element J_A, the clone of all projections defined on A and the greatest element $O(A)$ The lattice $\mathcal{L}_2$ for a two-element set $A = \{0, 1\}$ was completely described by E. L. Post in [29], [30]. The lattice $\mathcal{L}_2$ is countably infinite. Each subclone of $O(\{0, 1\})$ is finitely generated. The lattice is atomic and dually atomic. The dual atoms (or maximal classes) are the clones $C_3 = Pol(\{0\}), C_2 = Pol(\{1\}), M_1 = Pol(\{\leq\})$, where $\leq$ is the usual order relation on $\{0, 1\}$, $D_3 = Pol(\{(0, 1), (1, 0)\}), L_1 = Pol(\{(a, b, c, d) \mid a + b = c + d\})$, where $+$ is adddition modulo 2. The operation f^d is said to be dual to a given operation $f \in O(\{0, 1\})$ if $f^d(a_1, \ldots, a_n) := \neg f(\neg a_1, \ldots, \neg a_n))$ for all $a_1, \ldots, a_n \in \{0, 1\}$. A clone C^d is dual to the clone C if $C^d := \{f^d \mid f \in C\}$. In the following Hasse diagram of the lattice $\mathcal{L}_2$ clones which are dual to each other are symmetric with respect to the symmetry line.
Minimal clones are $O_4 = \langle\{\neg\}\rangle, O_6 = \langle\{c_0\}\rangle, O_5 = \langle\{c_1\}\rangle$, ($c_0, c_1$- constant operations with value 0 and 1, respectively) $P_1 = \langle\{\wedge\}\rangle, S_1 = \langle\{\vee\}\rangle, L_4 = \langle\{x+y+z\}\rangle$ ($+$ - addition mod m), $D_2 = \langle\{m\}\rangle(m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. For a short proof of the completeness of this lattice see [33] (see also [8], 10.3).

If $|A| > 2$, then there are uncountably many clones of operations defined on the finite set A. Not all of them are finitely generated. Since the largest clone $O(A)$ is finitely generated, it has only finitely many maximal subclones, i.e. dual atoms in the lattice $\mathcal{L}_A$. These clones were completely determined by I. G. Rosenberg ([35]). New proofs of Rosenberg's theorem were give by R. W. Quackenbush ([32]) and D. Lau ([20], see also [21]). The full description of the maximal subclones of $O(A)$ gives us the following completeness criterion:

Theorem 11.1 *Let A be a finite set and let F be a subset of $O(A)$. Then Fgenerates $O(A)$ (F is functionally complete) if and only if F is not contained in one of the maximal subclones of $O(A)$.*

Every maximal subclone of $O(A)$ is given in the form $Pol\varrho$ for some h-ary relation ϱ. So the description of the maximal clones amounts to the determination of the corresponding h-ary relations $\varrho \subseteq A^h$. We have the following six classes of relations:

(i) Let $\mathcal{S}_A$ be the full symmetric group of all permutations on A. Let $s \in \mathcal{S}_A$ be a fixed-point free permutation with $r = n/p$ cycles of equal prime length p. We set $\rho_s = \{(a, b) \in A^2 \mid s(a) = b\}$. Then $Pol\rho_s$ is a maximal clone.

(ii) Let $\rho \subseteq A^2$ be a partial order with least and greatest elements 0 and 1, respectively. Then $Pol\rho$ is a maximal clone.

(iii) Let $\mathcal{G} = (A; +, -, 0)$ be an abelian group. A function $f \in O^k(A)$ is called quasilinear with respect to $\mathcal{G}$ if for all $x_1, \ldots, x_k$ and $y_1, \ldots, y_k \in A$ we have $f(x_1, \ldots, x_k)+f(y_1, \ldots, y_k) = f(x_1+y_1, \ldots, x_k+y_k)+f(0, \ldots, 0)$. The set of all quasilinear functions is a clone which can be characterized as $Pol\chi_G$ for the relation $\chi_G := \{(x, y, z, u) \in A^4 : x+y = z+u\}$. This clone is maximal iff $\mathcal{G}$ is p-elementary (meaning that $px = 0$ for all $x \in A$) or, equivalently, if $\mathcal{G}$ is the additive group of an m-dimensional vector space over the p-element field $GF(p)$. Thus $|A| = p^m$ for some prime p and $m \in \mathbb{N}^+$.

(iv) Let $|A| \geq 3$. Let ϑ be a non-trivial equivalence relation on A, distinct from A^2 and the diagonal relation Δ_A. Then $Pol_A\vartheta$ is a maximal clone. For B a proper subset of A with at least two elements, let ϑ_B denote the (non-trivial) equivalence relation having B as unique non-singleton block (that is, the blocks of ϑ_B are B and all $\{c\}$ with $c \in A \setminus B$). Later we will distinguish the two cases $\vartheta = \vartheta_B$ and $\vartheta \neq \vartheta_B$.

(v) A relation $\varrho \subseteq A^h$ is called *totally reflexive* if ϱ contains each h-tuple $(a_1, \ldots, a_h) \in A^h$ with a repetition of coordinates (that is, with $a_i = a_j$ for some $1 \leq i < j \leq h$). A relation ρ is called *totally symmetric* if $(a_1, \ldots, a_h) \in \varrho \Leftrightarrow (a_{\pi(1)}, \ldots, a_{\pi(h)}) \in \varrho$ for every permutation π on the set $\{1, 2, \ldots, h\}$. The center $C(\varrho)$ of ϱ is the set of all elements $c \in A$ with the property that $(c, a_2, \ldots, a_h) \in \varrho$ for all $a_2, \ldots, a_h \in A$. A totally reflexive and totally symmetric relation ϱ with a non-trivial (not all of A) center is called *central*. $Pol_A\varrho$ is a maximal clone for every central relation ϱ. Later we distinguish two cases depending on whether ϱ is unary or not.

(vi) Let $|A| = n$. For each $3 \leq t \leq n$, let
$E_t := \{1, 2, \ldots, t\}$,
$\iota_t := \{(c_1, \ldots, c_t) \in E_t^t \mid c_i = c_j \text{ for some } 1 \leq i < j \leq t\}$, and

$$\iota_t^{\otimes m} := \iota_t \otimes \ldots \otimes \iota_t$$
$$= \{((c_{11}, \ldots, c_{1m}), \ldots, (c_{t1}, \ldots, c_{tm})) \in (E_t^m)^t \mid (c_{1i}, \ldots, c_{ti}) \in \iota_t \text{ for } i = 1, \ldots, m\}.$$

For $t \geq 3$, a t-ary relation $\varrho \subset A^t$ is called *t-universal* if there are an $m \geq 1$ and a surjective mapping $\mu : A \longrightarrow E_t^m$ such that

$$\varrho = \varrho(\mu) := \{(a_1, \ldots, a_t) \in A^t \mid (\mu(a_1), \ldots, \mu(a_t)) \in \iota_t^{\otimes m}\}.$$

There is only one non-trivial n-universal relation ($t = |A| = n, m = 1$), namely

$$\iota_n(A) = \{(a_1, \ldots, a_n) \in A^n \mid a_i = a_j \text{ for some } 1 \leq i < j \leq n\}.$$

The well-known Słupecki criterion says that $f \in Pol\iota_n$ iff f is not surjective or f depends essentially on at most one variable ([37]). $Pol\varrho$ is a maximal clone for every t-universal relation.

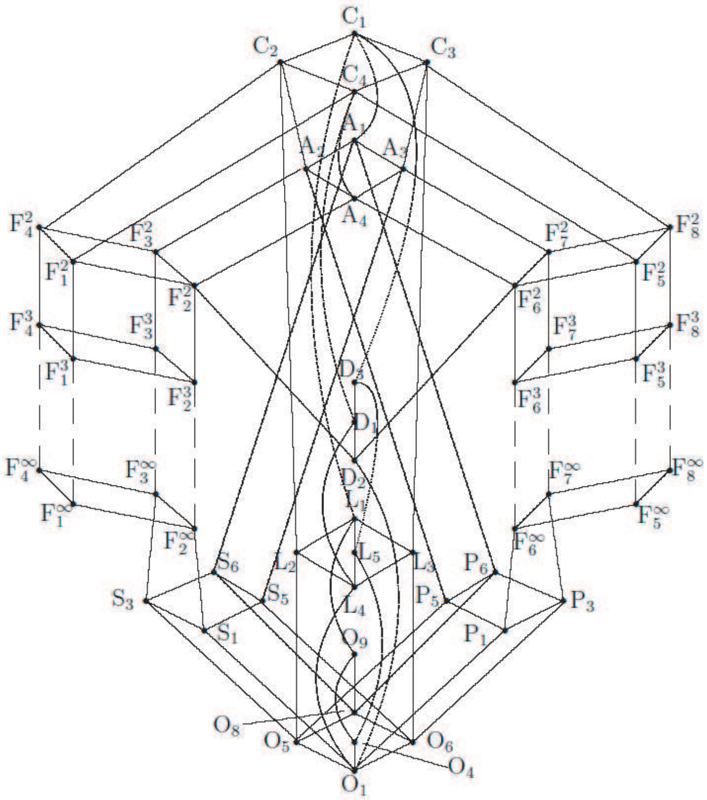


Figure 1: The Lattice of all Clones of Boolean Operations

For finite sets A there are only finitely many minimal subclones of $O(A)$ (atoms in the lattice $\mathcal{L}_A$). While the maximal subclones are completely known, there is no full description of the minimal clones. Every minimal clone is generated by each of its elements f different from a projection. I. G. Rosenberg ([34]) showed that if f generates a minimal clone, then f must fit one of the following descriptions:

(1) f is unary and $f^2 = f$ or $f^p = id_A$ (the identity function on A) for some prime number p, or

(2) f is binary and idempotent, or

(3) f is a ternary *majority function*, i.e. f satisfies $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$, or

(4) f is the ternary operation $x + y + z$ in a *Boolean group* (i.e. a ternary *minority operation* f satisfying $f(x, x, y) = f(x, y, x) = f(y, x, x) = y$), or

(5) f is a semiprojection, i.e. f has arity $n \geq 3$ and there exists an element $i \in \{1, \ldots, n\}$ such that $f(a_1, \ldots, a_n) = a_i$ whenever $a_1, \ldots a_n$ are not pairwise different.

For $|A| = 3$ B. Csákány determined 84 minimal clones. For more information on minimal clones see [21]. It was interesting to learn from P. P. Palfy (talk on the AAA75, CYA 23, Darmstadt, November 2007) that there exist operations f defined on a finite set $A, |A| > 2$, such that the algebra $(A; f)$ is functionally complete, while the clone $\langle f \rangle$ of its term operations is a minimal clone.

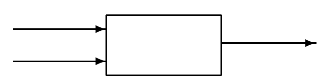
12 Clones on Infinite Sets

If A is finite and $|A| > 2$, then $\mathcal{L}_A$ has $2^{\aleph_0}$ elements. The lattice $\mathcal{L}_A$ has $2^{(2^{|A|})}$ elements if $|A| \geq \aleph_0$. By a result of I. G. Rosenberg for arbitrary sets A with $|A| \geq \aleph_0$ there are also $2^{(2^{|A|})}$ maximal classes. Four countably infinite A this means that $\mathcal{L}_A$ has as many elements as the power set of the real numbers. M. Goldstern proved in [15] that if the continuum hypothesis holds, then the clone lattice on a countably infinite base set is not dually atomic. For infinite clones not every clone contains a minimal one ([27]). As an example of a clone defined on an infinite base set we mention the clone C_{fin} of all operations which are either projections or have finite range.

A subset $C \subseteq O(A)$ is called a *local clone* if C is closed under interpolation, i.e. if for a given $f \in O(A)$ there is an operation $g \in C$ such that for any finite subset $B \subset A$ with $f|B = g|B$ there holds $f \in C$. This is equivalent to $C = \bigcap_{i \in I} Pol(R_i)$ for some family $(R_i)_{i \in I}$ of finitary relations. Many results and ideas from the theory of clones on finite sets carry over to the theory of local clones on infinite sets. The local clones on infinite sets A form also complete lattices $\mathcal{L}_{locA}$ which are not algebraic. The lattices $\mathcal{L}_{locA}$ have no compact elements.

13 Applications of Clone Theory

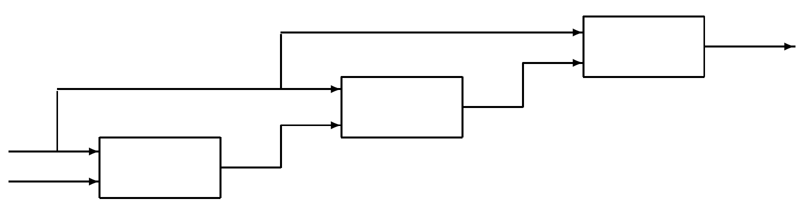
Clone theory has many applications, inside and outside of Mathematics. If n-ary operations are technically realized by switching cicuits, devices which have n inputs and one output, then composition of operations corresponds to combinations of those switching circuits. Let for instance the binary operation f defined on the finite set A be realized by



then the composition

$$S_2^2(f, S_2^2(f, S_2^2(f, x_1, x_2), x_2), x_2)$$

can be realized by



One of the problems which arises here is related with the functional completeness problem: to find a small set of elementary devices such that the complex circuits generated by these base set can realize any functions defined on A.

Another application of clones in Theoretical Computer Science is the so-called *constraint satisfaction problem* (CSP).
Let Γ, Σ', Σ' be sets of finitary relations on domains V, D. Then $CSP(\Gamma)$ is the set of problems of the form: Does there exists a relational homomorphism $(V, \Sigma) \to (D, \Sigma')$ between relational systems $(V, \Sigma), (D, \Sigma')$ of the same type if $\Sigma' \subseteq \Gamma$.
An important result is the following theorem:

Theorem 13.1 ([18]) *Let $R(A)$ be the collection of all relations defined on the set A and let $\Gamma_1 \subseteq \Gamma_2 \subseteq R(A)$. If $\Gamma_1 \subset InvPol\Gamma_2$, then $CSP(\Gamma_1)$ can be reduced to $CSP(\Gamma_2)$ in polynomial time.*

In particular, the complexity of $CSP(\Gamma)$ depends only on the clone $Pol\Gamma$.

Finally we want to mention one more algebraic application. Let V be a variety of algebras of type τ. Let Id_nV be the set of all identities satisfied in V consisting of n-ary terms. It is not difficult to see that $(Id_nV)_{n\in\mathbb{N}^+}$ is a sorted congruence on the many-sorted algebra $clone\tau$. The quotient algebra $cloneV := clone\tau/(Id_nV)_{n\in\mathbb{N}^+}$ is said to be the clone of the variety V. Algebraic properties of $cloneV$ correspond to properties of the variety V (see [38]) since $cloneV$ is the clone of the free algebra with respect to V generated by a countably infinite alphabet. We mention that $cloneV$ can be regarded as a category which is dual to Lawvere's algebraic theory ([22]). The following table shows corresponding properties of V and $cloneV$.

property of V	property of $cloneV$
reduct	subclone
equivalence	isomorphism
subvariety	congruence relation
hypervariety	variety
hyperidentity	identity

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Communications：

(Ⅰ) Conferences for Young Algebraists

Klaus Denecke*

The following list gives some information on the future conferences of young algebraists:

1. AAA Linz (Austria), May 22-25, 2008
2. CYA (77. AAA), Potsdam, Februar 2009
   (dedicated to K. Denecke)
3. AAA Bern (Switzerland), June 11-14, 2009
4. CYA (79. AAA), Olomouc (Czech Republic), February 2010

See also pages 10 and 11 of Notices from the ISMS, January 2007.

(Ⅱ)Call for Proposals and Organizers for Special Sessions in IVMS 2008 and IVMS 2009

The ISMS holds inter-regional videoconference via internet. The first videoconference was held in December 2003, the second in June 2004, and the third in March 2005. We are planning a videoconferencing system that will be able to connect up to four research sites, at present. Therefore presenters may be asked to travel to one of these local sites in order to present. The international videoconference consists of special sessions only. These sessions will be devoted to special fields of study, for example Fixed point theory and its applications. Each session's organizers will decide the type of the videoconference: presentation of original papers (contributed and/or invited papers)and/or expository articles, or tutorials. Speakers of the session can write on a white board or an OHP sheet, or can use Power Point. Participants can ask questions or make comments. All these are performed similarly to the traditional meetings. Organizers of the sessions chair their meeting at their co-ordination sites and can turn the speakers' sites.

Time differences between local sites will become an important factor. Please note the following possibilities: **There are three combinations of connections for inter-regional videoconferences:**

(1) Europe (morning) – Asia(evening) for 4hours from 08:30(GT) to 12:30(GT)
(2) Asia(morning)- West coast area of USA(evening) for 4hours from 23:30(GT) to 03:30(GT)
(3) Asia(around noon)-Asia(evening) for 4hours from 16:00(GT) to 20:30(GT)

Every IVMS is performed through three steps.

(1) Trial of link in advance between (1) organizer and Osaka Nakanoshima Center(ONC)and (2) co-organizer and ONC are recommended.
(2) In the first step, all papers are presented on the homepages of the ISMS. (http;//www.jams.or.jp/ivms/index-ivms.html)
(3) In the second step of the IVMS, all papers are presented similarly to the usual assembly type meeting via internet, often using CD-ROMs or DVDs which are sent beforehand, when the author can't use VC system via internet.

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Scheme of Videoconference

1)Videoconference System
In order to have a videoconference with us, your institutions should have one of the following in the descending order of desirability.

1. Videoconference room
2. Facilities for distance learning
3. Facilities in computer centers

Making use of one of the above, your institutions can be connected with our system(TANDBERG 6000 or TANDBERG 550) at Nakanoshima Center of Osaka University as far as your system satisfies the following ITU-T standards.

International Standards of Videoconference System

ITU-T	over IP H.323
Video Coding	H.261, H.263
Audio Coding	G.711, G.723 G.722, G.728
Multiplexing (Mux/Demux)	H.225
System Control	H.245
Transmission Rate	64kbps 128kbps 384kbps

Image dimensions: CIF:352×288, QCIF:176×144, SQCIF:128×96

The following products are assured to be able to be connected with our "TANDBERG 6000".

H.323 Endpoints(over IP)
Equipment Software Revision
Polycom View Station 512MP 7.0.1, 7.2.4, Polycom ViewStation FX 4.2,5.0
Polycom Via Video 5.0,PictureTel 970 5.0.0.415,PictureTel 680 5.0.0.415
PictureTel(Intel) TeamStation 4.0a,Microsoft NetMeeting 3.01
VCON Vigo 4.6, VCON Falcon 0300.M07.D28.H11, VCON Cruiser 384 4.6
VTEL Galaxy 2.2.0.070, Sony PCS-1600 3.10, Sony PCS-6000 5.00
D-Lonk i2Eye 2.0.0.20

2) Organizers
Organizers should appoint invited speakers and call for papers for their sessions. The selection of the papers is left to the organizers' own choice. They should inform the ISMS of their programs with the titles, author's names of the papers. They should send the following "Form of Application of Organizers" to the ISMS.

3) Application Form for Organizers
Every organizer should inform the following data to the ISMS program committee
(Through the homepage of the ISMS. http://www.jams.or.jp/hp/ivms_orhanizers.html)

1. Name of the organizer
2. E-mail address
3. Title of the planned special session

4. Names of the co-organizers of the joint universities/societies and their affiliations, if any.
5. Name(s) of the invited speakers, if any.

4) Participation

Individuals who wish to participate the videoconference can designate a session or sessions in which they are going to participate.

(1) For participants, these can be five sites that are connected with our key site simultaneously. The foreign participants can go to the nearest university announced on our web page as the joining sites.

(2) The application for participation can be made on the web inputting the required data in the registration form on the ISMS homepage.
(http://www.jams.or.jp/hp/ivms_organizers.html) or write items in the following forms and send by post to the International Society for Mathematical Sciences, 2-1-18 Minami Hanadaguchi, Sakai, Osaka 590-0075, Japan
A applicants can mail also by post the following data 5)1~5 both to the ISMS managing office and to the organizer of the session in which they wish to participate.

(3) The participation fee to cover the head office cost is free. Please note that local sites may request an additional fee to cover local costs.

(4) The ISMS will give a password to the participants to enter the session of the web.

5) Presentation on the WWW

Programs, abstracts of the papers, (and the full text of the papers)if the author wishes of each session will be located beforehand on the web.

6) Connection Test on March 29, 2008: 15:30-17:00 (Japan time) on March 29, 2008 ((See (III) on the page 17))

7) Publication in SCMJ

Authors should designate one of the editors and send their papers both to the editor and to our office. In the case of the papers to be presented in IVMS, the organizers, the co-organizers and the invited speakers can, upon their approval, referee in place of the editors. The editorial board expects this will ensure the papers to be refereed quickly and published adequately.

8) ISMS (JAMS) Prize Winners

Winners of ISMS (JAMS) Prize can give their lectures or speeches at IVMS if they wish.

9) FAQ about videoconference

(Question 1) Where do the authors (lectures, speakers) present his/her paper in the videoconference?
(Question 2) Is there any limitation to the length of the lecture?
(Question 3) Can the authors make lectures using OHP?
(Question 4) What should the authors do expect for making lectures?
(Question 5) Is there any limitation to the length of the papers?
(Question 6) Is there any limitation to the length of the abstract?
(Answer 1) As the conference sites are announced on the web before the videoconference, the authors go to the nearest site to present their papers.
(Answer 2) It depends on the organizers of the sessions. Please ask the organizers directly. We are thinking of 30 minutes as a rough standard including questions and answers.
(Answer 3) Yes, they can. They can also use white boards. However, please ask the organizers if it is possible at their site.
(Answer 4) They should submit the papers (including abstract) for the web, which will be located on the web in advance.
(Answer 5) It depends on the organizers of the sessions. Please ask the organizers directly. We are thinking of fifteen pages as a rough standard.
(Answer 6) It should be within 20 lines in Word style.

The ISMS
(Ⅰ) International Society for Mathematical Sciences
----- Contributions

Dear Colleagues and Friends,

In September 2007, we establish the following two funds.

(1) **International ISMS Prizes Fund**
in order to award the prizes for the original papers or survey works published in Scientiae Mathematicae Japonicae or Notices from the ISMS.

(2) **International Research Promoting Fund**
in order to promote and support international joint meetings by IVMS.

The contributions are classified into the following five categories.

(A) ¥ 500,000 (or $5,000) and above
(B) ¥ 100,000 (or $1,000) and above
(C) ¥ 50,000 (or $500) and above
(D) ¥ 10,000 (or $100) and above
(E) Less than ¥10,000 (or $100)

We deeply appreciate your generous contributions to support the above activities of our society.

Your remittance to the following accounts of ours will be much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai, Osaka 590-0075, Japan

(2) Through a bank, make remittance to the following account of JAMS.
A/C 94103518
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

Kiyoshi Iseki
Tadashige Ishihara

(Ⅱ)Business Meeting (2008) and IVMS

The business meeting (2008) will be held on March 29, at Osaka University Nakanoshima Center as follows.

Data: March 29, 2008
Place: 7F Seminar Room at Osaka University Nakanoshima Center
Time: 13:00-14:00(Japan Time)··· **business meeting Test and IVMS Test** among joining Universities
14:00-15:30(Japan Time)··· business meeting
Agenda: 1. Financial Report for the fiscal year 2007
2. Budget for the fiscal year 2008
3. Honorary Member
4. Miscellaneous
15:30-17:00(Japan Time)··· IVMS among proposed Universities

Call for Papers for SCMJ：

（Ⅰ）Call for Papers for SCMJ

Scientiae Mathematicae Japonicae(SCMJ) calls for excellent papers.
(1) Authors can choose one of the editors in the Editors List and send their papers directly to him/her for refereeing which promises **quick refereeing and publication.**
(2) If the SCMJ authors prepare their files in ISMS standard format (Js.), the lead time from acceptance to the online publication **will be extremely short or nil.**
(3) In the proofreading is made by the SCMJ (Paper or TeX) author, we will publish the paper on the Web as soon as we receive the corrected galley proof.
(4) The Journal is reviewed by **Mathematical Review** and **Zentralblatt from cover to cover.**

(A) Submission

Authors are requested to choose one of the editors in the SCMJ editors list and send their papers, satisfying all of the following conditions, **directly to the editor.** The editors list can be obtained from (i) URL:http://www.jams.or.jp/ (ii) " Editorial Board" of SCMJ(Vol.64, No.1, July 2006).

Prepare **e-mail Form for Submission and three** hard copies of your paper, **three** hard copies of Form for Submission, and send them as follows.

- **To the editor's e-mail address**; Form for Submission (with the abstract)
- **To the editor's postal address; Two** hard copies of your paper, **two** hard copies of the Form for Submission (with the abstract)
- **To the e-mail address of ISMS** (http://www.jams.or.jp/hp/submission f.html); Form for Submission (with the abstract)
- **To the postal address of ISMS; One** hard copy and **one** Form for Submission

The received date of the paper is the date when the editorial office receives the paper together with the Form for Submission, and not necessarily the date when the editors receive them.

To e-mail Form for Submission is mandatory to support the editor-receive-system, not to waste the precious reseach time of the editors and promote efficiency in the editorial procedure.

(B) Abstract

Every paper should contain an abstract. Try to limit your abstract to 20 lines when typed in TeX. The abstract should be a kind of mini research announcement which is **self-contained** and gives **the overview** of your paper. Abstracts of accepted papers are **very rapidly displayed** on ISMS home page and are announced **all over the world via Internet.** Abstracts in Paper Form and E-mail Form should be typed **in Text file.** If it is inevitable for you to use symbols in the abstract, you may make it in a TeX source file indicating **the kind of TeX** as notes, for example, (via LaTeX2e).

(C) Data

The full postal address, telephone and facsimile numbers, e-mail address of the author should be specified at the bottom of the last page of the manuscript. 2000 AMS Subject Classification and Keywords should be written both in Paper, E-mail Form and at the **footnote** on the first page of the manuscript.

(D) Receipt

ISMS will send a letter of receipt when we receive a hard copy, a Paper Form and E-mail Form (if the author has e-mail facility). The received date is to be specified in the letter.

(E) Revision

If revision of your paper is necessary, the editor informs you directly. When you revise abstract of your paper in that case, you should send new Paper Form with new abstract and E-mail Form with new abstract also.

(F) Acceptance or Rejection, Page Charges

ISMS will inform authors of **acceptance or rejection** of their papers **by e-mail.**

Authors should choose one of the following 3 types of his final draft he will send after acceptance of his paper, (1) **P:** Paper draft only (2) **T:** Paper prepared using TEX and its source file (3) Js: Paper prepared using TEX with ISMS style file, and its source file.

List of the page charges for SCMJ (2008 year)

Every accepted paper is charged ①**¥1,000(US$10, €7) as handling charges** plus ②**page charges.** The page charges per printed page are reduced as follows.

	ISMS members	Non-members
P	¥ 3,500 (US$35, € 23)	¥ 4,000 (US$40, €27)
Tex	¥ 2,000 (US$20, € 14)	¥ 2,500 (US$25, €17)
LateX2e, LaTeX	¥ 700 (US$ 7, € 4)	¥ 1,000 (US$10, € 7)
Js (ISMS style file)	¥ 500 (US$ 5, € 3)	¥ 800 (US$ 8, € 5)

The above page charges include 20 offprints. The additional page charge may be required for the figures contained in the papers. For more information, see our Web Page.

1) **Js** (ISMS style TeX) files mean the files which are ready for publication without any process by our Publication Dept.

Please note whether the file meets the requirement of the ISMS style or not **is judged by ISMS Publication Dept.**

Js files can be made using the ISMS style file for LaTeX, or LaTeX2e, which can be downloaded from ISMS Web Page.

The procedure to make Js files :

(a) Prepare your paper in LaTeX, or LaTeX 2e.

(b) Use the following ISMS style file to make your paper "ISMS style TeX" (Js). (The ISMS style files can be obtained from ISMS Web Pages.)

If your paper contains graphs or figures which cannot be processed even in LaTeX(2e), make them EPS (Encapsulated Post Script) files and then PDF files.

(G) After Acceptance

If the paper is accepted, P authors are requested to send the following (1) & (2), T and Js authors (1) – (4).

(1) A hard copy of the final draft(for publishing)
(2) Paper Form for WWW
(3) The source file of the final draft in TeX, by e-mail or on diskette.
(4) E-mail Form for WWW

(H) Proofreading

ISMS will send a galley proof to P and T authors only but **not to Js authors.** We regard the final files sent by Js authors as ready for publication.

(I) Offprints

Every author can obtain a **password to read his paper and** can make **as many offprints as they want,** using Acrobat Reader.

(J) Online version of SCMJ

The full texts of the accepted papers will be located on the online version of SCMJ in the following two manners from Vol.66, No. 1 (July 2007).

(1) A list of papears in the order of the accepted date.
(2) A list of accepted papers organized by filed of specialization with a link to (1). The field of specialization of the accepted papers will be chosen by the authors in the fields of f-1 - f-14.
(See a list of on page 24.)

(Ⅱ) Items required in “ Form for Submission ” (from 2008 March)

Give the following information by filling in the items on the list.

105. Write the name of the editors whom you will send your manuscript to be refereed.
110. Date of dispatching this form
120. Title of Paper
125. Abbreviated Title (Running Title) of Paper **(Less than 35 letters)**
130. The field of specialization in the fields of f-1〜f-14 (p.25) of the paper
132. Keywords and phrases
135. Number of Pages of the manuscript
140. Number of Authors
151.First author’s name, affiliation(institution /university) and its postal address
152.Second author’s name, affiliation (institution /university) and its postal address
153.Third author’s name, affiliation (institution /university) and its postal address

In typing author’s names the order should be Last (Family) name, First name, Middle name. In case there are four authors or more, please write their names, affiliations and their postal addresses as items 154, 155... If the authors are temporarily studying or teaching in Japan, please type the affiliations (or universities they graduated from) in their own countries.

170. Author's name to whom correspondences should be sent
180. Author's address of 170
190. E-mail address 195. Fax
200. Please type P or T or Js (see CALL FOR PAPERS (F)).
    For the following 201〜210 items, authors T and Js are requested to answer. Authors P may skip these items.
201. Do you send your files via e-mail or on diskette? Please type “e-mail” / “diskette” or both.
202. What kind of TeX; do you use?
    Please type, LaTeX(old version), LaTeX2e or other TeX
203. Please type you platform you are using. Type WINDOWS, Macintosh or others.
204. If you have used ISMS style file, then please type the name of the style file you have used.
207. Are you using figures in your paper?
    If yes, please type `a figure' or `figures', and type also they will be processed in LaTeX or not?
208. Is your electronic graphics file (if any) is saved in Encapsulated PostScript (EPS) format or in TIFF, PICT, GIF or other standard graphics format?
209. If you send the above files with attachment, you should send its extension name together. Please type the file name and the extension name.
210. If you use compress utility of your file, please type ZIP, gzip, tar, tar + gzip, tar + compress, or etc.
213. Are you a member of ISMS? If yes, type M. If you are not a member, type NM in case you apply for the membership together with an application form, and type NN in case you do not apply for the membership.
214. Is your faculty an Institutional Member of ISMS ? Please type YES or NO.
    In case Yes, are you an associate member of ISMS or not ? If you are an associate member, please type AM.
216. Are you sure that you or your institution will pay the required page charge when this paper is accepted for publication? ; YES or NO.
218. Signature
219. Type Abstract of your paper **in Text file**. (If you use TeX file, you should write the kind of TeX as notes.)

159, 179, 189. For Japanese Author(s), write 151~158, 170, 180 in Japanese.

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Special Fields (f-1 - f-14)

f-1. Mathematical logic, Set theory, Relative systems, Algebra systems
f-2. Classical algebra, Number theory, Combinatorics, Cryptology
f-3. Topology, Geometry, Imaging
f-4. Real analysis, Complex analysis
f-5. Functional analysis, Operator theory
f-6. Differential equations, Integral equations, Functional equation, Numerical analysis
f-7. Infinite dimensional dynamical systems, Inverse problems
f-8. Fluid dynamics, Atmospheric research, Rheology, Computer aided design, Control theory, Nanoscience
f-9. Probability theory, Statistics, Experimental Design, Quality control
f-10. Operations Research, Decision theory, Queuing theory, Scheduling, Mathematical finance, Mathematical economics
f-11. Informatics, Pattern recognition, Imaging, Computer science, Computer simulation
f-12. Biomathematics, Proteomics, Imaging, Bioscience, System biology
f-13. Mathematical education, History of mathematics
f-14. Over several fields (Ex. Fixed point theory)

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Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$300. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members do including the discount of the page charge. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$300) ☐Print ＋ Online (US$300)
University (Institution)	
Department	
Postal Address where SCMJ should be sent	
E-mail address	
Person in charge	Name: Signature:
Payment Check one of the two.	☐Bank transfer ☐Credit Card (Visa, Master)
Name of Associate Membership	1.
	2.

Call for regular Members

ISMS Membership Dues from 2008

A new category "life member" has been established and can be applied for from 2005. An eligible member may become a life member by making a one-time payment of dues. A member who has been an ISMS member for ten years or more is eligible for a life member. The amounts of dues are : ¥70,000 for the domestic members, US$ 600 (€480)for the foreign members, and US$ 500 (€400)for the members in developing countries.

We have reduced the ISMS membership dues since 2001 and copies of the printed journal have not been distributed to the members, free of charge. Instead, we give User Name and Password to each member so that he/she can view or print out the full text of the papers published in SCMJ except papers in the international plaza from our Web site (http://www.jams.or.jp).

The Membership Dues for each category is as follows. Applications for the 3-year members can be made only in 2005 and in every three years.

Membership Dues for this year

Categories	Domestic	Overseas	Developing countries
1-year member (1A)	A1: ￥9,000	F1: US$75 , €60	D1: US$45, €36
3-year member (3A)	A3: ￥24,000	F3: US$200, €160	D3: US$117, €93
1-year students or aged (1S)	SA1: ￥5,000	SF1: US$40, €32	SD1: US$27, €21
3-year students or aged (3S)	SA3: ¥12,000	SF3: US$100, €80	SD3: US$71, €57
Life member* (L)	AL: ￥90,000	FL: US$740, €592	DL:US$616, €493

*The members who have been the ISMS members for more than 10 years are eligible for this category.
The categories 1S and 3S are for students or persons over 70 years old. The figure 1 and 3 means a year and 3 years respectively.
Category D is for those who reside in the countries of Eastern Europe, CIS or developing countries.

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Payment Instructions

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.
(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance
(1) to our Post Office Transfer Account - 00930-3-73982 or
(2) to our account No.1565679 at SUMITOMO BANK, Sakai, Osaka, Japan.
All the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Membership Application Form (from 2008 January)

To determine what membership category you are eligible for, read “Join ISMS” on the inside of the back cover.

1. Name: Family Name, First Name, Middle Name (in this order)
2. Home Address
3. Name of Firm or Institution affiliation
4. Postal address to which correspondence should be sent
5. e-mail address
6. Telephone Number, Fax Number
7. Membership Category
8. Panel (Please choose one out of the following 14panels in the page 26 and write the panel number. You could choose one or more.)
9. Would you like to buy the printed copies of SCMJ, whose prices a year are US$60(6,000yen) for 1-year-members(A1, D1, S-A1, S-D1)and US$55(5,500yen) for 4-year-members(A4, D4, S-A4, S-D4) ? Type YES or NO.
10. If you apply for an aged member (70 years old or over), please type the year of your birth.
11. If you wish to be a student member, please verify.
12. Is your university (institution) an Academic or Institutional Member of the ISMS? Yes or No.
13. If the answer of 12 is Yes, please answer the following. Are you designated associate member by your university (institution)?
14. Date
15. Signature

For Japanese Applicants, please send two application forms, one in English and the <u>other in Japanese.</u>

I wish to enroll as a member of ISMS and will pay to International Society for Mathematical Sciences the annual dues upon presentation of an invoice. Copies of Mathematica Japonica, Scientiae Mathematicae and Scientiae Mathematicae Japonicae received as an ISMS member will be for my personal use and shall not be placed in institutional, university or other libraries or organizations, nor can membership subscriptions be used for library purposes.

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Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)**, which enjoyed an international reputation, for about sixty years in print and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The number of the annual total pages of the print version has been from 900 to 1,200 pages in six issues since January 1978. The whole number of SCMJ exceeds 240, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 90 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The submitted papers are received directly by the editors and are refereed quickly. The accepted papers are published online with no lead time after compiling or proofreading. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) The original papers and surveys of distinguished mathematical scientist appear in every issue of SCMJ. The section called "International Plaza" of SCMJ has very interesting expository papers written by the eminent mathematical scientist of the world. Presentations of recent research frontier including award lectures by the winners of the ISMS Prize or Shimizu Prize are made.
4) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.
5) The ISMS holds inter-regional videoconferences called **International Videoconference of Mathematical Sciences (IVMS)** via internet. There is no need for the participants to travel abroad.

Privileges to ISMS Members: (1) Free access (**including printing out**) to the online version of SCMJ, (2)Discounted price for the printed version of SCMJ (See **Table 1**), (3) Discounted page charges (See **Table 2**).

Privileges to Institutional Members: (1) Two associate members can be registered, free of charge, from an institution. (2) The discounted page charges (Table 2) are applied to the associate members.

Table 1: Subscription Price (from 2007)

	Individual 1-year mem.	Individual 3-year mem.	Institutional member	List Price
Print / year	¥6,000 US$60, €48	¥5,500 * US$55, €44	¥33,000 US$300, €240	¥45,000 US$400, €320
Online/year	Free	Free	—	—
Online+Print / year	¥6,000 US$60, €48	¥5,500 * US$55, €44	¥33,000 US$300, €240	¥45,000 US$400, €320

Postal charge is US$2 (€1.6) per issue. *In case three-year members make the payment at a time in advance, the price for 3 years is ¥15,000 (US$150, €120). The authors can buy a copy of the print version at a price of ¥1,200 (US$12) per issue including postage.

Table 2: Page charge for Printed Page

Every accepted paper is charged ①**¥1,000(US$10, €7) as handling charges** plus ②**page charges.** The page charges per printed page are reduced as follows.

	ISMS members	Non-members
p	¥ 3,500 (US$35, € 23)	¥ 4,000 (US$40, €27)
Tex	¥ 2,000 (US$20, € 14)	¥ 2,500 (US$25, €17)
LateX2e, LaTeX	¥ 700 (US$ 7, € 4)	¥ 1,000 (US$10, € 7)
Js (ISMS style file)	¥ 500 (US$ 5, € 3)	¥ 800 (US$ 8, € 5)

Table 3: Membership Dues for this year

Categories	Domestic	Overseas	Developing countries
1-year member (1A)	A1: ¥9,000	F1: US$75 , €60	D1: US$45, €36
3-year member (3A)	A3: ¥24,000	F3: US$200, €160	D3: US$117, €93
1-year students or aged (1S)	SA1: ¥5,000	SF1: US$40, €32	SD1: US$27, €21
3-year students or aged (3S)	SA3: ¥12,000	SF3: US$100, €80	SD3: US$71, €57
Life member* (L)	AL: ¥90,000	FL: US$740, €592	DL: US$616, €493

*The members who have been the ISMS members for more than 10 years are eligible for this category. The categories 1S and 3S are for students or persons over 70 years old.