Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Special Version Whole Number 281

2015

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

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FOREWORD

The Forum for Interdisciplinary Mathematics, FIM for short, is a registered trust in India to promote research in mathematics and its applications/relation in diverse branches of science and technology. Incepted in 1975 by a group of intellectuals in Delhi led by Dr. Bhu Dev Sharma, the forum has been holding annual international conferences in India and abroad alternately. The Twenty Second International Conference of the forum on interdisciplinary mathematics, statistics and computational techniques was held during 10-12, November 2013, in Kokura of Kitakyushu city blessed with both a rich natural environment and a high level of culture. The proceedings of the conference were held in the Kitakyushu International Conference Centre with excellent facilities and the conference itself was sponsored by the International Society of Management Engineers, the Graduate School of Information. Production Systems of the Waseda University, Kitakyushu, besides the International Society for Mathematical Sciences and the City of Kitakyushu.

Fifty abstracts were accepted for presentation in the conference comprising three key-note speeches and five plenary talks. Apart from paper reading sessions on applied mathematics, mathematical systems and network and computer systems, four invited sessions/symposia on Rough sets, computational techniques and Combinatorial Design were also held as part of this conference. Part of the submitted papers of a more theoretical/mathematical nature, numbering ... that have been refereed and accepted, are being published in this special issue of Scientiae Mathematicae Japonicae, published by the International Society for Mathematical Sciences. The guest-editors are grateful to the International Society for Mathematical Sciences and the Forum for Interdisciplinary Mathematics for this opportunity and hope that this collaboration would continue in the future as well.

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ON THE SPACE OF FUZZY NUMBERS

P.V. Subrahmanyam

28 January, 2014.

1 Introduction Ever since Frechet introduced them in 1928, metric spaces have come to stay as a basic aspect of abstract analysis. Several important classes of functions and their modes of convergence typify metric spaces and supplement our understanding of these classes. Zadeh[22], propounded the theory of fuzzy sets and fuzzy logic, in a path breaking publication in 1965 to study quantitatively problems involving uncertainty due to subjective considerations. Since then a number of attempts have been made to endow fuzzy sets with interesting metrics. A metric being a non-negative real-valued function it is natural to explore if it could take values in the set of fuzzy real numbers. Notable contributions along this line are due to Kaleva and Seikkala [12] followed by Felbin [7]. Kaleva [9] had also shown that a fuzzy metric space (in the sense of Kaleva and Seikkala [12]) has a completion unique up to isometry. In another direction Kramosil and Michalek [13] defined a fuzzy metric space in analogy with and equivalent to a statistical metric space as defined by Menger [14]. Inspired by an intermediate function considered by Hausdorff in defining the Hausdorff distance between closed and bounded subsets of a metric space, Erceg [6] defined a pseudo quasimetric as a map satisfying some natural conditions from $L^X \times L^X$ into $[0, \infty]$, L^X being the set of all maps from a set X into L, a completely distributive lattice with order-preserving involution. For fuzzy points, a pseudo metric was defined and studied by Deng [1]. Subsequently Peng Yu Wei [15] simplified the concept of Erceg's pseudo quasi metric and also related his concept and results to Erceg's theory. Later Rodabaugh [17] and subsequently Jian-Zhong Xiao and Xing-hua Zhu [20] examined $L-$ fuzzy real line for a completely distributive lattice L, vis-a-vis Erceg's pseudo metric.

Dubois and Prade [4] defined a fuzzy real number as a continuous function $\mu : \mathbb{R} \to [0, 1]$ vanishing outside a compact interval $[c, d]$ of real numbers such that for some real numbers a and b with $c \leq a \leq b \leq d$, μ increases on $[c, a]$ and decreases on $[b, d]$ and $\mu(x)$ is 1 on $[a, b]$. Goetschel and Voxman [8] modified the assumption of continuity in the definition of Dubois and Prade to upper semicontinuity to avoid any inconsistency, while including the characteristic functions of singleton real numbers. More importantly they defined a metric for this set of fuzzy real numbers, based on the Hausdorff distance between closed and bounded subsets. This metric has found applications in the study of fuzzy random variables (see Puri and Ralescue [16]), fuzzy differential equations (Kaleva [10]) and the calculus of fuzzy real variables (Kaleva [11]) and has been extensively studied by Diamond and Kloeden in their monograph [3]. Besides this metric, other metrics on fuzzy real numbers have also been studied by Voxman [18] (using reducing functions), Yang and Zhang [21] (endograph metric) and Wu Congxion, Hongliang and Xuekun [19] (sendograph metric). Diamond and Kloeden ([3], [2]) may be consulted for further details.

The purpose of this paper is to consider a wider class of fuzzy subsets of real numbers that can be topologized by a family (gauge) of pseudometrics and study its properties. These fuzzy numbers need not have bounded supports, though their supports intersect a

2010 *Mathematics Subject Classification.* subject classifications .
Key words and phrases. keywords .

fixed closed set. In this way this class of fuzzy numbers serves to supplement the existing theory of fuzzy real numbers.

2 The space $F_U(\mathbb{R})$

We recall the following

Definition 2.1. *A fuzzy subset u of a topological space (X, τ) is called upper semi-continuous if $u : X \to [0,1]$ is a mapping such that $[u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ is a closed subset X for each $\alpha \in [0,1]$. A fuzzy subset $u : X \to [0,1]$ is called normal if $\{x : u(x) = 1\} = \{x : u(x) \geq 1\}$ is nonempty.*

We denote the set of all normal upper semi-continuous fuzzy subsets of X by $F_U(X)$. In particular $F_U(\mathbb{R})$ is the set of all normal upper semi-continuous mappings of $\mathbb{R}$ (with the normal topology) into $[0,1]$.

We now prove a representation theorem for members of $F_U(X)$, X being a topological space.

Theorem 2.1. *Let (X, τ) be a topological space and $u \in F_U(X)$, the set of all normal upper semi-continuous fuzzy subsets of X. For each $\alpha \in I = [0,1]$, let $C_\alpha = [u]^\alpha = \{x \in X : u(x) \geq \alpha\}$. Then*

(i) for each $\alpha \in I$, C_α is a nonempty closed subset of X;

(ii) $C_\beta \subseteq C_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$;

(iii) $C_\alpha = \bigcap_{i=1}^{\infty} C_{\alpha_i}$, for each sequence α_i increasing to α in I.

Conversely, if in a topological space (X, τ), there is a family of nonempty closed subsets $\{C_\alpha : \alpha \in I = [0,1]\}$ satisfying properties (i), (ii) and (iii) above, then there is a unique $u \in F_U(X)$ such that $[u]^\alpha = C_\alpha$ for each $\alpha \in [0,1]$.

Proof. Since $u \in F_U(X)$ in an upper semi-continuous map of X into $[0,1]$, $C_\alpha = [u]^\alpha$ in a closed subset of X for each $\alpha \in [0,1]$. Since $C_1 = \{x : u(x) \geq 1\}$ is nonempty, $C_\alpha(\supseteq C_1)$ is nonempty for each $\alpha \in [0,1]$. For $0 \leq \alpha \leq \beta \leq 1$, $C_\beta \subseteq C_\alpha$ is obvious. Thus for $u \in F_U(X)$, (i), (ii) and (iii) are true.

Conversely, suppose $\{C_\alpha : \alpha \in I = [0,1]\}$ is a family of subsets of X satisfying (i)-(iii). Define $u : X \to [0,1]$ by

$$u(x) = sup\{\alpha \in I : x \in C_\alpha\}$$

Clearly u is a well-defined map of X into $[0,1]$, since $C_0 = X$. Since $C_1 \neq \emptyset$, $u(x) = 1$ for some $x \in X$ and so u is normal. For $\alpha \in I$, if $x \in [u]^\alpha$, then $u(x) \geq \alpha$. Let $I_x = \{\beta \in I : x \in C_\beta\}$ and $\alpha^{'} = sup\ I_x$, so that $\alpha^{'} = u(x)$. Clearly $\alpha^{'}\ (= u(x)) \geq \alpha$ and by hypothesis $x \in C_{\alpha'} \subseteq C_\alpha$. So $[u]^\alpha \subseteq C_\alpha$. On the other hand if $x \in C_\alpha$, then $u(x) = sup\ I_x = \alpha^{'} \geq \alpha$ and consequently $x \in [u^\alpha]$, so that $C_\alpha \subseteq [u]^\alpha$. Thus each $[u]^\alpha = C_\alpha$ for $\alpha \in I$ and hence u is an upper semi-continuous function. □

For topological spaces which are sums of an increasing family of proper closed subsets this representation theorem can be stated in a different form. For this we need the following

Definition 2.2. *A topological space (X, τ) is called $F-$ summable if $X = \bigcup\{F_t : t \in P\}$ satisfying the following conditions:*

(i) $(P, \leq)$ is a totally ordered set with a least element $\widehat{o}$;

(ii) every nonempty subset of P has a greatest lower bound in P;

(iii) *each F_t is a nonempty proper closed subset of X and $F_t \geq F_s$ for $t \geq s$, $t, s \in P$. Further $F_t \neq F_s$ for $t > s$.*

Theorem 2.2. *Let X be an $F-$ summable topological space as in Definition 2.2 and $u \in F_U(X)$. Then for each $t \in P$ and $\alpha \in I = [0,1]$, the sets $C_{\alpha,t} = u^{[\alpha]} \cap F_t$ satisfy the following:*

(i) *$C_{\alpha,t}$ is a nonempty closed subset of X for all $t \geq t_0 \in P$ for all $\alpha \in [0,1]$;*

(ii) *$C_{\beta,t} \subseteq C_{\alpha,t}$ for all $0 \leq \alpha \leq \beta \leq 1$ for all $t \in P$;*

(iii) *If $C_{\alpha,t} \neq \emptyset$ and $\alpha_i (\in [0,1]) \uparrow \alpha$ then $C_{\alpha,t'} = \cap_{i=1}^{\infty} C_{\alpha_i,t'}$ for all $t^{'} \geq t$;*

(iv) *$[u]^\alpha = \cup_{t \in P} C_{\alpha,t}$ is closed for each $\alpha \in I$.*

Conversly, if X is an $F-$ summable topological space (as in Definition 2.2) and $C_{\alpha,t}$, $\alpha \in [0,1]$, $t \in P$ is a family of closed subsets of X satisfying $(i) - (iv)$ above. Then there exists a unique $u \in F_U(X)$ such that for each $\alpha \in I$ and $t \in p$, $[\ u\ ]^\alpha \cap F_t = C_{\alpha,t}$.

Proof. While the proof of necessity part of the theorem is straight-forward, for proving the sufficiency part, define $u : X \to [0,1]$ by $u(x) = sup\{\alpha \in [0,1] :\ x \in C_{\alpha,t}\ for\ least\ t \in P\}$. Since $x \in X = \cup_{t \in P} F_t$, $x \in F_t$ for smallest $t \in P$ and $1 \geq u(x) \geq 0$. Let $\alpha_0 = u(x)$. Then $x \in C_{\alpha_0,t_0}$ clearly $[u]^\alpha \cap F_{t_0} = C_{\alpha_0,t_0}$. Further $[u]^\alpha = \cup_{t \geq t_0} C_{\alpha,t}$ is closed, by (iv). Thus u is upper semi-continuous. Since $C_{1,t}$ is a nonempty closed subset of X for some t_0, $[u]^1 = \cup_{t \geq t_0} C_{1,t}$ is a closed set by (iv) and u is normal. Thus $u \in F_U(X)$. □

3 A topology on a subspace of $F_U(\mathbb{R})$ Let (X, d) be a metric space and $F_U(X)$ the set of all normal upper semi-continuous fuzzy subsets of X. For a fixed element a of X, let B_n denote the closed ball in X centered at a and radius r_n and H_n be the Hausdorff metric on the nonempty closed subsets of B_n for each $n \in \mathbb{N}$. As B_n is bounded, the Hausdorff distance H_n induced by d is well-defined on the family of nonempty closed subsets of B_n. We recall the following

Definition 3.1. *Let d_λ be a pseudometric on a nonempty set X for each $\lambda \in \Lambda$. The family $D = \{d_\lambda : \lambda \in \Lambda\}$ is called separating if for $x, y \in X$ with $x \neq y$, there exists $\lambda_0 \in \Lambda$ such that $d_{\lambda_0}(x, y) > 0$. The topology $\tau(D)$ with the subbase $\{B(x; d_\lambda, t) :\ x \in X, \lambda \in \Lambda$ and $\epsilon > 0\}$ is called the topology on X induced by the family D. D is called a gauge and a topological space whose topology admits a gauge structure is called a gauge space.*

Definition 3.2. *Let (X, D) be a gauge space and (x_n), a sequence in D is called Cauchy if $lim_{n,m \to \infty} d_\lambda(x_n, x_m) = 0$ for each $d_\lambda \in D$. If every Cauchy sequence in (X, D) converges to a limit, (X, D) is called sequentially complete.*

We also recall the following

Theorem 3.1. *(see Dugundji [5]) A topological space is a gauge space if and only if it is completely regular (or Tychonoff). A gauge space is metrizable if and only if it has a countable gauge.*

With these preliminaries, we can provide a metric topology on $CL_1(X)$ for any metric space (X, d), that have a non-void intersection with $B(a; r_1)$.

Theorem 3.2. *Let (X,d) be a metric space. Then H_n is a pseudo-metric on $CL_1(X)$, the set of all nonvoid closed subsets of X, that have a non-void intersection with $B(a;r_1)$ for each $n \in \mathbb{N}$, where $H_n(A,B) = H(A \cap B_n,\ B \cap B_n)$ for $A, B \in CL_1(X)$, H being the Hausdorff distance induced by d. Then $CL_1(X)$ is a gauge space with the gauge $\{H_n : x \in \mathbb{N}\}$ and is metrizable assuming that $lt_{n\to\infty} r_n = \infty$ and $\sum \frac{r_n}{2^n}$ converges. If X is complete then $CL_1(X)$ is also complete.*

Proof. Since H_n is the Hausdorff metric on $CL(B_n)$, H_n is a pseudo-metric on $CL(X)$ for each $n \in \mathbb{N}$. For $A \neq B$ in $CL(X)$, $A \cap B_n \neq B \cap B_n$ for some $n = n_0$. So $H_{n_0}(A,B) = r > 0$. Thus $\{H_n : n \in \mathbb{N}\}$ is a countable separating family of pseudometrics on $CL_1(X)$. Thus $CL_1(X)$ is a metrizable gauge space, in view of Theorem 3.1. Since $H_n(A,B) \leq r_n$ for all $A, B \in CL_1(X)$ and $n \in \mathbb{N}$, and $\sum_1^\infty \frac{r_n}{2^n}$ converges. $H(A,B) = \sum_1^\infty \frac{H_n(A,B)}{2^n}$ defines a metric on $CL_1(X)$. Further, this metric topology is the same as the gauge topology (we take $H_n(A,B) = 0$ whenever $F_n \cap A$ or $F_n \cap B = \phi$).

Let $\{C_n\}$ be a Cauchy sequence in $CL_1(X)$. Without loss of generality, we can assume that $\sum_1^\infty H(C_i, C_{i+1}) < \infty$. For C_1 and C_2 we can find n_1 so that $C_1 \cap F_{n_1}$ and $C_2 \cap F_{n_1} \neq \phi$. So for $x_1 \in C_1 \cap F_{n_1}$, noting that $\frac{H_{n_1}}{2^{n_1}}$ is the Hausdorff metric on F_{n_1} induced by $\frac{d}{2^{n_1}}$, we can find $x_2 \in C_2 \cap F_{n_1}$ such that

$$\tfrac{d(x_1,x_2)}{2^{n_1}} < \tfrac{1}{2^{n_1}}(H_{n_1}(C_1,C_2) + 1).$$

For this n_1, we can find $n_2 > n_1$ so that $C_2 \cap F_{n_2}$ and $x_3 \in C_3 \cap F_{n_2} \neq \phi$. Since $\frac{1}{2^{n_2}} d$ induces $\frac{1}{2^{n_2}} H_{n_2}$, a Hausdorff metric on F_{n_2}, we can find $C_3 \cap F_{n_3}$ so that

$$\tfrac{d(x_2,x_3)}{2^{n_2}} < \tfrac{1}{2^{n_2}}(H_{n_2}(C_2,C_3) + \tfrac{1}{2}).$$

Thus proceeding we get a sequence of elements $(x_k) \in C_n \cap F_{n_k}$ so that

$$\frac{d(x_k,x_{k+1})}{2^{n_k}} < \frac{1}{2^{n_k}}(H_{n_k}(C_k,C_{k+1}) + \frac{1}{2^k}). \tag{1}$$

Since $\sum_{i=1}^\infty H(C_i,C_{i+1})$ is convergent,

$$\textstyle\sum_{k=1}^\infty H_{n_k}(C_{n_k}, C_{n_{k+1}}) + \frac{1}{2^k}$$

converges. From (1) it follows that $d(x_k,x_{k+1}) < H_{n_k}(C_{n_k}, C_{n_{k+1}}) + \frac{1}{2^k}$ and so $\sum_{k=1}^\infty d(x_k,x_{k+1})$ is finite. Hence $\{x_n\}$ is a Cauchy sequence that converges to some element x by the completeness of X. Since $x_k \in C_k$ for each k, $x_n \in \cup_{n\geq k} C_n$ for all $n \geq k$. So $x^* = lim_{n>k} x_n$, $x^* \in \overline{\bigcup_{k\geq n} C_k}$ for all k. Thus $x^* \in \cap_{n=1}^\infty (\overline{\bigcup_{k\geq n}^\infty C_k})$.

Define $C = \cap_{n=1}^\infty (\overline{\bigcup_{k\geq n}^\infty C_k})$. Then C is nonempty and closed and is the closure of the set of all limit points of $\{x_n\}$. We now show that $H(C,C_n) \to 0$ as $n \to \infty$. For any $\epsilon > 0$ given, let $n = N(\epsilon)$ be chosen so that $\sum_{n=N(\epsilon)}^\infty [H(C_n,C_{n+1}) + \frac{1}{2^n}] < \frac{\epsilon}{2}$. Let $x^* \in C$ and x_0 be the limit of sequence (x_n) so that $d(x^*,x_0) < \frac{\epsilon}{2}$. Then the distance of x^* from $C_k = d(x^*,C_k)$ is

$$\leq d(x^*,x_0) + \textstyle\sum_{n=k}^\infty d(x_n,x_{n+1})$$

$$< d(x^*,x_0) + \textstyle\sum_{n=k}^\infty [H(C_n,C_{n+1}) + \frac{1}{2^n}]$$

$$< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \text{ for } k \geq N(\epsilon).$$

Since any $x_k \in C_k$ can be the starting point of such a convergent sequence (x_n) converging to x_0,

$$d(x_k,x_0) \leq \textstyle\sum_{n\geq k}^\infty d(x_n,x_{n+1}) < \sum_{n\geq k}^\infty [H(C_n,C_{n+1}) + \frac{1}{2^n}]$$

$$< \frac{\epsilon}{2}, \text{ for all } k \geq N(\epsilon).$$

So it follows that $H(C, C_k) = max\{sup_{x^* \in C} d(x^*, C_k), sup_{x_k \in C_k} d(x_k, C)\} < \epsilon$ for $k \geq N(\epsilon)$. Thus $lim_{k\to\infty} C_k = C$ in $CL_1(X)$. Thus for a complete metric space (X, d) that is the countable union of closed spheres $B(a; r_n)$ where (r_n) increases to ∞ with $\sum \frac{r_n}{2^n} < \infty$, the set of all non-void closed subsets of X that intersect $B(a; r_1)$ (and hence $B(a; r_n)$ for all n) can be given a complete metric using the Hausdorff metric on $\subset B(a; r_n)$. Consider a subclass $F_U^1(\mathbb{R})$ of $F_U(\mathbb{R})$ comprising upper semicontinuous functions $u : \mathbb{R} \to [0, 1]$ such that $[u]^1 \subseteq [-r_1, r_1]$ where $\mathbb{R} = \cup_{n=1}^{\infty}[-r_n, r_n]$, $0 < r_n$, $lim_{n\to\infty r_n} = \infty$ and $\sum \frac{r_n}{2^n} < +\infty$. Clearly for such functions the level sets need not be compact nor convex. Although for such functions, the level of normality has to lie in $[-r_1, r_1]$, by choosing r_1 sufficiently large many fuzzy numbers with compact support can be found in $F_U^1(\mathbb{R})$. The following theorem shows that $F_U^1(\mathbb{R})$ and more generally $F_U^1(X)$ admits a complete metric so that analysis can be carried out in $F_U^1(X)$. □

Theorem 3.3. *Let (X, d) be a complete metric space. Suppose $X = \cup_{n=1}^{\infty} B(a; r_n)$ where $B(a; r_n)$ is the closed sphere centered at a and radius r_n with $lim_{n\to\infty} r_n = +\infty$ and $\sum_1^{\infty} \frac{r_n}{2^n} < \infty$. Let $F_U^1(X)$ be the set u of all normal upper semicontinuous fuzzy subsets of X, so that $[u]^1 \cap B(a, r_1) \neq \emptyset$. Then $F_U^1(X)$ is a complete metric space under the metric $\triangle$ defined by $\triangle(u, v) = sup_{0\leq\alpha\leq 1} H([u]^\alpha, [v]^\alpha)$ where $H(A, B) = \sum_{n=1}^{\infty} \frac{H_n(A,B)}{2^n}$ (as defined in Theorem 3.2), for $A, B \in CL_1(X)$.*

Proof. Clearly $F_U^1(X)$ is nonempty, as the characteristic function of $B(a, r_1)$ is in $F_U^1(X)$. For $u \in F_U^1(X)$, for all $\alpha \in [0, 1]$, the closed sets $[u]^\alpha \supseteq [u]^1$ and the nonempty set $[u]^1 \subseteq B(a, r_1)$. So for $u, v \in F_U^1(X)$, for $0 \leq \alpha \leq 1$,

$$H([u]^\alpha, [v]^\alpha) = \sum_{n=1}^{\infty} \frac{H_n([u]^\alpha, [v]^\alpha)}{2^n} \leq \sum_{n=1}^{\infty} \frac{r_n}{2^n} = k < \infty,$$

for $sup_{0\leq\alpha\leq 1} H([u]^\alpha, [v]^\alpha) = \triangle(u, v) \leq k$ is well-defined. Also for $0 \leq \alpha \leq 1$, $u, v, w \in F_U^1(X)$

$$H([u]^\alpha, [v]^\alpha) \leq H([u]^\alpha, [w]^\alpha) + H([w]^\alpha, [v]^\alpha)$$

and so $\triangle(u, v) \leq \triangle(u, w) + \triangle(w, v)$. Thus $(F_U^1(X), \triangle)$ is a metric space.

For proving the completeness of $F_U^1(X)$ under $\triangle$, consider a Cauchy sequence u_n in $F_U^1(X)$. So given $\epsilon > 0$, we can find $M(\epsilon) \in \mathbb{N}$ such that $\triangle(u_k, u_m) < \epsilon$ for all $k, m \geq M(\epsilon)$. Let $H_n^1(u, v) = sup_{0\leq\alpha\leq 1} H_n([u]^\alpha, [v]^\alpha)$ for each $n \in \mathbb{N}$. Since the gauge $\{H_n^1 : n \in \mathbb{N}\}$ generates $\triangle$ and $\{u_n\}$ is Cauchy with respect to $\{H_n^1 : n \in \mathbb{N}\}$, it follows that $\{[u]_n^\alpha \cap B_n\}$ is uniformly Cauchy in α for a fixed n and being a Cauchy sequence of closed sets in the complete space $CL(B_n)$, $[u]_n^\alpha \cap B_n$ converges to $C^\alpha \cap B_n$ for each n uniformly in α in $CL(B_n)$. Clearly the family of closed sets $\{C^\alpha \cap B_n : \alpha \in [0, 1], n \in \mathbb{N}\}$ satisfies the conditions of Theorem 2.2 and so there exists a function u in $F_U^1(X)$ for which $[u]^\alpha = \cup_{n=1}^{\infty} C^\alpha \cap B_n = C^\alpha$ is closed for $0 \leq \alpha \leq 1$. Further $H_n^1(u, u_m) \to 0$ as $m \to \infty$ for each $n \in \mathbb{N}$. Thus $F_U^1(X)$ is complete. □

Remark 3.1. *If we specialise X to $\mathbb{R}$ or $\mathbb{R}^n$ $(n > 1)$, the $F_U^1(X)$ is a special space of fuzzy numbers whose support can be unbounded. It will also contain all fuzzy numbers with support lying in a prescribed interval. In a sense this can supplement the space (E^n, d_∞) considered notably by Kaleva [10] and Kloeden and Diamond [3].*

4 An alternative approach While the space $F_U^1(X)$ complements E^1 or E^n with d_∞ for $X = \mathbb{R}^1$ *or* $\mathbb{R}^n$ respectively, $F_U^1(\mathbb{R})$ or $F_U^1(\mathbb{R}^n)$ does not contain E^1 or E^n. However, this situation can be remedied in the following manner: for a metric space (X, d), the metric topology induced is the same as the topology induced by the bounded metric d^* defined by $d^* = \ min\{1, d(x, y)\}$ for $x, y \in X$ so that (X, d^*) is complete whenever (X, d) is complete. The following theorem is easy to prove.

Theorem 4.1. *Let (X, d) be a metric space and d^* be defined by $d^*(x, y) = \ min\{1, d(x, y)\}$ for $x, y \in X$. Then (X, d^*) is a metric space and H^* be the Hausdorff metric induced by d^* on $CL(X)$, the set of all non-void closed subsets of X. If d is complete, then d^* is complete, Further $(CL(X), H^*)$ is also complete.*

This enables us to define a metric on $F_U(X)$, the space of normal upper semi-continuous fuzzy subsets of metric space (X, d) into $[0, 1]$. Again the proof of the following theorem is straight forward.

Theorem 4.2. *Let (X, d) be a complete metric space. Then $F_U(X)$, the space of all normal upper semi-continuous fuzzy subsets of X is a complete metric space with the metric D^* defined by $D^* = \ sup_{0 \le \alpha \le 1} H^*([u]^\alpha, [v]^\alpha)$ for $u, v \in F_U(X)$, H^* being the Hausdorff metric on $CL(X)$ induced by $d^* = \ min\{1, d\}$.*

Remark 4.1. *Besides D^*, other bounded metrics homeomorphic to d can be used to generate Hausdorff metrics on $CL(X)$. This, in turn can be used to metrize $F_U(X)$, the space of normal upper semi-continuous fuzzy subsets of X.*

Remark 4.2. *If (X, d) is a real normed linear space, then taking $r_n = n \in \mathbb{N}$ and $a = 0$, the zero vector, it can be seen that the maps ϕ_t defined by*

$$\phi_t(x) = \begin{cases} 1 & x = 1, \\ t & x \in [0, 1), \\ 0 & otherwise \end{cases} \tag{2}$$

are in $F_U^1(\mathbb{R})$ and $\triangle(\phi_t, \phi_s) \ge \frac{1}{2}$. Consequently $F_U^1(X)$ containing an isometric copy of $F_U^1(\mathbb{R})$ is not separable.

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Risk Analysis of Portfolio Based on Kernel Density Estimation-Maximum Likelihood Method and Monte Carlo Simulation

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Feb, 5, 2015

Abstract.

Nowadays one of the most studied issues in economic or finance field is to get the best possible return with the minimum risk. Therefore, the objective of the paper is to select the optimal investment portfolio from SP500 stock market and CBOE Interest Rate 10-Year Bond to obtain the minimum risk in the financial market.

For this purpose, the paper consists of: 1) the marginal density distribution of the two financial assets is described with kernel density estimation to get the "high-picky and fat-tail" shape; 2) the relation structure of assets is studied with copula function to describe the correlation of financial assets in a nonlinear condition; 3) value at Risk (VaR) is computed through the combination of Copula method and Monte Carlo simulation to measure the possible maximum loss better.

Therefore, through the above three steps methodology, the risk of the portifolio is described more accuratly than the conventional method, which always underestimates the risk in the finicial market.

So it is necessary to pay attention to the happening of extreme cases like "Black Friday 2008" and appropriate investment allocation is a wise strategy to make diversification and spread risks in financial market.

uzzy regression model, fuzzy random variable, expected value, variance, confidence interval.

1 Introduction

In finance market, with fierce volatility, the risk management has become a hot research issue in the study. Especially after the accident happened such as the closing down of Barings Bank and the bankrupt of Enron Corp, in the analysis of portfolio the emphasis has moved on the balance between profit and safety.

For the conventional methods, person coefficient is used to measure the correlation of variables and Risk metrics are common ways to calculate VaR. However, due to the assumption of the methods are based on normal distribution, the methods deviate from the real situation more or less.

Therefore, it is necessary to propose a new assets allocation method to evaluate the risk of portfolio in the financial market.

Firstly, according to Markowitz 1987[17]; Terrance. C. Mills 2002[18], the assumption that the distribution of assets return rate submits normal distribution always neglects the happening of extreme conditions, which results in lack of precaution and huge losses in the

2010 *Mathematics Subject Classification.* subject classifications .
Key words and phrases. Sensitivity analysis, portfolio selection model, value at risk (VaR).
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Table 1: Tabe l

	Distribution	Theory	Relation Structure	Risk at Value
Conventional	Normal Distribution	Central Limit Theorem	Person Coefficient	Risk Metrics
Burgeoning	High-picky; Fat-tail	Kernel Density Estimation	Copula Function	Monte Carlo S

end. Meanwhile, lots of experiments have indicated the return curve presents "high-picky" and "fat-tail". So it is necessary to estimate the probability distribution density of asset return with kernel smoothing under a wide precondition.

Secondly, from Embrechts 1999[7], based on figuring out the marginal density distribution of financial assets, the study of relation structure between two financial assets is an important step in the asset allocation and risk management. In the premise of normal distribution, Pearson correlation is a common option to describe the linear relationship. However, some defects such as restricted variance, and easy to be distorted show its bounded-ness in the nonlinear application.

Therefore, from Sklar1959 [30]; Nelsen 1999[19], Copula model is introduced and widely used as a link function $C(u_1, u_2, \cdots, u_N)$ to define the simultaneous distribution $F(x_1, x_2, \cdots, x_N)$ according to the marginal distribution $F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_N}(x_N)$ of random variables $X_1, X_2, \cdots, X_N$. Namely,

$$F(x_1, x_2, \cdots, x_N) = C[F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_N}(x_N)] \tag{1}$$

Copula function is not only the tool to build the joint probability of multi-dimensional random variables, but also the one to explore the relation structure among random variables.

Thirdly, after better fitting the joint distribution and describing the relation structure, we can obtain the value in risk of portfolio return more accurately, which has become main qualitative technology in risk degree.

From the definition of Philippe Jorion [13], Value at Risk (VaR) is aimed to compute the potential loss of financial assets using distribution function in a certain holding period and confidence level c. If z and VaR indicate the value of financial assets and the risk value respectively, then

$$P(z \leq VaR) = 1 - c \tag{2}$$

Here Monte Carlo simulation is applied to reckon the yield distribution of portfolio risk factors, hence the gains and losses could be constructed in the portfolio and the risk value is estimated in the light of given confidence level.

To sum up, the comparison of the conventional and burgeoning methodologies follows the next table:

Recently, from C.Perignon2010 [], D.Fantazzini2009 [] and J.Shin2009 [], the burgeoning methodology has an obvious effect on analyzing the risk of portfolio in the financial market.

The paper is organized as follows. Section 2 presents the kernel density estimation, the relation structure based on copula model and VaR calculation by Montel Carlo simulation. The combination of the three methods has an obvious advantage compared with the conventional one with linear premise. Section 3 discusses empirical results according to the past and present one, respectively. Section 4 discusses the empirical results. Section 5 is the conclusion.

2 Method

2.1 kernel density estimation (KDE) Experiences show that a large gap is formed between premises of the distribution of financial assets and the complexity in practice. So an approach like the kernel density estimator mitigates the rigidity of the function that belongs to a certain group and hence deserves to be applied in the financial issue.

Let $X_1, X_2, \cdots, X_n$ be independent samples obtained from an unknown density function $f(x)$. $f(x)$ is the formula of kernel density estimator (KDE) (M. Rosenblatt)[28]:

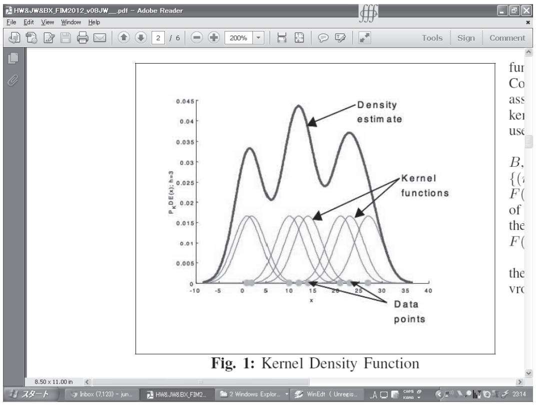

Figure 1: Kernel Density Function

Kernel Density Estimator:

$$P_{KDE}(x) : \hat{f}(x, h) = (nh)^{-1} \sum_{i=1}^{n} K(\frac{x - x_i}{h}) \tag{3}$$

K denotes kernel and h is bandwidth; The smooth kernel estimate is a sum of "bumps" and the kernel function K determines the shape of the bumps. Because of higher efficiency, Gaussian kernel

$K_G(u) = (\sqrt{2\pi})^{-1}\exp(-\frac{z^2}{2})$ is adopted; the parameter h, also called the "bandwidth," determines their width. (M.P.Wand; M.C.Johns)[31]

The bandwidth h plays the role of a scaling factor in determining the spread of the kernel. And it determines the amount of smoothing applied in estimating $f(x)$. The following is the "rule of thumb," which is the most widely used method. (Silveman)[29]

If $f(x)$ is a normal dessity function, then:

$$\int (f(x)^2 dx = \frac{3}{8}\pi^{-0.5}\sigma^{-5} \approx 0.212\sigma^{-5} \tag{4}$$

normal kernel

$$K(u) = (\sqrt{2\pi}^{-1} \exp(\frac{-u^2}{2}) \exp : h = 1.06\sigma n^{\frac{-1}{5}} \tag{5}$$

Hjort and Jones (1996)[10] proposed an improved rule obtained by using an Edgeworth expansion for $f(x)$ around the Gaussian density. Such a rule is given by:

$$\hat{h}^*_{opt} = h_{AMISE}(1 + \frac{35}{48}\hat{\gamma}_4 + \frac{35}{32}\hat{\gamma}_3 + \frac{385}{1024}\hat{\gamma}_4)^{\frac{-1}{5}} \tag{6}$$

2.2 Relation structure based on Copula function According to Sklar theorem, a multiple joint distribution function could be described with marginal distribution and Copula model. To portray the relation structure of financial assets, a kind of two phases method, which is named as kernel density estimation-maximum likelihood method, is used here. (Bouye 2000)[1]

When random variables are two financial assets A and B, whose observation series of return rate (r_A, r_B) is $\{(r_A^t, r_B^t)\}_{t=1}^T$, the simultaneous distribution function is $F(x, y)$ and the probability density and distribution function of r_A and r_B are $f_A(x)$, $F_A(x)$, and $g_B(x)$, $F_B(x)$, and the Copula $C : C_\alpha(u_t, v_t) = C(F_A(r'_A), F_B(r'_B)) = F(r'_A, r'_B)$.

1) Primarily, kernel density estimation is used to measure the unknown marginal density of the financial assets. (Devroye 1983[4]; Fan Yao 2003[32])

$$\begin{aligned} f_A(x) &= \frac{1}{Th_A} \sum_{t=1}^T K_A(\frac{x-r'_A}{h_A}); \\ g_B(x) &= \frac{1}{Th_B} \sum_{t=1}^T K_B(\frac{y-r'_B}{h_B}); \end{aligned} \tag{7}$$

When $K(\cdot)$ is the normal kernel:

$$\begin{aligned} u_i &= \frac{1}{T} \sum_{j=1}^T \phi(\frac{r_A^t - r_A^j}{h_A}); \\ v_i &= \frac{1}{T} \sum_{j=1}^T \phi(\frac{r_B^t - r_B^j}{h_B}); \end{aligned} \tag{8}$$

2) Next the unknown parameter a in Copula is estimated by maximum likelihood and examined by frequency histogram graph and Minimum Variance Test to choose a optimal copula function. (Genest, Rivest 1993)[23]

The partial derivative is taken to the two sides of formula 1

$$f(x, y) = c_\alpha(F_X(x; \theta_x), F_Y(y; \theta_y)) f_X(x; \theta_x) f_Y(y; \theta_y), \tag{9}$$

$f_X(x; \theta_x)$ and $f_Y(y; \theta_y)$ are the marginal density function of $f(x, y)$, θ_x and θ_y are the parameters of marginal density $f_X(x)$ and $f_Y(y)$, α is the parameter of Copula, c_{alpha} is the density function of Copula: $c_{alpha}(u, v) = \dfrac{\partial^2 C(u, v)}{\partial u \partial v}$

Then the formula 8 is taken logarithm:

$$\begin{aligned} \ln L(\theta_x, \theta_y : \alpha) = \quad & \ln c_\alpha(F_X, F_Y) + \ln f_X(x, \theta_x) + \\ & \ln f_Y(y; \theta_y) \end{aligned} \tag{10}$$

From maximum likelihood (ML) conception, the log-likelihood function is:

$$\begin{aligned} l(v) = \quad & \sum_{t=1}^T \ln c(F_X(X_t; \theta_x), F_Y) + \ln f_x(X; \theta_x) \\ & + \sum_{t=1}^T \ln f_Y(Y_t; \theta_y) \end{aligned} \tag{11}$$

(V. Durrleman 2000[6]; Roberto De Matteis 2001[3]; Claudio Romano 2002[27])

Then, the parameter of Copula C is estimated with ML method:

$$\hat{\alpha} = arg \max \sum_{t=1}^T \ln c(u_t, v_t; \alpha), \tag{12}$$

$c(u, v)$ is the density of Copula,

To sum up, in the above two illustrated steps of the method, the density distribution of financial assets could be estimated in a wide postulated condition and a relation structure especially the tail dependence between them could be described effectively.

2.3 VaR Calculation Analytical Methods such as Variance-Covariance Approach offer an instinctive comprehension of the driving factors of risk in a portfolio, which derives from the risk metrics and obeys the normal distribution. When there are only two assets, the portfolio variance is: (Harry Markowitz, 1952[15]; Peter Zangari, 1996[33])

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{13}$$

And the portfolio VAR is then:

$$\begin{aligned} VaR_p &= \alpha\sigma_p W \\ &= \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}W \end{aligned} \tag{14}$$

where α is quantile of confidence, w weight, σ the variance of assets, ρ correlation coefficient, W the original value, respectively.

Withal, for Monte Carlo simulation based on Copula-VaR, on the one hand, Copula function has the advantage of depicting nonlinear and asymmetric and especially capturing the tail dependence; on the other hand, an abundance of random data that conform to historical distribution is generated to simulate the behavior of the return rate of financial assets by Monte Carlo method.

So the process of portfolio VaR of two assets X and Y based on Copula model and Monte Carlo simulation is followed: (Rank J, Siegl T, 2003[22]; Romano C, 2002[27])

1) The copula model is chosen to describe the marginal distribution of assets and related structure $C(*, *)$.

2) The parameter of Copula model is estimated according to the historical data of return rate of asset X and Y, and hence the distribution function of assets return $F(*)$, $G(*)$ and $C(u, v)$ that are to demonstrate the relation structure between assets could be confirmed. Thereinto, $u = F(Rx)$, $v = G(Ry)$, which submit to $(0, 1)$ even distribution.

3) Two independent random numbers u and v, which submit $(0, 1)$ even distribution, are generated. u is the first simulated pseudo random numbers (PRN). For another thing, $C_u(v) = w$, another PRN v could be calculated through the reversion function of $C_u(v)$: $v = C_u^{-1}(w)$.

4) The values of corresponding assets return $R_X = F^{-1}(u)$, $R_Y = G^{-1}(v)$ are obtained according to the distribution function of assets return $F(\cdot)$, $G(\cdot)$ and u, v;

5) The weight w is given in the portfolio and the return Z of portfolio is calculated: $z = wR_X + (1 - w)R_Y$, which provides a possible perspective to the future yield of the portfolio.

6) (3)-(5) steps are repeated through K times, which means the k kinds of possible scenarios of the future yield of the portfolio are generated through simulation, which is amied to obtain the empirical distribution of the future return of the portfolio. For the given confidence $1 - \alpha$, the VaR in the portfolio is confirmed from $P[Z < -VaR_\alpha] = \alpha$.

3 Numerical Experiment In the empirical experiment, it is assumed that the portfolio just includes stock and bond. The analyzed data of the two selected financial assets is from Standard&Poor's500 and CBOE Internet Rate 10-Year Bond (2008.7.1-2012.7.3), and the following is the graph of return rate r: $r_{At} = log[P_{At}/P_{At-1}]$

First, Kolmogorov-Smirnov test is used to make the test of normality in SPSS, which shows they don't satisfy normality; Augmented Dickey-Fuller (ADF) unit root test is aimed to demonstrate whether it is the stationary time series data, which demonstrates the time series are the stationary ones.

1) According to the formula 5 6, the bandwidths of SP500 and 10-year bond are 0.0012 and 0.0024, respectively. Through the optimal bandwidth and default Gaussian kernel

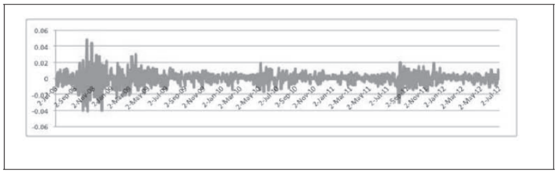

Figure 2: The time series of SP500 return rate

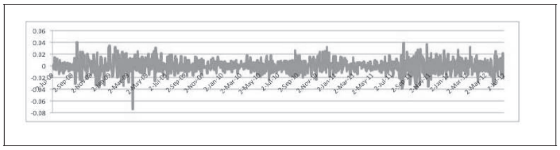

Figure 3: The time series of CBOE Internet Rate 10-Year Bond return rate

function, the density function and cumulative distribution function of the financial assets could be estimated through invoking KS density function in Matlab.

The following is the comparison of kernel density, frequency histogram and normal distribution density:

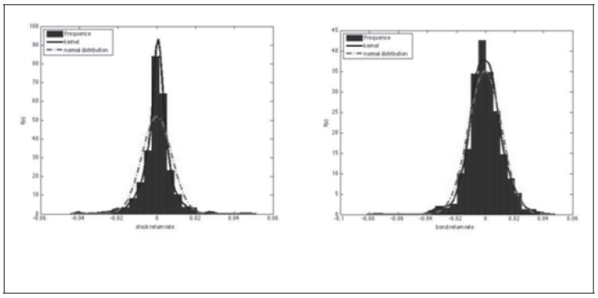

Figure 4: Frequency histogram, kernel density estimation and normal distribution density of the yield of SP500 stock and 10-year bond

The following is the comparison of the empirical, estimated and theoretical normal distribution function under the same conditions:

On the basis of the kernel density estimation to the unknown marginal density of the two financial assets, the parameter of copula model could be estimated.

2) The construction of the bi-variant copula model

Conventionally, Person correlation coefficient is written in the following:

$$\rho_{xy} = \frac{cov(x,y)}{(\sigma_x, \sigma_y)} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}} \tag{15}$$

[25]

It assumes the variables submit to the multi-variant normal distribution. Then, the correlation coefficient of SP500 and 10-year bond is 41.97%.

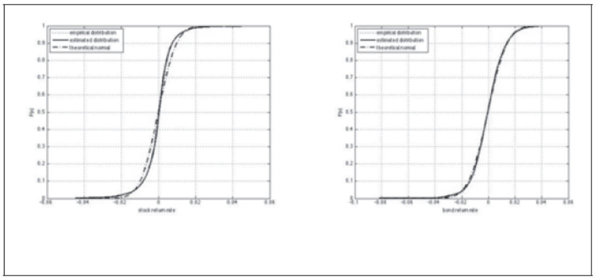

Figure 5: Empirical, estimated and theoretical normal distribution function graph of the return rate of SP500 and 10-year bond

According to the kernel density estimation-maximum likelihood method (8)-(11) and Minimum Variance Test Method

(16) $$Var(\alpha) \cong \frac{4}{n}\alpha^{\frac{3}{2}}(1+\sqrt{\alpha})^2$$

(Kendall and Stuart (1967)[14]; Mardia (1970)[16]), Gumbel and Clayton are adopted [19] and the corresponding Copula parameters are 1.4173 and 0.7515.

Then, the correlations of stock and bond could be obtained from function relationship between Kendall and Copula parameter: 29.44% and 27.3% respectively here, which is similar to 31.26% from Kendall rank correlation.

Then, through the parsing expression of the correlation coefficient in tail, the correlation coefficient in up-tail and low-tail could be measured according to Gumbel and Clayton function:

(17) $$Gumbel : \lambda^{up} = 2 - 2^{\frac{1}{\alpha}} = 0.37$$

(18) $$Clayton : \lambda^{lo} = 2 - 2^{\frac{1}{\alpha}} = 0.40$$

Fig 6 also shows the similar characteristic in the end of the diagonal.

Then, the VaR value could be computed by the combination of copula model and Montel Carlo simulation like the algorithm step (1)-(6) in 2.3.

3) VaR computation

For the analytical formula (13), the assumption is that $c = 95\%(a = 1.65)$ and the original value W is set to 1:

When $W_1 = W_2 = 0.5(c = 95\%, a = 1.65)$,

(19) `VaR value is equal to  0.01336;`

When VaR is minimum, the proportion of W_1 SP500 and W_2 10-year bond is respectively equal to 80.5% and 19.5%, and

(20) `VaR is 0.000055`

According to the Monte Carlo simulation (1)-(6), when $W_1 = W_2 = 0.5$, from Gumble or Clayton model:

(21) `VaR=0.0135;`

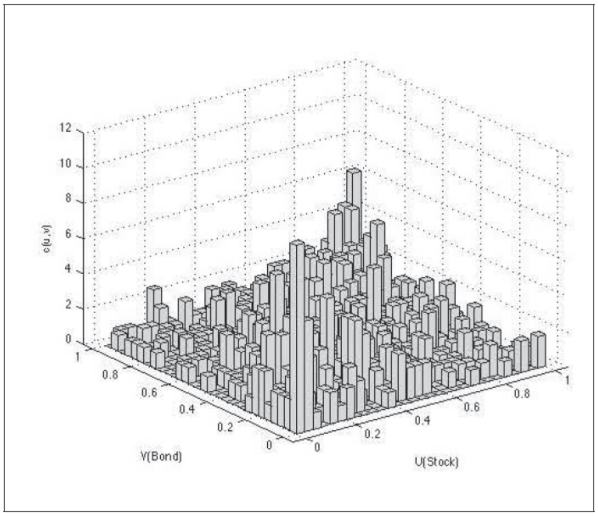

Figure 6: Bibariate frequency histogram

From the following graph, it is concluded that the ratio of stock and bond reaches 85% to 15%, the value at risk could be minimum,

(22) `which is about 0.0122;`

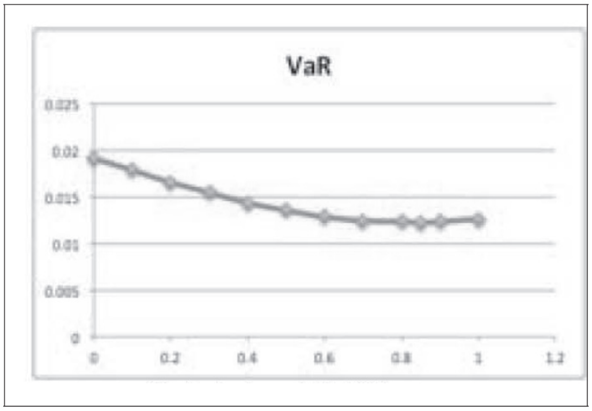

Figure 7: Stock weight-VaR

4 Discussions 1) From the time series graph Figs 2 and 3, the volatility of the two return rate series have the obvious "cluster" phenomenon, which means big fluctuations follow big ones and small fluctuation follow small ones, and there is a certain similarity between them, which shows some interaction exists in it.

From Figs 4 and 5, we can get the negative skewness and high kurtosis, which demonstrates falling days are less than rising days, but the falling average range is higher than

Table 2: VaR of different portions

W_1 (stock weight)	W_2 (bond weight)	VAR
0.00	1.00	0.0191
0.10	0.90	0.0178
0.20	0.80	0.0165
0.30	0.70	0.0154
0.40	0.60	0.0143
0.50	0.50	0.0135
0.60	0.40	0.0128
0.70	0.30	0.0124
0.80	0.20	0.0123
0.85	0.15	0.0122
0.90	0.10	0.0123
1.00	0.00	0.0126

the rising one and return rate happen near the separate average value. So compared with normal distribution, kernel density estimation is a better way to describe the feature of "fat tail and high picky" in the real situation.

2) Through the comparison between Person correlation coefficient and correlation coefficient from copula model, the value of Person one is higher than the one from copula model and Kendall correlation, which shows that the former overestimates the relation between stock market and bond

Contrary to the inability to capture the relevance in tail from linear perspective, the correlation coefficient in tail well describes the possibility of consistency in bond market when the exception situations happen in stock market such as boom or slump.

3) In the VaR comparsion part, it implies that 50% stock-50% bond portfolio has a 95% chance of losing the maximum value 0.01336 and 0.0135 under the above two methods when 1 is invested.

Through the contrast of the VaR results from analytical method and Monte Carlo simulation, it is found that the VaR value in assumption of the normal distribution is less than the one by Monte Carlo, which means the former underestimates the financial risk easily.

Meanwhile, to obtain the safest asset security, it is a wise strategy for a robust investor to allocate $80\%-85\%$ capital to stock market and $15\%-20\%$ one to 10-year bond theoretically according to results of the minimum VaR computation.

5 Conclusions In the analysis of portfolio, there is an importance in the study of relation structure between financial assets, which results in how to capture the principal of change between them especially in the tail with better correlation model.

In this paper, through kernel density estimation-maximum likelihood two steps, Gumbel and Clayton copula model are adopted to model the correlation between stock and bond. Then, VaR is analyzed based on it and the optimal allocation in the portfolio could be confirmed by Montel Carlo simulation.

By comparison between the present methods introduced in this paper and the conventional methods which is based on the normal distribution, it is concluded that the latter one always underestimate the happening of risk and the value of risk, which should be brought to the forefront.

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Hesitant Fuzzy Geometric Heronian Mean Operators and Their Application to Multi-Criteria Decision Making

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ABSTRACT. Aggregation is the process of fusing a large data in one representative value. This is done in different ways, through what may be called 'operators', every operator having special characteristics. Expanding study of vague phenomena, through hesitant fuzzy information of hesitant fuzzy set (HFS) theory and their applications has attracted useful aggregation techniques. Paper explores the geometric Heronian mean (GHM) under hesitant fuzzy environment and defines some new geometric Heronian mean operators such as the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and the weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM) operator. Further, we give definition of hesitant fuzzy geometric Heronian element (HFGHE), which is a basic calculation unit in HFGGHM and reflects the conjunction between two aggregated arguments. Properties of the new aggregation operators are reported and their special cases are considered. Furthermore, based on the WHFGGHM operator, an approach to deal with multi-criteria decision-making problems under hesitant fuzzy environment is developed. Finally, a practical example is provided to illustrate the multi-criteria decision-making process.
Keywords: fuzzy sets; fuzzy multi-sets; intuitionistic fuzzy set; hesitant fuzzy sets.

1. Introduction

Mathematics is known for its quantitative and logically sound foundations. It started with study of deterministic phenomena. However, the wider world phenomena, all the more those in man-made world, are not deterministic in nature. Ingenuity of mathematicians expanded mathematical study to a class of in-deterministic/uncertain phenomena that are statistical/probabilistic nature. Without sacrificing its quantitative and logically sound basis, a vast discipline of statistics developed. Moving thus a major step forward in the study of uncertain phenomena, it was observed that there are uncertain phenomena that are not statistically stable in which chances of happening of an event can be quantified in terms of probabilities and distribution-patterns. This presented mathematicians with a challenge to define phenomena that are uncertain in non-statistical ways. In general these may be called vague or imprecise. Zadeh [44] was the first to capture this idea in defining fuzzy sets. Several extensions and generalizations of Zadeh' fuzzy-sets have since been made as intuitionistic fuzzy sets [1, 2], interval-valued fuzzy sets [10, 19], type-2 fuzzy sets [45], type-n fuzzy sets [45], fuzzy multisets [6, 35], vague sets [9], and hesitant fuzzy sets [17, 18], etc. In a rather natural way, set operations were defined and it was found that these present a panorama of laws as the defining terms in these sets involve functions, which was not the case with theory of crisp sets. These studies enriched areas of applications in different ways [4, 5, 7, 8, 11-16, 20-34, 38-41, 45-50].

The vagueness/fuzziness that appeared to be diluting/loosing precise quantitative tenor of things in the process, Zadeh and thereafter others defined measures of fuzziness of various shades over family of fuzzy-sets. These measures of fuzziness are quantitative in nature and follow the pattern of measures defined in place Shannon's probabilistic information theory.

Another age old idea is that of 'aggregation,' a process of meaningfully fusing a collection of values into one representative value. It is, in fact, a multi-faceted avtar of the simple idea of arithmetical and other means/averages of a given set of numbers. In probabilistic-statistics, one encounters it at several places – 'statistical expectations,' correlation and regression analysis, etc.

Shannon's entropy of a probability distribution being average of self-information arguments of its elements is, generally speaking, an aggregation of self-information elements. With this background, information aggregation in hesitant fuzzy set theory has been studied with quite some interest by researchers and practitioners in recent years. Xia and Xu [27] developed some arithmetic and geometric aggregation operators under hesitant fuzzy environment, investigated the connections of these operators and applied them to multi-criteria decision making. To aggregate the hesitant fuzzy information under confidence levels, Xia et al. [26] developed a series of confidence-induced hesitant fuzzy aggregations operators. Xu et al. [30] developed several series of aggregation operators for hesitant fuzzy information using the quasi-arithmetic means. Gu et al. [11] utilized the hesitant fuzzy weighted average (HFWA) operator to investigate the evaluation model for risk investment with hesitant fuzzy information. Based on the prioritized weighted average (PWA) operator [37, 38], Yu [40] proposed the hesitant fuzzy prioritized weighted average (HFPWA) operator and the hesitant fuzzy prioritized weighted geometric (HFPWA) operator to aggregate the hesitant fuzzy information. Wei [22] also developed some prioritized aggregation operators for aggregating hesitant fuzzy information and then applied them to develop models for hesitant fuzzy multiple attribute decision making.

Reflecting on the concept of aggregation, it may be noted that the above discussed aggregation operators with hesitant fuzzy information are based on the assumption that all aggregating arguments are independent However, in real world situations there are always some degrees of interrelationships between arguments. To deal with this issue, Yu et al. [39] and Wei et al. [21] developed some hesitant fuzzy correlative operators, such as the hesitant fuzzy Choquet integral (HFCI) operator, the hesitant fuzzy Choquet ordered average (HFCOA) operator, the hesitant fuzzy Choquet ordered geometric (HFCOG) operator, the generalized hesitant fuzzy Choquet ordered average (GHFCOA) operator and the generalized hesitant fuzzy Choquet ordered geometric (GHFCOG) operator and found their application to multiple attribute decision making. Motivated by the idea of power average (PA) operator [36], Zhang [48] developed some hesitant fuzzy power average (HFPA) operators and hesitant fuzzy power geometric (HFPG) operators for aggregating hesitant fuzzy correlative information. Further, Zhu et al. [51] and Zhu and Hu [52] extended the Bonferroni mean (BM) to hesitant fuzzy environment and introduced some hesitant fuzzy Bonferroni means such as the hesitant fuzzy Bonferroni mean (HFBM), the weighted hesitant fuzzy Bonferroni mean (WHFBM), the hesitant fuzzy geometric Bonferroni mean (HFGBM), the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM).

The Heronian mean (HM) is another aggregation technique, which is better suited to aggregate the exact numerical values [3]. A prominent characteristic of HM is its capability to capture interrelationships between input arguments. This makes HM useful in various application fields, such as decision making, information retrieval, pattern recognition, and data mining etc. The HM is different from power average or Choquet integral. The HM operator focuses on the aggregated arguments while the Choquet integral or power average on changing the weight vector of the aggregation operators. Based on HM operator, Yu [43] defined some generalized HM operators such as generalized geometric Heronian mean (GGHM), the generalized geometric intuitionistic fuzzy Heronian mean (GGIFHM) and the weighted generalized geometric intuitionistic fuzzy Heronian mean (WGGIFHM).

In this paper, we extend the idea of generalized geometric Heronian mean operator to hesitant fuzzy environment. In order to do so, we propose the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and the weighted hesitant fuzzy generalized geometric Heronian mean (WHFGWBM) operator for aggregating the hesitant fuzzy correlative information. We study their properties and discuss special cases. We show that several aggregation operators on hesitant fuzzy sets studied earlier are special cases of our generalized operator. Also, there are others interesting particular cases that as well arise from it. Further, we develop an approach for multi-criteria decision making under hesitant fuzzy information environment.

The paper is organized as follows: In Section 2 some basic concepts related to fuzzy sets, hesitant fuzzy sets and Heronian mean operators are briefly given. In Section 3 we propose the hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and study some of their properties. Some special cases of HFGGHM are also discussed in this section. In Section 4 we introduce the weighted hesitant fuzzy generalized geometric Heronian mean (HFGGHM) operator and develop an approach for solving multi-criteria decision making under hesitant fuzzy environment. In Section 5 finally, a numerical example is presented to illustrate the proposed approach to multi-criteria decision-making and our conclusions are presented in Section 6.

2. Preliminaries

Definition 1. ***Fuzzy set*** [44]**:** A fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is defined as

$$A = \{\langle x, \mu_A(x)\rangle \,| x \in X\}\,, \tag{1}$$

where $\mu_A(x) : X \to [0, 1]$ is the membership function of A and the number $\mu_A(x)$ describing the degree of membership of $x \in X$ in the set A.
An step further, the concept of hesitant fuzzy sets (HFSs) was introduced by Torra and Narukawa [17] and Torra [18]. An HFS permits the membership degree of an element to be a set of several possible membership values between 0 and 1. This better describes the situations where a set of people have hesitancy in providing their preferences over objects in the process of decision making.

Definition 2. ***Hesitant Fuzzy Set***[18]**:** Let $X = \{x_1, x_2, ..., x_n\}$ be a reference set, a set E defined in X given by

$$E = \{\langle x, h_E(x)\rangle \,| x \in X\} \tag{2}$$

where $h_E\,(x)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E, is called a hesitant fuzzy set.

Further, Torra [18] defined the 'empty hesitant fuzzy set' and the 'full hesitant fuzzy set' as follows:

$$E^{\circ} = \{\langle x, h_{E^{\circ}}(x)\rangle \,| x \in X\},\ \text{where } h_{E^{\circ}}(x) = \{0\}\ \forall x \in X,$$
$$E^{*} = \{\langle x, h_{E^{*}}(x)\rangle \,| x \in X\},\ \ \text{where } h_{E^{*}}(x) = \{1\}\ \forall x \in X.$$

For convenience, Xia and Xu [27] named the set $h = h_E(x)$ as the hesitant fuzzy element (HFE) and let $HFE(X)$ represent the family of all hesitant fuzzy elements defined in X.

Definition 3.*Algebraic Operations on HFEs*: Let $h, h_1, h_2 \in HFE(X)$, Xia and Xu [26] defined the following operations:

1. $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$, $\lambda > 0$;
2. $\lambda h = \bigcup_{\gamma \in h} \{1 - (1-\gamma)^\lambda\}$, $\lambda > 0$;
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$.

Definition 4. ***Score Function*** [27]: Let h be a hesitant fuzzy element, the score function S of an HFE is defined as follows:

$$S(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma, \tag{3}$$

where $\#h$ is the number of elements in h.

To ranking any two h_i, $i = 1, 2$, we shall use the following definition of Xia & Xu [27]:

Definition 5: Let $h_1$and h_2 be two hesitant fuzzy elements with their respective scores $S(h_1)$ and $S(h_2)$, then

1. h_1 is larger than h_2, denoted by $h_1 > h_2$ if $S(h_1) > S(h_2)$.
2. $h_1 = h_2$, if $S(h_1) = S(h_2)$.

Heronian mean (HM), which is one of the aggregation methods, is characterized by the ability to capture the relevance between the input arguments. The definition of HM is as follows:

Definition 6. ***Heronian Mean***[3]**:** For a collection a_i, $i = 1, 2, \cdots, n$, of nonnegative real numbers, their Heronian mean (HM) is defined as:

$$HM(a_1, a_2, \cdots, a_n) = \frac{2}{n(n+1)} \sum_{i,j=1}^{n} \sqrt{a_i a_j} \tag{4}$$

Based on Definition 6, Yu [42] proposed the geometric Heronian mean (GHM) as follows:

Definition 7. ***Geometric Heronian Mean***[43]**:** For a collection a_i, $i = 1, 2, \cdots, n$, of nonnegative real numbers, their the geometric Heronian mean (GHM) is defined by:

$$GHM(a_1, a_2, \cdots, a_n) = \prod_{i,j=1}^{n} \left(\frac{a_i + a_j}{2}\right)^{\frac{2}{n(n+1)}} \tag{5}$$

Further, using the idea of geometric Bonferroni mean [31], Yu [43] also proposed the generalized geometric Heronian mean (GGHM) as follows:

Definition 8. ***Generalized Geometric Heronian Mean*** [43]**:** Let $p, q \geq 0$, p, q do not take the value 0 simultaneously and let a_i, $i = 1, 2, \cdots, n$, be a collection of nonnegative real numbers, then generalized geometric Heronian mean (GGHM) is given by:

$$GGHM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} (pa_i + qa_j)^{\frac{2}{n(n+1)}} \tag{6}$$

It may be noted that $GGHM^{p,q}$ have the following properties.

1. $GGHM^{p,q}(0, 0, \cdots, 0) = 0$ and $GGHM^{p,q}(1, 1, \cdots, 1) = 1$;
2. $GGHM^{p,q}(a_1, a_2, \cdots, a_n) = a$ if $a_i = a$, $\forall i$;
3. If $a_i \leq b_i$ $\forall i$, then $GGHM^{p,q}(a_1, a_2, \cdots, a_n) \leq GGHM^{p,q}(b_1, b_2, \cdots, b_n)$ i.e., $GGHM^{p,q}$ is monotonic;
4. $\min_i \{a_i\} \leq GGHM^{p,q}(a_1, a_2, \cdots, a_n) \leq \max_i \{a_i\}$.

In the next section, in respect of hesitant fuzzy environment, we extend the GGHM to hesitant fuzzy environment and propose:

(i) The hesitant fuzzy generalized geometric Heronian mean (HFGGHM);
(ii) The weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM).

3. Hesitant Fuzzy Generalized Geometric Heronian Means

We propose the following definition:

Definition 9.*Hesitant Fuzzy Generalized Geometric Heronian Mean*: Let $p, q > 0$ and h_i $i = 1, 2, \cdots, n$ be a collection of HFEs, the hesitant fuzzy generalized geometric Heronian mean ($HFGGHM^{p,q}$) is given by:

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2}{n(n+1)}} \tag{7}$$

Next, based on the operational laws of HFEs, we have the following theorem:

Theorem 1: Let $p, q > 0$ and h_i $i = 1, 2, \cdots, n$ be a collection of hesitant fuzzy elements, then the aggregated value by using the $HFGGHM^{p,q}$ operator is also a hesitant fuzzy element, and

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\eta_{i,j} \in \sigma_{i,j}; i \leq j} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \leq j}}^{n} (\eta_{i,j})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\} \tag{8}$$

where $\sigma_{i,j;\, i\le j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ reflects the interrelationship between h_i and h_j, $i, j = 1, 2, \ldots, n$.

Proof: Since

$$\sigma_{i,j;\, i\le j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)) = \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\le j}} \{\eta_{i,j}\} \tag{9}$$

which is also a HFE, then Equation (8) can be written as:

$$HFGGHM^{p,q}(h_1, h_2, \ldots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\le j}}^{n} (\sigma_{i,j})^{\frac{2}{n(n+1)}} \tag{10}$$

Furthermore, we have

$$\begin{aligned}
\bigotimes_{\substack{i,j=1\\ i\le j}}^{n} (\sigma_{i,j})^{\frac{2}{n(n+1)}} &= \left(\bigotimes_{\substack{i,j=1\\ i\le j}}^{n} (\sigma_{i,j}) \right)^{\frac{2}{n(n+1)}} \\
&= \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\le j}} \left\{ \left(\prod_{\substack{i,j=1\\ i\le j}}^{n} \eta_{i,j} \right)^{\frac{2}{n(n+1)}} \right\} \\
&= \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\le j}} \left\{ \left(\prod_{\substack{i,j=1\\ i\le j}}^{n} (\eta_{i,j})^{\frac{2}{n(n+1)}} \right) \right\}
\end{aligned}$$

and then

$$\frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\le j}}^{n} (\sigma_{i,j})^{\frac{2}{n(n+1)}} = \bigcup_{\eta_{i,j}\in\sigma_{i,j;i\le j}} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1\\ i\le j}}^{n} (\eta_{i,j})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}. \tag{11}$$

This completes the proof of the Theorem 1.

It is noted that, in Theorem 1, $\sigma_{i,j}$ is a basic element in (8), which we call a hesitant fuzzy geometric Heronian element (HFGHE). Apparently, $\sigma_{i,j}$ represents the interrelationship between the HFEs h_i and h_j by two types of conjunction calculations, i.e., "$\oplus$" and "$\otimes$".

Further, we discuss some properties of the $HFGGHM^{p,q}$:

1. Let h_i, $i = 1, 2, \cdots, n$, be collection of HFEs. If $h_i = h$ for all i, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = \frac{1}{p+q}((p+q)h)^2. \tag{12}$$

Proof: Since $h_i = h$ for all i, we have

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h, h, \cdots, h)$$

$$= \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} ((ph \oplus qh) \otimes (ph \oplus qh))^{\frac{2}{n(n+1)}}$$

$$= \frac{1}{p+q}((ph \oplus qh) \otimes (ph \oplus qh))$$

$$= \frac{1}{p+q}((p \oplus q)h)^2. \tag{13}$$

This proves the property.

Corollary 1: If h_i, $i = 1, 2, \cdots, n$, is a collection of the empty HFEs, i.e., $h_i = h^\circ = \{0\}$, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h^\circ, h^\circ, \cdots, h^\circ) = \{0\}. \tag{14}$$

Corollary 2: If h_i, $i = 1, 2, \cdots, n$ is a collection of the full HFEs, i.e., $h_i = h^* = \{1\}$, then

$$HFGGHM^{p,q}(h_1, h_2, \cdots, h_n) = HFGGHM^{p,q}(h^*, h^*, \cdots, h^*) = \{1\}. \tag{15}$$

2. (Monotonicity). Let $h_\alpha = (h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_n})$ and $h_\beta = (h_{\beta_1}, h_{\beta_2}, \cdots, h_{\beta_n})$ be two collections of HFEs, $\sigma_{\alpha_{i,j}} = ((ph_{\alpha_i} \oplus qh_{\alpha_j}) \otimes (ph_{\alpha_j} \oplus qh_{\alpha_i}))$ and $\sigma_{\beta_{i,j}} = ((ph_{\beta_i} \oplus qh_{\beta_j}) \otimes (ph_{\beta_j} \oplus qh_{\beta_i}))$, if for any $\gamma_{\alpha_i} \in h_{\alpha_i}$, $\gamma_{\beta_i} \in h_{\beta_i}$, we have $\gamma_{\alpha_i} \leq \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \leq \gamma_{\beta_j}$ for all $i, j = 1, 2, \ldots, n$, then

$$HFGGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) \leq HFGGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_n}). \tag{16}$$

Proof: Since $\gamma_{\alpha_i} \leq \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \leq \gamma_{\beta_j}$ for all $i, j = 1, 2, \ldots, n$, we have

$$(1 - (1-\gamma_{\alpha_i})^p (1-\gamma_{\alpha_j})^q) \leq (1 - (1-\gamma_{\beta_i})^p (1-\gamma_{\beta_j})^q), \tag{17}$$

$$(1 - (1-\gamma_{\alpha_j})^p (1-\gamma_{\alpha_i})^q) \leq (1 - (1-\gamma_{\beta_j})^p (1-\gamma_{\beta_i})^q). \tag{18}$$

Additionally, we obtain

$$\sigma_{i,j;\, i \leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$$

(19)

$$= \Big(\bigcup_{\gamma_{\alpha_i} \in h_{\alpha_i}, \gamma_{\alpha_j} \in h_{\alpha_j}} \{1 - (1-\gamma_{\alpha_i})^p + 1 - (1-\gamma_{\alpha_j})^q - (1 - (1-\gamma_{\alpha_i})^p)(1 - (1-\gamma_{\alpha_j})^q)\} \Big)$$

$$\otimes \Big(\bigcup_{\gamma_{\alpha_i} \in h_{\alpha_i}, \gamma_{\alpha_j} \in h_{\alpha_j}} \{1 - (1-\gamma_{\alpha_j})^p + 1 - (1-\gamma_{\alpha_i})^q - (1 - (1-\gamma_{\alpha_j})^p)(1 - (1-\gamma_{\alpha_i})^q)\} \Big)$$

Let $\eta_{\alpha_{i,j}} \in \sigma_{\alpha_{i,j;i\leq j}}$ and $\eta_{\beta_{i,j}} \in \sigma_{\beta_{i,j;i\leq j}}$, for all $i, j = 1, 2 \ldots, n$; $i \leq j$, then from Equations (17)-(19), we have

$$\begin{aligned}
\eta_{\alpha_{i,j}} &= \Bigg(\Bigg(\bigcup_{\gamma_{\alpha_i}\in h_{\alpha_i},\gamma_{\alpha_j}\in h_{\alpha_j}} \{1-(1-\gamma_{\alpha_i})^p+1-(1-\gamma_{\alpha_j})^q-(1-(1-\gamma_{\alpha_i})^p)(1-(1-\gamma_{\alpha_j})^q)\}\Bigg)\\
&\otimes \Bigg(\bigcup_{\gamma_{\alpha_j}\in h_{\alpha_j},\gamma_{\alpha_i}\in h_{\alpha_i}} \{1-(1-\gamma_{\alpha_j})^p+1-(1-\gamma_{\alpha_i})^q-(1-(1-\gamma_{\alpha_j})^p)(1-(1-\gamma_{\alpha_i})^q)\}\Bigg)\Bigg)\\
\leq \eta_{\beta_{i,j}} &= \Bigg(\Bigg(\bigcup_{\gamma_{\beta_i}\in h_{\beta_i},\gamma_{\beta_j}\in h_{\beta_j}} \{1-(1-\gamma_{\beta_i})^p+1-(1-\gamma_{\beta_j})^q-(1-(1-\gamma_{\beta_i})^p)(1-(1-\gamma_{\beta_j})^q)\}\Bigg)\\
&\otimes \Bigg(\bigcup_{\gamma_{\beta_j}\in h_{\beta_j},\gamma_{\beta_i}\in h_{\beta_i}} \{1-(1-\gamma_{\beta_j})^p+1-(1-\gamma_{\beta_i})^q-(1-(1-\gamma_{\beta_j})^p)(1-(1-\gamma_{\beta_i})^q)\}\Bigg)\Bigg)
\end{aligned} \tag{20}$$

thus

$$\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \geq \left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}. \tag{21}$$

According to Definition 9 and Equation (21), we get

$$\begin{aligned}
HFGGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) &= \frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(\sigma_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\\
&= \bigcup_{\eta_{\alpha_{i,j}}\in\sigma_{\alpha_{i,j;i\leq j}}} \left\{1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}\\
&\leq \bigcup_{\eta_{\beta_{i,j}}\in\sigma_{\beta_{i,j,i\leq j}}} \left\{1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(\eta_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}\\
&= \frac{1}{p+q} \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(\sigma_{\beta_{i,j}}\right)^{\frac{2}{n(n+1)}}\\
&= HFGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \ldots, h_{\beta_n}).
\end{aligned} \tag{22}$$

This proves the property.

3. (Commutativity). Let h_i, $i = 1, 2, \ldots, n$, be collection of HFEs, and $(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n)$ be any permutation of $(h_1, h_2, \ldots, h_n)$, then

$$HFGGHM^{p,q}(h_1, h_2, \ldots, h_n) \leq HFGGHM^{p,q}(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n). \tag{23}$$

Proof: Since $(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n)$ is a permutation of $(h_1, h_2, \ldots, h_n)$, then

$$\begin{aligned} HFGGHM^{p,q}\left(h_{\alpha_1}, h_{\alpha_2}, ..., h_{\alpha_n}\right) &= \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} \left(\sigma_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}} \\ &= \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \leq j}}^{n} \left(\dot{\sigma}_{\alpha_{i,j}}\right)^{\frac{2}{n(n+1)}} \\ &= HFGGHM^{p,q}(\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_n), \end{aligned} \tag{24}$$

where $\sigma_{i,j;\, i\leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ and $\dot{\sigma}_{i,j;\, i\leq j} = (p\dot{h}_i \oplus q\dot{h}_j) \otimes (p\dot{h}_j \oplus q\dot{h}_i)$, $i, j = 1, 2, \ldots, n$.

This proves the property.

4. (Boundedness). Let h_i, $i = 1, 2, \ldots, n$ be collection of HFEs, $h_i^+ = \bigcup_{\gamma_i \in h_i} \max\{\gamma_i\}$, $h_i^- = \bigcup_{\gamma_i \in h_i} \min\{\gamma_i\}$, $\gamma^+ \in h_i^+$, $\gamma^- \in h_i^-$, and $\sigma_{i,j} = (ph_i \oplus qh_j) = \bigcup_{\eta_{i,j} \in \sigma_{i.j}} \{\eta_{i.j}\} = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j} \{1 - (1-\gamma_i)^p (1-\gamma_j)^q\}$, then

$$\begin{aligned} &\bigcup_{\gamma^- \in h_i^-} \left\{ 1 - \left(1 - \left(1 - (1-\gamma^-)^{p+q}\right)^2\right)^{\frac{1}{p+q}} \right\} \\ &\qquad \leq HFGGHM^{p,q}(h_1, h_2, \ldots, h_n) \\ &\qquad\qquad \leq \bigcup_{\gamma^- \in h_i^-} \left\{ 1 - \left(1 - \left(1 - (1-\gamma^+)^{p+q}\right)^2\right)^{\frac{1}{p+q}} \right\}. \end{aligned} \tag{25}$$

Proof: Since $\gamma^- \leq \gamma_i \leq \gamma^+$ and $\gamma^- \leq \gamma_j \leq \gamma^+$ $\forall\, i, j = 1, 2, \ldots, n$, then

$$1 - \left(1-\gamma^-\right)^{p+q} \leq 1 - (1-\gamma_i)^p (1-\gamma_j)^q \leq 1 - \left(1-\gamma^+\right)^{p+q} \tag{26}$$

$$1 - \left(1-\gamma^-\right)^{p+q} \leq 1 - (1-\gamma_j)^p (1-\gamma_i)^q \leq 1 - \left(1-\gamma^+\right)^{p+q} \tag{27}$$

and

$$\begin{aligned} \sigma_{i,j;\, i\leq j} &= ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)) \\ &= \left(\bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \{1 - (1-\gamma_i)^p (1-\gamma_j)^q\} \right) \otimes \left(\bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \{1 - (1-\gamma_j)^p (1-\gamma_i)^q\} \right) \\ &\geq \left(\bigcup_{\gamma^- \in h_i^-} \left\{ \left(1 - (1-\gamma^-)^{p+q}\right)^2 \right\} \right). \end{aligned} \tag{28}$$

Similarly, we have

$$\sigma_{i,j;\,i\leq j} \leq \left(\bigcup_{\gamma^+\in h_i^+} \left\{ \left(1-\left(1-\gamma^+\right)^{p+q}\right)^2 \right\} \right). \tag{29}$$

According to Definition 9, Equations (28) and (29), we obtain

$$\bigcup_{\gamma^-\in h_i^-} \left\{ 1-\left(1-\left(1-\left(1-\gamma^-\right)^{p+q}\right)^2\right)^{\frac{1}{p+q}} \right\} \leq HFGGHM^{p,q}\left(h_1,h_2,\ldots,h_n\right)$$

$$\leq \bigcup_{\gamma^+\in h_i^+} \left\{ 1-\left(1-\left(1-\left(1-\gamma^+\right)^{p+q}\right)^2\right)^{\frac{1}{p+q}} \right\}. \tag{30}$$

This proves the property.

Some special cases of $HFGGHM^{p,q}$ for different values of parameters p and q.

(i) If $q\rightarrow 0$ (or $p\rightarrow 0$), then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\rightarrow 0} HFGHM^{p,q}\left(h_1,h_2,\ldots,h_n\right) = \frac{1}{p} \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(ph_i\otimes ph_j\right)^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\gamma_i\in h_i,\gamma_j\in h_i} \left\{ 1-\left(1-\prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\left(1-\left(1-\gamma_i\right)^p\right)\left(1-\left(1-\gamma_j\right)^p\right)\right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p}} \right\}, \tag{31}$$

which we call the generalized hesitant fuzzy geometric Heronian mean (GHFGHM).

(ii) If $p=1$ and $q\rightarrow 0$, then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\rightarrow 0} HFGHM^{1,q}(h_1,h_2,\ldots,h_n) = \bigotimes_{\substack{i,j=1\\ i\leq j}}^{n} \left(h_i\otimes h_j\right)^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\gamma_i\in h_i,\gamma_j\in h_i} \left\{ \prod_{\substack{i,j=1\\ i\leq j}}^{n} \left(\left(1-\left(1-\gamma_i\right)\right)\left(1-\left(1-\gamma_j\right)\right)\right)^{\frac{2}{n(n+1)}} \right\}, \tag{32}$$

which we call the hesitant fuzzy geometric Heronian mean (GHFGHM).

(iii) If $p = 2$ and $q \to 0$, then the $HFGGHM^{p,q}$ reduces to

$$\lim_{q\to 0} HFGHM^{2,q}(h_1, h_2, ..., h_n) = \frac{1}{2} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} (2h_i \otimes 2h_j)^{\frac{2}{n(n+1)}},$$

$$= \bigcup_{\gamma_i \in h_i, \gamma_j \in h_i} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \le j}}^{n} \left(\left(1 - (1-\gamma_i)^2\right) \left(1 - (1-\gamma_j)^2\right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\}, \tag{33}$$

which we call the hesitant fuzzy square geometric Heronian mean (HFSGHM).

(iv) If $p = q = 1$, let $\sigma^{1,1}_{i,j;i\le j} = ((h_i \oplus h_j) \otimes (h_j \oplus h_i)) = \bigcup_{\varepsilon_{i,j} \in \sigma^{1,1}_{i,j;i\le j}} \{\varepsilon_{i,j}\}$, the $HFGGHM^{p,q}$ reduces to

$$HFGHM^{1,1}(h_1, h_2, ..., h_n) = \frac{1}{2} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} ((h_i \oplus h_j) \otimes (h_j \oplus h_i))^{\frac{2}{n(n+1)}}$$

$$= \bigcup_{\varepsilon_{i,j} \in \sigma^{1,1}_{i,j;i\le j}} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \le j}}^{n} (\varepsilon_{i,j})^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right\} \tag{34}$$

which we call the hesitant fuzzy interrelated square geometric Heronian mean (HFISGHM).

Further, to consider the importance of aggregated arguments, we define a weighted hesitant fuzzy generalized geometric Heronian mean (WHFGGHM) operator as follows:

Definition 10: Let $h_i, i = 1, 2, \ldots, n$, be a collection of HFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_i where w_i indicates the importance degree of h_i, satisfying $w_i \ge 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. For any $p, q > 0$, the weighted hesitant fuzzy generalized geometric Heronian mean ($WHFGGHM^{p,q}$) is given by:

(35)
$$WHFGGHM^{p,q}(h_1, h_2, \ldots, h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2w_i w_j}{(1+w_i)}}.$$

In view of Equation (35), we prove a result in the following theorem:

Theorem 2. Let $p, q > 0$, and $h_i, i = 1, 2, \ldots, n$ be a collection of HFEs with weight vector $w = (w_1, w_2, \ldots, w_n)^T$ satisfying $w_i \ge 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated value using the WHFGGHM is also an HFE, and

$$WHFGGHM^{p,q}(h_1, h_2, ..., h_n) = \frac{1}{p+q} \bigotimes_{\substack{i,j=1 \\ i \le j}}^{n} ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))^{\frac{2w_i w_j}{(1+w_i)}},$$

$$(36) \qquad = \bigcup_{\eta_{i,j} \in \sigma_{i,j;i\leq j}} \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \leq j}}^{n} (\eta_{i,j})^{\frac{2w_i w_j}{(1+w_i)}} \right)^{\frac{1}{p+q}} \right\}$$

and $\sigma_{i,j;\,i\leq j} = ((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i))$ reflects the interrelationship between h_i and h_j, $i, j = 1, 2, \ldots, n$.

Proof: This theorem is easy to prove on lines similar to that of Theorem 1.

Note: If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then $WHFGGHM^{p,q}$ in (36) reduces to $HFGGHM$ in (8).

In the following section, we suggest application of the proposed $WHFGGHM^{p,q}$ operator to multi criteria decision making problems with hesitant fuzzy information and give an illustrative numerical example.

4. An Approach to Multi Criteria Decision Making under Hesitant Fuzzy Environment

For a multi criteria decision making problem, let $A = (A_1, A_2, \ldots, A_m)$ be a set of m alternatives and $C = (C_1, C_2, \ldots, C_n)$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_i = 1$. The decision makers provide all the possible values that the alternative A_i satisfies the criterion C_j represented by HFEs $h_{ij} = \bigcup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\}$, and all h_{ij}, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, construct the hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$:

Table 1: Hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$

	C_1	C_2	$\ldots$	C_n
A_1	h_{11}	h_{12}	$\ldots$	h_{1n}
A_2	h_{21}	h_{21}	$\ldots$	h_{2n}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
A_m	h_{m1}	h_{m2}	$\ldots$	h_{mn}

To harmonize the data,first step is to look at the criteria.These in general can be of different types. If all the criteria $C = (C_1, C_2, \ldots, C_n)$ are of the same type, then the criteria values do not need harmonization. However if these involve different scales and /or units, there is need to be convert them all to the same scale and/or unit. Just to make this point clear, let us consider two types of criteria, namely, (i) cost type and the (ii) benefit type. Considering their natures, a benefit criterion (the bigger the values better is it) and cost criterion (the smaller the values the better is it) are of rather opposite type. In such cases, we need to first transform the criteria values of cost type into the criteria values of benefit type. So, transform the hesitant fuzzy decision matrix $H = [h_{ij}]_{m\times n}$ into the normalized hesitant fuzzy decision matrix $B = [b_{ij}]_{m\times n}$ by the method given by Zhu and Xu [52], where

$$(37) \qquad b_{ij} = \begin{cases} h_{ij} & \text{for benefit criterion } C_j \\ h_{ij}^c, & \text{for cost criterion } C_j \end{cases}, \qquad i = 1, 2, ..., m; \quad j = 1, 2, ..., n,$$

where $h_{ij}^c = \bigcup_{\gamma_{ij} \in h_{ij}} \{1 - \gamma_{ij}\}$ is the complement of h_{ij}.

With criteria harmonized and using the $WHFGGHM$ operator, we now formulate an algorithm to solve multi criteria decision making problem with hesitant fuzzy information:

Algorithm:

Step 1: Use the $WHFGGHM$ operator to aggregate all the performance values $b_{ij}, j = 1, 2, \ldots, n$, of the i^{th} row, and get the overall performance value b_i corresponding to the alternative $A_i, i = 1, 2, \ldots, m$:

$$b_i = WHFGGHM_w^{p,q}(h_{i1}, h_{i2}, \ldots, h_{in}). \tag{38}$$

Step 2: By Definition 3, calculate the scores $S(b_i)$ of b_i and rank the overall performance values $b_i, i = 1, 2, \ldots, m$.

Step 3: Rank the alternatives$A_i, i = 1, 2, \ldots, m$, in accordance with $b_i, i = 1, 2, \ldots, m$, in descending order and select the most desirable alternative(s).

We demonstrate the above proposed algorithm to a real life multi-criteria decision making through following illustrative example.

Example[52]**:** Consider a factory site selection problem for new buildings. After pre-elimination process, only three alternatives $A_i, i = 1, 2, 3$, are being considered for further evaluation and selection. The decision makers take into account three criteria to decide the best site: C_1: price, C_2: environment, and C_3: location. The weights of criteria are $w = (0.5, 0.3, 0.2)^T$. Next let the characteristics of the alternative $A_i, i = 1, 2, 3$, with respect to the criteria $C_j, j = 1, 2, 3$, be represented by the HFEs $h_{ij} = \bigcup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\}$, where γ_{ij} indicates that the alternative A_i satisfies the criterion C_j. All $h_{ij}, i, j = 1, 2, 3$, are contained in shown in the following hesitant fuzzy decision matrix $H = [h_{ij}]_{3\times3}$:

Table 2: Hesitant fuzzy decision matrix $H = [h_{ij}]_{3\times3}$

	C_1	C_2	C_3
A_1	$\{0.6, 0.7, 0.8\}$	$\{0.25\}$	$\{0.4, 0.5\}$
A_2	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.3, 0.55, 0.6\}$
A_3	$\{0.2, 0.4\}$	$\{0.6, 0.5\}$	$\{0.7, 0.5\}$

Considering that all the criteria $C_j, j = 1, 2, 3$, are of the benefit type, then the preference values of the alternatives $A_i, i = 1, 2, 3$, do not need harmonization, therefore, $B = [b_{ij}]_{m\times n} = [h_{ij}]_{3\times3}$.

Step 1: Using the WHFGGHM operator (here, we take $p = q = 1$) to aggregate all the preference values $b_{ij}, j = 1, 2, 3$ of the i^{th} row and get the overall performance values b_i corresponding to the alternative A_i as

$$b_1 = \{0.2795, 0.2836, \ldots, 0.3803, 0.3849\},$$

$$b_2 = \{0.2052, 0.2145, \ldots, 0.2902, 0.2913\},$$

$$b_3 = \{0.2066, 0.2045, \ldots, 0.2941, 0.2883\}.$$

Step 2: We calculate the scores of all the alternatives according to $b_i, i = 1, 2, 3$:

$$S(b_1) = 0.3325, S(b_2) = 0.2542, S(b_3) = 0.2538.$$

Step 3: Since $S(b_1) > S(b_2) > S(b_3)$, by Definition 4, the ranking of the HFEs $b_i, i = 1, 2, 3$, that is, $b_1 > b_2 > b_3$, and thus, the ranking of the alternatives $A_i, i = 1, 2, 3$, is $A_1 > A_2 > A_3$. Hence A_1 is the best alternative.

Next, if we take $p = 1$ and $q = 3$ in WHFGGHM operator, then

$$b_1 = \{0.2805, 0.2911, \ldots, 0.3687, 0.3722\},$$

$$b_2 = \{0.2805, 0.2911, \ldots, 0.3644, 0.3644\},$$

$$b_3 = \{0.2262, 0.2254, \ldots, 0.3552, 0.3512\}.$$

and the scores of all the alternatives are

$$S(b_1) = 0.3449, S(b_2) = 0.3292, S(b_3) = 0.2907.$$

Thus, the ranking of the alternatives $A_i, i = 1, 2, 3$, now is $A_1 > A_2 > A_3$. Hence A_1 is still the best alternative.

4. Conclusions

In this paper, we extended the idea of aggregation and considering a wider range of aggregating operators, introduced Hesitant Fuzzy Generalized Geometric Heronian Mean (HFGGHM) operator and also that of Weighted Hesitant Fuzzy Generalized Geometric Heronian Mean (WHFGGHM) operator. Properties of the proposed operators are studied and their special cases are examined. Furthermore, we have applied the WHFGGHM operator to multi criteria decision making with hesitant fuzzy numbers. Finally, an illustrative example is given to verify the developed method and to demonstrate its practicality and effectiveness. The work has scope for extensive further application and results on these new measures.

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Algebra in Combinatorics of Statistical Dependence

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Abstract.

This paper proposes homological analysis of statistical dependency graph. If a dependency graph model satisfy the condition of a chain complex, homological algebra can be applied. Especially, the degree of freedom can be viewed as a dual space of an original complex.

Keywords: Statistical Independence, Pearson Residual, Homology, Cohomology

1 Introduction

Analysis of a contingency analysis has a long history [1], where χ^2-test play a central role in detecting statistical independence of two variables. The key idea of χ^2-test is a degree of freedom of χ^2-test statistics, the number of independent cells in a given table. If we assume that the marginal distributions of a column and a row are fixed, all the numbers in the cell will be determined by the values of independent cells. For example, since the degree of freedom of a 2×2 contingency table is equal to 1, if one cell is given, other three cells will be obtained under the given marginal distribution.

One interesting observation is that the formula of chi-square test statistics of a 2×2 contingency table includes the form of a determinant when a table is regarded as a 2×2 matrix. Tsumoto focuses on this observation and finds the interesting relations between linear algebra and statistical independence. from the viewpoint of granular computing[5, 6, 8, 7]. The important result is that a degree of freedom is equal to the number of 2×2 submatrices in a contingency table, which can be viewed as a granule of statistical independence. Interestingly, the results are generalized into mulivariate contingency tables [10], where combinatorics of independent variables is important to determine the degree of freedom [9, 11, 12]. Furthermore, symmetry of dependent variables gives classificaiton of a contingency table[13]. This paper gives further extension of this analysis, which shows that the degree of freedom corresponds to the number of outer products of dependent variables, which shows that the degree of freedom will give a dual space of statistical dependency graph when a graphical model satisfies the condition of a chain complex.

The paper is organized as follows: Section 2 gives the results of previous studies on Pearson residuals. Section 3 gives some mathematical discussions on geometrical and combinatorial structure of the above theory. Section 4 introduces homological algera as a tool for the analysis of statistical dependence. Section 5 discusses correspondence between boundary and coboundary operators and table operations. Finally, Section 6 concludes this paper.

2 Combinartorics of Peasrson Residuals

2.1 Multiway Contingency Table

Definition 2.1 *Let $R_1, R_2, \cdots, R_n$ denote $n(\in N)$ multinominal attributes in an attribute space A which have $m_1, m_2, \cdots, m_n$ values Let $|R_j = A_{j_i}|$ denote the set of data whose jth-attribute is equal to A_{j_i} (ith-partition of j). Then, an element of a multiway contigency table, which has n attributes, is defined as:*

$$x_{i_1 i_2 \cdots i_n} = \#\{x \in |R_1 = A_{i_1}| \wedge |R_2 = A_{i_2}| \cdots \wedge |R_n = A_{i_n}|\},$$

where their marginal sums are not included as elements. □

For example, in the two dimensional case, this table is arranged into the form shown in Table 1, where: $|[R_1 = A_j]_A| = \sum_{i=1}^{m} x_{1i} = x_{\cdot j}$, $|[R_2 = B_i]_A| = \sum_{j=1}^{n} x_{ji} = x_{i\cdot}$, $|[R_1 = A_j \wedge R_2 = B_i]_A| = x_{ij}$, $|U| = N = x_{\cdot\cdot}$ ($i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, m$).

Table 1: Contingency Table ($m \times n$)

	A_1	A_2	$\cdots$	A_n	Sum
B_1	x_{11}	x_{12}	$\cdots$	x_{1n}	$x_{1\cdot}$
B_2	x_{21}	x_{22}	$\cdots$	x_{2n}	$x_{2\cdot}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B_m	x_{m1}	x_{m2}	$\cdots$	x_{mn}	$x_{m\cdot}$
Sum	$x_{\cdot 1}$	$x_{\cdot 2}$	$\cdots$	$x_{\cdot n}$	$x_{\cdot\cdot} = \lvert U \rvert = N$

Let us denote the sum over one attribute a contingency table by "$\bullet$". Then, marginal sums over one attribute is defined as follows.

Definition 2.2 *Let a contigency table have m attributes. The marginal sum over $i_k (1 \leq k \leq m)$ is:*

$$x_{i_1 \cdots i_{k-1} \bullet i_k \cdots i_m} = \sum_{j=1}^{q_k} x_{i_1 \cdots i_{k-1} i_j i_{k+1} \cdots i_m}, \tag{1}$$

where q_k is the number of equivalence classes of i_k. □

Then, marginal sums over all the attributes is equal to the sample size:

$$x_{\bullet \cdots \bullet} = N,$$

2.2 Information Granule in a Contingency Table

2.2.1 Pearson Residual

Definition 2.3 *Pearson residual of the cell $i_1 \cdots i_m (m \geq 2)$, denoted by $\sigma_{i_1 \cdots i_m}$, is defined as the difference between the observed value $x_{i_1 \cdots i_m}$ and its expected value:*

$$\begin{aligned} \sigma_{i_1 \cdots i_m} = & x_{i_1 \cdots i_m} \\ & - \frac{x_{i_1 \bullet \cdots \bullet} \times x_{\bullet i_2 \cdots \bullet} \cdots \times x_{\bullet\bullet \cdots i_m}}{x_{\bullet\bullet\bullet}^{m-1}}. \end{aligned} \tag{2}$$

□

"Partial residuals" in which one of three attributes are summarized (marginalized) are defined as follows:

$$\sigma_{\bullet i_2 \cdots i_m} = x_{\bullet i_2 \cdots i_m} - \frac{x_{\bullet i_2 \cdots \bullet} \times x_{\bullet\bullet i_3 \cdots \bullet} \cdots \times x_{\bullet\bullet \cdots i_m}}{x_{\bullet\bullet\bullet}^{m-2}}. \tag{3}$$

Therefore, we obain the following theorem [12, 13]:

Theorem 2.1 *The Pearson residual of a m-way contingency table is reformulated as:*

$$\sigma_{i_1 \cdots i_m} = \frac{x_{i_1 \bullet \cdots \bullet}}{x_{\bullet\bullet \cdots \bullet}} \sigma_{\bullet i_2 \cdots i_m} + \frac{1}{x_{\bullet\bullet \cdots \bullet}} (x_{i_1 \cdots i_m} x_{\bullet\bullet \cdots \bullet} - x_{i_1 \bullet \cdots \bullet} x_{\bullet i_2 \cdots i_m}) \tag{4}$$

□

In the subsequent sections, the second part of Equation (4), $x_{i_1 \cdots i_m} x_{\bullet\bullet \cdots \bullet} - x_{i_1 \bullet \cdots \bullet} x_{\bullet i_2 \cdots i_m}$ is denoted by $\sigma^{i_1}_{i_2 \cdots i_m}$. More detailed examples are shown in [12].

2.3 Degree of Freedom

2.3.1 Formula of Degree of Freedom From Equation (4),

$$\sigma_{i_1 \cdots i_m} = \frac{x_{i_1 \bullet \cdots \bullet}}{x_{\bullet\bullet \cdots \bullet}} \sigma_{\bullet i_2 \cdots i_m} + \frac{1}{x_{\bullet\bullet \cdots \bullet}} (x_{i_1 \cdots i_m} x_{\bullet\bullet \cdots \bullet} - x_{i_1 \bullet \cdots \bullet} x_{\bullet i_2 \cdots i_m})$$

Although the first part includes the same number of the determinants as $\sigma_{i_2 \cdot 1_m}$ multiplied by the size of the first attribute, the weight:

$$\frac{x_{i_1 \bullet \cdots \bullet}}{x_{\bullet\bullet \cdots \bullet}} = \frac{1}{size \text{ of } the\ first\ attribute}$$

should be considered for estimation of the degree of freedom. In other words, the number of the subdeterminants can be estimated as the number of the subdeterminants of $(m-1)$-way contingency table. On the other hand, the second part is equal to:

$$\begin{aligned} \sigma^{i_1}_{i_2 \cdots i_m} &= x_{i_1 \cdots i_m} x_{\bullet\bullet \cdots \bullet} - x_{i_1 \bullet \cdots \bullet} x_{\bullet i_2 \cdots i_m} \\ &= \sum_{j_1 \neq i_1} (x_{i_1 \cdots i_m} x_{j_1 \bullet \cdots \bullet} - x_{i_1 \bullet \cdots \bullet} x_{j_1 i_2 \cdots i_m}) \end{aligned} \tag{5}$$

If the other terms $i_k (k = 2, \cdots, m)$ are not equal to j_k, the subdeterminant is not equal to 0 and the subscript of the summation of Equation (5) is equivalent to:

$$\bigvee_{k=2}^{m} (i_k \neq j_k),$$

Therefore, the following theorem is obtained [13]:

Theorem 2.2 *Let IND_p^m denote a set of index which is a p out of m attributes. Then, the total number of determinants of 2×2 submatrices in a residual of a m-way contingency table is given by:*

$$\zeta(1,2,\cdots,m) = \sum_{p=2}^{m} \prod_{t\in IND_p^m} (n_t - 1), \tag{6}$$

where n_t denote the number of partitions of an attribute t. □

Corollary 2.3 *The total numbers of determinants of 2×2 submatrices in m-way contingency table, denoted by $\zeta(1,2,\cdot,m)$ are equal to*

$$\zeta(1,2,\cdots,m) = n_1 n_2 \cdots n_m - \sum_{i=1}^{m} n_i + (m-1).$$

□

In this way, the degree of freedom summarizes information on combinatorial nature of Pearson residuals:

Corollary 2.4 *Let IND_p^m denote a set of index which is a p out of m attributes The degree of freedom of a m-way contingency table is given by:*

$$\zeta(1,2,\cdots,m) = \sum_{p=2}^{m} \prod_{t\in IND_p^m} (n_t - 1),$$

where n_t is the number of partitions in an attribute t, if all the variables are assumed to be independent. □

When some of attributes are dependent, the corresponding term will be eliminated. For example, m_2 and m_3 of three-way attribute is dependent, but m_1 is conditionally independent of this pair:

$$\begin{aligned}\zeta(1,2,3) &= (n_1-1)(n_2-1)(n_3-1)\\ &\quad +(n_3-1)(n_1-1)\\ &\quad +(n_1-1)(n_2-1)\\ &= n_1n_2n_3 - n_2n_3 - n_1 + 2, \end{aligned} \tag{7}$$

which is the same formula given in [1].

3 Symmetry in Pearson Residuals

3.1 Determinants as Pencil of Lines Equation (5) shows that 2×2-subdeterminants are information granules of statistical independence. Since a 2×2-subdeterminant gives a line in a projective plane in classic projective geometry, a set of the subdetermiants can be viewed a *pencil of lines* in a space of projective plane whose coordinates are given as a cell in a given contingency table: dependence and independence can be captured as geometrical structure of a pencil in a projective space. Thus, dependent or independent relations of a multivariate table gives complex geometrical structure, which suggests that a tool of algebra can be applied to analysis

3.2 Symmetry Let $a_1, a_2, \cdots, a_m$ denote m attributes in a m-way contingency table and e_i be equal to $n_i - 1$ where n_i is a number of partition in attribute a_i. Then, $e_j e_k$ gives the number of 2×2subdeterminants when a_j and a_k are dependent. In general, $e_{j_1} e_{j_2} \cdots e_{j_l}$ gives the number of $l - way$ subdeterminants when these l attributes are dependent.

Let $E^l_{12\cdots m}$ denote a set of $e_{j_1} e_{j_2} \cdots e_{j_l}$, each element of which is a product of l selected from m attributes. For each l dimension, a polynomial symmetric over S_m can be derived as:

$$s^l_m = \sum_{E_{12\cdots m}} e_{j_1} e_{j_2} \cdots e_{j_l},$$

Then, a polynomial symmetric over S_m is represented as:

$$\bigoplus_{k=2}^{l} s^k_m = \bigoplus_{k=2}^{l} \sum_{E_{12\cdots m}} e_{j_1} e_{j_2} \cdots e_{j_l} \tag{8}$$

Then, relations between symmetric group and geometrical structure can be discussed [13]. For example, since conditional dependence is defined as statistical dependence of a set of variables, denoted by V_1 under the assumptions where the values of the set of other variables (V_2) are fixed, the symmetry of V_2 will be lost, that is , "breakdown of symmetry".

Theorem 3.1 *Let $a_1, a_2, \cdots, a_m$ denote m attributes in a m-way contingency table and e_i be equal to $n_i - 1$ where n_i is a number of partition in attribute a_i. A formula $e_{j_1} e_{j_2} \cdots e_{j_l}$ is equal to the number of $l - way$ subdeterminants when these l attributes are dependent. Let $E^l_{12\cdots m}$ denote a set of $e_{j_1} e_{j_2} \cdots e_{j_l}$, each element of which is a product of l selected from m attributes. Then, a polynomial symmetric over S_m is given as:*

$$\bigoplus_{k=2}^{l} s^k_m = \bigoplus_{k=2}^{l} \sum_{E_{12\cdots m}} e_{j_1} e_{j_2} \cdots e_{j_l}. \tag{9}$$

□

Then,

Corollary 3.2 *A model with partial independence can be derived by removal of attributes whose values a fixed. For example,*

$$\bigoplus_{k=2}^{l} s^k_m - \sum_{k=2}^{m} \sum_{T^m_k} e_{j_1} e_{j_2} \cdot e_{j_k} \tag{10}$$

gives the number of subdeterminants where T^m_k gives a set of k-pair statistical independent attributes out of m attributes. □

Tsumoto [13] shows that structure of symmetric group will give some global information on statistical dependency model in a multivariate contingency table. However, this tool focuses on interchangeability of dependence relations among variables. In order to investigate other properties of geometrical structure, other tool is needed as shown in the next section.

4 Degree of Freedom and Homological Calculus Equation (10) is a little complicated and it is difficult to see the meaning except for the action of symmetric group. The most import problem is that we have to eliminate independence variables explicitly, which makes the representation power weak. The other point is that we may have a situation when three variables are statistical dependent (independent) although all the combinations of two of three variables are statistical independent (dependent), which analysis based on symmetric group cannot capture. Thus, by investigating the nature of statistical dependence much further, new representation should be explored.

4.1 Main Ideas The main idea is that geometrical structure of a statistical dependence model corresponds to its degree of freedom of a contingency table as shown in Figure 1. Let us assume three variables provided in a contingency table. In the case of one dimension, the degree of freedom of one attribute will be the number of its partition minus 1 if the number of examples is fixed. If we add one more dependent attribute, the degree of freedom is equal to $(n_1 - 1)(n_2 - 1)$, where n_1 and n_2 denote the number of partitions of the first and second attribute. In the same way, dependency graph of three attribute has $(n_1 - 1)(n_2 - 1)(n_3 - 1)$ as its degree of freedom. If we consider dependency of three attributes with full dependency, the degree of freedom is equal to:

$$\begin{aligned}(n_1-1)(n_2-1)(n_3-1) \quad &+(n_1-1)(n_2-1)\\ &+(n_2-1)(n_3-1)\\ &+(n_3-1)(n_1-1).\end{aligned}$$

The main idea is that decomposition of a dependency graph gives a homological sequence

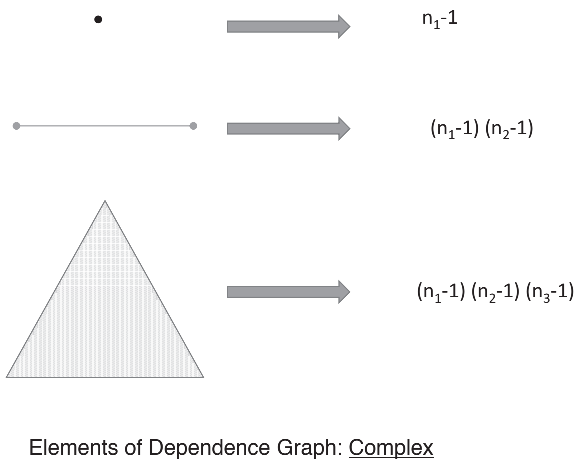

Figure 1: Correspondence between Dependency Graph and its Degree of Freedom

shown in Figure 2. Formal definition of mappings will be given in subsequent subsections.

4.2 Basic Definition The key components of the above section are the degree of freedom of each attribute. When two attributes are dependent, the degree of freedom is obtained by their product of the degree of freedom. Furthermore, the product is invariant over permutation.

Definition 4.1 *Let $A = \{a_1, a_2, \cdots, a_m\}$ denote a set of m attributes in a m-way contingency table and e_i be equal to $n_i - 1$ where n_i is a number of partition in attribute a_i. Then, a linear sum of m attributes gives a vector space spanned by A:*

$$vec(A) = \sum_{i=1}^{m} k_i a_i,$$

where $k_i \in \mathbb{Z}$. Thus, $vec(A)$ can be viewed as $\mathbb{Z}$-module. □

As a different type of operation, we define $a_1 \wedge a_2$ as a matrix generated by a_1 and a_2:

$$a_1 \wedge a_2 = Mat(a_1, a_2).$$

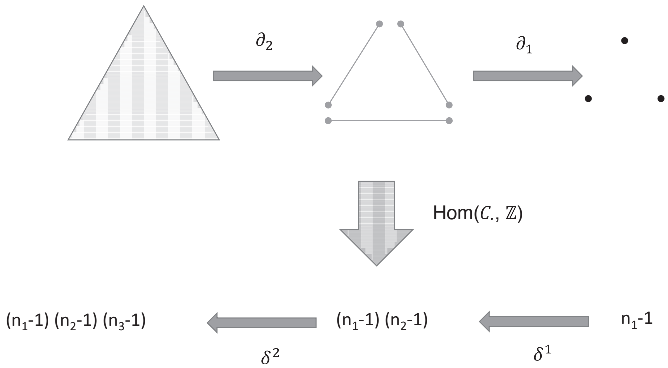

Figure 2: Homological Sequence of Three Attributes

It is notable that a_1 and a_2 are reversibly obtained by $Mat(a_1, a_2)$ as marginal sums. For example, when a matrix is given as:

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix},$$

corresponding two vectors are: $a_1 = (2+2, 2+3) = (4,4)$ and $a_2 = (2+1, 2+3) = (3,5)$.

It is notable that this transformation is linear and can be represented as a matrix. In the above example, the transformation is given by:

$$\begin{pmatrix} 4 & 3 \\ 4 & 5 \end{pmatrix} = X \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix},$$

In this case, $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ has a reverse, so X is obtained as:

$$X = \frac{1}{4} \begin{pmatrix} 9 & -2 \\ 7 & 2 \end{pmatrix}$$

However, if the determinant of original table is equal to 0, the situation is much more complex: we should use the generalized inverse for calculation. But, since this case is corresponding to statistical independence, let say $Mat(\bullet \wedge \bullet) = 0$. That is, for two attributes a_i, a_j $(i, j = 1, \cdots, m)$,

(11) $$a_i \wedge a_j = \begin{cases} Mat(a_i, a_j) & det(a_i, a_j) \neq 0 \\ 0 & det(a_i, a_j) = 0, \end{cases}$$

where $det(a_i, a_j)$ gives a determinant of a matrix (a_i, a_j) and Mat is a corresponding matrix operation. Thus, although there are many ways to make a matrix from a_1 and a_2, for a given table, a matrix corresponds to one function which partitions a_1 and a_2 as shown the above. Thus, a matrix can be viewed as a partition function of $\mathbb{Z} \times \mathbb{Z}$.

Since a vector and matrix can be viewed as a specific form of tensor, the above discussion can be discussed in the context of tensor calculus: it is easy to see that this framework satisfies the axiom of tensor space. Moreover, a space have elements of different grades, since a_1 and $a_1 \wedge a_2$ are the first and second grade, respectively.

4.3 Degree of Freedom as a Mapping Since the degree of freedom of a_i is the number of elementary partition of a_i minus 1, it can be viewed as a function of a_i:

$$df(a_i) = e_i.$$

It can be rewritten as:

$$df = Hom(a_i, \mathbb{Z}),$$

which is a homomorphism under addition. In the above example of matrix, $df(a_1 \wedge a_2) = (2-1)*(2-1) = 1$ and $df(a_1) = df(a_2) = 1 - 1 = 0$ because the second element of a_1 and a_2 can be described as a linear sum of the first element: $4 = 1 \times 4$, $5 = \frac{5}{3} \times 3$.

It is notable that df will give a dual space of a_i. Let us denote $df(a_i)$ by a_i^*. Then, it is easy to show that $A^* = df(A) = \{df(a_1), df(a_2), \cdots, df(a_m)\}$ gives a dual (tensor) space of $A = a_1, a_2, \cdots, a_m$.

Thus, dependence and independence can be easily described as an outer product, alternating tensor product of $df(a_i)$ $(i = 1, 2, \cdots, m)$. Since the calculation of the degree of freedom starts from two attributes as a matrix calculus, let us select two attributes first. If we take two dependent attributes a_i and a_j, then $a_i \wedge a_j$ gives:

$$a_i \wedge a_j = det(M(a_i, a_j))e_i \wedge e_j,$$

where a rectangular matrix $M(a_i, a_j)$ is generated by $\{a_i, a_j\}$ and e_i and e_j denote the orthonormal basis generated by a_i and a_j. Here, the determinant is given by Cullin's determinant, which is an extension of ordinary matrix.[2, 4] Then, when two attributes a_1 and a_2 are independent, since the rank of matrix is equal to 1, $M(a_i, a_j)$ is represented as (v_i, kv_i)[5].

Thus,

$$df(a_1 \wedge a_2) \quad = df(a_1 \wedge ka_1) \quad = 0,$$

where $k \in Z$. On the other hand, if both are dependent:

$$df(a_1 \wedge a_2) = df(a_1)df(a_2) = (n_1 - 1) * (n_2 - 1),$$

where n_1- and $n_2 - 1$ denote the degree of freedom of a_1 and a_2.

When we take 3 attributes, we can append this attribute as $a_1 \wedge a_2 \wedge a_3$. Then, full dependence can be described as:

$$a_1 \wedge a_2 \wedge a_3 + a_1 \wedge a_2 + a_2 \wedge a_3 + a_3 \wedge a_1.$$

The formula shown in Equation 7 is given by:

$$\begin{aligned} df(a_1 \wedge a_2 \wedge a_3) &= df(a_1 \wedge a_2 \wedge a_3 + a_1 \wedge a_2 \\ &\quad + a_2 \wedge a_3 + a_3 \wedge a_1) \\ &= df(a_1 \wedge a_2 \wedge a_3) + df(a_1 \wedge a_2) \\ &\quad + df(a_2 \wedge a_3) + df(a_3 \wedge a_1) \\ &= (n_1 - 1)(n_2 - 1)(n_3 - 1) \\ &\quad + (n_1 - 1)(n_2 - 1) \\ &\quad + (n_2 - 1)(n_3 - 1) \\ &\quad + (n_3 - 1)(n_1 - 1) \end{aligned}$$

If a_1 and a_2 is independent, since $df(a_1 \wedge a_2) = 0$, we should remove this term from the above equation.

Since the $df(a_i)$ can ve viewed as a dual vector, we can rewrite the above equation as:

$$\begin{aligned} a_1^* \wedge a_2^* \wedge a_3^* &= a_1^* \wedge a_2^* \wedge a_3^* + a_1^* \wedge a_2^* + a_2^* \wedge a_3^* \\ &\quad + a_3 \wedge a_1^* \\ &= a_1^* \wedge a_2^* \wedge a_3^* + a_1^* \wedge a_2^* + a_2^* \wedge a_3^* \\ &\quad + a_3^* \wedge a_1^* \\ &= (n_1 - 1)(n_2 - 1)(n_3 - 1) \\ &\quad + (n_1 - 1)(n_2 - 1) \\ &\quad + (n_2 - 1)(n_3 - 1) \\ &\quad + (n_3 - 1)(n_1 - 1) \end{aligned}$$

By using these ideas, Theorem 3.1 can be reformulated as follows.

Theorem 4.1 *Let $A = \{a_1, a_2, \cdots, a_m\}$ denote a set of m attributes in a m-way contingency table and e_i be equal to $n_i - 1$ where n_i is a number of partition in attribute a_i. Let DEP^m denote a set of dependent variables of A. Then, since the correspondence between a_i and e_i is given as a function: $f(a_i) = e_i$, dual tensor space can be defined by $df(A)$, which can be denoted by $A^* = \{a_1^*, a_2^*, \cdots, a_m^*\}$ Then, a polynomial symmetric over S_m is represented as:*

$$\bigoplus_{k=2}^{l} s_m^k = \bigoplus_{k=2}^{l} \sum_{DEP^k} (-1)^{\sigma(j)} a_{j_1}^* \wedge a_{j_2}^* \wedge \cdots \wedge a_{j_k}^*, \tag{12}$$

where $\sigma(j)$ denotes the the number of substitutions over $j_1, j_2, \cdots j_k$. □

4.4 Chain Complex Let A_n^* denote a space spanned by a set of dual of outer product of n attributes. Since this space is an Abelian group, we can consider a sequence $A_n, A_{n-1}, \cdots, A_2, A_1$. Let us define the boundary map from A_n^* to A_{n-1}^* as:

$$\partial^n : A_n^* \to A_{n-1}^* : a_1^* \wedge a_2^* \cdots \wedge a_n^* \mapsto \sum_{i=1}^{n} (-1)^{i-1} \sigma([a_1^* \wedge a_2^* \wedge \ldots \hat{a}_i^* \ldots \wedge a_n^*]), \tag{13}$$

where the hat denotes the omission of an attribute. Then, the following theorem is obtained.

Theorem 4.2

$$\partial^n \partial^{n-1} A_n^* = 0$$

Thus, $(A., \partial.)$ is a chain complex. Proof

$$\begin{aligned}
\partial_n \partial_{n-1} A_n^* &= \partial\partial a_1^* \wedge a_2^* \cdots \wedge a_n^* \\
&= \partial \left(\sum_{i=1}^{n} (-1)^{i-1} a_1^* \ldots \wedge \hat{a}_i^* \ldots \wedge a_n^* \right) \\
&= \sum_{j=1}^{n} (-1)^{j-1} \left(\sum_{i=1}^{n} (-1)^{i-1} a_1^* \ldots \wedge \hat{a}_i^* \right. \\
& \left. \wedge \ldots \wedge \hat{a}_j^* \ldots \wedge a_n^* \right) \\
&= \sum_{i,j=1}^{n} \left(a_1^* \ldots \wedge \hat{a}_i^* \ldots \wedge \hat{a}_j^* \ldots \wedge a_n^* \right. \\
& \left. - a_1^* \ldots \wedge \hat{a}_i^* \ldots \wedge \hat{a}_j^* \ldots \wedge a_n^* \right) \\
&= 0
\end{aligned}$$

□

Example 4.1

$$\begin{aligned}
\partial^3 \partial^2 (a_1^* \wedge a_2^* \wedge a_3^*) &= \partial\partial (n_1 - 1)(n_2 - 1)(n_3 - 1) \\
&= \partial \left((n_2 - 1)(n_3 - 1) \right. \\
& -(n_1 - 1)(n_3 - 1) \\
& \left. +(n_1 - 1)(n_2 - 1) \right) \\
&= (n_2 - 1) - (n_3 - 1) - (n_1 - 1) \\
& +(n_3 - 1) + (n_1 - 1) - (n_2 - 1) \\
& = 0
\end{aligned}$$

Since it is easy to see that the outer space of a_i gives a chain complex, the dual space a_i^* is its dual space. Thus, homology of a sequence $H_n(A)$can be considered:

$$H_n(A) = \ker \partial_n / \mathrm{im} \partial_{n+1}.$$

$$H^n(A) = \ker \partial^{n+1} / \mathrm{im} \partial^n$$

Example 4.2 *In the above example,*

$$\begin{aligned}
C_2 &= \{(n_1 - 1)(n_2 - 1)(n_3 - 1)\} \\
C_1 &= \{(n_1 - 1)(n_2 - 1), (n_2 - 1)(n_3 - 1), \\
& \quad (n_1 - 1)(n_3 - 1)\} \\
C_0 &= \{(n_1 - 1), (n_2 - 1), (n_3 - 1)\}
\end{aligned}$$

$$\begin{aligned}
im\partial^2(A) = & \quad (n_2 - 1)(n_3 - 1) \\
& -(n_1 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)
\end{aligned}$$

Then, $\partial^2(A)$, $\partial^1(A)$, *and* $\partial^0(A)$ *can be represented in a matrix form shown as below.*

(14)
$$\begin{aligned}
&\partial_2(n_1-1)(n_2-1)(n_3-1)\\
&=\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\begin{pmatrix}(n_1-1)(n_2-1)\\(n_2-1)(n_3-1)\\(n_1-1)(n_3-1)\end{pmatrix}\\
&\partial_1\begin{pmatrix}(n_1-1)(n_2-1)\\(n_2-1)(n_3-1)\\(n_1-1)(n_3-1)\end{pmatrix}\\
&=\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}(n_1-1)\\(n_2-1)\\(n_3-1)\end{pmatrix}\\
&\partial_0\begin{pmatrix}(n_1-1)\\(n_2-1)\\(n_3-1)\end{pmatrix}\\
&=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}
\end{aligned}$$

Since the rank of each matrix gives the dimension of ∂*, the difference between matrix size and its rank is equivalent of the dimension of* $\ker\partial$ *From these equations, we obtained the ranks of im and* ker *as follows.*

	Im	*Ker*
∂^3	*0*	*0*
∂^2	*1*	*0*
∂^1	*2*	*1*
∂^0	*0*	*3*

Thus, H^0, H^1 *and* H^2 *is obtained as follows.*

$$\begin{aligned}
H^2 &= \frac{0}{1} = 0\\
H^1 &= \frac{0}{\mathbb{Z}\oplus\mathbb{Z}} = 0\\
H^0 &= \frac{\mathbb{Z}}{0} = \mathbb{Z}
\end{aligned}$$

Thus, the cohomological sequence is obtained as follow.

$$0 \to \mathbb{Z} \to 0 \to 0$$

In a dual way, homological sequence of the outer product of dependency relations can be obtained as follows: since ∂_n *is given as the transpose of* ∂^n *in a matrix representation, the* *table of rank of im and* ker *is equivalent. Thus,* H_0*,* H_1 *and* H_2 *is obtained as follows.*

$$\begin{aligned} H_2 &= \frac{0}{0} = 0 \\ H_1 &= \frac{\mathbb{Z}}{\mathbb{Z}} = 0 \\ H_0 &= \frac{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\oplus\mathbb{Z}} = \mathbb{Z} \end{aligned}$$

Thus, the homological sequence is obtained as follow.

$$0 \to 0 \to \mathbb{Z} \to 0$$

□

In the same way, both the homological and cohomology sequences for the model with only two variables dependent are:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

which corresponds to the homology of a corresponding dependency graph.

5 Discussion Figure 2 illustrates how boundary and coboundary operators are used in the context of contingency table analysis. Boundary operators will reduce the degree of freedom, which corresponds to marginalization as shown in Figure 3. On the other hand, coboundary operators corresponds to partition as shown in Figure 4.

Although boundary and coboundary operators are dual to each other, corresponding operations show that although boundary is one choice, but coboundary may give many possible ways. In other words, partition or coboundary suffers from combinatorial problems. Thus, cohomological analysis may give insights to formal discussions on partition.

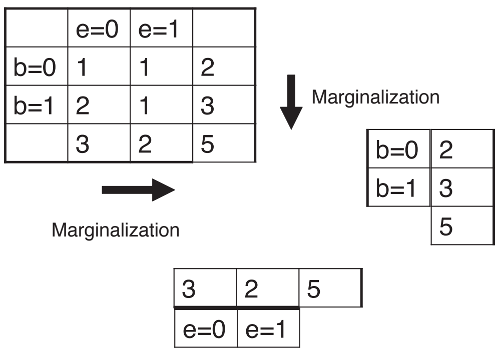

Figure 3: Marginalization as Boundary Operator

6 Conclusion This paper focuses on the formula of degree of freedom and investigate its nature. First, if we assume that a dependency graph satisfies the condition of a complex, a boundary operator ∂ for the formula of degree of freedom can be defined and the duplicated operation will be 0: $\partial\partial = 0$, which leads to the basic step to homological algebra. Second, the formula can be viewed as a homomorphism from structure to integer, denoted by $Hom(Structure, \mathbb{Z})$, thus the hierarchy of the formula of degree of freedom,

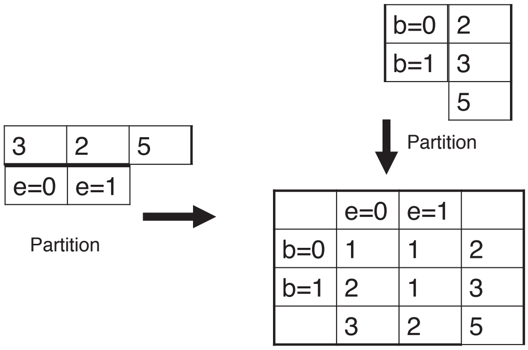

Figure 4: Partition as Coboundary Operator

which corresponds to the hierarchy of the dependency graph, generates a cocomplex. Thus, cohomology of the formula can be considered. By using this framework, the complex nature of dependency graph is translated into the algebraic structure of (co-)homological sequence, and (co)homology groups characterize the dependency graph. Thus, several tools in homological algebra can be applied to analysis of statistical independence

This study is a preliminary step of the analysis of statistical (in)dependence based on homological algebra. It will be our future work to investigate further the property a of contingency table from the viewpoint of algebra.

Acknowledgment This research is supported by Grant-in-Aid for Scientific Research (B) 15H2750 from Japan Society for the Promotion of Science(JSPS).

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FUZZY LINEAR PROGRAMS WITH OCTAGONAL FUZZY NUMBERS

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Abstract. Zimmermann [9] developed the decision making concept in a fuzzy environment which was proposed and analysed by Bellman and Zadeh [3] in 1970. Its application in fuzzy linear programming was well handled by Tanaka et al. [7] and by Maleki et.al in [6] . Later several kinds of fuzzy linear programming problems have been dealt with and various methodologies have been adopted to solve such problems using trapezoidal fuzzy numbers for example as in [4, 8]. The concept of octagonal fuzzy numbers was introduced by the authors in an earlier paper [5]. In this paper the octagonal fuzzy numbers are used to solve fuzzy linear programming problems (FLP) involving simplex method. A method for solving FLP involving symmetric octagonal fuzzy numbers is developed and it may be noted that it is solved without converting to crisp linear programming problem. The process is illustrated with a numerical example involving a real life problem.

The distinguishing factor which is innovative in the present study is the use of a new arithmetic on symmetrical octagonal fuzzy numbers. On this class is introduced a binary operation of multiplication denoted by $*$ defined in Definition 1.2 that is more natural having the desired property $\widetilde{A}*\widetilde{B} \approx -(-\widetilde{A})*\widetilde{B}$ and such a property is absent in the multiplication introduced by earlier authors in [4].

Keywords Fuzzy linear programming, symmetric octagonal fuzzy numbers, ranking.

1 Introduction We adhere to the concepts, notions and notations in [5]. Here we consider a subclass of octagonal fuzzy numbers called symmetrical octagonal fuzzy numbers using which a method for solving fuzzy linear programming problems without converting them to crisp linear programming problem has been discussed. The $*$ multiplication defined in this paper is more natural as it coincides with multiplication of real numbers in crisp case.

In section 1 octagonal fuzzy numbers that are symmetrical is considered and fuzzy arithmetic on this class and fuzzy measure of octagonal fuzzy numbers are defined. In section 2, a general fuzzy linear programming problem is cited and the theory related to simplex algorithm for solving FLP is dealt with. The same is illustrated by using a numerical example in section 3.

Definition 1.1. *A fuzzy number $\tilde{A}$ is called a symmetric octagonal fuzzy number if there exist real numbers $a_1, a_2, a_1 < a_2$ and $h > s > g > 0$ such that*

$$\mu_{\tilde{A}}(x) = \begin{cases} k\left[\frac{x}{h-s} + \frac{h-a_1}{h-s}\right], & x \in [a_1 - h, a_1 - s] \\ k, & x \in [a_1 - s, a_1 - g] \\ k + (1-k)\left[\frac{x}{g} + \frac{g-a_1}{g}\right], & x \in [a_1 - g, a_1] \\ 1, & x \in [a_1, a_2] \\ k + (1-k)\left[\frac{a_2+g}{g} - \frac{x}{g}\right], & x \in [a_2, a_2 + g] \\ k, & x \in [a_2 + g, a_2 + s] \\ k\left[\frac{a_2+h}{h-s} - \frac{x}{h-s}\right], & x \in [a_2 + s, a_2 + h] \\ 0, & otherwise \end{cases} \tag{1.1}$$

We denote it by $\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$ .When $h = s = g = 0$; $\tilde{A} \approx (a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2; k, 1)$ reduces to a trapezoidal fuzzy number. The set of all symmetric octagonal fuzzy numbers is denoted by $\mathcal{F}(S_O)$.

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Definition 1.2.

If $\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$ and
$\widetilde{B} \approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$ are two symmetric octagonal fuzzy numbers. Then

(i) Addition:

$$\begin{aligned}\tilde{A} + \widetilde{B} \approx\ & (a_1 + b_1 - (h + m), a_1 + b_1 - (s + l), a_1 + b_1 - (g + f), a_1 + b_1,\\ & a_2 + b_2, a_2 + b_2 + (g + f), a_2 + b_2 + (s + l), a_2 + b_2 + (h + m); k, 1)\end{aligned}$$

(ii)Subtraction:

$$\begin{aligned}\tilde{A} - \widetilde{B} \approx\ & (a_1 - b_2 - (h + m), a_1 - b_2 - (s + l), a_1 - b_2 - (g + f), a_1 - b_2,\\ & a_2 - b_1, a_2 - b_1 + (g + f), a_2 - b_1 + (s + l), a_2 - b_1 + (h + m); k, 1)\end{aligned}$$

(iii) Multiplication:

$$\begin{aligned}\widetilde{A} * \widetilde{B} \approx\ & \left(\left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) - p\right) - \left(\left|\frac{a_1+a_2}{2}\right| m + \left|\frac{b_1+b_2}{2}\right| h\right),\right.\\
& \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) - p\right) - \left(\left|\frac{a_1+a_2}{2}\right| l + \left|\frac{b_1+b_2}{2}\right| s\right),\\
& \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) - p\right) - \left(\left|\frac{a_1+a_2}{2}\right| f + \left|\frac{b_1+b_2}{2}\right| g\right),\\
& \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) - p\right), \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) + p\right)\\
& \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) + p\right) + \left(\left|\frac{a_1+a_2}{2}\right| f + \left|\frac{b_1+b_2}{2}\right| g\right),\\
& \left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) + p\right) + \left(\left|\frac{a_1+a_2}{2}\right| l + \left|\frac{b_1+b_2}{2}\right| s\right),\\
& \left.\left(\left(\frac{a_1+a_2}{2}\right)\left(\frac{b_1+b_2}{2}\right) + p\right) + \left(\left|\frac{a_1+a_2}{2}\right| m + \left|\frac{b_1+b_2}{2}\right| h\right); k, 1\right)\end{aligned}$$

where $p = \left(\frac{\beta - \alpha}{2}\right)$, $\alpha = \min(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$, $\beta = \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$

Also, it is clear from (iii) that for any real λ

$$\lambda\tilde{A} \approx \begin{cases} (\lambda(a_1 - h), \lambda(a_1 - s), \lambda(a_1 - g), \lambda a_1, \lambda a_2, \lambda(a_2 + g), \\ \qquad \lambda(a_2 + s), \lambda(a_2 + h); k, 1), \text{for } \lambda \geq 0 \\ (\lambda(a_2 + h), \lambda(a_2 + s), \lambda(a_2 + g), \lambda a_2, \lambda a_1, \lambda(a_1 - g), \\ \qquad \lambda(a_1 - s), \lambda(a_1 - h); k, 1), \text{for } \lambda < 0 \end{cases}$$

Remark 1.3. *Any real number $r \in \mathbb{R}$ can be expressed as $(r, r, r, r, r, r, r, r; k, 1)$. Continuing this view point consider two real numbers $r, s \in \mathbb{R}$ expressed as symmetric octagonal fuzzy numbers $(r, r, r, r, r, r, r, r; k, 1) * (s, s, s, s, s, s, s, s; k, 1)$. Using the Definition 1.2 we obtain its product as $(rs, rs, rs, rs, rs, rs, rs, rs; k, 1)$.*

Definition 1.4. *For any symmetric octagonal fuzzy number $\tilde{x}$, let us define $\tilde{x} \succcurlyeq \tilde{0}$ if there exist $a \geq 0$ and $h \geq s \geq g \geq 0$ such that*

$\tilde{x} \succcurlyeq (-(a+h), -(a+s), -(a+g), -a, a, (a+g), (a+s), (a+h); k, 1)$. Note that $(-(a+h), -(a+s), -(a+g), -a, a, (a+g), (a+s), (a+h); k, 1)$ is equivalent to $\tilde{0}$.

Remark 1.5. *$\tilde{x}$ is called zero symmetric octagonal fuzzy number if $\widetilde{x} \approx \widetilde{0}$. $\tilde{x}$ is said to be a non-zero symmetric octagonal fuzzy number, if $\tilde{x} \not\approx \tilde{0}$. If $\tilde{x}$ is a non-negative symmetric octagonal fuzzy number and $\tilde{x}$ is not equivalent to $\widetilde{0}$, then $\tilde{x}$ is called a positive symmetric octagonal fuzzy number denoted $\tilde{x} \succ \tilde{0}$. If $\widetilde{x}$ is a non-positive symmetric octagonal fuzzy number and is not equivalent to $\widetilde{0}$, then $\tilde{x}$ is called a negative symmetric octagonal fuzzy number denoted $\tilde{x} \prec \tilde{0}$.*

Definition 1.6. *Two symmetrical octagonal fuzzy numbers $\tilde{A}$, $\widetilde{B}$ are called equivalent denoted, $\tilde{A} \approx \widetilde{B}$ if and only if for*

$\tilde{A} \approx (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$ and
$\tilde{B} \approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$,
we have

$$\begin{aligned} & (a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1) \\ & -(b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1) \\ \approx\ & (-\alpha - (h + m), -\alpha - (s + l), -\alpha - (g + f), -\alpha, \alpha, \alpha + (g + f), \\ & \alpha + (s + l), \alpha + (h + m); k, 1) \\ \approx\ & \widetilde{0} \end{aligned}$$

i.e.$(a_1 - h, a_1 - s, a_1 - g, a_1, a_2, a_2 + g, a_2 + s, a_2 + h; k, 1)$
$\approx (b_1 - m, b_1 - l, b_1 - f, b_1, b_2, b_2 + f, b_2 + l, b_2 + m; k, 1)$
Note that this is possible even if $a_1 \neq b_1$ and $a_2 \neq b_2$.

Remark 1.7. *A particular case of Definition 1.2 taking $h = s = g$ and $m = l = f$ resulting in symmetric trapezoidal fuzzy numbers whose $*$ multiplication will be as follows:*

For $\tilde{A} \approx (a_1 - g, a_1, a_2, a_2 + g)$, $\tilde{B} \approx (b_1 - f, b_1, b_2, b_2 + f)$
Then

$$\begin{aligned} \widetilde{A} * \widetilde{B} = & \left(\left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - P \right) - \left(\left| \frac{a_1 + a_2}{2} \right| f + \left| \frac{b_1 + b_2}{2} \right| g \right), \right. \\ & \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - p \right), \left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p, \\ & \left. \left(\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + p \right) + \left(\left| \frac{a_1 + a_2}{2} \right| f + \left| \frac{b_1 + b_2}{2} \right| g \right) \right) \end{aligned}$$

where $p = \left(\frac{\beta - \alpha}{2} \right)$, $\alpha = \min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$, $\beta = \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)$

In this case, we note that $\tilde{A} * \tilde{B} \approx -(-\tilde{A}) * \tilde{B}$ is satisfied which falls in line with the classical problems in the crisp case.

Remark 1.8.

(i) Symmetric Octagonal fuzzy numbers satisfy the distributive property i.e. Let $\tilde{a}, \tilde{b}$ and $\tilde{c}$ be any three symmetric octagonal fuzzy numbers, then
$\tilde{c} * (\tilde{a} + \tilde{b}) \approx (\tilde{c} * \tilde{a} + \tilde{c} * \tilde{b})$ and $\tilde{c} * (\tilde{a} - \tilde{b}) \approx (\tilde{c} * \tilde{a} - \tilde{c} * \tilde{b})$,
where addition, subtraction and multiplication is defined by Definition 1.2

(ii) Multiplication operation given by Definition 1.2 asserts that product of two symmetrical octagonal fuzzy numbers is a symmetrical octagonal fuzzy number.

(iii) Insistence on a symmetric product is easier to handle for computational purposes.

Remark 1.9. *In the literature the following fuzzy numbers are considered - triangular fuzzy numbers, trapezoidal fuzzy numbers and hexagonal fuzzy numbers. In this concept we have used octagonal fuzzy numbers recently. The class of such numbers form a tower of subclasses. The last one namely octagonal fuzzy numbers forming the largest subclass. Therefore operations defined for the last class of octagonal fuzzy numbers evidently apply to the smaller classes.*

Definition 1.10. *[5] Let $\tilde{A}$ be a normal octagonal fuzzy number. The value $M_0^{Oct}(\tilde{A})$, called the measure of $\tilde{A}$ is calculated as follows:*

$$\begin{aligned} M_0^{Oct}(\tilde{A}) &= \frac{1}{2}\int_0^k (l_1(r)+l_2(r))dr + \frac{1}{2}\int_k^1 (s_1(t)+s_2(t))dt \quad \text{where} \quad 0<k<1 \\ &= \frac{1}{4}[k(a_1+a_2+a_7+a_8)+(1-k)(a_3+a_4+a_5+a_6)] \end{aligned}$$

Remark 1.11. *[5]*

1) If $a_1+a_2+a_7+a_8 = a_3+a_4+a_5+a_6$ we would get the measure of an octagonal number same for any value of k. $(0<k<1)$

2) In case of symmetric octagonal fuzzy numbers, the condition $a_1+a_2+a_7+a_8 = a_3+a_4+a_5+a_6$ holds and hence the measure is independent of the choice of k.

3) If $\tilde{A}$ and $\tilde{B}$ are two normal octagonal fuzzy numbers, then as in [5] we adhere to the following definitions:

i.If $M_0^{Oct}(\tilde{A}) \le M_0^{Oct}(\tilde{B})$ then $\tilde{A} \preccurlyeq \tilde{B}$
ii. If $M_0^{Oct}(\tilde{A}) = M_0^{Oct}(\tilde{B})$ then $\tilde{A} \approx \tilde{B}$
iii. If $M_0^{Oct}(\tilde{A}) \ge M_0^{Oct}(\tilde{B})$ then $\tilde{A} \succcurlyeq \tilde{B}$

4) Also $\tilde{A} \preccurlyeq \tilde{B}$ and $\tilde{B} \preccurlyeq \tilde{A} \nRightarrow \tilde{A} \approx \tilde{B}$

2 FUZZY LINEAR PROGRAM

The mathematical model

$$\left.\begin{array}{c} \min \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j \\ \text{Subject to constraints} \\ \sum_{j=1}^{n} a_{ij}\tilde{x}_j \preccurlyeq \tilde{b}_i, \ i=1,2,\ldots,m_0 \\ \sum_{j=1}^{n} a_{ij}\tilde{x}_j \succeq \tilde{b}_i, \quad i=m_0+1, m_0+2,\ldots,m \\ \text{and } \tilde{x}_j \succcurlyeq \tilde{0} \text{ for all } j=1,2,\ldots,n \end{array}\right\} \tag{2.1}$$

where $a_{ij} \in \mathbb{R}$, $\tilde{c}_j$, $\tilde{x}_j$, $\tilde{b}_i \in \mathcal{F}(S_O)\, i=1,2,\ldots,m, j=1,2,\ldots,n$ and $\mathcal{F}(S_O)$ the set of all symmetric octagonal fuzzy numbers, is called a fuzzy linear programming problem.

Definition 2.1. *Any $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,\ldots\tilde{x}_n) \in \mathcal{F}^n(S_O)(=\mathcal{F}(S_O)\times\mathcal{F}(S_O)\times\ldots\times\mathcal{F}(S_O)$: (n fold)), where each $\tilde{x}_i \in \mathcal{F}(S_O)$, which satisfies 2.1 is said to be a fuzzy feasible solution to equation 2.1.*

Definition 2.2. *A fuzzy feasible solution is called a fuzzy optimum solution to equation 2.1, denoted $(\tilde{x}_1^o, \tilde{x}_2^o, \ldots, \tilde{x}_n^o) \in Q$ if $\sum_{j=1}^{n} \tilde{c}_j\tilde{x}_j^o \preccurlyeq \sum_{j=1}^{n} \tilde{c}_j\tilde{x}_j \forall$ elements of Q ,where Q is the set of all fuzzy feasible solutions of equation 2.1 .*

Definition 2.3. *If $\tilde{x}_j \approx (-(\alpha_j+h_j), -(\alpha_j+s_j), -(\alpha_j+g_j), -\alpha_j, \alpha_j, (\alpha_j+g_j), (\alpha_j+s_j), (\alpha_j+h_j); k, 1)$ for some $\alpha_j \ge 0$ and $h_j \ge s_j \ge g_j \ge 0$, then $\tilde{x}$ is said to be a fuzzy basic solution, where $\tilde{x}$ solves $A\tilde{x} \approx \tilde{b}$, A being the appropriate Matrix (a_{ij}). If $\tilde{x}_j \not\approx (-(\alpha_j+h_j), -(\alpha_j+s_j), -(\alpha_j+g_j), -\alpha_j, \alpha_j, (\alpha_j+g_j), (\alpha_j+s_j), (\alpha_j+h_j); k, 1)$ for all $\alpha_j \ge 0$ and $h_j \ge s_j \ge g_j \ge 0$, then $\tilde{x}$ has some non-zero components which can be reordered if required, say $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_t, 1 \le t \le n$.Then $A\tilde{x} \approx \tilde{b}$ becomes*

$$\begin{aligned}
& a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_t\tilde{x}_t + a_{t+1}[(-(\alpha_{t+1}+h_{t+1}), -(\alpha_{t+1}+s_{t+1}),\\
& -(\alpha_{t+1}+g_{t+1}), -\alpha_{t+1}, \alpha_{t+1}, (\alpha_{t+1}+g_{t+1}), (\alpha_{t+1}+s_{t+1}),\\
& (\alpha_{t+1}+h_{t+1}); k, 1] + a_{t+2}[(-(\alpha_{t+2}+h_{t+2}), -(\alpha_{t+2}+s_{t+2}),\\
& -(\alpha_{t+2}+g_{t+2}), -\alpha_{t+2}, \alpha_{t+2}, (\alpha_{t+2}+g_{t+2}), (\alpha_{t+2}+s_{t+2}), (\alpha_{t+2}+h_{t+2}); k, 1]\\
& + \cdots + a_n[(-(\alpha_n+h_n), -(\alpha_n+s_n), -(\alpha_n+g_n), -\alpha_n,\\
& \alpha_n, (\alpha_n+g_n), (\alpha_n+s_n), (\alpha_n+h_n); k, 1]\\
\approx\ & \tilde{b}
\end{aligned}$$

And $\tilde{x}$ will become a fuzzy basic solution if the columns $a_1, a_2, \ldots, a_t$ corresponding to these non-zero components $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_t$ are linearly independent.

Remark 2.4. *Given a system of m simultaneous fuzzy linear equations involving symmetric octagonal fuzzy numbers in n unknowns $(m \leq n)$ $A\tilde{x} \approx \tilde{b}$; $\tilde{b} \in \mathcal{F}^m(S_O)$ where A is a $(m \times n)$ real matrix and rank of A is m. Let B be any $(m \times m)$ matrix formed by m linearly independent columns of A. Then the fuzzy basic solution is $\tilde{x}_B = B^{-1}\tilde{b}$, where $\tilde{x}_B \in \mathcal{F}^m(S_O)$. We will eventually prove that, if $\tilde{x}_B$ is a basic solution for the fuzzy linear programming problem equation 2.1, then a solution to the given system is $[\tilde{x}_B, \tilde{0}]$ where $\tilde{0} \in \mathcal{F}^{n-m}(S_O)$*

i.e.$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k, \tilde{0}, \tilde{0}, \ldots \tilde{0})$. In this case we also say that $\tilde{x}_B$ is a fuzzy basic solution.

We shall now give the fuzzy analogues of some important linear programming results.

The standard form of any fuzzy linear programming problem is given by:

$$\left.\begin{aligned}
\min \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j \\
\text{Subject to} \quad A\tilde{x} \approx \tilde{b} \ \text{ and } \tilde{x} \succcurlyeq \tilde{0}
\end{aligned}\right\} \tag{2.2}$$

where $A = (a_{ij})$ is an $(m \times n)$ real matrix, $\tilde{b}$, $\tilde{c}$, $\tilde{x}$ are $(m \times 1)$, $(1 \times n)$, $(n \times 1)$ fuzzy matrices consisting of symmetric octagonal fuzzy numbers.

Definition 2.5. *We say that a fuzzy vector $\tilde{x} \in \mathcal{F}(\mathbb{R})^n$ is a fuzzy feasible solution to the problem given by equation 2.2 if $\tilde{x}$ satisfies the constraints of the problem.*

Definition 2.6. *A fuzzy feasible solution $\tilde{x}_* \in \mathcal{F}(\mathbb{R})^n$ is a fuzzy optimal solution for equation 2.2, if for all fuzzy feasible solution $\tilde{x}$ for equation 2.2, we have $\tilde{c}\tilde{x} \preccurlyeq \tilde{c}\tilde{x}_*$*

Improving a fuzzy basic feasible solution

Let the basis for the columns of A be $B = (b_1, b_2, \ldots, b_m)$. Let a fuzzy basic feasible solution be $\tilde{x}_B \approx B^{-1}\tilde{b}$ and the fuzzy value of $\tilde{z}$ is given by $\tilde{z}_0 \approx \tilde{c}_B * \tilde{x}_B$, where

$\tilde{c}_B = (\tilde{c}_{B_1}, \tilde{c}_{B_2}, \tilde{c}_{B_3}, \ldots \tilde{c}_{B_m})$ is the corresponding cost vector of $\tilde{x}_B$.

Suppose that

$$a_j = \sum_{i=1}^{m} y_{ij} b_i = y_j B$$

and the symmetric octagonal fuzzy number $\tilde{z}_j = \sum_{i=1}^{m} \tilde{c}_{B_i} y_{ij} = \tilde{c}_B y_j$ are known for every column vector a_j in A, which is not in B. Let us now examine the possibility of finding another fuzzy feasible solution which will improve the fuzzy value of $\tilde{z}$, by replacing one of the columns in B by a_j.

Theorem 2.7. *Let $\tilde{x}_B$ be a fuzzy basic feasible solution of equation 2.2 such that $\tilde{x}_B \approx B^{-1}\tilde{b}$. If the condition $(\tilde{z}_j - \tilde{c}_j) \succ \tilde{0}$ hold for any column a_j in A which is not in B and $y_{ij} > 0$ for some i, $i \in \{1, 2, 3, \ldots, m\}$ then we can obtain a new fuzzy basic feasible solution by replacing one of the columns in B by a_j.*

Proof. Let $\tilde{x}_B \approx (\tilde{x}_{B_1}, \tilde{x}_{B_2}, \ldots, \tilde{x}_{B_m})$ be a fuzzy basic feasible solution with t positive components such that $B\tilde{x}_B = \tilde{b}$ or $\tilde{x}_B = B^{-1}\tilde{b}$ where $\tilde{x}_{B_i} \approx ((\alpha_i - h_i), (\alpha_i - s_i), (\alpha_i - g_i), \alpha_i, \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1), \alpha_i \leq \beta_i, h_i \geq s_i \geq g_i \geq 0, 0 < k < 1$and

$M_0^{Oct}(\tilde{x}_{B_i}) > 0$ for $i = 1, 2, \ldots, t$ and $M_0^{Oct}(\tilde{x}_{B_i}) = 0$ for $i = t+1, t+2, \ldots, m$

i.e., $\tilde{x}_{B_i} \succ \tilde{0}$ for $i = 1, 2, \ldots, t$ and

$\tilde{x}_{B_i} \approx (-(\beta_i+h_i), -(\beta_i+s_i), -(\beta_i+g_i), -\beta_i, \beta_i, (\beta_i+g_i), (\beta_i+s_i), (\beta_i+h_i); k, 1)$ for $i = t+1, t+2, \ldots, m; 0 < k < 1$ Then $B\tilde{x}_B \approx \tilde{b}$ is written as

$$\begin{aligned}
&\sum_{i=1}^{t} \tilde{x}_{B_i} b_i + (-(\beta_{t+1} + h_{t+1}), -(\beta_{t+1} + s_{t+1}), -(\beta_{t+1} + g_{t+1}), -\beta_{t+1}, \\
&\beta_{t+1}, (\beta_{t+1} + g_{t+1}), (\beta_{t+1} + s_{t+1}), (\beta_{t+1} + h_{t+1}); k, 1) b_{t+1} + \\
&(-(\beta_{t+2} + h_{t+2}), -(\beta_{t+2} + s_{t+2}), -(\beta_{t+2} + g_{t+2}), -\beta_{t+2}, \beta_{t+2}, (\beta_{t+2} + g_{t+2}), \\
&(\beta_{t+2} + s_{t+2}), (\beta_{t+2} + h_{t+2}); k, 1) b_{t+2} + \cdots \\
&+(-(\beta_m + h_m), -(\beta_m + s_m), -(\beta_m + g_m), -\beta_m, \beta_m, \\
&(\beta_m + g_m), (\beta_m + s_m), (\beta_m + h_m); k, 1) b_m \\
\approx\ &\tilde{b}
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\sum_{i=1}^{t} \tilde{x}_{B_i} b_i + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), -(\beta_i + g_i), -\beta_i, \\
&\beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ &\tilde{b} \qquad (2.3)
\end{aligned}$$

Now any column a_j of A not in B can be written as

$a_j = \sum_{i=1}^{m} y_{ij} b_i = y_{1j} b_1 + y_{2j} b_2 + \cdots + y_{rj} b_r + \cdots + y_{mj} b_m = \mathbf{y}_j B.$

Also if the basis vector b_r for which $y_{rj} \neq 0$ is replaced by a_j of A, then $(b_1, b_2, \ldots, b_{r-1}, a_j, b_{r+1}, \ldots, b_m)$ still forms a basis.

Now for $y_{rj} \neq 0$ and $r \leq t$, we can write

$b_r = \frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}} b_i = \frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i \neq r}}^{t} \frac{y_{ij}}{y_{rj}} b_i = \frac{a_j}{y_{rj}} - \sum_{i=t+1}^{m} \frac{y_{ij}}{y_{rj}} b_i$

Equation2.3 becomes

$$\begin{aligned}
&\sum_{\substack{i=1 \\ i \neq r}}^{t} \tilde{x}_{B_i} b_i + \tilde{x}_{B_r} b_r + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), -(\beta_i + g_i), -\beta_i, \\
&\beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ &\tilde{b}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &\sum_{\substack{i=1 \\ i \neq r}}^{t} \tilde{x}_{B_i} b_i + \frac{\tilde{x}_{B_r}}{y_{rj}} a_j - \frac{\tilde{x}_{B_r}}{y_{rj}} \sum_{\substack{i=1 \\ i \neq r}}^{t} y_{ij} b_i - \frac{\tilde{x}_{B_r}}{y_{rj}} \sum_{i=t+1}^{m} y_{ij} b_i + \sum_{i=t+1}^{m} (-(\beta_i + h_i), -(\beta_i + s_i), \\
&-(\beta_i + g_i), -\beta_i, \beta_i, (\beta_i + g_i), (\beta_i + s_i), (\beta_i + h_i); k, 1) b_i \\
\approx\ &\tilde{b}
\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & \sum_{\substack{i=1\\ i\neq r}}^{t}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j+\sum_{i=t+1}^{m}(-(\beta_i+h_i),-(\beta_i+s_i),-(\beta_i+g_i),-\beta_i,\\ & \beta_i,(\beta_i+g_i),(\beta_i+s_i),(\beta_i+h_i);k,1)-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij})b_i\\ \approx\ & \tilde{b}\end{aligned}$$

Since $\tilde{x}_{B_i}\approx(-(\beta_i+h_i),-(\beta_i+s_i),-(\beta_i+g_i),-\beta_i,\beta_i,(\beta_i+g_i),(\beta_i+s_i),(\beta_i+h_i);k,1)$ for $i=t+1,t+2,\dots,m$, we have

$$\sum_{\substack{i=1\\ i\neq r}}^{t}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j+\sum_{i=t+1}^{m}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i\approx\tilde{b}$$

$$\Rightarrow\sum_{\substack{i=1\\ i\neq r}}^{m}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)b_i+\frac{\tilde{x}_{B_r}}{y_{rj}}a_j\approx\tilde{b}$$

$$\Rightarrow\sum_{\substack{i=1\\ i\neq r}}^{m}\hat{\tilde{x}}_{B_i}b_i+\hat{\tilde{x}}_{B_r}a_j\approx\tilde{b}$$

where $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right),i\neq r$ and $\hat{\tilde{x}}_{B_r}=\frac{\tilde{x}_{B_r}}{y_{rj}}$ which is a improved fuzzy basic solution to $A\tilde{x}\approx\tilde{b}$. We shall now prove that the improved solution is also feasible. That is to prove that $\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)\succcurlyeq\tilde{0},i\neq r$ and $\frac{\tilde{x}_{B_r}}{y_{rj}}\succcurlyeq\tilde{0}$. To this end, select $y_{rj}>0$ such that $\frac{\tilde{x}_{B_r}}{y_{rj}}\approx min\left\{\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}>0\right\}$. Then $\frac{\tilde{x}_{B_r}}{y_{rj}}\preccurlyeq\frac{\tilde{x}_{B_i}}{y_{ij}}$

$$\begin{aligned}\Rightarrow\ & \left(\frac{\alpha_r-h_r}{y_{rj}},\frac{\alpha_r-s_r}{y_{rj}},\frac{\alpha_r-g_r}{y_{rj}},\frac{\alpha_r}{y_{rj}},\frac{\beta_r}{y_{rj}},\frac{\beta_r+g_r}{y_{rj}},\frac{\beta_r+s_r}{y_{rj}},\frac{\beta_r+h_r}{y_{rj}}\right)\\ \preccurlyeq\ & \left(\frac{\alpha_i-h_i}{y_{ij}},\frac{\alpha_i-s_i}{y_{ij}},\frac{\alpha_i-g_i}{y_{ij}},\frac{\alpha_i}{y_{ij}},\frac{\beta_i}{y_{ij}},\frac{\beta_i+g_i}{y_{ij}},\frac{\beta_i+s_i}{y_{ij}},\frac{\beta_i+h_i}{y_{ij}}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & \left(\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{h_r}{y_{rj}}+\frac{h_i}{y_{ij}}\right),\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{s_r}{y_{rj}}+\frac{s_i}{y_{ij}}\right),\frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}}-\left(\frac{g_r}{y_{rj}}+\frac{g_i}{y_{ij}}\right),\right.\\ & \frac{\alpha_i}{y_{ij}}-\frac{\beta_r}{y_{rj}},\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}},\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{g_r}{y_{rj}}+\frac{g_i}{y_{ij}}\right),\\ & \left.\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{s_r}{y_{rj}}+\frac{s_i}{y_{ij}}\right),\frac{\beta_i}{y_{ij}}-\frac{\alpha_r}{y_{rj}}+\left(\frac{h_r}{y_{rj}}+\frac{h_i}{y_{ij}}\right)\right)\\ \succcurlyeq\ & \tilde{0}\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & \left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}},\frac{\alpha_i-\beta_r}{y_{ij}},\right.\\ & \left.\frac{\beta_i-\alpha_r}{y_{ij}},\frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)\\ \succcurlyeq\ & \tilde{0}\end{aligned}$$

$$\Rightarrow$$

$$M_0^{Oct}\left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}},\frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}},\frac{\alpha_i-\beta_r}{y_{ij}},\right.$$
$$\left.\frac{\beta_i-\alpha_r}{y_{ij}},\frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}},\frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)$$
$$\geq \tilde{0}$$

$$\Rightarrow \frac{1}{4}\left(k\left(\frac{\alpha_i-\beta_r-(h_r+h_i)}{y_{ij}}+\frac{\alpha_i-\beta_r-(s_r+s_i)}{y_{ij}}+\frac{\beta_i-\alpha_r+(s_r+s_i)}{y_{ij}}+\frac{\beta_i-\alpha_r+(h_r+h_i)}{y_{ij}}\right)\right.$$
$$\left.+\ (1-k)\left(\frac{\alpha_i-\beta_r-(g_r+g_i)}{y_{ij}}+\frac{\alpha_i-\beta_r}{y_{ij}}+\frac{\beta_i-\alpha_r}{y_{ij}}+\frac{\beta_i-\alpha_r+(g_r+g_i)}{y_{ij}}\right)\right)$$
$$\geq 0$$

$$\Rightarrow \frac{1}{4}\left(\frac{\alpha_i-\beta_r}{y_{ij}}+\frac{\beta_i-\alpha_r}{y_{ij}}\right)\geq 0$$

$$\Rightarrow \left(\frac{\alpha_i+\beta_i}{y_{ij}}\right)-\left(\frac{\alpha_r+\beta_r}{y_{ij}}\right)\geq 0 \Rightarrow \left(\frac{\tilde{x}_{B_i}}{y_{ij}}-\frac{\tilde{x}_{B_r}}{y_{rj}}\right)\succcurlyeq \tilde{0}$$

and hence the improved solution is a fuzzy basic feasible solution and the theorem is proved. □

Remark 2.8. *The new basis matrix obtained after replacing the basis vectors is* $\hat{B}=(\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_m)$*, where* $\hat{b}_i=b_i$ *for* $i\neq r$ *and* $\hat{b}_r=a_j$*. The new fuzzy basic feasible solution is* $\hat{\tilde{x}}_B$*, where* $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right), i\neq r$ *and* $\hat{\tilde{x}}_{B_r}=\frac{\tilde{x}_{B_r}}{y_{rj}}$ *are the basic variables.*

Theorem 2.9. *If* $\tilde{x}_B$ *and* $\hat{\tilde{x}}_B$ *are the fuzzy basic feasible solutions of 2.2 having their objective values* $\tilde{z}_0\approx\tilde{c}_B*\tilde{x}_B$ *and* $\hat{\tilde{z}}\approx\hat{\tilde{c}}_B*\hat{\tilde{x}}_B$ *respectively and if* $\hat{\tilde{x}}_B$ *was the value obtained after admitting* a_j *in the basis, it being a non basic column vector and also for which* $(\tilde{z}_j-\tilde{c}_j)\succ\tilde{0}$ *and* $y_{ij}>0$ *for some* $i, i\in\{1,2,3,\ldots,m\}$*, then* $\hat{\tilde{z}}\preccurlyeq\tilde{z}_0$*.*

Proof. Given $\tilde{x}_B$ be a fuzzy basic feasible solution and $\tilde{z}_0\approx\tilde{c}_B\tilde{x}_B$. Let b_r be the column vector removed from the basis in place of which a_j is introduced. Also given that $\hat{\tilde{x}}_B$ is the new fuzzy basic feasible solution, then $\hat{\tilde{x}}_{B_i}\approx\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right), i\neq r$ and $\hat{\tilde{x}}_{B_r}=\frac{\tilde{x}_{B_r}}{y_{rj}}$.

Since $\hat{\tilde{c}}_{B_i}\approx\tilde{c}_{B_i}, i\neq r$ and $\hat{\tilde{c}}_{B_r}=\tilde{c}_j$, the modified fuzzy value of the objective function is

$$\hat{\tilde{z}}\approx\hat{\tilde{c}}_B\hat{\tilde{x}}_B\approx\sum_{i=1}^{m}\hat{\tilde{c}}_{B_i}\hat{\tilde{x}}_{B_i}\approx\sum_{\substack{i=1\\i\neq r}}^{m}\hat{\tilde{c}}_{B_i}\hat{\tilde{x}}_{B_i}+\hat{\tilde{c}}_{B_r}\hat{\tilde{x}}_{B_r}$$
$$\approx\sum_{\substack{i=1\\i\neq r}}^{m}\tilde{c}_{B_i}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\sum_{i=1}^{m}\tilde{c}_{B_i}\left(\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}y_{ij}\right)+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\sum_{i=1}^{m}\tilde{c}_{B_i}\tilde{x}_{B_i}-\frac{\tilde{x}_{B_r}}{y_{rj}}\sum_{i=1}^{m}\tilde{c}_{B_i}y_{ij}+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$
$$\approx\tilde{z}_0-\frac{\tilde{x}_{B_r}}{y_{rj}}\tilde{z}_j+\tilde{c}_j\frac{\tilde{x}_{B_r}}{y_{rj}}$$

$$\approx \tilde{z}_0 - \frac{\tilde{x}_{B_r}}{y_{rj}}(\tilde{z}_j - \tilde{c}_j) \tag{2.4}$$

Since $y_{rj} > 0$, $(\tilde{z}_j - \tilde{c}_j) \succ \tilde{0}$ and $\frac{\tilde{x}_{B_r}}{y_{rj}} \succcurlyeq \tilde{0}$, hence $\frac{\tilde{x}_{B_r}}{y_{rj}}(\tilde{z}_j - \tilde{c}_j) \succeq \tilde{0}$.

So Equation 2.4 implies $\hat{\tilde{z}} \preccurlyeq \tilde{z}_0$. Hence the new fuzzy basic feasible solution gives the improved fuzzy value of the objective function. □

Condition of optimality

Similar to classical linear programming problem, we can prove that the process of inserting and removing vectors from the basis matrix will lead to the following situations

i) unbounded solution

ii) infeasible solution

iii) optimal solution. In which case, $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$.

Hence we have the following theorem whose proof is immediate.

Theorem 2.10. *If $\tilde{x}_B = B^{-1}\tilde{b}$ is a fuzzy basic feasible solution of 2.2 and if $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$ for every column a_j of A, then $\tilde{x}_B$ is a fuzzy optimal solution to 2.2.*

Remark 2.11. *Here we solve a fuzzy linear programming problem whose optimal function is to be minimised and whose variables are non-negative. Hence we arrive at two situations and they are i) we obtain the optimal solution or ii)we obtain an infeasible solution in sense the optimality will not be reached inspite of repeated iterations.*

Remark 2.12. *Theorem2.9 and Theorem2.10 gives sufficient condition for existence of optimal solution. It is to be seen whether this sufficient condition ensures convergence of the iteration problem. Condition for convergence of iteration process needs to be studied separately.*

Remark 2.13. *If we consider the maximisation problem given by*

$$\begin{aligned} \max \tilde{z} &\approx \sum_{j=1}^{n} \tilde{c}_j * \tilde{x}_j \\ \textit{Subject to } A\tilde{x} &\approx \tilde{b} \;\; \textit{and} \;\; \tilde{x} \succcurlyeq \tilde{\mathbf{0}} \end{aligned}$$

then this problem can be converted into a minimisation problem given by

$$\begin{aligned} max\tilde{z} &\approx -min(-\tilde{z}) \approx \sum_{j=1}^{n} (-\tilde{c}_j) * \tilde{x}_j \\ \text{Subject to } A\tilde{x} &\approx \tilde{b} \;\; \text{and} \;\; \tilde{x} \succcurlyeq \tilde{\mathbf{0}} \end{aligned}$$

and solved as above.

3 Numerical Example

Food A contains 20 units of Proteins and 40 units of minerals per gram. Food B contains 30 units each of Proteins and minerals. The daily minimum human requirements of Protein and Mineral are 900 units and 1200 units respectively. How many grams of each type of food should be consumed so as to minimize the cost, if food A costs $Rs.6$ per gram and food B costs $Rs.8$ per gram.

Note that the daily minimum human requirements of proteins and minerals may vary from individual to individual. Also the cost of food may vary depending on the market condition. Due to these uncertain variations the problem is modelled as a fuzzy linear programming problem and symmetrical octagonal fuzzy numbers are used to describe these uncertain values.

Cost of Food A 6 is modeled as $(2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)$and the same is done for the other parameters also . Hence the mathmatical formulation of the above problem is given by fuzzy linear programming problem as

$\min \tilde{z} \approx (2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)\tilde{x}_1 + (3, 4, 5, 7, 9, 11, 12, 13; 0.3, 1)\tilde{x}_2$
Subject to $20\tilde{x}_1 + 30\tilde{x}_2 \succcurlyeq (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$
$40\tilde{x}_1 + 30\tilde{x}_3 \succcurlyeq (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$
$\tilde{x}_1 \succcurlyeq \tilde{0}, \tilde{x}_2 \succcurlyeq \tilde{0}.$

Hence the fuzzy linear programming problem in standard form is
$min\tilde{z} \approx (2, 3, 4, 5, 7, 8, 9, 10; 0.3, 1)\tilde{x}_1 + (3, 4, 5, 7, 9, 11, 12, 13; 0.3, 1)\tilde{x}_2$
Subject to $20\tilde{x}_1 + 30\tilde{x}_2 - \tilde{S}_1 + \tilde{A}_1 \approx (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$
$40\tilde{x}_1 + 30\tilde{x}_3 - \tilde{S}_2 + \tilde{A}_2 \approx (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$
$\tilde{x}_1, \tilde{x}_2, , \tilde{S}_1, \tilde{S}_2, \tilde{A}_1, \tilde{A}_2 \succcurlyeq \tilde{0}$

where $\tilde{S}_1, \tilde{S}_2$, are the surplus fuzzy variables and $\tilde{A}_1, \tilde{A}_2$, are artificial variables. That is $min\ \tilde{z} \approx \tilde{c}\tilde{x}$ subject to $A\tilde{x} \approx \tilde{b}$ and $\tilde{x} \succcurlyeq \tilde{0}$, where

$$A = \begin{pmatrix} \overset{a_1}{20} & \overset{a_2}{30} & \overset{a_3}{-1} & \overset{a_4}{0} & \overset{a_5}{1} & \overset{a_6}{0} \\ 40 & 30 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\tilde{b} \approx \begin{pmatrix} (885,886,888,890,910,912,914,915;0.3,1) \\ (1190,1191,1193,1195,1205,1207,1209,1210;0.3,1) \end{pmatrix}, \tilde{x} \approx (\tilde{x}_1\ \ \tilde{x}_2\ \ \tilde{S}_1\ \ \tilde{S}_2\ \ \tilde{A}_1\ \ \tilde{A}_2)$$

And $\tilde{c} \approx ((885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1),$
$(1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1), \tilde{0}, \tilde{0}, \tilde{M}, \tilde{M})$

Initial Iteration: The initial fuzzy basic feasible solution is given by $\tilde{x}_B \approx B^{-1}\tilde{b}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{x} = \begin{matrix} \tilde{A}_1 \\ \tilde{A}_2 \end{matrix}, \tilde{b} \approx \begin{pmatrix} (885,886,888,890,910,912,914,915;0.3,1) \\ (1190,1191,1193,1195,1205,1207,1210;0.3,1) \end{pmatrix} \text{ and}$$

$\tilde{x}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{x}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{A}_1 \approx (885, 886, 888, 890, 910, 912, 914, 915; 0.3, 1)$,
$\tilde{A}_2 \approx (1190, 1191, 1193, 1195, 1205, 1207, 1209, 1210; 0.3, 1)$
and the fuzzy objective value is
$\tilde{z} \approx (2075M, 2077M, 2081M, 2085M, 2115M, 2119M, 2123M, 2125M; 0.3, 1).$
Now $(\tilde{z}_1 - \tilde{c}_1) \succ \tilde{0}$ is one among the highest positive value among all $\tilde{z}_j - \tilde{c}_j$

First Iteration: By Theorem 2.7 and Theorem 2.9, we get a new fuzzy basic feasible solution
$\tilde{x}_1 \approx \left(\frac{1190}{40}, \frac{1191}{40}, \frac{1193}{40}, \frac{1195}{40}, \frac{1205}{40}, \frac{1207}{40}, \frac{1209}{40}, \frac{1210}{40}; 0.3, 1\right)$, $\tilde{x}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$, $\tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$
$\tilde{A}_1 \approx \left(\frac{560}{2}, \frac{563}{2}, \frac{569}{2}, \frac{575}{2}, \frac{625}{2}, \frac{631}{2}, \frac{637}{2}, \frac{640}{2}; 0.3, 1\right)$, $\tilde{A}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$
with the improved fuzzy objective value
$$\tilde{z} \approx \left(\frac{11200M+2320}{40}, \frac{11260M+3532}{40}, \frac{11380M+4751}{40}, \frac{11500M+5970}{40}, \frac{12500M+8430}{40}, \frac{12620M+9649}{40}, \frac{12740M+10868}{40}, \frac{12800M+12080}{40}; 0.3, 1\right).$$
Here $(\tilde{z}_2 - \tilde{c}_2) \succ \tilde{0}$ is the highest positive value among all $\tilde{z}_j - \tilde{c}_j$

Second Iteration: Proceeding in a similar way, we get a new fuzzy basic feasible solution
$\tilde{x}_1 \approx \left(\frac{550}{40}, \frac{554}{40}, \frac{562}{40}, \frac{570}{40}, \frac{630}{40}, \frac{638}{40}, \frac{646}{40}, \frac{650}{40}; 0.3, 1\right)$,
$\tilde{x}_2 \approx \left(\frac{560}{30}, \frac{563}{30}, \frac{569}{30}, \frac{575}{30}, \frac{625}{30}, \frac{631}{30}, \frac{637}{30}, \frac{640}{30}; 0.3, 1\right)$,
$\tilde{S}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1), \tilde{S}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$,
$\tilde{A}_1 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1), \tilde{A}_2 \approx (0, 0, 0, 0, 0, 0, 0, 0; 0.3, 1)$
with the improved fuzzy objective value,
$\tilde{z} \approx \left(\frac{7830}{120}, \frac{12412}{120}, \frac{17186}{120}, \frac{24460}{120}, \frac{35540}{120}, \frac{42814}{120}, \frac{47588}{120}, \frac{52170}{120}; 0.3, 1\right)$.

Here $(\tilde{z}_j - \tilde{c}_j) \preccurlyeq \tilde{0}$ for all j. Here by Theorem 2.10, the feasible solution obtained now is a fuzzy optimal solution.

Remark 3.1. *The parameter k is involved in calculating the ratio between the minimum requirements and the selected y_{ij} column elements.*

Remark 3.2. *The measure M_0^{oct} considered (as in Definition 1.10) is used in comparing the ratios mentioned in Remark.3.1*

Remark 3.3. *The choice of trapezoidal, hexagonal or octagonal fuzzy numbers for minimum feasible solution range seems to be dependent on the parameter k. Details of these investigations will be published after completion.*

Remark 3.4. *In [1] the authors have proved the following theorem, For a fixed partition P_m, the set $\mathcal{F}_{c,m}(\mathbb{R})$ is isomorphic to the convex closed cone $C = \{z \in \mathbb{R}^{2(m+1)} : Bz \geqslant 0\}$ with*

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & ... & 0 \\ 0 & -1 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & 1 & ... & 0 \\ . & & & & . & \\ . & & & & . & \\ . & & & & . & \\ 0 & 0 & ... & 0 & -1 & 1 \end{bmatrix}_{(2m+1)x(2m+2)}$$

where $\mathcal{F}_{c,m}(\mathbb{R})$ is the space of real fuzzy subsets of $\mathbb{R}$ with level sets belonging to the space of compact convex subsets of $\mathbb{R}$ endowed with suitable Hausdorff metric.This implies that trapezoidal fuzzy numbers correspond to the case $m = 4$ and octagonal fuzzy numbers correspond to the case $m = 8$ in the mentioned theorem. As these cones are distinct it is clear that these numbers also have distinct properties. Also trapezoidal fuzzy numbers can be viewed as special case of octagonal fuzzy numbers.

4 Conclusion In this paper FLP is solved without converting it to crisp linear programming problem. If

this problem is solved using several steps with trapezoidal fuzzy numbers, then there is a sizeable difference in the spread of the solution when compared to the solution obtained for octagonal fuzzy number problem and this diference is dependent on the choice of k. The concept of octagonal fuzzy numbers enables using more data relating to the problem and obtaining similarly more data about the solution. *Also the $*$ multiplication defined in this paper is more natural.* Approximation of any fuzzy number by trapezoidal fuzzy numbers was considered by Ban et.al. and symmetric trapezoidal approximation is cited as future work [2].We may have to consider whether the data given in our problem can be approximated to trapezoidal fuzzy numbers given in [2] which would give a solution, which is nearer to the solution given using octagonal fuzzy numbers as such. Octagonal Approximations in a fuzzy environment may be considered for future work.

Acknowledgement The authors wish to thank Dr. M.S. Rangachari and Dr. P.V. Subrahmanyam for their useful discussions and reviewers for their comments which were useful in shaping the paper.

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Reliability and Profit Analysis of a Single-Unit System with Inspection under Warranty

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Abstract. The purpose of the present paper is to carry out reliability and profit analysis of a single-unit system considering the concept of inspection under warranty. Within warranty, failures are rectified by the manufacturer at no cost to the users provided warranty does not apply to product failure due to user-induced damage such as cracked screen, accident, misuse, physical damage, damage due to liquid and unauthorized modifications, etc. The cost to rectify failures beyond the warranty is borne by the users. After failure, unit goes under inspection within warranty. There is single repairman, which is always available with the system to do repair, inspection and replacement of the unit. Repairman inspects the failed unit to see the feasibility of its repair or replacement. If repair of the unit is not feasible, it is replaced by new one.The time to failure of the system follows negative exponential distribution while inspection and repair time distributions are taken as arbitrary. By using supplementary variable technique, various measures of system performance such as reliability, mean time to system failure (MTSF), availability of the system and profit function have been determined. The numerical results for reliability and profit function are also obtained in the form of tables for particular values of various parameters and repair cost.

1 Introduction: Warranty is a key promotional tool for the seller since it has become an essentially compititive stretegy employed by sellers to boost their market share, profitability and corporate image. Item sold under warranty often require post sale support in terms of repair or replacement. Several authors including Kadyan et al. [3], Kaur et al. [4], Kharoufeh et al. [5] and Xiaoning Jin et al. [6] studied single unit systems without considering any warranty of the systems. But, in the modern age, most products are sold with

2010 *Mathematics Subject Classification.* Primary 90B25 and Secondary 60K10.
Key words and phrases. Reliability, Warranty, Inspection and Profit Analysis.

a warranty to gain some advantages in the highly competitive markets. Further, warranty plays an important role to assure reliability of a sold product and may increase sales. Also, repair of the failed unit is not always feasible due to its excessive use and increased cost of maintenance. In such cases, the failed unit may be replaced by new unit after getting necessary inspection in order to avoid unnecessary expenses on repair. Yeh et al. [7] have studied an inspection model with discount factor. But single-unit systems with warranty and inspection have not appeared in the literature so far.

Keeping in view of the above facts, here we studied a single unit reliability model with the concept of warranty and inspection. Within warranty, failures are rectified by the manufacturer at no cost to the users provided failures are not due to the negligence of users. After failure, unit goes under inspection within warranty. There is single repairman, which is always available with the system to do repair, inspection and replacement of the unit. Repairman inspects the failed unit to see the feasibility of repair. If repair of the unit is not feasible, it is replaced by new one.The time to failure of the system follows negative exponential distribution while inspection and repair time distributions are taken as arbitrary. The supplementary variable technique is adopted to derive the expressions for some economic measures such as reliability, MTSF, availability and profit function. The numerical results for reliability and profit function are also obtained in the form of tables for particular values of various parameters and repair cost.

2 Assumptions:

1. The system has a single unit.
2. There is single repairman, which is always available with the system to do repair, inspection and replacement of the unit.
3. The cost of repair of the failed unit during warranty is borne by the manufacturer provided failures are not due to the neglegence of users.
4. Under warranty, unit goes for inspection after failure.
5. Repairman inspects the failed unit to see the feasibility of repair or replacement.
6. The unit works as new after repair.
7. The distribution of failure time is taken as negative exponential while the inspection and repair time are considered as arbitrary.

3 State-Specification:

s_0/s_1 The unit is operative under warranty/ beyond warranty.

s_3/s_4 The unit is in failed state under warranty/ beyond warranty.

s_2 The failed unit is under inspection.

4 Notations:

λ/λ_1 Constant failure rate of the unit within warranty/beyond warranty.

α Constant rate of completion of warranty.

p/q Probability that repair is feasible/not feasible.

$\mu(x), s(x)$ Repair rate of the unit and probability density function, for the elapsed repair time 'x' in warranty.

$\mu_1(x), s_1(x)$ Repair rate of the unit and probability density function, for the elapsed repair time 'x' beyond warranty.

$h(y), s_2(y)$ Inspection rate of the failed unit and probability density function, for the elapsed inspection time 'y'.

$p_0(t)/p_1(t)$ The Probability that at time t the system is in good state in warranty/beyond warranty.

$p_3(x,t)\Delta$ The Probability that at time t the system is in failed state in warranty, the elapsed repair time lies in the interval $[x, x+\Delta)$.

$p_4(x,t)\Delta$ The Probability that at time t the system is in failed state beyond warranty, the elapsed repair time lies in the interval $[x, x+\Delta)$.

$p_2(y,t)\Delta$ The Probability that at time t the failed unit is under inspection, the elapsed inspection time lies in the interval $[y, y+\Delta)$.

$p(s)$ Laplace transform of function $p(t)$

$s(x) = \mu(x)e^{\left[-\int_0^x \mu(x)dx\right]}$

$s_1(x) = \mu_1(x)e^{\left[-\int_0^x \mu_1(x)dx\right]}$

$s_2(y) = h(y)e^{\left[-\int_0^y h(y)dy\right]}$

5 Formulation of Mathematical Model: Using the probabilistic arguments and limiting transitions, we have the following difference-differential equations (Cox D.R. [2]):

(1) $$\left[\frac{d}{dt}+\lambda+\alpha\right]p_0(t)=\int_0^\infty \mu(x)p_3(x,t)dx+\int_0^\infty qh(y)p_2(y,t)dy$$

(2) $$\left[\frac{d}{dt}+\lambda_1\right]p_1(t)=\alpha p_0(t)+\int_0^\infty \mu_1(x)p_4(x,t)dx$$

(3) $$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\ h(y)\right]p_2(y,t)=0$$

(4) $$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu(x)\right]p_3(x,t)=0$$

(5) $$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)\right]p_4(x,t)=0$$

Boundary conditions

(6) $$p_2(0,t)=\lambda p_0(t)$$

(7) $$p_3(0,t)=\int_0^\infty ph(y)p_2(y,t)dy$$

(8) $$p_4(0,t)=\lambda_1 p_1(t)$$

Initial conditions

$$p_i(0)=1;\quad \text{when}\quad i=0$$

(9) $$p_i(0)=0;\quad \text{when}\quad i\neq 0$$

6 Model analysis: The state transtion diagrame of the model is:

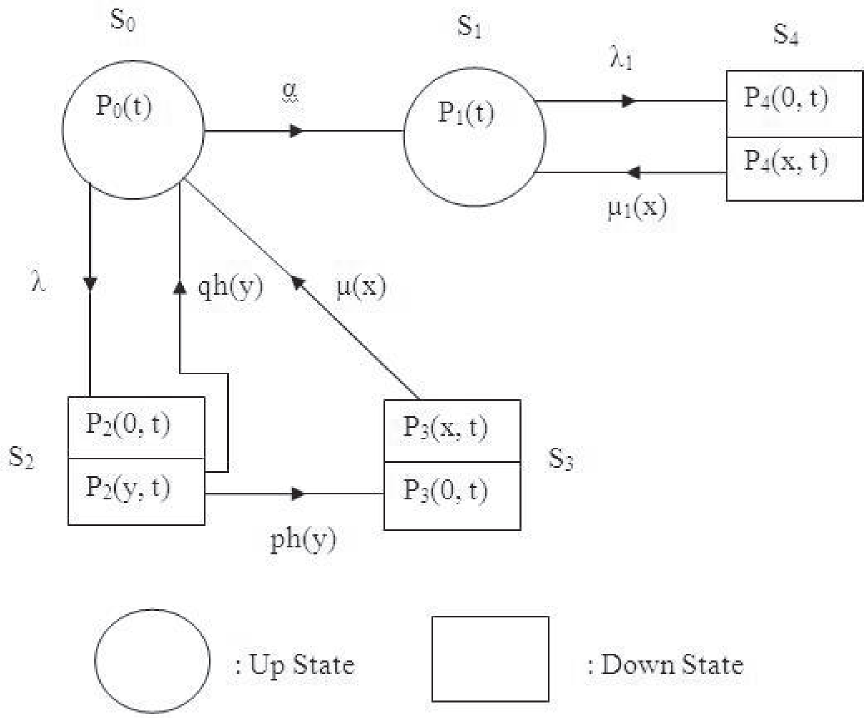

Figure 1

6.1 Solution of the equations: Taking Laplace transforms of equations (1)-(8) and using (9) we obtain

(10) $$[s+\lambda+\alpha]\,p_0(s) = 1 + \int_0^\infty \mu(x)p_3(x,s)dx + \int_0^\infty qh(y)p_2(y,s)dy$$

(11) $$[s+\lambda_1]\,p_1(s) = \alpha p_0(s) + \int_0^\infty \mu_1(x)p_4(x,s)dx$$

(12) $$\left[\frac{\partial}{\partial y} + s + \ h(y)\right] p_2(y,s) = 0$$

(13) $$\left[\frac{\partial}{\partial x} + s + \mu(x)\right] p_3(x,s) = 0$$

(14) $$\left[\frac{\partial}{\partial x} + s + \mu_1(x)\right] p_4(x,s) = 0$$

$$p_2(0,s) = \lambda p_0(s) \tag{15}$$

$$p_3(0,s) = \int_0^\infty p h(y) p_2(y,s) dy \tag{16}$$

$$p_4(0,s) = \lambda_1 p_1(s) \tag{17}$$

Integrating equations (12), (13) and (14), we get

$$p_2(y,s) = p_2(0,s) e^{\left[-sy-\int_0^y h(y)dy\right]} \tag{18}$$

$$p_3(x,s) = p_3(0,s) e^{\left[-sx-\int_0^x \mu(x)dx\right]} \tag{19}$$

and

$$p_4(x,s) = p_4(0,s) e^{\left[-sx-\int_0^x \mu_1(x)dx\right]} \tag{20}$$

Using equations (15) and (18), equation (16) yield

$$p_3(0,s) = \int_0^\infty p h(y) p_2(0,s) e^{\left[-sy-\int_0^y h(y)dy\right]}$$

$$p_3(0,s) = p\lambda p_0(s) S_2(s) \tag{21}$$

Using equation (21), equation (19) yields

$$p_3(x,s) = p\lambda p_0(s) S_2(s) e^{\left[-sx-\int_0^x \mu(x)dx\right]} \tag{22}$$

Using equations (15), (18) and (22), equation (10) yields

$$\begin{aligned}[s+\lambda+\alpha]p_0(s) &= 1 + p_3(0,s)\int_0^\infty \mu(x) e^{\left[-sx-\int_0^x \mu(x)dx\right]}dx \\ &\quad + p_2(0,s) q \int_0^\infty h(y) e^{\left[-sy-\int_0^y h(y)dy\right]}dy \\ &= 1 + p\lambda p_0(s) S(s) S_2(s) + q\lambda S_2(s) p_0(s)\end{aligned} \tag{23}$$

$$p_0(s) = \frac{1}{T(s)} \tag{24}$$

where

$$T(s) = s + \alpha + \lambda - \lambda p S(s) S_2(s) - q\lambda S_2(s) \tag{25}$$

Using equations (17) and (20), equation (11) yields

$$\begin{aligned}[s+\lambda]p_1(s) &= \alpha p_0(s) + p_4(0,s)\int_0^{\infty} \mu_1(x) e^{\left[-sx-\int_0^x \mu_1(x)dx\right]}dx \\ &= \alpha p_0(s) + \lambda p_1(s) S_1(s)\end{aligned} \tag{26}$$

$$p_1(s) = \frac{A(s)}{T(s)} \tag{27}$$

where

$$A(s) = \frac{\alpha}{(s + \lambda_1 - \lambda_1 S_1(s))} \tag{28}$$

Now, the Laplace transform of the probability that the system is in the failed state is given by

$$p_2(s) = \int_0^{\infty} p_2(s,y)dy = \lambda p_0(s)\frac{(1 - S_2(s))}{s} = \frac{\lambda B(s)}{T(s)} \tag{29}$$

where

$$B(s) = \frac{1 - S_2(s)}{s} \tag{30}$$

Similarly

$$p_3(s) = \int_0^{\infty} p_3(s,x)dx = \lambda p S_2(s) p_0(s)\frac{(1 - S(s))}{s} = \frac{\lambda p S_2(s) C(s)}{T(s)} \tag{31}$$

where

$$C(s) = \frac{1 - S(s)}{s} \tag{32}$$

similarly

$$p_4(s) = \int_0^{\infty} p_4(s,x)dx = \lambda_1 p_1(s)\frac{(1 - S_1(s))}{s} = \frac{\lambda_1 A(s) D(s)}{T(s)} \tag{33}$$

where

$$D(s) = \frac{1 - S_1(s)}{s} \tag{34}$$

It is worth noticing that

$$p_0(s) + p_1(s) + p_2(s) + p_3(s) + p_4(s) = \frac{1}{s} \tag{35}$$

6.2 Evaluation of Laplace transforms of up and down state probabilities: Let $Av(t)$ is the probability that the system is operating satisfactorily at time t. The Laplace transforms of $Av(t)$ or probabilities that the system is in up $P_{up}(t)$ (i.e. good) and down $P_{down}(t)$ (i.e. failed) state at time "t" are as follows

$$Av(s) \text{ or } P_{up}(s) = p_0(s) + p_1(s)$$

$$Av(s) = \frac{1 + A(s)}{T(s)} \tag{36}$$

$$P_{down}(s) = p_2(s) + p_3(s) + p_4(s)$$

$$P_{down}(s) = \frac{\lambda B(s) + \lambda p C(s) S_2(s) + \lambda_1 A(s) D(s)}{T(s)} \tag{37}$$

6.3 Steady-State Behaviour of the System: In the long run as t tends to infinity, the steady state behaviour of the system can be obtained by using Abel′s Lemma in Laplace transforms, viz.
$\lim_{s\to 0} s[Av(s)] = \lim_{n\to\infty}[Av(t)] = Av(\text{say})$, Provided the limit on the right hand side exists, the following time independent probabilities have been obtained.

$$Av = \frac{1}{1 - \lambda_1 S_1'(0)} \tag{38}$$

$$P_{down} = \frac{-\lambda_1 S_1'(0)}{1 - \lambda_1 S_1'(0)} \tag{39}$$

6.4 Reliability of the system: Let $R(t)$ is the probability that the system performs well in an interval $(0, t]$. Therefore in order to obtain $R(t)$, the differential-difference equations for reliability are:

$$\left[\frac{d}{dt} + \lambda + \alpha\right] p_0(t) = 0 \tag{40}$$

$$\left[\frac{d}{dt} + \lambda_1\right] p_1(t) = \alpha p_0(t) \tag{41}$$

Theorem 1. The reliability of the system is given by

$$R(t) = e^{-(\lambda+\alpha t)}\left[\frac{(\lambda - \lambda_1)}{(\lambda - \lambda_1 + \alpha)}\right] + e^{-(\lambda_1 t)}\left[\frac{\alpha}{(\lambda - \lambda_1 + \alpha)}\right]$$

Proof. Taking Laplace transforms of (40), (41) and using (9), we get

(42) $$[s+\lambda+\alpha]p_0(s)=1$$

(43) $$[s+\lambda_1]p_1(s)=\alpha p_0(s)$$

The solution can be written as

(44) $$p_0(s)=\frac{1}{(s+\lambda+\alpha)}$$

(45) $$p_1(s)=\frac{\alpha}{(s+\lambda+\alpha)(s+\lambda_1)}$$

$$R(s)=p_0(s)+p_1(s)=\frac{1}{(s+\lambda+\alpha)}+\frac{\alpha}{(s+\lambda+\alpha)(s+\lambda_1)}$$

Taking inverse Laplace transform, we get

(46) $$R(t)=e^{-(\lambda+\alpha t)}\left[\frac{(\lambda-\lambda_1)}{(\lambda-\lambda_1+\alpha)}\right]+e^{-(\lambda_1 t)}\left[\frac{\alpha}{(\lambda-\lambda_1+\alpha)}\right]$$

Corollary 1. The mean time to system failure (MTSF) is:

$$MTSF=\left[\frac{(\lambda-\lambda_1)}{(\lambda-\lambda_1+\alpha)(\lambda+\alpha)}\right]+\left[\frac{\alpha}{(\lambda-\lambda_1+\alpha)\lambda_1}\right]$$

Proof. As MTSF is the expected time for which the system is in operation before it completely fails.

$$\therefore \quad MTSF=\int_0^\infty R(t)dt$$

$$MTSF=\int_0^\infty\left\{e^{-(\lambda+\alpha)t}\left(\frac{(\lambda-\lambda_1)}{(\lambda-\lambda_1+\alpha)}\right)+e^{-(\lambda_1)t}\left(\frac{\alpha}{(\lambda-\lambda_1+\alpha)}\right)\right\}dt$$

(47) $$MTSF=\left(\frac{(\lambda-\lambda_1)}{(\lambda-\lambda_1+\alpha)(\lambda+\alpha)}\right)+\left(\frac{\alpha}{(\lambda-\lambda_1+\alpha)\lambda_1}\right)$$

7 Particular cases:

7.1 Availability of the system: When repair and inspection time follows exponential distribution i.e. setting

$$S(s) = \frac{\mu}{(s+\mu)}, S_1(s) = \frac{\mu_1}{(s+\mu_1)} \text{ and } S_2(s) = \frac{h}{(s+h)}$$

where μ and μ_1 are constant repair rates and h is constant inspection rate. Putting these values in equations (24)-(28), we get

$$p_0(s) = \frac{1}{I(s)} \tag{48}$$

where

(49)

$$I(s) = \frac{s^3 + s^2(\lambda+\alpha+\mu+h) + s(\mu h + \lambda h + \lambda\mu + \alpha h + \alpha\mu - q\lambda h) + \alpha\mu h}{(s+\mu)(s+h)}$$

$$p_1(s) = \frac{E(s)}{I(s)} \tag{50}$$

where

$$E(s) = \left[\frac{\alpha(s+\mu_1)}{s(s+\lambda_1+\mu_1)}\right] \tag{51}$$

$$Av(s) \text{ or } P_{up}(s) = p_0(s) + p_1(s)$$

$$Av(s) = \left[\frac{(s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0)}{s(s+\lambda_1+\mu_1)(s^3 + a_2 s^2 + a_1 s + a_0)}\right] \tag{52}$$

where $b_3 = (\lambda_1+\mu+\alpha+\mu_1+h), b_2 = (\lambda_1\mu+\mu\alpha+\alpha\mu_1+\mu_1 h+\mu h+\mu\mu_1+\lambda_1 h+h\alpha)$, $b_1 = (\mu\mu_1 h + \lambda_1\mu h + \alpha\mu\mu_1 + h\alpha\mu + h\alpha\mu_1)$ and $b_0 = (\alpha\mu\mu_1 h)$ and $a_2 = (\lambda+\mu+\alpha+h), a_1 = (\lambda\mu+\mu\alpha+\mu h+\lambda h+h\alpha-qh\lambda)$ and $a_0 = \mu\alpha h$

Taking inverse Laplace transforms of equation (52), we get

(53)

$$
\begin{aligned}
Av(t) &= \frac{-b_0}{(\lambda_1+\mu_1)z_1z_2z_3} \\
&+ \left\{\frac{(\lambda_1+\mu_1)^4 - b_3(\lambda_1+\mu_1)^3 + b_2(\lambda_1+\mu_1)^2 - b_1(\lambda_1+\mu_1) + b_0}{(\lambda_1+\mu_1)(\lambda_1+\mu_1+z_1)(\lambda_1+\mu_1+z_2)(\lambda_1+\mu_1+z_3)}\right\} e^{-(\lambda_1+\mu_1)t} \\
&+ \left\{\frac{(z_1^4 + b_3z_1^3 + b_2z_1^2 + b_1z_1 + b_0)}{z_1(\lambda_1+\mu_1+z_1)(z_1-z_2)(z_1-z_3)}\right\} e^{z_1t} \\
&+ \left\{\frac{(z_2^4 + b_3z_2^3 + b_2z_2^2 + b_1z_2 + b_0)}{z_2(\lambda_1+\mu_1+z_2)(z_2-z_2)(z_2-z_3)}\right\} e^{z_2t} \\
&+ \left\{\frac{(z_3^4 + b_3z_3^3 + b_2z_3^2 + b_1z_3 + b_0)}{z_3(\lambda_1+\mu_1+z_3)(z_3-z_1)(z_3-z_2)}\right\} e^{z_3t}
\end{aligned}
$$

z_1, z_2 and z_3 are three roots of the equation $s^3 + s^2a_2 + sa_1 + a_0 = 0$ (E. Balagurusamy [1])

7.2 Profit analysis of the user: Suppose that the warranty period of the system is $(0, w]$. Since the repairman is always available with the system, therefore beyond warranty period, it remains busy during the interval $(w, t]$. Let K_1 be the revenue per unit time and K_2 be the repair cost per unit time, then the expected profit $H(t)$ during the interval $(0, t]$ is given by

(54)
$$H(t) = K_1 \int_0^t Av(t)dt - K_2(t-w)$$

$$
\begin{aligned}
&= K_1 \Bigg[\frac{-b_0t}{(\lambda_1+\mu_1)z_1z_2z_3} \\
&+ \left\{\frac{(\lambda_1+\mu_1)^4 - b_3(\lambda_1+\mu_1)^3 + b_2(\lambda_1+\mu_1)^2 - b_1(\lambda_1+\mu_1) + b_0}{(\lambda_1+\mu_1)^2(\lambda_1+\mu_1+z_1)(\lambda_1+\mu_1+z_2)(\lambda_1+\mu_1+z_3)}\right\} (1 \\
&- e^{-(\lambda_1+\mu_1)t}) + \left\{\frac{(z_1^4 + b_3z_1^3 + b_2z_1^2 + b_1z_1 + b_0)}{z_1^2(\lambda_1+\mu_1+z_1)(z_1-z_2)(z_1-z_3)}\right\} (e^{z_1t} - 1) + \\
&\left\{\frac{(z_2^4 + b_3z_2^3 + b_2z_2^2 + b_1z_2 + b_0)}{z_2^2(\lambda_1+\mu_1+z_2)(z_2-z_2)(z_2-z_3)}\right\} (e^{z_2t} \\
&- 1) + \left\{\frac{(z_3^4 + b_3z_3^3 + b_2z_3^2 + b_1z_3 + b_0)}{z_3^2(\lambda_1+\mu_1+z_3)(z_3-z_1)(z_3-z_2)}\right\} (e^{z_3t} - 1)\Bigg] - K_2(t-w)
\end{aligned}
$$

8 Numerical Computations: In order to study the behaviours of reliability $R(t)$ and expected profit $H(t)$ mentioned in equations (46) and (54) respectively, some numerical results are presented in the form of tables for $R(t)$ and $H(t)$ for particular values of various parameters w.r.t. time t as:

Table 1: Effect of failure rates (λ and λ_1) on Reliability ($R(t)$)

Time(t)	$\lambda_1 = 0.02$, $\alpha = 0.003$		$\lambda = 0.01$, $\alpha = 0.003$	
	$R(t)$ (for $\lambda = 0.01$)	$R(t)$ (for $\lambda = 0.02$)	$R(t)$ (for $\lambda_1 = 0.01$)	$R(t)$ (for $\lambda_1 = 0.03$)
10	0.90353744	0.8187308	0.9048374	0.9023208
11	0.89428347	0.8025188	0.8958341	0.8928418
12	0.88510119	0.7866279	0.8869204	0.8834209
13	0.87599061	0.7710516	0.8780954	0.8740593
14	0.86695174	0.7557837	0.8693582	0.864758
15	0.85798456	0.7408182	0.860708	0.8555182
16	0.84908904	0.726149	0.8521438	0.8463406
17	0.84026512	0.7117703	0.8436648	0.8372263

Table 2: Effect of rate of completion of warranty (α) on Reliability ($R(t)$)

Time(t)	λ	λ_1	$R(t)$ (for $\alpha = 0.005$)	$R(t)$ (for $\alpha = 0.004$)	$R(t)$ (for $\alpha = 0.003$)
10	0.01	0.02	0.9026852	0.90310989	0.9035374
11	0.01	0.02	0.89326861	0.89377417	0.8942835
12	0.01	0.02	0.88391256	0.884504484	0.8851012
13	0.01	0.02	0.87461773	0.875301177	0.8759906
14	0.01	0.02	0.86538475	0.866164563	0.8669517
15	0.01	0.02	0.85621422	0.857094929	0.8579846
16	0.01	0.02	0.84710669	0.848092533	0.849089
17	0.01	0.02	0.83806267	0.838063	0.840265

Table 3: Effect of repair cost (K_2) on Expected Profit ($H(t)$)

Time(t)	$\lambda = 0.01, \lambda_1 = 0.02$, $h = 0.5$, $\alpha = 0.003, \mu = 0.2$, $\mu_1 = 0.1$, $q = 0.3$				
	K_1	W	$H(t)$ (for $K_2 = 150$)	$H(t)$ (for $K_2 = 100$)	$H(t)$ (for $K_2 = 50$)
10	500	3	3799.487	4149.487	4499.487
11	500	3	4125.804	4525.804	4925.804
12	500	3	4451.44	4901.44	5351.44
13	500	3	4776.5	5276.5	5776.5
14	500	3	5101.066	5651.066	6201.066
15	500	3	5425.205	6025.205	6625.205
16	500	3	5748.973	6398.973	7048.973
17	500	3	6072.412	6772.412	7472.412

Table 4: Effect of repair cost (K_2) and constant inspection rate (h) on Expected Profit ($H(t)$)

Time(t)	$\lambda = 0.01, \lambda_1 = 0.02$, $\alpha = 0.003$, $\mu = 0.2$, $\mu_1 = 0.1$, $q = 0.3$, $h = 0.5$		$\lambda = 0.01, \lambda_1 = 0.02$, $\alpha = 0.003$, $\mu = 0.2$, $\mu_1 = 0.1$, $q = 0.3$, $h = 0.6$	
	$H(t)$ (for $K_2 = 150$)	$H(t)$ (for $K_2 = 100$)	$H(t)$ (for $K_2 = 150$)	$H(t)$ (for $K_2 = 100$)
10	3799.487	4149.487	3805.808	4155.808
11	4125.804	4525.804	4133.338	4533.338
12	4451.44	4901.44	4460.236	4910.236
13	4776.5	5276.5	4786.594	5286.594
14	5101.066	5651.066	5112.49	5662.49
15	5425.205	6025.205	5437.982	6037.982
16	5748.973	6398.973	5763.121	6413.121
17	6072.412	6772.412	6087.945	6787.945

9 Interpretation and Conclusion Tables 1 and 2 show the behavior of system reliability. Table 1 indicates that the reliability of the system decreases with the increase of failure rates (λ and λ_1) with respect to (w.r.t.) time 't' and for fixed values of other parameters. From table 2, it is analyzed that the reliability of the system increases with the decrease of rate of completion of warranty (α) w.r.t. time 't'. It reveals that the system becomes more reliable for users as we increase time duration of warranty because any failure during warranty is rectified free of cost to the users. Table 3, shows that expected profit $H(t)$ during the interval $(0, t]$ increases with the decrease of repair cost (K_2) w.r.t. time 't' . Also, table 4 represents that the expected profit increase with the increase of inspection rate (h) of the failed unit w.r.t time

't'. This shows that inspection during warranty is profitable to manufacturer because it protected manufacturer about unnecessary expenses on repair of a continuously usage system or unit.

Hence, on the basis of the above discussion and the results obtained for a particular case (as mentioned in section 7), it is concluded that the concept of reliability and profit analysis of a single-unit system with inspection and warranty can be made more reliable and profitable to user and manufacturer both by decreasing the rate of completion of warranty, repair cost and increasing inspection rate.

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Partial Differential Equations Method to Analyze Nonlinear Resilience Force in a Long Clearance Seal and Rotor's Dynamics.

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Keywords: clearance seal, nonlinear elastic force, rotor, centrifugal machine, oscillation amplitude, critical frequency.

Abstract. The authors have developed a method of calculating the non-linear elastic force that arises in the clearance seal of a finite length. The authors have analyzed the force impact on the dynamics of the rotor of a centrifugal machine. In this work authors studied impact of nonlinear hydrostatic force in an arbitrary long seal, opposite to the previous literature where this force assumed to be linear, and seals are short (e.g. the ratio of the length of the clearance seal to the seal's radius is less than 0.5).

1 Introduction

1.1 Metodological novelty. In this work authors present a model of the non-linear hydrostatic force in an arbitrary long seal. After establishing the model authors present numerical experiments in order to illustrate obtained results. Mainly, the novelty of presented work consists of relaxing the linearity assumption made in previous works [1] and [2]. Also, authors generalized the model in order to fit arbitrary seal length, and only so called "short seal" as previously (e.g. the ratio of the length of the clearance seal to the seal's radius is less than 0.5).

1.2 The partial differential equation role in the model. Basically, the usage of partial differential equation for dynamic processes was established by Newton by his famous relation $\overline{F} = m \cdot \overline{a}$. Rewriting this relation in terms of X and Y projections we will get the following system:

$$\begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \end{cases}.$$

Solution of the system above will present rotor linear movements, or fractions. They should be kept as low as possible in order to keep the rotor well balanced.

1.3 Historical background. One of the pioneer investigators in this area was Prof. Dr. Lomakin. In 1953, he established a study of rotor seals frictions [1]. He has studied the problem connected with fast rebalancing of CBP-220-280. He resolved the problem connected with vibration by using the clearance seals of a different shape.

Similar problems encountered the NASA team headed by Dr. Childs, in [9] he mentioned that dynamic instability of main engine of the shuttle was explained by vibrations due to hydrodynamic forces in the rotor seal. Again, experimental change of the seals shape allowed resolving the issue.

1.4 Technical background. In the flow of the hydraulic machines, for the removal of significant flows of fluid from the high pressure zone to a zone of the lower pressure, clearance seals are used. Their sealing effect is determined by the large hydraulic resistance of the O-ring throttle with a small (0.1-0.35 mm) radial clearance. In literature there are numerous publications, which demonstrate that clearance seals of centrifugal machines significantly affect the rotor dynamic characteristics: arising hydrodynamic forces in the seal, depending on the design and operating conditions of the seal, may reduce vibroactivity of the rotor, or vice versa, lead to its dynamic imbalance. Most fully this problem is indicated in [1, 2, 3]. However, they consider a model of so-called "short" seals in which the circumferential component of fluid velocity, due to the pressure field, is neglected. When using the seals, wherein the circumferential component of the axial fluid is comparable to, or even exceeds it, the dynamic characteristics of the rotor vary significantly [4]. In [6, 7] they propose a model of a clearance seal of a finite length, for which there are obtained analytical expressions for the dimensionless coefficients of elastic and damping forces of the clearance seal.

It is shown that these coefficients depend only on two dimensionless parameters $\frac{l}{r}, \frac{r}{h_0}$, which are determined by geometry of the clearance (l- length, r- radius, h_0 - medium radial clearance). The ratio of the coefficients of elastic (K_c) and damping (K_b) forces, obtained by techniques of the short and the finite length clearance seals, are shown in Figure 1.

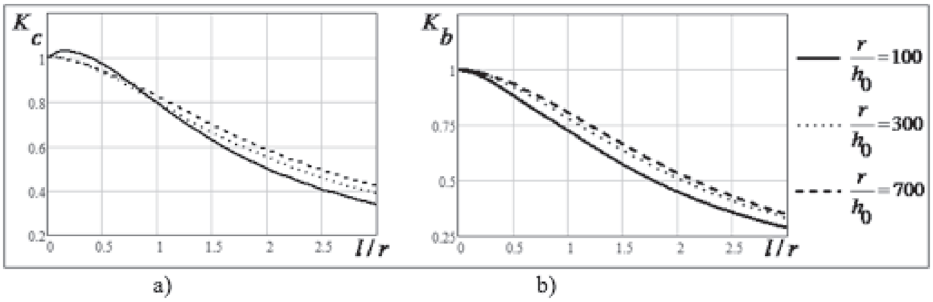

Figure 1: Dependence of the ratio of the coefficients:
a) for elastic force; b) for damping force

Apparently, with level growth of $\frac{l}{r}$ the rigidity and damping coefficients are significantly reduced (due to influence of circumferential overflows of fluid in the annular channel due to pressure field). That leads to significant deterioration of the vibratory state. Therefore, the problem of studying the impact of the circumferential overflows of fluid on hydrodynamic forces in the clearance seal of finite length is currently of great importance.

In this paper, we investigate the influence of fluid overflows in the circumferential gaps caused by pressure field on the nonlinearity of the elastic force, and the influence of the latter on the rotor dynamics.

The problem is solved with the following simplifying assumptions:

1) We considered the annular channels, for which the radial clearance is substantially less than the diameter.

2) The flow pattern across the gap is a self-similar region of the turbulent flow.

3) We consider the isothermal flow.

Assumptions 1-3) are quite natural for solving engineering problems connected to power machinery and rotor industry. Indeed, assumption 1) is applicable to general features of heavy industrial rotors. Assumption 2) reflects the best existing way to describe the fluid movements under high pressure and high rotation intensity. 3) Modern cooling systems give us abilities to assume that the fluid temperature will not change through the process. The rest of the paper will be presented as follows: in Part 2 authors will present and develop their model, Part 3 presents obtained numerical results and part 4 (Conclusions) finishes the paper.

2 The Model.

2.1 The study of the elastic force. Fluid motion in the clearance seal without inertial components is described by the system of equations introduced in [1]

$$\begin{cases} \dfrac{\partial p(z,\varphi)}{\partial \varphi} = -\dfrac{\lambda r}{2h_0}\dfrac{\rho w_0}{2}u(z,\varphi), \\ \dfrac{\partial p(z,\varphi)}{\partial z} = -\dfrac{\lambda l}{2h(\varphi)}\dfrac{\rho w^2(z,\varphi)}{2}, \\ \dfrac{\partial (w(z,\varphi)\cdot h(\varphi))}{\partial z} + \dfrac{l}{r}\dfrac{\partial (u(z,\varphi)\cdot h_0)}{\partial \varphi} = 0 \end{cases}$$

with boundary conditions

$$\begin{cases} p(0,\varphi) = p_{10} - \xi_1 \cdot \dfrac{\rho \cdot w^2(0,\varphi)}{2}, \\ p(1,\varphi) = p_{20} - \xi_2 \cdot \dfrac{\rho \cdot w^2(1,\varphi)}{2}, \end{cases}$$

where
$p(z,\varphi)$ - pressure of the fluid in the annular gap;
$w(z,\varphi)$ - axial velocity caused by pressure field;
$u(z,\varphi)$ - circumferential speed caused by pressure field;
$h(\varphi)$ - the value of the radial clearance;
ρ - density of the fluid;
λ - coefficient of the hydraulic friction;
p_{10} - pressure of the fluid in front of the clearance seal;
p_{20} - pressure of the fluid behind the clearance seal;
ξ_1 - coefficient of input losses;
ξ_2 - recovery ratio of the axial velocity downstream of the seal.

This system is transformed to quasilinear elliptic equation

$$\frac{(1-\varepsilon\cdot\cos\varphi)^2}{2\cdot l_r^2}\cdot\frac{\partial^2 p}{\partial z^2} + \sqrt{\frac{-(1-\varepsilon\cdot\cos\varphi)\cdot\xi_0}{\xi_1\cdot\Delta p}\cdot\frac{\partial p}{\partial z}}\cdot\frac{\partial^2 p}{\partial \varphi^2} = 0$$

with boundary conditions

$$\begin{cases} p(0,\varphi) = p_{10} + \xi_1 \cdot \dfrac{1-\varepsilon\cdot\cos\varphi}{\xi_1}\cdot\dfrac{\partial p(0,\varphi)}{\partial z}, \\ p(1,\varphi) = p_{20} + \xi_2 \cdot \dfrac{1-\varepsilon\cdot\cos\varphi}{\xi_1}\cdot\dfrac{\partial p(1,\varphi)}{\partial z}, \\ p(z,0) = p(z,2\pi), \end{cases}$$

where
$l_r = \dfrac{l}{r}$, $\varepsilon = \dfrac{e}{h_0}$ - dimensionless parameters;
ξ_1 - loss coefficient along the clearance seal;
ξ_0 - the total loss coefficient in the clearance seal;
Δ_p - pressure drop across the gap.

To solve this equation we used the grid method applying the method of successive approximations. The solution is the fluid pressure values at the mesh point at a predetermined relative eccentricity. After interpolation of the obtained values by two-dimensional cubic spline, we got the distribution of pressure in the gap. The elastic force is then determined by formula

$$F(\varepsilon) = -r\cdot l\cdot\int_0^{2\pi}\int_0^1 p(z,\varphi,\varepsilon)\cdot\cos\varphi dz d\varphi.$$

where
$P_i(\varepsilon)$ - the ith Legendre polynomial;
$a_i = \dfrac{2i+1}{2}\int_{-1}^{1} f(\varepsilon)\cdot P_i(\varepsilon)d\varepsilon$ - are expansion coefficients;

$f(\varepsilon)$ - spline of the table data.

This result can be represented as

$$F(\varepsilon) = -k_c(0) \cdot h_0 \cdot \varepsilon \cdot \alpha(\varepsilon),$$

where
$k_c(0)$ - stiffness coefficient of linearized elastic force;
$\alpha(\varepsilon) = 1 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \ldots + \alpha_n\varepsilon^n$ - dimensionless coefficient of nonlinear elastic force.

Figure 2 shows dependence of this ratio for some types of clearance seals. As one can see from the figure, the amount of elastic force decreases with increasing eccentricity. At the same time, the influence of parameter $\frac{r}{h_0}$ is irrelevant.

Analytical expression for stiffness of elastic force, depending on the displacement of the shaft is determined by formula

$$k_c(\varepsilon) = \frac{dF(\varepsilon)}{de} = k_c(0) \cdot \beta(\varepsilon),$$

where
$\beta(\varepsilon) = 1 + 2\alpha_1\varepsilon + 3\alpha_2\varepsilon^2 + \ldots + (n+1)\alpha_n\varepsilon^n$ - dimensionless coefficient of nonlinear stiffness.

Dependence of coefficient $\beta(\varepsilon)$ is shown in Figure 3. It's evident that stiffness of elastic force decreases with increasing displacement of the shaft, i.e. this system has a soft characteristic of stiffness. As far as we know, this fact deteriorates the vibration characteristics of the rotor.

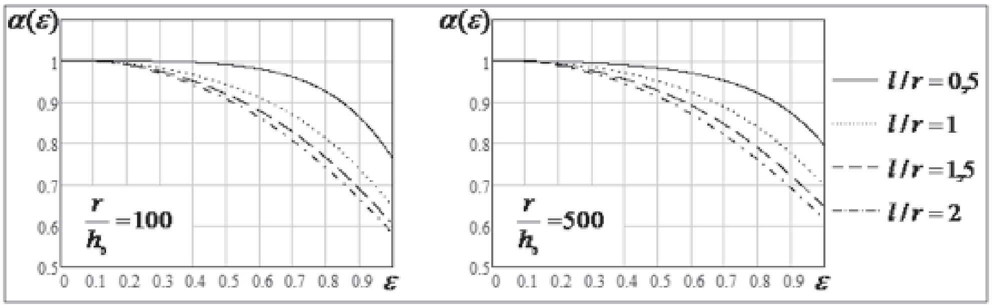

Figure 2: Dependence of nonlinearity coefficient of elastic force on relative eccentricity

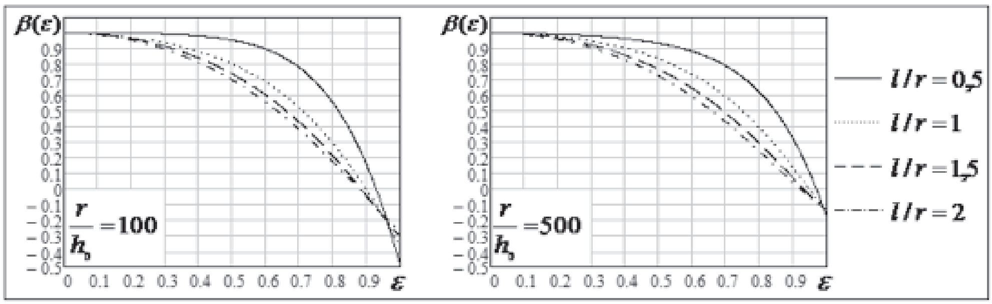

Figure 3: Dependence of nonlinearity coefficient of stiffness on relative eccentricity

2.2 Investigation of dynamic characteristics of the rotor To study the effect of nonlinear force on the dynamic characteristics of the rotor we consider a single-mass rotor model (Figure 4) with the parameters of the shaft: length $l = 520$ mm and diameter $d = 25$ mm; rotor mass $m = 18$ kg; clearance seal geometry: length $l = 48$ mm and radius $r = 25$ mm; average radial clearance $h_0 = 0.3$ mm, and pressure drop across the gap $\Delta_p = 1.25$ MPa.

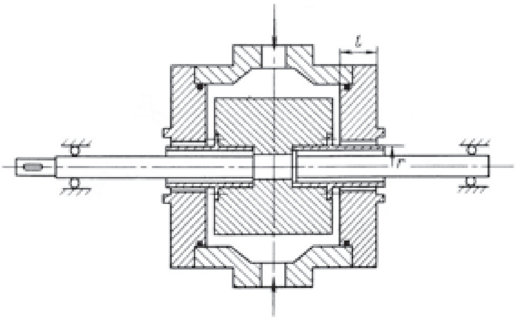

Figure 4: A single-mass rotor model

The structure of hydrodynamic forces, arising in the gap sealing, can be assumed as

$$\begin{cases} F_x = -b \cdot \dot{x} - k_c(0)\alpha(r) \cdot x - q \cdot y, \\ F_y = -b \cdot \dot{y} - k_c(0)\alpha(r) \cdot y - q \cdot x, \end{cases}$$

where
b - damping coefficient;
$q = 0.5b\omega$ - circulation ratio;
x, y - displacement coordinates of the shaft center in the fixed coordinate system;
$r = \sqrt{x^2 + y^2}$ - radius of the shaft movement orbit.

The damping coefficient was determined according to [6], nonlinear elastic force projection on fixed axes by the above method. For the considered clearance seals we have

$$b = 5079 \frac{H \cdot c}{\mathcal{M}}, \qquad k_c(0) = 1,035 \cdot 10^6 \frac{H}{\mathcal{M}},$$

$$\alpha(r) = 1 + 0,057 \cdot r - 0,515 \cdot r^2 + 0,076 \cdot r^3 - 0,047 \cdot r^4 - 0,104 \cdot r^5 - 0,068 \cdot r^6 + 0,075 \cdot r^7 + 0,032 \cdot r^8 - 0,02 \cdot r^9$$

Differential equations of motion of this model have the form

$$\begin{cases} m\ddot{x} + b\dot{x} + c_b x + k_c(0)\alpha(r) \cdot x + qy &= me_1\omega^2 \cos(\omega t); \\ m\ddot{y} + b\dot{y} + c_b y + k_c(0)\alpha(r) \cdot y - qx &= me_1\omega^2 \sin(\omega t); \end{cases} \tag{1}$$

where
m - rotor weight; c_b - stiffness of the shaft; me_1 - rotor imbalance; ω - speed of rotation.

Having entered the designations

$$\tau = t \cdot \omega_0, \ \frac{d}{d\tau} = \omega_0 \frac{d}{dt}, \ \omega_0 = \sqrt{\frac{c_b + k_c(0)}{m}}, \ \overline{x} = \frac{x}{h_0}, \ \overline{y} = \frac{y}{h_0}, \ \overline{b} = \frac{b\omega_0}{c_b + k_c(0)}, \ \overline{\omega} = \frac{\omega}{\omega_0}, \ \overline{e} = \frac{e_1}{h_0},$$

system (1) is written in the dimensionless form (hereinafter, for the convenience, $\overline{x}$ and $\overline{y}$ will be written in the form x and y).

$$\begin{cases} \ddot{x} + b\dot{x} + x\left(1 + \dfrac{k_c(0)}{c_b + k_c(0)}\left(\alpha_1 r + \alpha_2 r^2 + \ldots + \alpha_9 r^9\right)\right) + 0,5\bar{b}\overline{\omega}y = & \overline{e\omega}^2\cos(\overline{\omega}\tau) \\ \ddot{y} + b\dot{y} + y\left(1 + \dfrac{k_c(0)}{c_b + k_c(0)}\left(\alpha_1 r + \alpha_2 r^2 + \ldots + \alpha_9 r^9\right)\right) - 0,5\bar{b}\overline{\omega}x = & \overline{e\omega}^2\sin(\overline{\omega}\tau) \end{cases} \quad (2)$$

3 Results of numerical experiments Numerical solution of (2) was performed using software package Mathcad. We obtained the area of sustainable movement of the shaft, as well as its rotational speed with boundary condition of stability. In this case the rotational speed with boundary condition of stability was determined by the appearance of a subharmonic self-oscillating imposition when the relative speed of the rotor $\overline{\omega}$ varied.

As an example, Figure 5 shows the oscillation of the shaft center in a horizontal plane (a), the orbit of the shaft center (b), and the relevant spectrum (c) at the speed of rotation $\overline{\omega} = 1,5$. As can be seen, the shaft makes a steady circular motion with the rotational frequency.

Figure 6 shows the oscillation of the shaft center in the horizontal plane (a), the orbit of the shaft center (b), and a corresponding spectrum (c) at the boundary of the unstable region of rotation $\overline{\omega} = 2,01$. Thus, along with a synchronous component, there is a subharmonic one with an amplitude which exceeds the amplitude of a synchronous component.

Figure 7 shows the oscillations of the shaft center in the horizontal plane (a) and the orbit of the shaft center (b) in the unstable region of rotation. In this case, we observe a rapid growth of the shaft displacement, leading to an emergency mode.

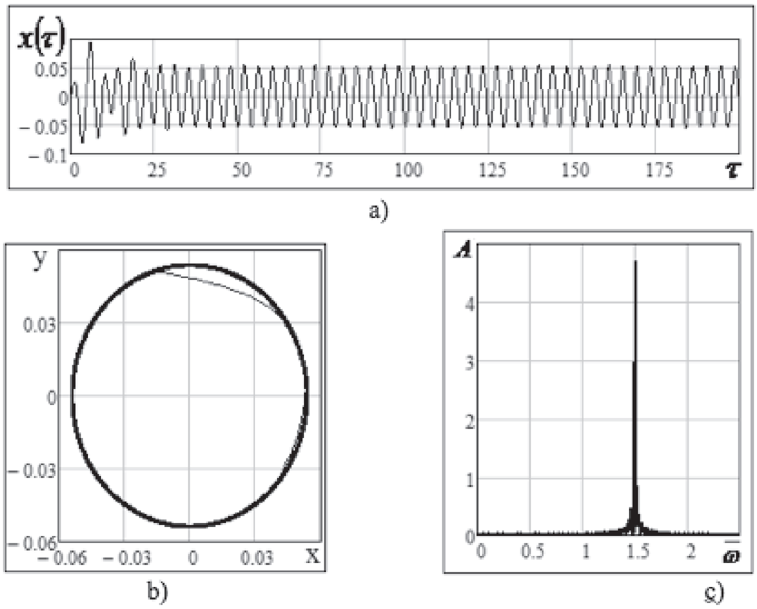

Figure 5: Oscillations of the shaft center in a horizontal plane (a), the orbit of the shaft movement (b), and the oscillation spectrum (c) in the stable region of rotation $\overline{\omega} = 1.5$

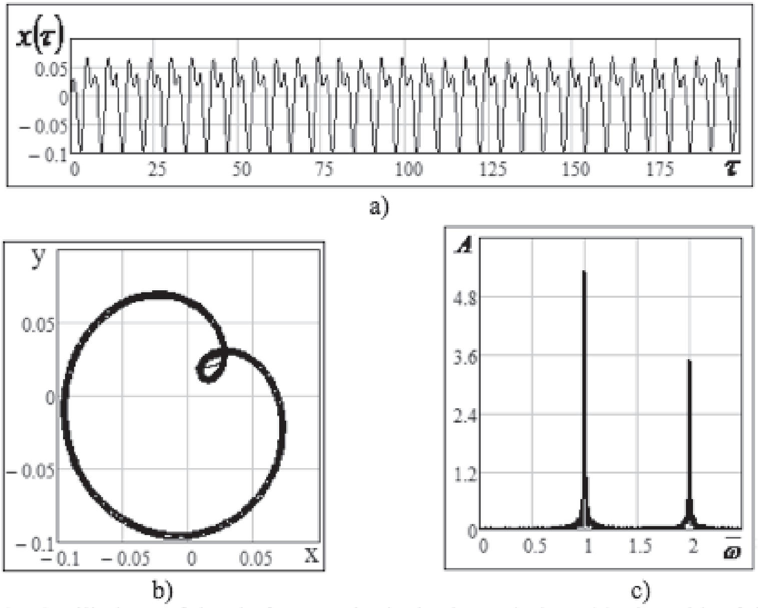

Figure 6: Oscillations of the shaft center in the horizontal plane (a), the orbit of the shaft movement (b) and the oscillation spectrum (c) at the stable region of rotation $\overline{\omega} = 2.01$

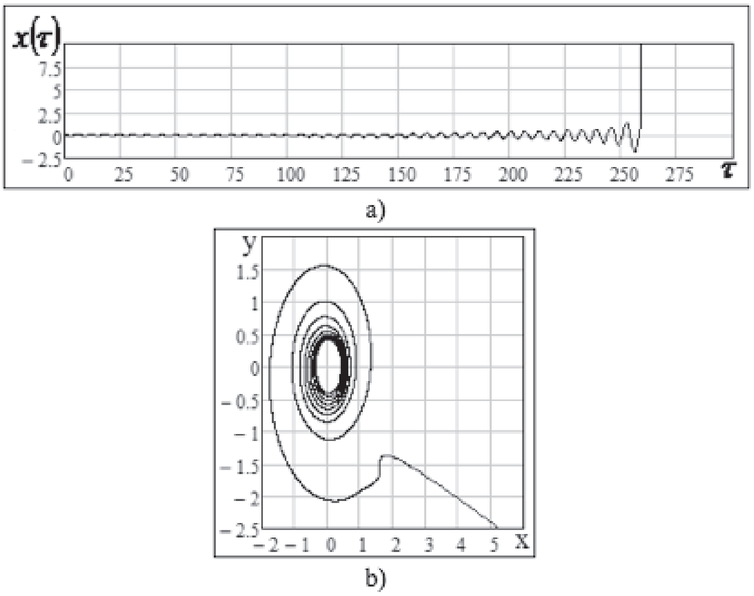

Figure 7: Oscillations of the shaft center in the horizontal plane (a), the orbit of the shaft movement (b) in the unstable region of rotation $\overline{\omega} = 2.05$

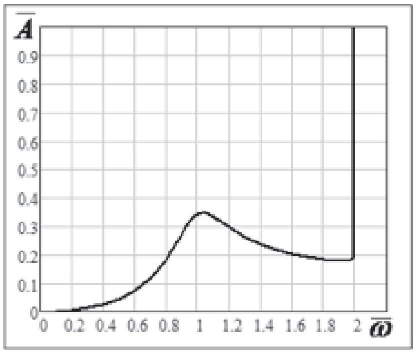

Figure 8: Dependence of the relative amplitude of the synchronous component of the rotor oscillations on the speed of rotation

Studies show that the differences between the calculations of the dynamic characteristics of the rotor according to the procedures of short clearance seals and clearance seals of finite length can be both quantitative and qualitative. For example, from Figure 8 we can see that in the unstable region of rotation the rotor performs oscillations with steady amplitude. Moreover, the total vibration level for speed of rotation above the boundary remains within acceptable limits. At the same time, Figures 6,7 and 8 show that for the investigated rotor model, which has a mild characteristic of stiffness, self-oscillating mode takes place only on the stability boundary; then shaft displacement increases rapidly, i.e. for the "long clearance" model the "emergency" effect occurs immediately after buckling.

4 Conclusions: There are two main points that make presented work different from previous ones, as [1] and [2]: First, a new method of calculating the *nonlinear* quasi-elastic force in a relatively long clearance seal is developed and some numerical results are presented. Assumption of linearity is too general and may lead to substantial problems why applied to the real world machinery.

Second, the rotor dynamics is investigated, taking into account the nonlinearity of the quasi-elastic force. It is shown that flows in the circumferential direction, due to the pressure field, reduce the elastic force in the gap sealing, which leads to deterioration of the dynamic characteristics of the rotor, which is quite interesting result and have some direct application in machinery industry involving rotor systems.

As for future plans: authors will do theoretical investigation of *nonlinear* damper and circular forces in short and long clearance seals, in order to evaluate rotor dynamics for these cases.

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Exponential information measures on pairs of fuzzy sets

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Abstract

Exponential fuzzy entropy with a single fuzzy set has been considered. Here we consider exponential information theoretic measures with pair of fuzzy sets. This leads to results that parallel the results as Shannon's joint, conditional and mutual information measures between two fuzzy sets.
Mathematics Subject Classification (2000): 94 A17
Keywords: Fuzzy entropy, fuzzy joint entropy, fuzzy conditional entropy, fuzzy mutual information

1. Introduction

Fuzzy sets proposed by Zadeh [7] in 1965 have gained extensive applications in many areas such as engineering, artificial intelligence, medical science, signal processing, decision making and so on because of its capability to represent/ model non-statistical imprecision or vague concepts.

In Fuzzy set theory, the entropy is defined as a measure of fuzziness which expresses the amount of ambiguity or difficulty in making a decision whether an element belongs to a set or not. The first measure of fuzziness associated with a fuzzy set also mentioned by Zadeh [8] in 1968. In 1972, De Luca and Termini [2] formulated axioms for the entropy of fuzzy sets and defined the measure of fuzzy entropy based on Shannon's function [5].

In addition, Yager [6] defined a measure of fuzzy entropy in terms of a lack of distinction between fuzzy set and its complement. In 1989, Pal and Pal [3] proposed a new measure of fuzzy entropy based on exponential function called '*exponential fuzzy entropy'*. Recently, Verma and Sharma [4] have introduced a parametric generalized entropy measure for fuzzy sets called '*exponential fuzzy entropy of order-α*'. In 2007, Ding et al. [1] extended the notion of fuzzy entropy to define conditional fuzzy entropy, joint fuzzy entropy and fuzzy mutual information corresponding to De Luca and Termini's fuzzy entropy and studied their relations also.

In this paper, we extend the idea of measure of exponential fuzzy entropy on pairs of fuzzy sets and propose some new exponential fuzzy entropy measures such as exponential fuzzy joint entropy, exponential fuzzy conditional entropies. Further, a measure of exponential fuzzy mutual information is defined here. Some relations among them are also studied.

This paper is organized as follows: In Section 2 basic definitions related to probability theory, fuzzy sets, and fuzzy entropy measures are briefly reviewed. In Section 3 exponential fuzzy joint entropy and exponential fuzzy conditional entropies are introduced and some of their properties are proved. In Section 4 the concept of exponential fuzzy mutual information measure is proposed and studied their properties. Our conclusions are presented in the final section.

2. Preliminaries

In this section we give some basic concepts and definitions related to probability theory, fuzzy sets, which will be used in the following analysis.

Let $\Delta_n = \left\{ P = (p_1, p_2, ..., p_n) : p_j \geq 0, \sum_{j=1}^{n} p_j = 1 \right\}$, $n \geq 2$ be a set of n-complete probability distributions. For any probability distribution $P = (p_1, p_2, ..., p_n) \in \Delta_n$, Shannon's entropy [5], is defined as

$$H_S(P) = -\sum_{j=1}^{n} p_j \log p_j \tag{1}$$

After the pioneering work of De Luca and Termini [2], various measures of fuzzy entropy have been proposed by many researchers and developed their applications in different areas. In 1989, Pal and Pal [3] analyzed the classical Shannon's entropy and proposed a new measure of probabilistic entropy based on exponential function as follows

$$_eH(P) = \sum_{j=1}^{n} p_j \left(e^{1-p_j} - 1 \right) \tag{2}$$

Pal and Pal also pointed out that, the measure of exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $P = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, exponential entropy has a fixed upper bound

$$\lim {_eH}\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right) = (e-1) \qquad \text{as } n \to \infty. \tag{3}$$

which is not the case for Shannon's entropy.

Definition 1. Fuzzy Set [5]: A fuzzy set A in a discrete universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is given by

$$A = \{ \langle x, \mu_A(x) \rangle \mid x \in X \} , \tag{4}$$

where $\mu_A : X \to [0,1]$ is the membership function of A. The number $\mu_A(x)$ describes the degree of membership of $x \in X$ in A.

Definition 2. Set Operations on Fuzzy Sets [7]: Let $FS(X)$ denote the family of all FSs in X and let $A, B \in FS(X)$ be given by

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$

$$B = \{\langle x, \mu_B(x)\rangle \mid x \in X\},$$

then set operations are defined as follows:

(i) **Containment:** $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x) \quad \forall x \in X$;

(ii) **Equality:** $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(iii) **Complement:** $A^C = \{ \langle x,\ 1-\mu_A(x)\rangle \mid x \in X\}$;

(iv) **Union:** $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$;

(v) **Intersection:** $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x))\rangle \mid x \in X\}$.

Definition 3. Sharpened Fuzzy Set[2]**:** A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\mu_{A^*}(x_j) \le \mu_A(x_j), \text{ if } \mu_A(x_j) \le 0.5 \ \ \forall\ j$$

$$\mu_{A^*}(x_j) \ge \mu_A(x_j), \text{ if } \mu_A(x_j) \ge 0.5 \ \ \forall\ j.$$

The first attempt to quantify the fuzziness was made in 1968 by Zadeh [8], who based on probabilistic framework, introduced the fuzzy entropy by combining probability and membership function of a fuzzy event as weighted Shannon entropy [5] given by

$$H_Z(A) = -\sum_{j=1}^{n} \mu_A(x_j)\ p_j\ \log p_j \tag{5}$$

In 1972, De Luca and Termini [2] defined the measure of fuzzy entropy for a fuzzy set A corresponding (1) by

$$H_{DT}(A) = -\frac{1}{n}\sum_{j=1}^{n}\left[\mu_A(x_j)\ \log\left(\mu_A(x_j)\right) + \left(1-\mu_A(x_j)\right)\log\left(1-\mu_A(x_j)\right)\right]. \tag{6}$$

Fuzzy exponential entropy for fuzzy set A corresponding to (2) has also been introduced by Pal and Pal [3] as

$${}_eH(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{j=1}^{n}\left[\mu_A(x_j)\,e^{1-\mu_A(x_j)} + \left(1-\mu_A(x_j)\right)e^{\mu_A(x_j)} - 1\right]. \tag{7}$$

In the next section, we define exponential fuzzy joint and exponential fuzzy conditional entropies and study their properties.

3. Exponential fuzzy joint and exponential fuzzy conditional entropies

We proceed with the following formal definitions:

Definition 4: Let A and B be two fuzzy sets defined in $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_j)$, $j = 1, 2, ..., n$, and $\mu_B(x_j)$, $j = 1, 2, ..., n$, respectively. Let

$$X^+ = \{x \mid x \in X,\ \mu_A(x_j) \ge \mu_B(x_j)\ \},$$

$$X^- = \{x \mid x \in X,\ \mu_A(x_j) < \mu_B(x_j)\}.$$

Based on the idea of Ding et al. [1], we propose the exponential fuzzy joint entropy and exponential fuzzy conditional entropies as follows:

Exponential Fuzzy Joint Entropy (EFJE):

$${}_eH(A\cup B) = \frac{1}{n(\sqrt{e}-1)}\sum_{j=1}^{n}\left[\left(\mu_{A\cup B}(x_j)\,e^{(1-\mu_{A\cup B}(x_j))} + \left(1-\mu_{A\cup B}(x_j)\right)e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$

$$(8)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^+}\left(\mu_A(x_j)\,e^{(1-\mu_A(x_j))}+(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-1\right) \\ +\sum_{x_j\in X^-}\left(\mu_B(x_j)\,e^{(1-\mu_B(x_j))}+(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-1\right) \end{array}\right].$$

Exponential Fuzzy Conditional Entropies (EFCE):

$${}_eH(A/B) = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^+}\left(\mu_A(x_j)\,e^{(1-\mu_A(x_j))}+(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-1\right) \\ -\sum_{x_j\in X^+}\left(\mu_B(x_j)\,e^{(1-\mu_B(x_j))}+(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-1\right) \end{array}\right]$$

$$(9)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^+}\left[\left(\begin{array}{c} \mu_A(x_j)\,e^{(1-\mu_A(x_j))}-\mu_B(x_j)\,e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-(1-\mu_B(x_j))\,e^{\mu_B(x_j)} \end{array}\right)\right],$$

$${}_eH(B/A) = \frac{1}{n\left(\sqrt{e}-1\right)}\left[\begin{array}{c} \sum_{x_j\in X^-}\left(\mu_B(x_j)\,e^{(1-\mu_B(x_j))}+(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-1\right) \\ -\sum_{x_j\in X^-}\left(\mu_A(x_j)\,e^{(1-\mu_A(x_j))}+(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-1\right) \end{array}\right]$$

$$(10)\quad = \frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X^-}\left[\left(\begin{array}{c} \mu_B(x_j)\,e^{(1-\mu_B(x_j))}-\mu_A(x_j)\,e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-(1-\mu_A(x_j))\,e^{\mu_A(x_j)} \end{array}\right)\right].$$

Some properties of these entropies are proved below:

Theorem 1: For $A, B \in FS(X)$,

(i) ${}_eH(A/B) \le {}_eH(A)$,

(ii) ${}_eH(B/A) \le {}_eH(B)$,

with equality if and only if $A = B$ i.e., $\mu_A(x_j) = \mu_B(x_j),\ \forall\ x_j \in X$.

Proof: **(i).** Let us consider the expression

$$(11)\qquad {}_eH(A) - {}_eH(A/B)$$

$$\begin{aligned} &= \tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{j=1}^{n}\left[\mu_A(x_j)\,e^{1-\mu_A(x_j)}+(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-1\right] \\ &\quad -\tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{x_j\in X^+}\left[\left(\begin{array}{c} \mu_A(x_j)\,e^{(1-\mu_A(x_j))}-\mu_B(x_j)\,e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j))\,e^{\mu_A(x_j)}-(1-\mu_B(x_j))\,e^{\mu_B(x_j)} \end{array}\right)\right] \\ &= \tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{x_j\in X^+}\left[\left(\mu_B(x_j)\,e^{(1-\mu_B(x_j))}+(1-\mu_B(x_j))\,e^{\mu_B(x_j)}\right)-1\right] \\ &\quad +\tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{x_j\in X^-}\left[\left(\mu_A(x_j)\,e^{(1-\mu_A(x_j))}+(1-\mu_A(x_j))\,e^{\mu_A(x_j)}\right)-1\right] \\ &\ge\ 0. \end{aligned}$$

This completes the proof.

(ii). Let us consider the expression

$$(12)\qquad {}_eH(B) - {}_eH(B/A)$$

$$\begin{aligned} &= \tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{j=1}^{n}\left[\mu_B(x_j)\,e^{1-\mu_B(x_j)}+(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-1\right] \\ &\quad -\tfrac{1}{n(\sqrt{e}-1)}\textstyle\sum_{x_j\in X^-}\left[\left(\begin{array}{c} \mu_B(x_j)\,e^{(1-\mu_B(x_j))}-\mu_A(x_j)\,e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j))\,e^{\mu_B(x_j)}-(1-\mu_A(x_j))\,e^{\mu_A(x_j)} \end{array}\right)\right] \end{aligned}$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$\geq 0.$$

This completes the proof.

Remark 1: Note that ${}_eH(A/B) \neq {}_eH(B/A)$ in general. However ${}_eH(A) - {}_eH(A/B) = {}_eH(B) - {}_eH(B/A)$.

The naturalness of the definition of exponential fuzzy joint entropy and exponential fuzzy conditional entropy is exhibited by the fact that the fuzzy entropy of a pair of fuzzy sets is the fuzzy entropy of one plus the fuzzy conditional entropy of the other. This is proved in the following theorem.

Theorem 2(*Chain rule*): For $A, B \in FS(X)$,

(i) ${}_eH(A \cup B) = {}_eH(A) + {}_eH(B/A)$;

(ii) ${}_eH(A \cup B) = {}_eH(B) + {}_eH(A/B)$;

(iii) ${}_eH(A \cup B) = {}_eH(A) + {}_eH(B/A) = {}_eH(B) + {}_eH(A/B)$.

Proof :(i) Let us consider the expression

$$(13) \qquad {}_eH(A) + {}_eH(B/A) - {}_eH(A \cup B)$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_A(x_j)\, e^{1-\mu_A(x_j)} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)} - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\begin{array}{c} \mu_B(x_j)\, e^{(1-\mu_B(x_j))} - \mu_A(x_j)\, e^{(1-\mu_A(x_j))} \\ + (1-\mu_B(x_j))\, e^{\mu_B(x_j)} - (1-\mu_A(x_j))\, e^{\mu_A(x_j)} \end{array}\right)\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j))\, e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$

$$= 0.$$

This proves (i).

(ii) Let us consider the expression

$$(14) \qquad {}_eH(B) + {}_eH(A/B) - {}_eH(A \cup B)$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\mu_B(x_j)\, e^{1-\mu_B(x_j)} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)} - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\begin{array}{c} \mu_A(x_j)\, e^{(1-\mu_A(x_j))} - \mu_B(x_j)\, e^{(1-\mu_B(x_j))} \\ + (1-\mu_A(x_j))\, e^{\mu_A(x_j)} - (1-\mu_B(x_j))\, e^{\mu_B(x_j)} \end{array}\right)\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j))\, e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$
$$+ \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right]$$
$$- \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right]$$

$= 0.$

This proves (ii).

(iii) It obvious follows (i) and (ii).

This completes the proof.

In the Shannon's theory, another important concept is that of trans-information or mutual information. It is the measure of the amount of information contains one random variable about another. Based on the idea of fuzzy mutual information (FMI) [1], in the next section, we propose the concept of exponential fuzzy mutual information (EFMI) and study their properties.

4. Exponential fuzzy mutual information

Definition 5: Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universe of discourse, and $A, B \in FS(X)$, then the difference value, ${}_eH(A) - {}_eH(A/B)$, is called the EFMI between fuzzy set A and B, denoted by ${}_eH(A \cap B)$ i.e.

$$(15) \qquad {}_eH(A \cap B) = {}_eH(A) - {}_eH(A/B)$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right].$$

Similarly, we define ${}_eH(B \cap A)$, given by

$$(16) \qquad {}_eH(B \cap A) = {}_eH(B) - {}_eH(B/A)$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j)\, e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j)\, e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)}\right) - 1\right].$$

Some properties on lines parallel to Shannon's mutual information are proved below:

Theorem 3: For $A, B \in FS(X)$,

(i) ${}_eH(A \cap B) \geq 0$ and ${}_eH(B \cap A) \geq 0$;

(ii) ${}_eH(A \cap B) = {}_eH(B \cap A)$;

(iii) ${}_eH(A \cap B) = {}_eH(A) + {}_eH(B) - {}_eH(A \cup B)$;

(iv) ${}_eH(A \cap B) = {}_eH(A \cup B) - {}_eH(A/B) - {}_eH(B/A)$;

(v) ${}_eH(A \cap A) = {}_eH(A)$.

Proof: **(i)** It follows straight forwardly from Theorem 2.

(ii) It follows directly from Definition.

(iii) We consider the expression

$$(17) \qquad {}_eH(A) + {}_eH(B) - {}_eH(A \cup B)$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_A(x_j)\, e^{1-\mu_A(x_j)} + (1-\mu_A(x_j))\, e^{\mu_A(x_j)} - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_B(x_j)\, e^{1-\mu_B(x_j)} + (1-\mu_B(x_j))\, e^{\mu_B(x_j)} - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j)\, e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j))\, e^{\mu_{A\cup B}(x_j)}\right) - 1\right]$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_A(x_j) e^{1-\mu_A(x_j)} + (1-\mu_A(x_j)) e^{\mu_A(x_j)} - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\mu_B(x_j) e^{1-\mu_B(x_j)} + (1-\mu_B(x_j)) e^{\mu_B(x_j)} - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j) e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j)) e^{\mu_A(x_j)}\right) - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j) e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j)) e^{\mu_B(x_j)}\right) - 1\right]$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j) e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j)) e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j) e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j)) e^{\mu_A(x_j)}\right) - 1\right]$$

$$= {}_eH(A \cap B).$$

This completes the proof.

(iv) Let us consider the following expression

$$_eH(A \cup B) - {}_eH(A/B) - {}_eH(B/A) \tag{18}$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{j=1}^{n} \left[\left(\mu_{A\cup B}(x_j) e^{(1-\mu_{A\cup B}(x_j))} + (1-\mu_{A\cup B}(x_j)) e^{\mu_{A\cup B}(x_j)}\right) - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\begin{array}{c} \mu_A(x_j) e^{(1-\mu_A(x_j))} - \mu_B(x_j) e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j)) e^{\mu_A(x_j)} - (1-\mu_B(x_j)) e^{\mu_B(x_j)} \end{array}\right)\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\begin{array}{c} \mu_B(x_j) e^{(1-\mu_B(x_j))} - \mu_A(x_j) e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j)) e^{\mu_B(x_j)} - (1-\mu_A(x_j)) e^{\mu_A(x_j)} \end{array}\right)\right]$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_A(x_j) e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j)) e^{\mu_A(x_j)}\right) - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_B(x_j) e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j)) e^{\mu_B(x_j)}\right) - 1\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\begin{array}{c} \mu_A(x_j) e^{(1-\mu_A(x_j))} - \mu_B(x_j) e^{(1-\mu_B(x_j))} \\ +(1-\mu_A(x_j)) e^{\mu_A(x_j)} - (1-\mu_B(x_j)) e^{\mu_B(x_j)} \end{array}\right)\right] - \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\begin{array}{c} \mu_B(x_j) e^{(1-\mu_B(x_j))} - \mu_A(x_j) e^{(1-\mu_A(x_j))} \\ +(1-\mu_B(x_j)) e^{\mu_B(x_j)} - (1-\mu_A(x_j)) e^{\mu_A(x_j)} \end{array}\right)\right]$$

$$= \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^+} \left[\left(\mu_B(x_j) e^{(1-\mu_B(x_j))} + (1-\mu_B(x_j)) e^{\mu_B(x_j)}\right) - 1\right] + \frac{1}{n(\sqrt{e}-1)} \sum_{x_j \in X^-} \left[\left(\mu_A(x_j) e^{(1-\mu_A(x_j))} + (1-\mu_A(x_j)) e^{\mu_A(x_j)}\right) - 1\right]$$

$$= {}_eH(A \cap B).$$

This completes the proof.

(v) $_eH(A \cap A) = {}_eH(A) - {}_eH(A/A) = {}_eH(A)$**.**

Thus the mutual information of a fuzzy set with itself is the entropy of the fuzzy set.

5. Conclusions

This work introduces some new information measures on pair of fuzzy sets called exponential fuzzy joint entropy, exponential fuzzy conditional entropy, and exponential fuzzy mutual information measure in the setting of fuzzy set theory. Some properties and relations of these measures have been studied.

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Bootstrap and Other Tests For Goodness of Fit.

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Abstract: Goodness of fit has a long time been a problem of research .It has received a considerable attention in the statistical literature. Goodness of fit techniques can be described as the method of how well a sample of data agrees with a given distribution as its population. Goodness of fit techniques is based on measuring in some way the conformity of the sample data to the hypothesized distribution or equivalently, its discrepancy from it. The techniques usually give formal statistical tests and the data based measures of conformity or discrepancy are referred to as test statistics. In this paper we have studied the performance of the bootstrap based procedure of EDF based tests for testing the goodness of fit for normality of the distribution using simulation technique. Some results are calculated to know the performance of bootstrap based technique and these are displayed in tables. Discussions and conclusions are made on the basis of results obtained.

Key words: Bootstrap procedure, Kolmogorov-Smirnov test, Anderson-Darling test, simulation, power.

1. Introduction:

Goodness-of-fit has been occupying an important place in statistical inference since long time. In short it is a technique of examining how well a sample of data agrees with a given distribution as its population. Measures of goodness of fit typically summarize the discrepancy between observed values and the expected values under the model in equation. Such measures can be used in statistical hypothesis testing, i.e. to test for normality of residuals, to test whether two samples are drawn from identical distributions, or whether the outcome frequencies follow a specified distribution. Although it is a cornerstone of modern statistical theory, but still no clear notion of optimality for more complicated situations. An important but difficult problem is evaluating the goodness-of -fit of a model and obtains the P-value. The normal theory of likelihood ratio test statistics whose distribution is approximated by a chi-square test is often used in practice. But it is known that the chi-square distribution is not so accurate for small sample sizes even when the latent factors are normally distributed. Similar results are also reported by Anderson-test too.

Fitting of a probability model to observed data is an important statistical problem from both theory and application point of view. There is a multitude of statistical models and procedures that rely on the validity of a given data hypothesis, being the normality of the data assumption one of the most commonly found in statistical studies. As observed in many models and in research on applied statistics and economics, following the normal distribution assumption blindly may affect the accuracy of inference and estimation procedures. The evaluation of this distributional assumption has been addresses, for example, in Min(2007) where the conditional normality assumption in the sample selection model applied to housing demand is examined, or in Lisenfeld and Jung(2000) where the normality assumption has been addressed in the context of stock market data, a type of data that has been found to be typically heavy –tailed in Gel and Gastwirth(2008). The analysis of the normality hypothesis can also be found in the characterization of error terms in the context of regression analysis models applied to economic time-series Giles(2007), Thadewald and Buning(2007), to probit models Wilde(2008) or to other types of time series Onder and Zaman(2005), Quddus(2008). In medical research the assumption of normality is also very common Schoder, Himmelmann and Wilhelm(2006) and Surucu and Koc(2007), but the suitability of this assumption must also be verified with adequate statistical tests. The definition of adequate normality tests can, therefore , be seen to be of much importance since the acceptance or rejection of the normality assumption of a given data set plays a central role in numerous research fields. As such, the problem of testing normality has gained considerable importance in both theoretical and empirical research and has led to the development of a large number of goodness of fit tests to detect departures from normality. Given the importance of this subject and the widespread development of normality tests over the years, comprehensive descriptions and power comparisons of such tests have also been the focus of attention, thus helping the analyst in the choice of suitable tests for his particular needs.

There have been a quite a few works on goodness-of-fit test based on bootstrap as compared to other test. Blake(2005) discussed the utility of bootstrap method in normally distributed data of violent crime across the states. Matthias von(1997) showed that the parametric bootstrap can be used for analyzing goodness-of-fit, even when the data are very sparse. Alberto and Harry (2008) provided an overview of the new developments in limited information goodness-of-fit assessment of categorical data models. The goodness-of-fit of latent trait models in attitude measurement was discussed by Bartholomew and Tzamourani (1999). There are two versions of the bootstrap, the (naïve) bootstrap and the parametric bootstrap, of which only the parametric bootstrap can be used for goodness-of-fit testing (Bollen & Stine, 1993,Langeheine-1996)

2. Goodness of Fit test Based on Empirical Distribution Functions

2.1 The Kolmogorov-Smirnov test modified by Lilliefors and Stephens

Kolmogorov and Smirnov (1933,1948) developed a one sample goodness of fit test based on empirical distribution function(EDF). Kolmogorov-Smirnov(K-S) statistic is popular, although other EDF based statistics such as the Cramer-von-Mises(C-vM) and Anderson –Darling(A-D) statistics have better sensitivity for some data-model differences. However, the goodness of fit probabilities derived from the K-S or other EDF statistics are usually not correct when applied in model fitting situations with estimated parameters.

K-S statistic is no longer distribution –free if some parameters are estimated from the data set under consideration. The K-S probabilities are only valid if the model being tested is derived independently of the data set at hand.

Lilliefors (1967) proposed a modification of Kolmogorov-Smirnov test for normality when the mean and the variance are unknown , and must be estimated from the data. The test statistic K-S is defined as

$$KS = \max_{1 \le i \le n} \left[\Phi(x_i;\bar{x};s^2) - \frac{(i-1)}{n}; \frac{i}{n} - \Phi(x_i;\bar{x},s^2) \right] \qquad (2.1)$$

Where $\Phi(x_i;\bar{x},s^2)$ is the cumulative distribution function of the normal distribution with parameters estimated from data. The normality hypothesis of the data is then rejected for large values of K-S. Table of percentage points are found in Lilliefors(1967). It can also be obtain give by Stephens(1969). Modification of K-S statistic given by Stephens(1969) from the Lilliefors form is as follows;

$$KS^* = KS(\sqrt{n} - 0.01 + 0.85/\sqrt{n}) \qquad (2.2)$$

Comparing with the upper tail significance points of the distribution on the null hypothesis; may be reject the null hypothesis if value of KS* exceeds the table value at corresponding significance levels. Table of percentage point is available in Stephens(1969).

2.2. The Anderson- Darling test

Anderson and Darling(1952,1954) introduced a new class of quadratic a test statistics . These are given by

$$Q_n(\psi) = n\int_{-\infty}^{\infty}[F_n(x)-\Phi(x)]^2\psi(x)dF(x)$$

Where $F_n(x)$ is empirical distribution function(EDF) , $\Phi(x)$ is the cumulative distribution function of the standard normal distribution and $\psi(x)$ is a weight given by

$[\Phi(x).(1-\Phi(x))]^{-1}$. It can be seen from Anderson-Darling(1954) that AD can be written as

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[In\Phi(z_{(i)}) + In(1-\Phi(z_{(n+1-i)})] \qquad (2.3)$$

Where $z_{(i)} = (x_{(i)} - \bar{x})/s$. In order to increase its power when μ and σ are estimated from the sample, a modification factor has proposed for AD by Stephens(1974) resulting in new statistic AD*:

$$AD^* = AD\left(1+\frac{0.75}{n}+\frac{2.25}{n^2}\right) \qquad (2.4)$$

The normality hypothesis of the data is then rejected for large values of the test statistic.

Table of percentage points of this statistic is given by D'Agostino(1986).

3. Goodness fit test Based on Bootstrap Resampling:

Fortunately, there is an alternative to the erroneous use of K-S procedure, although it require a numerically intensive calculation for each data set and model addressed. It is based on bootstrap resampling, a data-based Monte Carlo method that has been mathematically shown to give valid estimates of goodness of fit probabilities under a very wide range of situations.

We now outline the mathematics underlying bootstrap calculations. Let $\{F(. ; \theta) : \theta \in \Theta\}$ be a family of continuous distributions parameterized by θ. We want to test whether the univariate data set $X_1, X_2, \ldots, X_n$ comes from $F = F(. ;\theta)$ for some $\theta = \theta_0$. The K-S C-vM and A-D statistics (and a few other goodness of fit tests) are continuous functional of the process. $Yn(x; \hat{\theta}_n) = \sqrt{n}(F_n(x) - F(x;\hat{\theta}_n))$. Here F_n denotes the EDF of $X_1, X_2, \ldots, X_n$, $\hat{\theta}_n = (X_1, X_2, \ldots, X_n)$ is an estimator of θ derived from the dataset, and $F(x;\hat{\theta}_n)$ is the model being tested. For a simple example , if $\{F(. ; \theta) : \theta \in \Theta\}$ denotes the Gaussian family with $\theta = (\mu, \sigma^2)$, then $\hat{\theta}_n$ can be taken as $(\bar{X}_n, s_n^2)$ where $\bar{X}_n$ is the sample mean and s_n^2 is the sample variance based on the data $X_1, X_2, \ldots, X_n$.

In the case of evaluating goodness of fit for a model where the parameters have been estimated from the data, the bootstrap can be computed in two different ways: the parametric bootstrap and the nonparametric bootstrap. The parametric bootstrap may be familiar to a well established technique of creating fake datasets realizing the parametric model by Monte Carlo realizations of the observed EDF using a "random selection with replacement" procedure.

We now outline the mathematics underlying these techniques. Let $\hat{F}_n$ be an estimator of F , based on $X_1, X_2, \ldots, X_n$. In order to bootstrap, we generate data $X_1^*, X_2^*, \ldots, X_n^*$ from the estimated population $\hat{F}_n$ and then construct $\hat{\theta}_n^* = \theta_n(X_1^*, X_2^*, \ldots, X_n^*)$

Using the same functional form. For example, if $F(.;\theta)$ is Gaussian with $\theta = (\mu, \sigma^2)$ and if $\hat{\theta} = (\bar{X}, s_n^2)$, then $\hat{\theta}_n^*(\bar{X}_n^*, s_n^{*2})$.

3.1 Parametric Bootstrap

The bootstrapping procedure is called parametric if $\hat{F} = F(.;\hat{\theta}_n)$; that is, we generate data $X_1^*, X_2^*, \ldots, X_n^*$ from the model assuming the estimated parameter values $\hat{\theta}_n$. The process $Y_n^P(x) = \sqrt{n}(F_n^*(x) - F(x;\hat{\theta}_n^*))$ and the sample process $Y_n(x) - \sqrt{n}(F_n(x) - F(x;\hat{\theta}_n))$ converges to the same Gaussian process Y. Consequently , Ln = $\sqrt{n}\sup_x|F_n(x) - F(x;\hat{\theta})|$ and $L_n^* = \sqrt{n}\sup_x|F_n^*(x) - F(x;\hat{\theta}_n^*|$ have the same limiting distribution. For the K-S statistic, the critical values of L_n can be derived as follows: construct B resample based on the parametric model (B $\approx$ 1000) , arrange the the resulting L_n^* values in increasing order to obtain 90 or 99 percentile points for getting 90% or 99% critical values. This procedure replaces the incorrect use of the standard probability distribution.

3.2 Nonparametric Bootstrap

The nonparametric bootstrap involving resample from the EDF;

$$Y_n^N(x) = \sqrt{n}(F_n^*(x) - F(x;\hat{\theta}_n^*)) - B_n(x)$$

$$= \sqrt{n}(F_n^*(x) - F_n(x) - +F(x;\hat{\theta}_n) - F(x;\hat{\theta}_n^*))$$

Is operationally easy to perform but requires an additional step of bias correction

$$B_n(x) = \sqrt{n}(F_n(x) - F(x;\hat{\theta}_n))$$

The sample process Y_n and the bias corrected nonparametric process Y_n^N converges to the same Gaussian process Y. That is L_n = $\sqrt{n}\sup_x|F_n(x) - F(x;\hat{\theta})|$ and

$J_n^* = \sup_x \left| \sqrt{n} F_n^*(x) - F(x;\hat{\theta}_n^*)) - B_n(x) \right|$ have the same limiting distribution. The critical value of the distribution of L_n can then be derived as in the case of parametric bootstrap.

P Values based on Resampling Methods: P-values for goodness-of-fit statistics can be obtained by generating the empirical sampling distribution of goodness-of-fit statistics using a resampling method such as the parametric bootstrap method. However, there is strong evidence that parametric bootstrap procedures do not yield accurate P-values. Furthermore, resampling methods may be very time consuming if the researchers is interested in computing the fit of several models.

4. Simulation Study

A simulation study is presented in the following to estimate the level and power of the selected normality tests. The effects on the power of the tests due to the sample size, the selected significance level and the type of the alternative distribution are shown with the help of simulation method. The study carried out for seven (n = 10,15,20,25 ,50 and 100)sample sizes and considering significance levels 0.10,0.05 and 0.01 (for 1 percent level not shown in table due to space) considering the alternative of nonnormal symmetric and asymmetric . Results obtained are shown in different tables given below. Here, normal observations are generate using Box-Muller(1958) formula and for the other distribution method of inverse integral transformation are used. For each result 10,000 repetitions are made. The ratio of number of test statistic value greater than critical value divided by the total number of repetition gives the empirical level of test statistic under null case and power of the test statistic under the alternative hypothesis .

Table 1. Empirical levels of test under normal(0,1) Distribution at 0.05 and 0.10 levels

Sample Size n	K-S α=.10	K-S .05	AD .10	AD .05	KS(Bootstrap) .10	KS(Bootstrap) .05	AD(Bootstrap) .10	AD(Bootstrap) .05
10	.0997	.0467	.1025	.0521	.0945	.0475	.0925	.0500
15	.1036	.0430	.1030	.0527	.0935	.0435	.0900	.0535
20	.1002	.0548	.1043	.0510	.0950	.0480	.1050	.0485
25	.0950	.0494	.0976	.0518	.1010	.0530	.1055	.0520
30	.1086	.0598	.0984	.0526	.1040	.0565	.0980	.0535
50	.0920	.0480	.0960	.0450	.0870	.0380	.0845	.0410
100	.0942	.0492	.0970	.0525	.1100	.0545	.0995	.0495

Table 2 Empirical power of test Normal Vs Cauchy (0,1) Distribution at 0.05 and 0.10 levels

Sample Size n	K-S α=.10	K-S .05	AD .10	AD .05	KS(Bootstrap) .10	KS(Bootstrap) .05	AD(Bootstrap) .10	AD(Bootstrap) .05
10	5274	.4548	.7355	.6580	.5172	.4468	.7243	.4278
15	.5755	.5037	.8495	.7640	.5634	.4836	.8312	.7526
20	.6206	.5478	.9050	.8395	.6124	.5372	.8942	.8245
25	.6512	.5821	.9545	.9070	.6478	.5747	.9456	.8942
30	.6862	.6152	.9740	.9440	.6804	.6124	.9678	.9376
50	.7538	.6836	.9995	.9965	.7512	.6818	.9934	.9834
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3 Empirical power of test Normal Vs logistic (0,1) Distribution at 0.05 and 0.10 levels

Sample Size n	K-S α=.10	K-S .05	AD .10	AD .05	KS(Bootstrap) .10	KS(Bootstrap) .05	AD(Bootstrap) .10	AD(Bootstrap) .05
10	.1415	.0755	.5955	.4710	.1344	.0662	.5726	.4674
15	.1435	.0800	.7060	.5860	.1378	.0752	.6784	.5752
20	.1835	.1080	.8025	.6900	.1778	.0948	.7898	.6856
25	.1885	.1145	.8680	.7685	.1790	.1064	.8594	.7544
30	.2055	.1125	.9135	.8465	.2042	.8356	.9034	.8268
50	.2260	.1370	.9820	.9530	.2212	.1272	.9812	.9510
100	.2590	.1755	1.000	1.000	.2578	.1752	.9910	.9890

Table 4 Empirical power of test Normal Vs Exponential ($\lambda = 1$) Distribution at 0.05 and 0.10 levels

Sample Size n	K-S α=.10	K-S .05	AD .10	AD .05	KS(Bootstrap) .10	KS(Bootstrap) .05	AD(Bootstrap) .10	AD(Bootstrap) .05
10	.5410	.3920	1.000	.9880	.5328	.3841	1.000	.9765
15	.7040	.5310	1.000	1.000	.6878	.5244	1.000	1.000
20	.7920	.7195	1.000	1.000	.7856	.7054	1.000	1.000
25	.8660	.7805	1.000	1.000	.8566	.7784	1.000	1.000
30	.9275	.8745	1.000	1.000	.9128	.8646	1.000	1.000
50	.9900	.9705	1.000	1.000	.9884	.9674	1.000	1.000
100	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5. Empirical power of test Normal Vs lognormal (0,1) Distribution at 0.05 and 0.10 levels

Sample Size n	Test Statistics							
	K-S		AD		KS(Bootstrap)		AD(Bootstrap)	
	α =.10	.05	.10	.05	.10	.05	.10	.05
10	.6880	.5610	1.000	1.000	.6742	.5578	1.000	.9855
15	.8495	.7505	1.000	1.000	.8365	.7455	1.000	1.000
20	.9080	.8650	1.000	1.000	.8884	.8568	1.000	1.000
25	.9520	.9200	1.000	1.000	.9478	.9146	1.000	1.000
30	.9815	.9625	1.000	1.000	.9776	.9568	1.000	1.000
50	1.000	.9985	1.000	1.000	1.000	.9924	1.000	1.000
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

5. Conclusion

We investigate the KS statistics and Anderson-Darling statistics when location and scale parameters are unknown and studied the power of both the exact and bootstrap test against different alternatives, viz. Cauchy ,exponential, lognormal and logistic distribution. The tables presented here lead to the following conclusions.

1. The estimates of powers of exact tests are lightly larger than those of the bootstrap tests in case of K-S test, but it is hard to say which test is more powerful because the estimating are subject to variability. In case of Anderson –Darling test both the exact and bootstrap based test power are almost similar.

2. The difference between the powers of two tests gets smaller as the sample size n increases. When n>10 both tests have very similar powers. This conclusion is also verified by the strong correlation between the exact and bootstrap p-values.

3. Both kinds of tests appear to be unbiased when the sample sizes is large enough.

From the study we may conclude that test based on bootstrap technique be used for goodness of fit of normality without any hesitation. More work can be done using bootstrap technique to know the performance of the tests based on bootstrap with various situations.In future we hope extend more research on this line.

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(73 − age) × ¥3,000

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