REMARKS ON ω-CLOSED SETS IN SUNDARAM-SHEIK JOHN'S SENSE OF DIGITAL N-SPACES

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Abstract. The aim of this paper is to study some topological properties, especially, ω-closed sets (in Sundaram-Sheik John's sense) of digital lines and digital n-spaces ($n \geq 2$).

1 Introduction In 2000, the concept of *ω-closed sets* (in Sundaram-Sheik John's sense) of topological spaces was introduced and investigated by P. Sundaram and M. Sheik John [35] [36] [37] and some results on bitopological version were investigated by [12]. We note that, in 1982, Hdeibe [14] had defined the same named concept: *ω-closed sets* (e.g., [14]); but their definitions are different. Throughout the present paper, we call the ω-closed sets [35] *the ω-closed sets in Sundaram-Sheik John's sense* (cf. Definition 2.1). The concept of Λ_s-sets was introduced and investigated by [4]. In the present paper, for the digital n-space $(\mathbb{Z}^n, \kappa^n)(n \geq 1)$, we try to investigate properties on ω-closed sets in Sundaram-Sheik John's sense and Λ_s-sets. The concept of the digital line $(\mathbb{Z}, \kappa)$ is initiated by Khalimsky [15], [16] and sometimes it is called the *Khalimsky line* (cf. [17] and references there, [33], [19, p.905], [20, p.175]; e.g., [11], [18]). We reference the naming of the *digital n-space* $(\mathbb{Z}^n, \kappa^n)$ in [20, Definition 4]; $(\mathbb{Z}^n, \kappa^n)$ is the topological product of n copies of the digital line $(\mathbb{Z}, \kappa)$ (cf. Section 3).

The purpose of the present paper is to characterlize the ω-closedness in Sundaram-Sheik John's sense in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6). Namely, a subset A is an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ if and only if A is closed in $(\mathbb{Z}^n, \kappa^n)$ (Theorem 4.6). In order to prove the result, we investigate the concept of semi-kernels of subsets in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.5) after checking on some examples in $(\mathbb{Z}^n, \kappa^n)$ (cf. Example 4.2). In Section 2 we recall some definitions and properties on topological spaces which are used in the present paper; moreover in Section 3 we recall the definitions of the digital lines and digital n-spaces ($n \geq 2$) and we give a short survey of important properties which are used in the present paper. In Section 4 we give some examples and we prove a characterization of ω-closed sets in Sundaram-Sheik John's sense for $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6). In order to prove Theorem 4.6, we need the construction of semi-open sets containing a point of $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.4). In the end of Section 4, using Theorem 4.4 and Theorem 4.9, we give an alternative and direct proof of [30, Theorem 4.2] which shows $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2.

Throughout the present paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned.

2 Preliminaries We recall some concepts and properties on topological spaces.

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Definition 2.1 (i) ([22, Definition 2.1]) A subset A of a topological space (X, τ) is called *generalized closed* (shortly, g-closed) in (X, τ) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).

(ii) ([35], [36]) A subset A of a topological space (X, τ) is called *ω-closed in Sundaram-Sheik John's sense* in (X, τ) if $\mathrm{Cl}(A) \subset V$ whenever $A \subset V$ and V is semi-open in (X, τ). The complement of an ω-closed set is called an *ω-open set*.

A subset B of (X, τ) is said to be *semi-open* [21, Definition 1] in (X, τ), if there exists an open set U such that $U \subset B \subset \mathrm{Cl}(U)$. It is shown that [21, Theorem 1] a subset B is semi-open if and only if $B \subset \mathrm{Cl}(\mathrm{Int}(B))$ in (X, τ). A subset E of (X, τ) is said to be *preopen* [25] in (X, τ), if $E \subset \mathrm{Int}(\mathrm{Cl}(E))$ holds in (X, τ). Every open set is semi-open and preopen in (X, τ). The complement of a semi-open set (resp. preopen set) is said to be *semi-closed* (resp. *preclosed*). In the present paper, the famly of all semi-open sets (resp. preopen sets) of (X, τ) is denoted by $\mathrm{SO}(X, \tau)$ (resp. $\mathrm{PO}(X, \tau)$). Namely, for a topological space (X, τ), as notation,

- $SO(X, \tau) := \{B | B \subset \mathrm{Cl}(\mathrm{Int}(B)), B \subset X\}$, $PO(X, \tau) := \{E | E \subset \mathrm{Int}(\mathrm{Cl}(E)), E \subset X\}$; and $\tau \subset SO(X, \tau)$ and $\tau \subset PO(X, \tau)$ hold for any topological space (X, τ).

The following concept of *semi-kernels* is due to [4] and the concept of *kernels* is well known (e.g., [28]).

Definition 2.2 Let E be a subset of a topological space (X, τ).

(i) ([4, Definition 1]) The following set τ-sKer(E) (or shortly sKer(E)) is called a *semi-kernel* of E in (X, τ) (in [4], it is denoted by E^{Λ_s}):

- $\tau\text{-sKer}(E) = E^{\Lambda_s} := \bigcap\{V | E \subset V \text{ and } V \text{ is semi-open in } (X, \tau)\}$.

Note that, in the present paper, we use the symbol τ-sKer(E) or sKer(E).

(ii) (e.g., [28]) The following set τ-Ker(E) (or shortly Ker(E)) is called a *kernel of* E in (X, τ):

- $\tau\text{-Ker}(E) := \bigcap\{V | E \subset V \text{ and } V \text{ is open in } (X, \tau)\}$.

Note that, in [28] (resp. [24]), the set τ-Ker(E) above is denoted by $\mathrm{Ker}_\tau(E)$ (resp. $E^\wedge$).

Definition 2.3 ([4, Definition 2]) In a topological space (X, τ), a subset E is a *Λ_s-set* of (X, τ) if $E = E^{\Lambda_s}$ (i.e., $E = \mathrm{sKer}(E)$).

We recall the following property on semi-kernels.

Proposition 2.4 *For a family $\{E_i | i \in \Omega\}$ of subsets of a topological space (X, τ), where Ω is an index set,*

(i) ([4, Proposition 3.1]) $\mathrm{sKer}(\bigcup\{E_i | i \in \Omega\}) = \bigcup\{\mathrm{sKer}(E_i) | i \in \Omega\}$ *holds; and*

(ii) (e.g., [24, (2.5)]) $\mathrm{Ker}(\bigcup\{E_i | i \in \Omega\}) = \bigcup\{\mathrm{Ker}(E_i) | i \in \Omega\}$ *holds.*

Theorem 2.5 *t60* ([35], [36]) *A subset A is ω-closed* (*in Sundaram-Sheik John's sense*) *in a topological space (X, τ) if and only if* $\mathrm{Cl}(A) \subset \mathrm{sKer}(A)$.

Proposition 2.6 (i) ([4, Proposition 3.7]) *A topological space (X, τ) is semi-T_1 if and only if every subset is a Λ_s-set.*

(ii) ([4, Corollary 3.8]) *Every semi-T_1-space is a semi-R_0-space.*

We need the following notation.

Definition 2.7 (e.g., [10, p.166]; [39, Definition 2.1] [38, p.47] for the case where $E := \mathbb{Z}^n$) For a subset E of (X, τ), we define the following subsets E_τ and $E_\mathcal{F}$:

$E_\tau := \{x \in E \mid \{x\} \text{ is open in } (X, \tau), \text{ i.e., } \{x\} \in \tau\}$;

$E_\mathcal{F} := \{x \in E \mid \{x\} \text{ is closed in } (X, \tau)\}$.

3 Preliminaries-2 In the present section, we recall some foundamental definitions and topological properties on digital lines and digital n-spaces ($n \geq 2$); this includes a survey on digital lines and digital n-spaces ($n \geq 2$) on our topics. And the notation of Definition 3.11 and ($*$ 20) in (II) below are used in the proofs of results in Section 4.

(I) (digital lines):

• Let us recall some definitions and topological properties on digital lines (cf. ($*1$) - ($*11$) below).

Definition 3.1 (cf. [20, p.175], [19, p.905, p.908], [26, Section 2], [27, Example 4 in Section 2]; e.g., [11, Section 1], [33, Section 6 in p.9]) *The digital line* or so called *the Khalimsky line* $(\mathbb{Z}, \kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase.

Remark 3.2 We put $\mathcal{G} := \{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ in Definition 3.1.

(i) By the definition of κ, a subset U of $\mathbb{Z}$ is open in $(\mathbb{Z}, \kappa)$ (i.e., $U \in \kappa$) if and only if there exists a family of subsets of $(\mathbb{Z}, \kappa)$, say $\{B_i^{(U)} |\ i \in I^{(U)}\}$, where $I^{(U)}$ is an index set, such that $U = \bigcup\{B_i^{(U)} |\ i \in I^{(U)}\}$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ for some positive integer m and some subsets $V_j^{(i)} \in \mathcal{G} (1 \leq j \leq m)$, here we assume that $V_j^{(i)} \neq V_{j_1}^{(i)}$ if $j \neq j_1$, where $j, j_1 \in \{1, 2, ..., m\}$).

(ii) For the set $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ above, we note that:

$(*)_1$ if $m = 1$ (resp. $m = 2$), then $B_i^{(U)} = \{2t-1, 2t, 2t+1\}$ (resp. $=\{2u+1\}$ or $\emptyset$) for some $t \in \mathbb{Z}$ (resp. for some $u \in \mathbb{Z}$);

$(*)_2$ if $m \geq 3$, then $B_i^{(U)} = \bigcap\{V_j^{(i)} | j \in \{1, 2, ..., m\}\} = \emptyset$.

• For examples, we first have some properties on singletons and two-pointed sets of $(\mathbb{Z}, \kappa)$ (cf. ($*1$) - ($*3$) below): for an integer s,

· ($*1$) *a singleton* $\{2s+1\}$ *is open in* $(\mathbb{Z}, \kappa)$*;* $\{2s+1\}$ *is not closed in* $(\mathbb{Z}, \kappa)$.

· ($*2$) *a singleton* $\{2s\}$ *is not open in* $(\mathbb{Z}, \kappa)$*; but* $\{2s\}$ *is closed in* $(\mathbb{Z}, \kappa)$.

· ($*3$) *subsets* $\{2s, 2s+1\}$ *and* $\{2s-1, 2s\}$ *are not open in* $(\mathbb{Z}, \kappa)$*, where* $s \in \mathbb{Z}$ (cf. ($*8$)(iii) below).

(*Proof of* ($*1$)). (*Proof of the openness*) It is shown that $\{2s+1\} = V_1 \cap V_2$, where $V_1 := \{2s-1, 2s, 2s+1\} \in \mathcal{G}$ and $V_2 := \{2s+1, 2s+2, 2s+3\} \in \mathcal{G}$. Thus, $\{2s+1\}$ is open in $(\mathbb{Z}, \kappa)$.

(*Proof of the non-closedness*) Suppose that $\{2s+1\}$ is closed. Put $U := \mathbb{Z} \setminus \{2s+1\}$. Then, $U \in \kappa$ and so there exists a family of subsets: $\{B_i^{(U)} |\ i \in I^{(U)}\}$, where $I^{(U)}$ is an index set, such that $U = \bigcup\{B_i^{(U)} |\ i \in I^{(U)}\}$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ for some positive integer m and some subsets $V_j^{(i)} \in \mathcal{G} (1 \leq j \leq m)$ (cf. Definition 3.1,Remark 3.2(i)). Pick a point $2s \in U$, where $s \in \mathbb{Z}$. Then, we have

$(*)_a$ $2s \in B_{i'}^{(U)} = \bigcap\{V_j^{(i')} | j \in \{1, 2, ..., m'\}\}$ and $B_{i'}^{(U)} \subset U$ for some $i' \in I^{(U)}$ and positive integer m'.

By Remark 3.2(ii), it is shown that $m' = 1$ and $B_{i'}^{(U)} = \bigcap\{V_j^{(i')} | j \in \{1, 2, ..., m'\}\}$ $=\{2s-1, 2s, 2s+1\}$. Thus, using $(*)_a$, we have $2s+1 \in U$; but this contradicts the definition of U in the first setting. Therefore, the singleton $\{2s+1\}$ is not closed in $(\mathbb{Z}, \kappa)$. ($\circ$)

(*Proof of* ($*2$)). (*Proof of the non-openness*). Suppose that $\{2s\} \in \kappa$. We put $U := \{2s\}$. By the definition of κ (cf. Remark 3.2(i)), there exists subsets $B_i^{(U)} (i \in I^{(U)})$, where $I^{(U)}$ is an index set, such that $2s \in B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ and $B_i^{(U)} \subset U$ for some positive integer m and $V_j^{(i)} \in \mathcal{G} (1 \leq j \leq m)$. By using Remark 3.2(ii), it is shown that $m = 1$ and $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\} = \{2s-1, 2s, 2s+1\} \subset U$; and so $2s+1 \in U$.

This contradicts the definition of $U := \{2s\}$. Therefore, any singleton $\{2s\}$ is not open in $(\mathbb{Z}, \kappa)$.
(*Proof of the closedness*). It is shown that $\{2s\} = \mathbb{Z} \setminus E$, where $E := \bigcup\{\{2s-2j-1, 2s-2j, 2s-2j+1\} | j \in \mathbb{Z}$ and $j \neq 0\}$. Since $E \in \kappa$, $\mathbb{Z} \setminus E$ is closed; and so $\{2s\}$ is closed in $(\mathbb{Z}, \kappa)$. $(\circ)$

(*Proof of* $(*3)$) Suppose that $\{2s-1, 2s\} \in \kappa$. Then, we have a contradiction. Put $U := \{2s-1, 2s\}$. By Definition 3.1 (cf. Remark 3.2 (i)), there exists an index set $I^{(U)}$ and some subsets $B_i^{(U)}$ such that $U = \bigcup\{B_i^{(U)} |\ i \in I^{(U)}\}$, where $B_i^{(U)} = \bigcap\{V_j^{(i)} |\ j \in \{1, 2, ..., m\}\}$ for some positive integer m and $V_j^{(i)} \in \mathcal{G}(1 \leq j \leq m)$ (cf. Remark 3.2). It is noted that $B_k^{(U)} \subset U$ for any $k \in I^{(U)}$. Then, we have:
$(*)^a$ $2s \in B_a^{(U)}$ for some $a \in I^{(U)}$; $(*)^b$ $2s-1 \in B_b^{(U)}$ for some $b \in I^{(U)}$;
$(*)^c$ $B_a^{(U)} \cup B_b^{(U)} \subset U$, where $U := \{2s-1, 2s\}$.
Using $(*)^a$, $(*)^b$ and $(*)^c$, we have: $(*)^d$ $U = B_a^{(U)} \cup B_b^{(U)}$.
Using Remark 3.2(ii), $(*)^a$ and $(*)^b$ above, we have $B_a^{(U)} = \{2s-1, 2s, 2s+1\}$ and $B_b^{(U)} = \{2s-1\}, \{2s-1, 2s, 2s+1\}$ or $\{2s-3, 2s-2, 2s-1\}$. Thus, using $(*)^d$ above, we have $U = \{2s-1, 2s, 2s+1\}$ or $U = \{2s-3, 2s-2, 2s-1, 2s, 2s+1\}$. These properties above contradict the defininion of $U = \{2s-1, 2s\}$. Therefore, $\{2s-1, 2s\}$ is not open in $(\mathbb{Z}, \kappa)$. Similarly, it is proved that $\{2s+1, 2s\}$ is not open in $(\mathbb{Z}, \kappa)$. In $(*8)$(iii) below, we note that they are semi-open in $(\mathbb{Z}, \kappa)$. $(\circ)$
• For the digital line $(\mathbb{Z}, \kappa)$, the concept of *the smallest open set, say* $U(x)$*, containing a point* x *of* $(\mathbb{Z}, \kappa)$ is very important; throughout the present paper, we put:
$\cdot$ $U(2s) := \{2s-1, 2s, 2s+1\}$; $\cdot$ $U(2s+1) := \{2s+1\}$, where $s \in \mathbb{Z}$.
We first recall the definition of *the smallest open set containing a point* x for a topological space (X, τ).

Definition 3.3 (e.g., [29, Definition 2.4]) Let (X, τ) be a topological space and a point $x \in X$. A subset E is called *the smallest open set containing* x if $x \in E, E \in \tau$ and $A = E$ holds for any open set A such that $x \in A$ and $A \subset E$.

For an open set E and $x \in E, E$ is the smallest open set containing x if and only if $E \subset G$ holds for every open set G containing the point x (e.g., [29, Remark 2.5 (ii)]).
• For the digital line $(\mathbb{Z}, \kappa)$, we recall the concept of *the smallest open set, say* $U(x)$*, containing a point* x *of* $(\mathbb{Z}, \kappa)$. Obviously, every subset belonging to $\mathcal{G} =: \{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ is open in $(\mathbb{Z}, \kappa)$. Then, we have the following important property on $U(x)$, where $x \in \mathbb{Z}$:
$\cdot$**$(*4)$ (i)** *$U(2s) := \{2s-1, 2s, 2s+1\}$ is the smallest open set containing $2s$. Namely, $U(2s)$ is an open set containing the point $2s$ and if A is an any open set such that $2s \in A$ and $A \subset U(2s)$, then $A = U(2s)$. And, if G is any open set containing $2s$ in $(\mathbb{Z}, \kappa)$, then $U(2s) \subset G$.*

(ii) *$U(2s+1) := \{2s+1\}$ is the smallest open set containing $2s+1$.*

(iii) *For each point x of $(\mathbb{Z}, \kappa)$, there exists the smallest open set $U(x)$ containing the point x* (cf. [20, p.175]). *Namely, for the point $x \in \mathbb{Z}$, $U(x)$ is an open set containing the point x and if A is an any open set such that $x \in A$ and $A \subset U(x)$, then $A = U(x)$. And, if G is any open set containing x in $(\mathbb{Z}, \kappa)$, then $U(x) \subset G$.*

(*Proof of* $(*4)$). **(i)** By $(*2)$ and $(*3)$ above, it is shown that:
$(*^e)$ $U(2s)$ is open in $(\mathbb{Z}, \kappa)$ and $2s \in U(2s)$ (because of $U(2s) \in \mathcal{G}$); and
if A is any open subset of $U(2s)$ such that $2s \in A$, then $A = U(2s)$.
Indeed, if $A_1 \subset U(2s)$ such that $2s \in A_1$ and $A_1 \neq U(2s)$, then $A_1 = \{2s\}, \{2s-1, 2s\}$ or $\{2s, 2s+1\}$ and the subset A_1 is not open in $(\mathbb{Z}, \kappa)$ (cf. $(*\ 2)$, $(*\ 3)$ above). Thus, we have

$A = U(2s)$ for any open subset A such that $2s \in A$ and $A \subset U(2s)$. Moreover, we show: $(*^f)$ $U(2s) \subset G$ holds for any open set G containing the point $2s$ and $2s \in U(2s)$. (Indeed, let G be any open set containing the point $2s$. Then, we have $2s \in U(2s) \cap G$ and $U(2s) \cap G$ is an open set such that $U(2s) \cap G \subset U(2s)$; thus we have $U(2s) \cap G = U(2s)$ (cf. $(*^e)$ above). Namely, we have $U(2s) \subset G$.)

Therefore, by $(*^e)$ or $(*^f)$, it is shown that $U(2s)$ is the smallest open set containing $2s$ (cf. Definition 3.3).

(ii) For an odd integer $2s+1$, where $s \in \mathbb{Z}, U(2s+1) = \{2s+1\}$ is the smallest open set containing the point $2s+1$ (cf. $(*1)$). **(iii)** Using (i) and (ii) above, the set $U(x)$ is the smallest open set containing the point x. $(\circ)$

• We have the form of the *κ-closure* of $\{x\}$, the *κ-interior* of $\{x\}$ and the *κ-kernel* of $\{x\}$, respectively, (cf. $(*5)$, $(*6)$ below): for an integer s,

$\cdot$ **$(*5)$ (i)** $\kappa\text{-Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$, $\kappa\text{-Cl}(\{2s\}) = \{2s\}$;

(ii) $\kappa\text{-Int}(\{2s+1\}) = \{2s+1\}; \kappa\text{-Int}(\{2s\}) = \emptyset$;

(iii) $\kappa\text{-Ker}(\{2s+1\}) = \{2s+1\}; \kappa\text{-Ker}(\{2s\}) = \{2s-1, 2s, 2s+1\} = U(2s)$.

(*Proof of* $(*5)$). **(i)** They are shown by $(*4)$(i), $(*1)$ and $(*2)$ above, respectively. **(ii)** They are shown by $(*1)$ and $(*2)$ above, respectively. **(iii)** They are shown by $(*1)$ and $(*4)$(i) above. $(\circ)$

$\cdot$ **$(*6)$(i)** *In the digital line $(\mathbb{Z}, \kappa)$, a singleton $\{x\}$ is open if and only if the integer x is odd in $\mathbb{Z}$.*

(ii) *A singleton $\{x\}$ is closed in $(\mathbb{Z}, \kappa)$ if and only if the integer x is even in $\mathbb{Z}$.*

(*Proof of* $(*6)$) **(i)**. It is shown by $(*5)$(ii) above. **(ii)** By the closure form in $(*5)$(i) above, (ii) is shown. $(\circ)$

By $(*6)$ above, it is shown that:

$\cdot$ $(*7)$ **(i)** *Every singleton of $(\mathbb{Z}, \kappa)$ is open or closed* (cf. $(*6)$; or $(*1)$ and $(*2)$ above). *This shows that $(\mathbb{Z}, \kappa)$ is $T_{1/2}$* (e.g., [8, Example 4.6]; cf. [22, Definition 5.1], [9, Theorem 2.5]). We recall some topological properties; in general, the class of $T_{1/2}$-spaces is properly placed between the classes of T_0-spaces and T_1-spaces ([22, Corollary 5.6]). Furthermore, Dontchev and Ganster [8, Example 4.6] proved that $(\mathbb{Z}, \kappa)$ *is* $T_{3/4}$; in general, the class of $T_{3/4}$-spaces is properly placed between the classes of T_1-spaces and $T_{1/2}$-spaces ([8, Corollary 4.4 and Corollary 4.7]). For the digital plane $(\mathbb{Z}^2, \kappa^2)$ (cf. Definition 3.4 below), it is well known that $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$ ([26, Section 3]).

• We recall the *semi-openness* (resp. *semi-closedness*) (cf. Section 2) of singletons in $(\mathbb{Z}, \kappa)$ and the *semi-closure* of $\{x\}$, the *semi-interor* of $\{x\}$ and the *semi-kernel* (cf. Definition 2.2(i)) of $\{x\}$ (cf. $(*8)$ and $(*9)$ below): for an integer s,

$\cdot$**$(*8)$(i)** *every open singleton $\{2s+1\}$ is semi-open and semi-closed in $(\mathbb{Z}, \kappa)$;*

(ii) *every closed singleton $\{2s\}$ is semi-closed in $(\mathbb{Z}, \kappa)$; but $\{2s\}$ is not semi-open in $(\mathbb{Z}, \kappa)$;*

(iii) *the subsets $\{2s, 2s+1\}$ and $\{2s-1, 2s\}$ are semi-open on $(\mathbb{Z}, \kappa)$.*

(*Proof of* $(*8)$). **(i)** Every open set is semi-open and so $\{2s+1\}$ is semi-open in $(\mathbb{Z}, \kappa)$ (cf. $(*6)$(i) above). And, since $\kappa\text{-Int}(\kappa\text{-Cl}(\{2s+1\}))= \kappa\text{-int}(\{2s, 2s+1, 2s+2\}) = \{2s+1\}$ hold, $\{2s+1\}$ is semi-closed (cf. $(*5)$(i)(ii) above). **(ii)** Since $\kappa\text{-Int}(\kappa\text{-Cl}(\{2s\})) = \kappa\text{-Int}(\{2s\}) = \emptyset \subset \{2s\}, \{2s\}$ is semi-closed in $(\mathbb{Z}, \kappa)$. And, we have $\text{Cl}(\text{Int}(\{2s\})) = \text{Cl}(\emptyset) = \emptyset \not\supset \{2s\}$ and so $\{2s\}$ is not semi-open in $(\mathbb{Z}, \kappa)$. **(iii)** It is easily shown that $\kappa\text{-Cl}(\kappa\text{-Int}(\{2s, 2s+1\})) = \kappa\text{-Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\} \supset \{2s, 2s+1\}$; and so $\{2s, 2s+1\}$ is semi-open in $(\mathbb{Z}, \kappa)$. Similarly, the subset $\{2s-1, 2s\}$ is semi-open in $(\mathbb{Z}, \kappa)$. $(\circ)$

$\cdot$**$(*9)$** For an integer s, we have the following properties:

(i) $\kappa\text{-sCl}(\{2s+1\}) = \{2s+1\}$; $\kappa\text{-sCl}(\{2s\}) = \{2s\}$;

(ii) κ-sInt($\{2s+1\}$) $= \{2s+1\}$*;* κ-sInt($\{2s\}$) $= \emptyset$;

(iii) κ-sKer($\{2s+1\}$) $= \{2s+1\}$; κ-sKer($\{2s\}$) $= \{2s\}$.

(*Proof of* $(*9)$). **(i)** (resp. **(ii)**) They are proved by $(*8)$(i) (resp. $(*8)$(ii)) above. **(iii)** By $(*8)$(iii) (resp. $(*8)$(i)), it is obtained that κ-sKer($\{2s\}$) $= \{2s, 2s+1\} \cap \{2s-1, 2s\} = \{2s\}$ (resp. κ-sKer($\{2s+1\}$) $= \{2s+1\}$). $(\circ)$

• We recall more topological properties on $(\mathbb{Z}, \kappa)$:

· **$(*10)$ (i)** *For* $(\mathbb{Z}, \kappa)$*,* $\kappa = PO(\mathbb{Z}, \kappa), PO(\mathbb{Z}, \kappa) \subset SO(\mathbb{Z}, \kappa)$ *and* $\kappa^\alpha = \kappa$ *hold* ([10, Theorem 2.1 (i)(a)(b)]), where $\kappa^\alpha := \{V | \ V$ is α-open in $(\mathbb{Z}, \kappa)\}$. For topological spaces, the concept of the *α-open set* was introduced by Njåstad [31] who called it *the α-set*. A subset A of a topological space (X, τ) is said to be *α-open* in (X, τ) if $A \subset$Int(Cl(Int(A))) holds.

(ii) *The digital line* $(\mathbb{Z}, \kappa)$ *is submaximal*. This fact may be known in folklore; however, we are able to read one of the proof ([10, Theorem 1.1(i)]). Furthermore, it is noted that, by [10, Theorem 1.1(ii)(iii)], the digital plane $(\mathbb{Z}^2, \kappa^2)$ (cf. **(II)** below) is not submaximal but it is *quasi-submaximal*. Al-Nashef [1, Definition 3.2] introduced the concept of *quasi-submaximal* topological spaces which is weaker than one of submaximal spaces (e.g., [3, Definition 1.1], [13, p.137]).

(iii) *The digital line* $(\mathbb{Z}, \kappa)$ *is s-normal* ([11, Section 3, Theorem B]). In 1978, Maheshwari and Prasad [23] introduced the concept of s-normal topological spaces using semi-open sets. The digital plane is also a geometric example of s-normal spaces ([11, Section 5, Theorem D]).

• Using Definition 2.7 for $(X, \tau) = (\mathbb{Z}, \kappa)$, we can define the following subsets $\mathbb{Z}_\kappa := \{x \in \mathbb{Z} \mid \{x\} \in \kappa\}, \mathbb{Z}_\mathcal{F} := \{x \in \mathbb{Z} \mid \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$; for a nonempty subset E of $(\mathbb{Z}, \kappa)$, $E_\kappa := \{x \in E | \ \{x\} \in \kappa\}$ and $E_\mathcal{F} := \{x \in E | \ \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$.

· **$(*11)$ (i)** Let $A \subset \mathbb{Z}$. Then we have that $\mathbb{Z}_\kappa = \{2m+1 \in \mathbb{Z} \mid m \in \mathbb{Z}\}$; $A_\kappa = \{2m+1 \in A \mid m \in \mathbb{Z}\}$ (cf. $(*6)$(i) above);
$\mathbb{Z}_\mathcal{F} = \{2m \in \mathbb{Z} | \ m \in \mathbb{Z}\}$; $A_\mathcal{F} = \{2m \in A \mid m \in \mathbb{Z}\}$ (cf. $(*6)$(ii) above).

(ii) A_κ *is open in* $(\mathbb{Z}, \kappa)$ *for any subset* A *of* $(\mathbb{Z}, \kappa)$*; and* $A_\kappa = \mathbb{Z}_\kappa \cap A$.

(iii) $\mathbb{Z} = \mathbb{Z}_\kappa \cup \mathbb{Z}_\mathcal{F}$ $(\mathbb{Z}_\kappa \cap \mathbb{Z}_\mathcal{F} = \emptyset)$ *and* $A = A_\kappa \cup A_\mathcal{F}$ $(A_\kappa \cap A_\mathcal{F} = \emptyset)$ *for any subset* A *of* $(\mathbb{Z}, \kappa)$ (cf. $(*6)$ above).

(iv) *For any subset* A *of* $(\mathbb{Z}, \kappa)$*,* $A_\mathcal{F} = A \setminus A_\kappa$ *holds and* $A_\mathcal{F}$ *is closed in* $(\mathbb{Z}, \kappa)$*; and* $A_\mathcal{F} = \mathbb{Z}_\mathcal{F} \cap A$.

(v) If $E \subset F \subset \mathbb{Z}$, then $E_\kappa \subset F_\kappa$ and $E_\mathcal{F} \subset F_\mathcal{F}$ hold in $(\mathbb{Z}, \kappa)$.

(*Proof of* $(*11)$) **(iv)**. (*Proof of the closedness of* $A_\mathcal{F}$). Let $x \in \mathbb{Z} \setminus A_\mathcal{F}$.

Case 1. $x = 2s+1$, where $s \in \mathbb{Z}$: for this case, we have $x \in \mathbb{Z}_\kappa$ (cf. $(*6)$(i) above); and so $\{x\} \cap A_\mathcal{F} = \emptyset$ (cf. (iii) above). Thus, there exists an open set $\{x\}$, say U_x, containing x such that $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$.

Case 2. $x = 2t$, where $t \in \mathbb{Z}$: for this case, we have $x \in \mathbb{Z}_\mathcal{F}$ and $x \notin A_\mathcal{F}$ (cf. (iii) above and $(*6)$(ii) above). Hence, for the point $x \in \mathbb{Z}_\mathcal{F} \setminus A_\mathcal{F}$, there exists an open set $\{x-1, x, x+1\}$, say U_x, containing x and $\{x-1, x+1\} \subset \mathbb{Z}_\kappa$; and so $U_x \cap A_\mathcal{F} = \{x-1, x, x+1\} \cap A_\mathcal{F} = \emptyset$, i.e., $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$.

Thus, for each point $x \in \mathbb{Z} \setminus A_\mathcal{F}$, the subset U_x above is an open set containing x such that $U_x \subset \mathbb{Z} \setminus A_\mathcal{F}$. We have $\mathbb{Z} \setminus A_\mathcal{F} = \bigcup\{U_x | x \in \mathbb{Z} \setminus A_\mathcal{F}\}$ and so $\mathbb{Z} \setminus A_\mathcal{F} \in \kappa$. Namely, $A_\mathcal{F}$ is closed in $(\mathbb{Z}, \kappa)$. $(\circ)$

(II) (digital n-spaces $(n \geq 2)$**)**:

• In the final stage of the present section, we recall some structures of the digital n-space $(n \geq 2)$ ([20, Definition 4]; e.g., [26, Section 3], [39], [38], [11]; for $n = 2$, [10], [5, Section 6], [34, Section 5], [7, Section 7], [6], [32, Section 6]).

Definition 3.4 ([20, Definition 4]) Let n be an integer with $n \geq 2$. *The digital n-space* or *Khalimsky n-space* is the Cartesian product of n-copies of the digital line $(\mathbb{Z}, \kappa)$. This topological space is denoted by $(\mathbb{Z}^n, \kappa^n)$, where $\mathbb{Z}^n := \prod_{i=1}^n X_i$, where $X_i = \mathbb{Z}$ for all integers i with $1 \leq i \leq n$, and $\kappa^n := \prod_{i=1}^n \tau_i$, where $\tau_i := \kappa$ for all integers i with $1 \leq i \leq n$. For $n = 2$, $(\mathbb{Z}^2, \kappa^2)$ is called *the digital plane* or *Khalimsky plane.*

Since κ^n is the product topology of n-copies of κ, it is shown that: for a point $x := (x_1, x_2, ..., x_n)$ of $(\mathbb{Z}^n, \kappa^n)$,
· **($*$12)** (a) κ^n-Cl$(\{x\}) = \prod_{i=1}^n \kappa$-Cl$(\{x_i\})$; (b) κ^n-Int$(\{x\}) = \prod_{i=1}^n \kappa$-Int$(\{x_i\})$;
(c) κ^n-Ker$(\{x\}) = \prod_{i=1}^n \kappa$-Ker$(\{x_i\})$.

(*Note on* (c)). Let $(X, \tau) := \prod_{i=1}^n (X_i, \tau_i)$ be a product topological space of topological spaces $(X_i, \tau_i)(1 \leq i \leq n)$. In general, for a point $x := (x_1, x_2, ..., x_n)$ of (X, τ), it is shown that τ-Ker$(\{x\}) = \prod_{i=1}^n (\tau_i$-Ker$(\{x_i\}))$, where $\tau = \prod_{i=1}^n \tau_i$. $\circ$

We use the following well known property; we recall shortly the proof.

Proposition 3.5 *Let $x := (x_1, x_2, ..., x_n)$ be a point of $(\mathbb{Z}^n, \kappa^n)$.*

(i) *If all the coordinates of the point x is odd, say $x_i = 2s_i + 1 \in \mathbb{Z}$ $(s_i \in \mathbb{Z})$ for each integer i with $1 \leq i \leq n$, then for the point $x = (2s_1 + 1, 2s_2 + 1, ..., 2s_n + 1)$*

(a) κ^n-Cl$(\{x\})= \prod_{i=1}^n \{2s_i, 2s_i + 1, 2s_i + 2\}$.

(b) κ^n-Int$(\{x\})= \prod_{i=1}^n \{2s_i + 1\} = \{x\}$*; and so the singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$.*

(c) κ^n-Ker$(\{x\})= \prod_{i=1}^n \{2s_i + 1\} = \{x\}$.

(ii) *If all the coordinates of the point x is even, say $x_i = 2s_i \in \mathbb{Z}$ $(s_i \in \mathbb{Z})$ for each integer i with $1 \leq i \leq n$, then for the point $x = (2s_1, 2s_2, ..., 2s_n)$*

(a) κ^n-Cl$(\{x\})= \prod_{i=1}^n \{2s_i\} = \{x\}$*; and so the singleton $\{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)$.*

(b) κ^n-Int$(\{x\})= \prod_{i=1}^n \emptyset = \emptyset$.

(c) κ^n-Ker$(\{x\})= \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\} = \prod_{i=1}^n U(2s_i)$.

(iii) (a) *A singleton $\{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)$ if and only if all the coordinates of x, say $x_i (1 \leq i \leq n)$, are even.*

(b) *A singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ if and only if all the coordinates of x, say $x_i (1 \leq i \leq n)$, are odd.*

Proof. **(i) (ii)** The properties are shown by ($*$5) in (I), ($*$12) in (II) and definitions.

(iii) (a) (Necessity) It follows from assumption that κ^n-Cl$(\{x\}) = \{x\}$. Using ($*$12)(a) in (II), it is shown that κ-Cl$(\{x_i\}) = \{x_i\}$ for each integer i with $1 \leq i \leq n$. Then, using ($*$6)(ii) in (I), we have that x_i is even for each i with $1 \leq i \leq n$. **(Sufficiency)** It is obtained by (ii)(a) above. **(iii) (b) (Necessity)** By using ($*$12)(b) in (II) and ($*$6)(i) in (I) above, (iii)(b) is proved. **(Sufficiency)** It is obtained by (i)(b) above. $\square$

Example 3.6 (i) Especially, for the case where $n = 2$, we have the following forms of κ^2-closures of singletons: for integers $s, t \in \mathbb{Z}$,

κ^2-Cl$(\{(2s + 1, 2t + 1)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2t, 2t + 1, 2t + 2\}$;

κ^2-Cl$(\{(2s, 2t)\}) = \{(2s, 2t)\}$;

κ^2-Cl$(\{(2s, 2t + 1)\}) = \{2s\} \times \{2t, 2t + 1, 2t + 2\}$;

κ^2-Cl$(\{(2s + 1, 2t)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2t\}$.

(ii) By the following figure, the closure κ^2-Cl$(\{(2s+1, 2t+1)\})$ is illustrated; the singleton $\{(2s+1, 2t+1)\}$ is denoted by a symbol $\circ$ and the closure κ^2-Cl$(\{(2s+1, 2t+1)\})$ contains the 9-points only denoted by the symbols $\circ, \star, \bullet$:

$$\kappa^2\text{-Cl}(\{(2s+1,2t+1)\}) = \quad \text{Cl}(\circ) = \begin{array}{cccc} \bullet & \star & \bullet & \text{2t+2} \\ \star & \circ & \star & \text{2t+1} \\ \bullet & \star & \bullet & \text{2t} \\ \text{2s} & \text{2s+1} & \text{2s+2} & \end{array}$$

(iii) By the following figure, the closures κ^2-Cl$(\{(2s, 2t+1)\})$ is illustrated:

$$\kappa^2\text{-Cl}(\{(2s,2t+1)\}) = \quad \text{Cl}(\star) = \begin{array}{cc} \bullet & \text{2t+2} \\ \star & \text{2t+1} \\ \bullet & \text{2t} \\ \text{2s} & \end{array}$$

(iv) By the following figure, the closure κ^2-Cl$(\{(2s+1, 2t)\})$ is illustrated:

$$\kappa^2\text{-Cl}(\{(2s+1,2t)\}) = \quad \text{Cl}(\star) = \begin{array}{cccc} \bullet & \star & \bullet & \text{2t} \\ \text{2s} & \text{2s+1} & \text{2s+2} & \end{array}$$

We give the concept of the *smallest open set containing a point* of $(\mathbb{Z}^n, \kappa^n)$.

Definition 3.7 (e.g., [39, p.602], [38, p.47], [11, p.47]) For a point $x := (x_1, x_2, ..., x_n)$ of $(\mathbb{Z}^n, \kappa^n)$, the following subset $U^n(x)$ is called *the smallest open set containing the point* x (cf. Theorem 3.9, Definition 3.3):

$U^n(x) := \prod_{i=1}^n U(x_i)$, where $U(x_i)$ is the smallest open set (cf. $(*4)$ in (I)) in $(\mathbb{Z}, \kappa)$ containing the i-th coordinate x_i of $x (1 \leq i \leq n)$.

Example 3.8 (i) For examples, in the case where $n = 2$ of Definition 3.7, we have the following forms $U^2(x)$ for the following points $x \in \mathbb{Z}^2$:
$U^2((2s+1, 2t+1)) = \{(2s+1, 2t+1)\}$;
$U^2((2s, 2t)) = \{2s-1, 2s, 2s+1\} \times \{2t-1, 2t, 2t+1\}$;
$U^2((2s, 2t+1)) = \{2s-1, 2s, 2s+1\} \times \{2t+1\}$ and
$U^2((2s+1, 2t)) = \{2s+1\} \times \{2t-1, 2t, 2t+1\}$.

(ii) In the figure below, a subset $U^2((2s, 2t))$ is illustrated; the singleton $\{(2s, 2t)\}$ is denoted by a symbol $\bullet$ and $U^2((2s, 2t))$ is the set of the 9-points only denoted by the symbols $\bullet, \circ, \star$:

$$U^2((2s,2t)) = \quad U^2(\bullet) = \begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \circ & \star & \circ & \cdot & \text{2t+1} \\ \cdot & \star & \bullet & \star & \cdot & \text{2t} \\ \cdot & \circ & \star & \circ & \cdot & \text{2t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 2s-1 & 2s & 2s+1 & & \end{array}$$

(iii) In the figure below, a subset $U^2((2s, 2t+1))$ is illustrated; the singleton $\{(2s, 2t+1)\}$ is denoted by a symbol $\star$ and $U^2((2s, 2t+1))$ is the set of the 3-points only denoted by the symbols $\circ$ and $\star$:

$$U^2((2s,2t+1)) = \quad U^2(\star) = \begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \circ & \star & \circ & \cdot & \text{2t+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \text{2t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \text{2t-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 2s-1 & 2s & 2s+1 & & \end{array}$$

(iv) In the figure below, a subset $U^2((2s+1, 2t))$ is illustrated; the singleton $\{(2s+1, 2t)\}$ is denoted by a symbol $\star$ and $U^2((2s+1, 2t))$ is the set of the 3-points only denoted by the symbols $\circ$ and $\star$:

$$
U^2((2s+1,2t))= \quad U^2(\star) = \quad
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \circ & \cdot & \text{2t+1} \\
\cdot & \cdot & \cdot & \star & \cdot & \text{2t} \\
\cdot & \cdot & \cdot & \circ & \cdot & \text{2t-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
 & 2s-1 & 2s & 2s+1 & &
\end{array}
$$

The following property is folklore, but we give its proof. The following theorem shows the well definedness of $U^n(x)$ of Definition 3.7.

Theorem 3.9 *Let x be a point of $(\mathbb{Z}^n, \kappa^n)$ and $U^n(x)$ the subset defined by Definition 3.7. Then, we have the following properties.*

(i) *$x \in U^n(x)$ and $U^n(x) \in \kappa^n$.*

(ii) *If A is an open set containing the point x in $(\mathbb{Z}^n, \kappa^n)$ such that $A \subset U^n(x)$, then $A = U^n(x)$.*

(iii) *If G is any open set containing the point x in $(\mathbb{Z}^n, \kappa^n)$, then $U^n(x) \subset G$.*

Proof. We put $x := (x_1, x_2, ..., x_n)$. **(i)** By Definition 3.7, (i) is shown.
(ii) Since $x \in A$ and $A \in \kappa^n$, there exist open sets $A_i \in \kappa (1 \leq i \leq n)$ such that $\prod_{i=1}^n A_i \subset A$ and $x_i \in A_i$ for each integer i with $1 \leq i \leq n$. Since A_i is open in $(\mathbb{Z}, \kappa)$ such that $x_i \in A_i$, we have $x_i \in U(x_i) \subset A_i$ for each integer i with $1 \leq i \leq n$ (cf. $(*4)$(iii) in (I)); and so $U^n(x) := \prod_{i=1}^n U(x_i) \subset \prod_{i=1}^n A_i \subset A$. Therefore, we have $U^n(x) \subset A$. By using assumption that $A \subset U^n(x)$, it is shown that $A = U^n(x)$ holds. **(iii)** Since $G \in \kappa^n$ and $U^n(x) \in \kappa^n$, we see $G \cap U^n(x) \in \kappa^n$. Put $A := G \cap U^n(x)$. Then, we have $x \in A, A \in \kappa^n$ and $A \subset U^n(x)$. By (ii) above, it is shown that $A = G \cap U^n(x) = U^n(x)$ holds. Namely, we have $U^n(x) \subset G$. $\square$

Remark 3.10 Using Theorem 3.9, we can investigate topological properties of κ^n-Cl(A), κ^n-Int(A) and κ^n-Ker(A), where A is a subset of $(\mathbb{Z}^n, \kappa^n)$.

• **(Some notation)** In the present paper, we use the following notation (cf. Definition 3.11, $(*20)$ below) for $(\mathbb{Z}^n, \kappa^n)(n \geq 2)$ (they are used in [39], [38], [11] for an integer $n \geq 1$); cf. $(*11)$ in (I) for $n = 1$.

Definition 3.11 ([39, Definition 2.1], [38, Section 2], [11, Section 6])

(i) The following subsets $(\mathbb{Z}^n)_{\kappa^n}, (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $(\mathbb{Z}^n)_{mix(r)}$ of $(\mathbb{Z}^n, \kappa^n)$ are well defined, where $r \in \mathbb{Z}$ with $1 \leq r \leq n$:
(i-1) $(\mathbb{Z}^n)_{\kappa^n} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ x_i$ is odd for each integer i with $1 \leq i \leq n\}$; by Proposition 3.5(i)(b) in (II), it is shown that: $(\mathbb{Z}^n)_{\kappa^n} = \{x \in \mathbb{Z}^n | \ \{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)\}$.
(i-2) $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ x_i$ is even for each integer i with $1 \leq i \leq n\}$; by Proposition 3.5(ii)(a), it is shown that: $(\mathbb{Z}^n)_{\mathcal{F}^n} = \{x \in \mathbb{Z}^n | \ \{x\}$ is closed in $(\mathbb{Z}^n, \kappa^n)\}$.
(i-3) $(\mathbb{Z}^n)_{mix(r)} := \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n | \ \#\{i \in \{1, 2, ..., n\} | \ x_i$ is even$\} = r \ \}$, where $1 \leq r \leq n$ and $\#A$ denotes the cardinality of a set A. Especially, for the case where $r = n$, we note $(\mathbb{Z}^n)_{\mathcal{F}^n} = (\mathbb{Z}^n)_{mix(n)}$ holds.

(ii) For a nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$, the following subsets $E_{\kappa^n}, E_{\mathcal{F}^n}$ and $E_{mix(r)}$ of $(\mathbb{Z}^n, \kappa^n)$ are well defined, where $1 \leq r \leq n$:
(ii-1) $E_{\kappa^n} := E \cap ((\mathbb{Z}^n)_{\kappa^n})$ (cf. (i-1) above);
(ii-2) $E_{\mathcal{F}^n} := E \cap ((\mathbb{Z}^n)_{\mathcal{F}^n})$ (cf. (i-2) above);
(ii-3) $E_{mix(r)} := E \cap ((\mathbb{Z}^n)_{mix(r)})$ (cf. (i-3) above); we note $E_{mix(n)} = E_{\mathcal{F}^n}$.

It is well known that: for any nonempty subset E of $(\mathbb{Z}^n, \kappa^n)$,
· **$(*20)$ (i)** $E_{\kappa^n} = \{x \in E \mid \{x\}$ *is open in* $(\mathbb{Z}^n, \kappa^n)\}$ $=\{(x_1, x_2, ..., x_n) \in E \mid x_i$ *is odd for each* $i \in \mathbb{Z}$ *with* $1 \leq i \leq n\}$.

(ii) $E_{\mathcal{F}^n} = \{x \in E \mid \{x\}$ *is closed in* $(\mathbb{Z}^n, \kappa^n)\}$ $=\{(x_1, x_2, ..., x_n) \in E \mid x_i$ *is even for each* $i \in \mathbb{Z}$ *with* $1 \le i \le n\}$.

(iii) *The subset* $(\mathbb{Z}^n)_{\kappa^n}$ *and* E_{κ^n} *are open in* $(\mathbb{Z}^n, \kappa^n)$.

(iv) *We have the following decomposition of* $\mathbb{Z}^n$ *and one of a nonempty set* E*, respectively, as follows* (Note: $n \ge 2$),

· $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | \ 1 \le r \le n-1\})$ (disjoint union);

· $E = E_{\kappa^n} \cup E_{\mathcal{F}^n} \cup (\bigcup\{E_{mix(r)} | \ 1 \le r \le n-1\})$ (disjoint union).

(Note: in the above decomposition of $\mathbb{Z}^n$ (resp. E), we should take $(\mathbb{Z}^n)_{mix(r)}$ (resp. $E_{mix(r)}$) with $1 \le r \le n-1$.)

(v) *Especially, for* $n = 2$ *and* $r = 1$, $E_{mix(1)} = \{(x_1, x_2) \in E \mid x_1$ is even and x_2 is odd$\}$ $\cup\{(x_1, x_2) \in E \mid x_1$ is odd and x_2 is even$\}$; *we have the following decompositions*:

· $\mathbb{Z}^2 = (\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix(1)}$ (disjoint union) and $E = E_{\kappa^2} \cup E_{\mathcal{F}^2} \cup E_{mix(1)}$ (disjoint union).

(vi) *If* $E \subset F \subset \mathbb{Z}^n$, *then* $E_{\kappa^n} \subset F_{\kappa^n}, E_{\mathcal{F}^n} \subset F_{\mathcal{F}^n}$ *and* $E_{mix(r)} \subset F_{mix(r)} (1 \le r \le n-1)$ *hold in* $(\mathbb{Z}^n, \kappa^n)$.

In Section 4, we need the following property Theorem 3.12 (cf. Theorem 4.9, Corollary 4.10 below).

Theorem 3.12 ([39, Lemma 2.3]) *Let* $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(a')}$ *and* $y = (y_1, y_2, ..., y_n) \in (\mathbb{Z}^n)_{mix(a)}$, *where* a' *and* a *are integers such that* $a' \le a, 1 \le a' \le n$ *and* $1 \le a \le n$. *Suppose that* $U^n(x) \cap U^n(y)$ *contains exactly the* $2^{a'}$ *open singletons, say* $\{q^{(1)}, q^{(2)}, ..., q^{(2^{a'})}\}$. *Then, the following properties holds.*

(i) $\{q^{(1)}, q^{(2)}, ..., q^{(2^{a'})}\} = (U^n(x))_{\kappa^n} = (U^n(x) \cap U^n(y))_{\kappa^n} \subseteq (U^n(y))_{\kappa^n}$.

(ii) $\{i | \ x_i$ *is even* $(1 \le i \le n)\} \subseteq \{i | \ y_i$ *is even* $(1 \le i \le n)\}$.

(ii)' *If* $a' = a$ *especially, then* $\{i | \ x_i$ *is even* $(1 \le i \le n)\} = \{i | \ y_i$ *is even* $(1 \le i \le n)\}$.

(iii) $x \in U^n(y)$ *holds.*

(iii)' *If* $a' = a$ *especially, then* $x = y$.

4 ω-closed sets in Sundaram-Sheik John's sense and Λ_s-sets in $(\mathbb{Z}^n, \kappa^n)$

In the present section, we investigate the concept of ω-closed sets (in Sundaram-Sheik John's sense) in $(\mathbb{Z}^n, \kappa^n)$ and we give a characterization of the ω-closedness in the digital n-spaces (cf. Theorem 4.6). In $(\mathbb{Z}^n, \kappa^n)$, we first give an example of a Λ_s-set, say $B(n)$, where $n \ge 2$, (cf. Definition 2.3, Example 4.2) which is not ω-closed (in Sundaram-Sheik John's sense) (cf. Example 4.2(ii-1)); this example informs us general properties on $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.5). In order to explain the example, we prove the following proposition. We use the notations of Definition 3.11 and (II)($*$20) etc in Section 3, i.e., some notation and well known properties in $(\mathbb{Z}^n, \kappa^n)$.

Proposition 4.1 *Let* V *be an open set of* $(\mathbb{Z}^n, \kappa^n)$.

(i) *If* $n \ge 2$, *then* $V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} | \ 1 \le r \le n-1\}) \subset \mathrm{Cl}(V_{\kappa^n})$.

(ii) *If* $n = 1$, *then* $V_{\mathcal{F}^n} \subset \mathrm{Cl}(V_{\kappa^n})$.

Proof. **(i)** Let $y \in V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} | \ 1 \le r \le n-1\})$ (cf. Definition 3.11(ii), (II)($*$20) etc in Section 3 above). Since $y \in V$ and V is open in $(\mathbb{Z}^n, \kappa^n)$, there exists the smallest open set $U^n(y)$ (cf. Definition 3.7) containing y such that

($*^1$) $U^n(y) \subset V$ (cf. Theorem 3.9(iii)) and so $(U^n(y))_{\kappa^n} \subset V_{\kappa^n}$ (cf. Definition 3.11(ii)(ii-1), (II)($*$20)(vi) above).

Case 1. $y \in V_{\mathcal{F}^n}$, i.e., $y = (2s_1, 2s_2, ..., 2s_n)$ and $y \in V$, where $s_i \in \mathbb{Z}$ $(1 \leq i \leq n)$ (cf. Definition 3.11(ii)(ii-2)): since $U^n(y) = \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}$ for this point y , we have $\prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\} \subset V$ (cf. Definition 3.7, Theorem 3.9(iii) and (I)($*$4) in Section 3). We pick a point $p(y) := (2s_1 + 1, 2s_2 + 1, ..., 2s_n + 1) \in (U^n(y))_{\kappa^n}$ and so $p(y) \in V_{\kappa^n}$ (cf. Proposition 3.5(iii)(b)). Then, since $\mathrm{Cl}(\{p(y)\}) = \prod_{i=1}^n \{2s_i, 2s_i + 1, 2s_i + 2\}$ (cf. Proposition 3.5(i)(a)), we have $y = (2s_1, 2s_2, ..., 2s_n) \in \mathrm{Cl}(\{p(y)\}) \subset \mathrm{Cl}(V_{\kappa^n})$. It is proved that $V_{\mathcal{F}^n} \subset \mathrm{Cl}(V_{\kappa^n})$. We note that the above proof is done for the case where $n \geq 1$ (cf. (I)($*$1), ($*$4), ($*$11)(v) in Section3).

Case 2. $y \in V_{mix(r)}$, where $1 \leq r \leq n-1$ $(n \geq 2)$ (cf. Definition 3.11(ii)(ii-3)): for this point y, we set $y = (y_1, y_2, ..., y_n)$; then by definition, $r = \#\{i \mid y_i$ is an even integer $(1 \leq i \leq n)\}$. We put $I_r := \{i \mid y_i \text{ is even }\} = \{e(1), e(2), ..., e(r)\}$
$(e(1) < e(2) < ... < e(r))$ and $J_{n-r} := \{j \mid y_j \text{ is odd }\} = \{o(1), o(2), ..., o(n-r)\}$ $(o(1) < o(2) < ... < o(n-r))$; then $\{1, 2, ..., n\} = I_r \cup J_{n-r}$ (disjoint union). For the present case, we claim that $y \in Cl(V_{\kappa^n})$. Indeed, we recall that:
$(*^2)$ $U^n(y) = \prod_{i=1}^n U(y_i)$, where $U(y_e) := \{y_e - 1, y_e, y_e + 1\}$ if $e \in I_r$; and $U(y_o) := \{y_o\}$ if $o \in J_{n-r}$ (cf. (I)($*$4) in Section 3, Definition 3.7).
For this point $y \in V_{mix(r)}$ $(1 \leq r \leq n-1$ and $n \geq 2)$, we pick a point $p(y) \in U^n(y)$ such that $p(y) \in (U^n(y))_{\kappa^n}$ as follows:
$(*^3)$ let $p(y) := (p_1, p_2, ..., p_n)$, where $p_e := y_e - 1$ if $e \in I_r$; $p_o := y_o$ if $o \in J_{n-r}$.
Then by $(*^2)$ and $(*^3)$ above, it is shown that the components of the point $p(y)$ are odd and so $(*^4)$ $p(y) \in (U^n(y))_{\kappa^n}$, because the components have the forms of $y_e - 1 \in U(y_e)$ or $y_o \in U(y_o)$.
Thus, using $(*^1)$, $(*^4)$ above and (II)($*$20)(vi) above, we see that $p(y) \in V_{\kappa^n}$; and so
$(*^5)$ $\mathrm{Cl}(\{p(y)\}) \subset \mathrm{Cl}(V_{\kappa^n})$.
We note that : $\mathrm{Cl}(\{p(y)\}) = \mathrm{Cl}(\{(p_1, p_2, ..., p_n)\}) = \prod_{i=1}^n \mathrm{Cl}(\{p_i\})$ in $(\mathbb{Z}^n, \kappa^n)$, where $\mathrm{Cl}(\{p_e\}) = \{p_e - 1, p_e, p_e + 1\} = \{y_e - 2, y_e - 1, y_e\}$ if $e \in I_r$; and $\mathrm{Cl}(\{p_o\}) = \{p_o - 1, p_o, p_o + 1\} = \{y_o - 1, y_o, y_o + 1\}$ if $o \in J_{n-r}$ (cf. Proposition 3.5). Thus, we have $y = (y_1, y_2, ..., y_n) \in \mathrm{Cl}(\{p(y)\})$. Moreover, using $(*^5)$ above, we conclude that $y \in \mathrm{Cl}(V_{\kappa^n})$ for a point $y \in V_{mix(r)}$. Namely, it is proved that $V_{mix(r)} \subset \mathrm{Cl}(V_{\kappa^n})$ for each r with $1 \leq r \leq n-1$ $(n \geq 2)$.

Therefore we have the required inclusion: $V_{\mathcal{F}^n} \cup (\bigcup \{V_{mix(r)} | \ 1 \leq r \leq n-1\}) \subset \mathrm{Cl}(V_{\kappa^n})$
.

(ii) For the case where $n = 1$, we may consider the case 1 only of the proof of (i) above; the proof is omitted (cf. (I)($*$1), ($*$4), ($*$11)(v) in Section3). □

Example 4.2 Throughout the present example, let $B(n) := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup \{x(1),\ x(2),\ ...\ ,\ x(s)\}$ be an infinite subset of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$ and s is a positive integer, $\{x(j)\}$ is an open singleton of $(\mathbb{Z}^n, \kappa^n)$ for each integer j with $1 \leq j \leq s$. We have the following properties on the subset $B(n)$: namely,

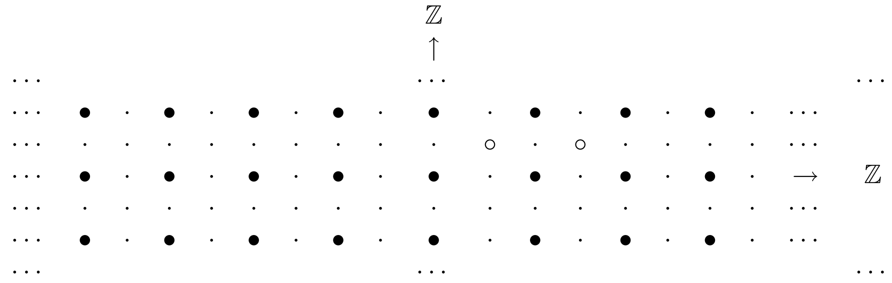
(i) *$B(n)$ is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$ for each $n \geq 1$* (cf. Proof of (i) below and Definition 2.3).

(ii) (ii-1) *If $n \geq 2$, then $B(n)$ is not an ω-closed set (in Sundaram-Sheik John's sense) of $(\mathbb{Z}^n, \kappa^n)$* (cf. Proof of (ii-1) below and Definition 2.1);

(ii-2) *For $n = 1, B(n)$ is a closed set of $(\mathbb{Z}, \kappa)$ and so it is an ω-closed set (in Sundaram-Sheik John's sense) in $(\mathbb{Z}, \kappa)$* (cf. Proof of (ii-2) below and Definition 2.1).

(iii) *Let A be a subset of $(\mathbb{Z}^n, \kappa^n)$ such that $B(n) \subset A \subset Cl(B(n))$. Then, A is not semi-open in $(\mathbb{Z}^n, \kappa^n)$.*

For the case where $n = 2$, the following figure illustrates the subset $B = (\mathbb{Z}^2)_{\mathcal{F}^2} \cup \{x(1), x(2)\}$ in $(\mathbb{Z}^2, \kappa^2)$; each symbol $\bullet$ means a point in $(\mathbb{Z}^2)_{\mathcal{F}^2}$ and two symbols $\circ$ mean $x(1) = (1, 1)$ and $x(2) = (3, 1)$ respectively.



In order to prove (i) above, we need the following property $(**)$:

$(**)$ *Suppose $n \geq 1$. Let $F_1(n) := B(n) \cup E_1(n)$ and $F_2(n) := B(n) \cup E_2(n)$, where $E_1(n) = \{(s_1, s_2, ..., s_n) \in \mathbb{Z}^n \mid s_i \equiv 1 \ mod\ 4\ (1 \leq i \leq n)\}$ and $E_2(n) := \{(s_1, s_2, ..., s_n) \in \mathbb{Z}^n \mid s_j \equiv 3 \ mod\ 4\ (1 \leq j \leq n)\}$. Then, $E_1(n) \cap E_2(n) = \emptyset$ holds and $F_1(n)$ and $F_2(n)$ are semi-open sets including $B(n)$ such that $F_1(n) \cap F_2(n) = B(n)$.*

Proof of $(**)$. We first recall the following expressions of $(\mathbb{Z}^n)_{\mathcal{F}^n} := \{(x_1, x_2, ..., x_n) \mid x_i$ is even $(1 \leq i \leq n)\}$ as follows:
$(*_1)$ $(\mathbb{Z}^n)_{\mathcal{F}^n}=\bigcup\{\prod_{i=1}^n\{x_i\} \mid x_i$ is even $(1 \leq i \leq n)\}=\bigcup\{\prod_{i=1}^n\{s_i-1, s_i+1\} \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\}$; and
$(*_1)'$ $(\mathbb{Z}^n)_{\mathcal{F}^n}= \bigcup\{\prod_{i=1}^n\{s_i-1, s_i+1\} \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 3 \bmod 4\ (1 \leq i \leq n)\}$.

We secondly claim that
$(*_2)$ $\mathrm{Cl}(E_i(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_i(n)$ for each $i \in \{1, 2\}$.
Indeed, we have $\mathrm{Cl}(E_1(n)) =\mathrm{Cl}(\bigcup\{\prod_{i=1}^n\{s_i\} \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\}) \supset \bigcup\{\mathrm{Cl}(\prod_{i=1}^n\{s_i\}) \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\} =\bigcup\{\prod_{i=1}^n \mathrm{Cl}(\{s_i\}) \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\}= \bigcup\{\prod_{i=1}^n\{s_i-1, s_i, s_i+1\} \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\} \supset \bigcup\{\prod_{i=1}^n\{s_i-1, s_i+1\} \mid (s_1, s_2, ..., s_n) \in \mathbb{Z}^n, s_i \equiv 1 \bmod 4\ (1 \leq i \leq n)\}=(\mathbb{Z}^n)_{\mathcal{F}^n}$ (cf. $(*_1)$ above, (I)$(*5)$(i) in Section 3) and $\mathrm{Cl}(E_1(n)) \supset E_1(n)$. Hence, we have $\mathrm{Cl}(E_1(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_1(n)$. In the same way, using $(*_1)'$ in place of $(*_1)$, we have $\mathrm{Cl}(E_2(n)) \supset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_2(n)$. Moreover, we claim that
$(*_3)$ $F_i(n)$ is semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each $i \in \{1, 2\}$.
Indeed, by using $(*_2)$ and definitions, it is shown that, for each $i \in \{1, 2\}$, $\mathrm{Cl}\,(\mathrm{Int}(F_i(n))) \supset\mathrm{Cl}\,(\mathrm{Int}((B(n))_{\kappa^n} \cup E_i(n)))= \mathrm{Cl}((B(n))_{\kappa^n} \cup E_i(n)) \supset (B(n))_{\kappa^n} \cup Cl(E_i(n)) \supset \{x(1), x(2), ..., x(s)\} \cup ((\mathbb{Z}^n)_{\mathcal{F}^n} \cup E_i(n)) = B(n) \cup E_i(n) = F_i(n)$. Namely, $F_i(n)$ is semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each $i \in \{1, 2\}$.

Finally, $(*_4)$ $F_1(n) \cap F_2(n) = B(n) \cup (E_1(n) \cap E_2(n)) = B(n)$ hold, because $E_1(n) \cap E_2(n) = \emptyset$. $(\circ)$

Proof of (i). We first claim that $\mathrm{sKer}(B(n)) \subset B(n)$. Indeed, we recall $(**)$ above and so $F_1(n)$ and $F_2(n)$ are semi-open sets in $(\mathbb{Z}^n, \kappa^n) (n \geq 1)$ such that $B(n) \subset F_i(n)$ for each $i \in \{1, 2\}$. Thus, by definitions, it is shown that $\mathrm{sKer}(B(n)) \subset F_1(n) \cap F_2(n)$ (cf. Definition 2.2(i)); and so $\mathrm{sKer}(B(n)) \subset B(n)$, because $F_1(n) \cap F_2(n) = B(n)$ (cf. $(**)$ above). This concludes that $\mathrm{sKer}(B(n)) = B(n)$, because $B(n) \subset sKer(B(n))$ holds. Namely, $B(n)$ is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$.

Proof of (ii)(ii-1). Suppose $n \geq 2$. We first show that:
$(*_5)$ $(\mathrm{Cl}(B(n)))_{mix(r)} \neq \emptyset$, for each integer r with $1 \leq r \leq n-1$. Indeed, since $\mathrm{Cl}(B(n)) = \mathrm{Cl}((\mathbb{Z}^n)_{\mathcal{F}^n}) \cup (\bigcup\{(\mathrm{Cl}(\{x(i)\})) \mid 1 \leq i \leq s\})$, it is shown that $(\mathrm{Cl}(B(n)))_{mix(r)} \supset (\mathrm{Cl}(\{x(1)\}))_{mix(r)}$ (cf. (II)$(*20)$ in Section 3). We can put $x(1) := (t_1, t_2, ..., t_n)$, where t_j is odd for each j with $1 \leq j \leq n$, because $x(1) \in (\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)). Then, we show $\mathrm{Cl}(\{x(1)\}) = \prod_{j=1}^n \mathrm{Cl}(\{t_j\}) = \prod_{j=1}^n\{t_j-1, t_j, t_j+1\}$ (cf. Proposition 3.5(i)(a)) and so
$(\mathrm{Cl}(\{x(1)\}))_{mix(r)} \neq \emptyset$ for each integer r with $1 \leq r \leq n-1$, because we can take a point

$p := (p_1, p_2, ..., p_n)$, where $p_j := t_j - 1$ is even for each j with $1 \leq j \leq r$ and $p_j := t_j$ is odd for each j with $r+1 \leq j \leq n$; and hence $p \in (\mathrm{Cl}(\{x(1)\}))_{mix(r)}$ (cf. Definition 3.11(i)(i-3)) and so $p \in (\mathrm{Cl}(B(n)))_{mix(r)}$ (cf. (II)(∗20) in Section 3). Thus, we prove the property $(*_5)$.

We secondly have the following property: $(*_6)$ $\mathrm{Cl}(B(n)) \not\subset F_1(n)$ holds.
Indeed, for a contradiction, we suppose $\mathrm{Cl}(B(n)) \subset F_1(n)$; then $(\mathrm{Cl}(B(n)))_{mix(r)} \subset (F_1(n))_{mix(r)}$ and so $(\mathrm{Cl}(B(n)))_{mix(r)} = \emptyset$ because of $(F_1(n))_{mix(r)} = \emptyset$ for each integer r with $1 \leq r \leq n-1$. This contradicts $(*_5)$ above.

For a contradiction, we finally suppose that $B(n)$ is ω-closed in Sundaram-Sheik John's sense, i.e., $\mathrm{Cl}(B(n)) \subset$sKer$(B(n))$ (cf. Theorem 2.5). Then, using $(**)$ above, we have sKer$(B(n)) \subset F_1(n)$ and so $\mathrm{Cl}(B(n)) \subset F_1(n)$; this contradicts $(*_6)$ above. Therefore, $B(n)$ is not ω-closed (in Sundaram-Sheik John's sense) in $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 2$.

Proof of (ii)(ii-2) Suppose $n = 1$. First, it is shown that $B(n) = B(1)$ is closed in $\mathbb{Z}^n$, where $n = 1$. Indeed, we have $\mathbb{Z} \setminus B(1) = \mathbb{Z}_\kappa \setminus \{x(j) | 1 \leq j \leq s\}$ and so $\mathbb{Z} \setminus B(1) = \bigcup\{\{z\} | z \in \mathbb{Z}_\kappa \text{ and } z \notin \{x(j) | 1 \leq j \leq s\}\}$, i.e., $\mathbb{Z} \setminus B(1)$ is the union of some open singletons $\{z\}$, and hence $\mathbb{Z} \setminus B(1) \in \kappa$ (cf. Definition 3.1). Thus, the set $B(1)$ is closed and so it is ω-closed in Sundaram-Sheik John's sense.

Proof of (iii). For a contradiction, we suppose that A is semi-open in $(\mathbb{Z}^n, \kappa^n)$. Then, there exists an open set V such that $V \subset A \subset \mathrm{Cl}(V)$ and so $V \subset \mathrm{Cl}(B(n))$. First we claim that: $(*_7)$ $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n})$ holds for each $n \geq 1$.

Proof of $(*_7)$. **Case (I)**. $n \geq 2$: for this case, we have $V = V_{\kappa^n} \cup V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} |\ 1 \leq r \leq n-1\})$ (cf. (II)(∗20)(iv) in Section 3). Since V is open, by Proposition 4.1(i), it is shown that $\mathrm{Cl}(V) = \mathrm{Cl}(V_{\kappa^n}) \cup \mathrm{Cl}(V_{\mathcal{F}^n} \cup (\bigcup\{V_{mix(r)} |\ 1 \leq r \leq n-1\})) \subset \mathrm{Cl}(V_{\kappa^n}) \cup \mathrm{Cl}(\mathrm{Cl}(V_{\kappa^n})) = \mathrm{Cl}(V_{\kappa^n})$; and so $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n})$.

Case (II). $n = 1$: for this case, we have $V = V_\kappa \cup V_{\mathcal{F}}$ (cf. (I)(∗11)(iii) in Section 3). Since V is open, by Proposition 4.1(ii), it is shown that

$\mathrm{Cl}(V) = \mathrm{Cl}(V_\kappa) \cup \mathrm{Cl}(V_{\mathcal{F}}) \subset \mathrm{Cl}(V_\kappa) \cup \mathrm{Cl}(\mathrm{Cl}(V_\kappa)) = \mathrm{Cl}(V_\kappa)$; and so $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_\kappa)$. $(\circ)$

We proceed the proof of (iii). We put $V_{\kappa^n} := \{p(k) \in V |\ \{p(k)\} \in \kappa^n, k \in \nu\}$, where $\nu \subset \mathbb{Z}$ is an index set (cf. Definition 3.11(i)(i-1)). Since $p(k) \in V_{\kappa^n} \subset V \subset \mathrm{Cl}(B(n))$ and so $p(k) \in \mathrm{Cl}(B(n))$, it is shown that $\{p(k)\} \cap B(n) \neq \emptyset$ and so $p(k) \in B(n)$ for each $k \in \nu$. Namely, we have:

$(*_8)$ $V_{\kappa^n} \subset (B(n))_{\kappa^n}$ (cf. Definition 3.11(i)(i-1),(ii)(ii-1) and (I)(∗11)(v), (II)(∗20)(vi)). Then, using $(*_7)$ and $(*_8)$ above, we conclude that $\mathrm{Cl}(V) \subset \mathrm{Cl}(V_{\kappa^n}) \subset \mathrm{Cl}((B(n))_{\kappa^n}) = \mathrm{Cl}(\{x(1), x(2), ..., x(s)\}) = \bigcup\{\mathrm{Cl}(\{x(j)\}) | 1 \leq j \leq s\}$; and hence $\mathrm{Cl}(V)$ is a finite subset of $(\mathbb{Z}^n, \kappa^n)$, because $\mathrm{Cl}(\{y\})$ is a finite subset of $\mathbb{Z}$ for every point $y \in \mathbb{Z}$ (cf. (I)(∗5)(i) in Section 3) and so $\mathrm{Cl}(\{x(j)\})$ is a finite subset of $\mathbb{Z}^n$ for each j with $1 \leq j \leq s$ (cf. (II)(∗12)(a) in Section 3). Therefore, we have A is a finite subset of $(\mathbb{Z}^n, \kappa^n)$, because of $V \subset A \subset \mathrm{Cl}(V)$; and so $B(n)$ is also finite, because of $B(n) \subset A$; this contradicts the definition of the set $B(n)$ (i.e., $B(n)$ is not finite). Therefore, A is not semi-open in $(\mathbb{Z}, \kappa)$.

In order to state Theorem 4.4, we need the following definition on $I_r(x)$ and $J_{n-r}(x)$, where $x \in \mathbb{Z}^n$.

Definition 4.3 (cf. Definition 3.11(i)(i-3),(II)(∗20)(iv) in Section 3; [39, Definiton 2.1(ii)]) Let $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$, where $n \geq 2$ and r is the cardinality of a set $\{k |\ x_k$ is even$\}$ with $1 \leq r \leq n-1$ (cf. Definition 3.11(i-3),(II)(∗20)(iv) in Section 3; in the present definition, we note the assumption that $1 \leq r \leq n-1$ and $n \geq 2$; and so $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset$). Let $x_{e(1)}, x_{e(2)}, ..., x_{e(r)}$ be all the components of x which are even; and $x_{o(1)}, x_{o(2)}, ..., x_{o(n-r)}$ be all the components of x which are odd, where $e(k) (1 \leq k \leq r)$ and $o(j) (1 \leq j \leq n-r)$ are positive integers with $1 \leq e(1) < e(2) < ... < e(r) \leq n$ and $1 \leq o(1) < o(2) < ... < o(n-r) \leq n$. Then, for this point $x = (x_1, x_2, ..., x_n)$, we define the following subsets $I_r(x)$ and $J_{n-r}(x)$ of $\{1, 2, ..., n\}$ as follows:

• $I_r(x) := \{k|\ x_k \text{ is even}\}$; and so $I_r(x) = \{e(1), e(2), ..., e(r)\}$ holds;
• $J_{n-r}(x) := \{j|\ x_j \text{ is odd}\}$; and so
$J_{n-r}(x) = \{o(1), o(2), ..., o(n-r)\}$, $\{1, 2, ..., n\} = I_r(x) \cup J_{n-r}(x)$ $(I_r(x) \cap J_{n-r}(x) = \emptyset)$, $I_r(x) \neq \emptyset$ and $J_{n-r}(x) \neq \emptyset$ hold, where $n \geq 2$ and $1 \leq r \leq n-1$.

We construct some semi-open sets containing a point of $(\mathbb{Z}^n, \kappa^n)$ where $n \geq 1$.

Theorem 4.4 *Let $x := (x_1, x_2, ..., x_n) \in \mathbb{Z}^n$.*

(i) *Suppose $n \geq 1$. If $x \in (\mathbb{Z}^n)_{\kappa^n}$,i.e., all the components $x_1, x_2, ..., x_n$ of the point x are odd* (cf. Definition 3.11(i)(i-1)), *then $\{x\}$ is a semi-open set containing x in $(\mathbb{Z}^n, \kappa^n)$.*

(ii) *Suppose $n \geq 1$ and $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, i.e., all the components $x_1, x_2, ..., x_n$ of the point x are even* (cf. Definition 3.11(i)(i-2)). *Then, we have the following properties.*

(ii-1) *We set $A(x) := \{(x_1 + i_1, x_2 + i_2, ..., x_n + i_n) \in \mathbb{Z}^n | i_k \in \{+1, -1\}(1 \leq k \leq n)\}$ for the point $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{\mathcal{F}^n}$. Then, $\#A(x) = 2^n$ holds. And, for each point of $A(x)$, say $p(x, u)(1 \leq u \leq 2^n)$, the singleton $\{p(x, u)\}$ is open in $(\mathbb{Z}^n, \kappa^n)$.*

(ii-2) *In $(\mathbb{Z}^n, \kappa^n)$, $\{p(x, u)|1 \leq u \leq 2^n\} = (U^n(x))_{\kappa^n}$ holds, where $U^n(x)$ is the smallest open set* (cf. Definition 3.7,Theorem 3.9) *containing the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$.*

(ii-3) *The subset $\{x\} \cup \{p(x, u)\}$ is a semi-open set containing the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ for each u with $1 \leq u \leq 2^n$.*

(iii) *Suppose $n \geq 2$ and $x := (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$ where $1 \leq r \leq n-1$* (cf. Definition 3.11(i)(i-3),(II)(∗20)(iv) in Section 3). *Let $I_r(x) = \{e(1), e(2), ..., e(r)\}$ and $J_{n-r}(x) = \{o(1), o(2), ..., o(n-r)\}$* (cf. Definition 4.3). *Then, we have the following properties.*

(iii-1) *We set $B(x) := \{(z_1, z_2, ..., z_n) \in \mathbb{Z}^n|\ z_{e(k)} \in \{x_{e(k)} - 1, x_{e(k)} + 1\}$ $(1 \leq k \leq r), z_{o(j)} = x_{o(j)}$ $(1 \leq j \leq n-r)\}$ for the point $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$. Then, $\#B(x) = 2^r$. And, for each point of $B(x)$, say $p(x, u)(1 \leq u \leq 2^r)$, the singleton $\{p(x, u)\}$ is open in $(\mathbb{Z}^n, \kappa^n)$.*

(iii-2) *In $(\mathbb{Z}^n, \kappa^n)$, $\{p(x, u)|1 \leq u \leq 2^r\} = (U^n(x))_{\kappa^n}$ holds, where $U^n(x)$ is the smallest open set containing the point $x \in (\mathbb{Z}^n)_{mix(r)}$.*

(iii-3) *The subset $\{x\} \cup \{p(x, u)\}$ is a semi-open set containing the point $x \in (\mathbb{Z}^n)_{mix(r)}$ for each u with $1 \leq u \leq 2^r$.*

Proof. **(i)** For the point $x \in (\mathbb{Z}^n)_{\kappa^n}$, the singleton $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Proposition 3.5(iii)(b)) ; and so it is semi-open.

(ii) (ii-1) Obviously, the cardinality of $A(x)$ is 2^n. The point $p(x, u)$, where $1 \leq u \leq 2^n$, is expressible as $p(x, u) = (x_1 + i_1, x_2 + i_2, ..., x_n + i_n)$ for some integers $i_k \in \{+1, -1\}(1 \leq k \leq n)$ and so all the components of $p(x, u)$ are odd, because all the components $x_1, x_2, ..., x_n$ are even. Thus, $\{p(x, u)\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Proposition 3.5(iii)(b)).

(ii-2) For the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, we set $x = (2s_1, 2s_2, ..., 2s_n)$ for some integers $s_i(1 \leq i \leq n)$. Then, $U^n(x) = \prod_{i=1}^n U(2s_i) = \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}$ is the smallest open set containing x (cf. Definition 3.7 and (I)(∗4)(i) in Section 3). Since $(U^n(x))_{\kappa^n} = \{z \in U^n(x)|\{z\} \text{ is open in } (\mathbb{Z}^n, \kappa^n)\} = \{(z_1, z_2, ..., z_n) \in \prod_{i=1}^n \{2s_i - 1, 2s_i, 2s_i + 1\}|z_1, z_2, ..., z_n$ are odd $\}$, we have $(U^n(x))_{\kappa^n} = \{(2s_1 + i_1, 2s_2 + i_2, ..., 2s_n + i_n) \in \mathbb{Z}^n | i_k \in \{+1, -1\}(1 \leq k \leq n)\} = A(x)$; and so we have $(U^n(x))_{\kappa^n} = \{p(x, u)|1 \leq u \leq 2^n\}$ (cf. Definition 3.11(i)(i-1),(ii)(ii-1) and (ii-1) above).

(ii-3) We first claim that $x \in \mathrm{Cl}(\{p(x, u)\})$ for each u with $1 \leq u \leq 2^n$. Indeed, we have $\mathrm{Cl}(\{p(x, u)\}) = \prod_{k=1}^n \mathrm{Cl}(\{x_k + i_k\}) = \prod_{k=1}^n \{x_k + i_k - 1, x_k + i_k, x_k + i_k + 1\}$ (cf. (II)(∗12)(a) in Section 3, Proposition 3.5(i)(a)); and so $x = (x_1, x_2, ..., x_n) \in \prod_{k=1}^n \mathrm{Cl}(\{x_k + i_k\}) = \mathrm{Cl}(\{p(x, u)\})$. Thus, we show that $\{x\} \cup \{p(x, u)\} \subset \mathrm{Cl}(\{p(x, u)\}) = \mathrm{Cl}(\mathrm{Int}(\{p(x, u)\})) \subset \mathrm{Cl}(\mathrm{Int}$

$(\{x\} \cup \{p(x,u)\}))$ (cf. (ii-1) above), i.e., $\{x\} \cup \{p(x,u)\} \subset \mathrm{Cl}(\mathrm{Int}(\{x\} \cup \{p(x,u)\}))$. Namely, $\{x\} \cup \{p(x,u)\}$ is semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each u with $1 \le u \le 2^n$.

(iii) (iii-1) By the definition of $B(x)$, it is obviously shown that $\#B(x) = 2^r$. A point $p(x,u)$ of $B(x)$ is expressible as $p(x,u) = (z(u)_1, z(u)_2, ..., z(u)_n)$, where $z(u)_{e(k)} \in \{x_{e(k)} - 1, x_{e(k)} + 1\}$ $(1 \le k \le r)$ and $z(u)_{o(j)} = x_{o(j)}$ $(1 \le j \le n-r)$. We recall that the r components $x_{e(1)}, x_{e(2)}, ..., x_{e(r)}$ are all even and the $n-r$ components $x_{o(1)}, x_{o(2)}, ..., x_{o(n-r)}$ are all odd, because we assume that $x = (x_1, x_2, ..., x_n) \in (\mathbb{Z}^n)_{mix(r)}$ where $1 \le r \le n-1 (n \ge 2)$ and $I_r(x) := \{k | \ x_k \text{ is even}\} = \{e(1), e(2), ..., e(r)\}$ $(e(1) < e(2) < ... < e(r))$; and
$J_{n-r}(x) := \{j | x_j \text{ is odd }\} = \{o(1), o(2), ..., o(n-r)\}$ $(o(1) < o(2) < ... < o(n-r))$ (cf. Definition 3.11(i)(i-3),(II)($*$20)(iv) in Section 3 and Definition 4.3 above). Then, since the integers $x_{e(k)} - 1, x_{e(k)} + 1$ and $x_{o(j)}$ are odd, all the components $z(u)_1, z(u)_2, ..., z(u)_n$ are odd for each u with $1 \le u \le 2^r$. We have that the singleton $\{p(x,u)\} = \{(z(u)_1, z(u)_2, ..., z(u)_n)\}$ is open in $(\mathbb{Z}^n, \kappa^n)$ (cf. Proposition 3.5(iii)(b)).

(iii-2) We recall that, for this point $x \in (\mathbb{Z}^n)_{mix(r)}$, $U^n(x) = \prod_{i=1}^n U(x_i)$, where $U(x_{e(k)}) = \{x_{e(k)} - 1, x_{e(k)}, x_{e(k)} + 1\} (1 \le k \le r)$ and $U(x_{o(j)}) = \{x_{o(j)}\} (1 \le j \le n-r)$ (cf. Definition 4.3,Definition 3.7,(I)($*$4)(i)(ii) in Section 3). Thus, we have that $(z_1, z_2, ..., z_n) \in (U^n(x))_{\kappa^n}$ if and only if $z_{e(k)} \in \{x_{e(k)} - 1, x_{e(k)} + 1\}$ and $z_{o(j)} = x_{o(j)}$ for integers $k,\ j$ with $1 \le k \le r$ and $1 \le j \le n-r$ (cf. Proposition 3.5(iii)(b), Definition 4.3). Namely, we have $(U^n(x))_{\kappa^n} = B(x)$ for the point $x \in (\mathbb{Z}^n)_{mix(r)}$ and so $(U^n(x))_{\kappa^n} = \{p(x,u) | 1 \le u \le 2^r\}$ (cf. (iii-1) above).

(iii-3) We first claim that $(*)$ $\{x\} \cup \{p(x,u)\} \subset \mathrm{Cl}(\{p(x,u)\})$ holds in $(\mathbb{Z}^n, \kappa^n)$ for each u with $1 \le u \le 2^r$. Indeed, for the point $p(x,u)$, we set $p(x,u) := (z(u)_1, z(u)_2, ..., z(u)_n)$ (cf. (iii-1) above). Then, for each positive integers k, j with $1 \le k \le r$ and $1 \le j \le n-r$, it is shown that: in $(\mathbb{Z}, \kappa)$,
if $z(u)_{e(k)} = x_{e(k)} - 1$, then $\mathrm{Cl}(\{z(u)_{e(k)}\}) = \{x_{e(k)} - 2, x_{e(k)} - 1, x_{e(k)}\}$ holds;
if $z(u)_{e(k)} = x_{e(k)} + 1$, then $\mathrm{Cl}(\{z(u)_{e(k)}\}) = \{x_{e(k)}, x_{e(k)} + 1, x_{e(k)} + 2\}$ holds;
if $z(u)_{o(j)} = x_{o(j)}$, then $\mathrm{Cl}(\{z(u)_{o(j)}\}) = \{x_{o(j)} - 1, x_{o(j)}, x_{o(j)} + 1\}$ holds, (cf. (I)($*$5)(i) in Section 3). Thus, we show that $x_{e(k)} \in \mathrm{Cl}(\{z(u)_{e(k)}\})$ and $x_{o(j)} \in \mathrm{Cl}(\{z(u)_{o(j)}\})$ $(1 \le k \le r$ and $1 \le j \le n-r)$; and so $\{x\} \subset \prod_{i=1}^n \mathrm{Cl}(\{z(u)_i\})$ holds in $(\mathbb{Z}^n, \kappa^n)$. Since $\mathrm{Cl}(\{p(x,u)\}) = \prod_{i=1}^n \mathrm{Cl}(\{z(u)_i\})$ in $(\mathbb{Z}^n, \kappa^n)$ (cf. (II)($*$12) in Section 3), we show that $\{x\} \subset \mathrm{Cl}(\{p(x,u)\})$ and $\{x\} \cup \{p(x,u)\} \subset$
$\mathrm{Cl}(\{p(x,u)\})$ hold in $(\mathbb{Z}^n, \kappa^n)$.

We finally finish the proof of (iii-3): there exists an open set $\{p(x,u)\}$ such that $\{p(x,u)\} \subset \{x\} \cup \{p(x,u)\} \subset \mathrm{Cl}(\{p(x,u)\})$, i.e., $\{x\} \cup \{p(x,u)\}$ is a semi-open in $(\mathbb{Z}^n, \kappa^n)$ for each u with $1 \le u \le 2^r$. □

Theorem 4.5 *For the digital n-space $(\mathbb{Z}^n, \kappa^n)$ where $n \ge 1$, we have the following properties.*

(i) *For any point x of $(\mathbb{Z}^n, \kappa^n)$, sKer$(\{x\}) = \{x\}$.*

(ii) *For any subset E of $(\mathbb{Z}^n, \kappa^n)$, sKer$(E) = E$.*

Proof. **(i)** We first note that: for the case where $n = 1$,
$\cdot\ \mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n}$ (disjoint union) holds, where $n = 1$ (cf. (I)($*$11)(iii) in Section 3); for the case where $n \ge 2$,
$\cdot\ \mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \le r \le n-1\})$ (disjoint union) and $(\mathbb{Z}^n)_{mix(r)} \ne \emptyset (1 \le r \le n-1)$ hold, where $n \ge 2$ (cf. Definition 3.11, (II)($*$20)(iv) in Section 3).

Let $x \in \mathbb{Z}^n$. It is enough to consider the following three cases for the point $x \in \mathbb{Z}^n$.

Case 1. $x \in (\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)): since $\{x\}$ is open in $(\mathbb{Z}^n, \kappa^n)$, it is semi-open. Then, it is obvious that sKer$(\{x\}) = \{x\}$ in $(\mathbb{Z}^n, \kappa^n)$ (cf. Definition 2.2(i)). We note this result is true for the case where $n \ge 1$.

Case 2. $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ (cf. Definition 3.11(i)(i-2)): we put $x = (2s_1, 2s_2, ..., 2s_n)$ where

$s_i \in \mathbb{Z}$ $(1 \leq i \leq n)$. Note that, for the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$, $U^n(x) := \prod_{i=1}^{n}\{2s_i-1, 2s_i, 2s_i+1\}$ is the smallest open set containing x (cf. Definition 3.7,(I)($*$4)(i) in Section 3,Theorem 3.9). Then, by Theorem 4.4(ii), there exist 2^n semi-open sets $\{x\} \cup \{p(x,u)\}(1 \leq u \leq 2^n)$ containing the point $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ such that $\{p(x,u)|1 \leq u \leq 2^n\} = (U^n(x))_{\kappa^n} = \{(2s_1 + i_1, 2s_2+i_2, ..., 2s_n+i_n)|i_k \in \{+1,-1\}(1 \leq k \leq n)\}$ and $\#((U^n(x))_{\kappa^n}) = 2^n$. Thus, we have $\mathrm{sKer}(\{x\}) \subset \bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^n\}$; moreover, $\bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^n\} = \{x\}$, because $\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^n\} = \emptyset$. We conclude that $\mathrm{sKer}(\{x\}) = \{x\}$ holds for this case. We note the result above is true for the case where $n \geq 1$.

Case 3. $x \in (\mathbb{Z}^n)_{mix(r)}$ where $1 \leq r \leq n-1 (n \geq 2)$ (cf. Definition 3.11(i)(i-3)): for this point x, we set $x = (x_1, x_2, ..., x_n)$; then by definition, $r = \#\{i|\ x_i$ is an even integer $(1 \leq i \leq n)\}$. We recall the following subsets $I_r(x)$ and $J_{n-r}(x)$ as follows (cf. Definition 4.3 above):

$I_r(x) := \{k|\ x_k$ is even$\}=\{e(1), e(2), ..., e(r)\}$ $(e(1) < e(2) < ... < e(r))$; and

$J_{n-r}(x) := \{j|x_j$ is odd $\}= \{o(1), o(2), ..., o(n-r)\}$ $(o(1) < o(2) < ... < o(n-r))$; and $\{1, 2, ..., n\} = I_r(x) \cup J_{n-r}(x)$ (disjoint union), $I_r(x) \neq \emptyset, J_{n-r}(x) \neq \emptyset$.

For the point $x \in (\mathbb{Z}^n)_{mix(r)}, U^n(x) = \prod_{i=1}^{n} U(x_i)$ is the smallest open set containing x, where $U(x_{e(k)}) = \{x_{e(k)}-1, x_{e(k)}, x_{e(k)}+1\}(1 \leq k \leq r)$ and $U(x_{o(j)}) = \{x_{o(j)}\}(1 \leq j \leq n-r)$ (cf. Definition 3.7,(I)($*$4) in Section 3,Theorem 3.9). Then, using Theorem 4.4(iii), there exist the 2^r semi-open sets $\{x\} \cup \{p(x,u)\}(1 \leq u \leq 2^r)$ containing the point $x \in (\mathbb{Z}^n)_{mix(r)}$ such that $\{p(x,u)|1 \leq u \leq 2^r\} = (U^n(x))_{\kappa^n} = \{(z_1, z_2, ..., z_n)|z_{e(k)} \in \{x_{e(k)}+1, x_{e(k)}-1\}(1 \leq k \leq r), z_{o(j)} = x_{o(j)}(1 \leq j \leq n-r)\}$ and $\#((U^n(x))_{\kappa^n}) = 2^r$. Thus, it is shown that $\mathrm{sKer}(\{x\}) \subset \bigcap\{\{x\} \cup \{p(x,u)\}|\ 1 \leq u \leq 2^r\} = \{x\} \cup (\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^r\}) = \{x\}$, because $\bigcap\{\{p(x,u)\}|\ 1 \leq u \leq 2^r\} = \emptyset$. Then, we show that $\mathrm{sKer}(\{x\}) = \{x\}$ holds for this case.

Therefore, for all cases above, we have proved that $\mathrm{sKer}(\{x\}) = \{x\}$ holds in $(\mathbb{Z}^n, \kappa^n)$, $n \geq 1$.

(ii) Since $E = \bigcup\{\{x\}|x \in E\}$, by Proposition 2.4(i.e., [4, Proposition 3.1]) and (i), it is shown that $\mathrm{sKer}(E) = \bigcup\{\mathrm{sKer}(\{x\})|x \in E\} = \bigcup\{\{x\}|x \in E\} = E$. □

The following result is a characterization of the ω-closed sets in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$.

Theorem 4.6 *For a subset A of $(\mathbb{Z}^n, \kappa^n)$, where $n \geq 1$, A is closed in $(\mathbb{Z}^n, \kappa^n)$ if and only if A is an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$.*

Proof. By Theorem 2.5, it is obtained that a subset A is an ω-closed in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ if and only if $\mathrm{Cl}(A) \subset \mathrm{sKer}(A)$. Then, by Theorem 4.5 (ii), it is well known that $A=\mathrm{sKer}(A)$ holds. Thus, A is ω-closed in Sundaram-Sheik John's sense if and only if $\mathrm{Cl}(A) \subset A$ (i.e., A is closed in $(\mathbb{Z}^n, \kappa^n)$). □

Remark 4.7 (i) Every subset of $(\mathbb{Z}^n, \kappa^n)$ is a Λ_s-set in $(\mathbb{Z}^n, \kappa^n)$. Indeed, let E be a subset of $(\mathbb{Z}^n, \kappa^n)$. By Theorem 4.5 (ii) and Definition 2.3, it is shown that $E=\mathrm{sKer}(E)$ holds, i.e., E is a Λ_s-set of $(\mathbb{Z}^n, \kappa^n)$.

(ii) By (i) and Proposition 2.6, it is obtained that $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_1 topological space. However, we note that, in 2004, S.I. Nada [30, Theorem 4.2, Theorem 4.1] proved that $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2; the proof is very elegantly done, using the semi-T_2 separation property of $(\mathbb{Z}, \kappa)$ and the product topology of κ; and hence their product space $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2; in 2006, present authors [11, Theorem 2.3, Theorem 4.8 (i)] proved that $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}^2, \kappa^2)$ are semi-T_2. But, in the end of the present paper (Corollary 4.10 below), we shall mention an alternative proof of the result ([30, Theorem 4.2]) using Theorem 4.4 and ideas in [39].

Example 4.8 In general, ω-closed sets in Sundaram-Sheik John's sense of a topological space are placed between closed sets and g-closed sets (cf. Definition 2.1(ii) (i.e.,[35])). The following example shows that there is a g-closed sets which is not an ω-closed set in Sundaram-Sheik John's sense of $(\mathbb{Z}^n, \kappa^n)$ (i.e., closed set in $(\mathbb{Z}^n, \kappa^n)$, cf. Theorem 4.6). Suppose $n \geq 2$. Let $A := \mathbb{Z}^n \setminus (\bigcup\{(\mathbb{Z}^n)_{mix(r)} |\ 1 \leq r \leq n-1\})$, i.e., $A = (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\mathbb{Z}^n)_{\kappa^n}$ and $A \neq \emptyset$. We consider the following figure which is shown by the symbols $\bullet \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $\circ \in (\mathbb{Z}^n)_{\kappa^n}$ in $\mathbb{Z}^2$. The figure shows the subset A above for $n = 2$.

$$
\begin{array}{cccccccccccccccccccc}
 & & & & & & & & & \mathbb{Z} & & & & & & & & & & \\
 & & & & & & & & & \uparrow & & & & & & & & & & \\
\cdots & & & & & & & & & \cdots & & & & & & & & & & \cdots \\
\cdots & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \cdots & & \\
\cdots & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdots & & \\
\cdots & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \rightarrow & & \mathbb{Z} \\
\cdots & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdot & \circ & \cdots & & \\
\cdots & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \bullet & \cdot & \cdots & & \\
\cdots & & & & & & & & & \cdots & & & & & & & & & & \cdots
\end{array}
$$

Let V be an open set containing A. Then, in below, it is proved that $V = \mathbb{Z}^n$; and hence the set A is g-closed in $(\mathbb{Z}^n, \kappa^n)$ (cf. Definition 2.1(i), i.e., [22, Definition 2.1]).
(*Proof of the property*: $V \supset \mathbb{Z}^n$). Let $x := (x_1, x_2, ..., x_n) \in \mathbb{Z}^n$ such that $x \notin A$. For this point x, we have $x \in (\mathbb{Z}^n)_{mix(r)}$ for some integer r with $1 \leq r \leq n-1$. The component $x_{e(k)}$ is even, where $e(k) \in I_r(x)$ $(1 \leq k \leq r)$ and $x_{o(j)}$ is odd, where $o(j) \in J_{n-r}(x)$ $(1 \leq j \leq n-r)$ (cf. the notation in Definition 4.3, the proof (Case 3) of Theorem 4.5(i) or in the proof (Case 2) of Proposition 4.1(i)). We pick a point $y := (y_1, y_2, ..., y_n)$ as follow: $y_{e(k)} := x_{e(k)}(1 \leq k \leq r)$ and $y_{o(j)} := x_{o(j)} + 1(1 \leq j \leq n-r)$. Then, $y \in (\mathbb{Z}^n)_{\mathcal{F}^n} \subset A$ and $x \in U^n(y)$. Since $y \in A \subset V$ and V is open, we have $U^n(y) \subset V$ (cf. Definition 3.7, (I)(∗4)(i)(ii) in Section 3,Theorem 3.9(iii)); and so $x \in V$. $(\circ)$
Thus, we have $\mathrm{Cl}(A) \subset \mathbb{Z}^n = V$ for an open set V such that $A \subset V$,i.e., A is g-closed. On the other hand, it is shown that $\mathrm{Cl}(A) = \mathbb{Z}^n$ and so A is not closed in $(\mathbb{Z}^n, \kappa^n)$ (cf. Theorem 4.6).

We mention an alternative proof of the result [30, Theorem 4.2] (cf. Remark 4.7(ii) above). For $(\mathbb{Z}^n, \kappa^n)$ $(n \geq 2)$, we can construct directly two disjoint semi-open sets separating two given distinct points (cf. Corollary 4.10). We need the following property Theorem 4.9 on the smallest open sets and Theorem 4.4.

Theorem 4.9 *Let $x, x' \in \mathbb{Z}^n$, where $1 \leq n$. If $x \neq x'$ in $(\mathbb{Z}^n, \kappa^n)$, then $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds.*

Proof. We first recall that $\mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ (disjoint union) holds and $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset (1 \leq r \leq n-1)$ if $n \geq 2$ (cf. (II)(∗20)(iv) in Section 3). Since $\{x, x'\} \subset \mathbb{Z}^n$, we should check the cases below, Case i (1 ≤ i≤ 3), in order to prove $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$. We secondly suppose, for a contradiction, that
$(*1)$ $(U^n(x))_{\kappa^n} = (U^n(x'))_{\kappa^n}$ holds.

Case 1. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\kappa^n}$ (cf. Definition 3.11(i)(i-1)): for these points x and x', we have that $\{x\}$ and $\{x'\}$ are open singletons and $U^n(x) = \{x\}$ and $U^n(x') = \{x'\}$ (cf. Definition 3.7, (I)(∗4)(ii) in Section 3); and so, by $(*1)$ above, $\{x\} = (U^n(x))_{\kappa^n} = (U^n(x'))_{\kappa^n} = \{x'\}$. This contradicts the first setting of the given points x and x' (i.e., $x' \neq x$).

Case 2. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r')} | 1 \leq r' \leq n-1\})$ (cf. Definition 3.11(i)): for this case, $\{x\} = U^n(x)$ holds (cf. Definition 3.7(I)(∗4)(ii) in Section 3); and by Theorem 4.4(ii)(iii), it is obtained that $\#(U^n(x'))_{\kappa^n} = 2^{R'}$, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$

and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1)$. And so, by $(*1)$, we have that $2^{R'} = 1$ holds, i.e., $2^n = 1$ or $2^{r'} = 1$. These contradict the first setting of the given integers n with $n \geq 1$ and r' with $1 \leq r' \leq n-1$.

Case 3. $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ (cf. Definition 3.11(i)(i-2)(i-3)):

By Theorem 4.4(ii) and (iii) for the point x, there exist the open singletons $\{p(x, u)\}(1 \leq u \leq R)$ such that $(U^n(x))_{\kappa^n} = \{p(x, u) | 1 \leq u \leq R\}$ holds, where $R := n$ if $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R := r$ if $x \in (\mathbb{Z}^n)_{mix(r)}(1 \leq r \leq n-1, n \geq 2)$. Moreover, for the point x', there exist the open singletons $\{p(x', u')\}(1 \leq u' \leq R')$ such that $(U^n(x'))_{\kappa^n} = \{p(x', u') | 1 \leq u' \leq R'\}$ holds, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1$ and $n \geq 2)$. We may assume that $R' \leq R$. Then, $\{p(x', u') | 1 \leq u' \leq 2^{R'}\} = (U^n(x'))_{\kappa^n} = (U^n(x))_{\kappa^n} \cap (U^n(x'))_{\kappa^n} = (U^n(x) \cap U^n(x'))_{\kappa^n} \subset U^n(x) \cap U^n(x')$. Namely, $U^n(x) \cap U^n(x')$ contains exactly the $2^{R'}$ open singletons $\{p(x', u')\}$ $(1 \leq u' \leq 2^{R'})$. This shows that the assumptions of Theorem 3.12 (i.e., [39, Lemma 2.3]) are satisfied. And, using $(*1)$ above, we have $2^{R'} = \#((U^n(x'))_{\kappa^n}) = \#((U^n(x))_{\kappa^n}) = 2^R$ and so $R' = R$. Then, under the assumption $(*1)$ above, we do not have the case where that $(R', R) = (r', n)$ or (n, r), because $r, r' \in \{1, 2, ..., n-1\}$ hold. Namely, under $(*1)$, the following case does not occurs : $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $x' \in (\mathbb{Z}^n)_{mix(r')}(1 \leq r' \leq n-1)$ (or $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $x \in (\mathbb{Z}^n)_{mix(r)}(1 \leq r \leq n-1)$). For other all cases where $(R', R) = (n, n)$ (i.e., $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n}$) or $(R', R) = (r', r)$ (i.e., $x \in (\mathbb{Z}^n)_{mix(r)}$ and $x' \in (\mathbb{Z}^n)_{mix(r')}$) with $r, r' \in \{1, 2, ..., n-1\}$, using Theorem 3.12(iii)' (i.e.,[39, Lemma 2.3 (iii)']), we have $x' = x$; this contradicts the first setting of the given points x and x' (i.e., $x' \neq x$).

Therefore, we show the required property that $(*2)$ $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds if $x \neq x'$ in $(\mathbb{Z}^n, \kappa^n)$. $\square$

Corollary 4.10 (Namda [30, Theorem 4.2] for any $n \geq 1$; [11] for $n = 1, 2$) *The digital n-space $(\mathbb{Z}^n, \kappa^n)$ is a semi-T_2-space.*

Proof. Suppose $n \geq 2$ in the present proof; and so we have $(\mathbb{Z}^n)_{mix(r)} \neq \emptyset$ for each integer r with $1 \leq r \leq n-1$ (cf. Definition 3.11(i)(i-3)). We use Theorem 4.4 on the construction of semi-open sets in $(\mathbb{Z}^n, \kappa^n)$ and Theorem 4.9; and we prove that $(\mathbb{Z}^n, \kappa^n)$ is semi-T_2, where $n \geq 2$, as follows.

Let x and x' be any distinct points of $(\mathbb{Z}^n, \kappa^n)$. We set $x = (x_1, x_2, ..., x_n)$ and $x' = (x'_1, x'_2, ..., x'_n)$, where $x_i \in \mathbb{Z}$ and $x'_i \in \mathbb{Z}(1 \leq i \leq n)$. Since $\{x, x'\} \subset \mathbb{Z}^n = (\mathbb{Z}^n)_{\kappa^n} \cup (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$ (disjoint union) (cf. (II)(*20)(iv) in Section 3), we consider the required proof for the following cases.

For the points x and x', we first use Theorem 4.9; we have that:
$(*2)$ $(U^n(x))_{\kappa^n} \neq (U^n(x'))_{\kappa^n}$ holds, where $U^n(y)$ is the smallest open set containing each point $y \in \{x, x'\}$. Namely, we have that:
- $(*a)$ there exists a point $z \in (U^n(x))_{\kappa^n}$ and $z \notin (U^n(x'))_{\kappa^n}$; or,
- $(*b)$ there exists a point $z' \in (U^n(x'))_{\kappa^n}$ and $z' \notin (U^n(x))_{\kappa^n}$.

Case 1. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\kappa^n}$: it is obviouse that $\{x\}$ and $\{x'\}$ are the required disjoint semi-open sets, because every open set is semi-open.

Case 2. $\{x, x'\} \subset (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \leq r \leq n-1\})$:

• For Case $(*a)$ above, by Theorem 4.4(ii) and (iii) for the point x, it is shown that $z = p(x, u_0)$ holds for some point $p(x, u_0) \in (U^n(x))_{\kappa^n}(1 \leq u_0 \leq 2^R)$, where $R := n$ if $x \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R := r$ if $x \in (\mathbb{Z}^n)_{mix(r)}$, because $(U^n(x))_{\kappa^n} = \{p(x, u) | 1 \leq u \leq 2^R\}$ holds. Moreover, we have that $\{x\} \cup \{z\}$ is a semi-open set containing the point x (cf. Theorem 4.4 (ii-3) and (iii-3)). Using Theorem 4.4 (ii) and (iii) for the point x', we can take any semi-open sets $\{x'\} \cup \{p(x', u')\}$ containing x', where $\{p(x', u') | 1 \leq u' \leq 2^{R'}\} = (U^n(x'))_{\kappa^n}$ and the integer R' is defined by $R' := n$ if $x' \in (U^n(x'))_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (U^n(x'))_{mix(r')}$ with $1 \le r' \le n-1$. Then, we have that $(\{x\} \cup \{z\}) \cap (\{x'\} \cup \{p(x',u')\}) =(\{x\} \cap \{x'\}) \cup (\{x\} \cap \{p(x',u')\}) \cup (\{z\} \cap \{x'\}) \cup (\{z\} \cap \{p(x',u')\}) \subset (V \cap (\mathbb{Z}^n)_{\kappa^n}) \cup ((U^n(x))_{\kappa^n} \cap V) \cup (\{z\} \cap (U^n(x'))_{\kappa^n}) = \emptyset$, where $V := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \le r \le n-1\})$, because of the decomposition of $\mathbb{Z}^n$ and the property in ($*$a) (i.e., $z \notin (U^n(x'))_{\kappa^n}$). Thus, for Case ($*$a), $\{x\} \cup \{z\}$ and $\{x'\} \cup \{p(x',u')\}$ are the required disjoint semi-open sets containing the points x and x', respectively.

• For Case ($*$b) above, by Theorem 4.4(ii) and (iii) for the point x', it is shown that $z' = p(x',u'_0)$ for some point $p(x',u'_0) \in (U^n(x'))_{\kappa^n}$, because $(U^n(x'))_{\kappa^n} = \{p(x',u') | 1 \le u' \le R'\}$ holds, where $R' := n$ if $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n}$ and $R' := r'$ if $x' \in (\mathbb{Z}^n)_{mix(r')}$ with $1 \le r' \le n-1$. Here we note that $z' \notin (U^n(x))_{\kappa^n}$. It is shown that $\{x'\} \cup \{z'\}$ (i.e., $\{x'\} \cup \{p(x',u'_0)\}$) is the required semi-open set containing x' (cf. Theorem 4.4(ii-3) and (iii-3) for the point x'). Using Theorem 4.4 (ii) and (iii) for the point x, we can take any semi-open sets $\{x\} \cup \{p(x,u)\}$ containing x, where $\{p(x,u) | 1 \le u \le 2^R\} = (U^n(x))_{\kappa^n}$ for the integer R with $R := n$ if $x \in (U^n(x))_{\mathcal{F}^n}$ and $R := r$ if $x \in (U^n(x))_{mix(r)}$ with $1 \le r \le n-1$. Thus, the above semi-open sets $\{x\} \cup \{p(x,u)\}$ and $\{x'\} \cup \{z'\}$ are the required disjoint semi-open sets containing the point x and x', respectively. Indeed, we have that $(\{x\} \cup \{p(x,u)\}) \cap (\{x'\} \cup \{z'\})= (\{x\} \cap \{x'\}) \cup (\{x\} \cap \{z'\}) \cup (\{p(x,u)\} \cap \{x'\}) \cup (\{p(x,u)\} \cap \{z'\}) \subset (V \cap (\mathbb{Z}^n)_{\kappa^n}) \cup ((U^n(x))_{\kappa^n} \cap V) \cup ((U^n(x))_{\kappa^n} \cap \{z'\}) = \emptyset$, where $V := (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \le r \le n-1\})$, because of the setting that $x \neq x'$, the decomposition of $\mathbb{Z}^n$ and $z' \notin (U^n(x))_{\kappa^n}$ for the Case ($*$b).

Case 3. $x \in (\mathbb{Z}^n)_{\kappa^n}$ and $x' \in (\mathbb{Z}^n)_{\mathcal{F}^n} \cup (\bigcup\{(\mathbb{Z}^n)_{mix(r)} | 1 \le r \le n-1\})$: for this case, we have that $\{x\} = U^n(x)$ and $\{x\} \cap (U^n(x'))_{\kappa^n} = \emptyset$ and so $\{x\}$ is the required semi-open set containing the point x. We can construct the required semi-open set containing x' using Theorem 4.4; the construction is done by an argument similar to that in Case 2.

Therefore, by Case 1, Case 2, Case 3 above for distinct points x and x', there exist disjoint semi-open sets containing the point x and x', respectively; and so $(\mathbb{Z}^n, \kappa^n)$ is semi-T$_2$. □

Remark 4.11 (cf. Remark 4.7(ii)) The digital n-space $(\mathbb{Z}^n, \kappa^n)$ is semi-T$_2$, where $n \ge 1$ [30]; $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}^2, \kappa^2)$ are semi-T$_2$ [11]. The results are confirmed directly by Corollary 4.10 above. Moreover, since the semi-T$_2$ separation axiom implies the semi-T$_1$ separation axiom, using Proposition 2.6(i), we have an alternative proof of Theorem 4.5(ii) (cf. Definition 2.3). The above proof of Corollary 4.10 is done constructively; the present authors believe that we applies the same method to other topological properties on $(\mathbb{Z}^n, \kappa^n)$ which are not proved by arguments preserving of topological products of $(\mathbb{Z}, \kappa)$ and we have further applications.

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GLOBAL EXISTENCE AND EXPONENTIAL ATTRACTOR OF SOLUTIONS OF FIX-CAGINALP EQUATION

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Abstract. We consider the Fix-Caginalp equation with the Neumann boundary condition in $\mathbf{R}^n$ with $n = 1, 2, 3$. We obtain a global solution by the existence of the Lyapunov function. After, we construct a dynamical system corresponding to the equation. By the existence of the Lyapunov function, the ω-limit set is included in the set of its stationary solution. We treat its dynamical properties such as a global attractor, absorbing set, exponential attractor and so on. It is important to obtain the estimate independent of the initial value. Finally, we construct an exponential attractor.

1 Introduction In this paper, we consider the following Fix-Caginalp equation:

$$\begin{cases} \tau\phi_t = \epsilon^2\Delta\phi + \phi - \phi^3 + 2u & x \in \Omega,\ t > 0, \\ u_t + \frac{l}{2}\phi_t = \kappa\Delta u & x \in \Omega,\ t > 0, \\ \frac{\partial\phi}{\partial\nu} = \frac{\partial u}{\partial\nu} = 0 & x \in \partial\Omega,\ t > 0, \\ \phi(x,0) = \phi_0(x) & x \in \Omega, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1}$$

where τ, l, κ and ϵ are positive constants, ν is the outer unit normal vector and Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ for $n = 1, 2, 3$. An equation (1) was proposed by Caginalp in [4] to describe the phase transitions with finite thickness interfaces. The unknown functions ϕ and u represent the phase function and the reduced temperature, respectively. The positive constants τ, l, κ and ϵ represent the relaxation time, the latent heat, the thermal diffusivity and a length scale which is a measure of the strength of the bonding at the microscopic level, respectively. In [12], they consider the historic background of the model and the derivation of a more general thermodynamically consistent model. At first in [4], he proved a global existence of a solution under the restriction $\frac{\epsilon^2}{\tau} < \kappa$. After in [7], [2], [3] and [16], they dropped the restriction and proved the global existence under the other boundary conditions

$$\phi(x,t) = \phi_{\partial\Omega}(x), \quad u(x,t) = u_{\partial\Omega}(x),$$

$$\frac{\partial\phi}{\partial\nu}(x,t) = 0, \quad u(x,t) = u_{\partial\Omega}(x)$$

and

$$\phi(x,t) = \phi_{\partial\Omega}(x), \quad \frac{\partial u}{\partial\nu}(x,t) = 0$$

for $x \in \partial\Omega,\ t > 0$, where $\phi_{\partial\Omega}(x)$ and $u_{\partial\Omega}(x)$ are given functions on $\partial\Omega$. In [7], they consider the stationary problem with the Neumann boundary condition, derive the existence and

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non-existence of nontrivial solutions and the multi-existence of trivial solutions according to the values of constants l, ϵ and $\int_\Omega u dx + \frac{l}{2}\int_\Omega \phi dx$ and deal with their stabilities. If $n = 1$, the stationary problem with the Dirichlet boundary condition is considered in [5] and [9]. They show that there exist exactly $2m + 1$ solutions with m being an integer determined by ϵ^2 and Ω. In [7], they consider the asymptotic behaviour of solution of (1). For results with initial data in different settings of spaces, see [7], [1] and [3]. Lately in [13], they consider non-local stationary problem and get some results on multiple existence, stability and bifurcation of the solution. For a system of reaction-diffusion equations in a bounded domain $\Omega \subset \mathbf{R}^2$, the existence of a global attractor and exponential attractor is proved in [11]. Their key fact is that its dynamical system has the squeezing property. Although the global existence for $(\phi_0, u_0) \in H^1(\Omega) \times L^2(\Omega)$ is known by [7] and [16], we treat more general space $H^\gamma(\Omega) \times H^\gamma(\Omega)$. For the definition of function space and notion of dynamical system, see Section 2 in this paper or [15], [6], [8], [14], [9]. In [16], he proves the dynamical properties with the Dirichlet boundary condition instead of the Neumann boundary condition. Since we can use the Poincaré inequality, the estimates of the Dirichlet boundary condition case are easier. In particular, since the solution (ϕ, u) with the Dirichlet boundary condition has the global dissipative property, we don't have to consider a space H_k mentioned in Theorem 4 in this paper in order to construct a global attractor. The purpose of this paper is to establish the existence of a global solution, the properties of ω-limit set and the exponential attractor in the dynamical system introduced by the Fix-Caginalp equation. The first theorem is concerned with the global existence.

Theorem 1 *Let $\Omega \subset \mathbf{R}^n (n = 1, 2, 3)$ be a bounded domain with smooth boundary $\partial\Omega$. We suppose that $\phi_0, u_0 \in H^\gamma(\Omega)$ for $\underline{\gamma} < \gamma < \overline{\gamma}$, where $\left(n, \underline{\gamma}, \overline{\gamma}\right) = \left(1, 0, \frac{1}{4}\right), \left(2, 0, \frac{1}{2}\right), \left(3, \frac{1}{2}, \frac{2}{3}\right)$. Then, the problem (1) admits a unique global solution (ϕ, u) such that*

$$\phi, u \in C\left((0,\infty); H^1(\Omega)\right) \cap C\left([0,\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,\infty); H^{-1}(\Omega)\right).$$

The associated nonlinear semigroup $T(t)$

$$T(t)\left(\phi_0(\cdot), u_0(\cdot)\right) = \left(\phi(\cdot, t), u(\cdot, t)\right)$$

defines a dynamical system in $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

To obtain the a priori estimate for H^1 norm, we use the Lyapunov function

$$L(\phi, u)(t) = \frac{1}{2}\int_\Omega u^2 dx + \frac{l\epsilon^2}{8}\int_\Omega |\nabla\phi|^2 dx + \frac{l}{4}\int_\Omega W(\phi) dx + \frac{\kappa\delta}{2}\int_\Omega |\nabla u|^2 dx$$

for $\delta < \frac{4\tau}{l}$, where

$$W(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2.$$

In the second theorem, we obtain the regularity of solution.

Theorem 2 *Under the same assumption as Theorem 1,*

$$\phi, u \in C^\infty\left((0, +\infty); C^\infty(\overline{\Omega})\right).$$

For any $\eta > 0$, the orbit $t \in [\eta, +\infty) \mapsto (\phi(\cdot, t), u(\cdot, t))$ is compact in $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

Combining the estimates obtained in Theorems 1 and 2 with the existence of the Lyapunov function, we consider the structure of ω-limit set in the third theorem. At first, by E we denote the set of stationary solution corresponding to (1). Since $\phi(t), u(t) \in H^1(\Omega)$ for $t > 0$, we assume that $\phi_0, u_0 \in H^1(\Omega)$. As proved in Theorem 1, it is also easy to show that the dynamical system is defined on $H^1(\Omega) \times H^1(\Omega)$.

Theorem 3 *We suppose that* $\phi_0, u_0 \in H^1(\Omega)$. *Then,* $\omega(\phi_0, u_0)$ *is nonempty, compact, invariant and connected in* $H^1(\Omega) \times H^1(\Omega)$. *And* $\omega(\phi_0, u_0)$ *is a single point and it holds that* $\omega(\phi_0, u_0) \subset E$.

We construct an exponential attractor in $H^1(\Omega) \times H^1(\Omega)$ in the last theorem. However, the solution (ϕ, u) of (1) does not have the global dissipative property. Thus, we restrict the initial function to

$$H_k = \left\{(\phi_0, u_0) \in H^1(\Omega) \times H^1(\Omega) \mid L(\phi_0, u_0) \le k\right\}$$

for fixed $k > 0$ and reduce a dynamical system to its subdynamical system $\{T(t) : H_k \to H_k\}$.

Theorem 4 *Under the same assumption as Theorem 3,* $T(t)$ *is dissipative in* H_k. *The dynamical system* $T(t)$ *has a global attractor* $\mathcal{A} \subset H_k$. *Then, there exists a compact absorbing and positively invariant set* $\mathcal{X} \subset H_k$ *such that its subdynamical system* $\{T(t) : \mathcal{X} \to \mathcal{X}\}$ *admits an exponential attractor* $\mathcal{E}$ *in* $H^1(\Omega) \times H^1(\Omega)$.

This paper is composed of 6 sections. In Section 2, we introduce the notions and theories of an abstract evolution equation and dynamical system. We also refer to the function space involved in this paper. In Section 3, we apply the existence theorem in Section 2 and establish the local solution of (1). In Section 4, we derive the a priori estimates and extend the local solution globally in time. In Section 5, we consider a nonlinear mapping from the initial function to the solution of (1) and define the dynamical system. The obtained estimates in Section 4 lead us to the proof of Theorems 1, 2 and 3. In section 6, we construct an exponential attractor and prove Theorem 4. Now that we restrict to H_k and have the Lyapunov function, our result follows at once.

2 Preliminaries We introduce the results and related facts in an abstract evolution equation. These results are mentioned in mainly [15] and [9], [8], [6]. Let X be a Banach space with the norm $\|\cdot\|$. Let A be a densely defined, closed linear operator in X. We assume that the spectrum of A is contained in an open sectorial domain such that

$$\sigma(A) \subset \Sigma_\omega \equiv \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \omega\}, \qquad \omega_A < \omega < \frac{\pi}{2} \tag{2}$$

and

$$\left\|(\lambda - A)^{-1}\right\| \le \frac{M_\omega}{|\lambda|}, \qquad \lambda \notin \Sigma_\omega, \omega_A < \omega < \frac{\pi}{2} \tag{3}$$

for $\omega_A \in [0, \frac{\pi}{2})$, where $M_\omega > 0$ is a constant depending on A and ω. We call A a sectorial operator of X with angle $0 \le \omega_A < \frac{\pi}{2}$. We consider the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} U_t + AU = F(U) & t > 0, \\ U(0) = U_0 \end{cases} \tag{4}$$

in X. Here, F is a nonlinear operator from $\mathcal{D}(A^\eta)$ into X, where $0 < \eta < 1$ and satisfies a Lipschitz condition of the form

$$\begin{aligned} \|F(U) - F(V)\| \quad &\le \quad \Phi\left(\left\|A^\beta U\right\| + \left\|A^\beta V\right\|\right) \times \\ &\quad \left\{\|A^\eta (U - V)\| + (\|A^\eta U\| + \|A^\eta V\|)\left\|A^\beta (U - V)\right\|\right\} \end{aligned} \tag{5}$$

for $U, V \in \mathcal{D}(A^\eta)$ with $0 < \beta \le \eta < 1$, where $\Phi(\cdot)$ is some increasing continuous function. We have the following global existence theorem.

Theorem 5 (Theorem 4.1 in [15]) *Let (2), (3) and (5) with $0 < \beta \le \eta < 1$ be satisfied. Then, for any $U_0 \in \mathcal{D}(A^\beta)$, (4) admits a unique local solution U in*

$$U \in C\left((0, T_{U_0}]; \mathcal{D}(A)\right) \cap C\left([0, T_{U_0}]; \mathcal{D}(A^\beta)\right) \cap C^1\left((0, T_{U_0}]; X\right),$$

where T_{U_0} denotes the maximal existence time depending only on the norm $\left\|A^\beta U_0\right\|$. Moreover, it holds that

$$\left\|A^\beta U\right\| + t^{1-\beta}\left\|U_t\right\| + t^{1-\beta}\left\|AU\right\| \le C_{U_0},$$

where C_{U_0} is a positive constant depending only on $\left\|A^\beta U_0\right\|$.

Here, we note that $\mathcal{D}(A^\beta) = X$ for $\beta = 0$. We can take $\beta = 0$ in the condition (5) throughout theorems in this section.

Theorem 6 (Corollary 4.1 in [15]) *Under the assumption of Theorem 5, we suppose that any local solution U satisfies the estimate*

$$\left\|A^\beta U(t)\right\| \le C_{U_0},$$

for $0 \le t \le T_{U_0}$ with some positive constant C_{U_0} depending only on $\left\|A^\beta U_0\right\|$ and independent of T_{U_0}. Then, (4) admits a unique global solution U for all $t > 0$.

Let $K(R)$ be a bounded ball in the space $\mathcal{D}(A^\beta)$

$$K(R) = \left\{U \in \mathcal{D}(A^\beta) \mid \left\|A^\beta U\right\| \le R\right\}$$

for $0 < R < \infty$. Then, for all $U_0 \in K(R)$, there exists a local solution of (4) on some interval $[0, T_{U_0}]$. There exists the time $T_R > 0$ such that $[0, T_R] \subset [0, T_{U_0}]$ for all $U_0 \in K(R)$. We have the theorem of the continuous dependence.

Theorem 7 (Theorem 4.3 and Corollary 4.2 in [15]) *Under the assumption of Theorem 5, let U and V be the solutions of (4) for the initial functions U_0 and V_0 in $K(R)$, respectively. Then, we have*

$$t^\eta\left\|A^\eta\left(U(t) - V(t)\right)\right\| + t^\beta\left\|A^\beta\left(U(t) - V(t)\right)\right\| + \left\|U(t) - V(t)\right\| \le L_R\left\|U_0 - V_0\right\|$$

and

$$t^{\eta-\beta}\left\|A^\eta\left(U(t) - V(t)\right)\right\| + \left\|A^\beta\left(U(t) - V(t)\right)\right\| \le L_R\left\|A^\beta\left(U_0 - V_0\right)\right\|$$

for $0 < t \le T_R$, where L_R is a positive constant depending only on R.

We assume that there exists an increasing continuous function $p(\cdot) > 0$ such that any local solution satisfies

$$\left\|A^\beta U(t)\right\| \le p(\left\|A^\beta U_0\right\|)$$

for $t \in [0, T_{U_0}]$ and $U_0 \in \mathcal{D}(A^\beta)$. Theorem 6 implies that there exists a global solution on $[0, +\infty)$ with the estimate

$$\left\|A^\beta U(t)\right\| \le p(\left\|A^\beta U_0\right\|) \tag{6}$$

for $t \in [0, +\infty)$ and $U_0 \in \mathcal{D}(A^\beta)$. We define a nonlinear operator $T(t) : \mathcal{D}(A^\beta) \to \mathcal{D}(A^\beta)$ by $T(t)U_0(\cdot) = U(\cdot, t)$. Let $\mathcal{M}$ be a subset of $\mathcal{D}(A^\beta)$, $\mathcal{M}$ being a metric space with the distance $d(U, V) = \left\|A^\beta\left(U - V\right)\right\|$ for $U, V \in \mathcal{M}$. A family of nonlinear operators $T(t)$ for $t \ge 0$ from $\mathcal{M}$ to itself is said to be a continuous semigroup on $\mathcal{M}$ provided that
(SG.1) $T(0)$ is an identity mapping on $\mathcal{M}$,
(SG.2) $T(t)T(s) = T(t+s)$ for $t, s \ge 0$,
(SG.3) $T(t)$ is continuous from $[0, +\infty) \times \mathcal{M}$ to $\mathcal{M}$.
To show the property (SG.3), we combine Theorem 7 with the estimate (6). We apply the estimate on the larger ball $K_{p(R)} \supset K_R$ because $\cup_{0 \le t < \infty} T(t)K_R \subset K_{p(R)}$.

Theorem 8 (Proposition 6.2 in [15]) *For any $0 < R < \infty$, it holds that*

$$\left\|A^{\beta}\left(T(t)U_0 - T(t)V_0\right)\right\| \leq L_{p(R)}^{n+1}\left\|A^{\beta}\left(U_0 - V_0\right)\right\|$$

for $t \in [nT_{p(R)}, (n+1)T_{p(R)}]$ with $n \in \mathbf{N} \cup \{0\}$ and $U_0, V_0 \in K_R$, where $L_{p(R)}^{n+1} > 0$ is a constant depending only on n and $p(R)$.

Henceforth, we write $X = \mathcal{D}(A^{\beta})$. We denote the totality of trajectories starting from the points in $\mathcal{M}$ by the triplet $(T(t), \mathcal{M}, X)$ and call it a dynamical system. A set $\Sigma \subset \mathcal{M}$ is said to be positively invariant under $T(t)$ if $T(t)\Sigma \subset \Sigma$ for all $t \geq 0$. A set $\Sigma \subset \mathcal{M}$ is said to be negatively invariant under $T(t)$ if $\Sigma \subset T(t)\Sigma$ for all $t \geq 0$. A set Σ is invariant under $T(t)$ if it satisfies both conditions. A set $A \subset \mathcal{M}$ is said to attract a set $B \subset \mathcal{M}$ under $T(t)$ if

$$\sup_{v \in T(t)B} \inf_{u \in A} \|v - u\| \to 0$$

as $t \to +\infty$. $T(t)$ is said to be dissipative if there exists a bounded set $C \subset \mathcal{M}$ such that attracts every point of $\mathcal{M}$ under $T(t)$. A set $\mathcal{A} \subset \mathcal{M}$ of $(T(t), \mathcal{M}, X)$ is said to be a global attractor if $\mathcal{A}$ is a maximal compact invariant set and attracts every bounded set $B \subset \mathcal{M}$. A set $D \subset \mathcal{M}$ is said to be an absorbing set if for every bounded set $B \subset \mathcal{M}$, there exists t_0 such that $\cup_{t \geq t_0} T(t)B \subset D$ holds. We take $t_1 \geq t_0$ so that $\cup_{t \geq t_1} T(t)D \subset D$ holds. Let $\mathcal{X} = \overline{\cup_{t \geq t_1} T(t)D} \subset D$. $\mathcal{E}$ is said to be an exponential attractor of $(T(t), \mathcal{X}, X)$, provided that
(EA.1) $\mathcal{A} \subset \mathcal{E} \subset \mathcal{X}$ holds, where $\mathcal{A}$ is a global attractor,
(EA.2) $\mathcal{E}$ is compact in X,
(EA.3) $\mathcal{E}$ is positively invariant under $T(t)$,
(EA.4) $\mathcal{E}$ has a finite fractal dimension $d_F(\mathcal{E})$,
(EA.5) $\sup_{u \in T(t)\mathcal{X}} \inf_{v \in \mathcal{E}} \|u - v\| \leq c_0 e^{-c_1 t}$, where c_0 and c_1 are positive constants. Here, if we denote by $N_r(\mathcal{E})$ the smallest number of $r-$balls necessary to cover $\mathcal{E}$, we define a fractal dimension by

$$d_F(\mathcal{E}) = \limsup_{r \to 0} \frac{\log N_r(\mathcal{E})}{\log \frac{1}{r}}.$$

Then, we have

Theorem 9 (Theorem 3.1 in [6]) *Let $F(U)$ satisfy the Lipschitz condition*

$$\|F(U) - F(V)\| \leq C_{\mathcal{X}} \left\|A^{\frac{1}{2}}\left(U - V\right)\right\|$$

for $U, V \in \mathcal{X}$, where $C_{\mathcal{X}} > 0$ depends only on $\mathcal{X}$. Moreover, we assume that the mapping $S(t, U_0) = T(t)U_0$ satisfies the Lipschitz condition

$$\|S(s, U_0) - S(t, V_0)\| \leq C_{\mathcal{X},T}\left(\|U_0 - V_0\| + |t - s|\right)$$

for $U_0, V_0 \in \mathcal{X}$ and $s, t \in [0, T]$ with any $T > 0$, where $C_{\mathcal{X},T}$ depends only on $\mathcal{X}$ and T. Then, the flow $\{T(t)\}$ admits an exponential attractor $\mathcal{E}$.

Finally, we introduce the function space treated in this paper. For $p \in \mathbf{N}$, $H^p(\Omega)$ denotes the usual Sobolev space with the norm

$$\|w\|_{H^p} = \left(\sum_{|\alpha| \leq p} \|D^{\alpha} w\|_2^2\right)^{\frac{1}{2}}$$

for $w \in H^p(\Omega)$, where $\|\cdot\|_p$ denotes the standard L^p norm in Ω, α is a multi index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}.$$

For $0 \le s_0 < s < s_1 < +\infty$, $H^s(\Omega)$ is the interpolation space between $H^{s_0}(\Omega)$ and $H^{s_1}(\Omega)$, denoted $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta, s = (1-\theta)\, s_0 + \theta s_1$ with $\theta \in [0,1]$. Then, the interpolation inequality

$$\|\cdot\|_{H^s} \le C \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta}$$

holds according to Theorem 1.15 in [15]. Moreover, we denote

$$H_N^m(\Omega) = \left\{ u \in H^m(\Omega) \mid \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega \right\}$$

for $m > \frac{3}{2}$. By $\mathcal{D}(\Omega)$, we denote the space of all infinitely differentiable functions on Ω with compact supports. $H_0^s(\Omega)$ is defined as the closure of the set $\mathcal{D}(\Omega)$ in the space $H^s(\Omega)$. $H^{-s}(\Omega)$ is defined as the dual space of $H_0^s(\Omega)$.

3 Local solution We prove the local existence and uniqueness of the solution by the theories of an abstract evolution equation. We show that the nonlinear term in (1) satisfies the condition (5).

Proposition 1 (Local existence in H^γ) *Suppose that $\phi_0, u_0 \in H^\gamma(\Omega)$ for $\underline{\gamma} < \gamma < \overline{\gamma}$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T_{\phi_0,u_0}^\gamma]; H^1(\Omega)\right) \cap C\left([0, T_{\phi_0,u_0}^\gamma]; H^\gamma(\Omega)\right) \cap C^1\left((0, T_{\phi_0,u_0}^\gamma]; H^{-1}(\Omega)\right),$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are defined in Theorem 1. In this paper, T_{ϕ_0,u_0}^s denotes the maximal existence time depending only on the norms $\|u_0\|_{H^s}$ and $\|\phi_0\|_{H^s}$ of initial functions.

Proof of Proposition 1: (1) can be written into

$$\begin{cases} U_t + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0 \equiv \begin{pmatrix} \phi_0 \\ u_0 \end{pmatrix}, & \end{cases}$$

where

$$U = \begin{pmatrix} \phi \\ u \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}, \quad F = \begin{pmatrix} \frac{1}{\tau}\left\{\left(\epsilon^2+1\right)\phi - \phi^3 + 2u\right\} \\ \left(\kappa - \frac{l}{\tau}\right) u + \frac{l}{2\tau}\left(\phi^3 - \phi\right) \end{pmatrix},$$

$$A_1 = -\frac{\epsilon^2}{\tau}\left(\Delta - 1\right), \quad A_2 = -\kappa\left(\Delta - 1\right) \quad \text{and} \quad B = \frac{l\epsilon^2}{2\tau}\Delta.$$

The two operators A_1 and A_2 are positive definite self-adjoint operators of $H^{-1}(\Omega)$ with domains $\mathcal{D}(A_1) = \mathcal{D}(A_2) = H^1(\Omega)$. We regard B as a linear and bounded operator from $H^1(\Omega)$ to $H^{-1}(\Omega)$. If necessary, we put $w(x,t) = pu(x,t)$ for small $p > 0$. Then, the second equation in (1) is converted into

$$w_t + \frac{lp}{2}\phi_t = \kappa\Delta w.$$

For sufficiently small $p > 0$, we can suppose that

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ pB & A_2 \end{pmatrix}$$

and hence A are strictly positive operators of $X \equiv H^{-1}(\Omega) \times H^{-1}(\Omega)$. Theorems 2.1 and 2.16 in [15] imply that A is a sectorial operator with angle $0 \le \omega_A < \frac{\pi}{2}$ in X. Then, it holds that

$$\mathcal{D}(A^\beta) = H^\gamma(\Omega) \times H^\gamma(\Omega)$$

for $\frac{1}{2} < \beta < 1$, where $\gamma = 2\beta - 1$ (for details, see Theorems 12.1 and 16.7 in [15]). Under our setting, we can apply Theorem 5 in Section 2 to (1). In fact, by the next lemma, we show that the nonlinear term in (1) satisfies the condition (5). We set

$$(n, \underline{\beta}, \overline{\beta}, \overline{\alpha}) = \left(n, \frac{\underline{\gamma}+1}{2}, \frac{\overline{\gamma}+1}{2}, \overline{\alpha}\right) = \begin{cases} \left(1, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}\right) & \text{for } n = 1, \\ \left(2, \frac{1}{2}, \frac{3}{4}, 1\right) & \text{for } n = 2, \\ \left(3, \frac{3}{4}, \frac{5}{6}, 1\right) & \text{for } n = 3. \end{cases}$$

Lemma 1 *Let $n = 1, 2, 3$. Then, there exist α and β satisfying $0 < \underline{\beta} < \beta < \overline{\beta} < \alpha < \overline{\alpha} \le 1$ such that*

$$\left\| (\phi - \psi)^3 \right\|_{H^{-1}} \le C \left\| A_1^\beta (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^\alpha (\phi - \psi) \right\|_{H^{-1}}$$

for $\phi, \psi \in H^\alpha(\Omega)$, where C is a positive constant depending only on α, β and Ω.

Proof of Lemma 1: In the case of $n = 1, 2$, we note that

$$\|w\|_q \le C \|w\|_{H^1}$$

for $w \in H^1(\Omega)$, where $q > 1$ and C is a positive constant depending only on q and Ω. Henceforth, we denote a positive embedding constant depending only on q and Ω by C. We take $0 < p < 2$ and $4 < q$ with $\frac{2}{2+p} + \frac{2}{q} = 1$. For $n = 1$, we have

$$\begin{aligned} \left\| (\phi - \psi)^3 \right\|_{H^{-1}} &= \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \le 1} \left| \int_\Omega (\phi - \psi)^3 w dx \right| \\ &\le \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \le 1} \|w\|_q \|\phi - \psi\|_{2+p}^2 \|\phi - \psi\|_q \\ &\le C \|\phi - \psi\|_{H^{\frac{p}{4+2p}}}^2 \|\phi - \psi\|_{H^{\frac{1}{2+p}}} \\ &\le C \left\| A_1^{\frac{4+3p}{4(2+p)}} (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^{\frac{3+p}{2(2+p)}} (\phi - \psi) \right\|_{H^{-1}}. \end{aligned}$$

Here, $\frac{1}{2} < \frac{4+3p}{4(2+p)} < \frac{5}{8} < \frac{3+p}{2(2+p)} < \frac{3}{4}$. For $n = 2$, we have

$$\begin{aligned} \left\| (\phi - \psi)^3 \right\|_{H^{-1}} &\le C \|\phi - \psi\|_{H^{\frac{p}{2+p}}}^2 \|\phi - \psi\|_{H^{\frac{2}{2+p}}} \\ &\le C \left\| A_1^{\frac{1+p}{2+p}} (\phi - \psi) \right\|_{H^{-1}}^2 \left\| A_1^{\frac{4+p}{2(2+p)}} (\phi - \psi) \right\|_{H^{-1}}. \end{aligned}$$

Here, $\frac{1}{2} < \frac{1+p}{2+p} < \frac{3}{4} < \frac{4+p}{2(2+p)} < 1$. In the case of $n = 3$, we note that

$$\|w\|_6 \le C \|w\|_{H^1}$$

for $w \in H^1(\Omega)$. We take $\frac{3}{2} < p < 3$ and $\frac{18}{5} < q < 6$ with $\frac{5}{6+p} + \frac{1}{q} = \frac{5}{6}$. We have

$$\begin{aligned} \left\| (\phi - \psi)^3 \right\|_{H^{-1}} &\leq \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \leq 1} \|w\|_6 \, \|\phi - \psi\|^2_{\frac{2}{5}(6+p)} \, \|\phi - \psi\|_q \\ &\leq C \, \|\phi - \psi\|^2_{H^{\frac{3(1+p)}{2(6+p)}}} \, \|\phi - \psi\|_{H^{\frac{9-p}{6+p}}} \\ &\leq C \left\| A_1^{\frac{5(3+p)}{4(6+p)}} (\phi - \psi) \right\|^2_{H^{-1}} \left\| A_1^{\frac{15}{2(6+p)}} (\phi - \psi) \right\|_{H^{-1}} . \end{aligned}$$

Here, $\frac{3}{4} < \frac{5(3+p)}{4(6+p)} < \frac{5}{6} < \frac{15}{2(6+p)} < 1$. □

For $U = \begin{pmatrix} \phi \\ u \end{pmatrix}, V = \begin{pmatrix} \psi \\ v \end{pmatrix} \in \mathcal{D}(A^\alpha)$ with $\overline{\beta} < \alpha < \overline{\alpha}$, we have

$$F(U) - F(V) = \begin{pmatrix} \frac{1}{\tau} \left\{ \left(\epsilon^2 + 1 - 3\phi\psi \right) (\phi - \psi) - (\phi - \psi)^3 + 2\,(u - v) \right\} \\ \left(\kappa - \frac{l}{\tau} \right) (u - v) + \frac{l}{2\tau} \left\{ (\phi - \psi)^3 + (3\phi\psi - 1)\,(\phi - \psi) \right\} \end{pmatrix}$$

and concentrate on the estimetes

$$\|\phi - \psi\|_{H^{-1}}, \quad \left\| (\phi - \psi)^3 \right\|_{H^{-1}}, \quad \|\phi\psi\,(\phi - \psi)\|_{H^{-1}}, \quad \|u - v\|_{H^{-1}} .$$

Now by the estimates as obtained in Lemma 1, we can apply Theorem 5 to our setting. □

Remark 1 (Local existence in L^2) *In the case of $n = 1$, We can take $\gamma = 0$ in Proposition 1. Now that it holds that $H^{\frac{1}{2}+r}(\Omega) \subset C(\overline{\Omega})$ for $r > 0$, we have*

$$\left\| (\phi - \psi)^3 \right\|_{H^{-1}} \leq \sup_{w \in H_0^1(\Omega), \|w\|_{H^1} \leq 1} \|w\|_C \, \|\phi - \psi\|_2^2 \, \|\phi - \psi\|_C \leq C \, \|\phi - \psi\|_2^2 \, \|\phi - \psi\|_{H^{\frac{1}{2}+r}} ,$$

where $r \in (0, \frac{1}{2})$ and $\| \cdot \|_C$ denotes the norm of the space of continuous functions in Ω. Hence, for $\phi_0, u_0 \in L^2(\Omega)$, (1) admits a unique local solution (ϕ, u) such that

$$\phi, u \in C\left((0, T^0_{\phi_0, u_0}]; H^1(\Omega) \right) \cap C\left([0, T^0_{\phi_0, u_0}]; L^2(\Omega) \right) \cap C^1\left((0, T^0_{\phi_0, u_0}]; H^{-1}(\Omega) \right) .$$

Proposition 2 (Local existence in H^1) *Suppose that $\phi_0, u_0 \in H^1(\Omega)$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^1_{\phi_0, u_0}]; H^2_N(\Omega) \right) \cap C\left([0, T^1_{\phi_0, u_0}]; H^1(\Omega) \right) \cap C^1\left((0, T^1_{\phi_0, u_0}]; L^2(\Omega) \right) .$$

Proof of Proposition 2: In Theorem 5, we take

$$X = L^2(\Omega) \times L^2(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^1(\Omega) \times H^1(\Omega) \quad \mathcal{D}(A) = H^2_N(\Omega) \times H^2_N(\Omega) \quad \beta = \eta = \frac{1}{2}.$$

We have

$$\left\| (\phi - \psi)^3 \right\|_2 = \|\phi - \psi\|_6^3 \leq C^3 \, \|\phi - \psi\|^3_{H^1}$$

for $\phi, \psi \in H^1(\Omega)$. Hence, we can apply Theorem 5 to our setting. □

Proposition 3 (Local existence in H^2) *Suppose that $\phi_0, u_0 \in H^2_N(\Omega)$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^2_{\phi_0,u_0}]; H^3_N(\Omega)\right) \cap C\left([0, T^2_{\phi_0,u_0}]; H^2_N(\Omega)\right) \cap C^1\left((0, T^2_{\phi_0,u_0}]; H^1(\Omega)\right).$$

Proof of Proposition 3: In Theorem 5, we take

$$X = H^1(\Omega) \times H^1(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^2_N(\Omega) \times H^2_N(\Omega) \quad \mathcal{D}(A) = H^3_N(\Omega) \times H^3_N(\Omega) \quad \beta = \eta = \frac{1}{2}.$$

Since it holds that

$$\|w\|_C \le C \|w\|_{H^2}$$

for $w \in H^2_N(\Omega)$, we have

$$\left\|\nabla(\phi - \psi)^3\right\|_2 = 3\left\|(\phi - \psi)^2 \nabla(\phi - \psi)\right\|_2 \le 3C^2 \|\phi - \psi\|^2_{H^2} \|\phi - \psi\|_{H^1} \le 3C^2 \|\phi - \psi\|^3_{H^2}$$

for $\phi, \psi \in H^2_N(\Omega)$, which proves the proposition. □

Proposition 4 (Local existence in H^3) *Suppose that $\phi_0, u_0 \in H^3_N(\Omega)$. Then, (1) admits a unique local solution (ϕ, u) such that*

$$\phi, u \in C\left((0, T^3_{\phi_0,u_0}]; H^4_N(\Omega)\right) \cap C\left([0, T^3_{\phi_0,u_0}]; H^3_N(\Omega)\right) \cap C^1\left((0, T^3_{\phi_0,u_0}]; H^2_N(\Omega)\right).$$

Proof of Proposition 4: In Theorem 5, we take

$$X = H^2_N(\Omega) \times H^2_N(\Omega) \quad \mathcal{D}(A^{\frac{1}{2}}) = H^3_N(\Omega) \times H^3_N(\Omega) \quad \mathcal{D}(A) = H^4_N(\Omega) \times H^4_N(\Omega) \quad \beta = \eta = \frac{1}{2}.$$

The following estimate shows the proposition.

$$\begin{aligned}
\left\|\Delta(\phi - \psi)^3\right\|_2 &\le 6\left\|(\phi - \psi)\left|\nabla(\phi - \psi)\right|^2\right\|_2 + 3\left\|(\phi - \psi)^2 \Delta(\phi - \psi)\right\|_2 \\
&\le 6C^2 \|\phi - \psi\|_{H^3} \|\phi - \psi\|_{H^2} \|\phi - \psi\|_{H^1} + 3C^2 \|\phi - \psi\|^3_{H^2} \\
&\le 9C^2 \|\phi - \psi\|^3_{H^3}
\end{aligned}$$

for $\phi, \psi \in H^3_N(\Omega)$. □

4 Global solution We derive the a priori estimates to obtain the global solution. The tools are the Lyapunov function and energy method.

Lemma 2 *For $\phi_0, u_0 \in H^1(\Omega)$ and $t \in [0, T^1_{\phi_0,u_0}]$,*

$$L(\phi, u)(t) = \frac{1}{2}\int_\Omega u^2 dx + \frac{l\epsilon^2}{8}\int_\Omega |\nabla\phi|^2 dx + \frac{l}{4}\int_\Omega W(\phi) dx + \frac{\kappa\delta}{2}\int_\Omega |\nabla u|^2 dx$$

is the Lyapunov function for (1), where $\delta < \frac{4\tau}{l}$ and $W(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2$.

Proof of Lemma 2: We have only to prove that $L(\phi, u)(t)$ is monotone decreasing with respect to t. Now that we have $(\phi(t), u(t)) \in H^1(\Omega) \times H^1(\Omega)$ for $t \in [0, T^1_{\phi_0, u_0}]$ from Proposition 2, $L(\phi, u)(t) < \infty$ because of the inclusion $H^1(\Omega) \subset L^4(\Omega)$. Note that

$$l\tau a^2 + 2l\delta ab + 4\delta b^2 = l\left(\sqrt{\tau}a + \frac{\delta}{\sqrt{\tau}}b\right)^2 + \delta\frac{4\tau - l\delta}{\tau}b^2 \geq 0$$

for $a, b \in \mathbf{R}$ and $\delta < \frac{4\tau}{l}$. We have

$$\begin{aligned}
L(\phi, u)(t) - L(\phi, u)(t') &= \int_{t'}^{t} \frac{d}{dt} L(\phi, u)(s) ds \\
&= \int_{t'}^{t}\int_\Omega u u_t dx ds + \frac{l\epsilon^2}{4}\int_{t'}^{t}\int_\Omega \nabla\phi\cdot\nabla\phi_t dx ds + \frac{l}{4}\int_{t'}^{t}\int_\Omega \left(\phi^2 - 1\right)\phi\phi_t dx ds \\
&\quad + \kappa\delta\int_{t'}^{t}\int_\Omega \nabla u \cdot \nabla u_t dx ds \\
&= \int_{t'}^{t}\int_\Omega u\left(\kappa\Delta u - \frac{l}{2}\phi_t\right) dx ds - \frac{l\epsilon^2}{4}\int_{t'}^{t}\int_\Omega \Delta\phi\phi_t dx ds \\
&\quad + \frac{l}{4}\int_{t'}^{t}\int_\Omega \left(\phi^2 - 1\right)\phi\phi_t dx ds - \delta\int_{t'}^{t}\int_\Omega u_t\left(u_t + \frac{l}{2}\phi_t\right) dx ds \\
&= -\kappa\int_{t'}^{t}\int_\Omega |\nabla u|^2 dx ds - \frac{1}{4}\int_{t'}^{t}\int_\Omega \left(l\tau\phi_t^2 + 2l\delta\phi_t u_t + 4\delta u_t^2\right) dx ds \leq 0
\end{aligned}$$

for $0 \leq t' < t \leq T^1_{\phi_0, u_0}$. In particular, we have

$$\kappa\int_0^t \|\nabla u\|_2^2 ds + \frac{\delta(4\tau - l\delta)}{4\tau}\int_0^t \|u_t\|_2^2 ds \leq L(\phi_0, u_0) - L(\phi, u)(t) \leq L(\phi_0, u_0). \tag{7}$$

On the other hand, since

$$l\tau a^2 + 2l\delta ab + 4\delta b^2 = \delta\left(2b + \frac{l}{2}a\right)^2 + l\frac{4\tau - l\delta}{4}a^2 \geq 0$$

for $a, b \in \mathbf{R}$ and $\delta < \frac{4\tau}{l}$, it also holds that

$$\kappa\int_0^t \|\nabla u\|_2^2 ds + \frac{l(4\tau - l\delta)}{16}\int_0^t \|\phi_t\|_2^2 ds \leq L(\phi_0, u_0) - L(\phi, u)(t) \leq L(\phi_0, u_0). \tag{8}$$

□

Proposition 5 (Global existence in H^1) *Suppose that $\phi_0, u_0 \in H^1(\Omega)$. Then, (1) admits a unique global solution (ϕ, u) such that*

$$\phi, u \in C\left((0, +\infty); H_N^2(\Omega)\right) \cap C\left([0, +\infty); H^1(\Omega)\right) \cap C^1\left((0, +\infty); L^2(\Omega)\right).$$

Proof of Proposition 5: By Proposition 2, there exists a unique local solution (ϕ, u) in the same function space. We have only to derive the a priori estimate thanks to Theorem

6. From Lemma 2, it holds that

$$\begin{aligned}
&\frac{1}{2}\|u\|_2^2+\frac{l\epsilon^2}{8}\|\nabla\phi\|_2^2+\frac{l}{16}\|\phi\|_2^2+\frac{l}{16}\int_\Omega\left(\phi^2-\frac{3}{2}\right)^2dx+\frac{\kappa\delta}{2}\|\nabla u\|^2-\frac{5l}{64}|\Omega| \\
&= L(\phi,u)(t) \\
&\leq L(\phi_0,u_0) \\
&\leq \frac{1}{2}\|u_0\|_2^2+\frac{l\epsilon^2}{8}\|\phi_0\|_{H^1}^2+\frac{l}{16}\|\phi_0\|_4^4+\frac{l}{16}|\Omega|+\frac{\kappa\delta}{2}\|\nabla u_0\|^2.
\end{aligned}$$

The Sobolev embedding theorem implies that the right-hand side is finite, which completes the proof of Proposition 5. □

Proposition 6 (Global existence in H^2) *Suppose that $\phi_0, u_0 \in H_N^2(\Omega)$. Then, (1) admits a unique global solution (ϕ, u) such that*

$$\phi,u\in C\left((0,+\infty);H_N^3(\Omega)\right)\cap C\left([0,+\infty);H_N^2(\Omega)\right)\cap C^1\left((0,+\infty);H^1(\Omega)\right).$$

Proof of Proposition 6: As mentioned in Proposition 5, we derive the a priori estimates for H^2 norm. In this paper, we denote by $C_{H^s}>0$ the constant depending only on the norms $\|u_0\|_{H^s}$ and $\|\phi_0\|_{H^s}$ of initial functions, the measure $|\Omega|$ and physical constants $\tau, l, \kappa, \epsilon$. We have the following two inequalities from (1):

$$\begin{aligned}
\frac{\tau}{2}\frac{d}{dt}\|\phi_t\|_2^2+\epsilon^2\|\nabla\phi_t\|_2^2+3\int_\Omega\phi^2\phi_t^2dx &= \int_\Omega\phi_t\left(\tau\phi_t-\epsilon^2\Delta\phi+\phi^3\right)_t dx \\
&= \|\phi_t\|_2^2+2\int_\Omega u_t\phi_t dx
\end{aligned}\tag{9}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\kappa\|\nabla u_t\|_2^2+\frac{l}{\tau}\|u_t\|_2^2-\frac{\epsilon^2 l}{2\tau}\int_\Omega\nabla u_t\cdot\nabla\phi_t dx \\
&\quad+\frac{l}{2\tau}\int_\Omega u_t\left(\phi_t-3\phi^2\phi_t\right)dx \\
&= \int_\Omega u_t\left\{u_{tt}-\kappa\Delta u_t+\frac{l}{\tau}u_t+\frac{\epsilon^2 l}{2\tau}\Delta\phi_t+\frac{l}{2\tau}\left(\phi_t-3\phi^2\phi_t\right)\right\}dx \\
&= \frac{l}{2\tau}\int_\Omega u_t\left(-\tau\phi_t+\epsilon^2\Delta\phi+\phi-\phi^3+2u\right)_t dx=0.
\end{aligned}\tag{10}$$

By integrating (9) over $(0,t)$ with respect to t, we have

$$\frac{\tau}{2}\|\phi_t\|_2^2+\epsilon^2\int_0^t\|\nabla\phi_t\|_2^2ds\leq\frac{\tau}{2}\|(\phi_0)_t\|_2^2+2\int_0^t\|\phi_t\|_2^2ds+\int_0^t\|u_t\|_2^2ds,\tag{11}$$

which implies that $\phi_t \in L^2(\Omega)$ by (7) and (8). Hence by (1), we have

$$\|\Delta\phi\|_2\leq C_{H^2}\quad\text{and}\quad\|\phi_t\|_2\leq C_{H^2}.\tag{12}$$

Next by integrating (10) over $(0,t)$ with respect to t, we have

$$\begin{aligned}&\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\|(u_0)_t\|_2^2+\kappa\int_0^t\|\nabla u_t\|_2^2\,ds+\frac{l}{\tau}\int_0^t\|u_t\|_2^2\,ds\\&-\frac{\epsilon^2 l}{2\tau}\int_0^t\int_\Omega\nabla u_t\cdot\nabla\phi_t dxds+\frac{l}{2\tau}\int_0^t\int_\Omega u_t\left(\phi_t-3\phi^2\phi_t\right)dxds=0.\end{aligned}\tag{13}$$

Here, it holds that

$$\begin{aligned}\int_0^t\int_\Omega\nabla u_t\cdot\nabla\phi_t dxds&=\frac{2}{l}\int_0^t\int_\Omega\nabla u_t\cdot\nabla\left(\kappa\Delta u-u_t\right)dxds\\&=-\frac{\kappa}{l}\left(\|\Delta u\|_2^2-\|\Delta u_0\|_2^2\right)-\frac{2}{l}\int_0^t\|\nabla u_t\|_2^2\,ds.\end{aligned}$$

From (12), $\|\phi\|_{H^2}$ is bounded, which implies $\phi\in C(\overline{\Omega})$ from the Sobolev embedding theorem. Then, it holds that

$$\left|\int_0^t\int_\Omega u_t\left(\phi_t-3\phi^2\phi_t\right)dxds\right|\le C_{H^2}\int_0^t\left(\|u_t\|_2^2+\|\phi_t\|_2^2\right)ds.$$

Thus (13) becomes

$$\begin{aligned}&\frac{1}{2}\|u_t\|_2^2+\frac{\epsilon^2\kappa}{2\tau}\|\Delta u\|_2^2+\left(\kappa+\frac{\epsilon^2}{\tau}\right)\int_0^t\|\nabla u_t\|_2^2\,ds\\\le\ &\frac{1}{2}\|(u_0)_t\|_2^2+\frac{\epsilon^2\kappa}{2\tau}\|\Delta u_0\|_2^2+\frac{lC_{H^2}}{2\tau}\int_0^t\left(\|u_t\|_2^2+\|\phi_t\|_2^2\right)ds.\end{aligned}\tag{14}$$

Finally, we obtain

$$\|\Delta u\|_2\le C_{H^2}\quad\text{and}\quad\|u_t\|_2\le C_{H^2}\tag{15}$$

by (7) and (8). After all, (12) and (15) imply the conclusion of proposition. □

Proposition 7 (Global existence in H^3) *Suppose that $\phi_0,u_0\in H_N^3(\Omega)$. Then, (1) admits a unique global solution (ϕ,u) such that*

$$\phi,u\in C\left((0,+\infty);H_N^4(\Omega)\right)\cap C\left([0,+\infty);H_N^3(\Omega)\right)\cap C^1\left((0,+\infty);H_N^2(\Omega)\right).$$

Proof of Proposition 7: We derive the a priori estimates for H^3 norm. We have

$$\begin{aligned}\frac{\tau}{2}\frac{d}{dt}\int_\Omega|\nabla\phi_t|^2\,dx&=\int_\Omega\nabla\phi_t\cdot\nabla\left(\epsilon^2\Delta\phi+\phi-\phi^3+2u\right)_t dx\\\le\ &\int_\Omega\left(2|\nabla\phi_t|^2+|\nabla u_t|^2\right)dx+3\int_\Omega\Delta\phi_t\phi^2\phi_t dx\\=\ &\int_\Omega\left(2|\nabla\phi_t|^2+|\nabla u_t|^2\right)dx+\frac{3}{\epsilon^2}\int_\Omega\phi^2\phi_t\left(\tau\phi_t-\phi+\phi^3-2u\right)_t dx\\\le\ &2\|\nabla\phi_t\|_2^2+\|\nabla u_t\|_2^2+C_{H^2}\|\phi_{tt}\|_2^2+C_{H^2}\|\phi_t\|_2^2+C_{H^2}\|u_t\|_2^2\end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_t|^2\,dx &= \int_\Omega \nabla u_t\cdot\nabla\left(\kappa\Delta u-\frac{l}{2}\phi_t\right)_t dx\\
&\leq \frac{l}{2}\int_\Omega \Delta u_t\phi_{tt}dx\\
&= \frac{l}{2\kappa}\int_\Omega\left(u_t+\frac{l}{2}\phi_t\right)_t\phi_{tt}dx\\
&\leq \frac{l}{4\kappa}\|u_{tt}\|_2^2+\frac{l(l+1)}{4\kappa}\|\phi_{tt}\|_2^2.
\end{aligned}
$$

We integrate these inequalities with respect to t and obtain

$$
\begin{aligned}
\frac{\tau}{2}\int_\Omega |\nabla\phi_t|^2\,dx &\leq \frac{\tau}{2}\|\nabla(\phi_0)_t\|_2^2+C_{H^2}\int_0^t\|\phi_{tt}\|_2^2\,ds\\
&\quad+\int_0^t\left(2\|\nabla\phi_t\|_2^2+\|\nabla u_t\|_2^2+C_{H^2}\|\phi_t\|_2^2+C_{H^2}\|u_t\|_2^2\right)ds
\end{aligned}
$$

and

$$
\frac{1}{2}\int_\Omega|\nabla u_t|^2\,dx\leq\frac{1}{2}\|\nabla(u_0)_t\|_2^2+\frac{l}{4\kappa}\int_0^t\|u_{tt}\|_2^2\,ds+\frac{l(l+1)}{4\kappa}\int_0^t\|\phi_{tt}\|_2^2\,ds.
$$

Now we have only to estimate $\int_0^t\|u_{tt}\|_2^2\,ds$ and $\int_0^t\|\phi_{tt}\|_2^2\,ds$ for $t>0$ owing to (7), (8), (11) and (14). It holds that

$$
\begin{aligned}
&\tau\int_0^t\int_\Omega\phi_{tt}^2dxds=\int_0^t\int_\Omega\phi_{tt}\left(\epsilon^2\Delta\phi+\phi-\phi^3+2u\right)_t dxds\\
&\leq \frac{\epsilon^2}{2}\|\nabla(\phi_0)_t\|_2^2+\frac{1}{2}\|\phi_t\|_2^2+\int_0^t\int_\Omega\sqrt{\frac{1}{\tau}}\left(3\phi^2|\phi_t|+2|u_t|\right)\cdot\sqrt{\tau}\,|\phi_{tt}|\,dxds\\
&\leq \frac{\epsilon^2}{2}\|\nabla(\phi_0)_t\|_2^2+\frac{1}{2}\|\phi_t\|_2^2+\frac{9}{\tau}\|\phi\|_\infty^4\int_0^t\|\phi_t\|_2^2\,ds+\frac{4}{\tau}\int_0^t\|u_t\|_2^2\,ds\\
&\quad+\frac{\tau}{2}\int_0^t\int_\Omega\phi_{tt}^2dxds.
\end{aligned}
$$

Hence, we have

$$
\int_0^t\|\phi_{tt}\|_2^2\,ds\leq\frac{\epsilon^2}{\tau}\|\nabla(\phi_0)_t\|_2^2+C_{H^2}
$$

from (7), (8) and (12). Next, we have

$$
\begin{aligned}
\int_0^t\int_\Omega u_{tt}^2dxds &= \int_0^t\int_\Omega u_{tt}\left(\kappa\Delta u-\frac{l}{2}\phi_t\right)_t dxds\\
&\leq -\frac{\kappa}{2}\int_0^t\frac{d}{ds}\|\nabla u_t\|_2^2\,ds+\int_0^t\int_\Omega|u_{tt}|\cdot\frac{l}{2}|\phi_{tt}|\,dxds\\
&\leq \frac{\kappa}{2}\|\nabla(u_0)_t\|_2^2+\frac{1}{2}\int_0^t\int_\Omega u_{tt}^2dxds+\frac{l^2}{8}\int_0^t\int_\Omega\phi_{tt}^2dxds
\end{aligned}
$$

and

$$
\int_0^t\|u_{tt}\|_2^2\,ds\leq\kappa\|\nabla(u_0)_t\|_2^2+\frac{l^2\epsilon^2}{4\tau}\|\nabla(\phi_0)_t\|_2^2+C_{H^2},
$$

which yields the desired estimates. □

5 Dynamical system For $(\phi_0, u_0) \in H^\gamma(\Omega) \times H^\gamma(\Omega)$, we show that (1) has a global solution

$$\phi, u \in C\left((0,+\infty); H^1(\Omega)\right) \cap C\left([0,+\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,+\infty); H^{-1}(\Omega)\right).$$

By $T(t)$, we denote a nonlinear semigroup $(\phi_0, u_0) \mapsto (\phi(t), u(t))$ acting on $H^\gamma(\Omega) \times H^\gamma(\Omega)$.

Proof of Theorem 1 By Proposition 1, we have a local solution ϕ, u in $[0, T^\gamma_{\phi_0,u_0}]$ with the estimate

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^\gamma}$$

for $t \in [0, T^\gamma_{\phi_0,u_0}]$ by Theorem 5. Let any small $t_1 \in (0, T^\gamma_{\phi_0,u_0})$ be fixed. Then, it holds that $\phi(t_1), u(t_1) \in H^1(\Omega)$. By Proposition 5, there exists a global solution

$$\phi, u \in C\left((t_1,+\infty); H^2_N(\Omega)\right) \cap C\left([t_1,+\infty); H^1(\Omega)\right) \cap C^1\left((t_1,+\infty); L^2(\Omega)\right)$$

with the estimate

$$\|\phi(t)\|_{H^1} + \|u(t)\|_{H^1} \le C_{H^1} \tag{16}$$

for $t \ge t_1$ with initial functions $\phi_0 = \phi(t_1), u_0 = u(t_1)$. Then, we have

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^1}$$

for $t \ge t_1$. Again, according to Theorem 5,

$$t_1^{1-\beta}\left(\|\phi(t_1)\|_{H^1} + \|u(t_1)\|_{H^1}\right) \le C_{H^\gamma}.$$

Finally, we have

$$\|\phi(t)\|_{H^\gamma} + \|u(t)\|_{H^\gamma} \le C_{H^\gamma}$$

for $t \ge 0$. By Theorems 6 and 8, we can extend a time local solution globally in the space

$$\phi, u \in C\left((0,+\infty); H^1(\Omega)\right) \cap C\left([0,+\infty); H^\gamma(\Omega)\right) \cap C^1\left((0,+\infty); H^{-1}(\Omega)\right)$$

and have a continuous mapping $T(t)$ from $[0,+\infty) \times H^\gamma(\Omega)$ to $H^\gamma(\Omega)$, which shows that $T(t)$ defines a dynamical system in $H^\gamma(\Omega) \times H^\gamma(\Omega)$. □

Proof of Theorem 2 For any $\eta > 0$, we have $\phi(\eta), u(\eta) \in H^1(\Omega)$. By the same argument as proof of Theorem 1, we have a global solution

$$\phi, u \in C\left((\eta,+\infty); H^2_N(\Omega)\right) \cap C\left([\eta,+\infty); H^1(\Omega)\right) \cap C^1\left((\eta,+\infty); L^2(\Omega)\right)$$

with the estimate (16) for $t \ge \eta$ with initial functions $\phi_0 = \phi(\eta), u_0 = u(\eta)$. Hence, the compactness of the orbit in $H^\gamma(\Omega) \times H^\gamma(\Omega)$ follows. Differentiating (1) with respect to t successively and making similar energy estimates to the proof of Proposition 7, we have the uniform boundedness of the orbit $\cup_{t\ge\eta} T(t)(\phi_0, u_0)$ in $H^m_N(\Omega) \times H^m_N(\Omega)$ for any small $\eta > 0$ and $m = 4, 5, \cdots$. We use the standard bootstrap argument to prove that

$$(\phi, u) \in C^\infty\left((0,+\infty); C^\infty(\overline{\Omega})\right) \times C^\infty\left((0,+\infty); C^\infty(\overline{\Omega})\right).$$

□

Proof of Theorem 3 We have a unique global solution $\phi, u \in H^1(\Omega)$ and Lyapunov function $L(\phi, u)(t)$. Therefore, the ω-limit set $\omega(\phi_0, u_0)$ of ϕ_0 and u_0 is nonempty, compact, invariant and connected in $H^1(\Omega) \times H^1(\Omega)$ according to Theorem 4.3.3 in [9]. And it holds that $\omega(\phi_0, u_0) \subset E$ by Theorem 4.3.4 in [9]. For any $\eta > 0$, we have $\phi(\eta), u(\eta) \in H^2_N(\Omega)$ by Proposition 2. By the estimates in Proposition 7, $\cup_{t\geq\eta} T(t)\,(\phi_0, u_0)$ is precompact in $H^2_N(\Omega) \times H^2_N(\Omega)$. As mentioned in Proposition 1, A is supposed to be a positive operator in $L^2(\Omega) \times L^2(\Omega)$ with domain $H^2_N(\Omega) \times H^2_N(\Omega)$. The similar computation to Lemma 2 shows that

$$-\frac{d}{dt} L\,(\phi, u)\,(t) \geq \frac{l(4\tau - l\delta)}{32} \int_\Omega \phi_t^2 dx + \frac{\delta(4\tau - l\delta)}{8\tau} \int_\Omega u_t^2 dx.$$

Hence, we can apply Theorem 1.1 in [10] to deduce that $\omega(\phi_0, u_0)$ is a single point in E. By the second equation in (1), (ϕ, u) satisfies

$$\frac{d}{dt} \int_\Omega \left(u + \frac{l}{2}\phi\right) dx = \kappa \int_\Omega \Delta u dx = 0.$$

Hence, we have

$$\int_\Omega \left(u + \frac{l}{2}\phi\right) dx = \int_\Omega \left(u_0 + \frac{l}{2}\phi_0\right) dx = m$$

for some $m \in \mathbf{R}$. The stationary solution $\Phi = \Phi(x)$ is satisfies

$$\begin{cases} \epsilon^2 \Delta\Phi + \Phi - \Phi^3 + \frac{2}{|\Omega|}\left(m - \frac{l}{2}\int_\Omega \Phi dx\right) = 0 & x \in \Omega, \\ \frac{\partial\Phi}{\partial\nu} = 0 & x \in \partial\Omega \end{cases}$$

because the stationary solution satisfies $\Delta U = 0$ in Ω and $U = U(x)$ is constant in Ω. □

6 Exponential attractor First, we derive the estimate for H^3 norm to obtain an absorbing set in H^3. Next, we construct an exponential attractor in $H^1 \times H^1$.

Proof of Theorem 4: If $(\phi_0, u_0) \in H_k$, then we have

$$\|\phi\|_{H^1} + \|u\|_{H^1} \leq \sqrt{\left(k + \frac{5l}{64}|\Omega|\right)} \left\{ \min\left(\frac{l\epsilon^2}{8}, \frac{l}{16}\right)^{-\frac{1}{2}} + \min\left(\frac{1}{2}, \frac{\kappa\delta}{2}\right)^{-\frac{1}{2}} \right\} \tag{17}$$

for all $t \geq 0$ by Proposition 5. By Theorem 5, Propositions 2 and 3, we have $\phi(\frac{t_1}{2}), u(\frac{t_1}{2}) \in H^2_N(\Omega)$ and $\phi(t_1), u(t_1) \in H^3_N(\Omega)$ for small $t_1 > 0$ with the estimate

$$\left(\frac{t_1}{2}\right)^{\frac{1}{2}} \left(\left\|\phi\left(\frac{t_1}{2}\right)\right\|_{H^2} + \left\|u\left(\frac{t_1}{2}\right)\right\|_{H^2} \right) \leq C_{H^1} \leq C_k$$

with initial functions $\phi_0 = \phi(0), u_0 = u(0)$ by (17) and

$$\left(\frac{t_1}{2}\right)^{\frac{1}{2}} (\|\phi(t_1)\|_{H^3} + \|u(t_1)\|_{H^3}) \leq C_{H^2}$$

with initial functions $\phi_0 = \phi\left(\frac{t_1}{2}\right), u_0 = u\left(\frac{t_1}{2}\right)$, where $C_k > 0$ is a constant depending only on the fixed k, the measure $|\Omega|$ and physical constants $\tau, l, \kappa, \epsilon$. Hence, we have

$$\|\phi(t)\|_{H^3} + \|u(t)\|_{H^3} \leq C_k$$

for all $t > t_1$ by Proposition 7. For any bounded set $B \subset H_k$, we have

$$\cup_{t\geq t_1} T(t)B \subset \mathcal{B} \equiv \{(\phi, u) \in H_k \mid \|\phi\|_{H^3} + \|u\|_{H^3} \leq C_k\}$$

for some $C_k > 0$. In particular, $T(t)\mathcal{B} \subset \mathcal{B}$ for all $t \geq t_1$. This set $\mathcal{B}$ shows us the existence of an absorbing set in H_k, which implies that the dynamical system $T(t)$ is dissipative in H_k. We apply Theorem 1.1 in [14] to guarantee the existence of global attractor $\mathcal{A} \subset H_k$. Let $\mathcal{X} = \overline{\cup_{t\geq t_1} T(t)\mathcal{B}}$. Then, $\mathcal{X}$ is a compact, invariant and absorbing set in $H^1(\Omega) \times H^1(\Omega)$. From now on, we consider the subdynamical system $T(t) : \mathcal{X} \to \mathcal{X}$. To construct an exponential attractor, we apply Theorem 9. Let $U = T(t)U_0 = \begin{pmatrix} \phi \\ u \end{pmatrix} \in \mathcal{X}, V = T(t)V_0 = \begin{pmatrix} \psi \\ v \end{pmatrix} \in \mathcal{X}$ and $s, t \in [0, T]$ for any $T > 0$. The first inequality follows at once from Propositions 2 and 3. Next, we prove the second inequality. We have

$$\begin{aligned}
\|U(t) - V(s)\|_{H^1} &\leq \|U(t) - V(t)\|_{H^1} + \|V(t) - V(s)\|_{H^1} \\
&\leq \|U(t) - V(t)\|_{H^1} + \int_s^t \left\| \frac{dV}{dt}(p) \right\|_{H^1} dp \\
&\leq \|U(t) - V(t)\|_{H^1} + \int_s^t \left(\|AV\|_{H^1} + \|F(V)\|_{H^1} \right) dp
\end{aligned}$$

for $s \leq t$. Since it holds the estimate in Theorem 8 and $AV, F(V) \in H^1(\Omega) \times H^1(\Omega)$ for $V(t) \in \mathcal{X}$,

$$\|U(s) - V(t)\|_{H^1} \leq C_k \|U_0 - V_0\|_{H^1} + C_k |t - s|,$$

which completes the proof of Theorem 4. □

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ON THE REPRESENTATION OF AN INTEGRATED GAUSS-MARKOV PROCESS

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Abstract. We find a representation of the integral of a Gauss-Markov process in the interval $[0, t]$, in terms of Brownian motion. In particular, such representation is used to analyze the temporal mean in a finite interval of a Gauss-Markov process. Finally, some example are explicitly reported.

1 Introduction In this short note, we consider a real continuous Gauss-Markov process $X(t)$ of the form:

$$X(t) = m(t) + h_2(t)B(\rho(t)), \ t \geq 0 \tag{1.1}$$

where:
• $B(t)$ is a standard Brownian motion (BM);
• $m(t) = E(X(t))$ is continuous for every $t \geq 0$;
• the covariance $c(s,t) := E[(X(s) - m(s))(X(t) - m(t))]$ is continuous for every $0 \leq s < t$, with $c(s,t) = h_1(s)h_2(t)$;
• $\rho(t) = h_1(t)/h_2(t)$ is a monotonically increasing function and $h_1(t)h_2(t) > 0$; moreover $\rho(0) = 0$.
Notice that a special case of Gauss-Markov process is the Ornstein-Uhlenbeck (OU) process, and in fact any Gauss-Markov process can be represented in terms of a OU process (see e.g. [13]).
Our aim is to find a representation of

$$Y(t) := \int_0^t X(s)ds, \ t > 0, \tag{1.2}$$

in terms of Brownian motion. Notice that the integrated process $Y(t)$ is equal to $\overline{X}_t \cdot t$, where $\overline{X}_t$ is the time average of $X(s)$ in the interval $[0, t]$.
The study of $Y(t)$ has interesting applications in Biology, for instance in the framework of diffusion models for neural activity; if one identifies $X(t)$ with the neuron voltage at time t, then, $Y(t)/t$ represents the time average of the neural voltage in the interval $[0, t]$. Another application can be found in Queueing Theory, if $X(t)$ represents the length of a queue at time t; then, $Y(t)$ represents the cumulative waiting time experienced by all the "users" till the time t. As for an example from Economics, let us suppose that the variable t represents the quantity of a commodity that producers have available for sale, then $Y(t)$ provides a measure of the total value that consumers receive from consuming the amount t of the product.
Among the papers concerning integrated Gauss-Markov processes, we cite, for instance [10], in which the author considered the integrated Brownian motion, which arises naturally in stochastic models for particle sedimentation in fluids. In [5] observations of integrated diffusion processes were used to estimate unknown parameters, by considering integrated data from the Ornstein-Uhlenbeck process and the CIR-model; in papers [7], [8], [9], the authors studied some properties for the statistical model obtained by the observation of local means of a diffusion process.

The first-passage time (FPT) for $Y(t)$ is an old and interesting problems in Probability; when $X(t)$ is Brownian motion, the two-dimensional process $(X(t), Y(t))$ was first studied by Kolmogorov ([12]). Useful references for FPT problems of integrated Markov processes are given by the paper [10] and the references therein; in particular, in [10] the conditional moments of the FPT of an integrated Brownian motion through a constant barrier were studied. Although the study of FPT problems for $Y(t)$ is not the purpose of the present article, since we aim mainly to give an explicit representation of $Y(t)$, we will outline as this representation can be useful to study the FPT of $Y(t)$ through a continuous boundary (see Example 5 in Section 3).

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2 Main Result

We begin with stating and proving the following:

Lemma 2.1 *Let $f(t)$ a continuous bounded deterministic function, then*

$$I(t) := \int_0^t f(s)B(s)ds \tag{2.1}$$

is normally distributed with zero mean and variance $\gamma(t)$, where $\gamma(t) = \int_0^t (R(t) - R(s))^2 ds$ and $R(t) = \int_0^t f(s)ds$. Moreover, if $\gamma(+\infty) = +\infty$, then there exists a BM $\widehat{B}(t)$ such that $I(t) = \widehat{B}(\gamma(t))$.

Proof. We observe that $I(t)$ is a Gaussian process with zero mean and variance

$$V(t) := Var(I(t)) = Cov\left(\int_0^t f(s)B(s)ds, \int_0^t f(u)B(u)du\right)$$

$$= E\left(\int_0^t f(s)B(s)ds \cdot \int_0^t f(u)B(u)du\right) = \int_0^t ds \int_0^t du E(f(s)B(s)f(u)B(u)).$$

Since $E(f(s)B(s)f(u)B(u)) = f(s)f(u)\min(s,u)$, we get:

$$V(t) = \int\int_{\Delta_1} f(s)f(u) \cdot u \; dsdu + \int\int_{\Delta_2} f(s)f(u) \cdot s \; dsdu,$$

where $\Delta_1 = \{(s,u) \in [0,+\infty) \times [0,+\infty) : 0 \le s \le t, \ 0 \le u \le s\}$ and $\Delta_2 = \{(s,u) \in [0,+\infty) \times [0,+\infty) : 0 \le s \le t, \ s \le u \le t\}$. Thus, by calculation, we obtain:

$$V(t) = 2\int_0^t f(s)ds \int_0^s f(u) \cdot u \; du.$$

As easily seen, $V(t)$ and $\gamma(t)$ have the same derivative, so the equality $V(t) = \gamma(t)$ follows for any $t \ge 0$, since $V(0) = \gamma(0) = 0$.
Moreover, by using Itô's formula we get:

$$I(t) = \int_0^t f(s)B(s)ds = R(t)B(t) - \int_0^t R(s)dB(s) = \int_0^t (R(t) - R(s))dB(s).$$

Notice that $I(t)$ is a continuous martingale and $\gamma(t)$ is its quadratic variation; therefore, if $\gamma(+\infty) = +\infty$, by the Dambis, Dubins-Schwarz Theorem (see e.g. [14]) we obtain that $I(t)$ can be written as $\widehat{B}(\gamma(t))$, where $\widehat{B}(t)$ is BM.

□

As a corollary of the previous lemma, we obtain our main result:

Proposition 2.2 *Let $X(t)$ be a Gauss-Markov process given by (1.1), and suppose that h_1, h_2 are continuous function and $\rho : [0,+\infty) \longrightarrow [0,+\infty)$ is a C^1 function with $\rho'(t) > 0 \ \forall t \ge 0$; then $Y(t) = \int_0^t X(s)ds$ is normally distributed with mean $M(t) = \int_0^t m(s)ds$ and variance $\gamma_1(\rho(t))$, where $\gamma_1(t) = \int_0^t (R_1(t) - R_1(s))^2 ds$ and $R_1(t) = \int_0^t h_2(\rho^{-1}(s))/\rho'(\rho^{-1}(s))ds$. Moreover, if $\gamma_1(+\infty) = +\infty$, then $Y(t)$ is Gauss-Markov and there exists a BM $\widehat{B}(t)$ such that $Y(t) = M(t) + \widehat{B}(\gamma_1(\rho(t)))$.*

Proof. We have:

$$Y(t) = \int_0^t X(s)ds$$

$$= \int_0^t m(s)ds + \int_0^t h_2(s)B(\rho(s))ds = M(t) + \int_0^{\rho(t)} h_2(\rho^{-1}(s))/\rho'(\rho^{-1}(s))B(s)ds,$$

where we have used a variable change in the integral. Then, the proof follows by using Lemma 2.1 with $f(t) = h_2(\rho^{-1}(t))/\rho'(\rho^{-1}(t))$. □

Example 1 (Brownian motion with drift)
Let be $X(t) = \mu t + B(t)$, then $m(t) = \mu t, \ h_1(t) = t, \ h_2(t) = 1$ and $\rho(t) = t$. Moreover, $R_1(t) = \int_0^t ds = t$ and $\gamma_1(t) = \int_0^t (t-s)^2 ds = t^3/3$. Thus, $Y(t) = \mu t^2/2 + \widehat{B}(t^3/3)$ (cf. [2]).

Remark 2.3 *If we consider the time average of $X(t)$ in the interval $[0,t]$, i.e. $\overline{X}_t = \frac{1}{t}\int_0^t X(s)ds$, we get by Proposition 2.2 $\overline{X}_t = \frac{1}{t}Y(t) = \frac{1}{t}\left[M(t) + \widehat{B}(\gamma_1(\rho(t)))\right]$, namely $\overline{X}_t$ is normally distributed with mean $M(t)/t$ and variance $\gamma_1(\rho(t))/t^2$. In particular, if $X(t)$ is BM, one has $\gamma_1(t) = t^3/3$, and so $\overline{X}_t \sim \mathcal{N}(0, t/3)$ (cf. [2]).*

3 A Few Examples

Example 2 (Ornstein-Uhlenbeck process)
Let $X(t)$ be the solution of the SDE:

$$dX(t) = -\mu(X(t) - \beta)dt + \sigma dB(t),\ X(0) = x,$$

where $\mu, \sigma > 0$ and $\beta \in (-\infty, +\infty)$. The explicit solution is (see e.g. [1]):

$$X(t) = \beta + e^{-\mu t}[x - \beta + \widetilde{B}(\rho(t))],$$

where $\widetilde{B}$ is Brownian motion and $\rho(t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$. So, $X(t)$ is a Gauss-Markov process with $m(t) = \beta + e^{-\mu t}(x - \beta)$, $h_1(t) = \frac{\sigma^2}{2\mu}\left(e^{\mu t} - e^{-\mu t}\right)$, $h_2(t) = e^{-\mu t}$ and $c(s,t) = h_1(s)h_2(t)$. By calculation, we obtain:

$$M(t) = \int_0^t \left(\beta + e^{-\mu s}(x - \beta)\right)\, ds = \beta t + \frac{(x-\beta)}{\mu}\left(1 - e^{-\mu t}\right),$$

$$R_1(t) = \int_0^t e^{-\mu\rho^{-1}(s)}(\rho^{-1})'(s)ds = \frac{1 - e^{-\mu\rho^{-1}(t)}}{\mu},$$

$$\rho^{-1}(s) = \frac{1}{2\mu}\ln\left(1 + \frac{2\mu}{\sigma^2}s\right),$$

$$\begin{aligned}\gamma_1(t) &= \frac{1}{\mu^2}\int_0^t \left(e^{-\mu\rho^{-1}(t)} - e^{-\mu\rho^{-1}(s)}\right)^2 ds = \frac{1}{\mu^2}\int_0^t \left(\frac{1}{\sqrt{1 + 2\mu t/\sigma^2}} - \frac{1}{\sqrt{1 + 2\mu s/\sigma^2}}\right)^2 ds \\ &= \frac{\sigma^2 t}{\mu^2(\sigma^2 + 2\mu t)} - \frac{2\sigma^2}{\mu^3\sqrt{1 + 2\mu t/\sigma^2}}\left(\sqrt{1 + 2\mu t/\sigma^2} - 1\right) + \frac{\sigma^2}{2\mu^3}\ln\left(1 + 2\mu t/\sigma^2\right).\end{aligned}$$

Then, by Proposition 2.2, we get that $Y(t) = \int_0^t X(s)ds$ is normally distributed with mean $M(t)$ and variance $\gamma_1(\rho(t))$. Moreover, since $\lim_{t\to+\infty}\gamma_1(t) = +\infty$, there exists a BM $\widehat{B}(t)$ such that $Y(t) = M(t) + \widehat{B}\left(\gamma_1(\rho(t))\right)$.

Example 3 (Brownian bridge)
For $T > 0$ and given a, b, let $X(t)$ be the solution of the SDE:

$$dX(t) = \frac{b - X(t)}{T - t}\, dt + dB(t),\ 0 \le t \le T,\ X(0) = a.$$

This is a transformed BM with fixed values at each end of the interval $[0, T]$, $X(0) = a$ and $X(T) = b$. The explicit solution is (see e.g. [14]):

$$\begin{aligned}X(t) &= a\left(1 - t/T\right) + bt/T + (T - t)\int_0^t \frac{1}{T - s}dB(s) \\ &= a\left(1 - t/T\right) + bt/T + (T - t)\widetilde{B}\left(\frac{t}{T(T - t)}\right),\ 0 \le t \le T,\end{aligned}$$

where $\widetilde{B}$ is BM. So, $X(t)$ is a Gauss-Markov process with:

$$m(t) = a\left(1 - t/T\right) + bt/T,\ c(s,t) = h_1(s)h_2(t),\ h_1(t) = t/T,\ h_2(t) = T - t,\ \rho(t) = \frac{t}{T(T - t)}.$$

By calculation, we obtain:

$$M(t) = at + \frac{b-a}{2T}t^2,$$

$$R_1(t) = \frac{T^3 t(2+Tt)}{2(1+Tt)^2},$$

$$\rho^{-1}(s) = \frac{T^2 s}{1+Ts},$$

$$\gamma_1(t) = \int_0^t \left(\frac{T^3 t(2+Tt)}{2(1+Tt)^2} - \frac{T^3 s(2+Ts)}{2(1+Ts)^2} \right)^2 ds.$$

Then, by Proposition 2.2, we get that $Y(t) = \int_0^t X(s)ds$ is normally distributed with mean $M(t)$ and variance $\gamma_1(\rho(t))$. Although it is straightforward to obtain the explicit form of $\gamma_1(t)$, we omit to write it (a numerical evaluation can be obtained by a computer). We limit ourselves to mention that $\lim_{t\to+\infty} \gamma_1(t) = +\infty$, as it can be verified by a boring calculation; so there exists a BM $\widehat{B}(t)$ such that $Y(t) = M(t) + \widehat{B}\left(\gamma_1(\rho(t))\right)$.

Example 4 (Generalized Gauss-Markov process)
Let us consider the diffusion $X(t)$ which is the solution of the SDE:

$$dX(t) = m'(t)dt + \sigma(X(t))dB(t),\ X(0) = m(0)$$

where $m(t)$ and $\sigma(x) > 0$ are regular enough deterministic functions. We suppose that $\rho(t) = \langle X \rangle_t = \int_0^t \sigma^2(X(s))ds$, i.e. the quadratic variation of $X(t)$, is increasing to $\rho(+\infty) = +\infty$. By using the Dambis, Dubins-Schwarz Theorem, it follows that $X(t) = m(t) + B(\rho(t)),\ t \geq 0$, where $\rho(t)$ is not necessarily deterministic, but it can be a random function. For this reason, we call $X(t)$ a generalized Gauss-Markov process. Denote by A the "inverse" of the random function ρ, that is, $A(s) = \inf\{t > 0 : \rho(t) > s\}$; since $\rho(t)$ admits derivative and $\rho'(t) = \sigma^2(X(t)) > 0$, also $A'(s)$ exists and $A'(s) = 1/\sigma^2(X(A(s)))$; we focus on the case when there exist deterministic continuous functions $\alpha(t),\ \beta(t)$ (with $\alpha(0) = \beta(0)$) and $\alpha_1(t),\ \beta_1(t)$, such that, for every $t \geq 0$:

$$\alpha(t),\ \beta(t) \text{ are increasing, } \alpha(t) \leq \rho(t) \leq \beta(t), \text{ and } \alpha_1(t) < A'(t) < \beta_1(t).$$

Since $\rho(t)$ is not deterministic, we cannot obtain exactly the distribution of $\int_0^t X(s)ds$, however we are able to find bounds to it. In fact, we have:

$$\int_0^t X(s)ds = \int_0^t m(s)ds + \int_0^t B(\rho(s))ds = \int_0^t m(s)ds + \int_0^{\rho(t)} B(v)A'(v)dv\ .$$

We can use the arguments of Lemma 2.1 with $f(v) = A'(v),\ R_1(t) = \int_0^t A'(s)ds$, and $\gamma_1(t) = \int_0^t (R_1(t) - R_1(s))^2 ds$; by assumptions we get $\int_0^t \alpha_1(s)ds \leq R_1(t) \leq \int_0^t \beta_1(s)ds$. Thus, we conclude that $\int_0^t X(s)ds$ is normally distributed with mean $M(t) = \int_0^t m(s)ds$ and variance $\gamma_1(\rho(t))$, which is bounded between $\gamma_1(\alpha(t))$ and $\gamma_1(\beta(t))$. The closer $\alpha(t)$ to $\beta(t)$, the better the approximation above; for instance, if $\sigma(x) = 1 + \epsilon \cos^2(x),\ \epsilon > 0$, we have $\rho(t) = \int_0^t (1 + \epsilon \cos^2(X(s)))^2 ds$ and so $\alpha(t) = t,\ \beta(t) = (1+\epsilon)^2 t,\ \alpha_1(t) = 1/(1+\epsilon)^2,\ \beta_1(t) = 1$. The smaller is ϵ, the closer $\gamma_1(\alpha(t))$ to $\gamma_1(\beta(t))$.

Example 5 (The FPT of $Y(t)$ over a continuous boundary)
Let $S(t) > 0$ a continuous boundary with $S(0) > 0$, and let us consider the FPT of $Y(t)$ over S i.e. $\tau_S = \inf\{t > 0 : Y(t) \geq S(t)\}$. If $\gamma_1(+\infty) = +\infty$, then τ_S is nothing but the FPT of $\widehat{B}(\overline{\gamma}_1(t))$ over $\overline{S}(t) = S(t) - M(t)$, where $\overline{\gamma}_1(t) = \gamma_1(\rho(t))$, or equivalently $\overline{\gamma}_1(\tau_S) = \inf\{u > 0 : \widehat{B}(u) > \overline{S}(\gamma_1^{-1}(u))\}$. Then, the distribution of τ_S can be easily obtained for a class of boundaries $S(t)$ for which the FPT of BM through the transformed boundaries is explicitly known (see e.g. [3], [6]). For instance, if $S(t) = M(t) + a + b\gamma_1(t)$ for some constants a and b, we get $\overline{S}(\gamma_1^{-1}(u)) = a + bu$; thus, the probability density of τ_S can be found in terms of the inverse Gaussian density, namely

the density of the first-crossing time of BM $\widehat{B}(u)$ through the linear boundary $g(u) = a + bu$, which is explicitly given by

$$\psi(u) = \frac{|a|}{u^{3/2}}\, \phi\left(\frac{a+bu}{\sqrt{u}}\right),\ u > 0 \tag{3.1}$$

where $\phi(y) = e^{-y^2/2}/\sqrt{2\pi}$ (see e.g. [11]).

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ELEMENTARY PROOFS OF OPERATOR MONOTONICITY OF SOME FUNCTIONS

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Abstract. Operator monotonicity of functions $\left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}$ $(-1 \le p \le 1)$ and $\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}$ $(-2 \le p \le 2)$ on $(0,\infty)$ are known. The former is the representing function of the power arithmetic mean and the latter is that of Stolarsky mean. We give somewhat elementary proofs of operator monotonicity of them and some other related functions.

1 Introduction. A (bounded linear) operator A acting on a Hilbert space H is said to be positive, denoted by $A \ge 0$, if $(Av, v) \ge 0$ for all $v \in H$. The definition of positivity induces the order $A \ge B$ for self-adjoint operators A and B on H. A real-valued function f on $(0,\infty)$ is *operator monotone,* if $f(A) \le f(B)$ for operators A and B such that $0 \le A \le B$. As a typical example, $x \mapsto x^p$ $(0 \le p \le 1)$ is an operator monotone function, which is well-known as *Löwner-Heinz (LH) theorem.*

Recently, Besenyei and Petz [1] showed the following two theorems by Löwner's theory:

Theorem 1.1 ([1, Theorem 3]). *The function*

$$f_p(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}, \ p \ne 0, 1 \quad \left(f_0(x)\left(= \lim_{p\to 0} f_p(x)\right) = \frac{x-1}{\log x}, \ f_1(x) = \frac{1}{e}x^{\frac{x}{x-1}}\right)$$

is operator monotone if $-2 \le p \le 2$.

Theorem 1.2 ([1, Theorem 4]). *The function*

$$w_p(x) = \left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}, \ p \ne 0 \quad \left(w_0(x) = x^{\frac{1}{2}}\right)$$

is operator monotone if (and only if) $-1 \le p \le 1$.

Theorem 1.2 is already known well ([3], [4], [5], [6], [9]). We shall give a simple proof of this fact by using the binomial expansion and (LH) theorem.

Now define

$$g_p(x) = \frac{p-1}{p} \cdot \frac{x^p-1}{x^{p-1}-1}, \ p \ne 0, 1 \quad \left(g_0(x) = \frac{x\log x}{x-1}, \ g_1(x) = \frac{x-1}{\log x}\right),$$

a function related to $w_p(x)$ or its extension (as stated afterward in the proof of Theorem 3.2). Using an integral representation of $g_p(x)$, Hiai and Kosaki [10] showed:

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Theorem 1.3 ([6, Proposition 4.2]). *$g_p(x)$ is operator monotone if $-1 \le p \le 2$.*

In [6] by Fujii and Seo, this fact had been shown essentially in virtue of Bendat-Sherman theorem. This fact was also shown, in [7] by Furuta, with a very elementary method, (LH) only used repeatedly, and in [3] by Fujii ([4] by Fujii-Fujii), with the notion of the integral mean of operator monotone functions.

In this paper, starting from the proof of Theorem 1.2, we give somewhat elementary proofs of Theorems 1.1, 1.3 and some other related results. As an application of Theorem 1.1, we give a proof of Petz-Hasegawa theorem [14], an elementary proof of which was recently presented by Furuta [8].

2 Preliminaries

By Kubo-Ando theory [12], an operator mean σ is defined as a binary relation of positive operators, satisfying the following properties in common:

(monotonicity)	$A \le C, B \le D \Longrightarrow A\sigma B \le C\sigma D$,
(transformer inequality)	$C(A\sigma B)C \le (CAC)\sigma(CBC)$,
(normality)	$A\sigma A = A$,
(strong operator semi-continuity)	$A_n \downarrow A, B_n \downarrow B \Longrightarrow A_n\sigma B_n \downarrow A\sigma B$.

As the basic operator means, we define: For $A, B \ge 0$

$$\begin{aligned} &\textit{arithmetic mean}: \quad A \nabla B = (A+B)/2, \\ &\textit{harmonic mean}: \quad A \,!\, B = \left\{\left(A^{-1}+B^{-1}\right)/2\right\}^{-1} \quad \text{and} \\ &\textit{geometric mean}: \quad A\#B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}. \end{aligned}$$

Sometimes for the definition of an operator mean we must assume operators to be invertible, say, for harmonic or geometric mean. Without any assumption for invertibility every mean is well-defined as the (strong operator) limits of $(A+\varepsilon I)\sigma(B+\varepsilon I)$ as $\varepsilon \downarrow 0$ instead of $A\sigma B$. (I is the identity operator.) For simplicity of discussions, from now on we assume that all positive operators are *invertible*.

To every operator mean σ corresponds a unique operator monotone function, that is, its representing function f_σ which is defined by $f_\sigma(x) = 1\sigma x$. Conversely, if f is an operator monotone function with $f(1) = 1$, then the definition of the operator mean corresponding to f is given by

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

for positive operators A and B.

For an operator mean σ and for two operator monotone functions g and h, we define $g\sigma h$ by

$$(g\sigma h)(A) = g(A)\sigma h(A)\left(= g(A)^{\frac{1}{2}} f_\sigma\left(g(A)^{-\frac{1}{2}}h(A)g(A)^{-\frac{1}{2}}\right)g(A)^{\frac{1}{2}}\right).$$

Then it is easy to see that $g\sigma h$ is operator monotone. In particular, if $f_{\#_\alpha}(x) = x^\alpha$ for $0 \le \alpha \le 1$, then $g\#_\alpha h \left(= g^{1-\alpha}h^\alpha\right)$ is also operator monotone.

Now to state another useful fact on an operator monotone function, let f be a strictly positive function on $(0,\infty)$. Define $f^\circ(x) := xf(1/x)$ (transpose), $f^*(x) := 1/f(1/x)$ (adjoint) and $f^\perp(x) := x/f(x)$ (dual). Then the following (i)-(iv) are equivalent [12]([11]):

(i) f is operator monotone,
(ii) f° is operator monotone,
(iii) f^* is operator monotone,

(iv) $f^{\perp}$ is operator monotone.

For a (continuous) path σ_t ($t \in [0,1]$) of operator means, its integral mean $\tilde{\sigma}$ is defined [3] ([4]) for positive operators A and B by

$$A\tilde{\sigma}B = \int_0^1 A\sigma_t B dt.$$

Correspondingly, for a path f_t of operator monotone functions, its integral mean $\tilde{f}$ can be defined by

$$\tilde{f}(x) = \int_0^1 f_t(x)dt,$$

which is an operator monotone function.

3 Main results To prove Theorem 1.2, we use the following fact: For integers m, n, q, r with $1 \le m \le n,\ n = mq + r,\ 0 \le r \le m-1$, and any $k = 1, 2, \dots, q$,

$$(x^{\frac{m}{n}})^k (1 + x^{\frac{m}{n}})^{\frac{r}{m}} = (1 + x^{\frac{m}{n}})^{\frac{r}{m}} \#_{\frac{k}{q}} x^{\frac{qm}{n}} (1 + x^{\frac{m}{n}})^{\frac{r}{m}} \tag{3.1}$$

holds. Now we show a proof of Theorem 1.2, borrowing Furuta's method, or applying the theorem (LH) repeatedly (say, in [7], [8]):

Proof of Theorem 1.2. It suffices to show the proof when p is rational, $p \neq 0, 1, -1$. First we assume that $0 < p < 1$, so put $p = \frac{m}{n}$, m, n are integers with $(m, n) = 1,\ 1 \le m < n$. Then $n = qm + r$ for some $1 \le r \le m - 1$, and

$$w_p(x) = \left(\frac{1 + x^{\frac{m}{n}}}{2}\right)^{\frac{n}{m}} = \left(\frac{1}{2}\right)^{\frac{n}{m}} \left(1 + x^{\frac{m}{n}}\right)^q \left(1 + x^{\frac{m}{n}}\right)^{\frac{r}{m}} = \left(\frac{1}{2}\right)^{\frac{n}{m}} \sum_{k=0}^{q} {}_qC_k \phi_k(x).$$

Here $\phi_k(x) = x^{\frac{km}{n}}(1 + x^{\frac{m}{n}})^{\frac{r}{m}}$. The notations ${}_qC_k$ for $k = 0, 1, \dots, q$ denote the binomial coefficients, i.e., ${}_qC_k = \frac{q!}{k!(q-k)!}$.

First note that $\phi_0(x) = (1 + x^{\frac{m}{n}})^{\frac{r}{m}}$, clearly, is operator monotone (by (LH)). Next for the last term

$$\phi_q(x) = x^{\frac{qm}{n}}(1 + x^{\frac{m}{n}})^{\frac{r}{m}} = \frac{x}{\left(\frac{x^{\frac{m}{n}}}{1+x^{\frac{m}{n}}}\right)^{\frac{r}{m}}},$$

so that the dual $\phi_q^{\perp}(x) = \frac{x}{\phi_q(x)}$ of $\phi_q(x)$ is

$$\phi_q^{\perp}(x) = \left(\frac{x^{\frac{m}{n}}}{1 + x^{\frac{m}{n}}}\right)^{\frac{r}{m}} = \left(\frac{x}{x^{\frac{n-m}{n}} + x}\right)^{\frac{r}{m}} = \left((x^{\frac{n-m}{n}} + x)^{\perp}\right)^{\frac{r}{m}}.$$

Hence $\phi_q^{\perp}(x)$, and $\phi_q(x)$ are both operator monotone by (iv) and (LH). Now recall (3.1) stated before. For the general k-th term of the sum, we see:

$$\phi_k(x) = \phi_0(x) \#_{\frac{k}{q}} \phi_q(x).$$

Hence all of $\phi_k(x)$ are operator monotone, so that the proof for $0 < p < 1$ is completed.

For $-1 < p < 0$, notice that

$$w_p^*(x) = \left(\frac{1 + x^{-p}}{2}\right)^{-\frac{1}{p}} = w_{-p}(x).$$

Hence we see that w_p^*, or, equivalently, w_p is operator monotone.

A property of an operator monotone function on $(0,\infty)$ is concavity [11] ([12]). If $p>1$ then we can see $w_p''(x)>0$, so that $w_p(x)$ is not concave, which implies that the function is not operator monotone. If $p<-1$ then since $w_p^*(x)=w_{-p}(x)$ is not operator monotone, so that $w_p(x)$ is not operator monotone. □

As a slight extension of Theorem 1.2, we can easily see the following:

Lemma 3.1. *For $a_i, b_i \geq 0$ $(i=1,2,\ldots,n)$, $-1\leq p\leq 1$, $p\neq 0$,*

$$\left(\sum_{i=1}^{n}(a_i+b_ix)^p\right)^{\frac{1}{p}} \text{ is operator monotone.} \tag{3.2}$$

More generally, if f_i $(i=1,2,\ldots,n)$, are positive operator monotone functions on $(0,\infty)$, $-1\leq p\leq 1$, $p\neq 0$, then

$$S_n := \left(\sum_{i=1}^{n} f_i^p\right)^{\frac{1}{p}} \text{ is operator monotone.} \tag{3.3}$$

Proof. We may show the general case. Denote by σ_p the operator mean corresponding to the power arithmetic mean $w_p = \left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}$.

Now we prove (3.3) by the mathematical induction. Let $n=2$, then

$$S_2 = (f_1^p+f_2^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}(f_1\sigma_p f_2).$$

Hence S_2 is operator monotone. Assume that S_n for $n\geq 2$ is operator monotone. We have to show S_{n+1} is also operator monotone. But this is clear since the both S_n and f_{n+1} are operator monotone, and

$$S_{n+1} = \left(\sum_{i=1}^{n+1} f_i^p\right)^{\frac{1}{p}} = \left(S_n^p + f_{n+1}^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}(S_n\sigma_p f_{n+1}).$$ □

Theorem 3.2 (cf. Fujii-Fujii [4], Fujii [3])**.** *For $-1\leq p\leq 1$, $0\leq s\leq 1$, the function*

$$u_{p,s}(x) = \frac{p}{p+s}\cdot\frac{x^{p+s}-1}{x^p-1},\ p\neq 0,-s \ \left(u_{0,s}(x)=\frac{x^s-1}{\log x^s},\ u_{-s,s}(x)=\frac{\log x^{-s}}{x^{-s}-1}\right) \tag{3.4}$$

is operator monotone.

Proof. We can see that for $p\neq 0$,

$$u_{p,s}(x) = \int_0^1 (1-t+tx^p)^{\frac{s}{p}}\,dt.$$

By Lemma 3.1, $(1-t+tx^p)^{\frac{1}{p}}$ for $t\in[0,1]$ is operator monotone, so that $(1-t+tx^p)^{\frac{s}{p}}$ is also operator monotone. Hence as its *integral mean*, $u_{p,s}(x)$ is operator monotone. We can see operator monotonicity of $u_{0,s}$, by taking the limits of $u_{p,s}$ as $p\to 0$. □

Corollary 3.3 (cf. Furuta [7])**.** *For* $-1 \le p \le 1$, *the function*

$$u_p(x) = \frac{p(x-1)}{x^p - 1},\ p \neq 0 \ \left(u_0(x) = \frac{x-1}{\log x}\right) \tag{3.5}$$

is operator monotone.

Proof. If $0 < p \le 1$, then put $s = 1-p$ in (3.4), and we have (3.5). If $-1 \le p < 0$, then put $p = -q,\ 0 < q \le 1$, and take the transpose of $u_p(x)$:

$$u_p^\circ(x) = \left(\frac{-q(x-1)}{x^{-q}-1}\right)^\circ = \frac{q(x-1)}{x^q - 1}.$$

We see that $u_p^\circ(x)$ is operator monotone from the previous discussion for $0 < p \le 1$, so that $u_p(x)$ is also operator monotone. □

Applying Theorem 3.2, we show:

Proof of Theorem 1.3. For $0 \le p \le 2$, replace p by $p-1$ in (3.4) of Theorem 3.2 and further put $s = 1$. Then we obtain $g_p(x)$, so that $g_p(x)$ is operator monotone. For $-1 \le p < 0$, put $p = -q$, then $0 < q \le 1$, and

$$g_p(x) = \frac{-q-1}{-q} \cdot \frac{x^{-q}-1}{x^{-q-1}-1} = \frac{q+1}{q} \cdot \frac{x(x^q - 1)}{x^{q+1} - 1}.$$

Hence

$$g_p^\perp(x) = \frac{x}{g_p(x)} = \frac{q}{q+1} \cdot \frac{x^{q+1}-1}{x^q - 1}.$$

By the previous paragraph (the proof for $0 \le p \le 2$), we then see that $g_p^\perp(x)$, and hence $g_p(x)$ are operator monotone. □

Using the above lemma, we also show:

Proof of Theorem 1.1. We may consider the case for $p \neq 1$. We can represent $f_p(x)$ as follows by using the integral:

$$f_p(x) = \left[\int_0^1 (1-t+tx)^{p-1} dt\right]^{\frac{1}{p-1}}.$$

First, we consider the case for $0 \le p \le 2$, or $-1 \le q := p-1 \le 1, (q \neq 0)$. Let

$$I(x) = \int_0^1 (1-t+tx)^{p-1} dt = \int_0^1 (1-t+tx)^q dt.$$

Then as its approximate sum, we have

$$\Sigma_n(x) := \sum_{i=1}^n (1-t_i+t_i x)^q \Delta t_i$$

$$\left(0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1,\ \Delta t_i = t_i - t_{i-1} < \frac{2}{n}\right).$$

From (3.2) in Lemma 3.1, $\{\Sigma_n(x)\}^{\frac{1}{q}}$ is operator monotone. Therefore $f_p(x) = I(x)^{\frac{1}{q}}$, as the limit of $\{\Sigma_n(x)\}^{\frac{1}{q}}$, is operator monotone.

Second, for $-2 \le p \le 0$, we put $q = -p$, so that $0 \le q \le 2$. (We may assume that $p \ne -1$ $(q \ne 1)$.) Then note that $\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}}$ is operator monotone from the previous argument. We now consider the following two cases:

(i) The case $0 < q < 1$ $(-1 < p < 0)$: We have

$$f_p(x) = f_{-q}(x) = \left(\frac{-q(x-1)}{x^{-q}-1}\right)^{\frac{1}{1+q}} = \left(\frac{q(x-1)x^q}{x^q-1}\right)^{\frac{1}{1+q}} = \left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_{\frac{2q}{1+q}} x^{\frac{1}{2}}.$$

Hence $f_p(x)$ is operator monotone.

(ii) The case $1 < q \le 2$ $(-2 \le p < -1)$: We may show that the adjoint $f_p^*(x) = f_p(x^{-1})^{-1}$ of $f_p(x)$ is operator monotone. We see:

$$f_p^*(x) = f_{-q}^*(x) = \left(\frac{-q(x^{-1}-1)}{x^q-1}\right)^{-\frac{1}{1+q}} = \left(\frac{q(x-1)}{(x^q-1)x}\right)^{-\frac{1}{1+q}} = \left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_{\frac{2}{1+q}} x^{\frac{1}{2}}.$$

Hence $f_p^*(x)$ is operator monotone. The proof is completed. □

As an application of Theorem 1.1, we show an alternative proof of the following result due to Petz and Hasegawa [14] ([8]):

Theorem 3.4. *For* $-1 \le p \le 2$

$$h_p(x) = p(1-p) \cdot \frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}, \; p \ne 0, 1 \; \left(h_0(x) = h_1(x) = \frac{x-1}{\log x}\right)$$

is operator monotone.

Proof. It is sufficient to consider the case for $p \ne 0, \pm 1, 2$. First notice that

$$h_p(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_p \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}.$$

(Here $\#_p$ also expresses an extended weighted mean if $p > 1$ or $p < 0$.) By Theorem 1.1, both $\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}$ and $\left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$ are operator monotone. Hence if $0 < p < 1$, then we, at once, see that $h_p(x)$ is operator monotone. Next if $1 < p < 2$, then putting $p = q+1$ $(0 < q < 1)$, we have

$$h_p(x) = h_{q+1}(x) = (-q)(q+1) \cdot \frac{(x-1)^2}{(x^{q+1}-1)(x^{-q}-1)} = \frac{q(q+1)x^q(x-1)^2}{(x^{q+1}-1)(x^q-1)}.$$

Now since $0 < q < 1$, we see that $\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}}$ is operator monotone by Theorem 1.1. Further, since $1 < q+1 < 2$, we see that

$$(\eta(x) :=) \; \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{1-(q+1)}} = \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{-\frac{1}{q}}$$

is operator monotone by Theorem 1.1, so that its dual $(\eta^{\perp}(x) =)\ x \cdot \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{q}}$ is operator monotone. Hence

$$\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_q\ x \cdot \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{q}} = h_p(x)$$

is operator monotone. Finally, if $-1 < p < 0$, then putting $p = -q\ (0 < q < 1)$, we have

$$h_p(x) = \frac{(-q)(q+1)(x-1)^2}{(x^{-q}-1)(x^{1+q}-1)} = \frac{q(q+1)x^q(x-1)^2}{(x^{q+1}-1)(x^q-1)}.$$

Hence $h_p(x)$ has the same expression as in case $1 < p < 2$, so that it is operator monotone. □

Remark 3.5. *For the (extended) weighted geometric mean, the identity*

$$A\#_{\alpha}(A\#_{\beta}B) = A\#_{\alpha\beta}B \qquad (\alpha, \beta : \textit{real})$$

holds for positive operators A and B (cf. the interpolationality [3]). Using this formula, we can get a slight extension of Theorem 3.4: Let $0 \le \alpha \le 1$. Then

$$\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha p} \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$$

$$= \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha} \left(\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_p \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}\right)$$

is operator monotone.

(*If $0 < p < 1$, then it is clear that*

$$\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha} \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$$

is also operator monotone.)

Concluding Remark. In this note we began with an elementary proof of operator monotonicity of the power arithmetic mean $w_p(x) = \left(\frac{1+x^p}{2}\right)^{\frac{1}{p}} \left(= \pi^{-1}\left(\frac{1+\pi(x)}{2}\right)\right)$ for $\pi(x) = x^p$. We now conclude with stating operator monotonicity of a very general extension of this fact [5]: Let f be a positive operator monotone function with $f(1) = 1$. Then $\hat{f}_t(x) = f^{-1}(1 - t + tf(x))$ is operator monotone for $0 \le t \le 1$.

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FRACTIONAL CALCULUS AND $L^{(\alpha)}$-CONJUGATES ON PARABOLIC HARDY SPACES

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ABSTRACT.
For $0 < \alpha \leq 1$, we consider the parabolic operator $L^{(\alpha)} = \partial/\partial t + (-\Delta_x)^\alpha$ on the upper half-space of the Euclidean space $\mathbb{R}^{n+1}$. For $1 \leq p < \infty$, the α-parabolic Hardy space $\boldsymbol{h}^p_\alpha$ is the set of all solutions u of $L^{(\alpha)}$ which have the finite $\boldsymbol{h}^p_\alpha$ norm. In this paper, we study fractional calculus on parabolic Hardy spaces. Also, we investigate properties of maximal functions and conjugate functions on parabolic Hardy spaces.

1. Introduction

Let $n \geq 1$ and H the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x,t) \in \mathbb{R}^{n+1} : x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 2). For $1 \leq p < \infty$, the Lebesgue space $L^p = L^p(\mathbb{R}^n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dV_n(x) \right)^{\frac{1}{p}} < \infty,$$

where dV_n is the Lebesgue volume measure on $\mathbb{R}^n$. The parabolic Hardy space $\boldsymbol{h}^p_\alpha$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with

$$\|u\|_{\boldsymbol{h}^p_\alpha} := \sup_{t>0} \|u(\cdot,t)\|_{L^p} < \infty.$$

We remark that $\boldsymbol{h}^p_{1/2}$ coincide with the harmonic Hardy spaces of [1, Chapter 7].

Our aim of this paper is the study of fractional calculus on parabolic Hardy spaces. In [3], we study fractional calculus on parabolic Bergman spaces, which are the Banach spaces consisting of all $L^p(H)$-solutions of the parabolic operator $L^{(\alpha)}$. Parabolic Bergman spaces are often studied by using fractional calculus (see [4], [5], and [7]). In this paper, we study properties of fractional calculus on parabolic Hardy spaces. Moreover, we investigate properties of α-parabolic maximal functions and $L^{(\alpha)}$-conjugates of parabolic Hardy functions, which are the extension of the non-tangential maximal functions and the harmonic conjugates, respectively.

To state our results of this paper, we give some notations. For a real number κ, let $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$ be the fractional differential operator with respect to t, and $\mathcal{FC}^\kappa$ the class of functions φ on $\mathbb{R}_+ = (0,\infty)$ such that $\mathcal{D}_t^\kappa \varphi$ is well defined (the explicit definitions of $\mathcal{D}_t^\kappa$ and $\mathcal{FC}^\kappa$ are described

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in Section 2). For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$. Theorem 1 shows basic properties of fractional calculus on parabolic Hardy spaces.

THEOREM 1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\gamma \in \mathbb{N}_0^n$, and $\nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha$. If $u \in \boldsymbol{h}_\alpha^p$, then the following statements hold*:

(1) *The derivative $\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ is well defined, and there exists a constant $C = C(n, \alpha, p, \gamma, \nu) > 0$ such that*

$$|\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)| \leq C t^{-(n/2\alpha)(1/p) - |\gamma|/2\alpha - \nu} \|u\|_{\boldsymbol{h}_\alpha^p}$$

for all $(x,t) \in H$. Furthermore, if $\nu > -(n/2\alpha)(1/p)$, then the derivative $\partial_x^\gamma \mathcal{D}_t^\nu u(x,t)$ is well defined, and the equation $\partial_x^\gamma \mathcal{D}_t^\nu u(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ holds.

(2) *If $\beta \in \mathbb{N}_0^n$, then the derivative $\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ is well defined, and*

$$\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma u(x,t) = \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} u(x,t).$$

(3) *If κ satisfies $\kappa + \nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha$, then the derivative $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ is well defined, and*

$$\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma u(x,t) = \mathcal{D}_t^{\kappa+\nu} \partial_x^\gamma u(x,t).$$

(4) *The derivative $\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

We present the definition of an $L^{(\alpha)}$-conjugate of functions on H, which is introduced in [7].

DEFINITION 1 ([7, Definition 1]). Let $0 < \alpha \leq 1$ and u a function on H. We shall say that an n-tuple of functions $(v_1, \ldots, v_n)$ on H is an $L^{(\alpha)}$-conjugate of u if $v_j(x, \cdot\,), u(x, \cdot\,) \in \mathcal{FC}^{1/2\alpha}$ and $(n+1)$-tuple $(v_1, \ldots, v_n, u)$ satisfies the following equations:

$$\partial_j v_k = \partial_k v_j, \qquad 1 \leq j, k \leq n, \tag{N.1}$$

$$\partial_j u = -\mathcal{D}_t^{1/2\alpha} v_j, \qquad 1 \leq j \leq n, \tag{N.2}$$

and

$$\mathcal{D}_t^{1/2\alpha} u = \sum_{j=1}^n \partial_j v_j. \tag{N.3}$$

We note that when $\alpha = 1/2$, the equations of Definition 1 coincide with the generalized Cauchy-Riemann equations for harmonic functions in [13]. As we see below, $u(x, \cdot\,) \in \mathcal{FC}^{1/2\alpha}$ for all $u \in \boldsymbol{h}_\alpha^p$. Theorem 2 shows the existence and the norm estimates of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}_\alpha^p$-functions.

THEOREM 2. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$, then the following statements hold*:

(1) *If $u \in \boldsymbol{h}_\alpha^p$, then there exists a unique $L^{(\alpha)}$-conjugate $(v_1, \ldots, v_n)$ of u such that $v_j \in \boldsymbol{h}_\alpha^p$.*

(2) *If an n-tuple of functions $(v_1, \ldots, v_n)$ with $v_j \in \boldsymbol{h}_\alpha^p$ satisfies Equation (N.1), then there exists a unique function $u \in \boldsymbol{h}_\alpha^p$ such that $(v_1, \ldots, v_n)$ is the $L^{(\alpha)}$-conjugate of u.*

(3) *There exists a constant $C > 0$ independent of $u \in \boldsymbol{h}_\alpha^p$ such that*

$$C^{-1} \|u\|_{\boldsymbol{h}_\alpha^p} \leq \sum_{j=1}^n \|v_j\|_{\boldsymbol{h}_\alpha^p} \leq C \|u\|_{\boldsymbol{h}_\alpha^p},$$

where $(v_1, \dots, v_n)$ is the $L^{(\alpha)}$-conjugate of u with $v_j \in \boldsymbol{h}^p_\alpha$.

We present the definition of an α-parabolic maximal function, which is the extension of the non-tangential maximal function. For $x \in \mathbb{R}^n$ and $\rho > 0$, let

$$C^{(\alpha)}_\rho(x) := \{(y,s) \in H : |y-x|^{2\alpha} \le \rho^{-1} s\}.$$

For a function u on H, we define an α-parabolic maximal function $\mathcal{N}^{(\alpha)}_\rho[u]$ on $\mathbb{R}^n$ by

$$\mathcal{N}^{(\alpha)}_\rho[u](x) := \sup\bigl\{|u(y,s)| : (y,s) \in C^{(\alpha)}_\rho(x)\bigr\}, \qquad x \in \mathbb{R}^n.$$

We remark that when $\alpha = 1/2$, the function $\mathcal{N}^{(1/2)}_\rho[u]$ coincides with the non-tangential maximal function of u. Theorem 3 shows that a function u on H belongs to $\boldsymbol{h}^p_\alpha$ if and only if an α-parabolic maximal function $\mathcal{N}^{(\alpha)}_\rho[u]$ belongs to L^p.

THEOREM 3. *Let $0 < \alpha \le 1$, $1 < p \le \infty$, $\rho > 0$, and u be an $L^{(\alpha)}$-harmonic function on H. Then, $u \in \boldsymbol{h}^p_\alpha$ if and only if $\mathcal{N}^{(\alpha)}_\rho[u] \in L^p$. Furthermore, the property $\mathcal{N}^{(\alpha)}_\rho[u] \in L^p$ is independent of ρ, that is, if $\mathcal{N}^{(\alpha)}_\rho[u] \in L^p$ for some ρ, then $\mathcal{N}^{(\alpha)}_\rho[u] \in L^p$ for all ρ.*

We note that Theorems 2 and 3 hold when $1 < p < \infty$. The investigations for the case $p = 1$ are more difficult, whose results will be described elsewhere.

We describe the construction of this paper. In Section 2, we recall definitions of the $L^{(\alpha)}$-harmonic functions and the fundamental solution of $L^{(\alpha)}$. Furthermore, some lemmas are presented. In Section 3, we introduce an $L^{(\alpha)}$-harmonic extension, which is defined by the convolution of the fundamental solution of $L^{(\alpha)}$. And we give several properties of $L^{(\alpha)}$-harmonic extensions. In Section 4, we study of fractional calculus on parabolic Hardy spaces, that is, we give the proof of Theorem 1. In Section 5, we show the existence of $L^{(\alpha)}$-conjugates on parabolic Hardy spaces. In Section 6, we estimate the norms of $L^{(\alpha)}$-conjugates of parabolic Hardy functions, that is, we give the proof of Theorem 2. In Section 7, we study properties of the α-parabolic maximal functions of parabolic Hardy functions, that is, we give the proof of Theorem 3. Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

In this section, we recall definitions of the $L^{(\alpha)}$-harmonic functions, the fundamental solution of $L^{(\alpha)}$ (for details, see [9]), and fractional differential operators. We begin with describing the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H and let $C^\infty_c(H) \subset C^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\varepsilon \to +0} \int_{|y|>\varepsilon} \frac{\psi(x+y,t) - \psi(x,t)}{|y|^{n+2\alpha}} dV_n(y) \tag{2.1}$$

for all $\psi \in C^\infty_c(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma\bigl((n+2\alpha)/2\bigr)/\Gamma(-\alpha) > 0$ and Γ is the gamma function. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function

$u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is,

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)\widetilde{L}^{(\alpha)}\psi(x,t)| dV_n(x)dt < \infty \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}^n} u(x,t)\widetilde{L}^{(\alpha)}\psi(x,t) dV_n(x)dt = 0$$

for all $\psi \in C_c^\infty(H)$. By (2.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that

$$\mathrm{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S = \mathbb{R}^n \times [t_1, t_2]$$

and

$$|\widetilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha} \text{ for } (x,t) \in S.$$

Hence, the condition $\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$ for all $\psi \in C_c^\infty(H)$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV_n(x)dt < \infty. \tag{2.2}$$

We present the explicit definition of the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x\cdot\xi)\, dV_n(\xi) & (t > 0) \\ 0 & (t \le 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is called the fundamental solution of $L^{(\alpha)}$. We also describe basic properties of $W^{(\alpha)}$. It is well known that

$$W^{(\alpha)}(x,t) > 0, \qquad (x,t) \in H \tag{2.3}$$

and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t) dV_n(x) = 1, \qquad 0 < t < \infty. \tag{2.4}$$

We also remark that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H and $W^{(\alpha)} \in C^\infty(H)$. The following estimate is [9, Lemma 3.1]: there exists a constant $C = C(n,\alpha) > 0$ such that

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t+|x|^{2\alpha})^{n/2\alpha+1}} \tag{2.5}$$

for all $(x,t) \in H$.

In case $\alpha = 1/2$, the function $W^{(1/2)}$ is the Poisson kernel, that is,

$$W^{(1/2)}(x,t) = \begin{cases} \dfrac{\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}} \dfrac{t}{(t^2+|x|^2)^{(n+1)/2}} & (t > 0) \\ 0 & (t \le 0). \end{cases}$$

In case $\alpha = 1$, the function $W^{(1)}$ is the Gauss kernel, that is,

$$W^{(1)}(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\Big(-\dfrac{|x|^2}{4t}\Big) & (t > 0) \\ 0 & (t \le 0). \end{cases}$$

In other cases, simple explicit expressions for $W^{(\alpha)}$ are not known.

We also present the following lemma, which is obtained from the proofs of [9, Theorem 4.1] and [14, Lemma 3.1] when $1 \le p < \infty$, and which is obtained from [10, Proposition 11] when $p = \infty$.

LEMMA 2.1. *Let $0 < \alpha \le 1$ and let u be $L^{(\alpha)}$-harmonic on H. If $1 \le p < \infty$ and u is p-th integrable on any strip domain of H, that is,*

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |u(x,t)|^p dV_n(x)dt < \infty \quad \textit{for all} \quad 0 < t_1 < t_2 < \infty, \tag{2.6}$$

then u satisfies the following Huygens property, that is,

$$u(x,t+s) = \int_{\mathbb{R}^n} u(x-y,t)W^{(\alpha)}(y,s)dV_n(y) = \int_{\mathbb{R}^n} u(y,t)W^{(\alpha)}(x-y,s)dV_n(y) \tag{2.7}$$

holds for all $x \in \mathbb{R}^n$, $0 < s < \infty$, and $0 < t < \infty$. Furthermore, if u is bounded on any strip domain of H, that is,

$$\sup\{|u(x,t)| : x \in \mathbb{R}^n, t \in [t_1,t_2]\} < \infty \quad \textit{for all} \quad 0 < t_1 < t_2 < \infty, \tag{2.8}$$

then u satisfies the Huygens property (2.7).

As in the proof of [9, Lemma 5.6], we clearly obtain by Lemma 2.1 the following.

LEMMA 2.2. *Let $0 < \alpha \le 1$ and let u be $L^{(\alpha)}$-harmonic on H. If $1 \le p < \infty$ and u satisfies the condition* (2.6), *then the function $t \mapsto \|u(\,\cdot\,,t)\|_{L^p}$ is non-increasing on $(0,\infty)$. Furthermore, If u satisfies the condition* (2.8), *then the function $t \mapsto \|u(\,\cdot\,,t)\|_{L^\infty}$ is non-increasing on $(0,\infty)$.*

Now, we recall definitions of the fractional integral and differential operators for functions on $\mathbb{R}_+ = (0,\infty)$. For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \{\varphi \in C(\mathbb{R}_+) : \varphi(t) = O(t^{-\kappa'})\ (t \to \infty) \text{ for some } \kappa' > \kappa\}.$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)}\int_0^\infty \tau^{\kappa-1}\varphi(\tau+t)d\tau, \qquad t \in \mathbb{R}_+. \tag{2.9}$$

We put $\mathcal{FC}^0 := C(\mathbb{R}_+)$ and $\mathcal{D}_t^0\varphi := \varphi$. Moreover, let

$$\mathcal{FC}^{\kappa} := \{\varphi\, ;\, \partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}\},$$

where $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. Then, we can also define the fractional derivative $\mathcal{D}_t^{\kappa}\varphi$ of $\varphi \in \mathcal{FC}^{\kappa}$ by

$$\mathcal{D}_t^{\kappa}\varphi(t) := \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)}\left((-\partial_t)^{\lceil\kappa\rceil}\varphi\right)(t), \qquad t \in \mathbb{R}_+. \tag{2.10}$$

Clearly, when $\kappa \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, the operator $\mathcal{D}_t^{\kappa}$ coincides with the ordinary differential operator $(-\partial_t)^{\kappa}$. For a real number κ, we may call both (2.9) and (2.10) *the fractional derivatives of φ with* *order* κ. And, we call $\mathcal{D}_t^\kappa$ *the fractional differential operator with order* κ. Some basic properties of the fractional differential operators are the following.

LEMMA 2.3. ([3, Proposition 2.1] and [4, Proposition 2.2]) *For real numbers* $\kappa, \nu > 0$, *the following statements hold.*

(1) If $\varphi \in \mathcal{FC}^{-\kappa}$, *then* $\mathcal{D}_t^{-\kappa}\varphi \in C(\mathbb{R}_+)$.

(2) If $\varphi \in \mathcal{FC}^{-\kappa-\nu}$, *then* $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\kappa-\nu}\varphi$.

(3) If $\partial_t^k\varphi \in \mathcal{FC}^{-\nu}$ *for all integers* $0 \le k \le \lceil\kappa\rceil - 1$ *and* $\partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)-\nu}$, *then* $\mathcal{D}_t^\kappa\mathcal{D}_t^{-\nu}\varphi = \mathcal{D}_t^{-\nu}\mathcal{D}_t^\kappa\varphi = \mathcal{D}_t^{\kappa-\nu}\varphi$.

(4) If $\partial_t^{k+\lceil\nu\rceil}\varphi \in \mathcal{FC}^{-(\lceil\nu\rceil-\nu)}$ *for all integers* $0 \le k \le \lceil\kappa\rceil - 1$, $\partial_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}$ *for all integers* $0 \le \ell \le \lceil\nu\rceil - 1$, *and* $\partial_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, *then* $\mathcal{D}_t^\kappa\mathcal{D}_t^\nu\varphi = \mathcal{D}_t^{\kappa+\nu}\varphi$.

(5) If $\partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-\lceil\kappa\rceil}$ *and* $\lim_{t\to\infty}\partial_t^k\varphi(t) = 0$ *for all integers* $0 \le k \le \lceil\kappa\rceil - 1$, *then* $\mathcal{D}_t^{-\kappa}\mathcal{D}_t^\kappa\varphi = \varphi$.

Here, we give some examples of fractional derivatives of elementary functions.

EXAMPLE 2.4. *Let* $\kappa > 0$ *and* ν *be real numbers. Then, we have the following.*

(1) $\mathcal{D}_t^\nu e^{-\kappa t} = \kappa^\nu e^{-\kappa t}$.

(2) *If* $-\kappa < \nu$, *then* $\mathcal{D}_t^\nu t^{-\kappa} = \dfrac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} t^{-\kappa-\nu}$.

We need the following lemma in our later arguments.

LEMMA 2.5. *Let* $0 < \alpha \le 1$ *and let* c *be a real number such that* $n/2\alpha - c < 0$. *Then, there exists a constant* $C = C(n, \alpha, c) > 0$ *such that*

$$\int_{\mathbb{R}^n} \frac{1}{(t + |x-y|^{2\alpha})^c} dV_n(y) = Ct^{n/2\alpha - c}$$

for all $(x, t) \in H$.

PROOF. Making a change of variable, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{(t + |x-y|^{2\alpha})^c} dV_n(y) &= \int_{\mathbb{R}^n} \frac{1}{(t + |y|^{2\alpha})^c} dV_n(y) \\ = \int_{\mathbb{R}^n} \frac{t^{n/2\alpha}}{(t + |t^{1/2\alpha}y|^{2\alpha})^c} dV_n(y) &= t^{n/2\alpha - c}\int_{\mathbb{R}^n} \frac{1}{(1 + |y|^{2\alpha})^c} dV_n(y).\end{aligned}$$

Since $n/2\alpha - c < 0$, we have

$$C = \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^{2\alpha})^c} dV_n(y) < \infty.$$

□

3. The $L^{(\alpha)}$-harmonic extensions

In this section, we study several properties of the $L^{(\alpha)}$-harmonic extensions. The results obtained in this section shall be used for investigations of parabolic Hardy spaces in the next section. We begin with recalling the definition of the Lebesgue spaces L^p. For $1 \le p \le \infty$, the Lebesgue space $L^p := L^p(\mathbb{R}^n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions f on $\mathbb{R}^n$ with $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} |f(x)|^p dV_n(x) \right)^{1/p} & (1 \le p < \infty) \\ \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| & (p = \infty). \end{cases}$$

Let $M := M(\mathbb{R}^n)$ be the set of all finite signed Borel measures on $\mathbb{R}^n$. We denote by $\|\mu\|$ the total variation norm of $\mu \in M$. Now, we present the definition of an $L^{(\alpha)}$-harmonic extension, which is introduced in [6].

DEFINITION 3.1 ([6, (1.2) and (1.3) of Section 1]). For $1 \le p \le \infty$, we define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_f^{(\alpha)}$ of $f \in L^p$ by

$$\mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t) f(y) dV_n(y), \qquad (x,t) \in H.$$

We also define an $L^{(\alpha)}$-harmonic extension $\mathcal{H}_\mu^{(\alpha)}$ of $\mu \in M$ by

$$\mathcal{H}_\mu^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t) d\mu(y), \qquad (x,t) \in H.$$

We note that $L^{(\alpha)}$-harmonic extensions of $f \in L^p$ and $\mu \in M$ are $L^{(\alpha)}$-harmonic on H (see [6, Theorem 5.2]).

First, we study derivatives or fractional derivatives of $L^{(\alpha)}$-harmonic extensions. For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$. We present some properties of fractional derivatives of the fundamental solution $W^{(\alpha)}$. The following lemma is [3, Theorem 3.1].

LEMMA 3.2. ([3, Theorem 3.1]) *Let $0 < \alpha \le 1$ and $\gamma \in \mathbb{N}_0^n$ a multi-index. If ν is a real number such that $\nu > -n/2\alpha$, then the following statements hold*:

(1) *The derivatives $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ and $\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ are well defined, and the equation $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ holds. Furthermore, there exists a constant $C = C(n,\alpha,\gamma,\nu) > 0$ such that*

$$|\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)| \le C \frac{1}{(t + |x|^{2\alpha})^{(n+|\gamma|)/2\alpha + \nu}}$$

for all $(x,t) \in H$.

(2) *If a real number κ satisfies $\kappa + \nu > -n/2\alpha$, then the derivative $\mathcal{D}_t^\kappa \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\kappa \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t) = \partial_x^\gamma \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(x,t).$$

(3) *The derivative $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

We define an auxiliary function on $\mathbb{R}$, which is used in our later arguments. For $\nu \in \mathbb{R}$, let

$$\omega(\nu) = \begin{cases} \lceil \nu \rceil & (\nu \ge 0) \\ 0 & (\nu < 0). \end{cases}$$

We give more general properties of fractional derivatives of $W^{(\alpha)}$.

LEMMA 3.3. *Let $0 < \alpha \le 1$ and $\gamma \in \mathbb{N}_0^n$ a multi-index. If ν is a real number such that $\nu > -(n+|\gamma|)/2\alpha$, then the following statements hold*:

(1) *The derivative $\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ is well defined. Furthermore, there exists a constant $C = C(n,\alpha,\gamma,\nu) > 0$ such that*

$$|\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)| \le C \frac{1}{(t+|x|^{2\alpha})^{(n+|\gamma|)/2\alpha+\nu}} \tag{3.1}$$

for all $(x,t) \in H$.

(2) *If $\beta \in \mathbb{N}_0^n$ is a multi-index, then the derivative $\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ is well defined, and*

$$\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} W^{(\alpha)}(x,t).$$

(3) *If a real number κ satisfies $\kappa+\nu > -(n+|\gamma|)/2\alpha$, then the derivative $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t) = \mathcal{D}_t^{\kappa+\nu} \partial_x^\gamma W^{(\alpha)}(x,t).$$

(4) *The derivative $\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

PROOF. (1) By Lemma 3.2 (1), we have $|\partial_x^\gamma W^{(\alpha)}(x,t)| \le C(t+|x|^{2\alpha})^{-(n+|\gamma|)/2\alpha}$ for all $(x,t) \in H$. It suffices to show the lemma for the case $\gamma \ne 0$ and $-n/2\alpha \ge \nu > -(n+|\gamma|)/2\alpha$. The proof is similar to that of [3, Theorem 3.1 (1)].

(2) Since $W^{(\alpha)} \in C^\infty(H)$, the case $\nu \in \mathbb{N}_0$ is trivial. Thus, suppose that $\nu \notin \mathbb{N}_0$. Then, the definitions of the fractional derivatives (2.9) and (2.10) imply that

$$\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t) = \frac{1}{\Gamma(\omega(\nu)-\nu)} \int_0^\infty \tau^{\omega(\nu)-\nu-1} \mathcal{D}_t^{\omega(\nu)} \partial_x^\gamma W^{(\alpha)}(x,\tau+t)d\tau.$$

Since we can differentiating under the integral sign by Lemma 3.3 (1), we obtain

$$\begin{aligned}\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t) &= \frac{1}{\Gamma(\omega(\nu)-\nu)} \int_0^\infty \tau^{\omega(\nu)-\nu-1} \mathcal{D}_t^{\omega(\nu)} \partial_x^{\beta+\gamma} W^{(\alpha)}(x,\tau+t)d\tau \\ &= \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} W^{(\alpha)}(x,t).\end{aligned}$$

(3) Using Estimate (3.1), we obtain the desired result from Lemma 2.3 (2), (3), (4), and (5).

(4) The proof is similar to that of [3, Theorem 3.1 (3)]. □

Now, we give properties of fractional derivatives of $L^{(\alpha)}$-harmonic extensions. We prepare the following interval. Let $n \ge 1$ and $0 < \alpha \le 1$ be fixed. For $\gamma \in \mathbb{N}_0^n$ and $1 \le p \le \infty$, define the interval $I(\gamma,p)$ by

$$I(\gamma,p) := \begin{cases} \{\nu \in \mathbb{R} : \nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha\} & (p \ne \infty) \\ \{\nu \in \mathbb{R} : \nu > -|\gamma|/2\alpha\} \cup \{0\} & (p = \infty). \end{cases}$$

THEOREM 3.4. *Let $0 < \alpha \le 1$, $1 \le p \le \infty$, and $\gamma \in \mathbb{N}_0^n$. If $f \in L^p$ and $\mu \in M$, then the following statements hold*:

(1) *If $\nu \in I(\gamma,p)$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t) f(y) dV_n(y). \tag{3.2}$$

Furthermore, there exists a constant $C = C(n, \alpha, p, \gamma, \nu) > 0$ such that

$$|\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)| \le Ct^{-(n/2\alpha)(1/p)-|\gamma|/2\alpha-\nu}\|f\|_{L^p}$$

for all $(x,t) \in H$. If $\nu \in I(0,p)$, then the derivative $\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_f^{(\alpha)}(x,t)$ is well defined, and the equation $\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_f^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ holds.

If $\nu \in I(\gamma,1)$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)$ is well defined, and

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t)d\mu(y).$$

Furthermore, there exists a constant $C = C(n,\alpha,\gamma,\nu) > 0$ such that

$$|\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)| \le Ct^{-n/2\alpha-|\gamma|/2\alpha-\nu}\|\mu\|$$

for all $(x,t) \in H$. If $\nu \in I(0,1)$, then the derivative $\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_\mu^{(\alpha)}(x,t)$ is well defined, and the equation $\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_\mu^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)$ holds.

(2) *If $\nu \in I(\gamma,p)$ and $\beta \in \mathbb{N}_0^n$, then the derivative $\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is well defined, and*

$$\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} \mathcal{H}_f^{(\alpha)}(x,t).$$

If $\nu \in I(\gamma,1)$ and $\beta \in \mathbb{N}_0^n$, then the derivative $\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)$ is well defined, and

$$\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} \mathcal{H}_\mu^{(\alpha)}(x,t).$$

(3) *If $\nu \in I(\gamma,p)$ and κ satisfies $\kappa+\nu \in I(\gamma,p)$, then the derivative $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) = \mathcal{D}_t^{\kappa+\nu} \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t).$$

If $\nu \in I(\gamma,1)$ and κ satisfies $\kappa+\nu \in I(\gamma,1)$, then the derivative $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)$ is well defined, and

$$\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t) = \mathcal{D}_t^{\kappa+\nu} \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t).$$

(4) *If $\nu \in I(\gamma,p)$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H. If $\nu \in I(\gamma,1)$ then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_\mu^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

PROOF. Since the proof of $\mathcal{H}_\mu^{(\alpha)}$ is analogous to that of $\mathcal{H}_f^{(\alpha)}$ with $f \in L^1$, we only show the assertion for $\mathcal{H}_f^{(\alpha)}$.

(1) Let $\nu \in \mathbb{N}_0$. Suppose $1 < p < \infty$ and let q be the exponent conjugate to p. Then, by the Hölder inequality, Lemma 3.3 (1), and Lemma 2.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t)f(y)|dV_n(y) \\
&\le \left(\int_{\mathbb{R}^n} |\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t)|^q dV_n(y)\right)^{1/q} \|f\|_{L^p} \\
&\le C\left(\int_{\mathbb{R}^n} \frac{1}{(t+|x-y|^{2\alpha})^{\{(n+|\gamma|)/2\alpha+\nu\}q}} dV_n(y)\right)^{1/q} \|f\|_{L^p} \\
&\le Ct^{(n/2\alpha)(1/q)-(n+|\gamma|)/2\alpha-\nu}\|f\|_{L^p}. \qquad (3.3)
\end{aligned}$$

We remark that (3.3) for the cases $p = 1$ and $p = \infty$ are also obtained by Lemma 3.3 (1), the property (2.4), and Lemma 2.5. Therefore, for $1 \le p \le \infty$, we can differentiate under the integral sign, and we get (3.2) for the case $\nu \in \mathbb{N}_0$. Furthermore, we also have

$$\begin{aligned} |\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)| &\le \int_{\mathbb{R}^n} |\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t) f(y)| dV_n(y) \\ &\le Ct^{-(n/2\alpha)(1/p)-|\gamma|/2\alpha-\nu} \|f\|_{L^p}. \end{aligned} \tag{3.4}$$

Let $\nu \in I(\gamma,p)\backslash\mathbb{N}_0$. Then, (3.2) for the case $\nu \in \mathbb{N}_0$ implies that

$$\begin{aligned} &\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) \\ &= \frac{1}{\Gamma(\omega(\nu)-\nu)} \int_0^\infty \tau^{\omega(\nu)-\nu-1} \mathcal{D}_t^{\omega(\nu)} \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,\tau+t) d\tau \\ &= \frac{1}{\Gamma(\omega(\nu)-\nu)} \int_0^\infty \tau^{\omega(\nu)-\nu-1} \int_{\mathbb{R}^n} \mathcal{D}_t^{\omega(\nu)} \partial_x^\gamma W^{(\alpha)}(x-y,\tau+t) f(y) dV_n(y) d\tau. \end{aligned} \tag{3.5}$$

We show that we can apply the Fubini theorem to (3.5). Indeed, Estimate (3.4) implies that

$$\begin{aligned} &\int_0^\infty \tau^{\omega(\nu)-\nu-1} \int_{\mathbb{R}^n} |\mathcal{D}_t^{\omega(\nu)} \partial_x^\gamma W^{(\alpha)}(x-y,\tau+t) f(y)| dV_n(y) d\tau \\ &\le C \int_0^\infty \tau^{\omega(\nu)-\nu-1} (\tau+t)^{-(n/2\alpha)(1/p)-|\gamma|/2\alpha-\omega(\nu)} d\tau < \infty, \end{aligned}$$

because $\nu \in I(\gamma,p)$. Therefore, we obtain (3.2) for the case $\nu \in I(\gamma,p)\backslash\mathbb{N}_0$. Furthermore, as in the proof of (3.3), we also get (3.4) for the case $\nu \in I(\gamma,p)\backslash\mathbb{N}_0$ by Lemma 3.3 (1).

Let $\nu \in I(0,p)$. Since we have already shown (3.2) for $\gamma = 0$ and $\nu \in I(0,p)$, we obtain

$$\mathcal{D}_t^\nu \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \mathcal{D}_t^\nu W^{(\alpha)}(x-y,t) f(y) dV_n(y).$$

Differentiating under the integral sign, we get

$$\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x-y,t) f(y) dV_n(y).$$

Hence, the equation $\partial_x^\gamma \mathcal{D}_t^\nu \mathcal{H}_f^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t)$ is obtained.

(2) Let $\nu \in I(\gamma,p)$ and $\beta \in \mathbb{N}_0^n$. Then, by the proof of Theorem 3.4 (1), we have

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t) f(y) dV_n(y)$$

By differentiating under the integral sign, Lemma 3.3 (2) implies that

$$\begin{aligned} \partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma \mathcal{H}_f^{(\alpha)}(x,t) &= \int_{\mathbb{R}^n} \partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x-y,t) f(y) dV_n(y) \\ &= \int_{\mathbb{R}^n} \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} W^{(\alpha)}(x-y,t) f(y) dV_n(y) \\ &= \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} \mathcal{H}_f^{(\alpha)}(x,t), \end{aligned}$$

because $\nu \in I(\beta+\gamma,p)$.

(3) Using the estimate of Theorem 3.4 (1), we obtain the desired result from Lemma 2.3 (2), (3), (4), and (5).

(4) We show that $\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{H}_f^{(\alpha)}$ satisfies (2.2). For any $0 < t_1 < t_2 < \infty$, Theorem 3.4 (1) implies that

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{R}^n} |\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{H}_f^{(\alpha)}(x,t)|(1+|x|)^{-n-2\alpha}dV_n(x)dt \\ &\leq C\int_{t_1}^{t_2} t^{-(n/2\alpha)(1/p)-|\gamma|/2\alpha-\nu}dt \int_{\mathbb{R}^n}(1+|x|)^{-n-2\alpha}dV_n(x) < \infty.\end{aligned}$$

Thus, $\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{H}_f^{(\alpha)}$ satisfies (2.2). The $L^{(\alpha)}$-harmonicity of $\mathcal{D}_t^{\nu}\partial_x^{\gamma}\mathcal{H}_f^{(\alpha)}$ follows from the Fubini theorem and Lemma 3.3 (4). □

Next, we shall give more properties of $L^{(\alpha)}$-harmonic extensions. The following lemma is shown in [6, Theorem 4.2].

LEMMA 3.5. ([6, Theorem 4.2]) *Let* $0 < \alpha \leq 1$. *Then, the following statements hold*:

(1) *Let* $1 \leq p \leq \infty$ *and* $f \in L^p$. *Then,*

$$\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)\|_{L^p} \leq \|f\|_{L^p}$$

for all $t > 0$.

(2) *Let* $\mu \in M$. *Then,*

$$\|\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,,t)\|_{L^1} \leq \|\mu\|$$

for all $t > 0$.

We note that $M = (C_0)^*$, where $C_0 := C_0(\mathbb{R}^n)$ is the set of all continuous functions on $\mathbb{R}^n$ that vanish at ∞. By properties (2.3), (2.4), and (2.5), the following results are obtained, which were shown in [6, Theorem 5.1].

LEMMA 3.6. ([6, Theorem 5.1]) *Let* $0 < \alpha \leq 1$. *Then, the following statements hold*:

(1) *If* $1 \leq p < \infty$ *and* $f \in L^p$, *then the functions* $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ *converge to* f *in the norm topology on* L^p *as* $t \to +0$.

(2) *If* $f \in L^\infty$, *then the functions* $\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)$ *converge to* f *in the weak-star topology on* L^∞ *as* $t \to +0$.

(3) *If* $\mu \in M$, *then the measures* $\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,,t)dV_n$ *converge to* μ *in the weak-star topology on* M *as* $t \to +0$.

We obtain the following theorem, which shall be used for investigations of parabolic Hardy spaces.

THEOREM 3.7. *Let* $0 < \alpha \leq 1$. *Then, the following statements hold*:

(1) *Let* $1 \leq p \leq \infty$ *and* $f \in L^p$. *Then,*

$$\sup_{t>0}\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)\|_{L^p} = \lim_{t\to 0}\|\mathcal{H}_f^{(\alpha)}(\,\cdot\,,t)\|_{L^p} = \|f\|_{L^p}.$$

(2) *Let* $\mu \in M$. *Then,*

$$\sup_{t>0}\|\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,,t)\|_{L^1} = \lim_{t\to 0}\|\mathcal{H}_\mu^{(\alpha)}(\,\cdot\,,t)\|_{L^1} = \|\mu\|.$$

PROOF. (1) Let $1 \le p < \infty$ and $f \in L^p$. Then, Lemma 3.6 (1) implies

$$\lim_{t \to 0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^p} = \|f\|_{L^p}.$$

By Lemma 3.5 (1) and Theorem 3.4 (4), the function $\mathcal{H}_f^{(\alpha)}$ satisfies (2.6) and is $L^{(\alpha)}$-harmonic on H. Thus, by Lemma 2.2, we have

$$\sup_{t>0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^p} = \lim_{t \to 0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^p} = \|f\|_{L^p}.$$

Let $f \in L^\infty$. Then, Lemma 3.6 (2) implies

$$\|f\|_{L^\infty} \le \liminf_{t \to 0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^\infty}.$$

By Lemma 3.5 (1) and Theorem 3.4 (4), the function $\mathcal{H}_f^{(\alpha)}$ satisfies (2.8) and is $L^{(\alpha)}$-harmonic on H. Thus, by Lemma 2.2, we have

$$\sup_{t>0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^\infty} = \lim_{t \to 0} \|\mathcal{H}_f^{(\alpha)}(\,\cdot\,, t)\|_{L^\infty} = \|f\|_{L^\infty}.$$

(2) The proof of (2) is similar to that of (1) when $p = \infty$. $\square$

4. The parabolic Hardy spaces

The parabolic Hardy spaces were introduced in [9, Remark 5.7]. Hardly properties of their spaces have been studied. In this section, we study properties of fractional derivatives of parabolic Hardy functions. Particularly, we give the proof of Theorem 1. We begin with recalling definition of the parabolic Hardy spaces. For $0 < \alpha \le 1$ and $1 \le p \le \infty$, the α-parabolic Hardy space $\boldsymbol{h}_\alpha^p$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with

$$\|u\|_{\boldsymbol{h}_\alpha^p} := \sup_{t>0} \|u(\,\cdot\,, t)\|_{L^p} < \infty.$$

By Lemma 2.2, we have

$$\|u\|_{\boldsymbol{h}_\alpha^p} = \lim_{t \to 0} \|u(\,\cdot\,, t)\|_{L^p}$$

for all $1 \le p \le \infty$ and $u \in \boldsymbol{h}_\alpha^p$. By Theorem 3.7 (1), the mapping $f \mapsto \mathcal{H}_f^{(\alpha)}$ is a linear isometry of L^p into $\boldsymbol{h}_\alpha^p$ when $1 < p \le \infty$. By Theorem 3.7 (2), the mapping $\mu \mapsto \mathcal{H}_\mu^{(\alpha)}$ is also a linear isometry of M into $\boldsymbol{h}_\alpha^1$. In Theorem 4.1 below, we show that the mappings are onto. Consequently, we obtain several properties of fractional derivatives of $\boldsymbol{h}_\alpha^p$-functions from Theorem 3.4. It also follows from Theorem 4.1 that $\boldsymbol{h}_\alpha^p$ are Banach spaces for all $1 \le p \le \infty$.

THEOREM 4.1. *Let* $0 < \alpha \le 1$. *Then, the following statements hold*:

(1) *For* $1 < p \le \infty$, *the mapping* $f \mapsto \mathcal{H}_f^{(\alpha)}$ *is a linear isometry of* L^p *onto* $\boldsymbol{h}_\alpha^p$.

(2) *The mapping* $\mu \mapsto \mathcal{H}_\mu^{(\alpha)}$ *is a linear isometry of* M *onto* $\boldsymbol{h}_\alpha^1$.

PROOF. (1) Let $1 < p \le \infty$ and $u \in \boldsymbol{h}_\alpha^p$. Also, let q be the exponent conjugate to p. Then, the set $\{u(\,\cdot\,, t) : t > 0\}$ is bounded in $L^p = (L^q)^*$. Since L^q is separable, there exist a sequence $\{t_j\}$

and a function $f \in L^p$ such that $t_j \to 0$ and $u(\,\cdot\,, t_j)$ converges to f in the weak-star topology. Let $(x,t) \in H$ be fixed. Then, by Lemma 2.1, we have

$$u(x, t+t_j) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)u(y,t_j)dV_n(y).$$

Here, we note that $W^{(\alpha)}(x-\,\cdot\,,t) \in L^q$. Indeed, when $q=1$, the conditions (2.3) and (2.4) imply that

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)dV_n(y) = \int_{\mathbb{R}^n} W^{(\alpha)}(y,t)dV_n(y) = 1.$$

Furthermore, when $1 < q < \infty$, Estimate (2.5) and Lemma 2.5 show that

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)^q dV_n(y) \le C \int_{\mathbb{R}^n} \frac{1}{(t+|x-y|^{2\alpha})^{nq/2\alpha}} dV_n(y) \le Ct^{n/2\alpha - nq/2\alpha}.$$

Hence, let $j \to \infty$, then we obtain $u(x,t) = \mathcal{H}_f^{(\alpha)}(x,t)$.

(2) Let $u \in \boldsymbol{h}^1_\alpha$. Then, the set $\{u(\,\cdot\,,t)dV_n : t > 0\}$ is bounded in $M = (C_0)^*$. Since C_0 is also separable, there exist a sequence $\{t_j\}$ and a measure $\mu \in M$ such that $t_j \to 0$ and $u(\,\cdot\,,t_j)dV_n$ converges to μ in the weak-star topology. Let $(x,t) \in H$ be fixed. Then, by Estimate (2.5), we have $W^{(\alpha)}(x-\cdot\,,t) \in C_0$. Thus, by similar arguments in the proof of (1), we obtain $u(x,t) = \mathcal{H}_\mu^{(\alpha)}(x,t)$. □

Now, we give the proof of Theorem 1.

PROOF OF THEOREM 1. The assertion immediately follows from Theorems 3.4 and 4.1. □

5. The existence of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}^p_\alpha$-functions

In this section, for given $\boldsymbol{h}^p_\alpha$-functions, we construct an conjugate system. In Theorem 5.2 below, we show the existence of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}^p_\alpha$-functions. We need the following lemma.

LEMMA 5.1. ([5, Lemma 4.1]) *Let* $0 < \alpha \le 1$. *Then,*

$$\left(\mathcal{D}_t^{1/\alpha} + \Delta_x\right) W^{(\alpha)}(x,t) = 0$$

for all $(x,t) \in H$.

Now, we show the existence of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}^p_\alpha$-functions.

THEOREM 5.2. *Let* $0 < \alpha \le 1$ *and* $1 \le p < \infty$, *then the following statements hold*:

(1) *If* $u \in \boldsymbol{h}^p_\alpha$, *then for each* $1 \le j \le n$, *we can define a function* v_j *on* H *by*

$$v_j(x,t) := -\mathcal{D}_t^{-1/2\alpha}\partial_j u(x,t), \qquad (x,t) \in H. \tag{5.1}$$

Also, each function v_j *is* $L^{(\alpha)}$*-harmonic on* H. *Furthermore, the* n*-tuple of functions* $(v_1, \dots, v_n)$ *is an* $L^{(\alpha)}$*-conjugate of* u.

(2) *If an* n*-tuple of functions* $(v_1, \dots, v_n)$ *with* $v_j \in \boldsymbol{h}^p_\alpha$ *satisfies Equation (N.1), then we can define a function* u *on* H *by*

$$u(x,t) := \sum_{j=1}^n \mathcal{D}_t^{-1/2\alpha}\partial_j v_j(x,t), \qquad (x,t) \in H. \tag{5.2}$$

Also, the function u is $L^{(\alpha)}$-harmonic on H. Furthermore, the n-tuple of functions $(v_1, \dots, v_n)$ is an $L^{(\alpha)}$-conjugate of u.

PROOF. (1) Let $u \in \boldsymbol{h}^p_\alpha$. By Theorem 1 (1), we can define a function v_j on H by (5.1), and we also have $v_j(x, \,\cdot\,) \in \mathcal{FC}^{1/2\alpha}$. The $L^{(\alpha)}$-harmonicity of v_j follows from Theorem 1 (4).

We show that the $(n+1)$-tuple $(v_1, \dots, v_n, u)$ satisfies Equations (N.1) and (N.2). By Theorem 1 (2), we obtain

$$\partial_j v_k(x,t) = -\partial_j \mathcal{D}_t^{-1/2\alpha} \partial_k u(x,t) = -\mathcal{D}_t^{-1/2\alpha} \partial_j \partial_k u(x,t), \qquad 1 \le j, k \le n. \tag{5.3}$$

Thus, Equation (N.1) is satisfied. Furthermore, by Theorem 1 (3), we have

$$-\mathcal{D}_t^{1/2\alpha} v_j(x,t) = \mathcal{D}_t^{1/2\alpha} \mathcal{D}_t^{-1/2\alpha} \partial_j u(x,t) = \partial_j u(x,t), \qquad 1 \le j \le n.$$

Therefore, Equation (N.2) is also satisfied.

We show that $(v_1, \dots, v_n, u)$ also satisfies Equation (N.3). First, we claim that

$$\left(\mathcal{D}_t^{1/\alpha} + \Delta_x\right) u(x,t) = 0 \tag{5.4}$$

for all $(x,t) \in H$. In fact, suppose that $1 < p < \infty$. By Theorem 4.1 (1), there exists a function $f \in L^p$ such that $u = \mathcal{H}_f^{(\alpha)}$. Thus, (3.2) of Theorem 3.4 and Lemma 5.1 imply that

$$\begin{aligned}\left(\mathcal{D}_t^{1/\alpha} + \Delta_x\right) u(x,t) &= \left(\mathcal{D}_t^{1/\alpha} + \Delta_x\right) \mathcal{H}_f^{(\alpha)}(x,t) \\ &= \int_{\mathbb{R}^n} \left(\mathcal{D}_t^{1/\alpha} + \Delta_x\right) W^{(\alpha)}(x-y,t) f(y) dV_n(y) = 0.\end{aligned}$$

The proof of the case $p = 1$ is similar to that of the case $1 < p < \infty$. Hence, we obtain (5.4) for all $1 \le p < \infty$. We show that $(n+1)$-tuple of functions $(v_1, \dots, v_n, u)$ satisfies Equation (N.3). By (5.3), we have

$$\sum_{j=1}^n \partial_j v_j(x,t) = -\sum_{j=1}^n \mathcal{D}_t^{-1/2\alpha} \partial_j^2 u(x,t) = -\mathcal{D}_t^{-1/2\alpha} \Delta_x u(x,t).$$

Therefore, (5.4) and Theorem 1 (3) imply that

$$\sum_{j=1}^n \partial_j v_j(x,t) = \mathcal{D}_t^{-1/2\alpha} \mathcal{D}_t^{1/\alpha} u(x,t) = \mathcal{D}_t^{1/2\alpha} u(x,t).$$

(2) Suppose that an n-tuple of functions $(v_1, \dots, v_n)$ with $v_j \in \boldsymbol{h}^p_\alpha$ satisfies Equation (N.1). By Theorem 1 (1), we can define a function u on H by (5.2), and we also have $u(x, \,\cdot\,) \in \mathcal{FC}^{1/2\alpha}$. The $L^{(\alpha)}$-harmonicity of u follows from Theorem 1 (4).

We show that the $(n+1)$-tuple $(v_1, \dots, v_n, u)$ satisfies Equations (N.2) and (N.3). By (5.2) and Theorem 1 (2), for each $1 \le k \le n$, Equation (N.1) implies that

$$\partial_k u(x,t) = \sum_{j=1}^n \mathcal{D}_t^{-1/2\alpha} \partial_k \partial_j v_j(x,t) = \mathcal{D}_t^{-1/2\alpha} \Big(\sum_{j=1}^n \partial_j^2 v_k(x,t) \Big) = \mathcal{D}_t^{-1/2\alpha} \Delta_x v_k(x,t).$$

Since (5.4) holds for all $\boldsymbol{h}^p_\alpha$-functions, Theorem 1 (3) shows that

$$\partial_k u(x,t) = -\mathcal{D}_t^{-1/2\alpha} \mathcal{D}_t^{1/\alpha} v_k(x,t) = -\mathcal{D}_t^{1/2\alpha} v_k(x,t),$$

so Equation (N.2) is satisfied. Moreover, (5.2) and Theorem 1 (3) imply that

$$\mathcal{D}_t^{1/2\alpha}u(x,t)=\sum_{j=1}^{n}\mathcal{D}_t^{1/2\alpha}\mathcal{D}_t^{-1/2\alpha}\partial_j v_j(x,t)=\sum_{j=1}^{n}\partial_j v_j(x,t),$$

so Equation (N.3) is also satisfied. □

6. The norms of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}_\alpha^p$-functions with $1<p<\infty$

In this section, we estimate the norms of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}_\alpha^p$-functions. Our estimates are given for the case $1<p<\infty$. Consequently, we give the proof of Theorem 2. For each $f\in L^1\cap L^2$, the Fourier transform of f is defined by (according to the definition of [12, p.28 (1.3 of Chapter II)])

$$\hat{f}(y):=\int_{\mathbb{R}^n} f(x)e^{2\pi i x\cdot y}dV_n(x),\qquad y\in\mathbb{R}^n,$$

and it is well known that the Fourier transform can be extended to all of L^2 by continuity. The following lemma is Theorem 1 of [12, p.29 (2.2 of Chapter II)].

LEMMA 6.1. (Theorem 1 of [12, p.29 (2.2 of Chapter II)]) *Let $K\in L^2$. We suppose:*

(a) *The Fourier transform of K is essentially bounded*

$$|\hat{K}(x)|\le B,\qquad x\in\mathbb{R}^n.$$

(b) *K is of class $C^1(\mathbb{R}^n)$ outside the origin and*

$$|\nabla_x K(x)|\le \frac{B}{|x|^{n+1}},\qquad x\in\mathbb{R}^n.$$

For $f\in L^1\cap L^p$, let us set

$$Tf(x)=\int_{\mathbb{R}^n} K(x-y)f(y)dV_n(y),\qquad x\in\mathbb{R}^n.$$

Then, the following statements hold:

(1) There exists a constant A_p, so that

$$\|Tf\|_{L^p}\le A_p\|f\|_{L^p}.$$

One can thus extend T to all of L^p by continuity. The constant A_p depends only on p, B, and n. In particular, it does not depend on the L^2 norm of K.

(2) There exists a constant $C=C(B,n)$, so that

$$V_n(\{x\in\mathbb{R}^n:|Tf(x)|>\lambda\})\le\frac{C}{\lambda}\|f\|_1.$$

The authors also describe the following remark. The assumption that $K\in L^2$ is made for the purpose of having direct definition of Tf on a dense subset of L^p (in this case $L^1\cap L^p$), and it could be replaced by other assumptions.

We show the following lemma.

LEMMA 6.2. *Let* $0 < \alpha \leq 1$ *and* $1 \leq j \leq n$. *We put*

$$K_j(x,t) := \mathcal{D}_t^{-1/2\alpha} \partial_j W^{(\alpha)}(x,t), \qquad (x,t) \in H.$$

Then, for fixed $t > 0$, *we have* $K_j(\,\cdot\,, t) \in L^q$ *for all* $1 < q < \infty$, *and the following conditions hold:*

(1) *There exists a constant* $B > 0$ *such that*

$$|\hat{K}_j(x,t)| \leq B$$

for all $(x,t) \in H$ *and* $1 \leq j \leq n$.

(2) *For fixed* $t > 0$, $K_j(\,\cdot\,, t)$ *is of class* $C^1(\mathbb{R}^n)$ *and there exists a constant* $B > 0$ *such that*

$$|\nabla_x K_j(x,t)| \leq \frac{B}{|x|^{n+1}}$$

for all $(x,t) \in H$ *and* $1 \leq j \leq n$.

PROOF. First, we show that $K_j(\,\cdot\,, t) \in L^q$ for all $1 < q < \infty$. Indeed, by Lemma 3.3 (1) and Lemma 2.5, we have

$$\int_{\mathbb{R}^n} |K_j(x,t)|^q dV_n(x) \leq C \int_{\mathbb{R}^n} \frac{1}{(t + |x|^{2\alpha})^{qn/2\alpha}} dV_n(x) = Ct^{n/2\alpha - qn/2\alpha}.$$

(1) Since we can differentiate under the integral sign, we get

$$\partial_j W^{(\alpha)}(x,t) = -2\pi i \int_{\mathbb{R}^n} \xi_j e^{-(2\pi)^{2\alpha} t|\xi|^{2\alpha}} e^{-2\pi i x\cdot\xi}\, d\xi.$$

Thus, we have

$$\begin{aligned} K_j(x,t) &= \mathcal{D}_t^{-1/2\alpha} \partial_j W^{(\alpha)}(x,t) \\ &= -2\pi i \frac{1}{\Gamma(1/2\alpha)} \int_0^\infty \tau^{1/2\alpha - 1} \int_{\mathbb{R}^n} \xi_j e^{-(2\pi)^{2\alpha}(t+\tau)|\xi|^{2\alpha}} e^{-2\pi i x\cdot\xi}\, d\xi d\tau. \end{aligned}$$

The Fubini theorem and Example 2.4 (1) imply that

$$\begin{aligned} K_j(x,t) &= -2\pi i \int_{\mathbb{R}^n} \xi_j \big(\mathcal{D}_t^{-1/2\alpha} e^{-(2\pi)^{2\alpha} t|\xi|^{2\alpha}}\big) e^{-2\pi i x\cdot\xi}\, d\xi \\ &= -i \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} e^{-(2\pi)^{2\alpha} t|\xi|^{2\alpha}} e^{-2\pi i x\cdot\xi}\, d\xi. \end{aligned}$$

Therefore, by the inversion theorem of the Fourier transform, we obtain

$$\hat{K}_j(x,t) = -i \frac{x_j}{|x|} e^{-(2\pi)^{2\alpha} t|x|^{2\alpha}},$$

so we get $|\hat{K}_j(x,t)| \leq 1$ for all $(x,t) \in H$ and $1 \leq j \leq n$.

(2) For each $1 \leq k \leq n$, Lemma 3.3 implies that

$$|\partial_k K_j(x,t)| = |\mathcal{D}_t^{-1/2\alpha} \partial_k \partial_j W^{(\alpha)}(x,t)| \leq \frac{C}{(t + |x|^{2\alpha})^{(n+1)/2\alpha}} \leq \frac{C}{|x|^{n+1}}$$

for all $(x,t) \in H$ and $1 \leq j \leq n$. $\square$

By Lemmas 6.1 and 6.2, we give the following estimate.

PROPOSITION 6.3. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. If $u \in \boldsymbol{h}^p_\alpha$, then there exists a constant $C = C(n, \alpha, \gamma, p) > 0$ independent of u such that*

$$\|\mathcal{D}_t^{-1/2\alpha} \partial_j u\|_{\boldsymbol{h}^p_\alpha} \leq C \|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $1 \leq j \leq n$.

PROOF. By Theorem 4.1 (1), there exists a function $f \in L^p$ such that $u = \mathcal{H}_f^{(\alpha)}$ and $\|u\|_{\boldsymbol{h}^p_\alpha} = \|f\|_{L^p}$. By Theorem 3.4 (1), we have

$$\begin{aligned}
\mathcal{D}_t^{-1/2\alpha} \partial_j u(x,t) &= \mathcal{D}_t^{-1/2\alpha} \partial_j \mathcal{H}_f^{(\alpha)}(x,t) \\
&= \int_{\mathbb{R}^n} \mathcal{D}_t^{-1/2\alpha} \partial_j W^{(\alpha)}(x-y,t) f(y) dV_n(y) \\
&= \int_{\mathbb{R}^n} K_j(x-y,t) f(y) dV_n(y),
\end{aligned}$$

where $K_j(x,t) = \mathcal{D}_t^{-1/2\alpha} \partial_j W^{(\alpha)}(x,t)$. Thus, by Lemmas 6.1 and 6.2, there exists a constant A_p so that

$$\left(\int_{\mathbb{R}^n} |\mathcal{D}_t^{-1/2\alpha} \partial_j u(x,t)|^p dV_n(x) \right)^{1/p} \leq A_p \|f\|_{L^p} = A_p \|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $t > 0$ and $1 \leq j \leq n$. Since the constant A_p is independent of $t > 0$, we obtain

$$\|\mathcal{D}_t^{-1/2\alpha} \partial_j u\|_{\boldsymbol{h}^p_\alpha} = \sup_{t>0} \left(\int_{\mathbb{R}^n} |\mathcal{D}_t^{-1/2\alpha} \partial_j u(x,t)|^p dV_n(x) \right)^{1/p} \leq A_p \|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $1 \leq j \leq n$. $\square$

Now, we give the proof of Theorem 2.

PROOF OF THEOREM 2. (1) Let $u \in \boldsymbol{h}^p_\alpha$. For each $1 \leq j \leq n$, let $(v_1, \ldots, v_n)$ be the n-tuple of functions defined by (5.1). Then, by Theorem 5.2 (1), the n-tuple $(v_1, \ldots, v_n)$ is an $L^{(\alpha)}$-conjugate of u. Furthermore, Proposition 6.3 implies that there exists a constant $C = C(n, \alpha, \gamma, p) > 0$ independent of u such that

$$\|v_j\|_{\boldsymbol{h}^p_\alpha} \leq C \|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $1 \leq j \leq n$, so we have $v_j \in \boldsymbol{h}^p_\alpha$. To show the uniqueness, we suppose that there exists an n-tuple of functions $(u_1, \ldots, u_n)$ with $u_j \in \boldsymbol{h}^p_\alpha$ such that $(u_1, \ldots, u_n)$ is an $L^{(\alpha)}$-conjugate of u. Then, by Lemma 3.3 (3) and Equation (N.2), we obtain

$$u_j = \mathcal{D}_t^{-1/2\alpha} \mathcal{D}_t^{1/2\alpha} u_j = -\mathcal{D}_t^{-1/2\alpha} \partial_j u = v_j.$$

(2) Let $(v_1, \ldots, v_n)$ be the n-tuple of functions with $v_j \in \boldsymbol{h}^p_\alpha$ such that $(v_1, \ldots, v_n)$ satisfies Equation (N.1). Let u be the function defined by (5.2). Then, by Theorem 5.2 (2), the n-tuple $(v_1, \ldots, v_n)$ is an $L^{(\alpha)}$-conjugate of u. Furthermore, Proposition 6.3 implies that there exists a constant $C = C(n, \alpha, \gamma, p) > 0$ such that

$$\|u\|_{\boldsymbol{h}^p_\alpha} = \| \sum_{j=1}^n \mathcal{D}_t^{-1/2\alpha} \partial_j v_j \|_{\boldsymbol{h}^p_\alpha} \leq \sum_{j=1}^n \|\mathcal{D}_t^{-1/2\alpha} \partial_j v_j\|_{\boldsymbol{h}^p_\alpha} \leq C \sum_{j=1}^n \|v_j\|_{\boldsymbol{h}^p_\alpha},$$

so we have $u \in \boldsymbol{h}^p_\alpha$. To show the uniqueness, we suppose that there exists an function $v \in \boldsymbol{h}^p_\alpha$ such that $(v_1, \dots, v_n)$ is an $L^{(\alpha)}$-conjugate of v. Then, by Lemma 3.3 (3) and Equation (N.3), we obtain

$$v = \mathcal{D}_t^{-1/2\alpha}\mathcal{D}_t^{1/2\alpha}v = \sum_{j=1}^{n} \mathcal{D}_t^{-1/2\alpha}\partial_j v_j = u.$$

(3) The desired result immediately follows form the proofs of (1) and (2). □

We close this section with making a remark for the case $p = 1$.

REMARK 6.4. Using Lemma 6.1 (2) instead of (1), we have the following weak $(1, 1)$ type relations between u and $(v_1, \dots, v_n)$ in Theorem 5.2 when $p = 1$:

(1) If $u \in \boldsymbol{h}^1_\alpha$, then we have for $1 \le j \le n$

$$\sup_{t>0} V_n(\{x \in \mathbb{R}^n : |v_j(x,t)| > \lambda\}) \le \frac{C}{\lambda}\|u\|_{\boldsymbol{h}^1_\alpha}$$

with some constant C, where $(v_1, \dots, v_n)$ is defined in Theorem 5.2 (1).

(2) If an n-tuple of functions $(v_1, \dots, v_n)$ with $v_j \in \boldsymbol{h}^1_\alpha$ satisfies Equation (N.1), then we have

$$\sup_{t>0} V_n(\{x \in \mathbb{R}^n : |u(x,t)| > \lambda\}) \le \frac{C}{\lambda}\sum_{j=1}^{n}\|v_j\|_{\boldsymbol{h}^1_\alpha}$$

with some constant C, where u is defined in Theorem 5.2 (2).

In fact, if $u \in \boldsymbol{h}^1_\alpha$, then for $t > s > 0$, we have

$$\mathcal{D}_t^{-1/2\alpha}\partial_j u(x,t) \le \int_{\mathbb{R}^n} K_j(x-y, t-s)u(y,s)dV_n(y).$$

Hence, by Lemma 6.1 (2), we have

$$V_n(\{x \in \mathbb{R}^n : |v_j(x,t)| > \lambda\}) \le \frac{C}{\lambda}\|u(\cdot, s)\|_{L^1}$$

with some constant C independent of $t > s > 0$, which shows (1). We obtain (2) similarly.

7. The α-parabolic maximal functions

In this section, we study properties of the α-parabolic maximal functions, that is, we give the proof of Theorem 3. We recall the definition of the α-parabolic maximal functions. For $x \in \mathbb{R}^n$ and $\rho > 0$, let

$$C^{(\alpha)}_\rho(x) := \{(y,s) \in H : |y-x|^{2\alpha} \le \rho^{-1}s\}.$$

The α-parabolic maximal function $\mathcal{N}^{(\alpha)}_\rho[u]$ of a function u on H is defined by

$$\mathcal{N}^{(\alpha)}_\rho[u](x) := \sup\bigl\{|u(y,s)| : (y,s) \in C^{(\alpha)}_\rho(x)\bigr\}, \qquad x \in \mathbb{R}^n.$$

Clearly, for a function u on H, we have

$$\|u\|_{\boldsymbol{h}^p_\alpha} \le \|\mathcal{N}^{(\alpha)}_\rho[u]\|_{L^p}$$

for all $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, and $\rho > 0$. We observe the relation with the Hardy-Littlewood maximal functions. For $1 \leq p \leq \infty$, the Hardy-Littlewood maximal function $\mathcal{M}_f$ of $f \in L^p$ is defined by

$$\mathcal{M}_f(x) = \sup_{r>0} \frac{1}{V_n(B(x,r))} \int_{|y-x|<r} |f(y)| dV_n(y), \qquad x \in \mathbb{R}^n,$$

where $B(x,r)$ is the ball of radius r centered at x. Furthermore, the Hardy-Littlewood maximal function $\mathcal{M}_\mu$ of $\mu \in M$ is defined by

$$\mathcal{M}_\mu(x) = \sup_{r>0} \frac{1}{V_n(B(x,r))} \int_{|y-x|<r} d|\mu|(y), \qquad x \in \mathbb{R}^n.$$

The following lemma concerning the Hardy-Littlewood maximal functions is well known.

LEMMA 7.1. ([11, Theorem 7.4] and [12, Theorem 1]) *If $1 < p \leq \infty$ and $f \in L^p$, then there exists a constant $C = C(n,p) > 0$ independent of f such that*

$$\|\mathcal{M}_f\|_{L^p} \leq C \|f\|_{L^p}. \tag{7.1}$$

Moreover, if $\mu \in M$ and $\lambda > 0$ is a real number, then there exists a constant $C = C(n) > 0$ independent of μ and λ such that

$$V_n(\{x \in \mathbb{R}^n : \mathcal{M}_\mu(x) > \lambda\}) \leq \frac{C}{\lambda} \|\mu\|. \tag{7.2}$$

Now we shall give a proof of Theorem 3. The following lemma is a generalization of [8, Lemma 5.2]. Here, we use the notation of $\mathcal{H}_\mu^{(\alpha)}$ and $\mathcal{M}_\mu^{(\alpha)}$ for general positive Borel measures μ on $\mathbb{R}^n$, which may take value $+\infty$.

LEMMA 7.2. *Let μ be a positive Borel measure on H. If we put $u := \mathcal{H}_\mu^{(\alpha)}$, then there exists a constant $C = C(n,\alpha,\rho) > 0$ independent of μ such that*

$$\mathcal{N}_\rho^{(\alpha)}[u](x) \leq C\, \mathcal{M}_\mu^{(\alpha)}(x)$$

for all $x \in \mathbb{R}^n$.

PROOF. Let $x \in \mathbb{R}^n$. Then for $(y,s) \in C_\rho^{(\alpha)}(x)$ and $z \in \mathbb{R}^n$, we have

$$\begin{aligned} s + |x-z|^{2\alpha} &\leq s + (|x-y| + |y-z|)^{2\alpha} \leq s + 2(|x-y|^{2\alpha} + |y-z|^{2\alpha}) \\ &\leq s + 2(\rho^{-1}s + |y-z|^{2\alpha}) \leq C\,(s + |y-z|^{2\alpha}), \end{aligned}$$

with some constant C. Now, let $\mu \geq 0$ be a Borel measure and put $u := \mathcal{H}_\mu^{(\alpha)}$. Then, (2.5) implies that there exists a constant $C = C(n,\alpha) > 0$ such that

$$\begin{aligned} |u(y,s)| &= \int_{\mathbb{R}^n} W^{(\alpha)}(y-z,s)\, d\mu(z) \leq C \int_{\mathbb{R}^n} \frac{s}{(s+|y-z|^{2\alpha})^{n/2\alpha+1}}\, d\mu(z) \\ &\leq Cs \int_{\mathbb{R}^n} \frac{1}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}\, d\mu(z) \end{aligned}$$

for all $(y,s) \in C_\rho^{(\alpha)}(x)$. Thus, putting

$$\tau_z := \frac{1}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}},$$

we have, by the Fubini theorem,

$$\begin{aligned}|u(y,s)| &\le Cs\int_{\mathbb{R}^n}\frac{1}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}\,d\mu(z) = Cs\int_{\mathbb{R}^n}\int_0^{\tau_z} dt\,d\mu(z)\\ &= Cs\int_0^{\tau_z}\int_{|z-x|<r_t} d\mu(z)\,dt \le Cs\int_0^{\tau_z} V_n(B(x,r_t))\,\mathcal{M}_\mu(x)\,dt\\ &= C\,\mathcal{M}_\mu(x)s\int_0^{\tau_z}\int_{|z-x|<r_t} dz\,dt\\ &= C\,\mathcal{M}_\mu(x)s\int_{\mathbb{R}^n}\frac{1}{(s+|x-z|^{2\alpha})^{n/2\alpha+1}}dV_n(z),\end{aligned}$$

where $r_t > 0$ denotes $|z|$ such that $\tau_z = t$. The proof is complete by Lemma 2.5. $\square$

THEOREM 7.3. *Let* $0 < \alpha \le 1$ *and* $\rho > 0$. *Then, the following statements hold*:
(1) *If* $1 < p \le \infty$, *then there exists a constant* $C = C(n, \alpha, p, \rho) > 0$ *such that*

$$\|u\|_{\boldsymbol{h}^p_\alpha} \le \|\mathcal{N}^{(\alpha)}_\rho[u]\|_{L^p} \le C\,\|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $L^{(\alpha)}$*-harmonic functions* u *on* H.

(2) *If* $p = 1$, *then there exists a constant* $C = C(n, \alpha, \rho) > 0$ *such that*

$$V_n(\{x \in \mathbb{R}^n : \mathcal{N}^{(\alpha)}_\rho[u](x) > \lambda\}) \le \frac{C}{\lambda}\,\|u\|_{\boldsymbol{h}^p_\alpha}$$

for all λ *and* $L^{(\alpha)}$*-harmonic functions* u *on* H.

PROOF. Let $1 \le p \le \infty$ and u an $L^{(\alpha)}$-harmonic function on H. If $\|u\|_{\boldsymbol{h}^p_\alpha} = \infty$, then the inequalities of (1) and (2) are always satisfied. Thus, suppose $\|u\|_{\boldsymbol{h}^p_\alpha} < \infty$, that is, $u \in \boldsymbol{h}^p_\alpha$.

We show the inequality of (1). It suffices to show the second inequality of (1). Suppose that $1 < p \le \infty$. Then, by Theorem 4.1 (1), there exists a function $f \in L^p$ such that $u = \mathcal{H}^{(\alpha)}_f$ and $\|u\|_{\boldsymbol{h}^p_\alpha} = \|f\|_{L^p}$. By Lemma 7.2 and (7.1), we obtain

$$\|\mathcal{N}^{(\alpha)}_\rho[u]\|_{L^p} \le C\|\mathcal{M}_f\|_{L^p} \le C\|f\|_{L^p} = C\|u\|_{\boldsymbol{h}^p_\alpha}.$$

We show the inequality of (2). By Theorem 4.1 (2), there exists a measure $\mu \in M$ such that $u = \mathcal{H}^{(\alpha)}_\mu$ and $\|u\|_{\boldsymbol{h}^1_\alpha} = \|\mu\|$. By Lemma 7.2, we have

$$\{x \in \mathbb{R}^n : \mathcal{N}^{(\alpha)}_\rho[u](x) > \lambda\} \subset \{x \in \mathbb{R}^n : \mathcal{M}_\mu(x) > \lambda/C\}$$

for all $\lambda > 0$. Therefore, (7.2) implies that

$$V_n(\{x \in \mathbb{R}^n : \mathcal{N}^{(\alpha)}_\rho[u](x) > \lambda\}) \le V_n(\{x \in \mathbb{R}^n : \mathcal{M}_\mu(x) > \lambda/C\}) \le \frac{C'}{\lambda/C}\|\mu\| = \frac{CC'}{\lambda}\|u\|_{\boldsymbol{h}^1_\alpha}$$

for all $\lambda > 0$. $\square$

Now, we give the proof of Theorem 3.

PROOF OF THEOREM 3. The assertion immediately follows from Theorem 7.3 (1). $\square$

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A CRITERION ON APPORTIONMENT METHODS MINIMIZING THE RÉNYI'S DIVERGENCE

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Abstract. For the Rényi's divergence with an index α, we propose a criterion with respect to the index α on apportionment methods, which the criterion is the sum of the Rényi's divergence and a proposed function of α as a kind of penalty. Under the criterion, we also obtain appropriate house seats in the House of Representatives and the House of Councillors in Japan.

1 Introduction In democratic states, seats are contested in the election. The seats are allocated by a rule which is regulated by the Diet, the Congress, or the Parliament. In principle, their seats should be proportional to the populations or the voters in their election districts, but it is difficult to determine them exactly because the seats are integers and the ratios of populations are usually rationals. To dissolve the gap between them, a lot of researches have been done from several research areas, for example, sociology, economics, operations research, and statistics.

[1] is a nice reference for determining methods with respect to proportional representation systems in the seats. Among the famous five divisor methods, i.e., the Adams method, the Dean method, the Hill method, the Webster (or Saint-Laguë) method, and the Jefferson (or D'Hondt) method, they propose that the Webster method is the best one because it is to minimize a bias, but there is counterviews against this proposal. [2] propose the primal problem and dual problem based on a rounding rule with respect to signposts as an optimization approach to vector and matrix apportionment problems. [4] proposes a divisor apportionment method based on the Kolm-Atkinson social welfare function. [3] shows that apportionment methods maximizing the Rényi's entropy are included in the divisor methods and that his approach with the index α is corresponding to the famous five methods. But it is not clear which α we should use among the apportionment method maximizing the Rényi's entropy.

In this paper, we propose a criterion with respect to the index α on apportionment methods, which the criterion is the sum of the Rényi's divergence and a proposed function of α as a kind of penalty and investigate the index α minimizing our proposed criterion. Under the criterion, we also investigate appropriate house seats in the House of Representatives (295 seats for single-seat constituency electoral system (2013)) and the House of Councillors (73 seats for that in re-election of half the members (2013)) in Japan.

2 Divisor methods We have to allocate the seats in proportion to the population to realize equivalent value of votes. The typical methods are the method of greatest remainders and the divisor method. The former has unfavorable properties like Alabama paradox and population paradox, but the latter is the only method which does not run such paradoxes.

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Suppose that a country has s states, the population in the state i is $p_i > 0$, and the total population is $\pi = \sum_{i=1}^{s} p_i$. Also suppose that the seat allocated in the state i is $a_i \geq 0$ by an appropriate divisor method and the total seats is $h = \sum_{i=1}^{s} a_i$, where any a_i is non-negative integer and $h > s$. If single-seat constituencies in the state i are a_i with the same size, we can define the value of a voter in the state i as a_i/p_i and it is obvious that the sum of the value of voters is equivalent to the total seats, i.e.,

$$\sum_{i=1}^{s} \left(\frac{a_i}{p_i}\right) \times p_i \;=\; \sum_{i=1}^{s} a_i \;=\; h.$$

For non-negative integer $a \in \mathbf{N}_0$, we define a function $d(a)$ as a rounding function. Then it is a strictly increasing function of integers a and it satisfies that $a \leq d(a) \leq a+1$. A divisor method is decided by determining a rounding function, respectively, as follows:

Divisor method	Adams	Dean	Hill	Webster	Jefferson
Rounding function $d(a)$	a	$\dfrac{a(a+1)}{a+\frac{1}{2}}$	$\sqrt{a(a+1)}$	$a+\frac{1}{2}$	$a+1$

Based on a rounding function $d(a)$ for positive integers $z > 0$, an integer $[z]$ is determined by the following rules:

- If $z < d(0)$, then $[z] = 0$.
- If $d(a) < z < d(a+1)$, then $[z] = a+1$.
- If $z = d(a)$, then $[z] = a$ or $[z] = a+1$.

Since $0 \leq d(0) < d(1) < d(2) < \cdots$, the value of the integer $[z]$ is uniquely determined for arbitrary real number $z > 0$ except for the last rule. If we decide a rounding function $d(a)$ for arbitrary non-negative integer a and we select a positive real number x appropriately, then, by the third rule, the total seats h is allocated as follows:

$$\sum_{i=1}^{s} \left[\frac{p_i}{x}\right] \;=\; h$$

and the seats in the state i is $a_i = [p_i/x]$. This method is called a divisor method based on a rounding function $d(a)$. If an allocated set is defined as follows:

$$\left\{ \boldsymbol{a} \;\middle|\; a_i \;=\; \left[\frac{p_i}{x}\right] \quad \text{where } x \text{ satisfies} \quad \sum_{i=1}^{s} \left[\frac{p_i}{x}\right] \;=\; h \right\},$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_s)$, this set is equivalent to the following set:

$$A_0 \;=\; \left\{ \boldsymbol{a} \;\middle|\; \max_{i \in S | a_i \geq 1} \left\{ \frac{d(a_i - 1)}{p_i} \right\} \;\leq\; \min_{j \in S} \left\{ \frac{d(a_j)}{p_j} \right\};\; \boldsymbol{a} \in F \right\}, \tag{2.1}$$

where $F = \{\boldsymbol{a} \,|\, a(S) = h;\; a_i \in \mathbf{N}_0 \ (i \in S)\}$, $S = \{1, 2, \ldots, s\}$, and $a(S) = \sum_{i=1}^{s} a_i$. [1](page 100).

3 Two types of the Rényi's divergences We consider the value of a voter and the size of constituency with respect to the Rényi's entropy.

When we can make a probability distribution by $\{a_i/(hp_i)\}$ as follows:

$$\mathcal{U} \;=\; \left(\overbrace{\frac{a_1}{hp_1},\ldots,\frac{a_1}{hp_1}}^{p_1},\ldots,\overbrace{\frac{a_s}{hp_s},\ldots,\frac{a_s}{hp_s}}^{p_s}\right),$$

the Rényi's entropy with respect to an index α is as follows:

$$H_\alpha(\mathcal{U}) \;=\; \begin{cases} \dfrac{1}{1-\alpha}\log_2\left(\displaystyle\sum_{i=1}^{s}\left(\frac{a_i}{h\,p_i}\right)^{\alpha} p_i\right), & \alpha>0,\ \alpha\neq 1,\\[2ex] -\displaystyle\sum_{i=1}^{s}\left(\frac{a_i}{h}\log_2\frac{a_i}{h\,p_i}\right), & \alpha = 1. \end{cases}$$

It is easy to obtain that the range of this entropy is $\min_i \log_2 p_i \le H_\alpha(\mathcal{U}) \le \log_2 \pi$, but that all hp_i/π are integers is extremely rare, so that we have to consider an apportionment which maximizes this entropy under some restrictions. Then the problem that the seats apportionment $\boldsymbol{a}$ maximizing the Rényi's entropy is equivalent to

$$\max_{\mathcal{U}} H_\alpha(\mathcal{U}) \quad \text{subject to} \quad a(S)=h,\ \ a_i\in\mathbf{N}_0\ (i\in S). \tag{3.2}$$

Let $\mathcal{A}$ be a distribution of proportion of each state's seats for the total seats,

$$\mathcal{A} \;=\; \left(\frac{a_1}{h},\ldots,\frac{a_s}{h}\right),\quad \frac{a_i}{h}\ge 0,\quad \sum_{i=1}^{s}\frac{a_i}{h} \;=\; 1,$$

and $\mathcal{P}$ be that of proportion of each state's population for the total population,

$$\mathcal{P} \;=\; \left(\frac{p_1}{\pi},\ldots,\frac{p_s}{\pi}\right),\quad \frac{p_i}{\pi}> 0,\quad \sum_{i=1}^{s}\frac{p_i}{\pi} \;=\; 1,$$

and $I_\alpha(\mathcal{A}\,\|\,\mathcal{P})$ be the Rényi's divergence of $\mathcal{A}$ based on $\mathcal{P}$,

$$I_\alpha(\mathcal{A}\,\|\,\mathcal{P}) \;=\; \frac{1}{\alpha-1}\log_2\left(\sum_{i=1}^{s}\left(\frac{a_i}{h}\right)^{\alpha}\left(\frac{p_i}{\pi}\right)^{1-\alpha}\right),\quad \alpha>0,\quad \alpha\neq 1.$$

Since the Rényi's entropy with respect to $\mathcal{U}$ is transformed into

$$H_\alpha(\mathcal{U}) \;=\; -\frac{1}{\alpha-1}\log_2\left(\sum_{i=1}^{s}\left(\frac{a_i}{h}\right)^{\alpha}\left(\frac{p_i}{\pi}\right)^{1-\alpha}\right)+\log_2\pi,$$

we have the following relationship:

$$H_\alpha(\mathcal{U})+I_\alpha(\mathcal{A}\,\|\,\mathcal{P}) \;=\; \log_2\pi,\quad \alpha>0,\quad \alpha\neq 1.$$

Note that, when $\alpha=1$, $I_1(\mathcal{A}\,\|\,\mathcal{P})$ is the Kullback-Leibler divergence of $\mathcal{A}$ based on $\mathcal{P}$. Thus [3] shows that the maximization of the Rényi's entropy $H_\alpha(\mathcal{U})$ is equivalent to the minimization of the Rényi's divergence $I_\alpha(\mathcal{A}\,\|\,\mathcal{P})$, so that the problem (3.2) to obtain the apportionment $\boldsymbol{a}$ maximizing the Rényi's entropy is reformulated as

$$\min_{\boldsymbol{a}} F_\alpha(\boldsymbol{a}) \quad \text{subject to} \quad a(S)=h,\ \ a_i\in\mathbf{N}_0\ (i\in S), \tag{3.3}$$

where

$$F_\alpha(\boldsymbol{a}) \;=\; \begin{cases} \dfrac{1}{\alpha-1}\displaystyle\sum_{i=1}^{s} a_i^\alpha p_i^{1-\alpha}, & \alpha > 0,\ \alpha \neq 1, \\ \displaystyle\sum_{i=1}^{s} a_i \log_2 \frac{a_i}{p_i}, & \alpha = 1. \end{cases} \tag{3.4}$$

Let $\alpha \neq 1$. $F_\alpha(\boldsymbol{a})$ is reformulated by $F_\alpha(\boldsymbol{a}) = \sum_{i=1}^{s} f_i(a_i)$, where $f_i(a_i) = a_i^\alpha/((\alpha-1)p_i^{\alpha-1})$ and the difference of $\{f_i(a_i)\}$ which are strictly convex is

$$u_i(a_i) \;=\; f_i(a_i+1) - f_i(a_i) \;=\; \frac{(a_i+1)^\alpha - a_i^\alpha}{(\alpha-1)\,p_i^{\alpha-1}}, \tag{3.5}$$

so that it is a strictly increasing function of a_i and $f_i(a_i) = \sum_{k=0}^{a_i-1} u_i(k)$. For the set $F = \{\boldsymbol{a} \,|\, a(S) = h;\ a_i \in \mathbf{N}_0\ (i \in S)\}$, a set of integer vector $\boldsymbol{a}$ is defined by

$$A \;=\; \left\{ \boldsymbol{a} \;\middle|\; \max_{i\in S|a_i\geq 1}\{u_i(a_i-1)\} \;\leq\; \min_{j\in S}\{u_j(a_j)\}\,;\, \boldsymbol{a}\in F \right\}. \tag{3.6}$$

For $\alpha = 1$, we define the followings:

$$f_i(a_i) \;=\; a_i \log_2 \frac{a_i}{p_i}, \quad u_i(a_i) \;=\; \log_2 \frac{(a_i+1)^{a_i+1}}{a_i^{a_i}\,p_i}.$$

[3] shows that, for $\alpha > 0$, $\boldsymbol{a}$ is the optimal solution of the problem (3.3) if and only if $\boldsymbol{a} \in A$.

For the set A which is characterized by the differences $\{u_i(a_i)\}$, we have rewritten forms

$$u_i(a_i) \;=\; \begin{cases} \dfrac{\alpha}{\alpha-1}\left(\dfrac{d(a_i)}{p_i}\right)^{\alpha-1}, \\ \log_2 \dfrac{d(a_i)}{p_i} + \log_2 e, \end{cases} \qquad d(a_i) \;=\; \begin{cases} \left(\dfrac{(a_i+1)^\alpha - a_i^\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}}, & (\alpha \neq 1,\ \alpha > 0), \\ \dfrac{1}{e}\,\dfrac{(a_i+1)^{a_i+1}}{a_i^{a_i}}, & (\alpha = 1), \end{cases}$$

so that the set A (3.6) is characterized by both the rounding function $d(a_i)$ and the ratios $d(a_i)/p_i$ like the set A_0 (2.1). [3] shows that an apportionment method maximizing the Rényi's entropy $H_\alpha(\mathcal{U})$ $(\alpha > 0)$ is a divisor method.

The value of a voter a_i/p_i means the number of seats per person in the state because the allocated seats in the state i is a_i. On the other hand, we can also consider an apportionment which satisfies the equality of p_i/a_i, which means the population per seat, i.e., the size of constituency. In this case $a_i \geq 1$ is assumed, which guarantees that every state has at least one seat. We regard $p_i/(\pi a_i)$ as a probability whose distribution is

$$\mathcal{W} \;=\; \left(\overbrace{\frac{p_1}{\pi a_1}, \ldots, \frac{p_1}{\pi a_1}}^{a_1}, \ldots, \overbrace{\frac{p_s}{\pi a_s}, \ldots, \frac{p_s}{\pi a_s}}^{a_s} \right),$$

so that the Rényi's entropy with respect to an index β is as follows:

$$H_\beta(\mathcal{W}) \;=\; \begin{cases} \dfrac{1}{1-\beta} \log_2 \left(\displaystyle\sum_{i=1}^{s} \left(\frac{p_i}{\pi\,a_i}\right)^\beta a_i \right), & (\beta > 0,\ \beta \neq 1), \\ -\displaystyle\sum_{i=1}^{s} \left(\frac{p_i}{\pi} \log_2 \frac{p_i}{\pi\,a_i} \right), & (\beta = 1). \end{cases}$$

As a similar way in the value of a voter, the Rényi's divergence of $\mathcal{P}$ based on $\mathcal{A}$ is

$$I_\beta(\mathcal{P}\,\|\,\mathcal{A}) \;=\; \frac{1}{\beta-1}\log_2\left(\sum_{i=1}^{s}\left(\frac{p_i}{\pi}\right)^{\beta}\left(\frac{a_i}{h}\right)^{1-\beta}\right), \quad \beta>0, \quad \beta\neq 1$$

and we have the following relationship:

$$H_\beta(\mathcal{W}) + I_\beta(\mathcal{P}\,\|\,\mathcal{A}) \;=\; \log_2 h, \quad \beta>0, \quad \beta\neq 1.$$

Note that, when $\beta=1$, $I_1(\mathcal{P}\,\|\,\mathcal{A})$ is the Kullback-Leibler divergence of $\mathcal{P}$ based on $\mathcal{A}$. The problem (3.2) to obtain the apportionment $\boldsymbol{a}=(a_1,a_2,\ldots,a_s)$ maximizing the Rényi's entropy is reformulated as

$$\min_{\boldsymbol{a}} F_\beta(\boldsymbol{a}) \quad \text{subject to} \quad a(S)=h, \;\; a_i\in\mathbf{N}\ (i\in S), \tag{3.7}$$

where $\mathbf{N}=\{1,2,\ldots\}$ and

$$F_\beta(\boldsymbol{a}) \;=\; \begin{cases} \dfrac{1}{\beta-1}\displaystyle\sum_{i=1}^{s} p_i^{\beta} a_i^{1-\beta}, & \beta>0,\ \beta\neq 1,\\ \displaystyle\sum_{i=1}^{s} p_i \log_2\frac{p_i}{a_i}, & \beta=1. \end{cases} \tag{3.8}$$

As the same way, we obtain the optimal set for the problem (3.7) and that an apportionment method based on the optimal set is a divisor method. In this case, the rounding function for $a_i\in\mathbf{N}$ is

$$d(a_i) \;=\; \begin{cases} \left(\dfrac{(a_i+1)^{1-\beta}-a_i^{1-\beta}}{1-\beta}\right)^{-\frac{1}{\beta}}, & \beta\neq 0,1,\\ \dfrac{1}{\log((a_i+1)/a_i)}, & \beta=1. \end{cases}$$

[3] shows that an apportionment method maximizing the Rényi's entropy $H_\beta(\mathcal{W})$ ($\beta>0$) is a divisor method and that the following relationship between an apportionment method maximizing the Rényi's entropy and the six popular divisor methods:

$\alpha>0$	$\alpha=1$	$\alpha=2$	$\alpha\to\infty$
$d(a)$	$\frac{(a+1)^{a+1}}{e\,a^a}$	$a+0.5$	$a+1$
method	Theil	Webster	Jefferson
$\beta>0$	$\beta=1$	$\beta=2$	$\beta\to\infty$
$d(a)$	$\frac{1}{\log((a+1)/a)}$	$\sqrt{a(a+1)}$	a
method	Theil & Schrage	Hill	Adams

4 A criterion on apportionment methods We understand that an apportionment method minimizing the Rényi's divergence is a divisor method and we have a question as follows: Which value in α or β should we use in the apportionment method as the best choice?

Before selecting the best value α,β, we consider properties of the Rényi's divergence.

LEMMA **4.1** *For $I_\alpha(\hat{\mathcal{A}}\|\mathcal{P})$ given $\hat{\mathcal{A}}$ with $\alpha\neq 1$, $I_\alpha(\hat{\mathcal{A}}\|\mathcal{P})$ is a monotone-increasing function with respect to α.*

Proof: Since

$$I_\alpha(\hat{\mathcal{A}}\|\mathcal{P}) \;=\; \frac{1}{\alpha-1}\log_2 G(\alpha) \;=\; \frac{\log G(\alpha)}{(\alpha-1)\,\log 2}$$

where

$$G(\alpha) \;=\; \sum_{i=1}^{s}\left(\frac{a_i}{h}\right)^{\alpha}\left(\frac{p_i}{\pi}\right)^{1-\alpha} \;=\; \sum_{i=1}^{s}\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right)^{\alpha}\frac{p_i}{\pi},$$

the derivative of $G(\alpha)$ with respect to α is

$$\dot{G}(\alpha) \;=\; \frac{\partial}{\partial\,\alpha}G(\alpha) \;=\; \sum_{i=1}^{s}\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right)^{\alpha}\frac{p_i}{\pi}\cdot\log\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right).$$

Letting

$$g_i(\alpha) \;=\; \left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right)^{\alpha-1}\frac{a_i}{h}$$

gives the relationships as follows:

$$G(\alpha) \;=\; \sum_{i=1}^{s} g_i(\alpha) \quad\text{and}\quad \dot{G}(\alpha) \;=\; \sum_{i=1}^{s} g_i(\alpha)\cdot\log\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right),$$

so that the derivative of $I_\alpha(\hat{\mathcal{A}}\|\mathcal{P})$ with respect to α is

$$\frac{\partial}{\partial\,\alpha}I_\alpha(\hat{\mathcal{A}}\|\mathcal{P}) \;=\; \frac{\dot{G}(\alpha)(\alpha-1) - G(\alpha)\log G(\alpha)}{G(\alpha)\,(\alpha-1)^2\,\log 2}. \tag{4.9}$$

Since the denominator is positive without $\alpha = 1$ and the property of Kullback-Leibler divergence, the numerator is

$$\begin{aligned}
\dot{G}(\alpha)(\alpha-1) - G(\alpha)\log(G(\alpha)) \;&=\; \sum_{i=1}^{s} g_i(\alpha)\left[(\alpha-1)\log\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right) - \log(G(\alpha))\right] \\
&=\; \sum_{i=1}^{s} g_i(\alpha)\left[\log\left(\frac{a_i}{h}\,\frac{\pi}{p_i}\right)^{\alpha-1} + \log\frac{a_i}{h} - \log\frac{a_i}{h} - \log(G(\alpha))\right] \\
&=\; \sum_{i=1}^{s} g_i(\alpha)\left[\log g_i(\alpha) - \log\frac{a_i}{h} - \log(G(\alpha))\right] \\
&=\; \sum_{i=1}^{s} g_i(\alpha)\left[\log\frac{g_i(\alpha)}{G(\alpha)} - \log\frac{a_i}{h}\right] \\
&=\; G(\alpha)\sum_{i=1}^{s}\frac{g_i(\alpha)}{G(\alpha)}\left[\log\frac{g_i(\alpha)}{G(\alpha)} - \log\frac{a_i}{h}\right] \;\geq\; 0, \qquad (4.10)
\end{aligned}$$

so that the Rényi's divergence $I_\alpha(\hat{\mathcal{A}}\|\mathcal{P})$ given $\hat{\mathcal{A}}$ is a monotone-increasing function of α. □

This means that the Rényi's divergence itself does not work as a criterion to choose the best index α. Then, in order to improve the Rényi's divergence as a criterion, we will use a kind of shrinkage function as follows:

$$r(\alpha) \;=\; \left(\frac{1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} \qquad (\alpha > 0), \tag{4.11}$$

which is a unimodal which is less than or equal to 1 and maximizes at $\alpha = 1$.

PROPOSITION 4.1 *Combining the monotonicity of the Rényi's divergence $I_\alpha(\mathcal{A}\|\mathcal{P})$ and the unimodality of the shrinkage function (4.11) with respect to α, we propose a following criterion based on the Rényi's divergence:*

$$
\begin{aligned}
IC(\alpha) \quad &:= \quad \frac{1}{\alpha-1}\log_2\left(\sum_{i=1}^{s}\left(\frac{a_i}{h}\right)^{\alpha}\left(\frac{p_i}{\pi}\right)^{1-\alpha} r(\alpha)\right) \\
&= \quad I_\alpha(\mathcal{A}\|\mathcal{P}) + \log_2\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}}. \qquad (4.12)
\end{aligned}
$$

The second term in (4.12) is interpreted as a penalty term, which is monotone-decreasing as α goes to $\exp(1)$ *and is monotone-increasing as α goes from* $\exp(1)$ *to the infinity.* □

We might consider that this formulation (4.12) is very similar to a usual information criterion which consists of both the likelihood and the penalty term, but the second term in this formulation corresponding to the penalty term is not the dimension of parameters, so that this formulation is not an information criterion exactly. It is, however, useful to obtain the index α which minimizes this criterion $IC(\alpha)$ given $\hat{\mathcal{A}}$ and we can use a following algorithm to obtain an appropriate index α and the appropriate apportionment.

<u>With respect to the seat per voter:</u>

- (Step 1) We choose arbitrary $\alpha > 0$, put it $\hat{\alpha}$, and set a permissible error $\varepsilon > 0$.
- (Step 2) For $\hat{\alpha} > 0$, we determine an allocated seats $\hat{\mathcal{A}}$ minimizing the Rényi's divergence:
$$\hat{\mathcal{A}} \ = \ \arg\min_{\mathcal{A}} I_{\hat{\alpha}}(\mathcal{A}\|\mathcal{P}).$$
- (Step 3) We determine an index $\tilde{\alpha}$ minimizing our proposed criterion:
$$\tilde{\alpha} \ = \ \arg\min_{\alpha} IC(\alpha) \ = \ \arg\min_{\alpha}\left(I_\alpha(\hat{\mathcal{A}}\|\mathcal{P}) + \log_2\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}}\right).$$
- (Step 4) If $|\hat{\alpha} - \tilde{\alpha}| < \varepsilon$, we output $\hat{\mathcal{A}}$ as a desired allocation. Otherwise, we replace $\tilde{\alpha}$ with $\hat{\alpha}$ and go to (Step 2).

<u>With respect to the population per seat:</u>

As the same way with respect to the seat per voter, we determine $\tilde{\beta}$ minimizing our proposed criterion:

$$\tilde{\beta} \ = \ \arg\min_{\beta} IC(\beta) \ = \ \arg\min_{\beta}\left(I_\beta(\mathcal{P}\|\hat{\mathcal{A}}) + \log_2\left(\frac{1}{\beta}\right)^{\frac{1}{\beta}}\right).$$

THEOREM 4.1 *For our proposed criterion $IC(\alpha)$, there exists α $(0 < \alpha < \exp(1), \alpha \neq 1)$ such that it attains the minimum of $IC(\alpha)$. As the same way, it holds for $IC(\beta)$ $(0 < \beta < \exp(1), \beta \neq 1)$.*

Proof: From (4.9) and (4.10), the derivative of our proposed criterion $IC(\alpha)$ is

$$\begin{aligned}
\frac{\partial}{\partial\,\alpha} IC(\alpha) &= \frac{\frac{\dot{G}(\alpha)}{G(\alpha)}(\alpha-1) - \log(G(\alpha))}{(\alpha-1)^2\,\log 2} - \frac{1-\log\alpha}{\alpha^2\,\log 2} \\
&= \frac{\alpha^2\,\gamma(\alpha) + (\alpha-1)^2(\log\alpha - 1)}{\alpha^2\,(\alpha-1)^2\,\log 2},
\end{aligned}$$

where

$$\gamma(\alpha) \;=\; \sum_{i=1}^{s} \frac{g_i(\alpha)}{G(\alpha)} \left[\log\frac{g_i(\alpha)}{G(\alpha)} - \log\frac{a_i}{h}\right] \;\geq\; 0,$$

so that the solution of $\partial IC(\alpha)/\partial\alpha = 0$ is α which satisfies the equation

$$\left(\frac{\alpha}{\alpha-1}\right)^2 \gamma(\alpha) \;=\; 1-\log\alpha \quad \text{for} \quad 0<\alpha<\exp(1),\ \alpha\neq 1.$$

Note that $\partial IC(\alpha)/\partial\alpha \geq 0$ for $\alpha \geq \exp(1)$. As the same way for $IC(\beta)$, we put the terms as follows:

$$\begin{aligned}
\tilde{G}(\beta) &= \sum_{i=1}^{s}\left(\frac{p_i}{\pi}\right)^{\beta}\left(\frac{a_i}{h}\right)^{1-\beta} = \sum_{i=1}^{s}\left(\frac{p_i}{\pi}\,\frac{h}{a_i}\right)^{\beta-1}\frac{p_i}{\pi}, \\
\tilde{g}_i(\beta) &= \left(\frac{p_i}{\pi}\,\frac{h}{a_i}\right)^{\beta-1}\frac{p_i}{\pi}, \\
\dot{\tilde{G}}(\beta) &= \frac{\partial}{\partial\,\beta}\tilde{G}(\beta) = \sum_{i=1}^{s}\tilde{g}_i(\beta)\,\log\left(\frac{p_i}{\pi}\,\frac{h}{a_i}\right), \\
\tilde{\gamma}(\beta) &= \sum_{i=1}^{s}\frac{\tilde{g}_i(\beta)}{\tilde{G}(\beta)}\left[\log\frac{\tilde{g}_i(\beta)}{\tilde{G}(\beta)} - \log\frac{p_i}{\pi}\right] \;\geq\; 0,
\end{aligned}$$

so that we have the similar result for $0<\beta<\exp(1), \beta\neq 1$. □

5 The single-seat constituencies of the Japanese Diet We consider the single-seat constituencies of the Japanese Diet based on 47 prefectures' population and voters in 2011. The Japanese Diet consists of the House of Representatives and the House of Councillors. The total seats in the former is 295 and that in the latter is 73 every reelection. The High Courts in Japan adjudges that the election of the House of Representatives in 2012 is unconstitutional under the present allocation with respect to the size of constituency. For this problem, we show the best solution under our proposed criterion in Table 1 which includes the results by the Webster and Hill methods for reference.

Table 1: Estimated seats by our proposed criterion: The term 'all' means the population in thousands and and 'voter' the voter in thousands. In the estimation, the term 'est.a' in the middle area in the 295 seats is an estimation with $\hat{\alpha} = 2.7$ and $IC(\hat{\alpha}) = -0.52308$, 'a2' is an estimation with $\alpha = 2$, 'est.b' with $\hat{\beta} = 2.7$ and $IC(\hat{\beta}) = -0.52321$, and 'b2' is an estimation with $\beta = 2$. The term 'est.a' in the right area in the 73 seats is an estimation with $\hat{\alpha} = 2.4$ and $IC(\hat{\alpha}) = -0.399774$, 'a2' is an estimation with $\alpha = 2$ which is the Webster method, 'est.b' with $\hat{\beta} = 2.3$ and $IC(\hat{\beta}) = -0.388826$, and 'b2' is an estimation with $\beta = 2$ which is the Hill method. Bold figures in the table indicate the difference of seats between $IC(\hat{\alpha})$ and $IC(\hat{\beta})$.

Prefecture	all	voter	seat	est.a	a2	est.b	b2	seat.C	est.a	a2	est.b	b2
Hokkaido	5486	4582	12	13	13	13	13	2	3	3	3	3
Aomori	1363	1127	4	3	3	3	3	1	1	1	1	1
Iwate	1314	1083	4	3	3	3	3	1	1	1	1	1
Miyagi	2327	1908	6	5	5	5	5	2	1	1	1	1
Akita	1075	906	3	**2**	2	**3**	3	1	1	1	1	1
Yamagata	1161	957	3	3	3	3	3	1	1	1	1	1
Fukushima	1990	1624	5	5	5	5	5	1	1	1	1	1
Ibaraki	2958	2417	7	7	7	7	7	2	2	2	2	2
Tochigi	2000	1638	5	5	5	5	5	1	1	1	1	1
Gumma	2001	1629	5	5	5	5	5	1	1	1	1	1
Saitama	7207	5908	15	**17**	17	**16**	16	3	4	4	4	4
Chiba	6214	5127	13	14	14	14	14	3	**4**	4	**3**	3
Tokyo	13196	11173	25	30	30	30	30	5	**8**	8	**6**	6
Kanagawa	9058	7461	18	21	21	21	21	4	**5**	5	**4**	4
Niigata	2362	1949	6	5	5	5	5	2	1	1	1	1
Toyama	1088	898	3	3	3	3	3	1	1	1	1	1
Ishikawa	1166	951	3	3	3	3	3	1	1	1	1	1
Fukui	803	652	2	2	2	2	2	1	**0**	0	**1**	1
Yamanashi	857	699	2	2	2	2	2	1	**0**	0	**1**	1
Nagano	2142	1747	5	5	5	5	5	2	1	1	1	1
Gifu	2071	1681	5	5	5	5	5	1	1	1	1	1
Shizuoka	3749	3066	8	9	9	9	9	2	2	2	2	2
Aichi	7416	5992	15	17	17	17	17	3	4	4	4	4
Mie	1847	1505	5	4	4	4	4	1	1	1	1	1
Shiga	1414	1130	4	3	3	3	3	1	1	1	1	1
Kyoto	2632	2173	6	6	6	6	6	2	1	1	1	1
Osaka	8861	7283	19	20	20	20	20	4	**5**	5	**4**	4
Hyogo	5582	4554	12	13	13	13	13	2	3	3	3	3
Nara	1396	1142	4	3	3	3	3	1	1	1	1	1
Wakayama	995	820	3	2	2	2	2	1	1	1	1	1
Tottori	585	478	2	1	1	1	1	1	**0**	0	**1**	1
Shimane	712	587	2	2	2	2	2	1	**0**	0	**1**	1
Okayama	1941	1582	5	4	4	4	4	1	1	1	1	1
Hiroshima	2855	2332	7	7	7	7	7	2	**2**	2	**1**	1
Yamaguchi	1442	1193	4	3	3	3	3	1	1	1	1	1
Tokushima	780	649	2	2	2	2	2	1	**0**	0	**1**	1
Kagawa	992	814	3	2	2	2	2	1	1	1	1	1
Ehime	1423	1171	4	3	3	3	3	1	1	1	1	1
Kochi	758	631	2	2	2	2	2	1	**0**	0	**1**	1
Fukuoka	5079	4140	11	12	12	12	12	2	**3**	3	**2**	2
Saga	847	679	2	2	2	2	2	1	**0**	0	**1**	1
Nagasaki	1417	1154	4	3	3	3	3	1	1	1	1	1
Kumamoto	1813	1474	5	4	4	4	4	1	1	1	1	1
Oita	1191	979	3	3	3	3	3	1	1	1	1	1
Miyazaki	1131	916	3	3	3	3	3	1	1	1	1	1
Kagoshima	1699	1382	5	4	4	4	4	1	1	1	1	1
Okinawa	1401	1068	4	3	3	3	3	1	1	1	1	1
total	127797	105011	295	295	295	295	295	73	73	73	73	73

In this Japanese case, the indexes α, β which are obtained by our criterion $IC(\alpha), IC(\beta)$ are equivalent to those corresponding to the famous Webster and Hill apportionment methods. Thus we could consider that our criterion as a unified method is appropriate among the famous apportionment methods.

6 Conclusion Based on the Rényi's entropy corresponding to the previous famous apportionment methods, we propose a criterion $IC(\alpha), IC(\beta)$ in order to select the best index α, β given an allocated seats, respectively, and an algorithm to select the totally best index. Under this criterion, we obtain appropriate seats in the single-seat constituencies of the Japanese Diet.

We might need more theoretical derivation with respect to our criterion because this does not exactly correspond to a usual information criterion.

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QUANTITATIVE INVESTIGATIONS FOR ODE MODEL DESCRIBING FISH SCHOOLING

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Abstract. This paper is devoted to investigating quantitatively the ODE model for fish schooling which was introduced in the paper [15]. First, we will study how each parameter in the model equations contributes to the geometrical structure of the school created by fish such as school diameter, connectedness, graph, etc. Second, we will concentrate on studying effects of the noise imposed to the model equations. In particular, it will be shown that, if the noise's magnitude is larger than a certain threshold, then fish can no longer form a school.

1 Introduction In the preceding paper [15], we have introduced an ordinary differential equation model:

$$(1.1)\qquad \begin{cases} dx_i(t) = & v_i dt + \sigma_i dw_i(t), \qquad i = 1, 2, \ldots, N, \\ dv_i(t) = & \Big[-\alpha \displaystyle\sum_{j=1,\, j\neq i}^{N} \left(\frac{r^p}{\|x_i - x_j\|^p} - \frac{r^q}{\|x_i - x_j\|^q} \right) (x_i - x_j) \\ & -\beta \displaystyle\sum_{j=1,\, j\neq i}^{N} \left(\frac{r^p}{\|x_i - x_j\|^p} + \frac{r^q}{\|x_i - x_j\|^q} \right) (v_i - v_j) \\ & +F_i(t, x_i, v_i) \Big] dt, \qquad i = 1, 2, \ldots, N, \end{cases}$$

for describing the process of schooling of N-fish system. Each fish is regarded as a moving particle in the Euclidean space $\mathbb{R}^d$, where $d = 2$ or 3. The unknown $x_i(t)$ is a stochastic process with values in $\mathbb{R}^d$ denoting a position of the i-th fish of system at time t; meanwhile, $v_i(t)$ is a stochastic process with values in $\mathbb{R}^d$ denoting a velocity of the i-th fish at time t. The fish are allowed to swim in the unbounded, continuous and homogeneous space $\mathbb{R}^d$.

The first equations of (1.1) are stochastic equations concerning x_i, where $\sigma_i dw_i$ denote noise resulting from the imperfectness of information-gathering and action of the i-th fish. In fact, $\{w_i(t), t \geqslant 0\}$ $(i = 1, 2, ..., N)$ are independent d-dimensional Brownian motions defined on a complete probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ satisfying the usual conditions. The second equations are deterministic equations on v_i, where $1 < p < q < \infty$ are fixed exponents, α, β are positive coefficients for interaction between fish and velocity matching, respectively, and $r > 0$ is a fixed distance. Since $1 < p < q < \infty$, if $\|x_i - x_j\| > r$ then the i-th fish moves toward the j-th; to the contrary, if $\|x_i - x_j\| < r$, then the i-th fish acts in order to avoid collision with the j-th fish. The number $r > 0$ therefore denotes a critical distance. Finally, the functions $F_i(t, x_i, v_i)$ denote external forces at time t which are given functions defined for (x_i, v_i) with values in $\mathbb{R}^d$. It is assumed that $F_i(t, x_i, v_i)$ $(i = 1, 2, \ldots, N)$ are locally Lipschitz continuous. In building up such a differential equation model we have referred to the fish's behavioral rules:

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1. The school has no leaders and each fish follows the same behavioral rules.
2. To decide where to move, each fish uses some form of weighted average of the position and orientation of its nearest neighbors.
3. There is a degree of uncertainty in the individual's behavior that reflects both the imperfect information-gathering ability of a fish and the imperfect execution of the fish's actions.

introduced by Camazine-Deneubourg-Franks-Sneyd-Theraulaz-Bonabeau [4, Chapter 11]. We have also referred to the idea due to Reynolds [14]. For the details, however, consult the paper [15].

The objective of the present paper is to investigate geometrical structures of the fish school when the fish move by obeying the kinematic equations (1.1) and create a swarm. For this purpose, we intend to introduce several quantitative notions: Distance to School Mates, Minimum Distance, Mean Distance to School Mates, Diameter of School, Variance of Velocity, and ε-Graph, to measure the geometrical structure of school. We in addition introduce a notion of ε-schooling where ε is fixed almost equally to r. We then perform many numerical computations to clarify effects of each parameter or exponent of the equations in determining geometry of structures of school. These will be presented in Section 2 with absence of noise. Next, in Section 3, we focus on studying effects of the noise which is an indispensable factor in the real world.

Empirical study on fish schooling has been done in [1, 3, 5, 8, 13]. As for the theoretical approach we want to quote [7, 10, 11, 16]. Vicsek et al. [16] introduced a simple difference model, assuming that each particle is driven with a constant absolute velocity and the average direction of motion of the particles in its neighborhood together with some random perturbation. Oboshi et al. [10] presented another difference model in which an individual selects one basic behavioral pattern from four based on the distance between it and its nearest neighbor. Olfati-Saber [11] and D´Orsogna et al. [7] constructed deterministic differential models using a generalized Morse and attractive/repulsive potential functions, respectively. We use the ODE model mentioned above. Such a model can describe the fish's behavior precisely. Moreover, an ODE model is tractable for making numerical simulations. In this paper, we will use the Euler scheme for stochastic differential equations which has been introduced by Kloeden and Platen [6].

2 Various Measures for Geometrical Structures In this section we want to introduce various measures to study the geometrical structures of school. Using these measures we will also clarify contributions of exponents and parameters included in (1.1) to the geometrical structure of school by examining many numerical examples.

For simplicity, we consider throughout this section the deterministic case, i.e., $\sigma_i = 0$ for all i. Therefore, $(x_i(t), v_i(t))$ denotes a trajectory of the i-th fish in the phase space $\mathbb{R}^d \times \mathbb{R}^d$.

2.1 Distance to School Mates For each fish i, put

$$\mathrm{DS}_i(t) = \min_{1 \leqslant j \leqslant N,\, j \neq i} \|x_j(t) - x_i(t)\|, \qquad 0 < t < \infty,\ i = 1, 2, \dots, N.$$

By definition, $\mathrm{DS}_i(t)$ denotes the distance between the i-th individual to its nearest mates at time t. We call $\mathrm{DS}_i(t)$ the distance of i to the school mates. It is observed that $\mathrm{DS}_i(t)$ depends on the position $x_i(t)$ considerably. If $x_i(t)$ is near the center of school, i.e., $\bar{x}(t) =$

$\frac{1}{N}\sum_{j=1}^{N} x_i(t)$, then $\mathrm{DS}_i(t)$ is much smaller than r; on the contrary, if $x_i(t)$ is in the periphery of school, then $\mathrm{DS}_i(t)$ can be almost equal to the maximum value r.

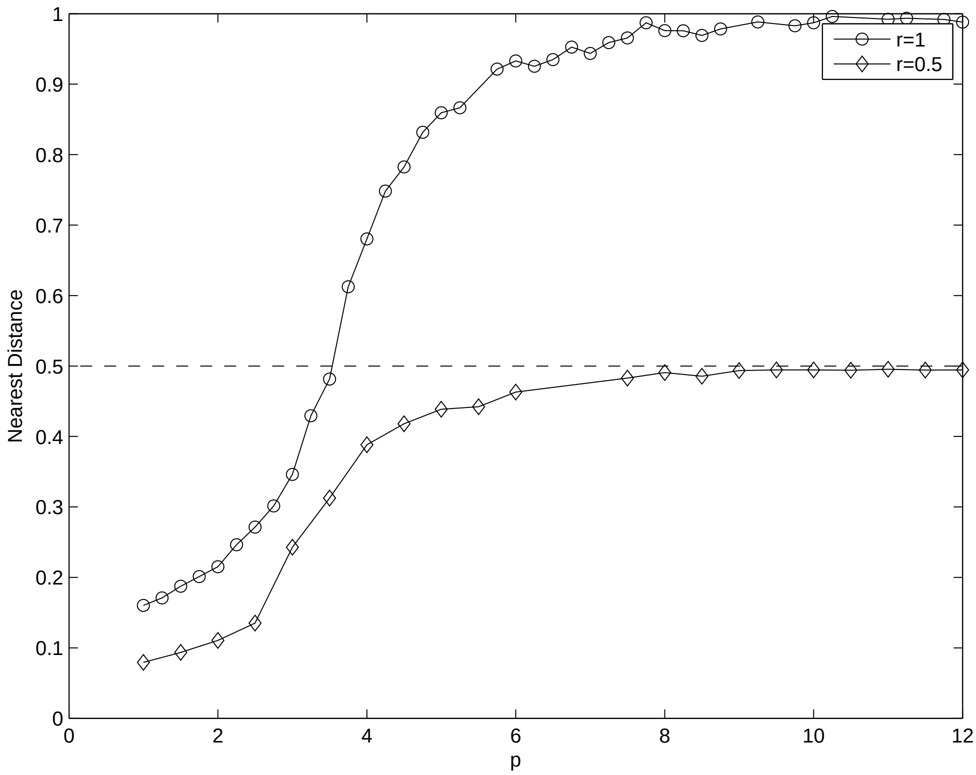

Figure 1: Dependence of MiDS on the exponent p

2.2 Minimum Distance We define

$$\mathrm{MiDS}(t) = \min_{1\leqslant i\leqslant N} \mathrm{DS}_i(t), \qquad 0 < t < \infty,$$

and call this value the minimum distance of school. This is the nearest distance between two fish in a group of N individuals at time t. Basically, $\mathrm{MiDS}(t)$ is dependent on r. But, it is seen that $\mathrm{MiDS}(t)$ depends on the exponents p and q, too. For example, we have

$$\lim_{p\to\infty} \mathrm{MiDS}(T) = r,$$

provided that T is a sufficiently large time. That is the nearest distance tends to the critical distance as power p tends to infinity for sufficiently large time T. By simulations, we shall find such a relationship between r and $\mathrm{MiDS}(T)$.

We consider a 100-fish system in the 2-dimensional space with $F_i = -5.0v_i$, which is often used to present the resistance against the moving particles. We fix two initial positions for two examples of 100-individual system (the initial positions $x_i(0)$, $1 \leqslant i \leqslant 100$, are randomly distributed in the square domain $[0,10]^2 \subset \mathbb{R}^2$) with all null initial velocities

$v_i = (0, 0)$, $(1 \leqslant i \leqslant 100)$. Taking the critical distance $r = 1$ for the first example and $r = 0.5$ for the second, we tune the exponent p from 1 to 12 and always keep the relation $q = p + 1$. Other parameters are chosen as follows: $\alpha = 1$, $\beta = 0.5$, step size $\delta = 0.001$. The result is got after 30.000 running steps, that is at time $T = 30$. Figure 1 illustrates dependence of MiDS(T) on the exponent p.

Remark 2.1. The model we consider contains many parameters, but we can find that the powers p and q are especially meaningful. p and q are concerned with a range of interactions among fish. As p and q increase, the range shortens and approaches sharply to the critical length r, namely, if $\|x_i - x_j\| > r$ the attraction between i and j is weak and if $\|x_i - x_j\| < r$ the repulsive is very strong. $\square$

In order to simplify our arguments, in what follows, we will always take q so that $q = p+1$. This assures the condition $q > p$ in modeling and the difference is similar to that of the Van der Waals and the Newton's law, where $p = 3$ and $q = 4$.

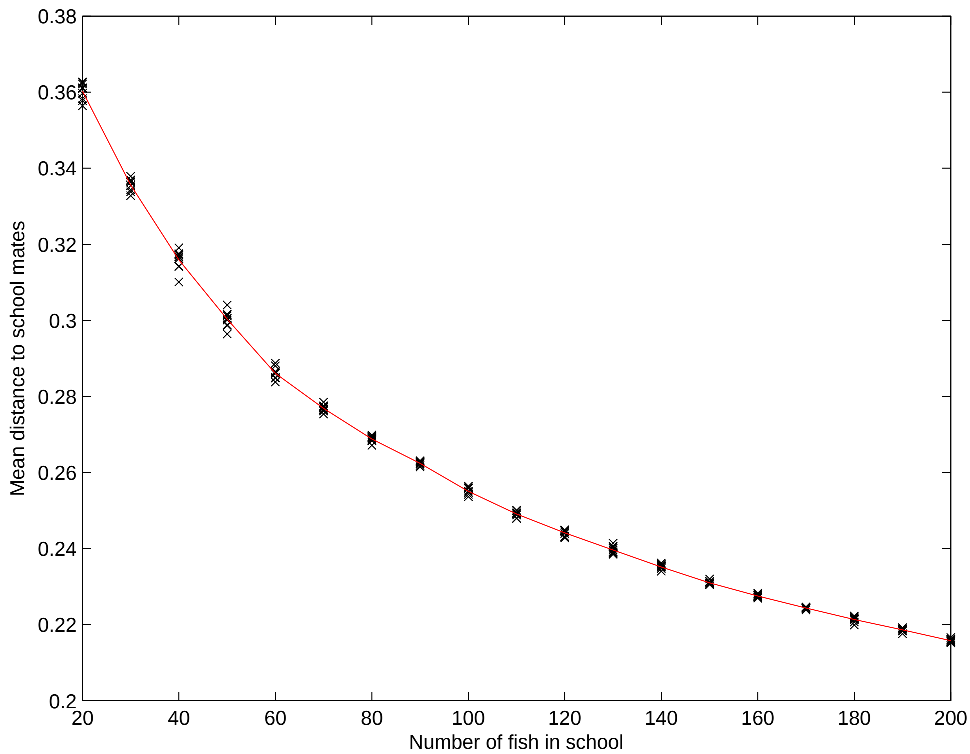

Figure 2: Dependence of MDS on the total number of fish

2.3 Mean Distance to School Mates We consider the mean of $\mathrm{DS}_i(t)$, i.e.,

$$\mathrm{MDS}(t) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{DS}_i(t), \qquad 0 < t < \infty.$$

This quantity is called the mean distance of school mates and is one of quantitative measures which are used to study the internal structure of the fish school.

It may be a very interesting question to know how MDS(t) depends on the total number of fish. In order to examine this, we consider an N-fish system in the 3-dimensional space with $F_i = -5.0v_i,\ 1 \leqslant i \leqslant N$. Let $\alpha = 5, \beta = 1, p = 3, q = 4$ and $r = 0.5$. We take various values N between 20 and 200. Initial positions $x_i(0)$, $1 \leqslant i \leqslant N$, are randomly distributed in the cubic domain $[0, 20]^3$ with all null initial velocities $v_i(0) = (0, 0, 0)$. The time T is fixed as $T = 120$ throughout the simulations. Figure 2 then shows dependence of MDS(T) on the total number N. In order to reduce the effect of the random initial positions to the result, for each value of N, we run 10 simulations each with different random initial positions in $[0, 20]^3 \subset \mathbb{R}^3$. The mean distance for each N is drawn by a cross $\times$. After that we take the mean value of these and then interpolate these values by a smooth curve.

As seen, MDS(T) decreases monotonically as N increases. This means that the school becomes "more condensed" as N is larger. This agrees with the results stated in a number of works, such as [2, 8, 9, 12] in which the authors show that the mean distance to school mates decreases as a function of the number of fish. From Figure 2, we also see that the range of the simulation results for MDS decreases as N increases.

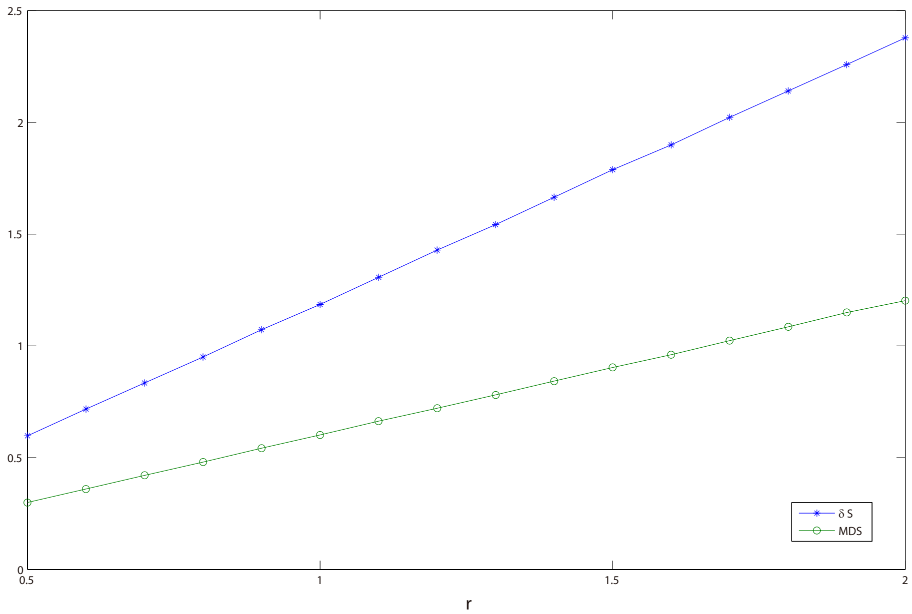

Figure 3: Dependence of MDS and δS on the critical distance r

2.4 Diameter of School The diameter of school is defined by

$$\delta S(t) = \sup_{1 \leqslant i \leqslant N} \|x_i(t) - \bar{x}(t)\|, \qquad 0 < t < \infty,$$

where $\bar{x}(t) = \frac{1}{N} \sum\limits_{i=1}^{N} x_i(t)$ is the center of the group at time t.

The diameter of school is, by definition, the radius of the minimal ball centered at $\bar{x}(t)$ and containing all the individuals at time t.

The following numerical example shows that MDS(T) and $\delta S(T)$ are linearly dependent on r for sufficiently large time T. We consider a 50-fish system in the 3-dimensional space with $F_i = -5v_i$. Let $\alpha = 5$, $\beta = 1$, $p = 3$ with $q = p + 1$. Now, r is a tuning parameter which varies from 0.5 to 2. Initial positions $x_i(0)$, $1 \leqslant i \leqslant 50$, are randomly distributed in the cubic domain $[0, 20]^3$ with null initial velocities $v_i(0) = (0, 0, 0)$. The time T is fixed as $T = 150$. Figure 3 then illustrated the dependence of MDS(T) and δS(T) on the critical distance r. The plots of these values are approximately on linear lines $\delta S(T) = ar$ and MDS(T) $= br$, respectively. In this parameter setting we observe that $a = 1.18984$ and $b = 0.60158$.

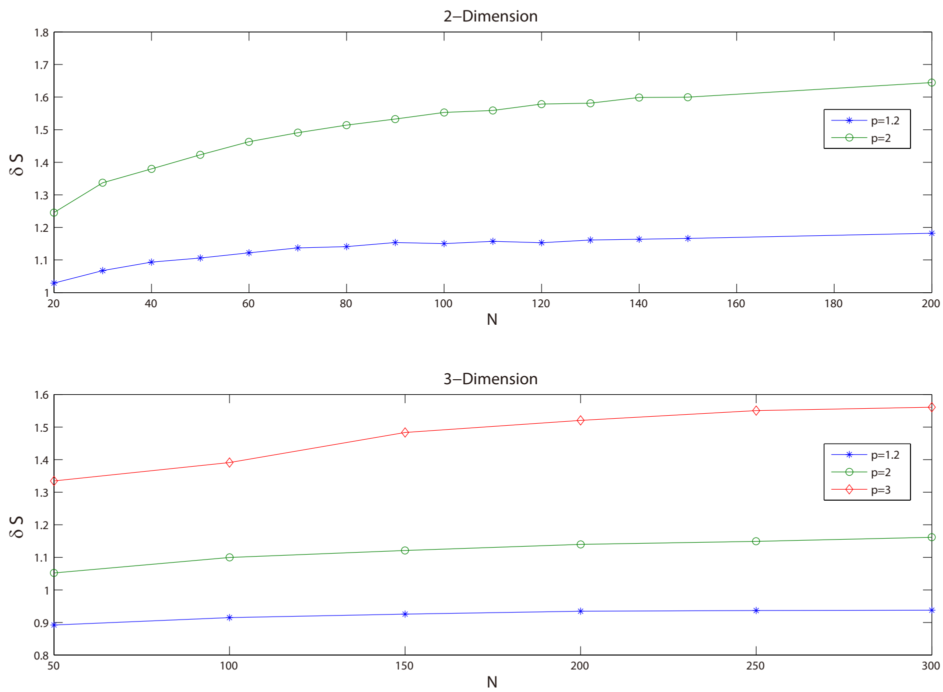

Figure 4: Dependence of δS on the total number N

How does $\delta S(T)$ respond when the total number N increases? To examine this question, we consider an N-fish system in the 2 or 3-dimensional space with $F_i = -5.0v_i$, and set $\alpha = 1$, $\beta = 0.5$, $p = 3$, $q = p + 1$, $r = 1$ and $T = 20$. As stated before, in order to simplify the arguments, each value shown in the figure is calculated by taking the mean value of the corresponding values for 10 simulations with different initial positions. Figure 4 shows that the diameter of school typically increases with the fish number. This is generally true in animal flocks, cf. also [7].

By observing the figure we find that the slope of the school radius as function of N is larger when p becomes larger.

2.5 Variance of Velocity In order to measure matching of velocity each other, we will use the ordinary variance

$$\sigma\mathrm{VS}(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\|v_i(t) - \bar{v}(t)\|^2}, \qquad 0 < t < \infty,$$

where $\bar{v}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} v_i(t)$ is the average of all velocities of fish at time t.

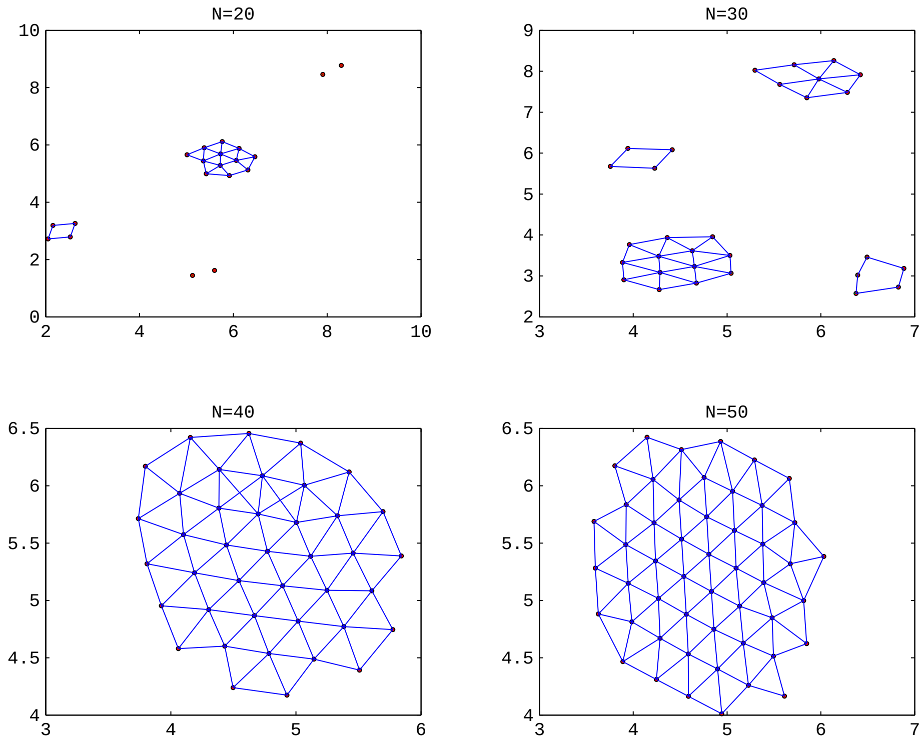

Figure 5: Effect of the total number N for N_ε

2.6 ε-Graph of School We finally introduce the ε-graph. Let $\varepsilon > 0$ be a fixed length. The vertices of graph at time t are all the positions of particles, $x_i(t)$, $1 \leqslant i \leqslant N$. Two vertices $x_i(t)$ and $x_j(t)$ are connected by the edge of graph if and only if $\|x_i(t) - x_j(t)\| \leqslant \varepsilon$. This graph is called the ε-graph of school at time t and is denoted by $\mathrm{GS}_\varepsilon(t)$. We also denote by $N_\varepsilon(t)$ the number of connected components of $\mathrm{GS}_\varepsilon(t)$. When $N_\varepsilon(t) = 1$, we consider that the fish have created a school with $\max\limits_{1\leqslant i\leqslant N} \mathrm{DS}_i(t) \leqslant \varepsilon$. If $N_\varepsilon(t) \geqslant 2$, $N_\varepsilon(t)$ denotes the number of sub-schools.

Let us now examine effects of the total population N on $N_\varepsilon(t)$ for sufficiently large time t. To create a single school, N must be sufficiently large. To see this fact, consider an

N-fish system in the 2-dimensional space with $F_i = -5.0v_i$. Let $\alpha = 1$, $\beta = 0.5$, $p = 4$, $q = p+1$, $r = \varepsilon = 0.5$. Initial positions $x_i(0)$, $1 \leqslant i \leqslant N$, are randomly distributed in $[0, 10]^2$ with null initial velocities $v_i(0) = (0, 0)$. The population number N changes from 20 to 50. Figure 5 illustrates the graph $\mathrm{GS}_{0.5}(400)$ for each N. Up to $N = 39$, $N_{0.5}(400) \geqslant 2$ and so the fish are divided into a few sub-schools. But after a threshold number $N = 40$, they can create a single school.

3 Robustness of ε, θ-Schooling against Noise In this section, we consider the stochastic model (1.1). Under $\sigma_i > 0$, we want to study how the terms $\sigma_i dw_i(t)$ affect the geometrical structure of school. Can the fish system still create a school?

Let us here give a mathematical definition of school.

Definition 3.1 (ε, θ-Schooling)**.** For a given length $\varepsilon > 0$ and a tolerance $\theta > 0$, we say that the fish system is in ε, θ-*schooling* if there exists a time $T > 0$ such that $N_\varepsilon(t) = 1$ and $\sigma VS(t) < \theta$ for every $t \geqslant T$.

According to the above definition, a system forms a school only if velocities of all the fish tend to their average with the error less than tolerance θ. Therefore, the distance $\|x_i(t) - x_j(t)\|$ between any pair (i, j) will almost remain unchanged for $t \geqslant T$. So, the school structure remains unchanged, too. The second condition ensures that all the fish keep the relation $\mathrm{DS}_i(t) \leqslant \varepsilon$ for $t \geqslant T$. As a consequence, $N_\varepsilon(t) = 1$ remains to hold for $t \geqslant T$.

Assume that a system is in ε, θ-schooling for $t \geqslant T$. According to Remark 2.1 (cf. also Figure 1), if $\|x_i(t) - x_j(t)\| > \varepsilon$, then i and j keep their distance far away and consquently

$$\left(\frac{r^p}{\|x_i(t) - x_j(t)\|^p} - \frac{r^q}{\|x_i(t) - x_j(t)\|^q} \right) (x_i(t) - x_j(t)) \tag{3.1}$$

is sufficiently small. In the meantime, if $\|x_i(t) - x_j(t)\| \approx \varepsilon$, then their distance is $\|x_i(t) - x_j(t)\| \approx r$ and consequently (3.1) is again sufficiently small. In addition, it is clear that

$$\left(\frac{r^p}{\|x_i(t) - x_j(t)\|^p} + \frac{r^q}{\|x_i(t) - x_j(t)\|^q} \right) (v_i(t) - v_j(t))$$

is sufficiently small because of $\|v_i(t) - v_j(t)\| \approx 0$. We thus verify that

$$\sum_{i=1}^{n} dv_i \approx \sum_{i=1}^{N} F_i(t, x_i, v_i) dt.$$

In particular, if we take $F_i(t, x_i, v_i) = -cv_i$ $(1 \leqslant i \leqslant N)$, then

$$\sum_{i=1}^{N} dv_i \approx -c \left(\sum_{i=1}^{N} v_i \right) dt.$$

Consequently, $\sum\limits_{i=1}^{N} v_i(t)$ decays exponentially as $t \to \infty$ and the system converges to a steady state.

Figure 6 shows an example of ε, θ-schooling generated by (1.1). 100 fish are situated at random positions in $[0, 10]^2 \subset \mathbb{R}^2$ with null velocities at time $t = 0$. Then they interact

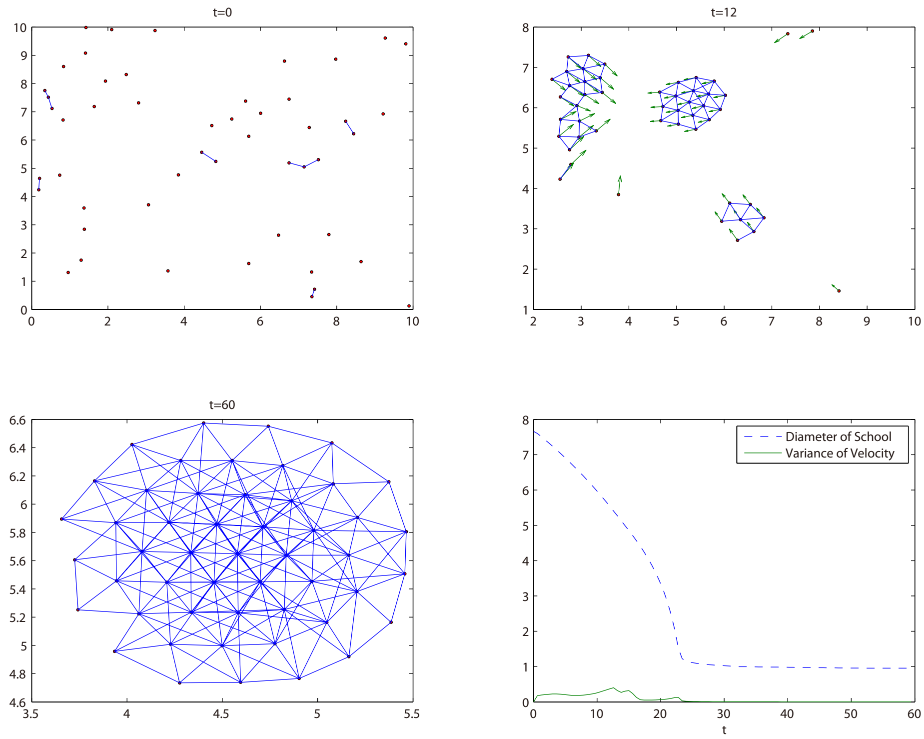

Figure 6: Example of ε, θ-schooling

with each other with $\alpha = 5$, $\beta = 1$, $p = 3$, $q = 4$, $r = 0.5$, $\sigma = 0$, $F_i = -5v_i$, $(1 \leqslant i \leqslant 100)$, we set $\varepsilon = 0.5 = r$ and $\theta = 10^{-6}$.

In the first three subfigures, we show ε-graphs of the system at different instants t. Each of these figure shows the positions of fish by points, their velocities by vectors and ε-graph edges by lines. The last subfigure draws the variance of velocity and the radius of school as functions of t.

Of course whether a system creates a school or not depends strongly on initial positions. It is also observed that 3-dimensional systems can create schools much easier than 2-dimensional ones.

Let us next study effects of the noise. We set $\sigma_i(t) = \sigma$, for $i = 1, 2, \ldots, N$. Simulations are implemented in the 3-dimensional space. We fix initial positions taking randomly in $[0, 5]^3 \subset \mathbb{R}^3$ with 50 fish, run 10 simulations with different realizations of the Wiener process for each value of σ. We observe the end point of each trajectories of $\sigma\mathrm{VS}(T)$ and $\delta\mathrm{S}(T)$ at $T = 50$. Other parameters are set as $p = 3$, $q = p + 1$, $\alpha = 5$, $\beta = 1$, $r = 0.5$, $F_i = -5.0v_i$, step size $\delta = 0.001$. Figure 7 shows that the fish can keep schooling against the noises when their magnitude σ is small enough. To the contrary, when it is large, the noises prevent the fish from creating a single school. It might be allowed, however, to insist that the swarming behavior described by our model (1.1) possesses the robustness for schooling. Figure 8 shows the expectation of school diameter as a function of σ. From this figure, too, we can find a similar tendency.

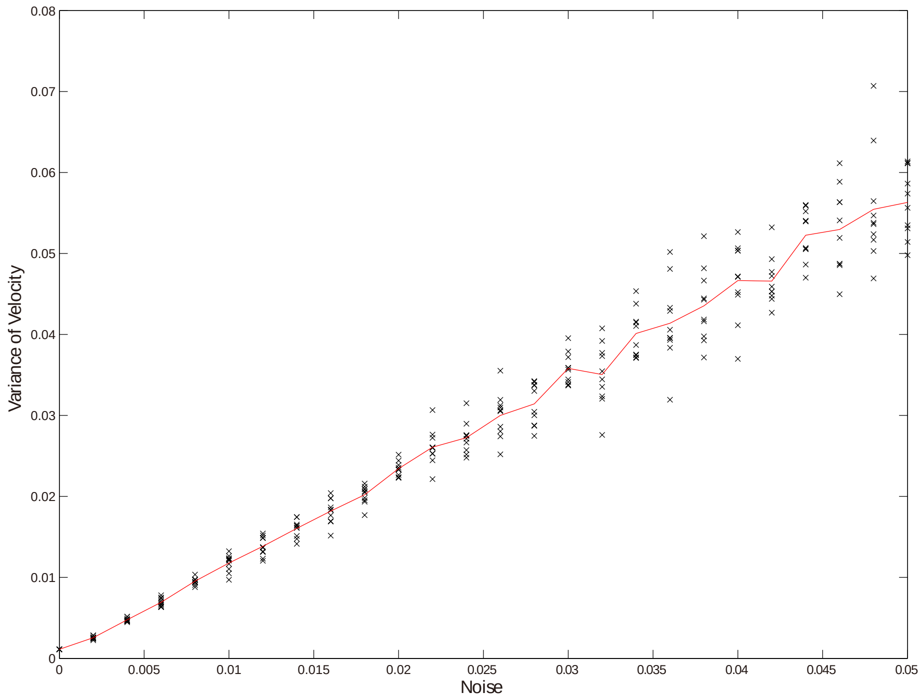

Figure 7: Influence of the noise magnitude σ for schooling

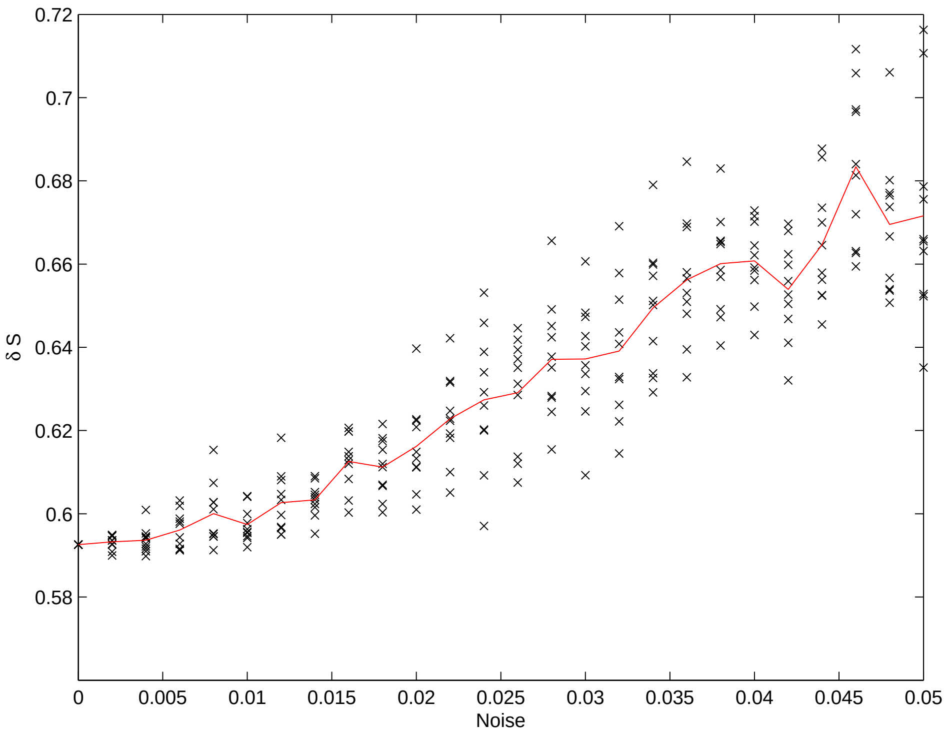

Figure 8: Influence of the noise magnitude σ to δS

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LOWER DECAY ESTIMATES FOR NON-DEGENERATE DISSIPATIVE WAVE EQUATIONS OF KIRCHHOFF TYPE

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Abstract. Consider the initial-boundary value problem for non-degenerate dissipative wave equations of Kirchhoff type. Using the energy method, we see that the energies have exponential decay rates. Also, we show that the decay rates from below of the solutions are exponentially.

1 Introduction In this paper, we study on the asymptotic behavior of solutions to the initial boundary value problem for the following non-degenerate dissipative wave equations of Kirchhoff type :

$$\tag{1.1} \begin{cases} \rho u'' + \left(1 + \|A^{1/2}u(t)\|^{2\gamma}\right) Au + u' = 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty)\,, \end{cases}$$

where $u = u(x,t)$ is an unknown real value function, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the usual norm of $L^2 = L^2(\Omega)$, and $0 < \rho \le 1$ and $\gamma > 0$ are constants.

In the case of $N = 1$, Equation (1.1) describes a small amplitude vibration of an elastic string (see Kichhoff [7] for the original equation ; also see [4], [5], [10]).

Many authors have shown the local in time solvability for initial data in suitable Sobolev spaces (see [1], [2], [6], [18], [19]).

By help of dissipation we can show the global in time solvability for initial data in certain Sobolev spaces (see [3], [17] for small data and $\gamma \ge 1$), and we can derive some exponential decay estimates for energies.

In previous paper [13], when $\gamma \ge 1$, we have derive some exponential decay estimates, that is,

$$\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le Ce^{-\theta t}$$

with some constant $\theta > 0$ under the small data condition (see Theorem 5.1).

Ghisi and Gobbino [9] have given some decay estimates of the solutions of (1.1) :

$$\begin{aligned} &C'e^{-\theta_2 t} \le \|A^{1/2}u(t)\|^2 \le Ce^{-\theta_1 t}\,, \\ &C'e^{-\theta_2 t} \le \|Au(t)\|^2 \le Ce^{-\theta_1 t}\,, \\ &\|u'(t)\|^2 \le Ce^{-\theta t} \quad \text{for} \quad t \ge 0\,. \end{aligned}$$

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under the smallness condition for the coefficient $\rho > 0$. However, from their results we can not know the lower decay estimate of the norm $\|u(t)\|^2$ (cf. [8], [9], [11], [14] and the references cited therein for mildly degenerate cases).

The purpose of this paper is to give the condition for the global solvability of (1.1) for any $\gamma > 0$ (see Theorem 3.1), and to derive a lower decay estimate of the L^2 norm of the solution $u(t)$ (see Theorem 4.6).

The notations we use in this paper are standard. The symbol $(\cdot, \cdot)$ means the inner product in $L^2 = L^2(\Omega)$ or sometimes duality between the space X and its dual X'. Positive constants will be denoted by C and will change from line to line.

2 A-priori Estimate

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem. The proof is standard and we omit it here (see [1], [2], [15], [16]).

Proposition 2.1 *If the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, *then the problem* (1.1) *admits a unique local solution* $u(t)$ *in the lass* $C^0([0,T);\mathcal{D}(A)) \cap C^1([0,T);\mathcal{D}(A^{1/2})) \cap C^0([0,T);L^2(\Omega))$ *for some* $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$. *Moreover,* $\|Au(t)\| + \|A^{1/2}u(t)\| < \infty$ *for* $t \geq 0$, *then we can take* $T = \infty$.

In what follows in this section, let $u(t)$ be a solution of (1.1) and we assume that

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{\gamma + 1} . \tag{2.1}$$

By fundamental calculation, we have the energy identity

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \quad \text{or} \quad E(t) + 2\int_0^t \|u'(s)\|^2 \, ds = E(0) , \tag{2.2}$$

where $E(t)$ is defined by

$$E(t) \equiv \rho\|u'(t)\|^2 + \left(1 + \frac{1}{\gamma+1}M(t)^\gamma\right)M(t) \quad \text{with} \quad M(t) \equiv \|A^{1/2}u(t)\|^2 . \tag{2.3}$$

Proposition 2.2 *Under the assumption* (2.1), *it holds that*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0) \tag{2.4}$$

where

$$G(t) \equiv \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t) , \tag{2.5}$$

$$Q(t) \equiv \frac{1}{(1 + M(t)^\gamma)M(t)^2} \left(\|A^{1/2}u'(t)\|^2 M(t) - \frac{1}{4}|M'(t)|^2\right) . \tag{2.6}$$

Proof. From Equation (1.1) we observe

$$
\begin{aligned}
&\frac{d}{dt}\frac{\|Au(t)\|^2}{M(t)}\\
&= \frac{1}{(1+M(t)^\gamma)M(t)^2}\left(2\left((1+M(t)^\gamma)Au, Au'\right)M(t) - \left((1+M(t)^\gamma)Au, Au\right)M'(t)\right)\\
&= \frac{-1}{(1+M(t)^\gamma)M(t)^2}\Bigg(2\left(\|A^{1/2}u'(t)\|^2 + \rho(A^{1/2}u'', A^{1/2}u')\right)M(t)\\
&\quad - \left(\frac{1}{2}|M'(t)|^2 + \rho\left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)\right)\Bigg)\\
&= -2Q(t) + \rho R(t) \qquad (2.7)
\end{aligned}
$$

where

$$
R(t) \equiv \frac{1}{(1+M(t)^\gamma)M(t)^2}\left(2(A^{1/2}u'', A^{1/2}u')M(t) + \left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t)\right)M'(t)\right).
$$

On the other hand, by simple calculation we have

$$
\frac{d}{dt}Q(t) = -\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma}Q(t) - R(t)\,. \tag{2.8}
$$

Thus, from (2.7) and (2.8) we obtain

$$
\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)} + \rho Q(t)\right) + 2\left(1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma}\right)Q(t) = 0\,.
$$

Since it follows from (2.1) and (2.5) that

$$
1 + \frac{\rho}{2}\frac{M'(t)}{M(t)}\frac{2+(\gamma+2)M(t)^\gamma}{1+M(t)^\gamma} \geq 0 \quad \text{and} \quad Q(t) \geq 0\,,
$$

we conclude the desired estimate (2.5). □

Proposition 2.3 *Under the assumption* (2.1), *it holds that*

$$
\frac{\|u'(t)\|^2}{M(t)} \leq B(0) \tag{2.9}
$$

where

$$
B(0) = \max\left\{\frac{\|u_1\|^2}{M(0)}\,,\ \frac{\gamma+1}{\gamma}G(0)(1+E(0)^\gamma)^2\right\}. \tag{2.10}
$$

Proof. Multiplying (1.1) by $2M(t)^{-1}u'$ and integrating it over Ω, we have

$$
\begin{aligned}
\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)} + 2\left(1+\frac{\rho}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)} &= -\frac{1+M(t)^\gamma}{M(t)}M'(t)\\
&\leq 2\frac{\|u'(t)\|}{M(t)^{\frac{1}{2}}}\frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}}(1+M(t)^\gamma)\,.
\end{aligned}
$$

Since it follows from (2.1) that

$$1+\frac{\rho}{2}\frac{M'(t)}{M(t)}\geq\frac{2\gamma+1}{2(\gamma+1)}\,, \tag{2.11}$$

the Young inequality yields

$$\begin{aligned}\rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)}+\frac{\gamma}{\gamma+1}\frac{\|u'(t)\|^2}{M(t)} &\leq \frac{\|Au(t)\|^2}{M(t)}(1+M(t)^\gamma)^2\\ &\leq G(0)(1+E(0)^\gamma)^2\end{aligned}$$

where we used the estimates (2.2) and (2.4) at the last inequality. Thus, by standard calculation for ODE, we obtain the desired estimate (2.9). □

3 Global Solvability for $\gamma > 0$

Theorem 3.1 *Let the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$. *Suppose that the coefficient* $\rho > 0$ *and the initial data* $[u_0, u_1]$ *satisfy*

$$2\rho B(0)^{\frac{1}{2}}G(0)^{\frac{1}{2}}<\frac{1}{\gamma+1} \tag{3.1}$$

where $G(0)$ *and* $B(0)$ *are given by* (2.5) *and* (2.10), *respectively. Then, the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty);\mathcal{D}(A))\cap C^1([0,\infty);\mathcal{D}(A^{1/2}))\cap C^0([0,\infty);L^2(\Omega))$, *and moreover, the solution* $u(t)$ *satisfies*

$$\rho\frac{|M'(t)|}{M(t)}<\frac{1}{\gamma+1}\quad\text{and}\quad M(t)\leq E(t)\leq E(0)\,, \tag{3.2}$$

$$\frac{\|Au(t)\|^2}{M(t)}\leq G(0)\quad\text{and}\quad\frac{\|u'(t)\|^2}{M(t)}\leq B(0) \tag{3.3}$$

for $t\geq 0$.

Proof. Let $u(t)$ be a solution of (1.1) on $[0, T]$. Since it follows from (2.5), (2.10), and (3.1) that

$$\rho\frac{|M'(0)|}{M(0)}\leq 2\rho\frac{\|u_1\|}{M(0)^{\frac{1}{2}}}\frac{\|Au_0\|}{M(0)^{\frac{1}{2}}}\leq 2\rho B(0)^{\frac{1}{2}}G(0)^{\frac{1}{2}}<\frac{1}{\gamma+1}\,,$$

putting

$$T_1\equiv\sup\left\{\,t\in[0,\infty)\;\Big|\;\rho\frac{|M'(s)|}{M(s)}<\frac{1}{\gamma+1}\text{ for }0\leq s<t\,\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$\rho\frac{|M'(t)|}{M(t)}<\frac{1}{\gamma+1}\quad\text{for}\quad 0\leq t<T_1\quad\text{and}\quad\rho\frac{|M'(T_1)|}{M(T_1)}=\frac{1}{\gamma+1}\,. \tag{3.4}$$

On the other hand, from Proposition 2.2 and Proposition 2.3, we observe

$$\rho\frac{|M'(t)|}{M(t)}\leq 2\rho\frac{\|u'(t)\|}{M(t)^{\frac{1}{2}}}\frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}}\leq 2\rho B(0)^{\frac{1}{2}}G(0)^{\frac{1}{2}}<\frac{1}{\gamma+1}\quad\text{for}\quad 0\leq t\leq T_1 \tag{3.5}$$

which is a contradiction to (3.4), and hence, we have that $T_1 \geq T$.

Moreover, for $0 \leq t \leq T$, multiplying (1.1) by $2(1+M(t)^\gamma)^{-1}Au'$ and integrating it over Ω, we have

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}+\|Au(t)\|^2\right)+2\left(1+\frac{\gamma}{2}\rho\frac{M(t)^\gamma}{1+M(t)^\gamma}\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}=0\,.$$

Since it follows from (3.5) that

$$1+\frac{\gamma}{2}\rho\frac{M(t)^\gamma}{1+M(t)^\gamma}\frac{M'(t)}{M(t)}\geq 1-\frac{\gamma}{2}\rho\frac{|M'(t)|}{M(t)}\geq\frac{\gamma+2}{2(\gamma+1)}\,,$$

we observe

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{1+M(t)^\gamma}+\|Au(t)\|^2\right)\leq 0\,,$$

and hence, we see that $\|Au(t)\|+\|A^{1/2}u'(t)\|\leq C$ for $0\leq t\leq T$. Therefore, by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from Proposition 2.2 and Proposition 2.3, we obtain the desired estimate (3.3). □

4 Decay

Proposition 4.1 *Under the assumption of Theorem 3.1, it holds that,*

$$M(t)\leq E(t)\leq\frac{2\alpha}{\rho}E(0)e^{-k_1 t} \tag{4.1}$$

with

$$\alpha=\max\left\{\frac{3}{2}\rho\,,\,\rho+c_*^2\right\}\quad\text{and}\quad k_1=\alpha^{-1}=\min\left\{\frac{2}{3\rho}\,,\,\frac{1}{\rho+c_*^2}\right\}\,, \tag{4.2}$$

where c_* is the Sobolev-Poincaré constant such that $\|\phi\|\leq c_*\|A^{1/2}\phi\|$.

Proof. We define $E_1(t)$ by

$$E_1(t)\equiv E(t)+\frac{1}{2\rho}\|u(t)\|^2+(u'(t),u(t))$$

with $E(t)$ given by (2.3). Since $|(u',u)|\leq(\rho/2)\|u'\|^2+(1/2\rho)\|u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2}E(t)\leq E_1(t)\leq\frac{\alpha}{\rho}E(t)\quad\text{with}\quad\alpha=\max\left\{\frac{3}{2}\rho\,,\,\rho+c_*^2\right\}\,. \tag{4.3}$$

Multiplying (1.1) by $2u'+\rho^{-1}u$ and integrating it over Ω, we have

$$\frac{d}{dt}E_1(t)+\|u'(t)\|^2+\frac{1}{\rho}(1+M(t)^\gamma)M(t)=0\,,$$

and moreover, it follows from (4.3) that

$$\frac{d}{dt}E_1(t)+k_1E_1(t)\leq 0\quad\text{with}\quad k_1=\alpha^{-1}\,.$$

Thus, we obtain that $E_1(t)\leq E_1(0)e^{-k_1 t}$, and hence, from (4.3) we arrive at the desired estimate. □

Proposition 4.2 *Under the assumption of Theorem 3.1, it holds that*

$$H(t) \equiv \rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \le \frac{m_1}{\rho^2} \tag{4.4}$$

with $m_1 = 2\alpha \max\{\rho H(0)\,,\, \gamma^{-1}(\rho(\gamma+1)E(0)^\gamma G(0)+1)\}$.

Proof. We define $H_1(t)$ by

$$H_1(t) \equiv H(t) + \frac{1}{2\rho} + \frac{(A^{1/2}u'(t), A^{1/2}u(t))}{M(t)}\,.$$

Since $|(A^{1/2}u', A^{1/2}u)| \le (\rho/2)\|A^{1/2}u'\|^2 + (1/2\rho)\|A^{1/2}u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2}H(t) \le H_1(t) \le \frac{\alpha}{\rho}H(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}. \tag{4.5}$$

Multiplying (1.1) by $M(t)^{-1}(2Au' + \rho^{-1}Au)$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \left(1 + \rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \\
&\quad = -\left(1 - (\gamma-1)M(t)^\gamma\right)\frac{M'(t)}{M(t)}\frac{\|Au(t)\|^2}{M(t)} - \frac{1}{2\rho}\frac{M'(t)}{M(t)} - \frac{1}{2}\frac{|M'(t)|^2}{M(t)^2}\,.
\end{aligned}$$

Since it follows from (3.2) that

$$1 + \rho\frac{M'(t)}{M(t)} \ge \frac{\gamma}{\gamma+1}\,, \tag{4.6}$$

we have from (3.2) and (3.3) that

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + \frac{\gamma}{\gamma+1}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|Au(t)\|^2 \\
&\quad \le \frac{|M'(t)|}{M(t)}\left((\gamma+1)M(t)^\gamma\frac{\|Au(t)\|^2}{M(t)} + \frac{1}{2\rho} + \frac{1}{2}\frac{|M'(t)|}{M(t)}\right) \\
&\quad \le \frac{1}{\rho(\gamma+1)}\left((\gamma+1)E(0)^\gamma G(0) + \frac{1}{\rho}\right),
\end{aligned}$$

and moreover, we observe from (4.5) that

$$\frac{d}{dt}H_1(t) + \frac{\gamma}{(\gamma+1)\alpha}H_1(t) \le \frac{\gamma}{\rho^2(\gamma+1)}I(0)$$

with $I(0) \equiv \gamma^{-1}\left(\rho(\gamma+1)E(0)^\gamma G(0)+1\right)$. Thus, we obtain

$$H_1(t) \le \max\left\{H_1(0)\,,\, \frac{\alpha}{\rho^2}I(0)\right\}$$

and from (4.5) we conclude the desired estimate (4.4). □

Proposition 4.3 *Under the assumption of Theorem 3.1, it holds that*

$$P(t) \equiv \rho \frac{\|u''(t)\|^2}{M(t)} + \frac{1+M(t)^\gamma}{M(t)}\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2}M(t)^\gamma \frac{|M'(t)|^2}{M(t)^2} \le \frac{m_2}{\rho^3} \tag{4.7}$$

with $m_2 = 2\alpha \max\{\rho^2 P(0)\,,\, \gamma^{-1}(6(\gamma+1)^2 E(0)^\alpha m_1 + \rho(\gamma+1)\gamma^{-1}B(0))\}$.

Proof. We define $P_1(t)$ by

$$P_1(t) \equiv P(t) + \frac{1}{2\rho}\frac{\|u'(t)\|^2}{M(t)} + \frac{(u''(t), u'(t))}{M(t)}\,.$$

Since $|(u'', u')| \le (\rho/2)\|u''\|^2 + (1/2\rho)\|u'\|^2$, we observe from the Sobolev-Poincaré inequality

$$\frac{1}{2}P(t) \le P_1(t) \le \frac{\alpha}{\rho}P(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}\,. \tag{4.8}$$

Multiplying (1.1) differentiated with respect to t by $M(t)^{-1}(2u''+\rho^{-1}u')$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}P_1(t) + \left(1 + \rho\frac{M'(t)}{M(t)}\right)\frac{\|u''(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2\rho}M(t)^\gamma\frac{|M'(t)|^2}{M(t)^2} \\
&= -(1-(3\gamma-1)M(t)^\gamma)\frac{M'(t)}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \frac{\gamma(\gamma-2)}{2}M(t)^\gamma\frac{(M'(t))^3}{M(t)^3} \\
&\quad - \frac{1}{2\rho}\frac{M'(t)}{M(t)}\frac{\|u'(t)\|^2}{M(t)} - \frac{M'(t)}{M(t)}\frac{(u''(t), u'(t))}{M(t)}\,.
\end{aligned}$$

From the Young inequality and (4.6) (or (3.2)) we observe

$$\begin{aligned}
&\frac{d}{dt}P_1(t) + \frac{\gamma}{2(\gamma+1)}\frac{\|u''(t)\|^2}{M(t)} + \frac{1}{\rho}\frac{1+M(t)^\gamma}{M(t)}\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2\rho}M(t)^\gamma\frac{|M'(t)|^2}{M(t)^2} \\
&\le 3(\gamma+1)^2 M(t)^\gamma \frac{|M'(t)|}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \\
&\quad + \frac{1}{2\rho}\frac{|M'(t)|}{M(t)}\frac{\|u'(t)\|^2}{M(t)} + \frac{\gamma+1}{2\gamma}\frac{|M'(t)|^2}{M(t)^2}\frac{\|u'(t)\|^2}{M(t)} \\
&\le \frac{1}{\rho(\gamma+1)}\left(3(\gamma+1)^2E(0)^\gamma\frac{m_1}{\rho^2} + \frac{\gamma+1}{2\rho\gamma}B(0)\right)
\end{aligned}$$

where we used the estimates (3.2) and (3.3), and moreover, we have from (4.8) that

$$\frac{d}{dt}P_1(t) + \frac{\gamma}{2(\gamma+1)\alpha}P_1(t) \le \frac{\gamma}{2\rho^3(\gamma+1)}J(0)$$

with $J(0) \equiv \gamma^{-1}(6(\gamma+1)^2E(0)^\gamma m_1 + \rho(\gamma+1)\gamma^{-1}B(0))$. Thus, we obtain

$$P_1(t) \le \max\left\{P_1(0)\,,\, \frac{\alpha}{\rho^3}J(0)\right\}$$

and from (4.8) we conclude the desired estimate (4.7). □

Proposition 4.4 *Under the assumption of Theorem 3.1, it holds that if* $u_0 \ne 0$,

$$M(t) \ge C'e^{-k_2 t} \quad \textit{with} \quad k_2 = \rho^{-1}\max\{2\,,\, \gamma-2\}(1+E(0)^\gamma)^{\frac{1}{2}}G(0)^{\frac{1}{2}}\,, \tag{4.9}$$

where C' *is some positive constant.*

Proof. Multiplying by $2M(t)^{-2}u'$ and integrating it over Ω, we have

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\right)+2\left(1+\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^2}=-\frac{2-(\gamma-2)M(t)^\gamma}{M(t)^2}M'(t)\,,$$

and from (3.2), (3.3), and the Young inequality we observe

$$\begin{aligned}&\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\right)\\&\le 2\max\{2,\gamma-2\}(1+M(t)^\gamma)^{\frac12}\frac{\|Au(t)\|}{M(t)^{\frac12}}\left(\frac{1+M(t)^\gamma}{M(t)}\right)^{\frac12}\frac{\|u'(t)\|}{M(t)}\\&\le \rho^{-1}\max\{2,\gamma-2\}(1+E(0)^\gamma)^{\frac12}G(0)^{\frac12}\left(\frac{1+M(t)^\gamma}{M(t)}+\rho\frac{\|u'(t)\|^2}{M(t)^2}\right).\end{aligned}$$

Thus, we obtain

$$\rho\frac{\|u'(t)\|^2}{M(t)^2}+\frac{1+M(t)^\gamma}{M(t)}\le Ce^{k_2t}\quad\text{with}\quad k_2=\rho^{-1}\max\{2\,,\,\gamma-2\}(1+E(0)^\gamma)^{\frac12}G(0)^{\frac12}$$

which gives the desired estimate (4.9). □

Proposition 4.5 *Under the assumption of Theorem 3.1, it holds that if* $u_0\neq 0$*,*

$$\|u(t)\|^2\ge C'e^{-k_3t}\quad\textit{with}\quad k_3=k_2+m_2/\rho^2\,, \tag{4.10}$$

where C' *is some positive constant.*

Proof. From Equation (1.1), we observe

$$\begin{aligned}\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}&=\frac{-2\rho}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),u''(t)\right)\\&\quad-\frac{2(1+M(t)^\gamma)}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),Au(t)\right)\end{aligned}$$

or

$$\begin{aligned}&\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}+\frac{2(1+M(t)^\gamma)}{\|u(t)\|^2}\|Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t)\|^2\\&=\frac{-2\rho}{\|u(t)\|^2}\left(Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t),u''(t)\right)\\&\le 2\rho\frac{1}{\|u(t)\|}\|Au(t)-\frac{M(t)}{\|u(t)\|^2}u(t)\|\frac{\|u''(t)\|}{\|u(t)\|}\,.\end{aligned}$$

The Young inequality yields

$$\frac{d}{dt}\frac{M(t)}{\|u(t)\|^2}\le\rho^2\frac{\|u''(t)\|^2}{\|u(t)\|^2}=\rho^2\frac{\|u''(t)\|^2}{M(t)}\frac{M(t)}{\|u(t)\|^2}\le\frac{m_2}{\rho^2}\frac{M(t)}{\|u(t)\|^2}$$

where we used the estimate (4.7). Thus, we have

$$\frac{M(t)}{\|u(t)\|^2}\le Ce^{\frac{m_2}{\rho^2}t}\,,$$

and hence, from (4.9) we obtain the desired estimate (4.10). □

From Propositions 4.1–4.5, we arrive at the following theorem.

Theorem 4.6 *Under the assumption of Theorem 3.1, the solution $u(t)$ of* (1.1) *satisfies that if $u_0 \neq 0$,*

$$C' e^{-k_3 t} \leq \|u(t)\|^2 \leq C e^{-k_1 t}\,, \tag{4.11}$$

$$C' e^{-k_2 t} \leq \|A^{1/2} u(t)\|^2 \leq C e^{-k_1 t}\,, \tag{4.12}$$

$$C' e^{-k_2 t} \leq \|A u(t)\|^2 \leq C e^{-k_1 t}\,, \tag{4.13}$$

$$\|A^{1/2} u'(t)\|^2 + \|u''(t)\|^2 \leq C e^{-k_1 t} \quad \textit{for} \quad t \geq 0 \tag{4.14}$$

with constants k_1, k_2, k_3 given by (4.1), (4.9), (4.10), *where C and C' are some positive constants.*

Proof. (4.12) follows from (4.1) and (4.9). (4.11) follows from (4.12) and (4.10). (4.13) follows from (4.12) and (3.3). (4.14) follows from (4.12) and (4.7). □

5 Appendix : Global Sovability for $\gamma \geq 1$ When $\gamma \geq 1$, if the initial energy $E(0)$ is small, then there exists a unique global solution and the solution decays exponentially. We intoroduce the function $F(t)$ as

$$F(t) \equiv \rho \|A^{1/2} u'(t)\|^2 + (1 + M(t)^\gamma) \|A u(t)\|^2\,. \tag{5.1}$$

Theorem 5.1 *Let the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. Suppose that the initial energy $E(0)$ is small such that*

$$2^5 \gamma^2 \alpha E(0)^{2\gamma - 1} F(0) < 1 \tag{5.2}$$

with $\alpha = \max\{3\rho/2\,,\, \rho + c_*\}$. Then, the problem (1.1) *admits a unique global solution $u(t)$ in the class $C^0([0,\infty); \mathcal{D}(A)) \cap C^1([0,\infty); \mathcal{D}(A^{1/2})) \cap C^0([0,\infty); L^2(\Omega))$, and moreover, the solution $u(t)$ satisfies*

$$\|A u(t)\|^2 + \|A^{1/2} u'(t)\|^2 + \|u''(t)\|^2 \leq C e^{-\theta t} \quad \textit{for} \quad t \geq 0 \tag{5.3}$$

with $\theta = (4\rho)^{-1}$, where C is some positive constant.

Proof. Let $u(t)$ be a solution of (1.1) on $[0,T]$. We define $F_1(t)$ by

$$F_1(t) \equiv F(t) + \frac{1}{2\rho} \|A^{1/2} u(t)\|^2 + (A^{1/2} u'(t), A^{1/2} u(t))\,,$$

Since $|(A^{1/2} u', A^{1/2} u)| \leq (\rho/2) \|A^{1/2} u'\|^2 + (1/2\rho) \|A^{1/2} u\|^2$, we observe from the Sobolev-Poincaré inequality that

$$\frac{1}{2} F(t) \leq F_1(t) \leq \frac{\alpha}{\rho} F(t) \quad \text{with} \quad \alpha = \max\left\{\frac{3}{2}\rho\,,\, \rho + c_*^2\right\}\,. \tag{5.4}$$

Multiplying (1.1) by $2Au' + \rho^{-1} Au$ and integrating it over Ω, we have

$$\frac{d}{dt} F_1(t) + \|A^{1/2} u'(t)\|^2 + \frac{1}{\rho} (1 + M(t)^\gamma) \|A u(t)\|^2 = \gamma M'(t) M(t)^{\gamma - 1} \|A u(t)\|^2\,. \tag{5.5}$$

We observe from (2.2) and (5.1) that

$$\gamma M'(t) M(t)^{\gamma - 1} \leq 2\gamma M(t)^{\gamma - 1} \|A u(t)\| \leq 2\gamma \rho^{-\frac{1}{2}} E(0)^{\gamma - \frac{1}{2}} F(t)^{\frac{1}{2}}\,. \tag{5.6}$$

Since $2^4\gamma^2\rho E(0)^{2\gamma-1}F(0) < 1$ (by (5.2)), putting

$$T_1 \equiv \sup\left\{t \in [0,\infty) \mid \mu(s) \equiv 2^4\gamma^2\rho E(0)^{2\gamma-1}F(s) < 1 \text{ for } 0 \le s < t\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$\mu(t) < 1 \quad \text{for} \quad 0 \le t < T_1 \quad \text{and} \quad \mu(T_1) = 1 \tag{5.7}$$

or

$$\gamma M'(t)M(t)^{\gamma-1}\|Au(t)\|^2 \le \frac{1}{2\rho}\|Au(t)\|^2 . \tag{5.8}$$

Thus, for $0 \le t \le T_1$, it follows from (5.4), (5.5), and (5.8) that

$$\frac{d}{dt}F_1(t) + \theta F_1(t) \le 0 \quad \text{with} \quad \theta = (4\rho)^{-1},$$

and hence,

$$F_1(t) \le F_1(0)e^{-\theta t} \quad \text{or} \quad F(t) \le \frac{2\alpha}{\rho}F(0)e^{-\theta t} . \tag{5.9}$$

Then, we observe

$$\|u''(t)\| \le \|\rho^{-1}(1+M(t)^\gamma)Au(t) + \rho^{-1}u'(t)\|^2 \le CF(t) \le Ce^{-\theta t} \tag{5.10}$$

and

$$\mu(t) \equiv 2^4\gamma^2\rho E(0)^{2\gamma-1}F(t) \le 2^5\gamma^2\alpha E(0)^{2\gamma-1}F(0) < 1 \quad \text{for} \quad 0 \le t \le T_1$$

which is a contradiction to (5.7), and hence, we have that $T_1 \ge T$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| \le C$ for $0 \le t \le T$. Therefore, by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from (5.9) and (5.10) we obtain the desired estimate (5.3). □

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Optimal Maintenance Policy for Fixed Operating Time Horizon

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Abstract

We consider an optimal stopping problem for the operation of system that deteriorates with age and fails stochastically until the fixed time limit in advance. When the system fails unexpectedly, we choose one of two actions, repair or stop. The optimal stopping time which minimizes the total expected cost is derived by means of a simple mathematical model and dynamic programming technique. Some numerical examples are presented to illustrate our results in detail when the failure and the repair distributions are given specifically.

1 Introduction

In practice, most system operational periods are fixed in advance. For instance, consider the management of some airline company with B747 jumbo jet. From the view point of running cost, the company takes into consideration of replacing B747 with B787 carbon fiber aircraft. The deliver time of a new aircraft is 3 years from now on. If the B747 jumbo jet fails unexpectedlly, there are two alternatives, repair and revolve service or stop flighting service until the delivery time. It is clear that if the failure such as engine trouble occurs just before the fixed time limit, then it will be better not to repair sevice. Hence, it is an important problem to find a critical point in time between repairing and stopping.

Another example is concerned with the operation of atomic power plants in Japan. As a turning point with the Fukushima's nuclear accident in 2011, the Japanese government has established the operating time limit of all atomic power plants in 2030. In this case, the same problem happens, because the voluntary moratorium on one atomic plant will loss about 1 billion dollar/year. So, one of important problems to the electric power company is to find the optimal operating and stopping policy for existing atomic power plant.

In general, all the system will deteriorate with age and will fail stochastically. When the system fails, it is repaired with a specified repair time distribution or left as it is until the fixed time limit in advance. From the view point of cost, if the system fails close to the time limit, we should stop and not repair the system.

As a result of stopping action, an idle time occurs and a cost is incurred due to the failed system remaining idle[2,7]. It is an interesting problem to find a critical point in time to repair or to leave the failed system as it is. Such problems have been investigated by some authors in the fields of operations research and reliability engineering[1,2,6]. Kijima and et al [4] discussed the periodic replacement problem and Nair and Hopp[5] gave a simple and efficient algorithm for finding the optimal stopping rule of an equipment replacement. A recent survey paper on maintenance strategy has been written by Wang[8].

In the next section, we provide a simple model to derive the optimal operating and stopping rule for the system with arbitrary failure and repair distributions. In section 3, numerical examples with some failure and repair distributions are given to derive the critical point in time explicitly. Section 4 includes our conclusion.

2 Model and Formulation

Consider a system that deteriorates with age and fails stochastically. When the system fails, we can choose one of two actions, repair or stop. If the repair action with repair distribution $R(t)$ is chosen, the setup cost K_2 and the idle time cost per unit time C are incurred. On the other hand, if the stop action is chosen, the system will be idle until the fixed time limit and the fixed cost K_1 (cost of decommissioning) and the idle time cost per unit time C are incurred. Our problem is to find the optimal action in order to minimize the total expected cost and to derive the critical point in time to repair or to stop the failed system.

Concentrating our model, we define the following notation:

- $F(y)$ and $f(y)$ = failure distribution and its density function
- $\lambda(y) = f(y)/(1 - F(y))$ = failure rate. So $\lambda(y)\Delta y$ represents the probability that the system aged y fails between y and $y + \Delta y$.
- $U(x, y)$ = minimum expected cost up to the fixed time limit when there is still a time x to go and the system aged y is in the state of failure
- $V(x, y)$ = minimum expected cost up to the fixed time limit when there is still a time x to go and the system aged y is in the operable state.

Under these notation, consider the situation in which the system aged y is failed when there is still a time x to go and let us compare the system at two closely spaced remaining times x and $x - \Delta x$. In this case, we have two alternatives, repair the system or stop the system. If the repair action is chosen at x, either the system turns out to be an operable state with probability $R(t)$ or the repair action does not finish until the fixed time limit with probability $1 - R(t)$. If we choose stop action, then the next state is still failure state and the cost $K_1 + Cx$ is incurred.

On the other hand, if the current state is operable, then after the small time interval Δy, the state remains as operable with probability $1 - \lambda(y)\Delta y$ and the state will run into the failure state with probability $\lambda(y)\Delta y$. When the repair action is over, the age of the system is A, a given value which may not exceed the system age prior to failure. It should be noted that $A = y$ corresponds to the minimal repair and $A = 0$ major repair. Then, we have the following functional equation:

$$U(x,y) = \min \begin{cases} K_1 + Cx, & : \text{stop} \\ K_2 + \int_0^x \{Ct + V(x-t, A)\} dR(t) & \\ +(K_1 + Cx) \int_x^\infty dR(t), & : \text{repair} \end{cases} \tag{1}$$

For simplicity, we assume that the repair is minimal $A = y$. The first line in the bracket represents the cost of stopping action and the second one the total expected cost of repair service. If x is small enough, it is clear that the stopping action is preferable. Thus, for small x,

$$U(x,y) = K_1 + Cx. \tag{2}$$

On the other hand, for small Δy, $V(x,y)$ is expressed as

(3)

$$\begin{cases} V(x,y) & = \lambda(y)\Delta y U(x - \Delta y, y + \Delta y) + (1 - \lambda(y)\Delta y) V(x - \Delta y, y + \Delta y) \\ V(0,y) & = K_1 \end{cases}$$

Using a Taylor expansion for U and V and $\Delta y \to 0$, we have a quasi-linear partial differential equation with the boundary condition $V(0,y) = K_1$.

$$\frac{\partial V(x,y)}{\partial x} - \frac{\partial V(x,y)}{\partial y} = \lambda(y)(K_1 + Cx - V(x,y)). \tag{4}$$

Applying the standard method, the solution for this equation is given by

$$\begin{aligned} V(x,y) &= K_1 + Ce^{-\int_0^x \lambda(x+y-z)dz} \int_0^x \lambda(x+y-z) z e^{\int_0^z \lambda(x+y-\xi)d\xi} dz \\ &= K_1 + C \int_0^x (1 - e^{-\int_y^{y+\xi} \lambda(z)dz}) d\xi \\ &= K_1 + \frac{C}{1 - F(y)} [\int_0^x (F(y+\xi) - F(y)) d\xi]. \end{aligned} \tag{5}$$

Therefore, the functional equation (1) for $U(x,y)$ can be written as

$$U(x,y) = K_1 + Cx + \min\{\ 0\ ;\ G_y(x)\}, \tag{6}$$

where $V(x,y)$ is given by (5) and $G_y(x)$ expresses the optimal stopping time function as

$$\begin{aligned} G_y(x) &= K_2 - K_1 R(x) - C \int_0^x R(t)dt + \int_0^x V(x-t, y) dR(t) \\ &= K_2 - K_1 R(0) + \frac{C}{1 - F(y)} [\int_0^x \int_0^{x-t} F(y+\xi) d\xi dR(t) \\ &\quad - \int_0^x R(t)dt + F(y)R(0)x]. \end{aligned} \tag{7}$$

Note that for each $y > 0$,

$$G_y(0) = K_2 - K_1 R(0).$$

So, if $G_y(0) = K_2 - K_1 R(0) > 0$, then the stopping action should be made, where $K_1 R(0)$ shows the expected stopping cost at $x = 0$. And if $K_2 - K_1 R(0) < 0$, then the repair action is preferential.

Since $R(0)$ means the probability that finishes the repair action in a moment, we assume that $R(0) = 0$ without a special case. Under this assumption, $G_y(x)$ is given by

$$G_y(x) = K_2 + \frac{C}{1 - F(y)}[\int_0^x \int_0^{x-t} F(y + \xi)d\xi dR(t) - \int_0^x R(t)dt]$$

and from this result we can observe that the solution of $G_y(x^*) = 0$ does not depend on K_1.

(Proposition) *If $R(0) = 0$, then the optimal stopping time $x^*(y)$ does not depend on the stopping cost K_1.*

It should be noted that the relation

$$U(x, y) = K_1 + Cx$$

is valid for the preferential region of stopping and

$$U(x, y) = K_1 + Cx + G_y(x)$$

gives the expected cost for repair action, that is

$$G_y(x) > 0 \Rightarrow \text{stop action} \quad G_y(x) < 0 \Rightarrow \text{repair action}$$

Thus, the critical value of x, for which the repair action should be made, is given by the minimum positive root of

$$G_y(x) = 0.$$

Moreover,

$$x^*(y) = \inf_{x>0}\{\ x\ :\ G_y(x) \le 0\}$$

represents the critical value for which the repair action should be made. It is intuitively clear, and can be easily demonstrated, that the optimal region is provided by the simple form as

$$\begin{cases} \text{stop} & \text{for} \quad 0 < x \le x^*(y) \\ \text{repair} & \text{for} \quad x \ge x^*(y). \end{cases}$$

3 Simple Examples

In this section, we show some simple examples to find a critical value $x^*(y)$ explicitly.

(1) General Failure Distribution and Negligible Repair Time
The first example is shown by an instanteneous repair time distribution $R(0) = 1$ and a genenral failure distribution $F(t)$. By equation (7), we have

$$G_y(x) = K_2 - Cx + \frac{C}{1-F(y)}[\int_0^x \{F(y+\xi) - F(y)\}d\xi].$$

Especially, if the failure distribution $F(t)$ is given by the exponential distribution $F(t) = 1 - e^{-\lambda t}, \quad \lambda > 0$, then

$$G_y(x) = K_2 + \frac{C}{\lambda}(e^{-\lambda x} - 1).$$

Under the condition $C > \lambda K_2$, we have

$$x^*(y) = -\frac{1}{\lambda}\ln(\frac{C - \lambda K_2}{C}).$$

On the other hand, if the repair is maximal(that is, after the repair the system's age is always $y = 0$) and $F(0) \neq 1$, we have

$$G_y(x) = K_2 - Cx + \frac{C}{1-F(0)}[\int_0^x \{F(\xi) - F(0)\}d\xi]$$

and

$$G_y(0) = K_2 > 0, \quad G_y(\infty) = -\infty < 0.$$

Therefore, the optimal stopping time equation $G_y(x) = 0$ has at least one root for $x > 0$.

1. If $F(0) = 1$, then the optimal rule is always stop since $G_y(x) = \infty > 0$ and the repaired system fails in a moment.

2. If $F(0) \neq 1$, then

   $$G_y(x) = K_2 - Cx + \frac{C}{1-F(0)}[\int_0^x F(\xi)d\xi - F(0)x].$$

   So, the optimal stopping time $x^*(y)$ satisfies the following equation:

   $$(1 - F(0))K_2 = C[\int_0^{x^*} (1 - F(\xi))d\xi].$$

   Especially, if $F(0) = 0$, then

   $$G_y(x) = K_2 - C[\int_0^x (1 - F(\xi))d\xi] = K_2 - C[m - T_F(x)]$$

where m is the mean time to failure and

$$T_F(x) = \int_x^\infty (x - \xi) dF(\xi).$$

Note that the transform $T_F(x)$ is a nonnegative convex and strictly decreasing function of x as was pointed out by DeGroot[2]. So, the optimal stopping time x^* is given by

$$x^* = T_F^{-1}(m - \frac{K_2}{C})$$

as shown in Figure 1.

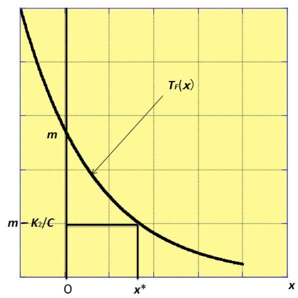

Figure 1: Graph of $T_F(x)$

(2) Gamma Type Failure Distribution

Suppose that the failure distribution $F(t)$ is Gamma type as

$$F(t) = \int_0^t \frac{\lambda^k}{(k-1)!} e^{-\lambda\xi} \xi^{k-1} d\xi.$$

Let

$$\Gamma_k(a, b) = \int_a^b e^{-\lambda t} t^{k-1} dt,$$

then

$$\lambda\Gamma_k(a, b) = k\Gamma_k(a, b) + a^k e^{-\lambda a} - b^k e^{-\lambda b}$$

and equation (5) can be denoted as

$$V(x, y) = K_1 + \frac{C}{\Gamma_k(y, \infty)}[(x + y)\Gamma_k(y, x + y) - \Gamma_{k+1}(y, x + y)].$$

It is difficult to carry out the operation of integral explicitly except for $k = 1$. Let $k = 1$, then the failure distribution is reduced to an exponential distribution and we have

$$V(x, y) = K_1 + Cx - \frac{C}{\lambda}(1 - e^{-\lambda x}).$$

Thus,

$$G_y(x) = K_2 - Ce^{-\lambda x} \int_0^x e^{-\lambda(x-t)} R(t)dt.$$

From this relationship, the critical value x* is given by the solution of

$$\frac{K_2}{C} = f(x) * R(x)$$

where the symbol * denotes the convolution integral.

In addition to the assumption that the failure distribution is exponential, we suppose that the repair time is subject to an exponential distribution $R(t) = 1 - \exp(-\mu t)$ and $\mu/\lambda = \rho > 1$. Then the optimal stopping time function can be written as

$$G_y(x) = K_2 - \frac{C}{\lambda}(1 - e^{-\mu x}) - \frac{C\mu}{\lambda(\mu - \lambda)}(e^{-\mu x} - e^{-\lambda x}).$$

Letting $e^{-\lambda x} = z$, we can write $G_y(x) = 0$ as

$$(\frac{\rho}{\mu - \lambda} - \frac{1}{\lambda})z^\rho - \frac{\rho}{\mu - \lambda}z = \frac{K_2}{C} - \frac{1}{\lambda}.$$

For $\lambda = 2, \mu = 1, C/K_2 = 8$, this equation yields a quardratic equation in z which has the solution $x^* = \ln 2$.

Especially, if $C > \lambda K_2$, we can easily obtain the analytical form of this value x^* for two extreme cases $\mu = \infty$ and $\mu = 0$. The assumption of $\mu = \infty$ shows a negligible repair time. Thus, the above equation is expressed as

$$G_y(x) = K_2 - \frac{C}{\lambda}(1 - e^{-\lambda x}).$$

Since $G_y(x)$ is a decreasing function of x, there exists the unique value

$$x^* = -\frac{1}{\lambda}\ln(\frac{C - \lambda K_2}{C})$$

as was derived above.

On the other hand, we consider the case of $\mu = 0$. This means that the repair action never finishes in the finite horizon. Then we have $G_y(x) = K_2 > 0$ and

$$U(x, y) = K_1 + Cx + \min\{0 : K_2\} = K_1 + Cx.$$

The result shows that the optimal policy is to be always idle for any x.

As the last example of repair time, we consider it as constant in time D. The distribution function $R(t)$ is written as

$$R(t) = \begin{cases} 0, & \text{for } 0 \le t < D \\ 1, & \text{for } t \ge D \end{cases}$$

Accordingly we have

$$G_y(x) = \begin{cases} K_2 & \text{for } \ 0 \le x < D \\ K_2 - \frac{C}{\lambda}(1 - e^{-\lambda(x-D)}) & \text{for } \ x \ge D \end{cases}$$

From the equation the optimal policy is described as follows:

(i) $C > \lambda K_2$

$$\text{stop for } \ 0 \le x < x^*$$

$$\text{repair for } \ x \ge x^* (> D)$$

where x^* is given by

$$x^* = D - \frac{1}{\lambda} \ln(\frac{C - \lambda K_2}{C})$$

(ii) $C \le \lambda K_2$ idle for all x since the second term is positive for all x.

(3) Linear Failure Distribution

Let

$$F(t) = \begin{cases} \beta t, & 0 \le t \le 1/\beta \\ 1, & t \ge 1/\beta, \end{cases}$$

then the failure rate is given by

$$\lambda(t) = \frac{\beta}{1 - \beta t}, \quad (0 \le t \le 1/\beta)$$

To derive an explicit expression of $V(x,y)$ and $G_y(x)$, we consider the following three cases:

Case(i) $x + y \le 1/\beta$

$$V(x,y) = K_1 + \frac{C\beta x^2}{2(1 - \beta y)}$$

and

$$G_y(x) = K_2 + \frac{C\beta}{1 - \beta y} \int_0^x (x + y - t - \frac{1}{\beta}) R(t) dt.$$

Note that the condition $x + y \le 1/\beta$ suggests that the time remaining until the fixed time limit is short and the system is in the nearly new state.

Case (ii) $x + y \ge 1/\beta$ and $y \le 1/\beta$

This case means that the time remaining is long enough and the system is nearly new. Then we have

$$V(x,y) = K_1 + C\{x + \frac{1}{2}(y - \frac{1}{\beta})\}$$

and

$$G_y(x) = K_2 + \frac{CR(x)}{2}(y - \frac{1}{\beta}).$$

Case(iii) $y \ge 1/\beta$

It is clear that $V(x,y) = \infty$. It follows that the optimal action should be always idle since the system aged $y(\geq 1/\beta)$ fails with probability 1. To study the optimal policy in detail, we specify that distribution of repair time $R(t)$ as follows:

(A) Exponential Repair Distribution $R(t) = 1 - \exp(-\mu t)$

From the results of three cases mentioned above, it follows that

$$G_y(x) = \begin{cases} K_2 + \frac{C\beta}{1-\beta y}[\frac{x^2}{2} + (y - \frac{1}{\beta} - \frac{1}{\mu})\{x - \frac{1}{\mu}(1 - e^{-\mu x})\}] & \text{for } x + y \leq 1/\beta \\ K_2 + \frac{C}{2}(y - \frac{1}{\beta})(1 - e^{-\mu x}) & \text{for } x + y \geq 1/\beta \text{ and } y \leq 1/\beta \\ \infty & \text{for } y \geq 1/\beta \end{cases}$$

It is clear that the critical point x^* depends on x and y. We can find the critical point by the numerical calculation and the following figure 2 and figure 3 are useful.

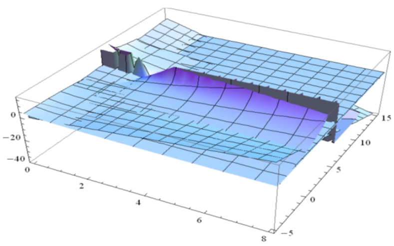

Figure 2: 3 dimensional graph of $G_y(x)$

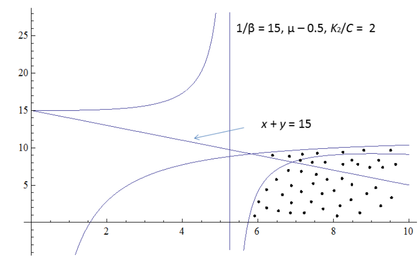

Figure 3: Graph of $G_y(x)$

(B) Straight Line Repair Distribution

Let the repair time distribution be a linear function as

$$R(t) = \begin{cases} \alpha t, & 0 \leq t \leq 1/\alpha \\ 1, & t \geq 1/\alpha \end{cases}$$

To avoid unnecessary complications, we assume that $\alpha \geq \beta$. Then we have the following result:

$$G_y(x) = K_2 + \begin{cases} \frac{C\alpha x^2}{2}\{\frac{\beta x}{3(1-\beta y)} - 1\} & \text{for } x + y \leq 1/\beta, 0 \leq x \leq 1/\alpha \\ C[\frac{1}{2\alpha} - x + \frac{\beta}{2(1-\beta y)}(x^2 - \frac{x}{\alpha} + \frac{1}{3\alpha^2})] & \text{for } x + y \leq 1/\beta, 1/\alpha \leq x \\ \frac{C\alpha x}{2}(y - \frac{1}{\beta}) & \text{for } x + y \geq 1/\beta, y \leq 1/\beta, x \geq 1/\alpha \\ \frac{C}{2}(y - \frac{1}{\beta}) & \text{for } x + y \geq 1/\beta, y \leq 1/\beta, x \leq 1/\alpha \\ \infty & \text{for } y \geq 1/\beta \end{cases}$$

The shaded portion in the figure shows a preferential region of repair service for this example. Note that the optimal stopping time $x^*(y)$ depends on the

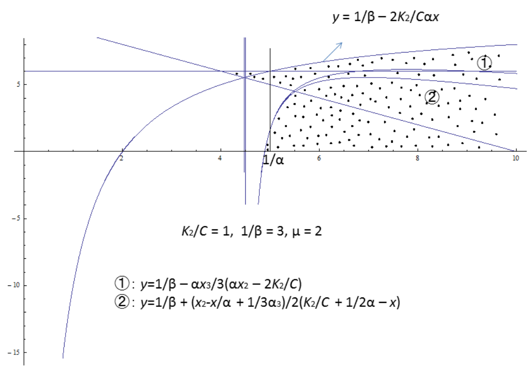

Figure 4: Repair Region

remaining time x and the system age y.

(4) Weibull Failure Distribution
Let

$$f(t) = \alpha\beta(\alpha t)^{\beta-1}e^{-(\alpha t)^{\beta}}.$$

Then

$$\lambda(t) = \beta\alpha^{\beta}t^{\beta-1}$$

1. If $\alpha = 1, \beta = 2$, then the failure distribution shows an increasing failure rate(IFR). In this case we have

$$G_y(x) = K_2 - C\sqrt{\pi}e^{y^2}\int_0^x \{\Phi(\sqrt{2}(x+y-t)) - \Phi(\sqrt{2}y)\}dR(t).$$

2. If $\alpha = 1, \beta = 1/2$, then the failure distribution shows a decreasing failure rate(DFR). In this case we have

$$G_y(x) = K_2 - 2Ce^{\sqrt{y}}\int_0^x \{e^{-y}(1+\sqrt{y}) - e^{-\sqrt{x+y-t}}(1-\sqrt{x+y-t})\}dR(t).$$

Unfortunately, it is difficult to carry out the operation of integrals explicitly.

4 Conclusion

The present paper is concerned with an optimal maintenance policy for the system with repair and idle time during the fixed time limit. An optimal policy and a critical point in time to repair or to leave the failed system as it is are provided by the method of dynamic programming technique. We show that the optimal policy depends not only the time until the fixed time limit but on the system age. It is difficult to obtain an explicit form of optimal policy for arbitrary distributions of failure and repair. The interesting results are that the critical value $x^*(y)$ does not depend on the system's age y for the exponential ditribution family by the memoryless property. Except the exponential distribution, the critical value depends on the remaining time x and the system's age y. A numerical calculation presents a solution to this difficult problem. For some simple examples, convenient figures which specify the critical point and the preferential region of repair action are easily described by the numerical calculation. The results will be useful to solve the practical problems.

5 Acknowledgement

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SOME COALITIONAL GAMES WITH THE SHAPLEY VALUE

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Abstract. In this paper, a characteristic function depending on the state of a pair's relationship is introduced to a coalitional game with the Shapley value. By applying the characteristic function, we draw some theorems where a specific player makes his reward the maximum or the minimum. Furthermore, some properties in two concrete models are shown and various strategies of each player are discussed in two simulations. Especially, it is investigated how a player with low original reward should cooperate with other players in order to make his portion the maximum.

1 Introduction

If three people obtain reward when they cooperate, there exists a problem how to divide reward to them. Ordinary person simply thinks it should be divided reward by three evenly. But with different potential or skills, that division way is not proper from the perspective of each person's satisfaction. In real life, if you think about your wage in the company, this wages are divided by your experience, your role, and the significance of your position. We think that the system is rational in our real life.

In this research, based on the Shapley value L.S. Shapley introduced, we will discuss how to divide reward in a coalitional game depending on the state of player's relationship. It was well known that L.S. Shapley won the 2012 Nobel Memorial Prize in Economic Sciences. In the coalitional game, it is clear that when relationship among all players should be good, their sum of reward becomes the maximum. But we are not sure that a specific player can get the most reward from that relationship. For a specific player, there exists the strategy what kind of relationship the player makes to other players. Here, we give each relationship between two players, and we define the value of characteristic function depending on the state of that relationship. If the relationship is good, the value of characteristic function goes high. If the relationship is bad, that value goes down. We define a characteristic function being like this situation and discuss the strategy of each player.

2 Definition of a characteristic function

Let S be the set of relationship between two players, $S=\{s_1, s_2, s_3,\ldots, s_m\}$.
For any $i < j$, let $s_i \gg s_j$. $\gg$ means that the relationship of s_i is better than that of s_j.

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All elements of S are the following relations, $s_1 \gg s_2 \gg s_3 \gg \ldots\ldots \gg s_i \gg \ldots\ldots \gg s_m$. Of course, s_1 means the best relationship and s_m means the worst relationship. Let P be the set of players, $P = \{p_1, p_2, p_3, \ldots, p_n\}$.

When the relationship of two players p_i and p_j is s_h, we make the characteristic function giving reward based on the relationship.

[Definition 1]

That characteristic function is defined as

$v(p_i \cup p_j, s_h)$, where for k<l, $v(p_i \cup p_j, s_k) \geqq v(p_i \cup p_j, s_l)$.

When we have the characteristic function with three people, we define

$v(p_i \cup p_j \cup p_k) = \frac{1}{2}\{v(p_i \cup p_j, s') + v(p_i \cup p_k, s'') + v(p_k \cup p_j, s''')\}$,

where s' is the relationship between p_i and p_j, s'' is the relationship between p_i and p_k, and s''' is the relationship between p_j and p_k.

For four players, we define as follows,

$$v(p_i \cup p_j \cup p_k \cup p_l) = \frac{1}{3}\{v(p_i \cup p_j \cup p_k) + v(p_j \cup p_k \cup p_l) + v(p_i \cup p_k \cup p_l) + v(p_i \cup p_j \cup p_l)\}.$$

We can define the same things to others following this.

By applying the characteristic function depending on the relationship, we discuss the strategy that each player makes the Shapley value the maximum.

[Definition 2]

For the coalitional game (P, v), the Shapley value of player p_i is given

$$f(p_i) = \sum_{P'} \frac{j!\,(n-j-1)!}{n!}\{v(P' \cup P_i) - v(P')\}$$

where n is the number of players in the set P, P' represents any set except player p_i, $P' \cup p_i$ is the set P' adding player p_i, j represents the number of players in the set P', and $\sum_{P'}$ can give us the sum of all of the combination of P'.

[Example]

In the coalitional game of three players, let be $P = \{p_1, p_2, p_3\}$ and $S = \{s_1, s_2, s_3, \ldots, s_m\}$.

It is assumed the reward to be able to get alone as follows,

$v(p_1) = q_1$, $v(p_2) = q_2$, $v(p_3) = q_3$.

We can get the Shapley value of each player as follows,

$$f(p_1) = \frac{2!}{3!}\{v(p_1) - v(\phi)\} + \frac{1}{3!}\{v(p_1 \cup p_2, s') - v(p_2)\} + \frac{1}{3!}\{v(p_1 \cup p_3, s'') - v(p_3)\} + \frac{2!}{3!}\{v(p_1 \cup p_2 \cup p_3) - v(p_2 \cup p_3, s''')\}$$

$$= \frac{1}{6}\{2q_1 - (q_2, +q_3)\} + \frac{1}{6}\{2(v(p_1 \cup p_2, s') + v(p_1 \cup p_3, s'')) - v(p_2 \cup p_3, s''')\}$$

$$f(p_2) = \frac{1}{6}\{2q_2 - (q_1 + q_3)\} + \frac{1}{6}\{2(v(p_1 \cup p_2, s') + v(p_2 \cup p_3, s''')) - v(p_1 \cup p_3, s'')\}$$

$$f(p_3) = \frac{1}{6}\{2q_3 - (q_1 + q_2)\} + \frac{1}{6}\{2(v(p_2 \cup p_3, s''') + v(p_1 \cup p_3, s'')) - v(p_1 \cup p_2, s')\}$$

where s', s'' and s''' are elements of S= $\{s_1, s_2, s_3,\ldots, s_m\}$.

Since the property of $v(p_i \cup p_j, s_k) \geqq v(p_i \cup p_j, s_l)$ for k<l, the strategy to make $f(p_1)$ be the maximum is $(s', s'', s''') = (s_1, s_1, s_m)$.

We get as follows similarly, the strategy to make $f(p_2)$ be the maximum is $(s', s'', s''') = (s_1, s_m, s_1)$,the strategy to make $f(p_3)$ be the maximum is $(s', s'', s''') = (s_m, s_1, s_1)$.

Even if it extends a player to n persons from three persons, it is clear that the same structure is held.

[Theorem 1]

When the relationship of $p_i \cup p_j$ for every j ($j \neq i$) is s_1 and the relationship of $p_j \cup p_k$ is s_m ($j \neq i$ and $k \neq i$), $f(p_i)$ becomes the maximum.

[Theorem 2]

When the relationship of $p_i \cup p_j$ for every j ($j \neq i$) is s_m and the relationship of $p_j \cup p_k$ is s_1 ($j \neq i$ and $k \neq i$), $f(p_i)$ becomes the minimum.

[Proof]

Two theorems can be quickly derives from Definition 1 and Definition 2.

From Theorem 1, when one chooses good relationship of a pair with oneself and does worse relationship of other pair except oneself, one can make one's reward the maximum. Conversely, to make reward of specific player the minimum is by having a bad relationship of a pair with the player, and also relationship of others except the player needs to be good.

3 Simulation Model I

Let P= {A,B,C} be a set of 3 players, S= {g,n,w} be the state set of relationship between two players. Let g be "good" of relationship, n be "neutral", and w be "worse".

Each of the 3 players can choose a element of the state set and the selection is carried out to their strategies.

The characteristic function v is defined as follows.

$v(A)=a, \quad v(B)=b, \quad v(C)=c,$

$$v(A\cup B,s)=\begin{cases} 2(a+b), & s=g \\ \frac{3}{2}(a+b), & s=n \\ a+b, & s=w \end{cases}$$

$$v(A\cup C,t)=\begin{cases} 2(a+c), & t=g \\ \frac{3}{2}(a+c), & t=n \\ a+c, & t=w \end{cases}$$

$$v(B\cup C,u)=\begin{cases} 2(b+c), & u=g \\ \frac{3}{2}(b+c), & u=n \end{cases}$$

b+c, u=w

$v(A\cup B\cup C)=\frac{1}{2}\{v(A\cup B,s)+v(A\cup C,t)+v(B\cup C,u)\}$

The Shapley value of each player can be calculated by using the dividend.

$f(A)=\frac{1}{6}(2a-b-c)+\frac{1}{6}\{2v(A\cup B,s)+2v(A\cup C,t)-v(B\cup C,u)\}$

$f(B)=\frac{1}{6}(2b-a-c)+\frac{1}{6}\{2v(A\cup B,s)+2v(B\cup C,u)-v(A\cup C,t)\}$

$f(C)=\frac{1}{6}(2c-a-b)+\frac{1}{6}\{2v(A\cup C,t)+2v(B\cup C,u)-v(A\cup B,s)\}$

[Property Ⅰ]

When relationships between three players are all "good", the sum of all players reward becomes the maximum. But the strategy which makes each individual's reward the maximum can be expressed by Theorem Ⅰ.

Each strategy of player A, player B, and player C is (s,t,u)=(g,g,w), (s,t,u)=(g,w,g), and (s,t,u)=(w,g,g), respectively.

[Property Ⅱ]

When relationships between three players are all "worse", the sum of all players reward becomes the minimum. But the strategy which makes each individual's reward the minimum can be expressed by Theorem Ⅱ.

Each strategy of player A, player B and player C is (s,t,u)=(w,w,g),(s,t,u)=(w,g,w), or (s,t,u)=(g,w,w), respectively.

[Property Ⅲ]

When (s,t,u)=(n,*,g) or (s,t,u)=(w,*,w), f(A) does not depend on b.

When (s,t,u)=(*,n,g) or (s,t,u)=(*,w,w), f(A) does not depend on c.

The symbol* denotes an arbitrary state of relationship.

Especially, when (s,t,u)=(n,n,g), $f(A)=\frac{3}{4}a$ does not depend on both b and c.

[Property Ⅳ]

When (s,t,u)=(n,g,*) or (s,t,u)=(w,w,*), f(B) does not depend on a.

When (s,t,u)=(*,g,n) or (s,t,u)=(*,w,w), f(B) does not depend on c.

Especially, when (s,t,u)=(n,g,n), $f(B)=\frac{3}{4}b$ does not depend on both a and c.

[Property Ⅴ]

When (s,t,u)=(g,*,n) or (s,t,u)=(w,*,w), f(C) does not depend on b.

When (s,t,u)=(g,n,*) or (s,t,u)=(w,w,*), f(C) does not depend on a.

Especially, when (s,t,u)=(g,n,n), $f(C)=\frac{3}{4}c$ does not depend on both a and b.

[Property Ⅵ]
When (s,t,u)=(w,w,w), f(A)=a, f(B)=b, and f(C)=c.

[Property Ⅶ]
When (s,t,u)=(w,*,g) or (s,t,u)=(*,w,g), f(A) is decreasing in b and is decreasing in c, respectively. Especially, when (s,t,u)=(w,w,g), f(A) is decreasing in both b and c.

[Property Ⅷ]
When (s,t,u)=(g,w,*) or (s,t,u)=(*,g,w), f(B) is decreasing in a and is decreasing in c, respectively. Especially, when (s,t,u)=(w,g,w), f(B) is decreasing in both a and c.

[Property Ⅸ]
When (s,t,u)=(g,w,*) or (s,t,u)=(g,*,w), f(C) is decreasing in b and is decreasing in c, respectively. Especially, when (s,t,u)=(g,w,w), f(B) is decreasing in both a and b.

[Numerical analysis of Model Ⅰ]
When v(A)=a=4, v(B)=b=3, and v(C)=c=2, the Shapley value of each player can be calculated in each strategy.
From Property Ⅰ, when you choose good relationship of a pair with yourself and does worse relationship of a pair except yourself, you can make your reward the maximum.
The number of strategies which 3 players take the state is 27. Figure 1-1 represents that it arranges in many order with reward f(A) of player A.
Figure 1-2 and Figure 1-3 are also the same.

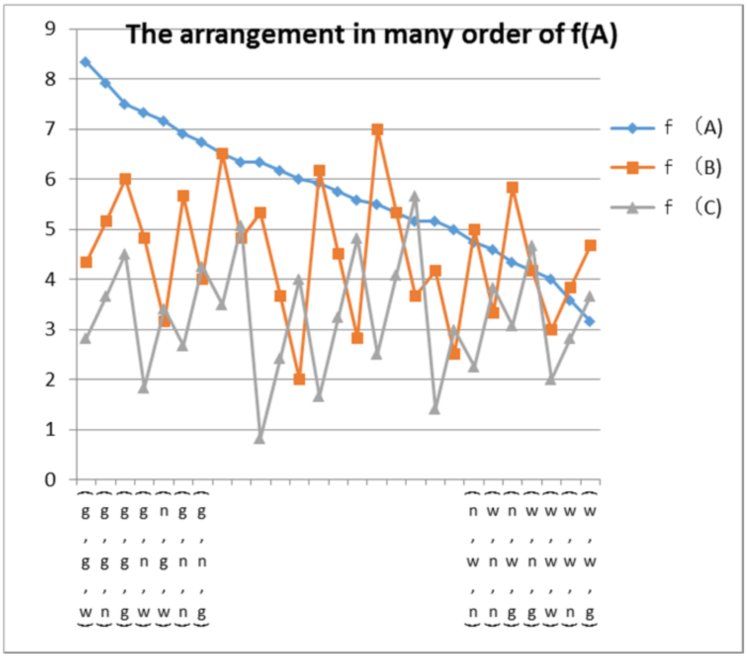

Figure 1-1

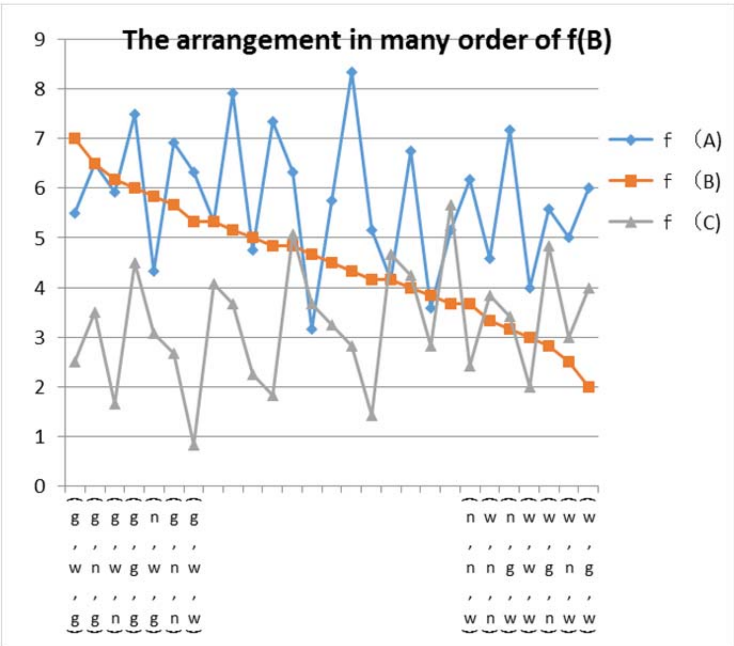

Figure 1-2

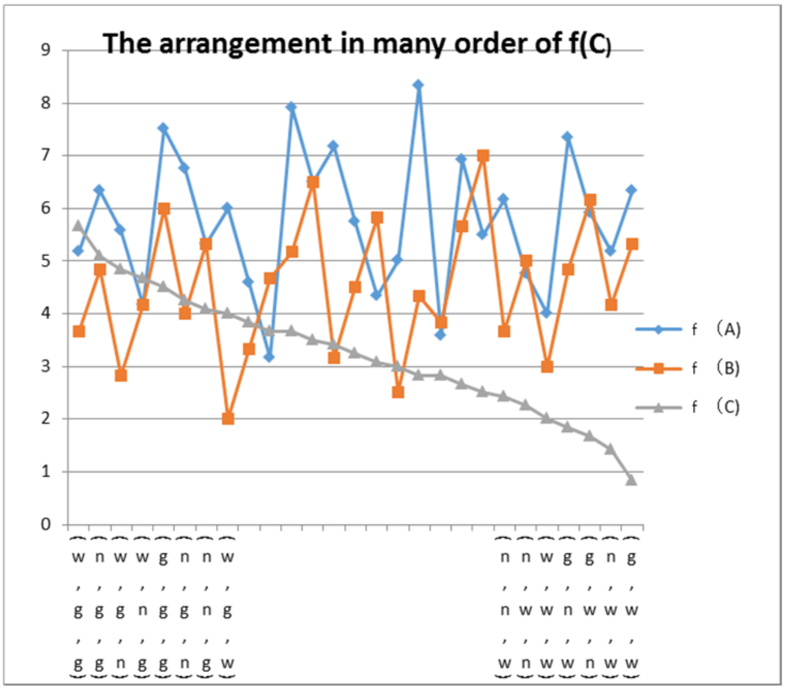

Figure 1-3

When relationships among three players are all “good”, the strategy (s,t,u)=(g,g,g) is not necessarily best for each player. In the arrangement of f(A),f(B), and f(C), the ranking of the strategy (s,t,u)=(g,g,g) is 3rd,4th and 5th,respectively. When original reward v(C) of player C is the lowest value of three players, a player like player C is called “low potential player”. For low potential player like player C, (s,t,u)=(g,g,g) is not so an important strategy. If the value of v(C) becomes small, the importance of the strategy (s,t,u)=(g,g,g) will fall for player C.

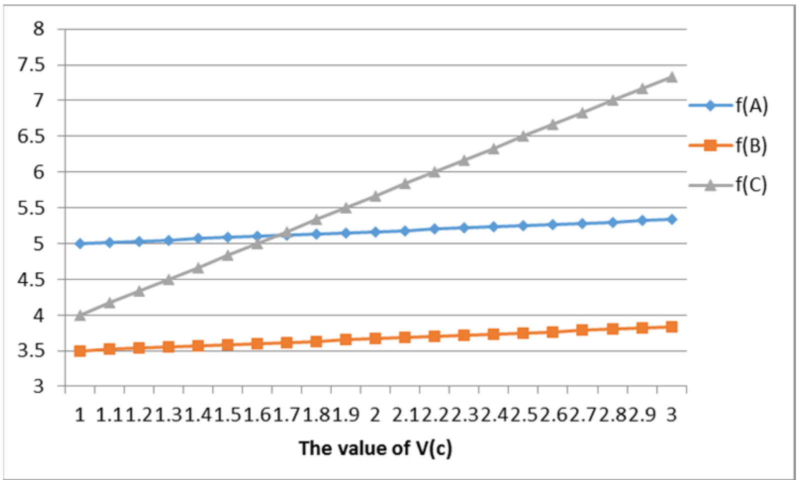

Figure 1-4

If player C takes the strategy (s,t,u)=(w,g,g) which makes one’s reward the maximum, it is investigated how f(A), f(B), and f(C) will change by the variable of v(C)=c.

Figure 1-4 represents the changes of f(A), f(B), and f(C) where v(A)=4, v(B)=3 and v(C) is changed to 3 from 1. If v(C) exceeds 1.7, f(C) will become the maximum among three players.

4 Simulation Model Ⅱ

In Model Ⅰ, the characteristic function v depends on the sum of 2 player's reward and is linear function of v(A), v(B) and v(C). In Model Ⅱ, the characteristic function v changes the product of 2 player's reward. The reward of each player will become large if a good relationship is chosen.

The characteristic function v is defined as follows.

$$v(A)=a,\quad v(B)=b,\quad v(C)=c,$$

$$v(A\cup B,s)=\begin{cases} 2ab, & s=g \\ \frac{3}{2}ab, & s=n \\ a+b, & s=w \end{cases}$$

$$v(A\cup C,t)=\begin{cases} 2ac, & t=g \\ \frac{3}{2}ac, & t=n \\ a+c, & t=w \end{cases}$$

$$v(B\cup C,u)=\begin{cases} 2bc, & u=g \\ \frac{3}{2}bc, & u=n \\ b+c, & u=w \end{cases}$$

$$v(A\cup B\cup C)=\frac{1}{2}\{v(A\cup B,s)+v(A\cup C,t)+v(B\cup C,u)\}$$

Since this characteristic function v satisfies the conditions of Theorem Ⅰ and Ⅱ, Model Ⅱ is keeping the same properties as Property Ⅰ, Property Ⅱ and Property Ⅵ in Model Ⅰ.

[Numerical analysis of Model Ⅱ]

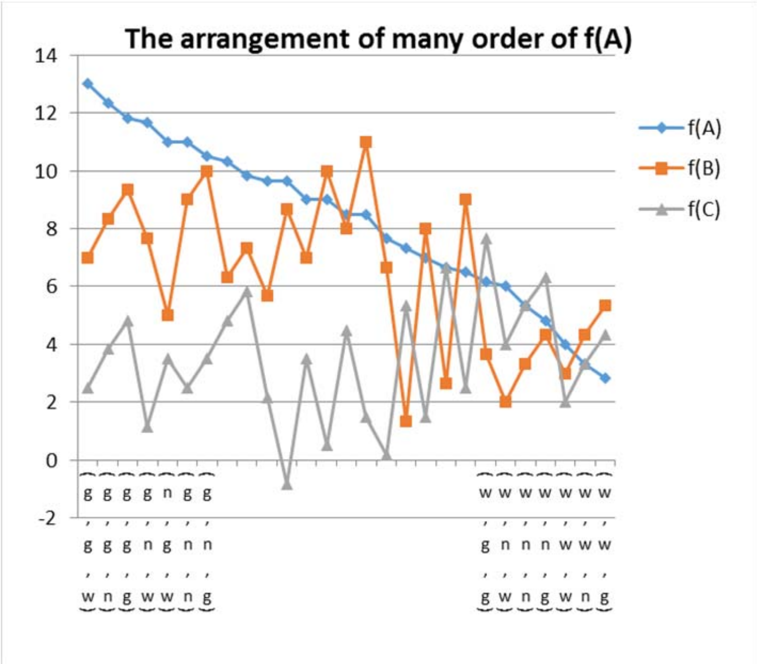

Figure 2-1

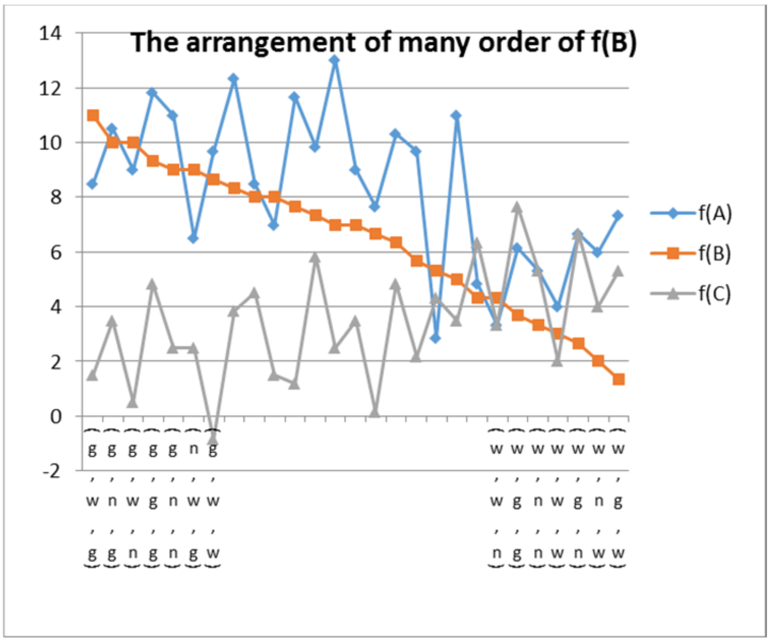

Figure 2-2

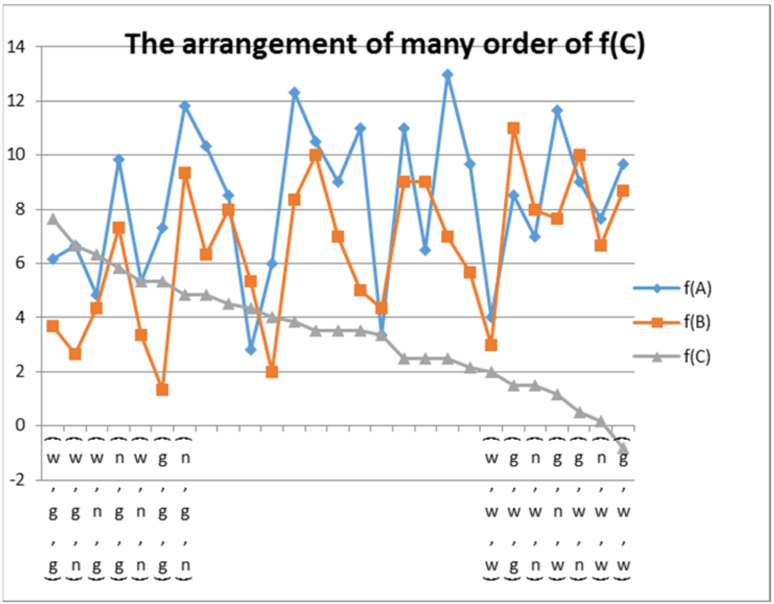

Figure 2-3

In the arrangement of many order of f(C), the ranking of the strategy (s,t,u)=(g,g,g) is 6th. Furthermore, the importance of this strategy will fall for player C.

[Property X]

When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(w,w,g), f(A) is decreasing in both b and c.

When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(w,g,w), f(B) is decreasing in both a and c.

When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(g,w,w), f(C) is decreasing in both a and b.

[Property XI]

Except (s,t,u)=(w,w,g) ,(s,t,u)=(w,g,w), and (s,t,u)=(g,w,w), let $a \geqq b \geqq c > \frac{1}{2}$,then f(A), f(B), and f(C) are increasing in all a, b and c.

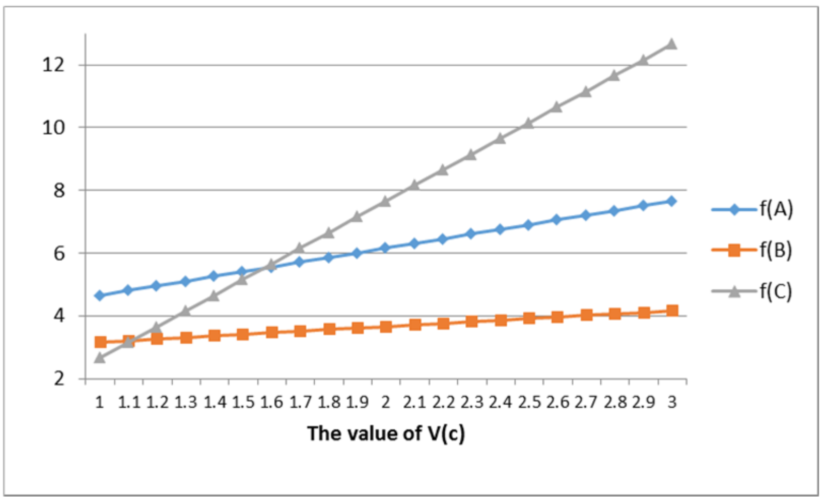

Figure 2-4

It is investigated how f(A), f(B) and f(C) will change by the variable of v(C)=c like Model

Ⅰ. In spite of the best strategy (s,t,u)=(w,g,g) for player C, it becomes the lowest at v(C)=1. When v(C) exceeds 1.6, f(C) will become the top of three players. The structure of characteristic function v is disadvantageous for the low potential player.

5 Comparison of Model Ⅰ and Model Ⅱ

In Model Ⅰ and Model Ⅱ,since 3 players can choose three kinds of relationships each one ,the number of their strategies is 27. We investigate the reward distribution of 3 players in all strategies. In Model Ⅰ and Model Ⅱ, the order of strategies make a small difference for each player. In particular, the change of the order is large for the low potential player. In all strategies, the average of each player's distribution percentage is as follows.

In Model Ⅰ,when v(A)=4, v(B)=3, and v(C)=2, we can see the share of f(A) in v(A∪B∪C). We take the average of that and let it be ASR(Average Share Rate).

ASR of f(A): ASR of f(B): ASR of f(C) =42.6% : 33.3% : 24.1% , and

the average of all of f(A): the average of all of f(B): the average of all of f(C) = 5.75 : 4.5 : 3.25 .

Compared with an original reward, the average of each player becomes large comparatively. Especially, when an original reward, v(C)=2 becomes the average 3.25, the satisfaction of player C may be high.

When relationships among three players are all "worse", f(A),f(B), and f(C) depend on each original reward only from Property Ⅵ. The structure of cooperative relation will not exist at all.

In Model Ⅱ, when v(A)=4, v(B)=3 and v(C)=2,

ASR of f(A): ASR of f(B): ASR of f(C) = 44.9%:34.9%:20.2%,

the average of all of f(A): the average of all of f(B): the average of all of f(C) =8.2777 : 6.4444 : 3.4444.

Compared with an original reward, the average of each player becomes large comparatively like Model Ⅰ. On the contrary to Model Ⅰ, the satisfactions of player A and player B will be high. The structure of the characteristic function is disadvantageous for the low potential player like player C.

6 Conclusion

It is clear that a high potential player is advantageous in the coalitional game with the Shapley value. Since a characteristic function depending on the state of a pair's relationship is introduced to a coalitional game, there exists the strategy where a specific player makes his reward the maximum or the minimum.

Since two models hold $v(p_i \cup p_j, s_k) \geqq v(p_i) + v(p_j)$ for any s_k, the Shapley value of each player becomes more than an original reward in a coalitional game. A low potential player is disadvantageous in two models. But when the low potential player C takes two strategies (s,t,u)=(w,g,g) or (s,t,u)=(w,n,g) in Model Ⅰ,four strategies (s,t,u)=(w,g,g), (s,t,u)=(w,g,n), (s,t,u)=(w,n,g), or (s,t,u)=(w,n,n) in Model Ⅱ,respectively, reward of the player will become the top of three players. The choice is increasing in spite of the disadvantageous structure of Model Ⅱ for the low potential player C.

In the future, we can extend to 4 players and 5 players from three players and may draw many properties from these models. In this paper, the characteristic function of more than three players was made from two players' relationship. We can make directly the relationship of more than three players and will discuss the structure of a complicated relationship.

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