Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 79, No. 2 Whole Number 283

October 2016

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meeetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21^{st} Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM). MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centurial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, begining from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences " as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathmatical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system** (**IVMS**) which can connect up to twenty sites of a reserch group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit http://www.jams.or.jp.

FUZZY HYPER BCK-IMPLICATIVE IDEALS OF HYPER BCK-ALGEBRAS

MUHAMMAD TOUQEER *AND M. ASLAM MALIK†

Received December 9, 2014

Abstract

The fuzzification of (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras is considered. It is shown that every fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is a fuzzy (weak, strong, reflexive) hyper BCK-ideal. We have discussed the properties of (fuzzy) weak hyper BCK-implicative ideals, (fuzzy) hyper BCK-implicative ideals, (fuzzy) strong hyper BCK-implicative ideals and (fuzzy) reflexive hyper BCK-implicative ideals and also their relations are given. Characterization of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals is given. The hyper homomorphic pre-image of a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is discussed. Lastly the properties of product of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals are discussed.

Keywords: Hyper BCK-algebra; (fuzzy) hyper BCK-implicative ideal; (fuzzy) weak hyper BCK-implicative ideal; (fuzzy) strong hyper BCK-implicative ideal; (fuzzy) reflexive hyper BCK-implicative ideal.

*touqeer-fareed@yahoo.com
†malikpu@yahoo.com

1 Introduction

In 1966, Imai and Iseki [7] introduced the notion of BCK-algebra. In the same year, Iseki introduced another notion called BCI-algebra. Liu et al. [13] discussed the concept of BCI-implicative ideals in BCI-algebras. Dudek [2] introduced the class of medial BCI-algebras. In 1983, Komori [11] introduced the notion of BCC-algebras as a new class of algebras. Then Dudek [3, 5] studied BCC-algebras and discussed the number of subalgebras of finite BCC-algebras. Dudek in [4] also gave the construction of BCC-algebras. After the introduction of the concept of fuzzy sets by Zadeh [16], various researchers discussed the idea of fuzzification of ideals in BCK/BCI/BCC-algebras. Khalid and Ahmad [10] considered the fuzzification of H-ideals in BCI-algebras. Mustafa [15] introduced the concept of fuzzy implicative ideals in BCK-algebras. Zhan and Jun [17] discussed generalized fuzzy ideals in BCI-algebras. Dudek and Jun [6] applied the idea of fuzzy sets to ideals in BCC-algebras. Marty [14], in 1934 introduced the hyper structure theory at the 8th Congrass of Scandinavian Mathematicians. Jun et al. [9] applied the hyper structures to BCK-algebras by introducing the concept of a hyper BCK-algebras, which is a generalization of BCK-algebras. In this paper, we introduce the concept of fuzzification of (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras and discuss some of their properties.

2 Preliminaries

Let H be a non-empty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $P(H) - \emptyset$. For two subset A and B of H, denote by $A \circ B$ the set $\bigcup\{a \circ b \mid a \in A, b \in B\}$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition 2.1. [9] By a hyper BCK-algebra we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:
$(HK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$
$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$
$(HK3)$ $x \circ H \ll \{x\}$
$(HK4)$ $x \ll y$ and $y \ll x$ imply $x = y$
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A,\ B \subseteq H$, $A \ll B$ is defined by $\forall\ a \in A$, $\exists\ b \in B$ such that $a \ll b$. In such case we call "$\ll$" the hyper order in H.

Proposition 2.2. [9] *In any hyper BCK-algebra H, the following hold:*

(i) $x \circ 0 = \{x\}$
(ii) $x \circ y \ll x$
(iii) $0 \circ A = \{0\}$
(iv) $A \ll A$
(v) $A \subseteq B$ *implies* $A \ll B$
(vi) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$
(vii) $0 \ll x$
(viii) $0 \circ x = \{0\}$
(ix) $0 \circ 0 = \{0\}$
(x) $y \ll z$ *implies* $x \circ z \ll x \circ y$

for all $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H.

Let I be a non-empty subset of hyper BCK-algebra H and $0 \in I$. Then I is called a hyper BCK-subalgebra of H if $x \circ y \subseteq I$, for all $x, y \in I$, a weak hyper BCK-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a hyper BCK-ideal of H if $x \circ y \ll I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a strong hyper BCK-ideal of H if $x \circ y \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$, for all $x, y \in H$. I is said to be reflexive if $x \circ x \subseteq I$ for all $x \in H$.

Lemma 2.3. [9] *Let H be a hyper BCK-algebra. Then*
- *any reflexive hyper BCK-ideal of H is a strong hyper BCK-ideal of H.*
- *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H.*
- *any hyper BCK-ideal of H is a weak hyper BCK-ideal of H.*

Lemma 2.4. [8] *Let I be a reflexive hyper BCK-ideal of a hyper BCK-algebra H. Then $x \circ y \cap I \neq \emptyset$ implies $x \circ y \ll I$, $\forall x, y \in H$.*

Proposition 2.5. [8] *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$ then $A \subseteq I$.*

Definition 2.6. Let H be a hyper BCK-algebra. A non-empty subset $I \subseteq H$ containing 0 is called
- a weak hyper BCK-implicative ideal of H if
$$((x \circ y) \circ y) \circ z \subseteq I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$
- a hyper BCK-implicative ideal of H if
$$((x \circ y) \circ y) \circ z \ll I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$
- a strong hyper BCK-implicative ideal of H if
$$(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$

Theorem 2.7. *Every (weak, strong, reflexive) hyper BCK-implicative ideal of a hyper BCK-algebra H is a (weak, strong, reflexive) hyper BCK-ideal of H.*

Proof. Suppose that I is a hyper BCK-implicative ideal of H. Then for any $x, y, z \in H$
$$((x \circ y) \circ y) \circ z \ll I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$

Putting $y = 0$ and $z = y$ we get

$$((x \circ 0) \circ 0) \circ y \ll I \text{ and } y \in I \text{ imply } x \circ (0 \circ (0 \circ x)) \subseteq I.$$

$$\Rightarrow (x \circ y) \ll I \text{ and } y \in I \Rightarrow x \in I.$$

Hence I is a hyper BCK-ideal of H. □

The converse of theorem 2.7 is not true in general. It can be observed by the following example

Example 2.8. Let $H = \{0, 1, 2, 3\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1\}$

Take $I = \{0, 1\}$. Then I is a hyper BCK-ideal of H but it is not a hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \ll I$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
It can be observed from the above example that I is a weak hyper BCK-ideal of H but it not a weak hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \subseteq I$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
Also I is a strong hyper BCK-ideal of H but it is not a strong hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \cap I \neq \emptyset$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
Moreover it is clear that I is a reflexive hyper BCK-ideal of H but it is not a reflexive hyper BCK-implicative ideal of H.

Theorem 2.9. *Let H be a hyper BCK-algebra. Then*
(i) Every hyper BCK-implicative ideal of H is a weak hyper BCK-implicative ideal of H.
(ii) Every strong hyper BCK-implicative ideal of H is a hyper BCK-implicative ideal of H.
(iii) Every reflexive hyper BCK-implicative ideal of H is a strong hyper BCK-implicative ideal of H.

Proof. (i) Suppose that I is a hyper BCK-implicative ideal of H.
For any $x, y, z \in H$, let $((x \circ y) \circ y) \circ z \subseteq I$ and $z \in I$. Then $((x \circ y) \circ y) \circ z \subseteq I$ implies $((x \circ y) \circ y) \circ z \ll I$ (by Proposition 2.2(v)), which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a weak hyper BCK-implicative ideal of H.

(ii) Suppose that I is a strong hyper BCK-implicative ideal of H. Let $((x \circ y) \circ y) \circ z \ll I$ and $z \in I$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\, b \in I$ such that $a \ll b$. This implies $0 \in a \circ b$ and thus $(a \circ b) \cap I \neq \emptyset$. By Theorem 2.7, I is also a strong hyper BCK-ideal of H, therefore $(a \circ b) \cap I \neq \emptyset$ along with $b \in I$ implies $a \in I$, that is $((x \circ y) \circ y) \circ z \subseteq I$. Therefore $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset$, which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a hyper BCK-implicative ideal of H.

(iii) Suppose that I is a reflexive hyper BCK-implicative ideal of H. For any $x, y, z \in H$, let $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset$ and $z \in I$. Being a reflexive hyper BCK-implicative ideal, I is also a reflexive hyper BCK-ideal of H (by Theorem 2.7), therefore by Lemma 2.4, $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset \Rightarrow ((x \circ y) \circ y) \circ z \ll I$, which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a strong hyper BCK-implicative ideal of H. □

The converse of Theorem 2.9 may not be true. It can be observed by the following examples:

Example 2.10. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Take $I = \{0, 2\}$. Then I is a weak hyper BCK-implicative ideal of H but it is not a hyper BCK-implicative ideal of H because

$$((1 \circ 0) \circ 0) \circ 2 = \{0, 1\} \ll I \text{ and } 2 \in I \text{ but } 1 \circ (0 \circ (0 \circ 1)) = \{1\} \nsubseteq I.$$

Example 2.11. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Take $I = \{0, 1\}$. Then I is a hyper BCK-implicative ideal of H but it is not a strong hyper BCK-implicative ideal of H because

$$(((2 \circ 0) \circ 0) \circ 1) \cap I = \{1, 2\} \cap I \neq \emptyset \text{ and } 1 \in I \text{ but } 2 \circ (0 \circ (0 \circ 2)) = \{2\} \nsubseteq I.$$

Zadeh [16] defined fuzzy set μ in H as a function

$$\mu : H \to [0, 1]$$

Definition 2.12. [8] A fuzzy set μ of a hyper BCK-algebra H is called

- a fuzzy weak hyper BCK-ideal of H if for all $x, y \in H$,
$$\mu(0) \geq \mu(x) \geq min\ \{inf_{a \in x \circ y}\ \mu(a),\ \mu(y)\}$$
- a fuzzy hyper BCK-ideal of H if $x \ll y$ implies $\mu(x) \geq \mu(y)$ and for all $x, y \in H$,
$$\mu(x) \geq min\ \{inf_{a \in x \circ y}\ \mu(a),\ \mu(y)\}$$
- a fuzzy strong hyper BCK-ideal of H if for all $x, y \in H$,
$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(x) \geq min\ \{sup_{b \in x \circ y}\ \mu(b),\ \mu(y)\}$$
- a fuzzy reflexive hyper BCK-ideal of H if for all $x, y \in H$,
$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(y) \text{ and } \mu(x) \geq min\ \{sup_{b \in x \circ y}\ \mu(b),\ \mu(y)\}$$

Theorem 2.13. [8] *Let H be a hyper BCK-aglebra. Then*

- *Every fuzzy hyper BCK-ideal of H is a fuzzy weak hyper BCK-ideal of H.*
- *Every fuzzy strong hyper BCK-ideal of H is a fuzzy hyper BCK-ideal of H.*
- *Every fuzzy reflexive hyper BCK-ideal of H is a fuzzy strong hyper BCK-ideal of H.*

3 Fuzzy hyper BCK-implicative ideals

Now we introduce the notions of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras and discuss some of their properties.

Definition 3.1. Let H be hyper BCK-algebra . A fuzzy set μ in H is called

- a fuzzy weak hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

  $\mu(0) \geq \mu(x)$ and for all $t \in x \circ (y \circ (y \circ x))$,
  $$\mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\}$$

- a fuzzy hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

  $x \ll y$ implies $\mu(x) \geq \mu(y)$ and for all $t \in x \circ (y \circ (y \circ x))$,
  $$\mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\}$$

- a fuzzy strong hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

  $inf_{a\in x\circ x}\ \mu(a) \geq \mu(x)$ and for all $t \in x \circ (y \circ (y \circ x))$,
  $$\mu(t) \geq min\ \{sup_{b\in((x\circ y)\circ y)\circ z}\ \mu(b),\ \mu(z)\}$$

- a fuzzy reflexive hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

  $inf_{a\in x\circ x}\ \mu(a) \geq \mu(y)$ and for all $t \in x \circ (y \circ (y \circ x))$,
  $$\mu(t) \geq min\ \{sup_{b\in((x\circ y)\circ y)\circ z}\ \mu(b),\ \mu(z)\}$$

Theorem 3.2. *Let H be a hyper BCK-algebra. Then every fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H is a fuzzy (weak, strong, reflexive) hyper BCK-ideal of H.*

Proof. Let μ be a fuzzy hyper BCK-implicative ideal of H. Then for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$ we have,
$$\mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\}$$
Putting $y = 0$ and $z = y$ we get,
$$\mu(x) \geq min\ \{inf_{a\in((x\circ 0)\circ 0)\circ y}\ \mu(a),\ \mu(y)\}$$
which gives,
$$\mu(x) \geq min\ \{inf_{a\in x\circ y}\ \mu(a),\ \mu(y)\}$$
Thus μ is a fuzzy hyper BCK-ideal of H. □

The converse of Theorem 3.2 may not be true. It can be observed by considering the hyper BCK-algebra $H = \{0, 1, 2, 3\}$ defined by the table given in example (2.8). Define a fuzzy set μ in H by:

$$\mu(0) = \mu(1) = 1,\ \mu(2) = 0.5,\ \mu(3) = 0.3$$

Then μ is a fuzzy hyper BCK-ideal of H but it is not a fuzzy hyper BCK-implicative ideal of H because for $2 \in (2 \circ (3 \circ (3 \circ 2)))$

$$\mu(2) = 0.5 < 1 = min\ \{inf_{a\in((2\circ 3)\circ 3)\circ 0}\ \mu(a),\ \mu(0)\}$$

From above example it can be observed that μ is a fuzzy weak hyper BCK-ideal of H but it is not a fuzzy weak hyper BCK-implicative ideal of H.

Also μ is a fuzzy strong hyper BCK-ideal of H but it is not a fuzzy strong hyper BCK-implicative ideal of H because for $2 \in (2 \circ (3 \circ (3 \circ 2)))$

$$\mu(2) = 0.5 < 1 = min\ \{sup_{a\in((2\circ 3)\circ 3)\circ 0}\ \mu(a),\ \mu(0)\}$$

Moreover it is clear that μ is a fuzzy reflexive hyper BCK-ideal of H but it is not a fuzzy reflexive hyper BCK-implicative ideal of H.

Theorem 3.3. *Let H be a hyper BCK-algebra. Then*
(i) Every fuzzy hyper BCK-implicative ideal of H is a fuzzy weak hyper BCK-implicative ideal of H.
(ii) Every fuzzy Strong hyper BCK-implicative ideal of H is a fuzzy hyper BCK-implicative ideal of H.
(iii) Every fuzzy reflexive hyper BCK-implicative ideal of H is a fuzzy strong hyper BCK-implicative ideal of H.

Proof. (i) Let μ be a fuzzy hyper BCK-implicative ideal of H. Since every fuzzy hyper BCK-implicative ideal is a fuzzy hyper BCK-ideal (By Theorem 3.2) and every fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal (By Theorem 2.13), therefore μ is a fuzzy weak hyper BCK-ideal of H. Hence μ satisfies $\mu(0) \geq \mu(x)$ for all $x \in H$. Also being a fuzzy hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies:

$$\mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\}$$

Hence μ is a fuzzy weak hyper BCK-implicative ideal of H.

(ii) Suppose that μ is a fuzzy strong hyper BCK-implicative ideal of H. Since every fuzzy strong hyper BCK-implicative ideal is a fuzzy strong hyper BCK-ideal (by Theorem 3.2) and every fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal (by Theorem 2.13), therefore

μ is a fuzzy hyper BCK-ideal of H. Hence for any $x, y \in H$, if $x \ll y$ then $\mu(x) \geq \mu(y)$.

Also being a fuzzy strong hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{sup_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

Since $sup_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) \geq \mu(b)$, for all $b \in ((x \circ y) \circ y) \circ z$, therefore we get,

$$\mu(t) \geq min\ \{\mu(b),\ \mu(z)\}, \text{ for all } b \in ((x \circ y) \circ y) \circ z$$

Since $\mu(b) \geq inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c)$ for all $b \in ((x \circ y) \circ y) \circ z$, therefore we have, $\mu(t) \geq min\ \{\mu(b),\ \mu(z)\} \geq min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\}$, that is

$$\mu(t) \geq min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\}$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H.

(iii) Let μ be a fuzzy reflexive hyper BCK-implicative ideal of H. Then μ satisfies

$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(y), \text{ for all } x, y \in H$$

$$\Rightarrow inf_{a \in x \circ x}\ \mu(a) \geq \mu(x), \text{ for all } x \in H$$

Hence the first condition for μ to be a fuzzy strong hyper BCK-implicative ideal of H is satisfied. Also being a fuzzy reflexive hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{sup_{b \in ((x \circ y) \circ y) \circ z}\ \mu(b),\ \mu(z)\}$$

Hence μ is a fuzzy strong hyper BCK-implicative ideal of H. □

The converse of Theorem 3.3 may not be true. Consider the hyper BCK-algebra $H = \{0, 1, 2\}$ defined by the table given in example (2.10). Define a fuzzy set μ in H by:

$$\mu(0) = \mu(2) = 1,\ \mu(1) = 0$$

Then μ is a fuzzy weak hyper BCK-implicative ideal of H but it is not a fuzzy hyper BCK-implicative ideal of H because:

$$1 \ll 2 \text{ but } \mu(1) = 0 < 1 = \mu(2)$$

Example 3.4. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Define a fuzzy set μ in H by:

$$\mu(0) = \mu(1) = 1,\ \mu(2) = 0$$

Then μ is a fuzzy hyper BCK-implicative ideal of H but it is not a fuzzy strong hyper BCK-implicative ideal of H because for $2 \in (2 \circ (2 \circ (2 \circ 2)))$

$$\mu(2) = 0 < 1 = min\ \{sup_{a\in(((2\circ 2)\circ 2)\circ 0)}\ \mu(a),\ \mu(0)\}$$

Let μ be a fuzzy set in a hyper BCK-algebra H. Then the set defined by $\mu_t = \{x \in H : \mu(x) \geq t\}$, where $t \in [0, 1]$, is called a level subset of H.

Theorem 3.5. *Let μ be a fuzzy set in a hyper BCK-algebra H. Then μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a (weak, strong, reflexive) hyper BCK-implicative ideal of H.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Since $\mu_t \neq \emptyset$, so for any $x \in \mu_t$, $\mu(x) \geq t$. Since every fuzzy hyper BCK-implicative ideal is also a fuzzy weak hyper BCK-implicative ideal (by Theorem 3.3(i)), so μ is also a fuzzy weak hyper BCK-implicative ideal of H. Thus $\mu(0) \geq \mu(x) \geq t$, for all $x \in H$, which implies $0 \in \mu_t$.

Let $((x \circ y) \circ y) \circ z \ll \mu_t$ and $z \in \mu_t$, for some $x, y, z \in H$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\ b \in \mu_t$ such that $a \ll b$. So $\mu(a) \geq \mu(b) \geq t$, for all $a \in ((x \circ y) \circ y) \circ z$. Thus $inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a) \geq t$. Also $\mu(z) \geq t$, as $z \in \mu_t$. Therefore for all $v \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(v) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\} \geq min\ \{t, t\} = t$$
$$\Rightarrow v \in \mu_t, \text{ for all } v \in x \circ (y \circ (y \circ x))$$
$$\Rightarrow x \circ (y \circ (y \circ x)) \subseteq \mu_t$$

Hence μ_t is hyper BCK-implicative ideal of H.

Conversely suppose that $\mu_t \neq \emptyset$ is a hyper BCK-implicative ideal of H for all $t \in [0,1]$. Let $x \ll y$ for some $x, y \in H$ and put $\mu(y) = t$. Then $y \in \mu_t$. So $x \ll y \in \mu_t \Rightarrow x \ll \mu_t$. Being a hyper BCK-implicative ideal, μ_t is also a hyper BCK-ideal of H (by Theorem (2.7)) therefore by Proposition 2.5, $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. That is $x \ll y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in H$.

Moreover for any $x, y, z \in H$, let $d = min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu(c),\ \mu(z)\}$. Then $\mu(z) \geq d \Rightarrow z \in \mu_d$ and for all $e \in ((x \circ y) \circ y) \circ z$, $\mu(e) \geq inf_{c\in((x\circ y)\circ y)\circ z}\ \mu(c) \geq d$, which implies $e \in \mu_d$. Thus $((x \circ y) \circ y) \circ z \subseteq \mu_d$. By Proposition 2.2(v), $((x \circ y) \circ y) \circ z \subseteq \mu_d \Rightarrow ((x \circ y) \circ y) \circ z \ll \mu_d$, which along with $z \in \mu_d$ implies $x \circ (y \circ (y \circ x)) \subseteq \mu_d$. Hence for all $u \in x \circ (y \circ (y \circ x))$, we get

$$\mu(u) \geq d = min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu(c),\ \mu(z)\}.$$

Thus μ is a fuzzy hyper BCK-implicative ideal of H. □

Theorem 3.6. *If μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H then the set $A = \{x \in H \mid \mu(x) = \mu(0)\}$ is a (weak, strong, reflexive) hyper BCK-implicative ideal of H.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Clearly $0 \in A$. Let $((x \circ y) \circ y) \circ z \ll A$ and $z \in A$ for any $x, y, z \in H$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\, b \in A$ such that $a \ll b$. Therefore $\mu(a) \geq \mu(b) = \mu(0)$. But being a fuzzy hyper BCK-implicative ideal, μ is also a fuzzy weak hyper BCK-implicative ideal of H (by Theorem 3.3(i)), so μ satisfies $\mu(0) \geq \mu(v)$, for all $v \in H$. This implies $\mu(0) \geq \mu(a)$, for all $a \in ((x \circ y) \circ y) \circ z$. Therefore $\mu(a) = \mu(0)$, for all $a \in ((x \circ y) \circ y) \circ z$, that is, $inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a) = \mu(0)$. Also $\mu(z) = \mu(0)$. Being a fuzzy hyper BCK-implicative ideal, for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\} = min\ \{\mu(0),\ \mu(0)\} = \mu(0)$$

Since $\mu(0) \geq \mu(v)$, for all $v \in H$, therefore $\mu(t) = \mu(0)$, for all $t \in x \circ (y \circ (y \circ x))$. Thus $x \circ (y \circ (y \circ x)) \subseteq A$.

Hence A is a hyper BCK-implicative ideal of H. □

The transfer principle for fuzzy sets described in [12] suggest the following theorem.

Theorem 3.7. *For any subset A of a hyper BCK-algebra H, let μ be a fuzzy set in H defined by:*

$$\mu(x) = \begin{cases} t & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

for all $x \in H$, where $t \in (0,1]$. Then A is a (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H.

Proof. Suppose that A is a hyper BCK-implicative ideal of H. Let $x \ll y$ for some $x, y \in H$ and put $\mu(y) = t$. Then $y \in \mu_t$. So $x \ll y \in \mu_t \Rightarrow x \ll \mu_t$. Being a hyper BCK-implicative ideal, μ_t is also a hyper BCK-ideal of H (by Theorem (2.7)) therefore by Proposition 2.5, $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. That is $x \ll y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in H$

Moreover for any $x, y, z \in H$,
If $((x \circ y) \circ y) \circ z \ll A$ and $z \in A$ then $x \circ (y \circ (y \circ x)) \subseteq A$. Since A is a hyper BCK-implicative ideal of H, so by Proposition 2.5, $((x \circ y) \circ y) \circ z \subseteq A$. Thus $\mu(a) = t$, for all $a \in ((x \circ y) \circ y) \circ z$ which implies $inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) = t$. Also $\mu(z) = t$. Since $x \circ (y \circ (y \circ x)) \subseteq A$, for all $u \in x \circ (y \circ (y \circ x))$, we have

$$\mu(u) = t = min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

If $((x \circ y) \circ y) \circ z \not\ll A$ and $z \notin A$ then

$$min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} = 0 \leq \mu(u), \text{ for all } u \in x \circ (y \circ (y \circ x))$$

If $((x \circ y) \circ y) \circ z \not\ll A$ and $z \in A$ (OR) If $((x \circ y) \circ y) \circ z \ll A$ and $z \notin A$

Then in both of these cases we have

$$min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} = 0 \leq \mu(u), \text{ for all } u \in x \circ (y \circ (y \circ x))$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H.

Conversely suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Then by Theorem 3.5, for all $t \in (0,1]$, $\mu_t = A$ is a hyper BCK-implicative ideal of H. □

For a family $\{\mu_i \mid i \in I\}$ of fuzzy sets in a non-empty set X, define the join $\vee_{i\in I}\ \mu_i$ and meet $\wedge_{i\in I}\ \mu_i$ as follows:

$$(\vee_{i\in I}\ \mu_i)(x) = sup_{i\in I}\ \mu_i(x)$$
$$(\wedge_{i\in I}\ \mu_i)(x) = inf_{i\in I}\ \mu_i(x)$$

for all $x \in X$, where I is any indexing set.

Theorem 3.8. *The family of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals of a hyper BCK-algebra H is a completely distributive lattice with respect to join and meet.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy hyper BCK-implicative ideals of H. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\vee_{i\in I}\ \mu_i$ and $\wedge_{i\in I}\ \mu_i$ are fuzzy hyper BCK-implicative ideals of H.

For any $x, y \in H$, if $x \ll y$ then

$$(\vee_{i\in I}\ \mu_i)(x) = sup_{i\in I}\ \mu_i(x) \geq sup_{i\in I}\ \mu_i(y) = (\vee_{i\in I}\ \mu_i)(y)$$
$$\Rightarrow (\vee_{i\in I}\ \mu_i)(x) \geq (\vee_{i\in I}\ \mu_i)(y)$$

Moreover, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, we have

$$(\vee_{i\in I}\ \mu_i)(t) = sup_{i\in I}\ \mu_i(t) \geq sup_{i\in I}\ [min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu_i(a),\ \mu_i(z)\}]$$
$$= min\ \{sup_{i\in I}\ (inf_{a\in((x\circ y)\circ y)\circ z}\ \mu_i(a)),\ sup_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ (sup_{i\in I}\ \mu_i(a)),\ sup_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ ((\vee_{i\in I}\ \mu_i)(a)),\ (\vee_{i\in I}\ \mu_i)(z)\}$$
$$\Rightarrow (\vee_{i\in I}\ \mu_i)(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ ((\vee_{i\in I}\ \mu_i)(a)),\ (\vee_{i\in I}\ \mu_i)(z)\}$$

Hence $\vee_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.

Now we prove that $\wedge_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.
For any $x, y \in H$ we have, if $x \ll y$ then

$$(\wedge_{i\in I}\ \mu_i)(x) = inf_{i\in I}\ \mu_i(x) \geq inf_{i\in I}\ \mu_i(y) = (\wedge_{i\in I}\ \mu_i)(y)$$
$$\Rightarrow (\wedge_{i\in I}\ \mu_i)(x) \geq (\wedge_{i\in I}\ \mu_i)(y)$$

Moreover, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, we have

$$(\wedge_{i\in I}\ \mu_i)(t) = inf_{i\in I}\ \mu_i(t) \geq inf_{i\in I}\ [min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ \mu_i(b),\ \mu_i(z)\}]$$
$$= min\ \{inf_{i\in I}\ (inf_{b\in((x\circ y)\circ y)\circ z}\ \mu_i(b)),\ inf_{i\in I}\ (\mu_i(z))\}$$

$$= min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ (inf_{i\in I}\ \mu_i(b)),\ inf_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ ((\wedge_{i\in I}\ \mu_i)(b)),\ (\wedge_{i\in I}\ \mu_i)(z)\}$$
$$\Rightarrow (\wedge_{i\in I}\ \mu_i)(t) \geq min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ ((\wedge_{i\in I}\ \mu_i)(b)),\ (\wedge_{i\in I}\ \mu_i)(z)\}$$

Hence $\wedge_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.

Thus the family of fuzzy hyper BCK-implicative ideals of H is a completely distributive lattice with respect to join and meet. □

Let X and Y be hyper BCK-algebras. A mapping $f : X \to Y$ is called a hyper homomorphism if

(i) $f(0) = 0$
(ii) $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in X$.

Theorem 3.9. *Let $f : X \to Y$ be an onto hyper homomorphism from a hyper BCK-algebra X to a hyper BCK-algebra Y. If ν is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of Y then the hyper homomorphic pre-image μ of ν under f is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of X.*

Proof. Suppose that ν is a fuzzy hyper BCK-implicative ideal of Y. Since μ is a hyper homomorphic pre-image of ν under f then μ is defined by $\mu = \nu \circ f$ that is $\mu(x) = \nu(f(x))$ for all $x \in X$.

For any $x, y \in X$ and $f(x), f(x) \in Y$
If $x \ll y$ then $0 \in x \circ y$, which implies $f(0) \in f(x \circ y)$

$$\Rightarrow 0 \in f(x) \circ f(y) \Rightarrow f(x) \ll f(y)$$
$$\Rightarrow \nu(f(x)) \geq \nu(f(y)) \Rightarrow \mu(x) \geq \mu(y)$$
$$\text{that is, } x \ll y \Rightarrow \mu(x) \geq \mu(y), \text{ for all } x, y \in X$$

Now for all $t \in x \circ (y \circ (y \circ x))$, $f(t) \in f(x \circ (y \circ (y \circ x))) = f(x) \circ (f(y) \circ (f(y) \circ f(x)))$, where $x, y \in X$ and $f(x), f(y) \in Y$, we have

$$\mu(t) = \nu(f(t)) \geq min\ \{inf_{f(a)\in((f(x)\circ f(y))\circ f(y))\circ z'}\ \nu(f(a)),\ \nu(z')\}$$

where $z' \in Y$. Since $f : X \to Y$ is an onto hyper homomorphism, so for $z' \in Y$, $\exists\ z \in X$

such that $f(z) = z'$. Hence we get

$$\mu(t) \geq min\ \{inf_{f(a)\in((f(x)\circ f(y))\circ f(y))\circ f(z)=f(((x\circ y)\circ y)\circ z)}\ \nu(f(a)),\ \nu(f(z))\}$$

$$\Rightarrow \mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\} \text{ for all } x, y, z \in X$$

Hence μ is a fuzzy hyper BCK-implicative ideal of X. □

4 Product of fuzzy hyper BCK-implicative ideals

Definition 4.1. [1] Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ are hyper BCK-algebras and $H = H_1 \times H_2$. We define a hyper operation " $\circ$ " on H by

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ a_2,\ b_1 \circ b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$ by (A, B) we mean

$$(A, B) = \{(a, b) : a \in A, b \in B\}$$

and $0 = (0_1, 0_2)$ and a hyper order " $\ll$ " on H by

$$(a_1, b_1) \ll (a_2, b_2) \Leftrightarrow a_1 \ll a_2 \text{ and } b_1 \ll b_2$$

Thus $(H, \circ, 0)$ is a hyper BCK-algebra.

Let μ and ν be fuzzy sets in hyper BCK-algebras H_1 and H_2 respectively. Then $\mu \times \nu$, the product of μ and ν of $H = H_1 \times H_2$ is defined as

$$(\mu \times \nu)((x, y)) = min\ \{\mu(x), \nu(y)\}$$

From now on, let H_1 and H_2 are hyper BCK-algebras and let $H = H_1 \times H_2$.

Definition 4.2. Let μ be a fuzzy set in H. Then fuzzy sets μ_1 and μ_2 on H_1 and H_2 respectively, are defined as

$$\mu_1(x) = \mu((x, 0)), \quad \mu_2(y) = \mu((0, y))$$

Theorem 4.3. *Let μ be a fuzzy set in H. If μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H, then $\mu = \mu_1 \times \mu_2$, where μ_1 and μ_2 are fuzzy sets on H_1 and H_2 respectively.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H.
Then for any $(x, u),\ (y, v),\ (z, w) \in H$, where $x, y, z \in H_1$ and $u, v, w \in H_2$ and for all

$(a,b) \in (x,u) \circ ((y,v) \circ ((y,v) \circ (x,u))) = (x \circ (y \circ (y \circ x)),\ u \circ (v \circ (v \circ u)))$, we have

$$\mu((a,b)) \geq min\ \{inf_{(c,d)\in(((x,u)\circ(y,v))\circ(y,v))\circ(z,w)}\ \mu((c,d)),\ \mu((z,w))\}$$

Putting $y = v = z = d = 0$ and $w = u$, we get

$$\begin{aligned} \mu((x,u)) &\geq min\ \{inf_{(c,0)\in(((x,u)\circ(0,0))\circ(0,0))\circ(0,u)}\ \mu((c,0)),\ \mu((0,u))\} \\ \Rightarrow \mu((x,u)) &\geq min\ \{inf_{(c,0)\in(x,\ u\circ u)}\ \mu((c,0)),\ \mu((0,u))\} \\ \Rightarrow \mu((x,u)) &\geq min\ \{\mu_1(x),\ \mu_2(u)\} \\ \Rightarrow \mu((x,u)) &\geq (\mu_1 \times \mu_2)((x,u)) \\ \Rightarrow \mu_1 \times \mu_2 &\subseteq \mu \qquad (1) \end{aligned}$$

Conversely, since $(x,0) \ll (x,u)$ and $(0,u) \ll (x,u)$

$$\Rightarrow \mu((x,0)) \geq \mu((x,u)) \text{ and } \mu((0,u)) \geq \mu((x,u))$$

Thus we have

$$\begin{aligned} (\mu_1 \times \mu_2)((x,u)) &= min\ \{\mu_1(x),\ \mu_2(u)\} = min\ \{\mu(x,0),\ \mu(0,u)\} \\ &\geq min\ \{\mu(x,u),\ \mu(x,u)\} = \mu(x,u) \\ \Rightarrow (\mu_1 \times \mu_2)((x,u)) &\geq \mu(x,u) \\ \Rightarrow \mu &\subseteq \mu_1 \times \mu_2 \qquad (2) \end{aligned}$$

Hence from (1) and (2) we have, $\mu_1 \times \mu_2 = \mu$ □

Theorem 4.4. *Let $\mu = \mu_1 \times \mu_2$ be a fuzzy set in H. Then $\mu = \mu_1 \times \mu_2$ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ_1 and μ_2 are fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals of H_1 and H_2 respectively.*

Proof. Let μ be a fuzzy hyper BCK-implicative ideal of H and let $x_1 \ll x_2$ for some $x_1, x_2 \in H_1$. Then $(x_1, 0) \ll (x_2, 0)$ which implies $\mu((x_1,0)) = \mu_1(x_1) \geq \mu((x_2,0)) = \mu_1(x_2)$, that is, $\mu_1(x_1) \geq \mu_1(x_2)$

Moreover for any $x_1, y_1, z_1 \in H_1$, let $t = min\ \{inf_{a\in((x_1\circ y_1)\circ y_1)\circ z_1}\ \mu_1(a),\ \mu_1(z_1)\}$

Then for all $b \in ((x_1 \circ y_1) \circ y_1) \circ z_1$, $\mu_1(b) \geq inf_{a\in((x_1\circ y_1)\circ y_1)\circ z_1}\ \mu_1(a) \geq t$ and $\mu_1(z_1) \geq t$

$\Rightarrow \mu((b,0)) \geq t$ and $\mu((z_1,0)) \geq t$, for all $(b,0) \in (((x_1,0) \circ (y_1,0)) \circ (y_1,0)) \circ (z_1,0)$

$\Rightarrow (b,0) \in \mu_t$ and $(z_1,0) \in \mu_t$, for all $(b,0) \in (((x_1,0) \circ (y_1,0)) \circ (y_1,0)) \circ (z_1,0)$

$$\Rightarrow (((x_1,0)\circ(y_1,0))\circ(y_1,0))\circ(z_1,0)\subseteq \mu_t \text{ and } (z_1,0)\in\mu_t$$

Since by Theorem 3.5, $\mu_t \neq \emptyset$ is a hyper BCK-implicative ideal of H and so is a weak hyper BCK-implicative ideal of H (by Theorem 2.9(i)). Thus

$$((((x_1,0)\circ(y_1,0))\circ(y_1,0))\circ(z_1,0))\subseteq\mu_t \text{ and } (z_1,0)\in\mu_t \text{ imply}$$
$$(x_1,0)\circ((y_1,0)\circ((y_1,0)\circ(x_1,0)))\subseteq\mu_t$$

Therefore $\mu((s,0)) \geq t$, for all $(s,0) \in (x_1,0)\circ((y_1,0)\circ((y_1,0)\circ(x_1,0))) = (x_1\circ(y_1\circ(y_1\circ x_1)),\ 0)$

$$\Rightarrow \mu_1(s) \geq t = min\ \{inf_{a\in((x_1\circ y_1)\circ y_1)\circ z_1}\ \mu_1(a),\ \mu_1(z_1)\},$$
$$\text{for all } s\in x_1\circ(y_1\circ(y_1\circ x_1))$$

Hence μ_1 is a fuzzy hyper BCK-implicative ideal of H_1.

Similarly we can prove that μ_2 is a fuzzy hyper BCK-implicative ideal of H_2.

Conversely suppose that μ_1 and μ_2 are fuzzy hyper BCK-implicative ideals of H_1 and H_2 respectively.

For any $(x,u),(y,v)\in H$, where $x,y\in H_1$ and $u,v\in H_2$, let $(x,u)\ll(y,v)$
Since $(x,u)\ll(y,v)\Leftrightarrow x\ll y$ and $u\ll v$

$$\Rightarrow \mu_1(x)\geq\mu_1(y) \text{ and } \mu_2(u)\geq\mu_2(v)$$
$$\Rightarrow min\ \{\mu_1(x),\ \mu_2(u)\}\geq min\ \{\mu_1(y),\ \mu_2(v)\}$$
$$\Rightarrow (\mu_1\times\mu_2)((x,u))\geq(\mu_1\times\mu_2)((y,v))$$
$$\Rightarrow \mu((x,u))\geq\mu((y,v))$$
$$\text{Thus } (x,u)\ll(y,v)\Rightarrow\mu((x,u))\geq\mu((y,v))$$

Moreover for any $(x,u),\ (y,v),\ (z,w)\in H$, where $x,y,z\in H_1$ and $u,v,w\in H_2$ and for all $(a,b)\in(x,u)\circ((y,v)\circ((y,v)\circ(x,u)))=(x\circ(y\circ(y\circ x)),\ u\circ(v\circ(v\circ u)))$, we have

$$\mu((a,b))=(\mu_1\times\mu_2)((a,b))=min\ \{\mu_1(a),\ \mu_2(b)\}$$

$$\geq min\ [min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu_1(c),\ \mu_1(z)\},\ min\ \{inf_{d\in((u\circ v)\circ v)\circ w}\ \mu_2(d),\ \mu_2(w)\}]$$

$$= min\ [min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu_1(c),\ inf_{d\in((u\circ v)\circ v)\circ w}\ \mu_2(d)\},\ min\ \{\ \mu_1(z),\ \mu_2(w)\}]$$

$$= min\ [inf_{c\in((x\circ y)\circ y)\circ z,\ d\in((u\circ v)\circ v)\circ w}\ \{min\ \{\mu_1(c),\ \mu_2(d)\}\},\ min\ \{\ \mu_1(z),\ \mu_2(w)\}]$$

$$= min\ \{inf_{(c,d)\in(((x\circ y)\circ y)\circ z,\ ((u\circ v)\circ v)\circ w)}\ (\mu_1\times\mu_2)((c,d)),\ (\mu_1\times\mu_2)((z,w))\}$$

$$= min\ \{inf_{(c,d)\in(((x\circ y)\circ y)\circ z,\ ((u\circ v)\circ v)\circ w)}\ \mu((c,d)),\ \mu((z,w))\}$$

$$\Rightarrow \mu((a,b)) \geq min\ \{inf_{(c,d)\in(((x,u)\circ(y,v))\circ(y,v))\circ(z,w)}\ \mu((c,d)),\ \mu((z,w))\}$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H. □

5 CONCLUSION

Every (fuzzy) reflexive hyper BCK-implicative ideal of a hyper BCK-algebra H is a (fuzzy) strong hyper BCK-implicative ideal of H and every (fuzzy) strong hyper BCK-implicative ideal of H is a (fuzzy) hyper BCK-implicative ideal of H, each of which in turn is a (fuzzy) weak hyper BCK-implicative ideal of H. Moreover a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H is a fuzzy (weak, strong, reflexive) hyper BCK-ideal of H. The hyper homomorphic pre-image of a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is also a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal in any onto hyper homomorphism of two hyper BCK-algebras. The product of two fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals is also a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal.

References

[1] R.A. Borzooei, A. Hasankhani, M.M. Zahedi, Y.B. Jun: On hyper K-algebras, Math. Japonica 52 (1) (2000), 113-121.

[2] W.A. Dudek: On medial BCI-algebras, Prace Naukowe WSP w Czestochowie, ser. Matematyka 1 (1987), 25-33.

[3] W.A. Dudek: On BCC-algebras, Logique et Analyse 129-130 (1990), 103-111.

[4] W.A. Dudek: On constructions of BCC-algebras, Selected Papers on BCK and BCI-algebras 1 (1992), 93-96.

[5] W.A. Dudek: Subalgebras in finite BCC-algebras, Bul. Inst. Math. Acad. Sinica 28 (2000), 201-206.

[6] W.A. Dudek and Y.B. Jun: Fuzzifications of ideals in BCC-algebras, Glasnik Mat. 36(56) (2001), 127-138.

[7] Y. Imai, K. Iseki: On axioms of propositional calculi, XIV Proc., Jpn. Acad. 42 (1966), 19-22.

[8] Y.B. Jun, X.L. Xin: Fuzzy hyper BCK-ideals of hyper BCK-algebras, Sci. Math. Jpn. 53 (2) (2001), 415-422.

[9] Y.B. Jun, M.M. Zahedi, X.L. Xin, R. A. Borzooei: On hyper BCK-algebras, Ital. J. Pure Appl. Math. 8 (2000), 127-136.

[10] H.M. Khalid, B. Ahmad: Fuzzy H-ideals in BCI-algebras, Fuzzy Sets and Systems 101 (1999), 153-158.

[11] Y. Komori: The variety generated by BCC-algebras is finitely based, Reports Fac. Sci. Shizuoka Univ. 17 (1983), 13-16.

[12] M. Kondo, W.A. Dudek: On the transfer principle in fuzzy theory, Mathware and Soft Computing 12 (2005), 41-55.

[13] Y.L. Liu, Y. Xu, J. Meng: BCI-implicative ideals of BCI-algebras, Information Sciences 177 (2007), 49874996.

[14] F. Marty: Sur une generalization de la notion de groupe, 8th Congress Math. Scandinaves, Stockholm 1934, 45-49.

[15] S.M. Mostafa: Fuzzy implicative ideals in BCK-algebras, Fuzzy Sets and Systems 87 (1997), 361-368.

[16] L. A Zadeh: Fuzzy sets, Inf. Control 8 (1965), 338-353.

[17] J. Zhan and Y.B. Jun: Generalized Fuzzy Ideals of BCI-Algebras, Bull. Malays. Math. Sci. Soc. (2) 32 (2) (2009), 119130.

Communicated by *Hiroaki Ishii*

Muhammad Touqeer
touqeer-fareed@yahoo.com
M. Aslam Malik
malikpu@yahoo.com

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus,
Lahore-54590, Pakistan.

DEFORMATIONS OF FINITE HYPERGROUPS

Satoshi Kawakami, Tatsuya Tsurii and Satoe Yamanaka
Received 14 November 2014

Abstract. The purpose of the present paper is to introduce q-deformations of finite groups of low order, for examples, cyclic groups, symmetric groups, dihedral groups and the quaternion group in the category of hypergroups. Moreover we discuss q-deformations of certain finite hypergroups.

1 Introduction

We investigate q-deformations of finite groups and finite hypergroups in the category of hypergroups. It is known that there is no q-deformations of finite groups in the category of quantum groups ([24]). However we introduce that there are many q-deformations of finite groups in the category of hypergroups.

Hypergroups $\mathbb{Z}_q(2)$ of order two with a parameter q $(0 < q \leq 1)$ are interpreted as q-deformations of the cyclic group $\mathbb{Z}_2$. This fact is our motivation that we started to investigate q-deformations of finite groups and finite hypergroups.

In section 3, we discuss q-deformations of the cyclic group $\mathbb{Z}_3$ of order three and the cyclic group $\mathbb{Z}_4$ of order four. In section 4, we discuss q-deformations of the symmetric group S_3, the dihedral group D_4 and quaternion group Q_4. These q-deformations are obtained by applying a notion of a semi-direct product hypergroup introduced by H. Heyer and S. Kawakami (see [5]).

Moreover we study q-deformations of certain finite hypergroups of low order, the orbital hypergroups $\mathcal{K}^\alpha(\mathbb{Z}_3)$ of $\mathbb{Z}_3$ and $\mathcal{K}^\alpha(\mathbb{Z}_4)$ of $\mathbb{Z}_4$, the character hypergroups $\mathcal{K}(\widehat{S_3})$ of S_3, $\mathcal{K}(\widehat{D_4})$ of D_4 and $\mathcal{K}(\widehat{Q_4})$ of Q_4, the conjugacy class hypergroups $\mathcal{K}(S_3)$ of S_3, $\mathcal{K}(D_4)$ of D_4 and $\mathcal{K}(Q_4)$ of Q_4 in section 5.

2 Preliminaries

For a finite set $K = \{c_0, c_1, \cdots, c_n\}$, we denote by $M^b(K)$ and $M^1(K)$, the set of all complex valued measures on K and the set of all non-negative probability measures on K respectively, namely

$$M^b(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} \ : \ a_j \in \mathbb{C} \ \ (j = 0, 1, 2, \cdots, n) \right\},$$

$$M^1(K) := \left\{ \sum_{j=0}^{n} a_j \delta_{c_j} \ : \ a_j \geq 0 \ \ (j = 0, 1, 2, \cdots, n), \ \sum_{j=0}^{n} a_j = 1 \right\}$$

where the symbol δ_c stands for the Dirac measure in $c \in K$. For $\mu = a_0\delta_{c_0} + a_1\delta_{c_1} + \cdots + a_n\delta_{c_n} \in M^b(K)$, the *support* of μ is

$$\mathrm{supp}(\mu) := \{c_j \in K \ : \ a_j \neq 0 \ (j = 0, 1, 2, \cdots, n)\}.$$

2010 *Mathematics Subject Classification.* 20N20, 20B05, 20C05.
Key words and phrases. Deformation, Finite group, Hypergroup.

Axiom A *finite hypergroup* $K = (K, M^b(K), \circ, *)$ consists of a finite set $K = \{c_0, c_1, \cdots, c_n\}$ together with an associative product (called convolution) $\circ$ and an involution $*$ in $M^b(K)$ satisfying the following conditions.

(1) The space $(M^b(K), \circ, *)$ is an associative $*$-algebra with unit δ_{c_0}.
(2) For $c_i, c_j \in K$, the convolution $\delta_{c_i} \circ \delta_{c_j}$ belongs to $M^1(K)$.
(3) There exists an involutive bijection $c_i \mapsto c_i^*$ on K such that $\delta_{c_i^*} = \delta_{c_i}^*$.
Moreover $c_j = c_i^*$ if and only if $c_0 \in \mathrm{supp}(\delta_{c_i} \circ \delta_{c_j})$ for all $c_i, c_j \in K$.

A finite hypergroup K is called *commutative* if the convolution $\circ$ on $M^b(K)$ is commutative.

Let K and L be finite hypergroups. A mapping $\varphi : K \to L$ is called a (*hypergroup*) *homomorphism* of K into L if there exists a $*$-homomorphism $\tilde{\varphi}$ of $M^b(K)$ into $M^b(L)$ as $*$-algebras such that $\delta_{\varphi(c)} = \tilde{\varphi}(\delta_c)$. If $\tilde{\varphi}$ is bijective, φ is called an *isomorphism* of K onto L. In the case that $L = K$, an isomorphism $\varphi : K \to K$ is called an *automorphism* of K. The set of all automorphisms of K becomes a group and it is denoted by $\mathrm{Aut}(K)$. Let G be a finite group. A homomorphism $\alpha : G \to \mathrm{Aut}(K)$ is called an action of G on K.

For a commutative hypergroup K, a complex-valued function χ on K is called a *character* if χ is linearly extendable on $M^b(K)$ to be $\tilde{\chi}(\delta_{c_i}) = \chi(c_i)$ and satisfying that $\tilde{\chi}(\delta_{c_0}) = 1$, $\tilde{\chi}(\delta_{c_i} \circ \delta_{c_j}) = \tilde{\chi}(\delta_{c_i})\tilde{\chi}(\delta_{c_j})$ and $\tilde{\chi}(\delta_{c_i}^*) = \overline{\tilde{\chi}(\delta_{c_i})}$ for all $c_i, c_j \in K$. We denote the trivial character by χ_0. Let $\hat{K}$ be the set of all characters of K. A convolution on $\hat{K}$ is defined by multiplication of functions on K. Then $\hat{K}$ becomes a signed hypergroup and the duality $\hat{\hat{K}} \cong K$ holds.

Conjugacy class hypergroup Let G be a finite group. For $g \in G$, put $\alpha_g(k) = Ad_g(k) = gkg^{-1}$ $(k \in G)$. Then α is an action of G on G. Hence we obtain the orbital hypergroup $\mathcal{K}^\alpha(G)$ which we denote by $\mathcal{K}(G)$ which is called a conjugacy class hypergroup of G.

Character hypergroup For a finite group G, $\hat{G} = \{\pi_0, \pi_1, \cdots, \pi_m\}$ is the set of the all equivalence classes of irreducible representations of G. For $\pi_j \in \hat{G}$, a character χ_j associated with π_j is defined by

$$\chi_j(g) = \frac{1}{\dim \pi_j} \mathrm{tr}(\pi_j(g)).$$

Then $\mathcal{K}(\hat{G}) = \{\chi_0, \chi_1, \cdots, \chi_m\}$ becomes a commutative hypergroup with unit χ_0 by the multiplication of functions on G.

Hypergroup join For two finite hypergroups $H = \{h_0, h_1, \cdots, h_m\}$ and $L = \{\ell_0, \ell_1, \cdots, \ell_k\}$, a hypergroup join

$$H \vee L = \{h_0, h_1, \cdots, h_m, \ell_1, \cdots, \ell_k\}$$

is defined by the convolution $\diamond$ whose structure equations are

$$\delta_{h_i} \diamond \delta_{h_j} = \delta_{h_i} \circ \delta_{h_j}, \quad \delta_{h_i} \diamond \delta_{\ell_j} = \delta_{\ell_j},$$
$$\delta_{\ell_i} \diamond \delta_{\ell_j} = \delta_{\ell_i} \circ \delta_{\ell_j} \text{ when } \ell_j \neq \ell_i^*,$$
$$\delta_{\ell_i} \diamond \delta_{\ell_i}^* = n_i^0 \omega(H) + \sum_{j=1}^{k} n_i^j \delta_{\ell_j}$$

where $\delta_{\ell_i} \circ \delta_{\ell_i}^* = n_i^0 \delta_{\ell_0} + \sum_{j=1}^k n_i^j \delta_{\ell_j}$ and $\omega(H)$ is the normalized Haar measure of H.

3 Deformations of finite abelian groups

Let $K = \{c_0, c_1\}$ be a hypergroup of order two. Then the structure of K is determined by

$$\delta_{c_1} \circ \delta_{c_1} = q\delta_{c_0} + (1-q)\delta_{c_1}$$

where $0 < q \leq 1$. We denote it by $\mathbb{Z}_q(2)$ which is interpreted as a q-deformation of $\mathbb{Z}_2$. Stimulating by this fact, we have started to study q-deformations of finite groups.

3.1 Deformation $\mathbb{Z}_q(3)$ of $\mathbb{Z}_3$

First of all we discuss a q-deformation of $\mathbb{Z}_3$. It is easy to check the following proposition directly and this fact is also described in the paper ([19], [23] and [25]).

Proposition 3.1 Let $K = \{c_0, c_1, c_2\}$ be a hypergroup of order three. For each q $(0 < q \leq 1)$ there exists a unique hypergroup of order three such that $\delta_{c_1}^* = \delta_{c_2}$ and $\delta_{c_1} \circ \delta_{c_2} = q\delta_{c_0} + a_1\delta_{c_1} + a_2\delta_{c_2}$.

We denote the above K by $\mathbb{Z}_q(3)$, which is interpreted as a q-deformation of $\mathbb{Z}_3$. The structure equations of $\mathbb{Z}_q(3) = \{c_0, c_1, c_2\}$ $(0 < q \leq 1)$ are determined by

$$\begin{aligned}
\delta_{c_1} \circ \delta_{c_2} &= q\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}, \\
\delta_{c_1} \circ \delta_{c_1} &= \frac{1-q}{2}\delta_{c_1} + \frac{1+q}{2}\delta_{c_2}, \\
\delta_{c_2} \circ \delta_{c_2} &= \frac{1+q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}.
\end{aligned}$$

Put $\widehat{\mathbb{Z}_q(3)} = \{\chi_0, \chi_1, \chi_2\}$. Then the character table of $\mathbb{Z}_q(3)$ is

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	ω_q	$\overline{\omega_q}$
χ_2	1	$\overline{\omega_q}$	ω_q

where $\omega_q = \frac{-q+i\sqrt{q^2+2q}}{2}$.

By the symmetry of the character table we see that $\widehat{\mathbb{Z}_q(3)} \cong \mathbb{Z}_q(3)$.

3.2 Deformation $\mathbb{Z}_{(p,q)}(4)$ of $\mathbb{Z}_4$

We investigated several kinds of extension problem in the category of commutative hypergroups, refer to [6], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The cyclic group $\mathbb{Z}_4$ of order four is a non-splitting extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. Then one can consider a non-splitting extension $\mathbb{Z}_{(p,q)}(4)$ $(0 < p \leq 1,\ 0 < q \leq 1)$ of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2)$ as follows.

Proposition 3.2 (Example 4.2 in [14]) For (p, q) $(0 < p \leq 1,\ 0 < q \leq 1)$ there exists a unique hypergroup $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ of order four, which is an extension hypergroup of $\mathbb{Z}_q(2)$ by $\mathbb{Z}_p(2) = \{c_0, c_2\}$ such that $c_1^* = c_3$.

The structure of $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ $(0 < p \leq 1, 0 < q \leq 1)$ is given by

$$\begin{aligned}
\delta_{c_1} \circ \delta_{c_1} &= \delta_{c_3} \circ \delta_{c_3} = \frac{1-q}{2}\delta_{c_1} + q\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_2} &= p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_1} \circ \delta_{c_2} = \frac{1-p}{2}\delta_{c_1} + \frac{1+p}{2}\delta_{c_3}, \\
\delta_{c_1} \circ \delta_{c_3} &= \frac{2pq}{1+p}\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{q-pq}{1+p}\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\
\delta_{c_2} \circ \delta_{c_3} &= \frac{1+p}{2}\delta_{c_1} + \frac{1-p}{2}\delta_{c_3}.
\end{aligned}$$

Put $\widehat{\mathbb{Z}_{(p,q)}(4)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then the character table of $\mathbb{Z}_{(p,q)}(4)$ is

	c_0	c_1	c_2	c_3
χ_0	1	1	1	1
χ_1	1	$i\sqrt{pq}$	$-p$	$-i\sqrt{pq}$
χ_2	1	$-q$	1	$-q$
χ_3	1	$-i\sqrt{pq}$	$-p$	$i\sqrt{pq}$

It is easy to see that $\mathbb{Z}_{(p,q)}(4)$ is interpreted as a (p,q)-deformation of $\mathbb{Z}_4$ and $\widehat{\mathbb{Z}_{(p,q)}(4)} \cong \mathbb{Z}_{(q,p)}(4)$.

4 Deformations of non-abelian finite groups

Let α be an action of a finite group G on a finite hypergroup $H = (H, M^b(H), \circ, *)$. Then a semi-direct product hypergroup $S := H \rtimes_\alpha G$ is introduced in [5]. A convolution $\circ_\alpha$ in $M^b(S)$ is defined by

$$(\varepsilon_{h_1} \otimes \delta_{g_1}) \circ_\alpha (\varepsilon_{h_2} \otimes \delta_{g_2}) := (\varepsilon_{h_1} \circ \varepsilon_{\alpha_{g_1}(h_2)} \otimes \delta_{g_1 g_2}),$$

where ε and δ stand for Dirac measures in $M^b(H)$ and $M^b(G)$ respectively. Unit element is $\varepsilon_e \otimes \delta_e$. An involution $^-$ is

$$(\mu \otimes \delta_g)^- := \alpha_g^{-1}(\mu^*) \otimes \delta_{g^{-1}}$$

for all $\mu \in M^b(H)$ and $g \in G$.

4.1 Deformation $S_q(3)$ of the symmetric group S_3

The symmetric group S_3 is a semi-direct product $\mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ where α is an action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$.

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on a hypergroup $\mathbb{Z}_q(3) = \{h_0, h_1, h_2\}$ $(0 < q \leq 1)$ such that

$$\alpha_g(h_1) = h_2, \quad \alpha_g(h_2) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$S_q(3) := \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$$

which is a q-deformation of the symmetric group $S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$.

4.2 Deformation $D_{(p,q)}(4)$ of the dihedral group D_4

The dihedral group D_4 is written by a semi-direct product $\mathbb{Z}_4 \rtimes_\alpha \mathbb{Z}_2$.

Let $H = \mathbb{Z}_{(p,q)}(4) = \{h_0, h_1, h_2, h_3\}$ $(0 < p \leq 1, 0 < q \leq 1)$ be the (p,q)-deformation of $\mathbb{Z}_4$ and α an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_{(p,q)}(4)$ given by

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Then we obtain a semi-direct product hypergroup

$$D_{(p,q)}(4) := \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2.$$

Hence, we obtain a (p,q)-deformation $D_{(p,q)}(4)$ of the dihedral group D_4.

4.3 Another deformation $W_q(4)$ of the dihedral group D_4

The dihedral group D_4 is also written by a semi-direct product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2$ where β is a flip action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $\mathbb{Z}_q(2)\times\mathbb{Z}_q(2) = \{(h_0, h_0), (h_0, h_1), (h_1, h_0), (h_1, h_1)\ ;\ h_0, h_1 \in \mathbb{Z}_q(2)\}$ be a q-deformation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let β be a flip action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ given by

$$\beta_g((h_i, h_j)) = (h_j, h_i) \quad (i, j = 0 \text{ or } 1).$$

Then we obtain a semi-direct product hypergroup

$$W_q(4) := (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2.$$

The hypergroup $W_q(4)$ is another q-deformation of D_4.

4.4 Deformation $Q_q(4)$ of the quaternion group Q_4

The structure of the quaternion group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ is determined by

$$i^2 = j^2 = k^2 = -1, \quad ij = k.$$

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ on $\mathbb{Z}_4 = \{h_0, h_1, h_2, h_3\}$ such that

$$\alpha_g(h_1) = h_3, \quad \alpha_g(h_2) = h_2, \quad \alpha_g(h_3) = h_1.$$

Let c be a $\mathbb{Z}_4$-valued 2-cocycle of $\mathbb{Z}_2$ which is also given by

$$c(e,e) = c(e,g) = c(g,e) = h_0 \text{ and } c(g,g) = h_2.$$

Then a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$ is defined by the product

$$(h, g)(h', g') = (h\alpha_g(h')c(g, g'), gg')$$

for $h, h' \in \mathbb{Z}_4$ and $g, g' \in \mathbb{Z}_2$. The quaternion group Q_4 is isomorphic to $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$. Hence we interpret Q_4 as a twisted semi-direct product group $\mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$.

Let $\mathbb{Z}_{(1,q)}(4) = \{h_0, h_1, h_2, h_3\}$ be a q-deformation of $\mathbb{Z}_4$ with a subgroup $\{h_0, h_2\}$ and c a $\mathbb{Z}_{(1,q)}(4)$-valued 2-cocycle which is also given by

$$c(e,e) = c(e,g) = c(g,e) = h_0 \text{ and } c(g,g) = h_2.$$

Then, we obtain a twisted semi-direct product hypergroup

$$Q_q(4) := \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2.$$

The hypergroup $Q_q(4)$ is a q-deformation of the quaternion group $Q_4 = \mathbb{Z}_4 \rtimes_\alpha^c \mathbb{Z}_2$.

5 Deformations of finite hypergroups

In this section we discuss q-deformations of several kinds of finite hypergroups in a similar way to the case of finite groups.

5.1 Deformations of orbital hypergroups

Given an action α of a finite group G on a commutative hypergroup H, we obtain a orbit $O = \{\alpha_g(h)\ ;\ g \in G\}$ of $h \in H$ under the action α. Let $\{O_0, O_1, \cdots, O_m\}$ be the set of all orbits in H. We denote an element c_j which is corresponding to each orbit O_j and put $H^\alpha = \{c_0, c_1, \cdots, c_m\}$. Let $M^b(H)^\alpha$ denote the fixed point algebra of $M^b(H)$ under the action α, namely

$$M^b(H)^\alpha = \{\mu \in M^b(H)\ ;\ \alpha_g(\mu) = \mu \text{ for all } g \in G\}.$$

We note that $M^b(H)^\alpha$ is a $*$-subalgebra of $M^b(H)$. For $c_j \in H^\alpha$, put

$$\delta_{c_j} = \frac{1}{|O_j|}\sum_{h\in O_j}\delta_h = \frac{1}{|G|}\sum_{g\in G}\alpha_g(\delta_h).$$

Then $\delta_{c_j} \in M^b(H)^\alpha \cap M^1(H)$. $\mathcal{K}^\alpha(H) = (H^\alpha, M^b(H)^\alpha, \circ, *)$ becomes a hypergroup which is called an orbital hypergroup of H by the action α.

Example 1 The orbital hypergroup $\mathcal{K}^\alpha(\mathbb{Z}_q(3)) = \{c_0, c_1\}$ is a q-deformation of $\mathcal{K}^\alpha(\mathbb{Z}_3)$.

The structure equations are

$$\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + \left(1 - \frac{q}{2}\right)\delta_{c_1}.$$

Remark $\mathcal{K}^\alpha(\mathbb{Z}_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2)$.

Example 2 The orbital hypergroup $\mathcal{K}^\alpha(\mathbb{Z}_{(p,q)}(4)) = \{c_0, c_1, c_2\}$ is a q-deformation of $\mathcal{K}^\alpha(\mathbb{Z}_4)$.

The structure equations are

$$\delta_{c_1} \circ \delta_{c_1} = p\delta_{c_0} + (1-p)\,\delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_2} = \delta_{c_2},$$
$$\delta_{c_2} \circ \delta_{c_2} = \frac{pq}{1+p}\delta_{c_0} + \frac{q}{1+p}\delta_{c_1} + (1-q)\delta_{c_2}.$$

Remark $\mathcal{K}^\alpha(\mathbb{Z}_{(p,q)}(4)) = \mathbb{Z}_p(2) \vee \mathbb{Z}_q(2)$.

5.2 Deformations of character hypergroups of semi-direct product hypergroups

Let $S = H \rtimes_\alpha G$ be a semi-direct product hypergroup defined by an action α of a finite abelian group G on a finite commutative hypergroup H (Refer to [5]). $\hat{S} = \widehat{H \rtimes_\alpha G}$ is the set of all equivalence classes of irreducible representations of S. For $(\pi, \mathcal{H}(\pi)) \in \hat{S}$, the character $ch(\pi)$ of π is defined by

$$ch(\pi)((h,g)) = \frac{1}{\dim \pi}\mathrm{tr}(\pi(h,g))$$

where $(h,g) \in H \rtimes_\alpha G$ and tr is the trace of $B(\mathcal{H}(\pi))$. Put $\mathcal{K}(\hat{S}) = \{ch(\pi)\ ;\ \pi \in \hat{S}\}$.

Proposition 5.1 ([5] and [7]) If the action α satisfies the regularity condition, then $\mathcal{K}(\widehat{H \rtimes_\alpha G})$ becomes a commutative hypergroup by the product of functions on $S = H \rtimes_\alpha G$.

This hypergroup is called a character hypergroup of the semi-direct product hypergroup $S = H \rtimes_\alpha G$.

Example 3 The character hypergroup $\mathcal{K}(\widehat{S_q(3)})$ of $S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$ is a q-deformation of $\mathcal{K}(\widehat{S_3})$.

$\widehat{S_q(3)} = \widehat{H \rtimes_\alpha G} = \{\chi_0 \odot \tau_0, \chi_0 \odot \tau_1, \pi\}$, where π is a two-dimensional irreducible representation of $S_q(3)$. $\mathcal{K}(\widehat{S_q(3)}) = \{ch(\chi_0 \odot \tau_0), ch(\chi_0 \odot \tau_1), ch(\pi)\}$. The character table is

	(h_0, e)	(h_1, e)	(h_2, e)	(h_0, g)	(h_1, g)	(h_2, g)
$\gamma_0 = ch(\chi_0 \odot \tau_0)$	1	1	1	1	1	1
$\gamma_1 = ch(\chi_0 \odot \tau_1)$	1	1	1	-1	-1	-1
$\gamma_2 = ch(\pi)$	1	$-\frac{q}{2}$	$-\frac{q}{2}$	0	0	0

and the structure equations of $\mathcal{K}(\widehat{S_q(3)})$ are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{q}{4}\gamma_0 + \frac{q}{4}\gamma_1 + \left(1 - \frac{q}{2}\right)\gamma_2, \quad \gamma_1\gamma_2 = \gamma_2.$$

Example 4 The character hypergroup $\mathcal{K}(\widehat{D_{(p,q)}(4)})$ of $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2$ is a (p, q)-deformation of $\mathcal{K}(\widehat{D_4})$.

The structure equations of $\mathcal{K}(\widehat{D_{(p,q)}(4)}) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ are

$$\begin{aligned}
&\gamma_1\gamma_1 = \gamma_0, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_2, \\
&\gamma_2\gamma_2 = \gamma_3\gamma_3 = q\gamma_0 + (1-q)\gamma_2, \quad \gamma_2\gamma_3 = q\gamma_1 + (1-q)\gamma_3, \\
&\gamma_4\gamma_4 = \frac{pq}{2(1+q)}\gamma_0 + \frac{pq}{2(1+q)}\gamma_1 + \frac{p}{2(1+q)}\gamma_2 + \frac{p}{2(1+q)}\gamma_3 + (1-p)\gamma_4, \\
&\gamma_1\gamma_4 = \gamma_4, \quad \gamma_2\gamma_4 = \gamma_4, \quad \gamma_3\gamma_4 = \gamma_4.
\end{aligned}$$

Example 5 The character hypergroup $\mathcal{K}(\widehat{Q_q(4)})$ of $Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2$ is a q-deformation of $\mathcal{K}(\widehat{D_4})$.

The structure equations of $\mathcal{K}(\widehat{Q_q(4)}) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ are

$$\begin{aligned}
&\gamma_1\gamma_1 = \gamma_0, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_2, \\
&\gamma_2\gamma_2 = \gamma_3\gamma_3 = q\gamma_0 + (1-q)\gamma_2, \quad \gamma_2\gamma_3 = q\gamma_1 + (1-q)\gamma_3, \\
&\gamma_4\gamma_4 = \frac{q}{2(1+q)}\gamma_0 + \frac{q}{2(1+q)}\gamma_1 + \frac{1}{2(1+q)}\gamma_2 + \frac{1}{2(1+q)}\gamma_3, \\
&\gamma_1\gamma_4 = \gamma_4, \quad \gamma_2\gamma_4 = \gamma_4, \quad \gamma_3\gamma_4 = \gamma_4.
\end{aligned}$$

5.3 Deformations of generalized conjugacy class hypergroups

Let $S = H \rtimes_\alpha G$ be a semi-direct product hypergroup. Then there exists the canonical conditional expectation E from $M^b(S)$ onto the center $Z(M^b(S))$ of $M^b(S)$. Put

$$\mathcal{K}(H \rtimes_\alpha G) := \{E(\delta_{(h,g)}) \ ; \ (h, g) \in H \rtimes_\alpha G\}.$$

Proposition 5.2 ([6]) If the action α satisfies the regularity condition, then $\mathcal{K}(H \rtimes_\alpha G)$ becomes a commutative hypergroup with the convolution in the center $Z(M^b(S))$. Moreover $\hat{\mathcal{K}}(H \rtimes_\alpha G) \cong \mathcal{K}(\widehat{H \rtimes_\alpha G})$ holds.

We call $\mathcal{K}(H \rtimes_\alpha G)$ a generalized conjugacy class hypergroup of $H \rtimes_\alpha G$.

Example 6 The generalized conjugacy class hypergroup $\mathcal{K}(S_q(3))$ of $S_q(3)$ is a q-deformation of $\mathcal{K}(S_3)$.

The structure equations of $\mathcal{K}(S_q(3)) = \{c_0, c_1, c_2\}$ are

$$\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + \left(1 - \frac{q}{2}\right)\delta_{c_1}, \quad \delta_{c_2} \circ \delta_{c_2} = \frac{q}{q+2}\delta_{c_0} + \frac{2}{q+2}\delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_2} = \delta_{c_2}.$$

Example 7 The generalized conjugacy class hypergroup $\mathcal{K}(D_{(p,q)}(4))$ of $D_{(p,q)}(4)$ is a (p,q)-deformation of $\mathcal{K}(D_4)$.

The structure equations of $\mathcal{K}(D_{(p,q)}(4)) = \{c_0, c_1, c_2, c_3, c_4\}$ are

$$\begin{aligned}
&\delta_{c_1} \circ \delta_{c_1} = \delta_{c_4} \circ \delta_{c_4} = \frac{pq}{1+p}\delta_{c_0} + (1-q)\delta_{c_1} + \frac{q}{1+p}\delta_{c_2}, \\
&\delta_{c_2} \circ \delta_{c_2} = p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_3} \circ \delta_{c_3} = \frac{p}{1+p}\delta_{c_0} + \frac{1}{1+p}\delta_{c_2}, \\
&\delta_{c_1} \circ \delta_{c_2} = \delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_3} = \delta_{c_4}, \quad \delta_{c_1} \circ \delta_{c_4} = q\delta_{c_3} + (1-q)\delta_{c_4}, \\
&\delta_{c_2} \circ \delta_{c_4} = \delta_{c_4}, \quad \delta_{c_2} \circ \delta_{c_3} = \delta_{c_3}, \quad \delta_{c_3} \circ \delta_{c_4} = \delta_{c_1}.
\end{aligned}$$

Example 8 The generalized conjugacy class hypergroup $\mathcal{K}(Q_q(4))$ of $Q_q(4)$ is a q-deformation of $\mathcal{K}(Q_4)$.

The structure equations of $\mathcal{K}(Q_q(4)) = \{c_0, c_1, c_2, c_3, c_4\}$ are

$$\begin{aligned}
&\delta_{c_1} \circ \delta_{c_1} = \delta_{c_4} \circ \delta_{c_4} = \frac{q}{2}\delta_{c_0} + (1-q)\delta_{c_1} + \frac{q}{2}\delta_{c_2}, \\
&\delta_{c_2} \circ \delta_{c_2} = \delta_{c_0}, \quad \delta_{c_3} \circ \delta_{c_3} = \frac{1}{2}\delta_{c_0} + \frac{1}{2}\delta_{c_2}, \\
&\delta_{c_1} \circ \delta_{c_2} = \delta_{c_1}, \quad \delta_{c_1} \circ \delta_{c_3} = \delta_{c_4}, \quad \delta_{c_1} \circ \delta_{c_4} = q\delta_{c_3} + (1-q)\delta_{c_4}, \\
&\delta_{c_2} \circ \delta_{c_4} = \delta_{c_4}, \quad \delta_{c_2} \circ \delta_{c_3} = \delta_{c_3}, \quad \delta_{c_3} \circ \delta_{c_4} = \delta_{c_1}.
\end{aligned}$$

By the above structure equations, we have the following theorem.

Theorem There are deformations $S_q(3) = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2$ of the symmetric group S_3, $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\alpha \mathbb{Z}_2$ and $W_q(4) = (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2$ of the dihedral group D_4 and $Q_q(4) = \mathbb{Z}_{(1,q)}(4) \rtimes_\alpha^c \mathbb{Z}_2$ of the quaternion group Q_4 in the category of hypergroups. These deformations have the following properties.

(1) $\mathcal{K}(\widehat{S_q(3)}) = \mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$ and $\mathcal{K}(S_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2) \vee \mathbb{Z}_2$.

(2) $\mathcal{K}(\widehat{D_{(p,q)}(4)})$ is a (q,p)-deformation of $\mathcal{K}(\widehat{D_4})$ and $\mathcal{K}(D_{(p,q)}(4))$ is a (p,q)-deformation of $\mathcal{K}(D_4)$. $\mathcal{K}(\widehat{Q_q(4)})$ is a q-deformation of $\mathcal{K}(\widehat{Q_4})$ and $\mathcal{K}(Q_q(4))$ is a q-deformation of $\mathcal{K}(Q_4)$. Moreover $\mathcal{K}(\widehat{D_{(1,q)}(4)}) \cong \mathcal{K}(\widehat{Q_q(4)})$ and $\mathcal{K}(D_{(1,q)}(4)) \cong \mathcal{K}(Q_q(4))$ although $D_{(1,q)}(4)$ is not isomorphic to $Q_q(4)$.

(3) $\mathcal{K}(\widehat{W_q(4)})$ is not a hypergroup when $q \neq 1$.

Proof (1) We put $\mathbb{Z}_2 = \{b_0, b_1\}$ and $\mathbb{Z}_{\frac{q}{2}}(2) = \{c_0, c_1\}$, where $\delta_{b_1} \circ \delta_{b_1} = \delta_{b_0}$ and $\delta_{c_1} \circ \delta_{c_1} = \frac{q}{2}\delta_{c_0} + (1 - \frac{q}{2})\delta_{c_1}$ $(0 < q \leq 1)$. The structure of $\mathcal{K}(\widehat{S_q(3)})$ in Example 3 is the same of the hypergroup join $\mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$. Hence $\mathcal{K}(\widehat{S_q(3)}) = \mathbb{Z}_2 \vee \mathbb{Z}_{\frac{q}{2}}(2)$. In a similar way we get $\mathcal{K}(S_q(3)) = \mathbb{Z}_{\frac{q}{2}}(2) \vee \mathbb{Z}_2$ as in Example 6.

(2) The former properties follow directly from above examples 4, 7, 5 and 8. Both of $D_{(1,q)}(4)$ and $Q_q(4)$ are extension hypergroups of $\mathbb{Z}_2$ by $\mathbb{Z}_{(1,q)}(4)$. However $D_{(1,q)}(4)$ is of splitting type but $Q_q(4)$ is of non-splitting type. Hence $D_{(1,q)}(4)$ is not isomorphic to $Q_q(4)$.

(3) We put $\widehat{\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ and $\widehat{\mathbb{Z}_2} = \{\tau_0, \tau_1\}$. Then

$$\widehat{W_q(4)} = \{\chi_0 \odot \tau_0, \chi_0 \odot \tau_1, \chi_3 \odot \tau_0, \chi_3 \odot \tau_1, \pi\},$$

where π is the two-dimensional irreducible representation of $W_q(4)$ given by

$$\pi = \mathrm{ind}_{\mathbb{Z}_q(2)\times\mathbb{Z}_q(2)}^{W_q(4)}(\chi_1 \odot \tau_0).$$

Hence,

$$\mathcal{K}(\widehat{W_q(4)}) = \{ch(\chi_0 \odot \tau_0), ch(\chi_0 \odot \tau_1), ch(\chi_3 \odot \tau_0), ch(\chi_3 \odot \tau_1), ch(\pi)\}.$$

Assume that $\mathcal{K}(\widehat{W_q(4)})$ is a hypergroup for $q \neq 1$. Then

$$ch(\chi_3\odot\tau_0)ch(\chi_3\odot\tau_0) = a_0 ch(\chi_0\odot\tau_0)+a_1 ch(\chi_0\odot\tau_1)+a_2 ch(\chi_3\odot\tau_0)+a_3 ch(\chi_3\odot\tau_1)+a_4 ch(\pi),$$

where $\sum_{j=0}^{4} a_j = 1$ and $a_j \geq 0$ $(j = 0, 1, 2, 3, 4)$. Since

$$ch(\chi_0 \odot \tau_1)(h_0, g) = -1, \quad ch(\chi_3 \odot \tau_1)(h_0, g) = -1,$$

$$ch(\pi)(h_0, g) = 0 \quad \text{and} \quad ch(\chi_3 \odot \tau_0)ch(\chi_3 \odot \tau_0)(h_0, g) = 1$$

where h_0 is the unit of $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$ and $\mathbb{Z}_2 = \{e, g\}$, $g^2 = e$, we see that

$$a_0 - a_1 + a_2 - a_3 = 1.$$

This implies that $a_1 = 0, a_3 = 0, a_4 = 0$. Hence, we get

$$ch(\chi_3 \odot \tau_0)ch(\chi_3 \odot \tau_0) = a_0 ch(\chi_0 \odot \tau_0) + a_2 ch(\chi_3 \odot \tau_0).$$

Restricting this equality to $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)$, we obtain

$$\chi_3\chi_3 = a_0\chi_0 + a_2\chi_3.$$

This contradicts with the fact :

$$\chi_3\chi_3 = q^2\chi_0 + q(1-q)\chi_1 + q(1-q)\chi_2 + (1-q)^2\chi_3.$$

Hence, $\mathcal{K}(\widehat{W_q(4)})$ is not a hypergroup when $q \neq 1$. □

Acknowledgement The authors are thankful to Professors Yoshimichi Ueda, Shigeru Yamagami, Herbert Heyer, Yoshikazu Katayama and Moto O'uchi for their stimulating discussions and valuable suggestions.

REFERENCES

[1] Bloom, W.R. and Heyer, H. : *Harmonic Analysis of Probability Measures on Hypergroups*, 1995, Walter de Gruyter, de Gruyter Studies in Mathmatics 20.

[2] Heyer, H., Jimbo, T., Kawakami, S. and Kawasaki, K. : *Finite commutative hypergroups associated with actions of finite abelian groups*, Bull. Nara Univ. Educ., Vol. 54 (2005), No. 2, pp.23-29.

[3] Heyer, H. and Kawakami, S. : *Extensions of Pontryagin hypergroups*, PMS(Probability and Mathematical Statstics), Vol. 26 (2006), Fasc. 2, pp.245-260.

[4] Heyer, H. and Kawakami, S. : *A cohomology approach to the extension problem for commutative hypergroups*, Semigroup Forum, Vol. 83 (2011), No. 3, pp.371-394.

[5] Heyer, H. and Kawakami, S. : *An Imprimitivity Theorem for Representations of a Semi-direct Product Hypergroup*, Journal of Lie Theory, Vol. 24 (2014), pp.159-178.

[6] Heyer, H., Katayama, Y., Kawakami, S. and Kawasaki, K. : *Extensions of finite commutative hypergroups*, Scientiae Mathematicae Japonicae, Vol.65 (2007), No. 3, pp.373-385.

[7] Ito, W. and Kawakami, S. : *Crossed Products of Commutative Finite Hypergroups*, Bull. Nara Univ. Educ., Vol. 48 (1999), No. 2, pp.1-6.

[8] Ichihara, R. and Kawakami, S. : *Strong hypergroups of order four arising from extensions*, Scientiae Mathematicae Japonicae, Vol.68 (2008), No. 3, pp.371-381.

[9] Ichihara, R. and Kawakami, S. : *Estimation of the orders of hypergroup extensions*, Scientiae Mathematicae Japonicae, Vol.72 (2010), No. 1, pp.307-317.

[10] Ichihara, R., Kawakami, S. and Sakao, M. : *Hypergroup extensions of finite abelian groups by hypergroups of order two*, Nihonkai Mathematical Journal,Vol. 21 (2010), No. 1, pp.47-71.

[11] Kawakami, S. : *Extensions of commutative hypergroups*, Infinite Dimesional Harmonic Analysis, World Scientific, 2008, pp.135-148.

[12] Kawakami, S., Kawasaki, K. and Yamanaka, S. : *Extensions of the Golden hypergroup by finite abelian groups*, Bull. Nara Univ. Educ., Vol. 57 (2008), No. 2, pp.1-10.

[13] Kawakami, S. and Mikami, I. : *Hypergroup extensions of the group of order two by the Golden hypergroup*, Bull. Nara Univ. Educ., Vol. 57 (2008), No. 2, pp.11-16.

[14] Kawakami, S., Mikami, I., Tsurii, T. and Yamanaka, S. : *Actions of Finite Hypergroups and Applications to Extension Problem*, Bull. Nara Univ. Educ., Vol. 60 (2011), No. 2, pp.19-28.

[15] Kawakami, S., Sakao, M. and Takeuchi, T. : *Non-splitting extensions of hypergroups of order two*, Bull. Nara Univ. Educ., Vol. 56 (2007), No. 2, pp.7-13.

[16] Kawakami, S., Sakao, M., Tsurii, T. and Yamanaka, S. : *Signed Actions of Finite Hypergroups and the Extension Problem*, Bull. Nara Univ. Educ., Vol. 61 (2012), No. 2, pp.13-24.

[17] Kawakami, S., Sakao, M. and Yamanaka, S. : *Extensions of hypergroups of order two by locally compact abelian groups*, Bull. Nara Univ. Educ., Vol. 59 (2010), No.2, pp.1-6.

[18] Kawakami, S. and Yamanaka, S. : *Extensions of the Golden hypergroup by locally compact abelian groups*, Acta Sci. Math., Vol. 78 (2012), pp.351-368.

[19] K.A.Ross and D.Xu : *Hypergroup Deformations and Markov Chains*, Journal of Theoretical Probability, Vol7(1994) , No. 4, pp.813-830.

[20] Sunder, V.S. and Wildberger, N.J : *Action of hypergroups and examples*, Harmonic Analysis and Hypergroups, Birkhauser, 1998, pp.145-163.

[21] M.Voit : *Substitution of open subhypergroups*, Hokkaido Math. J. Vol.23 (1994), pp.143-183.

[22] Wildberger, N.J. : *Finite commutative hypergroups and applications from group theory to conformal field theory*, Applications of Hypergroups and Related Measure Algebras, Amer. Math. Soc., Providence, 1994, pp.413-434.

[23] Wildberger, N.J. : *Strong hypergroups of order three*, Journal of Pure and Applied Algebra 174(2002), No. 1, pp.95-115.

[24] S.L.Woronowicz : *Compact Matrix Pseudogroups*, Comm. Math. Phys. Volume 111, Number 4 (1987), pp.613-665.

[25] D.Xu : *Hypergroup Deformations of $Z(p)$ and finite ensembles*, Preprint 1992.

Communicated by *Moto O'uchi*

Addresses

Satoshi Kawakami : Nara University of Education
Department of Mathematics
Takabatake-cho
Nara, 630-8528
Japan

e-mail : kawakami@nara-edu.ac.jp

Tatsuya Tsurii : Graduate School of Science of the Osaka Prefecture University
1-1 Gakuen-cho, Nakaku, Sakai
Osaka, 599-8531
Japan

e-mail : dw301003@edu.osakafu-u.ac.jp

Satoe Yamanaka : Nara University of Education
Department of Mathematics
Takabatake-cho
Nara, 630-8528
Japan
e-mail : s.yamanaka516@gmail.com

CLASSIFICATION OF IDEMPOTENTS AND SQUARE ROOTS IN THE UPPER TRIANGULAR MATRIX BANACH ALGEBRAS AND THEIR INDUCTIVE LIMIT ALGEBRAS

Takahiro Sudo

Received February 18, 2015

Abstract. We study idempotents and square roots in the upper triangular matrix Banach algebras over real or complex numbers. We compute explicitly and determine algebraically the idempotents and the square roots in the cases of size: two by two, three by three, and four by four. We also consider their equivalence classes by homotopy and classify topologically the upper triangular matrix algebras in those cases and in general by the groups generated by the homotopy classes. Moreover, we consider some infinite dimensional, Banach algebras obtained as inductive limits of the upper triangular matrix algebras and obtain several topological classification results for the inductive limits.

1 Introduction We begin to study idempotents and square roots in the upper triangular matrix Banach algebras over real or complex numbers. The upper triangular matrix algebras are typical examples of finite dimensional non self-adjoint Banach algebras over real or complex numbers. We compute explicitly and determine algebraically idempotents and square roots of the upper triangular matrix algebras in the cases of size: two by two, three by three, and four by four. The statements as lists as examples obtained should be useful and convenient for the readers. We also consider the equivalence classes of the idempotents and the square roots by homotopy and classify topologically the upper triangular matrix algebras in the cases and in the general case by the groups generated by the homotopy classes. Moreover, we consider some infinite dimensional, Banach algebras obtained as inductive limits of the upper triangular matrix algebras, and obtain several (topological) classification results for the inductive limits by our V-theory groups mentioned below and also by the scales for the units

As a contrast, C^*-algebras are self-adjoint Banach algebras over complex numbers with the C^*-norm condition. The full matrix algebras over complex numbers are typical examples of finite dimensional C^*-algebras. Projections of C^*-algebras, that are self-adjoint idempotents, and unitaries of C^*-algebras, with adjoints as inverses, play main roles in the K-theory for C^*-algebras, and their associated K-theory classes generate K-theory groups of C^*-algebras ([1], [4] and [5]). By lack of self-adjointness for non self-adjoint Banach algebras, as candidates as substitute, we consider idempotents and square roots and their homotopy classes, that generate our named V-theory groups, first introduced in this paper.

As for inductive limit algebras, AF (approximately finite dimensional) C^*-algebras, that are inductive limits of finite direct sums of full matrix algebras, are classified by K-theory groups (but K_0 only since K_1 trivial) as ordered groups with the scales (see the corresponding results in [1], [4], or [5], due to [2]). Our V-theory groups (V_0 of V_0 and V_1) just correspond to the scales in the C^*-algebra K-theory, in which the symbol V is used for indicating the sets of equivalence classes of projections of matrix algebras over a C^*-algebra

2000 *Mathematics Subject Classification.* Primary 46K50, 46H20, 46L80, 19K14.
Key words and phrases. Idempotent, square root, matrix algebra, upper triangular matrix, non self-adjoint algebra, Banach algebra, inductive limit.

and the symbol Σ is used for indicating the scales for a C^*-algebra (see [1]). Note that, by lack of self-adjointness, there are no non self-adjoint unitaries, and no unitary equivalence and no stably unitary equivalence as for idempotents in non self-adjoint Banach algebras, but can be used homotopy in the algebras.

However, the scaled ordered, C^*-algebra K-theory groups (K_0) for the inductive limits of non self-adjoint Banach subalgebras obtained in inductive limits of C^*-algebras, such as AF-algebras and UHF-algebras, have been already used to classify those non self-adjoint inductive limit algebras, containing the case we consider here (see [3]). Therefore, our classification results in application to inductive limit non self-adjont algebras are not new, but our formulation in terms of non self-adjoint algebras only, V_0 as well as V_1 (non-trivial while K_1 trivial in that case) seems to be new in this sense, and anyhow to be an equivalent replacement as another method or attempt.

2 The two by two case We denote by $T_2(\mathbb{R})$ the algebra of all upper triangular 2×2 matrices over the real field $\mathbb{R}$ and by $T_2(\mathbb{C})$ the same algebra over the complex field $\mathbb{C}$. We give the topology on the algebras by the Euclidean norm, for convenience, via $T_2(\mathbb{R}) \cong \mathbb{R}^3$ and $T_2(\mathbb{C}) \cong \mathbb{C}^3$ as a space. Let F be either $\mathbb{R}$ or $\mathbb{C}$.

Recall that a matrix element A of $T_2(F)$ is said to be an idempotent if $A^2 = A$.

Proposition 2.1. *All idempotents of $T_2(F)$ are listed up as*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for any $b \in F$.

Proof. Let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(F)$$

with $A^2 = A$, so that $a^2 = a$, $c^2 = c$, and $b(a + c) = b$. Hence $a = 0$ or 1, and $c = 0$ or 1. □

Denote by $P_2(F)$ the set of all idempotents of $T_2(F)$. Define the equivalence relation for elements of $P_2(F)$ by that two elements of $P_2(F)$ are equivalent if there is a continuous path within $P_2(F)$ between the two elements. Write by $E_0(T_2(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_2(F)$.

Corollary 2.2. *All classes of $E_0(T_2(F))$ are listed up as*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As, possibly, a new notion to simplify the situation, we may introduce, as an attempt,

Definition 2.3. We now define the anti-diagonal transpose A^{at} of $A \in T_2(F)$ by

$$A^{at} = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

We denote by $T_2(F)/\sim_{at}$ the set of matrices of $T_2(F)$ identified under the anti-transpose.

Note that the anti-diagonal transpose corresponds to a permutation on $T_2(F) \cong F^3$. Also, one has

$$\{J_2AJ_2\}^t \equiv \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}^t = \begin{pmatrix}c & b\\ 0 & a\end{pmatrix} = A^{at}$$

with $\{\cdots\}^t$ the usual transpose, but in $M_2(F)$ the 2×2 matrix algebra over F.

Corollary 2.4. *All idempotents of $T_2(F)/\sim_{at}$ are listed up as*

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & b\\ 0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

for any $b \in F$.

Corollary 2.5. *All classes of $E_0(T_2(F)/\sim_{at})$ are listed up as*

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Recall that a matrix element A of $T_2(F)$ is said to be a square root if $A^2 = I_2$ the 2×2 identity matrix.

Proposition 2.6. *All square roots of $T_2(F)$ are listed up as*

$$\begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix} \quad \text{(compound 4 cases)}, \quad \begin{pmatrix}\pm\mathbf{1} & b\\ 0 & \mp\mathbf{1}\end{pmatrix} \quad \text{(not compound 2 cases)}$$

for any $b \in F$ non-zero.

Remark. In what follows, we make the difference of the compound (or composite) in order case or not by denoting ± 1 usual or $\pm\mathbf{1}$ bold as in the statement above.

Proof. Let

$$A = \begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \in T_2(F)$$

with $A^2 = I_2$, so that $a^2 = 1$, $c^2 = 1$, and $b(a + c) = 0$. Hence $a = 1$ or -1, and $c = 1$ or -1, and if b is non-zero, then $a = -c$. □

Denote by $R_2(F)$ the set of all square roots of $T_2(F)$. Define the equivalence relation for elements of $R_2(F)$ by that two elements of $R_2(F)$ are equivalent if there is a continuous path within $R_2(F)$ between the two elements. Write by $E_1(T_2(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_2(F)$.

Corollary 2.7. *All classes of $E_1(T_2(F))$ are listed up as*

$$\begin{bmatrix}\pm 1 & 0\\ 0 & \pm 1\end{bmatrix} \quad \text{(compound in order 4 cases)}.$$

3 The three by three case We denote by $T_3(F)$ the algebra of all upper triangular 3×3 matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_3(F) \cong F^6$ as a space.

Proposition 3.1. *All idempotents of $T_3(F)$ are listed up as*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & -xz \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $x, y, z \in F$.

Proof. Let

$$A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix} \in T_3(F)$$

with $A^2 = A$, so that $a^2 = a$, $b^2 = b$, $c^2 = c$, and $(a+b)x = x$, $(b+c)z = z$, $(a+c)y+xz = y$. Hence $a = 0$ or 1, and $b = 0$ or 1, and $c = 0$ or 1.

If $a = b = c = 0$, then $x = y = z = 0$.
If $a = 1$ and $b = c = 0$, then $z = 0$.
If $b = 1$ and $a = c = 0$, then $y = xz$.
If $c = 1$ and $a = b = 0$, then $x = 0$.
Moreover, if $a = b = 1$ and $c = 0$, then $x = 0$.
If $a = c = 1$ and $b = 0$, then $y = -xz$.
If $a = 0$ and $b = c = 1$, then $z = 0$.
If $a = b = c = 1$, then $x = y = z = 0$.
These cases correspond to the matrices in the statement in this order. □

Denote by $P_3(F)$ the set of all idempotents of $T_3(F)$. Define the equivalence relation for elements of $P_3(F)$ by that two elements of $P_3(F)$ are equivalent if there is a continuous path within $P_3(F)$ between the two elements. Write by $E_0(T_3(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_3(F)$.

Corollary 3.2. *All classes of $E_0(T_3(F))$ are listed up as*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 3.3. We now define the anti-diagonal transpose A^{at} of $A \in T_3(F)$ by

$$A^{at} = \begin{pmatrix} c & z & y \\ 0 & b & x \\ 0 & 0 & a \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix}.$$

We denote by $T_3(F)/\sim_{at}$ the set of matrices of $T_3(F)$ identified under the anti-transpose.

Note that A^{at} is just the transpose of J_3AJ_3:

$$A^{at} = \{J_3AJ_3\}^t = J_3^tA^tJ_3^t = J_3A^tJ_3,$$

with J_3 the 3×3 matrix of $(1,3),(2,2),(3,1)$ components as 1 and other components as 0.

Corollary 3.4. *All idempotents of $T_3(F)/\sim_{at}$ are listed up as*

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&x&y\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&x&xz\\0&1&z\\0&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&y\\0&1&z\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&x&-xz\\0&0&z\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

for any $x, y, z \in F$.

Corollary 3.5. *All classes of $E_0(T_3(F)/\sim_{at})$ are listed up as*

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Proposition 3.6. *All square roots of $T_3(F)$ are listed up as*

$$\begin{pmatrix}\pm1&0&0\\0&\pm1&0\\0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&0\\0&\mp\mathbf{1}&0\\0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0\\0&\pm\mathbf{1}&z\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&0&y\\0&\pm1&0\\0&0&\mp\mathbf{1}\end{pmatrix},$$
$$\begin{pmatrix}\pm\mathbf{1}&x&0\\0&\mp\mathbf{1}&z\\0&0&\pm\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&y\\0&\mp\mathbf{1}&0\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&0&y\\0&\pm\mathbf{1}&z\\0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x&y\\0&\mp\mathbf{1}&z\\0&0&\pm\mathbf{1}\end{pmatrix}$$

for any non-zero $x, y, z \in F$, where x, y, z satisfy the equation $2(\pm\mathbf{1})y + xz = 0$ in the last case, so that $y = 2^{-1}(\mp\mathbf{1})xz$. There are compound or not $8+4+4+4+2+2+2+2 = 28$ cases.

Proof. Let

$$A = \begin{pmatrix}a&x&y\\0&b&z\\0&0&c\end{pmatrix} \in T_3(F)$$

with $A^2 = I_3$ the 3×3 identity matrix, so that $a^2 = 1$, $b^2 = 1$, $c^2 = 1$, and $(a+b)x = 0$, $(b+c)z = 0$, $(a+c)y + xz = 0$. Hence $a = \pm1$, and $b = \pm1$, and $c = \pm1$.

If $y = z = 0$ and x is non-zero, $a = -b$.

If $x = y = 0$ and z is non-zero, $b = -c$.

If $x = z = 0$ and y is non-zero, $a = -c$.

If $y = 0$ and $xz \neq 0$, then $a = -b$ and $b = -c$.

If $y \neq 0$ and $a = -c$, then $x = 0$ or $z = 0$.

The rest case is that $xyz \neq 0$ with $2ay + xz = 0$.

These cases correspond to the matrices in the statement in this order. □

Denote by $R_3(F)$ the set of all square roots of $T_3(F)$. Define the equivalence relation for elements of $R_3(F)$ by that two elements of $R_3(F)$ are equivalent if there is a continuous path within $R_3(F)$ between the two elements. Write by $E_1(T_3(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_3(F)$.

Corollary 3.7. *All classes of $E_1(T_3(F))$ are listed up as*

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \quad \text{(compound in order 8 cases).}$$

4 The four by four case We denote by $T_4(F)$ the algebra of all upper triangular 4×4 matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_4(F) \cong F^{1+2+3+4} = F^{10}$ as a space.

Proposition 4.1. *All idempotents of $T_4(F)$ are listed up as the zero matrix and*

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{12}x_{24} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & x_{13} & x_{13}x_{34} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & x_{14} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & x_{13} & -x_{12}x_{24} - x_{13}x_{34} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{12}x_{24} + x_{13}x_{34} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{14} \\ 0 & 1 & x_{23} & -x_{23}x_{34} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & -x_{12}x_{24} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & x_{13} & -x_{13}x_{34} \\ 0 & 1 & x_{23} & -x_{23}x_{34} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for any $x_{i,j} \in F$ $(i < j)$, and the identity matrix.

Proof. Let

$$A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix} \in T_4(F)$$

with $A^2 = A$, so that $a_j^2 = a_j$ $(1 \le j \le 4)$, and $(a_1 + a_2)x_{12} = x_{12}$, $(a_2 + a_3)x_{23} = x_{23}$, $(a_3 + a_4)x_{34} = x_{34}$, $(a_1 + a_3)x_{13} + x_{12}x_{23} = x_{13}$, $(a_2 + a_4)x_{24} + x_{23}x_{34} = x_{24}$, $(a_1 + a_4)x_{14} + x_{12}x_{24} + x_{13}x_{34} = x_{14}$. Hence $a_j = 0$ or 1 $(1 \le j \le 4)$.

If $a_j = 0$ $(1 \le j \le 4)$, then $x_{ij} = 0$ $(1 \le i < j \le 4)$.

If $a_1 = 1$ and $a_j = 0$ $(2 \le j \le 4)$, then $x_{23} = x_{34} = 0$, $x_{24} = 0$
If $a_2 = 1$ and $a_j = 0$ $(j \ne 2)$, then $x_{34} = 0$ and $x_{12}x_{23} = x_{13}$, $x_{12}x_{24} = x_{14}$.
If $a_3 = 1$ and $a_j = 0$ $(j \ne 3)$, then $x_{12} = 0$ and $x_{23}x_{34} = x_{24}$, $x_{13}x_{34} = x_{14}$.
If $a_4 = 1$ and $a_j = 0$ $(j \ne 4)$, then $x_{12} = x_{23} = x_{13} = 0$.
Moreover, if $a_1 = a_2 = 1$ and $a_3 = a_4 = 0$, then $x_{12} = x_{34} = 0$.
If $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, then $x_{13} = -x_{12}x_{23}$ and $x_{24} = x_{23}x_{34}$.
If $a_1 = a_4 = 1$ and $a_2 = a_3 = 0$, then $x_{23} = 0$ and $x_{14} = -x_{12}x_{24} - x_{13}x_{34}$.
If $a_2 = a_3 = 1$ and $a_1 = a_4 = 0$, then $x_{23} = 0$ and $x_{14} = x_{12}x_{24} + x_{13}x_{34}$.
If $a_2 = a_4 = 1$ and $a_1 = a_3 = 0$, then $x_{13} = x_{12}x_{23}$ and $x_{24} = -x_{23}x_{34}$.
If $a_3 = a_4 = 1$ and $a_1 = a_2 = 0$, then $x_{12} = x_{34} = 0$.
Furthermore, if $a_1 = a_2 = a_3 = 1$ and $a_4 = 0$, then $x_{12} = x_{23} = x_{13} = 0$.
If $a_1 = a_3 = a_4 = 1$ and $a_2 = 0$, then $x_{34} = 0$, $x_{13} = -x_{12}x_{23}$, $x_{14} = -x_{12}x_{24}$.
If $a_1 = a_2 = a_4 = 1$ and $a_3 = 0$, then $x_{12} = 0$, $x_{24} = -x_{23}x_{34}$, $x_{14} = -x_{13}x_{34}$.
If $a_1 = 0$ and $a_2 = a_3 = a_4 = 0$, then $x_{23} = x_{34} = x_{24} = 0$.
Finally, if $a_j = 1$ $(1 \le j \le 4)$, then $x_{ij} = 0$ $(1 \le i < j \le 4)$.
These cases correspond to the matrices in the statement in this order. □

Denote by $P_4(F)$ the set of all idempotents of $T_4(F)$. Define the equivalence relation for elements of $P_4(F)$ by that two elements of $P_4(F)$ are equivalent if there is a continuous path within $P_4(F)$ between the two elements. Write by $E_0(T_4(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_4(F)$.

Corollary 4.2. *All classes of $E_0(T_4(F))$ are listed up as the zero class and*

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

and the identity class.

Definition 4.3. We now define the anti-diagonal transpose A^{at} of $A \in T_4(F)$ by

$$A^{at} = \begin{pmatrix} a_4 & x_{34} & x_{24} & x_{14} \\ 0 & a_3 & x_{23} & x_{13} \\ 0 & 0 & a_2 & x_{12} \\ 0 & 0 & 0 & a_1 \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

We denote by $T_4(F)/\sim_{at}$ the set of matrices of $T_4(F)$ identified under the anti-transpose.

Note that A^{at} is just the transpose of J_4AJ_4:

$$A^{at} = \{J_4AJ_4\}^t = J_4A^tJ_4,$$

with J_4 the 4×4 matrix of $(1,4),(2,3),(3,2),(4,1)$ components as 1 and other components as 0.

Corollary 4.4. *All idempotents of $T_4(F)/\sim_{at}$ are listed up as the zero matrix and*

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{12}x_{23} & x_{12}x_{24} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & x_{14} \\ 0 & 0 & x_{23} & x_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & x_{12} & x_{13} & -x_{12}x_{24} - x_{13}x_{34} \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{12}x_{24} + x_{13}x_{34} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_{12} & -x_{12}x_{23} & -x_{12}x_{24} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for any $x_{i,j} \in F$ $(i < j)$, and the identity matrix.

Corollary 4.5. *All classes of $E_0(T_4(F)/\sim_{at})$ are listed up as the zero class and*

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

and the identity class.

Proposition 4.6. *All square roots of $T_4(F)$ are listed up as, for any nonzero $x_{ij} \in F$,*

$$\begin{pmatrix}\pm1&0&0&0\\0&\pm1&0&0\\0&0&\pm1&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x_{12}&0&0\\0&\mp\mathbf{1}&0&0\\0&0&\pm1&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0&0\\0&\pm\mathbf{1}&x_{23}&0\\0&0&\mp\mathbf{1}&0\\0&0&0&\pm1\end{pmatrix},$$
$$\begin{pmatrix}\pm1&0&0&0\\0&\pm1&0&0\\0&0&\pm\mathbf{1}&x_{34}\\0&0&0&\mp\mathbf{1}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x_{12}&x_{13}&0\\0&\pm\mathbf{1}&x_{23}&0\\0&0&\pm\mathbf{1}&0\\0&0&0&\pm1\end{pmatrix},\quad \begin{pmatrix}\pm1&0&0&0\\0&\pm\mathbf{1}&x_{23}&x_{24}\\0&0&\mp\mathbf{1}&x_{34}\\0&0&0&\pm\mathbf{1}\end{pmatrix},$$
$$\begin{pmatrix}\pm\mathbf{1}&x_{12}&0&0\\0&\mp\mathbf{1}&0&0\\0&0&\pm\mathit{1}&x_{34}\\0&0&0&\mp\mathit{1}\end{pmatrix}\ \begin{pmatrix}\text{not compound in}\\\text{each bold and italic,}\\\text{but compound}\\\text{between both}\end{pmatrix},\quad \begin{pmatrix}\pm\mathbf{1}&x_{12}&x_{13}&0\\0&\mp\mathbf{1}&x_{23}&x_{24}\\0&0&\pm\mathbf{1}&x_{34}\\0&0&0&\mp\mathbf{1}\end{pmatrix}$$

with $x_{12}x_{24}+x_{13}x_{34} = 0$, where possible cases in the following are written as matrix forms as above, and there are impossible case as tuples as below, with non-zero components (x_{12}, x_{23}) but $x_{13} = 0$ or with non-zero components (x_{23}, x_{34}) but $x_{24} = 0$ or with more other non-zero components:

$$(x_{12}, x_{23}; x_{14}),\quad (x_{12}, x_{23}; x_{24}),\quad (x_{12}, x_{23}; x_{34}),$$
$$(x_{12}, x_{23}; x_{14}, x_{24}),\quad (x_{12}, x_{23}; x_{14}, x_{34}),\quad (x_{12}, x_{23}; x_{24}, x_{34}),$$
$$or\quad (x_{12}, x_{23}; x_{14}, x_{24}, x_{34}),$$

and

$$(x_{23}, x_{34}; x_{13}),\quad (x_{23}, x_{34}; x_{14}),\quad (x_{23}, x_{34}; x_{13}, x_{14}),$$
$$(x_{23}, x_{34}; x_{12}, x_{13}),\quad or\quad (x_{23}, x_{34}; x_{12}, x_{13}, x_{14});$$

and moreover,

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

and

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and there is an impossible case with non-zero components $(x_{13}, x_{23}, x_{24}, x_{34})$ *but* $x_{14} = 0$; *and*

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and there is an impossible case with non-zero compoents $(x_{12}, x_{13}, x_{23}, x_{24})$ *but* $x_{14} = 0$; *and furthermore,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \quad \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$
$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathit{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathit{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix} \quad \begin{pmatrix} \text{not compound in} \\ \text{each bold and italic,} \\ \text{but compound} \\ \text{between both} \end{pmatrix},$$

and there are the cases which do not exist, with non-zero components:

$$(x_{12}, x_{13}, x_{14}, x_{24}, x_{34}) \quad \text{or} \quad (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$$

(*the full case*); *and moreover,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathit{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathit{1} \end{pmatrix} \begin{pmatrix} \text{not compound in} \\ \text{each bold and italic,} \\ \text{but compound} \\ \text{between both} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & 0 \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and the two impossible cases with non-zero components $(x_{12}, x_{13}, x_{23}, x_{24})$ *and* $(x_{13}, x_{23}, x_{24}, x_{34})$; *and furthermore,*

$$\begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

and there are five impossible cases with non-zero components:

$$\begin{aligned} &(x_{12}, x_{13}, x_{14}, x_{24}), \quad (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}), \quad (x_{12}, x_{13}, x_{14}, x_{24}, x_{34}), \\ &(x_{13}, x_{14}, x_{24}, x_{34}), \quad \textit{or} \quad (x_{13}, x_{14}, x_{23}, x_{24}, x_{34}); \end{aligned}$$

and

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \pm\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \mp\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & 0 & x_{13} & x_{14} \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

and finally,

$$\begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & x_{23} & 0 \\ 0 & 0 & \pm\mathbf{1} & 0 \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & x_{13} & x_{14} \\ 0 & \mp\mathbf{1} & 0 & 0 \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \pm\mathbf{1} & 0 & 0 & x_{14} \\ 0 & \mp\mathbf{1} & x_{23} & x_{24} \\ 0 & 0 & \pm\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \mp\mathbf{1} \end{pmatrix}, \begin{pmatrix} \pm\mathbf{1} & x_{12} & 0 & x_{14} \\ 0 & \mp\mathbf{1} & 0 & x_{24} \\ 0 & 0 & \mp\mathbf{1} & x_{34} \\ 0 & 0 & 0 & \pm\mathbf{1} \end{pmatrix}.$$

In total, with respect to the off diagonal part, there are 41 *distinct possible cases and* 23 *distinct impossible cases in* $64 = 2^6$ *all the cases. In more details, together with the diagonal part of components compound or not, there are possible* $\{2^4+3(2^3)+2(2^2)+2\cdot 2+2\}+\{2(2^3)+3(2^2)+2\}+\{3(2^2)+2\}+\{2^3+4(2^2)+2\cdot 2\}+\{2\cdot 2+4(2)\}+\{2(2)\}+\{4(2)+2(2^2)+4(2)\} = 166$ *cases.*

Proof. Let

$$A = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ 0 & a_2 & x_{23} & x_{24} \\ 0 & 0 & a_3 & x_{34} \\ 0 & 0 & 0 & a_4 \end{pmatrix} \in T_4(F)$$

with $A^2 = I_4$ the 4×4 identity matrix, so that $a_j^2 = 1$ $(1 \le j \le 4)$, and $(a_1 + a_2)x_{12} = 0$, $(a_2 + a_3)x_{23} = 0$, $(a_3 + a_4)x_{34} = 0$, $(a_1 + a_3)x_{13} + x_{12}x_{23} = 0$, $(a_2 + a_4)x_{24} + x_{23}x_{34} = 0$, $(a_1 + a_4)x_{14} + x_{12}x_{24} + x_{13}x_{34} = 0$. Hence $a_j = \pm 1$ $(1 \le j \le 4)$.

If $x_{12} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_2 = 0$.

If $x_{23} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_2 + a_3 = 0$.

If $x_{34} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_3 + a_4 = 0$.

If $x_{12} \neq 0$, $x_{23} \neq 0$, then $(a_1 + a_3)x_{13} \neq 0$ and $a_1 + a_2 = 0$ and $a_2 + a_3 = 0$.

If $x_{23} \neq 0$, $x_{34} \neq 0$, then $(a_2 + a_4)x_{24} \neq 0$ and $a_2 + a_3 = 0$ and $a_3 + a_4 = 0$.

If $x_{12} \neq 0$, $x_{34} \neq 0$, and $x_{ij} = 0$ otherwise, then $a_1 + a_2 = 0$ and $a_3 + a_4 = 0$.

There is another case with $x_{12}x_{23}x_{34} \neq 0$, so that $(a_1 + a_3)x_{13} \neq 0$ and $(a_2 + a_4)x_{24} \neq 0$.

Note that $x_{12}x_{23} \neq 0$ implies $x_{13} \neq 0$ and also that $x_{23}x_{34} \neq 0$ implies $x_{24} \neq 0$, so that several impossible cases with $x_{12}x_{23} \neq 0$ but $x_{13} = 0$ and with $x_{23}x_{34} \neq 0$ but $x_{24} = 0$ are obtained.

Moreover, if $x_{13} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_3 = 0$. Also, if $x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_2 + a_4 = 0$.

And if $x_{13} \neq 0$, $x_{12}x_{23} = 0$, and $x_{ij} = 0$ otherwise, then $a_1 + a_3 = 0$ and either $a_1 + a_2$ or $a_2 + a_3 = 0$. In addition, there are two possible cases with $x_{34} \neq 0$ and another impossible case with $x_{34} \neq 0$.

And if $x_{24} \neq 0$, $x_{23}x_{34} = 0$, and $x_{ij} = 0$ otherwise, then $a_2 + a_4 = 0$ and either $a_2 + a_3 = 0$ or $a_3 + a_4 = 0$. In addition, there are two possible cases with $x_{12} \neq 0$ and another impossible case with $x_{12} \neq 0$.

Furthermore, if $x_{14} \neq 0$ and $x_{ij} = 0$ otherwise, so that $x_{12}x_{24} + x_{13}x_{34} = 0$, then $a_1 + a_4 = 0$. In addition, there are some other cases with $x_{12} \neq 0$ or $x_{24} \neq 0$; $x_{13} \neq 0$ or $x_{34} \neq 0$; $x_{23} \neq 0$ and more in what follows. But if $(a_1 + a_4)x_{14} \neq 0$, then $x_{12}x_{24} \neq 0$ if and only if $x_{13}x_{34} \neq 0$, which implies a contradiction in sigh on the diagonal, so that impossible are the case with $(x_{12}, x_{13}, x_{14}, x_{24}, x_{34})$ and the full case.

Moreover, if $x_{13}x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_3 = 0$ and $a_2 + a_4$. In addition, there are other four possible cases with several other non-zero components and two impossible cases.

Furthermore, if $x_{13}x_{14}x_{24} \neq 0$ and $x_{ij} = 0$ otherwise, then $a_1 + a_4 = 0$, $a_1 + a_3 = 0$ and $a_2 + a_4 = 0$. In addition, there are one more possible case with $x_{23} \neq 0$ and five impossible cases by the contradiction of signs on the diagonal.

And there are the possible cases with $x_{12}x_{13}x_{14} \neq 0$ or $x_{14}x_{24}x_{34} \neq 0$ and with $x_{13}x_{14}x_{23} \neq 0$ or $x_{14}x_{23}x_{24} \neq 0$, and the possible cases with $x_{13}x_{14}x_{34} \neq 0$ or $x_{12}x_{14}x_{24} \neq 0$, so that $a_1 + a_4 \neq 0$.

Finally, there are four cases that complement the list above in all the cases, with $a_1 + a_3 \neq 0$, $a_1 + a_4 \neq 0$, $a_2 + a_4 \neq 0$, and $a_1 + a_4 \neq 0$, respectively.

These possible and impossible cases correspond respectively to the matrices and the tuples in the statement in this order. □

Denote by $R_4(F)$ the set of all square roots of $T_4(F)$. Define the equivalence relation for elements of $R_4(F)$ by that two elements of $R_4(F)$ are equivalent if there is a continuous path (or a homotopy) within $R_4(F)$ between the two elements. Write by $E_1(T_4(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_4(F)$.

Corollary 4.7. *All classes of $E_1(T_4(F))$ are listed up as*

$$\begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{bmatrix} \quad \text{(compound in order 16 cases).}$$

Proof. Note that any square root in the list of Proposition 4.6 has a homotopy class within $R_4(F)$, equal to one of the $2^4 = 16$ homotopy classes in the statement, by deforming off-diagonal components to zero. □

5 The general case by homotopy We denote by $T_n(F)$ the algebra of all upper triangular $n \times n$ matrices over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. We give the topology on the algebra by the Euclidean norm, for convenience, via $T_n(F) \cong F^{1+2+3+4+\cdots+n} = F^{2^{-1}n(n+1)}$ as a space.

Denote by $P_n(F)$ the set of all idempotents of $T_n(F)$. Define the equivalence relation for elements of $P_n(F)$ by that two elements of $P_n(F)$ are equivalent if there is a continuous path (or a homotopy) within $P_n(F)$ between the two elements. Write by $E_0(T_n(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of an idempotent $P = (\cdots) \in P_n(F)$.

Let $\{e_{ij}\}_{i,j=1,i\leq j}^n$ be the matrix unit for $T_n(F)$.

Theorem 5.1. *All classes of $E_0(T_n(F))$ are listed up as the zero class and the classes $[e_{ii}]$ for $1 \leq i \leq n$, and $[e_{ii} + e_{jj}]$ for $1 \leq i < j \leq n$, and $[e_{ii} + e_{jj} + e_{kk}]$ for $1 \leq i < j < k$, and moreover, in general, $[e_{i_1 i_1} + e_{i_2 i_2} + \cdots + e_{i_s, i_s}]$ for $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ with $3 \leq s \leq n-1$, and the class of the $n \times n$ identity matrix, and there are 2^n homotopy classes in all.*

Proof. One can prove the claim by induction. Indeed, let $P \in P_n(F)$. Then there are two cases of the block decomposition for P:

$$P = \begin{pmatrix} 1 & c \\ 0_{n-1} & Q \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} 0 & c \\ 0_{n-1} & Q \end{pmatrix}$$

with $Q \in P_{n-1}(F)$, c a $1 \times (n-1)$ row vector and 0_{n-1} the $(n-1) \times 1$ column zero vector, such that $Q^2 = Q$ and $cQ = 0_{n-1}^t$ the transpose of 0_{n-1}. By induction, the class $[Q]$ for Q is one of the classes listed as in the statement in the case of $n-1$. And then in both of two cases, the class $[P]$ can be one of the classes listed as in the statement just in the case of n, by deforming c to the $1 \times (n-1)$ row zero vector within $P_n(F)$ by a continuous path (i.e. a homotopy). □

We define the semigroup $\langle E_0(T_n(F))\rangle$ generated by $E_0(T_n(F))$ with the addition given by $[p]+[q] = [p+q]$ for $p, q \in P_n(F)$ if p is orthogonal to q, i.e. if $pq = 0$ and by $[p]+[p] = 2[p]$ and by $[p]+[q] = [p-p\wedge q]+2[p\wedge q]+[q-p\wedge q]$ if $pq \neq 0$, where $p\wedge q$ means the projection corresponding to the intersection of their ranges. It follows that the semigroup $\langle E_0(T_n(F))\rangle$ becomes an additive semigroup with the zero class as the identity element by this operation.

We define an abelian group $V_0(T_n(F))$ to be the Grothendieck group of the semigroup $\langle E_0(T_n(F))\rangle$. We say that $V_0(T_n(F))$ is the V_0-group of $T_n(F)$.

Corollary 5.2. *We obtain*

$$V_0(T_n(F)) \cong \mathbb{Z}^n.$$

Proof. Indeed, the group $V_0(T_n(F))$ is generated by the classes $[e_{11}]$, $[e_{22}], \cdots$, and $[e_{nn}]$, and the isomorphism is given by the correspondence:

$$\sum_{j=1}^{n} a_j[e_{jj}] \leftrightarrow (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n.$$

□

Corollary 5.3. *The class of Banach algebras of all upper triangular matrices over real or complex numbers is classified by their V-groups in the sense that $T_n(F) \cong T_m(F)$ as a Banach algebra if and only if $V_0(T_n(F)) \cong V_0(T_m(F))$ as a group.*

Denote by $R_n(F)$ the set of all square roots of $T_n(F)$. Define the equivalence relation for elements of $R_n(F)$ by that two elements of $R_n(F)$ are equivalent if there is a continuous path (or a homotopy) within $R_n(F)$ between the two elements. Write by $E_1(T_n(F))$ the set of all equivalence classes by the equivalence relation. Denote by $[\cdots]$ the class of a square root $R = (\cdots) \in R_n(F)$.

Theorem 5.4. *All classes of $E_1(T_n(F))$ are listed up as*

$$[(\pm e_{11}) + (\pm e_{22}) + \cdots + (\pm e_{nn})],$$

2^n classes in all.

Proof. One can prove the claim by induction. Indeed, let $R \in R_n(F)$. Then there are two cases of the block decomposition for R:

$$R = \begin{pmatrix} \pm 1 & c \\ 0_{n-1} & S \end{pmatrix}$$

with $S \in R_{n-1}(F)$, c a $1 \times (n-1)$ row vector and 0_{n-1} the $(n-1) \times 1$ column zero vector, such that $S^2 = I_{n-1}$ the $(n-1) \times (n-1)$ identity matrix and $\pm c + cS = 0_{n-1}^t$ the transpose of 0_{n-1}. By induction, the class $[S]$ for S is one of the classes listed as in the statement in the case of $n-1$. And then in both of two cases, the class $[R]$ can be one of the classes listed as in the statement just in the case of n, by deforming c to the $1 \times (n-1)$ row zero vector within $R_n(F)$ by a continuous path (i.e. a homotopy). □

We define the group $V_1(T_n(F))$ generated by $E_1(T_n(F))$ with the multiplication given by $[r] \cdot [s] = [rs]$ for $r, s \in R_n(F)$. It follows that the group $V_1(T_n(F))$ is an abelian group with the class of the $n \times n$ identity matrix I_n as the unit. We say that $V_1(T_n(F))$ is the V_1-group of $T_n(F)$.

Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2. Denote by $\oplus$ the diagonal sum.

Corollary 5.5. *We obtain*

$$V_1(T_n(F)) \cong (\mathbb{Z}_2)^n \equiv \Pi^n \mathbb{Z}_2.$$

Proof. Indeed, the group $V_1(T_n(F))$ is generated by the classes $[-1 \oplus I_{n-1}]$, $[1 \oplus -1 \oplus I_{n-2}], \cdots$, and $[I_{n-1} \oplus -1]$, and the isomorphism is given by the correspondence:

$$\Pi_{j=1}^{n}[I_{j-1} \oplus a_j \oplus I_{n-j}] = [a_1 \oplus a_2 \oplus \cdots \oplus a_n]$$
$$\leftrightarrow (a_1, a_2, \cdots, a_n) \in (\mathbb{Z}_2)^n,$$

where each a_j is 1 or -1 and I_0 is removed to be empty. □

Corollary 5.6. *The class of Banach algebras of all upper triangular matrices over real or complex numbers is classified by their V-groups in the sense that $T_n(F) \cong T_m(F)$ as a Banach algebra if and only if $V_1(T_n(F)) \cong V_1(T_m(F))$ as a group.*

6 Inductive limits by V-theory There is a canonical unital inclusion map $i_{n,m}$ from $T_n(F)$ to $T_m(F)$ if n divides m as $m = kn$ for some k a positive integer, defined as $i_{n,m}(p) = \oplus^k p$ the k-fold diagonal sum. Note that there are some other embeddings in the diagonal.

Let $T_\infty(F) = \varinjlim T_{n_j}(F)$ be the inductive limit of $\{T_{n_j}(F)\}_{j=1}^\infty$ with unital connecting maps $i_{n_j,n_{j+1}}$ for an increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that n_j divides n_{j+1} with $n_{j+1} = k_j n_j$ for some k_j a positive integer. Then $T_\infty(F)$ becomes a unital Banach algebra as a Banach algebra completion of the infinite union $\cup_{j=1}^\infty T_{n_j}(F)$ of $T_{n_j}(F)$. Note that $T_\infty(F)$ does depend on the choice of a family of connecting maps in general, as a non-trivial known fact (see [3, Exercises 6.3]).

Proposition 6.1. *Let $T_\infty(F) = \varinjlim T_{n_j}(F)$. We obtain*

$$V_0(T_\infty(F)) \cong \varinjlim \mathbb{Z}^{n_j} \cong \varinjlim \{\mathbb{Z}[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}[\frac{n_j}{n_j}]\}$$

as inductive limits of scaled ordered groups, with $[1] = \lim_{j\to\infty}\{[\frac{1}{n_j}] + \cdots + [\frac{n_j}{n_j}]\}$ as scale. Also,

$$V_1(T_\infty(F)) \cong \varinjlim \Pi^{n_j}\mathbb{Z}_2 \cong \varinjlim \{\mathbb{Z}_2[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_j}{n_j}]\}$$

with $[1] = \lim_{j\to\infty}\{[\frac{1}{n_j}] + \cdots + [\frac{n_j}{n_j}]\}$.

Proof. The inclusion map $i_{n_j,n_{j+1}}$ induces the injective group homomorphism:

$$(i_{n_j,n_{j+1}})_* : V_0(T_{n_j}(F)) \to V_0(T_{n_{j+1}}(F)),$$

so that $(i_{n_j,n_{j+1}})_*$ maps $\mathbb{Z}^{n_j} \cong \mathbb{Z}[\frac{1}{n_j}]\oplus\cdots\oplus\mathbb{Z}[\frac{n_j}{n_j}]$ injectively to $\mathbb{Z}^{n_{j+1}} \cong \mathbb{Z}[\frac{1}{n_{j+1}}]\oplus\cdots\oplus\mathbb{Z}[\frac{n_{j+1}}{n_{j+1}}]$ by Corollary 5.2, where we have $(i_{n_j,n_{j+1}})_*([p]) = [\oplus^{k_j} p]$ for $[p] \in V_0(T_{n_j}(F))$, so that the class $[p]$ is identified with $[\oplus^{k_j} p]$ and with their limit class in $V_0(T_\infty(F))$, and each k-th coordinate base for $\mathbb{Z}^{n_j}$ is identified with $\frac{k}{n_j}$ for $1 \le k \le n_j$. Therefore,

$$V_0(T_\infty(F)) \cong \varinjlim \mathbb{Z}^{n_j} \cong \varinjlim \{\mathbb{Z}[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}[\frac{n_j}{n_j}]\}.$$

Also, induced is the injective group homomorphism:

$$(i_{n_j,n_{j+1}})_* : V_1(T_{n_j}(F)) \to V_1(T_{n_{j+1}}(F)),$$

so that $(i_{n_j,n_{j+1}})_*$ maps $\Pi^{n_j}\mathbb{Z}_2 \cong \mathbb{Z}_2[\frac{1}{n_j}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_j}{n_j}]$ injectively to $\Pi^{n_{j+1}}\mathbb{Z}_2 \cong \mathbb{Z}_2[\frac{1}{n_{j+1}}] \oplus \cdots \oplus \mathbb{Z}_2[\frac{n_{j+1}}{n_{j+1}}]$ by Corollary 5.5 and by the same reason as above. □

Next, let $\varinjlim \oplus_{j=1}^k T_{n_{j,k}}(F)$ be a unital inductive limit of finite direct sums $\oplus_{j=1}^k T_{n_{j,k}}(F)$ with unital connecting maps $i_{k,k+1}$ such that each $n_{j,k+1}$ is a weighted sum of $n_{j,k}$, so that $n_{j,k+1} = \sum_{s=1}^k m_{s,k} n_{s,k}$ for some integers $m_{s,k} \ge 0$ and $i_{k,k+1}(x_l) = \oplus_{s=1}^k [\oplus_{j=1}^{m_{s,k}} x_l]$ for $x_l \in T_{n_{l,k}}(F)$. The diagram for such connecting maps is known as the Bratteli diagram (cf. [3] and [4]).

Proposition 6.2. *We obtain*

$$V_0(\varinjlim \oplus_{j=1}^k T_{n_{j,k}}(F)) \cong \varinjlim \oplus_{j=1}^k \mathbb{Z}^{n_{j,k}} \cong \varinjlim \{\oplus_{j=1}^k (\oplus_{s=1}^{n_{j,k}} \mathbb{Z}[\frac{s}{n_{j,k}}])\}$$

as inductive limits of scaled ordered groups, with $[1] = \lim_{k\to\infty}\{\oplus_{j=1}^k(\oplus_{s=1}^{n_{j,k}}[\frac{s}{n_{j,k}}])\}$ *as scale. Also,*

$$V_1(\varinjlim \oplus_{j=1}^k T_{n_{j,k}}(F)) \cong \varinjlim \oplus_{j=1}^k (\mathbb{Z}_2)^{n_{j,k}} \cong \varinjlim\{\oplus_{j=1}^k(\oplus_{s=1}^{n_{j,k}}\mathbb{Z}_2[\frac{s}{n_{j,k}}])\}$$

with $[1] = \lim_{k\to\infty}\{\oplus_{j=1}^k(\oplus_{s=1}^{n_{j,k}}[\frac{s}{n_{j,k}}])\}$.

Moreover, the V-theory groups V_0 *or* V_1 *with the scaled unit classes are complete invariants for unital inductive limits of finite direct sums of upper triangular matrix Banach algebras.*

Proof. The last consequence follows from the classification theorem for unital AF C^*-algebras which contain canonically those non self-adjoint inductive limits as only subalgebras, by the same way as in [3]. □

On the other hand, let $\{n_j\}_{j=1}^\infty$ be an increasing sequence of positive integers. We now denote by $K_\infty(F)$ the inductive limit of $T_{n_j}(F)$ by the non-unital inclusion maps given by $x \mapsto x \oplus O_{n_{j+1}-n_j}$ for $x \in T_{n_j}(F)$, where $O_{n_{j+1}-n_j}$ is the zero square matrix of size $n_{j+1} - n_j$. Then $K_\infty(F)$ becomes a Banach algebra as a Banach algebra completion of the infinite union of $T_{n_j}(F)$. Note that $K_\infty(F)$ does not depend on the choice of a family of connecting maps. Also, $K_\infty(F)$ is a non-unital algebra, so that it has no square roots.

For a non-unital Banach algebra B, one may define its V_1-group to be that of the unitization B^+ by F, so that $V_1(B) = V_1(B^+)$, as one way.

But, on the other way, for a non-unital Banach algebra which can be written as an inductive limit of unital Banach algebras, which may or not depend on a choice of a family of connecting maps, we this time define its V-theory group V_1 to be inductive limits of their V-theory groups V_1, so that the continuity in inductive limits do hold even in the non-unital case, depending on the choice.

Proposition 6.3. *We obtain*

$$V_0(K_\infty(F)) \cong \lim_{j\to\infty} \oplus^{n_j}\mathbb{Z},$$

and $V_1(K_\infty(F)) = \varinjlim V_1(T_{n_j}(F)) = \varinjlim(\mathbb{Z}_2)^{n_j}$.

Proof. Note that

$$V_0(K_\infty(F)) \cong V_0(\varinjlim T_{n_j}(F)) \cong \varinjlim \oplus^{n_j}\mathbb{Z},$$

where $n_j \to \infty$ as $j \to \infty$. □

In general,

Proposition 6.4. *Our* V_0*-theory group is always continuous, with respect to inductive limit Banach algebras, and the* V_1*-theory group is continuous only for unital inductive limit Banach algebras with unital connecting maps.*

Proof. It should follows from continuity of K-theory groups for inductive limits of C^*-algebras (see [5]), by the similar way. But omitted. □

Remark. A more general theory for V-theory groups may be continued to be investigated somewhere else in the future.

Let $\{N_j\}_{j=1}^\infty$ be an increasing sequence of positive integers. Denote by $\varinjlim \oplus_{j=1}^k T_{n_j}(F)$ a canonical inductive limit of the block diagonal sums $\oplus_{j=1}^k T_{n_j}(F)$ of $T_{n_j}(F)$ $(1 \le j \le k)$ in $T_{N_k}(F)$ with $\sum_{j=1}^k n_j = N_k$ and $n_k = N_k - N_{k-1}$ and $n_1 = N_1$, where the non-unital connecting maps are given by $x \mapsto x \oplus 0_{n_{k+1}}$ for x in the k-fold diagonal sum. Then the inductive limit is non-unital and is an infinite direct sum of block diagonal components, so that $\varinjlim \oplus_{j=1}^{k} T_{n_j}(F) \cong \oplus_{j=1}^{\infty} T_{n_j}(F)$. As well, let $\varinjlim T_{N_k}(F)$ be a non-unital inductive limit of $T_{N_k}(F)$ by the same way as above.

Proposition 6.5. *We obatin*

$$V_0(\varinjlim \oplus_{j=1}^{k} T_{n_j}(F)) \cong \varinjlim \oplus_{j=1}^{k} \mathbb{Z}^{n_j} \cong \varinjlim \mathbb{Z}^{N_k} \cong V_0(\varinjlim T_{N_k}(F)),$$

as an inductive limit of groups, but not as an inductive limit of scaled ordered groups, with

$$[1] = \lim_{k\to\infty} \{[1_{n_1}] + \cdots + [1_{n_k}]\} \quad \text{and} \quad [1] = \lim_{k\to\infty} \{[1_{N_k}]\}$$

as the scales of the respective (extended) unit classes. Also,

$$V_1(\varinjlim \oplus_{j=1}^{k} T_{n_j}(F)) = \varinjlim \oplus_{j=1}^{k} (\mathbb{Z}_2)^{n_j} \cong \varinjlim (\mathbb{Z}_2)^{N_k} \cong V_1(\varinjlim T_{N_k}(F))$$

as an inductive limit of groups, but not as an inductive limit of scaled oredered groups.

Consequently,

$$\varinjlim \oplus_{j=1}^{k} T_{n_j}(F) \not\cong \varinjlim T_{N_k}(F).$$

Proof. The last consequence follows from the classification theory for non self-adjoint Banach algebras viewed as sub-Banach algebras of AF C^*-algebras and UHF-algebras (see [3] in details). □

To distinguish non-unital inductive limits of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ for any sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers, we introduce a notion as follows. We may say that the sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers is a sequence of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$. We define that two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of positive integers is equivalent up to inductive permutation if for any m a positive interger, there are positive intergers k and k' such that $k, k' \geq m$ and the finite sequence $\{a_1, \cdots, a_k\}$ is the same sequence as $\{b_1, \cdots, b_{k'}\}$ by subtracting finitely many l and l' elements so that $k - l = k' - l' = m$ and by a permutation of m elements left, so that the respective unions of left elements are the respective sequences.

Proposition 6.6. *Two non-unital inductive limits of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$ for two sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ of positive integers, respectively, are isomorphic as Banach algebras if and only if these sequences are equivalent up to inductive permutation.*

Proof. The equivalence between those sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ implies that there exist isomorphisms of corresponding finite block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$ by permutation of their direct summands, inductively. Therefore, there exists an isomorphism between those inductive limits by the density of unions of isomorphic finite block diagonal sums in the inductive limits.

Conversely, the isomorphism denoted by Φ between the inductive limits denoted by $\mathfrak{I}$ and $\mathfrak{K}$ implies that each finite block diagonal sum of $\mathfrak{I}$ is mapped into a finite block diagonal sum of $\mathfrak{K}$ by Φ. Therefore, it follows that there is the equivalence between the sequences. □

Let $\mathfrak{I}$ be a non-unital inductive limit of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ for a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers. Then the (inductive or extended) unit I associated to $\mathfrak{I}$ (but not in $\mathfrak{I}$) is equal to

$$\varinjlim \oplus_{j=1}^{k} I_{n_j}, \quad I_{n_j} \in T_{n_j}(F) \text{ the units.}$$

We say that the limit is the inductive partition of the (extended) unit I. Or we may call it the scale of the inductive limit $\mathfrak{I}$, and write $\Sigma_{\mathfrak{I}}$. Similarly as in the case of sequences above, we define that two inductive partitions $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ of the respective units associated to two inductive limits $\mathfrak{L}$ and $\mathfrak{K}$ of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$, respectively, are equivalent up to inductive permutation if for any m a positive interger, there are positive intergers k and k' such that $k, k' \geq m$ and the element $\oplus_{j=1}^{k} I_{n_j}$ is identified with the element $\oplus_{j=1}^{k'} I_{m_j}$ by subtracting finitely many l and l' diagonal sum components so that $k - l = k' - l' = m$ and by a permutation of m diagonal sum components left, so that the respective left components add up to the respective units. In this case, we write $\Sigma_{\mathfrak{I}} \sim \Sigma_{\mathfrak{K}}$.

Corollary 6.7. *Non-unital inductive limits $\mathfrak{I}$ and $\mathfrak{K}$ of block diagonal sums of $\{T_{n_j}(F)\}_{j=1}^{\infty}$ and $\{T_{m_j}(F)\}_{j=1}^{\infty}$ for two sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ of positive integers, respectively, are isomorphic as Banach algebras if and only if the respective inductive partitions of units $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ are equivalent up to inductive permutation, i.e., $\Sigma_{\mathfrak{I}} \sim \Sigma_{\mathfrak{K}}$.*

Proof. The respective inductive partitions of units $\varinjlim \oplus_{j=1}^{k} I_{n_j}$ and $\varinjlim \oplus_{j=1}^{k} I_{m_j}$ are by definition, equivalent up to inductive permutation if and only if the sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ are equivalent up to inductive permutation. □

Remark. Those isomorphisms between the inductive limits are given by permutations, that are essentially equivalent to taking unitary equivalences, that are not allowed in the inductive limits. Namely, the isomorphisms exist in the self-adjoint world. If not allowed, i.e., in the non self-adjoint world, the inductive limits can not be isomorphic except the trivial cases. Note also that block diagonal sums are essentially equivalent to direct sums.

We may call the unital or non-unital, inductive limits of finite direct sums of the upper triangular matrix algebras as ATM algebras, in the sense of being approximately triangular matrix algebras. As a summary,

Corollary 6.8. *Two non-unital ATM algebras are isomorphic if and only if their scales are equivalent in our sense, where we suppose that permutations are allowed in isomorphisms.*

As well,

Corollary 6.9. *Two unital or non-unital ATM algebras are isomorphic if and only if their scaled V-theory groups are isomorphic.*

Proof. Note that the unital case can be proved within the same context as in the non-unital case above, without using the classification result in C^*-algebras. □

Remark. This is a sort of classification result in non self-adjoint Banach algebras corresponding to that of AF C^*-algebras. However, our method for the classification is similar to that of the C^*-algebra case, and the results should be the same essentially as contents.

REFERENCES

[1] B. BLACKADAR, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[2] G. A. ELLIOTT, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29-44.

[3] STEPHEN C POWER, *Limit algebras: an introduction to subalgebras of C^*-algebras*, Pitman Research Notes in Math. Ser. **278**, Longman Scientific & Technical (1992).

[4] M. RØRDAM, F. LARSEN AND N. J. LAUSTSEN, *An Introduction to K-Theory for C^*-Algebras*, LMSST **49**, Cambridge (2000).

[5] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Univ. Press (1993).

Communicated by *Moto O'uchi*

Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara, Okinawa 903-0213, Japan.
Email: sudo@math.u-ryukyu.ac.jp

$g(x)$-NIL CLEAN RINGS

HANI A. KHASHAN AND ALI H. HANDAM

Received July 9,2014

Abstract. An element in a ring R with identity is called nil clean if it is the sum of an idempotent and a nilpotent, R is called nil clean if every element of R is nil clean. Let $C(R)$ be the center of a ring R and $g(x)$ be a fixed polynomial in $C(R)[x]$. Then R is called $g(x)$-nil clean if every element in R is a sum of a nilpotent and a root of $g(x)$. In this paper, we investigate many properties and examples of $g(x)$-nil clean rings. Moreover, we characterize nil clean rings as $g(x)$-nil clean rings where $g(x) \in (x-(a+1))(x-b)C(R)[x]$, $a, b \in C(R)$ and $b - a \in N(R)$.

1. Introduction

Throughout this paper R denotes an associative ring with identity and all modules are unitary. The group of units, the set of idempotents and the set of nilpotent elements in R are denoted by $U(R)$, $Id(R)$ and $N(R)$ respectively. Following Han and Nicholson [11], an element $r \in R$ is called clean if $r = e + u$ for some $e \in Id(R)$ and $u \in U(R)$. A ring R is called clean if every element of R is clean. The notion of clean rings was first introduced by Nicholson [14] in 1977 in his study of lifting idempotents and exchange rings. Since then, some stronger concepts have been considered (e.g. uniquely clean, strongly clean and some special clean rings), see [4, 6, 15, 17, 18, 19, 20]. As well as some weaker ones (e.g. almost clean and weakly clean rings), see [1]. Recently, in 2013, Diesl [9] studied a stronger concept than clean rings, namely, nil-clean rings. They are rings in which every element is a sum of an idempotent element and a nilpotent element. In fact, nil clean rings were firstly presented in [12] as a special case of rings in which every element is a sum of nilpotent and potent elements.

Let $C(R)$ denotes the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Then following Camillo and Simón [5], R is called $g(x)$-clean if for each $r \in R$, $r = s + u$ where $u \in U(R)$ and $g(s) = 0$. Of course $(x^2 - x)$-clean rings are precisely the clean rings.

Nicholson and Zhou [16] proved that if $g(x) \in (x - a)(x - b)C(R)[x]$ with $a, b \in C(R)$ and $b, b - a \in U(R)$ and ${}_RM$ is a semisimple left R-module, then $End({}_RM)$ is $g(x)$-clean. Recently, Fan and Yang [10], studied more properties of $g(x)$-clean rings. Among many conclusions, they prove that if $g(x) \in (x - a)(x - b)C(R)[x]$ where $a, b \in C(R)$ with $b - a \in U(R)$, then R is a clean ring if and only if R is $(x - a)(x - b)$-clean.

In this paper, we define and study $g(x)$-nil clean rings as a special class of $g(x)$-clean rings. For a ring R and $g(x) \in C(R)[x]$, an element $r \in R$ is called $g(x)$-nil clean if $r = s+b$ for some $b \in N(R)$ and $g(s) = 0$. Moreover, R is called $g(x)$-nil clean if every element in R is $g(x)$-nil clean.

2000 *Mathematics Subject Classification.* 16N40, 16U99.
Key words and phrases. clean ring, $g(x)$-clean ring, nil clean ring, $g(x)$-nil clean ring.

In section 2, we study many properties of $g(x)$-nil clean rings analogous to those of nil clean and $g(x)$-clean rings. In particular, for a commutative ring R, we justify a condition under which the the amalgamated duplication $R \bowtie I$ of a ring R along an ideal I is $g(x)$-nil clean. Also, we consider the idealization $R(M)$ of any R-module M and prove that $R(M)$ is $g(x)$-nil clean ring if and only if R is so.

In section 3, we study $(x^2 + cx + d)$-nil clean rings where $c, d \in C(R)$. We give many characterizations for a nil clean ring R in terms of some $g(x)$-nil clean rings. In particular for $n \in \mathbb{N}$, we focus on $(x^2 - (n-1)x)$-nil clean and $(x^n - x)$-nil clean rings.

2. $g(x)$- NIL CLEAN RINGS

In this section, we give some properties of $g(x)$-nil clean rings which are similar to those of $g(x)$-clean rings.

Definition 2.1. Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element

$r \in R$ is called $g(x)$-nil clean if $r = b + s$ where $g(s) = 0$ and b is a nilpotent of R. We say that R is $g(x)$-nil clean if every element in R is $g(x)$-nil clean.

Clearly, nil clean rings are $(x^2 - x)$-nil clean. However, there are $g(x)$-nil clean rings which are not nil clean. For example, it can be easily proved that $\mathbb{Z}_3$ is an $(x^3 + 2x)$-nil clean ring which is not nil clean. For a non commutative $g(x)$-nil clean ring we have the following example.

Example 2.2. Consider the ring $R = \left\{ \begin{bmatrix} a & 2b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}_4 \right\}$.Then one can see that for any $x, y \in R$, $(x - x^2)(y - y^2) = 0$. Hence, R is $(x - x^2)^2$-nil clean.

Proposition 2.3. *Every $g(x)$-nil clean ring is $g(x)$-clean ring.*

Proof. Suppose R is a $g(x)$-nil clean ring and let $x \in R$. Then $x - 1 = b + s$ where b is nilpotent and $g(s) = 0$. Thus, $x = (b + 1) + s$ where $b + 1 \in U(R)$. Therefore, R is $g(x)$-clean. □

The converse of Proposition 2.3 is not be true in general. For example, one can verify that $\mathbb{Z}_{10}$ is $(x^7 - x)$-clean ring which is not $(x^7 - x)$-nil clean ring.

Let R and S be rings and $\phi : C(R) \to C(S)$ be a ring homomorphism with $\phi(1_R) = 1_S$. For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, we let $g^\star(x) := \sum_{i=0}^{n} \phi(a_i) x^i \in C(S)[x]$. In particular, if $g(x) \in \mathbb{Z}[x]$, then $g^\star(x) = g(x)$.

Proposition 2.4. *Let $\theta : R \to S$ be a ring epimorphism. If R is $g(x)$-nil clean, then S is $g^\star(x)$-nil clean.*

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ and consider $g^\star(x) := \sum_{i=0}^{n} \theta(a_i) x^i \in C(S)[x]$. For every $\alpha \in S$, there exist $r \in R$ such that $\theta(r) = \alpha$. Since R is $g(x)$-nil clean, there exist $s \in R$ and $u \in N(R)$ such that $r = u + s$ and $g(s) = 0$. So $\alpha = \theta(r) = \theta(u + s) = \theta(u) + \theta(s)$ with

$\theta(u) \in N(S)$ and $g^\star(\theta(s)) = \sum_{i=0}^{n}\theta(a_i)(\theta(s))^i = \sum_{i=0}^{n}\theta(a_i)\theta(s^i) = \sum_{i=0}^{n}\theta\left(a_i s^i\right) = \theta\left(\sum_{i=0}^{n} a_i s^i\right) = \theta(g(s)) = \theta(0) = 0$. Therefore, S is $g^\star(x)$-nil clean. □

Proposition 2.5. *If R is a $g(x)$-nil clean ring and I is an ideal of R, then $\overline{R} = R/I$ is $g^\star(x)$-nil clean. Moreover, The converse is true if I is nil and the roots of $g^\star(x)$ lift modulo I.*

Proof. For the first statement, we use Proposition 2.4 and the fact that $\theta : R \to R/I$ defined by $\theta(r) = \overline{r} = r + I$ is an epimorphism. Now, suppose R/I is $g^\star(x)$-nil clean and let $r \in R$. Then $\overline{r} = \overline{s} + \overline{b}$ where $\overline{b} \in N(\overline{R})$ and $g^\star(\overline{s}) = \overline{0}$. Since the roots of $g^\star(x)$ lift modulo I, we may assume that $s \in R$ with $g(s) = 0$. Now, $r - s$ is nilpotent modulo I and I is nil imply that $r - s$ is nilpotent. Therefore, R is $g(x)$-nil clean. □

Proposition 2.6. *Let $R_1, R_2, ..., R_k$ be rings and $g(x) \in \mathbb{Z}[x]$. Then $R = \prod_{i=1}^{k} R_i$ is $g(x)$-nil clean if and only if R_i is $g(x)$-nil clean for all $i \in \{1, 2, ..., n\}$.*

Proof. $\Rightarrow$) : For each $i \in \{1, 2, ..., k\}$, R_i is a homomorphic image of $\prod_{i=1}^{k} R_i$ under the projection homomorphism. Hence, R_i is $g(x)$-nil clean by Proposition 2.4.

$\Leftarrow$) : Let $(x_1, x_2, ..., x_k) \in \prod_{i=1}^{k} R_i$. For each i, write $x_i = n_i + s_i$ where $n_i \in N(R_i)$, $g(s_i) = 0$. Let $n = (n_1, n_2, ..., n_k)$ and $s = (s_1, s_2, ..., s_k)$. Then it is clear that $n \in N(R)$and $g(s) = 0$. Therefore, R is $g(x)$-nil clean. □

In general, the ring of polynomials $R[t]$ over a ring R is not $g(x)$-clean. This is also true for commutative $g(x)$-nil clean rings.

Proposition 2.7. *If R is any commutative ring, then the ring of polynomials $R[t]$ is not nil clean (and hence not $(x^2 - x)$-nil clean).*

Proof. Since R is commutative, $N(R[t]) = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k \mid a_0, a_1, \cdots, a_k \in N(R)$ and $k \in \mathbb{N}\}$. If t is nil clean, we may write $t = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + e$ where $e \in Id(R[t]) = Id(R)$ and $a_0, a_1, \cdots, a_k \in N(R)$. Hence, $1 = a_1 \in J(R)$ which is a contradiction. Therefore $R[t]$ is not nil clean. □

Let $\theta : R[[t]] \to R$ be defined by $\theta(f) = f(0)$. As a consequence of Proposition 2.3, if $R[[t]]$ is $g^\star(x)$-nil clean, then R is $g(x)$-nil clean.

Let R be a commutative ring and M an R-module. Nagata [13] introduced the idealization $R(M)$ of R and M. The idealization of R and M is the ring $R(M) = R \oplus M$ with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. This construction has been extensively studied and has many applications in different contexts, see [2] and [3].

Note that if $(r, m) \in R(M)$, then $(r, m)^k = (r^k, k r^{k-1} m)$ for any $k \in \mathbb{N}$. The proof of the following lemma is immediate.

Lemma 2.8. *Let R be a commutative ring and M an R-module. Then (b, m) is nilpotent in $R(M)$ if and only if b is nilpotent in R.*

We recall that R naturally embeds into $R(M)$ via $r \to (r, 0)$. Thus any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^{n} (a_i, 0)x^i \in R(M)[x]$ and conversely.

Theorem 2.9. *Let R be a commutative ring and M an R-module. Then the idealization $R(M)$ of R and M is $g(x)$-nil clean if and only if R is $g(x)$-nil clean.*

Proof. $\Rightarrow)$: Note that $R \simeq R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence R is $g(x)$-nil clean by Proposition 2.4.

$\Leftarrow)$: Let $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Write $r = b + s$ where $b \in N(R)$ and $g(s) = 0$. Then for $m \in M$, $(r, m) = (b, m) + (s, 0)$ where $(b, m) \in N(R(M))$ by Lemma 2.8. Moreover, we have

$$\begin{aligned} g(s,0) &= a_0(1,0) + a_1(s,0) + a_2(s,0)^2 + ... + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + a_2(s^2,0) + ... + a_n(s^n,0) \\ &= (a_0 + a_1 s + a_2 s^2 + ... + a_n s^n, 0) = (g(s),0) = (0,0). \end{aligned}$$

Therefore, $R(M)$ is $g(x)$-nil clean. □

Let R be a commutative ring with identity 1 and let I be a proper ideal of R. The amalgamated duplication of R along I is defined as $R \bowtie I = \{(a, a+i) : a \in R \text{ and } i \in I\}$. It is easy to check that $R \bowtie I$ is a subring with identity $(1,1)$ of $R \times R$ (with the usual componentwise operations). Moreover, $\varphi : R \to R \bowtie I$ defined by $\varphi(a) = (a, a)$ is a ring monomorphism and so $R \cong \{(a,a) : a \in R\} \subseteq R \bowtie I$. For more properties of $R \bowtie I$, one can see [7] and [8]. In the following theorem, we investigate the $g(x)$-nil cleanness of $R \bowtie I$.

Theorem 2.10. *Let R be a commutative ring, I be a proper ideal of R and $g(x) = \sum_{k=0}^{n} a_k x^k \in R[x]$. If R is $g(x)$- nil clean ring and $I \subseteq N(R)$, then $R \bowtie I$ is $g(x)$- nil clean ring. Moreover, the converse is true if $R \bowtie I$ is domain-like (every zero divisor of $R \bowtie I$ is nilpotent).*

Proof. Assume R is $g(x)$-nil clean. Let $(a, a+i) \in R \bowtie I$ and write $a = b + s$ where $b \in N(R)$ and $g(s) = 0$. Then $(a, a+i) = (b+s, b+s+i) = (b, b+i) + (s, s)$. Since $I \subseteq N(R)$, then $(b, b+i) \in N(R \bowtie I)$. Moreover, we have $g((s,s)) = \sum_{k=0}^{n} (a_k, a_k)(s,s)^k = \sum_{k=0}^{n} (a_k, a_k)(s^k, s^k) = (\sum_{k=0}^{n} a_k s^k, \sum_{k=0}^{n} a_k s^k) = (0,0)$. Therefore, $R \bowtie I$ is $g(x)$-nil clean.

Conversely, suppose that $R \bowtie I$ is domain-like $g(x)$-nil clean. Let $(0) \times I = \{(0,a) : a \in I\}$. Then clearly $(0) \times I$ is an ideal of $R \bowtie I$ with $R \bowtie I/(0) \times I \simeq R$. Thus, R is $g(x)$-nil clean by Proposition (2.3). Let i be a nonzero element in I and consider $(0, i) \in R \bowtie I$. Then $(0,i)(i,0) = (0,0)$ and so $(0,i)$ is a zero divisor in $R \bowtie I$. By assumption, $(0,i) \in N(R \bowtie I)$ and so $(0,i)^m = (0,0)$ for some $m \geq 1$. Therefore, $i^m = 0$ and $I \subseteq N(R)$. □

The proof of the following Lemma is straightforward.

Lemma 2.11. *Let R be a ring. For any $n \in \mathbb{N}$, we have*

$$N(T_n(R)) = \begin{bmatrix} Nil(R) & R & R & \cdots & R & R \\ 0 & Nil(R) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Nil(R) & R \\ 0 & 0 & 0 & \cdots & 0 & Nil(R) \end{bmatrix}$$ *where $T_n(R)$ is the upper triangular matrix ring over R.*

Theorem 2.12. *Let R be a ring, $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$ and $n \in \mathbb{N}$. Then R is $g(x)$-nil clean if and only if $T_n(R)$ is $g(x)$-nil clean.*

Proof. $\Leftarrow)$: Define $f : T_n(R) \longrightarrow R$ by $f(A) = a_{11}$ where $A = (a_{ij}) \in T_n(R)$. Then clearly f is a ring epimorphism and R is $g(x)$-nil clean.

$\Rightarrow)$: Suppose that R is $g(x)$-nil clean and let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix} \in T_n(R).$$ Since R is $g(x)$-nil clean, then for every $1 \leq i \leq n$, there exist $u_{ii} \in N(R)$ and $s_{ii} \in R$ such that $a_{ii} = u_{ii} + s_{ii}$ with $g(s_{ii}) = 0$. Write $A = B + C$ where $$B = \begin{bmatrix} u_{11} & b_{12} & b_{13} & \dots & b_{1,n-1} & b_{1n} \\ 0 & u_{22} & b_{23} & \dots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix}$$ and

$$C = \begin{bmatrix} s_{11} & 0 & \dots & 0 \\ 0 & s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & s_{nn} \end{bmatrix}.$$ Then B is nilpotent in $T_n(R)$ and $g(C) = a_0 I_n + a_1 C + ... +$ $$a_m C^m = \begin{bmatrix} g(s_{11}) & 0 & \dots & 0 \\ 0 & g(s_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & g(s_{nn}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$ Therefore, $T_n(R)$ is $g(x)$-nil clean. □

Theorem 2.13. *Let A and B be rings and let $M =_B M_A$ be a bimodule. If the formal triangular matrix $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ is $g(x)$-nil clean, then both, A and B are $g(x)$-nil clean.*

Proof. Let $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ be $g(x)$-nil clean. For every $a \in A$, $b \in B$ and $m \in M$, write $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = \begin{bmatrix} n_1 & 0 \\ n_2 & n_3 \end{bmatrix} + \begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix}$ where $\begin{bmatrix} n_1 & 0 \\ n_2 & n_3 \end{bmatrix} \in N(T)$ and $g\left(\begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix}\right) = 0$. Then $a = n_1 + s_1$ and $b = n_3 + s_3$. It is easy to see that $n_1 \in N(A)$, $n_2 \in N(B)$ and $g(s_1) = g(s_3) = 0$. Therefore, A and B are $g(x)$-nil clean. □

3. (x^2+cx+d)-nil Clean Rings

In this section we first consider $g(x)$-nil clean rings where $g(x) = (x-(a+1))(x-b)$, $a, b \in C(R)$. Then we turn to some special types of polynomials such as $x^n - 1$, $x^n - x$ and $x^n + x$.

For a ring R, a semisimple R-module ${}_RM$ and $a, b \in C(R)$, Nicholson and Zhou [16] proved that if $g(x) \in (x-a)(x-b)C(R)[x]$ where $b, b-a \in U(R)$, then $End({}_RM)$ is $g(x)$-clean. More recently, Fan and Yang proved the following.

Lemma 3.1. [10]. *Let R be a ring, $a, b \in C(R)$ and $g(x) \in (x-a)(x-b)C(R)[x]$ where $b-a \in U(R)$. Then*

(1) R is clean if and only if R is $(x-a)(x-b)$-clean

(2) If R is clean, then R is $g(x)$-clean.

Now, we prove the following main result.

Theorem 3.2. *Let R be a ring and $a, b \in C(R)$.Then R is nil clean and $b-a \in N(R)$ if and only if R is $(x-(a+1))(x-b)$-nil clean.*

Proof. $\Rightarrow$) : Let $r \in R$. Since R is nil clean, then $\frac{r-(a+1)}{(b-a)-1} = e + u$, where $e^2 = e$ and $u \in N(R)$. Hence, $r = e((b-a)-1)+(a+1)+u((b-a)-1) = t+v$ where t is a root of $(x-(a+1))(x-b)$ and $v \in N(R)$. Indeed,

$$\begin{aligned} & [e(b-a)-1)+(a+1)-(a+1)][e(b-a)-1)+(a+1)-b] \\ = \; & e^2((b-a)-1)^2 - e((b-a)-1)((b-a)-1) = 0 \end{aligned}$$

Thus, R is $(x-(a+1))(x-b)$-nil clean.

$\Leftarrow$) : Conversely, suppose R is $(x-(a+1))(x-b)$-nil clean. Then $a = s+u$ where $(s-(a+1))(s-b) = 0$ and $u \in N(R)$. Thus, $s-a \in N(R)$ and so $s-a-1 \in U(R)$. It follows that $s = b$ and $b-a \in N(R)$. Now, let $r \in R$. Since R is nil $(x-(a+1))(x-b)$-clean, then $r((b-a)-1)+(a+1) = s+u$ where s is a root of $(x-(a+1))(x-b)$ and $u \in N(R)$. Hence, $r = \frac{s-(a+1)}{(b-a)-1} + \frac{u}{(b-a)-1}$ where $\frac{u}{(b-a)-1} \in N(R)$ and

$$\begin{aligned} \left(\frac{s-(a+1)}{(b-a)-1}\right)^2 &= \frac{(s-(a+1))(s-b+b-(a+1))}{((b-a)-1)^2} \\ &= \frac{(s-(a+1))(s-b)+(s-(a+1))(b-(a+1))}{((b-a)-1)^2} = \frac{s-(a+1)}{(b-a)-1}. \end{aligned}$$

Therefore, R is nil clean. □

Next, we give some special cases of Theorem 3.2.

Corollary 3.3. *Let R be a ring and $a \in C(R)$. Then R is nil clean if and only if R is $(x^2-(2a+1)x+a(a+1))$-nil clean.*

Proof. We just take $a = b$ in Theorem 3.2. □

For example, we conclude that (x^2-3x+2)-nil clean rings, (x^2-5x+6)-nil clean rings and $(x^2-7x+12)$-nil clean rings are equivalent to nil clean rings.

Lemma 3.4. [9]. *If a ring R is nil clean, then 2 is a (central) nilpotent element in R.*

As 2 is a central nilpotent in any nil clean ring R, then $2n \in N(R)$ for any integer n. So the, previous lemma provides us with more characterizations of nil clean rings.

Corollary 3.5. *Let R be a ring and n be any integer. For any $b \in C(R)$, the following are equivalent*

(1) R is nil clean.

(2) R is $(x^2 - (2b+1-2n)x + (b^2 + b(1-2n)))$-nil clean.

(3) R is $(x^2 - (2b+1+2n)x + (b^2 + b(1+2n)))$-nil clean.

Proof. In Theorem 3.2, we take $a = b-2n$ to get (1)⇔(2) and $a = b+2n$ to get (1)⇔(3). □

In particular, a ring R is nil clean if and only if R is $(x^2 - (2n+1)x)$-nil clean ($(x^2+(2n-1)x)$-nil clean). For example, (x^2+x)-nil clean, (x^2+3x)-nil clean, (x^2-3x)-nil clean and (x^2-5x)-nil clean rings are all equivalent to nil clean rings.

Remark 3.6. The equivalence of (x^2+x)-nil clean rings and nil clean rings is a global property. That is, it holds for a ring R but it may fail for a single element. For example, $1 \in \mathbb{Z}_{12}$ is nil clean but it is not (x^2+x)-nil clean in $\mathbb{Z}_{12}$.

Remark 3.7. In [10], The authors give more characterizations of clean rings in terms of $g(x)$-clean rings under the additional assumption that 2 is a unit. But in a nil clean ring R, if we assume that $2n+1 \in N(R)$ for some integer n, then $1 \in N(R)$ by lemma 3.4. Thus, $1 = 0$ and $R = \{0\}$.

Definition 3.8. A ring R is called $g(x)$-nil*clean if every $0 \neq r \in R$, $r = s + b$ where $b \in N(R)$ and $g(s) = 0$.

Of course, every $g(x)$-nil clean ring is $g(x)$-nil*clean. On the other hand, the following are examples of $g(x)$-nil*clean rings which are not $g(x)$-nil clean.

Example 3.9. Let p be a prime integer. Then the field Z_p is $(x^{p-1}-1)$-nil*clean which is not $(x^{p-1}-1)$-nil clean.

Proof. Let $0 \neq r \in Z_p$. Then $r = 0 + r$ where $0 \in N(R)$ and $r^{p-1} - 1 = 0$ in Z_p by Fermat Theorem. Hence, Z_p is $(x^{p-1}-1)$-nil*clean. On the other hand, since Z_p is reduced, then 0 can't be written as a sum of a nilpotent and a root of $x^{p-1}-1$. Therefore Z_p is not $(x^{p-1}-1)$-nil clean. □

Next, we give a general example.

Example 3.10. Let R be a non zero ring, $n \in \mathbb{N}$ and $g(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in C(R)[x]$ where $a_0 \in U(R)$. Then R is not $g(x)$-nil clean. In particular, If R is any non zero ring and $n \in \mathbb{N}$, then R is not (x^n-1)-nil clean.

Proof. Suppose R is $g(x)$-nil clean and write $0 = s+b$ where $b \in N(R)$ and $g(s) = 0$. Then $s(s^{n-1} + a_{n-1}s^{n-2} + \ldots + a_1) = -a_0 \in U(R)$ and so $s \in U(R)$. Since also $s = -b \in N(R)$, then $R = \{0\}$, a contradiction. □

Remark 3.11. Let R be a ring and $g(x) \in C(R)[x]$, The concepts of $g(x)$-nil clean and $g(x)$-nil*clean coincide if there is a non unit root of $g(x)$ such that $0 = s + b$ for some $b \in N(R)$. In particular, they coincide if all roots of $g(x)$ are non units.

Proposition 3.12. *Let R be a ring and $n \in \mathbb{N}$. Then R is $(x^n - 1)$-nil*clean if and only if for every $0 \neq r \in R$, $r = v + b$ where $b \in N(R)$ and $v^n = 1$.*

Proof. $\Rightarrow$) Let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $s^n - 1 = 0$. Then $s^n = 1$ and the result follows.

$\Leftarrow$) Conversely, let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $v^n = 1$. Then clearly v is a root of $x^n - 1$ and R is $(x^n - 1)$-nil*clean. ☐

It is well known that if a ring R is commutative, then the sum of a nilpotent element and a unit in R is again a unit. Thus, we have the following Corollary.

Corollary 3.13. *Any commutative $(x^n - 1)$-nil*clean is a field.*

Proposition 3.14. *Let R be a ring and $2 \leq n \in \mathbb{N}$. If R is $(x^{n-1} - 1)$-nil*clean, then R is $(x^n - x)$-nil clean.*

Proof. If $r = 0$, then clearly r is an $(x^n - x)$-nil clean element. Suppose $0 \neq r \in R$. Then $r = v + b$ where $b \in N(R)$ and $v^{n-1} = 1$ and so v is a root of $x^n - x$. Therefore, R is $(x^n - x)$-nil clean. ☐

The converse of Proposition 3.14 is true under a certain condition.

Theorem 3.15. *Let R be a ring and let $0 \neq a \in R$ such that $(a+1)R$ or $R(a+1)$ contain no non trivial idempotents. Then a is $(x^n - x)$-nil clean if and only if a is $(x^{n-1} - 1)$-nil clean. In particular, if for every $a \in R$, $(a + 1)R$ or $R(a + 1)$ contain no non trivial idempotents, then R is $(x^n - x)$-nil clean if and only if R is $(x^{n-1} - 1)$-nil*clean*

Proof. $\Leftarrow$) : We use Proposition 3.14.

$\Rightarrow$) : Suppose a is $(x^n - x)$-nil clean and $(a + 1)R$ contains no non trivial idempotents. Then $a = s + b$ where $b \in N(R)$ and $s^n = s$. Now, $as^{n-1} = s + bs^{n-1}$ and so $a(1 - s^{n-1}) = b(1 - s^{n-1})$. Set $y = 1 + b$. Then $y \in U(R)$ and $(a + 1)(1 - s^{n-1}) = (b + 1)(1 - s^{n-1}) = y(1 - s^{n-1})$. This implies that $y(1 - s^{n-1})y^{-1} = (a+1)(1 - s^{n-1})y^{-1} \in (a+1)R$. obviously, $y(1 - s^{n-1})y^{-1}$ is an idempotent. If $1 - s^{n-1} \neq 0$, then $y(1 - s^{n-1})y^{-1} \neq 0$. Thus, $(a+1)R$ contains a non trivial idempotent, a contradiction. If $R(a + 1)$ contains no non trivial idempotents, then we get a similar contradiction. Therefore, $1 - s^{n-1} = 0$ and s is a root of $x^{n-1} - 1$. Thus, a is $(x^{n-1} - 1)$-nil clean. The other part of the Theorem follows clearly. ☐

Recall that for a ring R and $n \in \mathbb{N}$, $U_n(R)$ denotes the set of elements in R that can be written as a sum of no more than n units. If R is $(x^n - 1)$-nil*clean and $1 \neq r \in R$, then $r - 1 = v + b$ where $b \in N(R)$ and $v^n = 1$ and so $r = v + (b + 1) \in U_2(R)$. Since also clearly $1 \in U_2(R)$, then $R = U_2(R)$. This result can be generalized as follows.

Proposition 3.16. *let R be a ring, $n \in \mathbb{N}$ and $g(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in C(R)[x]$ where $1 \pm a_0 \in N(R)$. If R is $g(x)$-nil*clean, then $R = U_2(R)$. In particular, if R is $(x^{n-2} + x^{n-3} + ... + x + 1)$-nil*clean, then $R = U_2(R)$ is $(x^n - x)$-nil clean.*

Proof. Let $1 \neq r \in R$ and write $r-1 = s+b$ where $b \in N(R)$ and $s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = 0$. Then $r = s + (b+1)$ where $b+1 \in U(R)$. Moreover, $s(s^{n-1} + a_{n-1}s^{n-2} + ... + a_1) = -a_0 \in U(R)$ and so $s \in U(R)$. Thus, $r \in U_2(R)$. Since also $1 \in U_2(R)$, then $R = U_2(R)$. In particular, suppose R is $(x^{n-2} + x^{n-3} + ... + x + 1)$-nil* clean, then $R = U_2(R)$ by taking $a_0 = 1 \in U(R)$. Now, if $r = 0$, then r is clearly an $(x^n - x)$-nil clean element. Let $0 \neq r \in R$ and write $r = s + b$ where $b \in N(R)$ and $s^{n-2} + s^{n-3} + ... + s + 1 = 0$. Then $s^n - s = s(s-1)(s^{n-2} + s^{n-3} + ... + s + 1) = 0$ and so R is $(x^n - x)$-nil clean. □

By choosing $n = 4$ in the previous proposition, we conclude that if R is $(x^2 + x + 1)$-nil*clean, then $R = U_2(R)$ is $(x^4 - x)$-nil clean.

In the next Proposition, we determine conditions under which the group ring RG is $(x^n - x)$-nil clean for some integer n.

Proposition 3.17. *Let R be a Boolean ring and G any cyclic group of order p (prime). Then RG is $(x^{2^{p-1}} - x)$- nil clean ring.*

Proof. Let $G =< g >$ be a cyclic group of order p and $x = a_0 + a_1 g + a_2 g^2 + ... + a_{m-1} g^{m-1} \in RG$. Using mathematical induction, it can be shown that $x^{2^k} = \sum_{i=0}^{m-1} a_i g^{2^k * i}, k = 1, 2, ...$. It follows from Fermat theorem that $2^{p-1} = 1 + np$ for some $n \in \mathbb{N}$. So, $x^{2^{p-1}} = \sum_{i=0}^{m-1} a_i g^{2^{p-1} * i} = \sum_{i=0}^{m-1} a_i g^{(1+np)*i} = \sum_{i=0}^{m-1} a_i g^i = x$. Thus, RG is $(x^{2^{p-1}} - x)$-nil clean ring. □

Next we give examples showing that $(x^n - x)$-nil cleanness of a ring R does not imply nil cleanness of R whether n is odd or even.

Example 3.18. The field $\mathbb{Z}_3$ is $(x^3 - x)$-nil clean which is not nil clean. Also, by Proposition 3.17 the group ring $\mathbb{Z}_2(C_3)$ is $(x^4 - x)$-nil clean which is not nil clean.

Proposition 3.19. *Let R be a ring and $n \in \mathbb{N}$. Then R is $(ax^{2n} - bx)$-nil clean if and only if R is $(ax^{2n} + bx)$-nil clean.*

Proof. ⇒) : Suppose R is $(ax^{2n} - bx)$-nil clean and let $r \in R$. Then $-r = u + s$ where $u \in N(R)$ and $as^{2n} - bs = 0$. Thus, $r = (-u) + (-s)$ where $-u \in N(R)$ and $a(-s)^{2n} + b(-s) = as^{2n} - bs = 0$. Therefore, R is $(ax^{2n} + bx)$-nil clean.

⇐) : Suppose R is $(ax^{2n} + bx)$-nil clean and let $r \in R$. Then $-r = u + s$ where $u \in N(R)$ and $as^{2n} + bs = 0$. Thus, $r = (-u) + (-s)$ where $-u \in N(R)$ and $a(-s)^{2n} - b(-s) = as^{2n} + bs = 0$. Therefore, R is $(ax^{2n} - bx)$-nil clean. □

By Proposition 3.19, we conclude that $\mathbb{Z}_2(C_3)$ is also $(x^4 + x)$-nil clean. On the other hand, the equivalence in Proposition 3.19 need not be true if we replace the even power $2n$ by an odd power $2n+1$. By a simple calculations, we can see that the field $\mathbb{Z}_3$ is $(x^3 - x)$-nil clean ($(x^5 - x)$-nil clean) but not $(x^3 + x)$-nil clean ($(x^5 + x)$-nil clean). However, we don't know whether $(x^n + x)$-nil cleanness implies the $(x^n - x)$-nil cleanness of R or not.

Recall that a ring R is called unit n-regular if for any $a \in R$, $a = a(ua)^n$ for some $u \in U(R)$. In [10], the authors ask about the relation between the following conditions on a ring R

(1) R is $(x^n - x)$-clean for all $n \geq 3$.

(2) R is a unit n-regular.

In general, condition (1) does not imply condition (2) for odd or even integer n. For example, the ring $\mathbb{Z}_4$ is $(x^3 - x)$-clean which is not unit 3-regular and the ring $\mathbb{Z}_8$ is $(x^4 - x)$-clean which is not unit 4-regular. However, we still don't know whether condition (2) implies condition (1) or not. On the other hand if we replace $(x^n - x)$-cleanness by $(x^n - x)$-nil cleanness in condition (1), then non of the two conditions implies the other. For example, $\mathbb{Z}_4$ is also $(x^4 - x)$-nil clean which is not unit 4-regular and $\mathbb{Z}_3$ is unit 4-regular which is not $(x^4 - x)$-nil clean.

References

[1] M. -S. Ahn and D. D. Anderson, Weakly clean rings and almost clean rings, Rocky Mount. J. Math. 36 (2006), 783–798.
[2] M. M. Ali, Idealization and Theorems of D. D. Anderson, Comm. Algebra, 34 (2006), 4479-4501.
[3] D. D. Anderson and M. Winders, Idealization of a module, J. Comm. Algebra, 1(1) (2009), 3-56.
[4] G. Borooah, A. J. Diesl, T. J. Dorsey, Strongly clean matrix rings over commutative local rings, J. Pure Appl. Algebra 212 (1)(2008), 281–296.
[5] V. P. Camillo, J. J. Simón, The Nicholson-Varadarajan theorem on clean linear transformations, Glasgow Math. J. 44 (2002), 365-369.
[6] H. Chen, On uniquely clean rings, Comm. Algebra, 39 (2011), 189-198.
[7] M. Chhiti, N. Mahdou, Homological dimension of the amalgamated duplication of a ring along a pure ideal, Afr. Diaspora J. Math. (N. S.) 10, no. 1, (2010), 1–6.
[8] M. D'Anna, M. Fontana, An amalgamated duplication of a ring along an ideal, J. Algebra Appl. 6 (3)(2007), 443–459.
[9] A. J. Diesl, Nil clean rings, Journal of Algebra, 383 (2013), 197-211.
[10] L. Fan, X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, Comm. Algebra. 36 (2008), 269-278.
[11] J. Han and W. K. Nicholson, Extensions of clean rings, Communications in Algebra 29(6) (2001), 2589–2595.
[12] Y. Hirano, H. Tominaga and A. Yaqub, On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element, 30 (1988).
[13] A. Lambert, T. G. Lucas, Nagata's principle of idealization in relation to module homomorphisms and conditional expectations, Kyungpook Math. J. 40(2000), 327-337.
[14] W. K. Nicholson, Lifting idempotents and exchange rings, Transactions of the American Mathematical Society 229 (1977), 269–278.
[15] W. K. Nicholson, Strongly clean rings and Fitting's lemma. Comm. Alg. 27 (1999), 3583–3592.
[16] W. K. Nicholson, Y. Zhou, Endomorphisms that are the sum of a unit and a root of a fixed polynomial, Canad. Math. Bull. 49 (2006), 265-269.
[17] Z. Wang and J. Chen, A note on clean rings, Algebra Colloquium, 14 (3)(2007), 537-540.
[18] G. Xiao, W. Tong, n-clean rings and weakly unit stable range rings. Comm. Alg. 33 (2003),1501-1517.
[19] G. Xiao, W. Tong, n-clean rings, Algebra Colloquium. 13 (4)(2006), 599-606.
[20] Y. Q. Ye, Semiclean rings, Comm. Algebra 31 (2003), 5609-5625.

Communicated by Klaus Denecke

Department of Mathematics, Al al-Bayt University, P.O.Box: 130095, Al Mafraq, Jordan

E-mail address: hakhashan@aabu.edu.jo

Department of Mathematics, Al al-Bayt University, P.O.Box: 130095, Al Mafraq, Jordan

E-mail address: ali.handam@windowslive.com

CRITERIA FOR THE C-INTEGRAL

Toshiharu Kawasaki Ichiro Suzuki

Received October 31, 2013 ; revised March 7, 2014

Abstract. The C-integral was introduced by Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. Moreover Bongiorno, Piazza and Preiss gave some criteria for the C-integral. On the other hand, Nakanishi gave some criteria for the restricted Denjoy integral. In this paper we will give new criteria for the C-integral in the style of Nakanishi.

1 Introduction and preliminaries

Throughout this paper we denote by $\mathbf{(L)}(S)$ and $\mathbf{(D^*)}(S)$ the class of all Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a,b]$ into $(0,\infty)$ and a δ-fine McShane partition is a collection $\{(I_k, x_k) \mid k = 1,\ldots,k_0\}$ of non-overlapping intervals $I_k \subset [a,b]$ satisfying $I_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$ and $\sum_{k=1}^{k_0} |I_k| = b-a$. If $\sum_{k=1}^{k_0} |I_k| \le b-a$, then we say that the collection is a δ-fine partial McShane partition.

In [3] Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. It is given as follows:

Definition 1.1. A function f from an inteval $[a,b]$ into $\mathbb{R}$ is said to be C-integrable if there exists a number A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1,\ldots,k_0\}$ with $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$, where $d(I_k, x_k) = \inf_{x\in I_k} d(x, x_k)$. The constant A is denoted by

$$A = (C)\int_{[a,b]} f(x)dx.$$

We denote by $\mathbf{(C)}([a,b])$ the class of all C-integrable functions from $[a,b]$ into $\mathbb{R}$.

We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b]$. We denote by $\mathbf{(N)}([a,b])$ the class of all Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [3] they also gave a criterion for the C-integral as follows:

Theorem 1.1. *Let f be a function from an inteval $[a,b]$ into $\mathbb{R}$. Then $f \in \mathbf{(C)}([a,b])$ if and only if there exists $h \in \mathbf{(N)}([a,b])$ such that $f - h \in \mathbf{(L)}([a,b])$.*

2010 *Mathematics Subject Classification.* 26A36, 26A39.
Key words and phrases. C-integral, Denjoy integral, Lebesgue integral, Newton integral.

By the theorem above $\mathbf{(C)}([a,b])$ is the minimal class which contains $\mathbf{(L)}([a,b])$ and $\mathbf{(N)}([a,b])$. Moreover it is contained in the class of all restricted Denjoy integrable functions. Now we refer to the following theorems given by Bongiorno [1, 2].

Theorem 1.2. *Let $f \in \mathbf{(C)}([a,b])$. Then for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C)\int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ with $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a,b]$ is an interval function on $[a,b]$. If an interval function F on $[a,b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a,b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive. For an interval function F on $[a,b]$, for a positive number ε, for a gauge δ and $E \subset [a,b]$ let

$$V_\varepsilon(F,\delta,E) = \sup\left\{ \sum_{k=1}^{k_0} |F(I_k)| \;\middle|\; \begin{array}{l} \{(I_k,x_k) \mid k=1,\ldots,k_0\} \text{ is a } \delta\text{-fine partial McShane} \\ \text{partition with } x_k \in E \text{ and } \sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon} \end{array} \right\}.$$

Moreover let

$$V_C F(E) = \sup_\varepsilon \inf_\delta V_\varepsilon(F,\delta,E).$$

Theorem 1.3. *An interval function F on $[a,b]$ is the primitive of a C-integrable function if and only if $V_C F$ is absolutely continuous with respect to the Lebesgue measure, that is, for any Lebesgue measurable set E, if $|E| = 0$, then $V_C F(E) = 0$.*

Definition 1.2. Let F be an interval function on $[a,b]$. Then F is said to be C-absolutely continuous on $E \subset [a,b]$ if for any positive number ε there exist a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(α_1) $x_k \in E$ for any k;

(α_2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(α_3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_C(E)$ the class of all C-absolutely continuous interval functions on E. Moreover F is said to be C-generalized absolutely continuous on $[a,b]$ if there exists a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^\infty E_m = [a,b]$ and $F \in \mathbf{AC}_C(E_m)$ for any m. We denote by $\mathbf{ACG}_C([a,b])$ the class of all C-generalized absolutely continuous interval functions on $[a,b]$.

Theorem 1.4. *For any $F \in \mathbf{ACG}_C([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in \mathbf{(C)}([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (C)\int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in \mathbf{(C)}([a,b])$ satisfies $F \in \mathbf{ACG}_C([a,b])$.

On the other hand, in [7, 10] Nakanishi gave the following criteria for the restricted Denjoy integral. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that (A) implies (B) and (C) implies (D). Next Nakanishi gave the following theorems for the restricted Denjoy integral.

Theorem 1.5. *Let $f \in \mathbf{(D^*)}([a,b])$ and let F be an additive interval function on $[a,b]$ defined by*

$$F(I) = (D^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies (A).

Theorem 1.6. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (A), *then the pair of f and F satisfies* (C). *Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (B), *then the pair of f and F satisfies* (D).

Theorem 1.7. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* (D)*, then $f \in \mathbf{(D^*)}([a,b])$ and*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

By Theorems 1.5, 1.6 and 1.7 we obtain the following criteria for the restricted Denjoy integral.

Theorem 1.8. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* (A), (B), (C) *and* (D)*. Moreover, if the pair of f and F satisfies one of* (A), (B), (C) *and* (D)*, then*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

In this paper, motivated by the results above, we will give new criteria for the C-integral similar to Theorems 1.5, 1.6, 1.7 and 1.8.

2 Main results Firstly we consider the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

$(A)_C$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0+1, \ldots, k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$.

$(B)_C$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_1}\left(F(I_k)-(L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \varepsilon_n$$

for any finite family $\{I_k \mid k=1,\ldots,k_0,k_0+1,\ldots,k_1,\ 0\le k_0\le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k=1,\ldots,k_0\}$ with $I_k\cap M_n\neq\emptyset$ and a δ-fine partial McShane partition $\{(I_k,x_k) \mid k=k_0+1,\ldots,k_1\}$ satisfying

(4.1) $x_k\in M_n$ for any $k=k_0+1,\ldots,k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k,x_k) < \frac{1}{\varepsilon_n}$.

(C)$_C$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n=[a,b]$;

(2) $f\in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k,x_k) \mid k=1,\ldots,k_0\}$ in $[a,b]$ satisfying

(3.1) $x_k\in F_n$ for any k;

(3.2) $\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$;

(3.3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I\subset[a,b]$

$$F(I)=(L)\int_{I\cap F_n} f(x)dx+\sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p=1,2,\ldots\}$ is the sequence of all connected components of $I^i\setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D)$_C$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n=[a,b]$;

(2) $F_n\subset M_n$ for any n and $|[a,b]\setminus\bigcup_{n=1}^{\infty} F_n|=0$;

(3) $f\in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k,x_k) \mid k=1,\ldots,k_0\}$ in $[a,b]$ satisfying

(4.1) $x_k \in M_n$ for any k;
(4.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(4.3) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(\mathrm{A})_C$ implies $(\mathrm{B})_C$ and $(\mathrm{C})_C$ implies $(\mathrm{D})_C$. Now we give the following theorems for the C-integral.

Theorem 2.1. *Let $f \in \mathbf{(C)}([a, b])$ and let F be an additive interval function on $[a, b]$ defined by*

$$F(I) = (C)\int_I f(x)dx$$

for any interval $I \subset [a, b]$. Then the pair of f and F satisfies $(\mathrm{A})_C$.

Proof. Since f is C-integrable, it is restricted Denjoy integrable. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. By Theorem 1.5 for $\left\{\frac{\varepsilon_n}{2}\right\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a, b]$ with $I_k \cap F_n \neq \emptyset$. Since f is C-integrable, $f - f\chi_{F_n}$ is also C-integrable, where χ_{F_n} is the characteristic function of F_n. Since $f - f\chi_{F_n} = 0$ on F_n, by Theorem 1.2 there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1} (C)\int_{I_k} (f - f\chi_{F_n})(x)dx\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1}\left((C)\int_{I_k} (f - f\chi_{F_n})(x)dx - (f - f\chi_{F_n})(x_k)|I_k|\right)\right| \\
&\quad < \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0 + 1, \ldots, k_1\}$ in $[a, b]$ satisfying (3.1) and (3.2). Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad \le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n
\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying (3.1) and (3.2), that is, (3) holds. □

Theorem 2.2. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{A})_C$*, then the pair of f and F satisfies* $(\mathrm{C})_C$*. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{B})_C$*, then the pair of f and F satisfies* $(\mathrm{D})_C$*.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_C$ hold. By Theorem 1.6 (4) of $(\mathrm{C})_C$ holds. Next we show (3) of $(\mathrm{C})_C$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n,\varepsilon)$ such that, if $|E| < \rho(n,\varepsilon)$, then

$$\left|(L)\int_{E\cap F_n} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n,\varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n,\varepsilon) \ge n$, and put $\eta = \rho(m(n,\varepsilon),\varepsilon)$. By (3) of $(\mathrm{A})_C$ for $m(n,\varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a,b]$ satisfying (3.1), (3.2) and (3.3) of $(\mathrm{C})_C$. Then we obtain

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0} |I_k| < \eta = \rho(m(n,\varepsilon),\varepsilon)$, we obtain

$$\left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} F(I_k)\right| &\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| + \left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_C$, then the pair of f and F satisfies $(\mathrm{D})_C$. □

Theorem 2.3. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{D})_C$*, then $f \in \mathbf{(C)}([a,b])$ and*

$$F(I) = (C)\int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

Proof. By (1) and (4) we obtain $F \in \mathbf{ACG}_C([a,b])$. By Theorem 1.4 there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $g \in \mathbf{(C)}([a,b])$ such that $g(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and

$$F(I) = (C)\int_I g(x)dx$$

for any interval $I \subset [a,b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [12, Theorem (5.1)] $g_n \in \mathbf{(D^*)}(I)$ for any interval $I \subset [a,b]$ and by (3)

$$\begin{aligned} (D^*)\int_I g_n(x)dx &= (D^*)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(D^*)\int_{\overline{J_p}} g(x)dx \\ &= (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(C)\int_{\overline{J_p}} g(x)dx \\ &= (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}), \end{aligned}$$

where $\{J_p \mid p = 1,2,\ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*)\int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a,x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a,x]) = f(x)$ for almost every $x \in [a,b]$. □

By Theorems 2.1, 2.2 and 2.3 we obtain the following new criteria for the C-integral.

Theorem 2.4. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is C-integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_C$, $(\mathrm{B})_C$, $(\mathrm{C})_C$ *and* $(\mathrm{D})_C$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_C$, $(\mathrm{B})_C$, $(\mathrm{C})_C$ *and* $(\mathrm{D})_C$, *then*

$$F(I) = (C)\int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

References

[1] B. Bongiorno, *Un nuovo integrale per il problema della primitive*, Le Matematiche **51** (1996), 299–313.

[2] ———, *On the C-integral*, AMS Special Session on Nonabsolute Integration (University of Toronto, Toronto, September, 2000).

[3] B. Bongiorno, L. Di Piazza, and D. Preiss, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc. **62** (2000), 117–126.

[4] A. M. Bruckner, R. J. Freissner, and J. Foran, *The minimal integral which includes Lebesgue integrable functions and derivatives*, Coll. Math. **50** (1986), 289–293.

[5] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., Providence, 1994.

[6] R. Henstock, *The general theory of integration*, Clarendon Press, Oxford, 1991.

[7] S. Nakanishi (formerly S. Enomoto), *Sur une totalisation dans les espaces de plusieurs dimensions, I*, Osaka Math. J. **7** (1955), 59–102.

[8] ———, *Sur une totalisation dans les espaces de plusieurs dimensions, II*, Osaka Math. J. **7** (1955), 157–178.

[9] S. Nakanishi, *The Denjoy integrals defined as the completion of simple functions*, Math. Japon. **37** (1992), 89–101.

[10] ———, *A new definition of the Denjoy's special integral by the method of successive approximation*, Math. Japon. **41** (1995), 217–230.

[11] W. F. Pfeffer, *The Riemann approach to integration*, Cambridge University Press, Oxford, 1993.

[12] S. Saks, *Theory of the integral*, Warsaw, 1937.

Communicated by *Jun Kawabe*

T. Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail: toshiharu.kawasaki@nifty.ne.jp

I. Suzuki
Nihon Bunka University, Tokyo 192–0986, Japan
E-mail: i-suzuki@pd5.so-net.ne.jp

REMARK ON THE TRIEBEL-LIZORKIN SPACE BOUNDEDNESS OF ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES

Kôzô Yabuta

Received March 30, 2015 ; revised August 19, 2015

Abstract. In the present paper, we consider the boundedness of the rough singular integral operator $T_{\Omega,h,\phi}$ along a surface $\Gamma = \{x = \phi(|y|)y/|y|)\}$ on the Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in \mathbb{R}$, $1 < p, q < \infty$ for $\Omega \in H^1(S^{n-1})$ and Ω belonging to some class $W\mathcal{F}_\alpha(S^{n-1})$, which relates to the Grafakos-Stefanov class. We improve recent results about these operators.

1 Introduction. The purpose of this paper is to improve recent results in [10].

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$. Suppose $\Omega \in L^1(S^{n-1})$ satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0, \tag{1}$$

where $y' = y/|y|$.

For a suitable function ϕ and a measurable function h on $[0, \infty)$, we denote by $T_{\Omega,\phi,h}$ the singular integral operator along the surface

$$\Gamma = \{x = \phi(|y|)y' : y \in \mathbb{R}^n\}$$

defined as follows:

$$T_{\Omega,h,\phi}f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - \phi(|y|)y')\, dy, \tag{2}$$

for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. If $\phi(t) = t$, then $T_{\Omega,h,\phi}$ is the classical singular integral operator $T_{\Omega,h}$, which is defined by

$$T_{\Omega,h}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - y)\, dy. \tag{3}$$

When $h \equiv 1$, we denote simply $T_{\Omega,h,\phi}$ and $T_{\Omega,h}$ by $T_{\Omega,\phi}$ and T_Ω, respectively.

Let us recall the definitions of some function spaces. First recall the definitions of the *homogeneous Triebel-Lizorkin spaces* $\dot{F}^{\alpha}_{p,q} = \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and the *homogeneous Besov spaces* $\dot{B}^{\alpha}_{p,q} = \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$. For $0 < p, q \leq \infty$ $(p \neq \infty)$ and $\alpha \in \mathbb{R}$, $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) : ||f||_{\dot{F}^{\alpha}_{p,q}} = \left\| \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} |\Psi_k * f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\} \tag{4}$$

2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25, 47G10 .

Key words and phrases. Singular integrals; Triebel-Lizorkin spaces; rough kernel .

*The author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 23540228), Japan Society for the Promotion of Science.

and $\dot{B}_p^{\alpha,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) : ||f||_{\dot{B}^{\alpha}_{p,q}} = \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|\Psi_k * f\|^q_{L^p} \right)^{1/q} < \infty \right\}, \tag{5}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, and $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$, $\widehat{\Psi_k}(\xi) = \Phi(2^{-k}\xi)$ for $k \in \mathbb{Z}$ and $\Phi \in C_c^\infty(\mathbb{R}^n)$ is a radial function satisfying the following conditions: (i) $0 \le \Phi \le 1$; (ii) $\operatorname{supp}\Phi \subset \{\xi : 1/2 \le |\xi| \le 2\}$; (iii) $\Phi > c > 0$ if $3/5 \le |\xi| \le 5/3$; (iv) $\sum_{j\in\mathbb{Z}} \Phi(2^{-j}\xi) = 1$ $(\xi \ne 0)$. Note that the space $\mathcal{S}_\infty(\mathbb{R}^n)$ given by

$$\mathcal{S}_\infty(\mathbb{R}^n) := \bigcap_{\alpha\in(\mathbb{N}\cup\{0\})^n} \left\{ f \in \mathcal{S}(\mathbb{R}^n) \, : \, \int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0 \right\}$$

is dense in $\dot{F}^{\alpha}_{pq}(\mathbb{R}^n)$ and $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ as long as $\alpha \in \mathbb{R}$ and $p, q \in (0, \infty)$ ([9, Theorem 5.1.5]).

The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F^{\alpha}_{p,q}(\mathbb{R}^n)$ and $B^{\alpha}_{p,q}(\mathbb{R}^n)$ respectively, are obtained by adding the term $\|\Phi_0 * f\|_p$ to the right-hand side of (4) or (5) with $\sum_{k\in\mathbb{Z}}$ replaced by $\sum_{k=0}^{\infty}$, where $\Phi_0 \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\widehat{\Phi_0} \subset \{\xi : |\xi| \le 2\}$, and $\widehat{\Phi_0}(\xi) > c > 0$ if $|\xi| \le 5/3$.

The following properties of the Triebel-Lizorkin space and Besov space are well known. Let $1 < p, q < \infty, \alpha \in \mathbb{R}$, and $1/p + 1/p' = 1, 1/q + 1/q' = 1$.

(6)

(a) $\dot{F}^0_{2,2} = \dot{B}^0_{2,2} = L^2, \dot{F}^0_{p,2} = L^p$ and $\dot{F}^{\alpha}_{p,p} = \dot{B}^{\alpha}_{p,p}$ for $1 < p < \infty$, and $\dot{F}^0_{\infty,2} = \mathrm{BMO}$;

(b) $F^{\alpha}_{p,q} \sim \dot{F}^{\alpha}_{p,q} \cap L^p$ and $||f||_{F^{\alpha}_{p,q}} \sim ||f||_{\dot{F}^{\alpha}_{p,q}} + ||f||_{L^p}$ $(\alpha > 0)$;

(c) $B^{\alpha}_{p,q} \sim \dot{B}^{\alpha}_{p,q} \cap L^p$ and $||f||_{B^{\alpha}_{p,q}} \sim ||f||_{\dot{B}^{\alpha}_{p,q}} + ||f||_{L^p}$ $(\alpha > 0)$;

(d) $(\dot{F}^{\alpha}_{p,q})^* = \dot{F}^{-\alpha}_{p',q'}$ and $(F^{\alpha}_{p,q})^* = F^{-\alpha}_{p',q'}$;

(e) $(\dot{B}^{\alpha}_{p,q})^* = \dot{B}^{-\alpha}_{p',q'}$ and $(B^{\alpha}_{p,q})^* = B^{-\alpha}_{p',q'}$;

(f) $\left(\dot{F}^{\alpha_1}_{p,q_1}, \dot{F}^{\alpha_2}_{p,q_2}\right)_{\theta,q} = \dot{B}^{\alpha}_{p,q}$ $(\alpha_1 \ne \alpha_2, 0 < p < \infty, 0 < q, q_1, q_2 \le \infty,$
$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2, 0 < \theta < 1)$..

See [9] for more properties of $\dot{F}^{\alpha}_{p,q}$ and $\dot{B}^{\alpha}_{p,q}$.

Next, we give the definition of the *Hardy space* $H^1(S^{n-1})$.

$$H^1(S^{n-1}) = \left\{ \omega \in L^1(S^{n-1}) \middle| \, \|f\|_{H^1(S^{n-1})} = \left\| \sup_{0\le r<1} \left| \int_{S^{n-1}} \omega(y') P_{r(\cdot)}(y') d\sigma(y') \right| \right\|_{L^1(S^{n-1})} < \infty \right\},$$

where $P_{ry'}(x')$ denotes the Poisson kernel on S^{n-1} defined by $P_{ry'}(x') = (1-r^2)/|ry' - x'|^n$, $0 \le r < 1$ and $x', y' \in S^{n-1}$.

Besides $H^1(S^{n-1})$, there are two important function spaces $L(\log L)(S^{n-1})$ and the block spaces $B_q^{(0,0)}(S^{n-1})$ in the theory of singular integrals. Let $L(\log L)^\alpha(S^{n-1})$ (for $\alpha > 0$) denote the class of all measurable functions Ω on S^{n-1} which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(S^{n-1})} = \int_{S^{n-1}} |\Omega(y')| \log^\alpha(2 + |\Omega(y')|)\, d\sigma(y') < \infty.$$

Denote by $L(\log L)(S^{n-1})$ $L(\log L)^1(S^{n-1})$. A well-known fact is $L(\log L)(S^{n-1}) \subset H^1(S^{n-1})$, cf. [8].

We turn to the block space $B_q^{(0,v)}(S^{n-1})$. Let $1 < q \le \infty$ and $v > -1$. A q-block on S^{n-1} is an $L^q(S^{n-1})$ function b which satisfies $\operatorname{supp} b \subset$ and $\|b\|_q \le |I|^{-1/q'}$, where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) \cap S^{n-1}$ is a cap on S^{n-1} for some $x_0' \in S^{n-1}$ and $\theta_0 \in (0, 1]$. The *block space* $B_q^{(0,v)}(S^{n-1})$ is defined by

$$B_q^{(0,v)}(S^{n-1}) = \Big\{\Omega \in L^1(S^{n-1});\, \Omega = \sum_{j=1}^\infty \lambda_j b_j,\, M_q^{(0,v)}(\{\lambda_j\}) < \infty\Big\}, \tag{7}$$

where $\lambda_j \in \mathbb{C}$ and b_j is a q-block supported on a cap I_j on S^{n-1}, and

$$M_q^{(0,v)}(\{\lambda_j\}) = \sum_{j=1}^\infty |\lambda_j| \big\{1 + \log^{(v+1)}\big(|I_j|^{-1}\big)\big\}. \tag{8}$$

For $\Omega \in B_q^{(0,v)}(S^{n-1})$, denote

$$\|\Omega\|_{B_q^{(0,v)}(S^{n-1})} = \inf\Big\{M_q^{(0,v)}(\{\lambda_j\}); \Omega = \sum_{j=1}^\infty \lambda_j b_j, b_j \text{ is a } q\text{-block}\Big\}.$$

Then $\|\cdot\|_{B_q^{(0,v)}(S^{n-1})}$ is a norm on the space $B_q^{(0,v)}(S^{n-1})$, and $\Big(B_q^{(0,v)}(S^{n-1}), \|\cdot\|_{B_q^{(0,v)}(S^{n-1})}\Big)$ is a Banach space. The following inclusion relations are known.

(9)

(a) $B_q^{(0,v_1)}(S^{n-1}) \subset B_q^{(0,v_2)}(S^{n-1})$ if $v_1 > v_2 > -1$;

(b) $B_{q_1}^{(0,v)}(S^{n-1}) \subset B_{q_2}^{(0,v)}(S^{n-1})$ if $1 < q_2 < q_1$ for any $v > -1$;

(c) $\bigcup_{p>1} L^p(S^{n-1}) \subset B_q^{(0,v)}(S^{n-1})$ for any $q > 1$, $v > -1$;

(d) $\bigcup_{q>1} B_q^{(0,v)}(S^{n-1}) \not\subset \bigcup_{q>1} L^q(S^{n-1})$ for any $v > -1$;

(e) $B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+v}(S^{n-1})$ for any $q > 1$, $v > -1$;

(f) $\bigcup_{q>1} B_q^{(0,0)}(S^{n-1}) \subset H^1(S^{n-1})$.

Besides them, there is another class of kernels which lead L^p and Triebel-Lizorkin space boundedness of singular integral operators $T_{\Omega,h}$. It is closely related to the class $\mathcal{F}_\alpha$ introduced by Grafakos and Stefanov [4].

For $\beta > 0$ we say $\Omega \in \mathcal{F}_\beta(S^{n-1})$ if

$$\sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \log^\beta \frac{2}{|y' \cdot \xi'|} d\sigma(y') < \infty, \tag{10}$$

and $\Omega \in W\mathcal{F}_\beta(S^{n-1})$ ($\tilde{\mathcal{F}}_\beta(S^{n-1})$ in [6]) if

$$\sup_{\xi' \in S^{n-1}} \left(\int_{S^{n-1}} \int_{S^{n-1}} |\Omega(y')\Omega(z')| \log^\beta \frac{2e}{|(y'-z')\cdot\xi'|} d\sigma(y') d\sigma(z') \right)^{\frac{1}{2}} < \infty. \tag{11}$$

We note that $\cup_{r>1} L^r(S^{n-1}) \subset W\mathcal{F}_{\beta_2}(S^{n-1}) \subset W\mathcal{F}_{\beta_1}(S^{n-1})$ for $0 < \beta_1 < \beta_2 < \infty$.

About the inclusion relation between $\mathcal{F}_{\beta_1}(S^{n-1})$ and $W\mathcal{F}_{\beta_2}(S^{n-1})$, the following is known: when $n = 2$, Lemma 1 in [3] shows $\mathcal{F}_\beta(S^1) \subset W\mathcal{F}_\beta(S^1)$. It is also known that $W\mathcal{F}_{2\alpha}(S^1) \setminus \big(\mathcal{F}_\alpha(S^1) \cup H^1(S^1)\big) \neq \emptyset$. cf. [7].

To state our claims, we need one more function space. For $1 \le \gamma \le \infty$, $\Delta_\gamma(\mathbb{R}_+)$ is the collection of all measurable functions $h : [0, \infty) \to \mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty.$$

Note that

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\alpha(\mathbb{R}_+) \quad \text{for } \alpha < \beta,$$

and all these inclusions are proper.

In this short note, we report that Theorems 1.1, 1.2 and 1.3 in [10] are improved essentially in the following form. In the following theorems, the statement "$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$" means that

$$\|T_{\Omega,h,\phi} f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C \|T_{\Omega,h,\phi} f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)},$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n)$. In any case, by density we can extend the above inequality and have them for all $f \in \dot{F}^\alpha_{pq}(\mathbb{R}^n)$. We use similar abbreviation to $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$.

Theorem 1. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0,\infty)$ satisfying*

$$\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0,\infty). \tag{12}$$

Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Suppose $\Omega \in H^1(S^{n-1})$, satisfying the cancellation condition (1). Then

(i) *$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$ (hexagon $P_1P_2P_3P_4P_5P_6$ in the case $1 < \gamma \le 2$), where $P_1 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_2 = (\frac{1}{2}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_3 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $P_4 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_5 = (\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_6 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $R_2 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, and $R_4 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$.*

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q satisfying $|\frac{1}{2} - \frac{1}{p}| < \min\{\frac{1}{2}, \frac{1}{\gamma'}\}$ and $1 < q < \infty$.*

See the following Figures 1-1 and 1-2 for the conclusion (i) of Theorem 1.

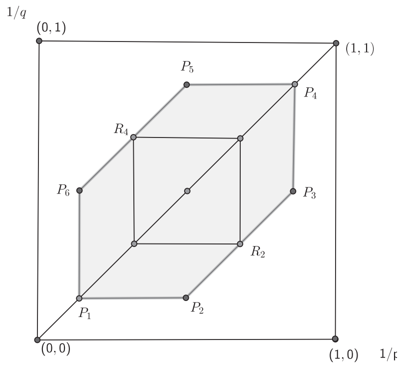

Figure 1 − 1 $(1 < \gamma < 2)$

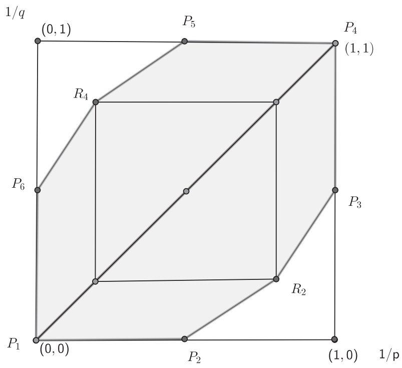

Figure 1 − 2 $(2 \le \gamma \le \infty)$

The following theorem shows that if Ω belongs to $L \log L(S^{n-1})$ or block spaces, then we can get better results than Theorem 1.

Theorem 2. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0, \infty)$ satisfying the same condition as in Theorem 1. Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, and $\Omega \in L^1(S^{n-1})$ satisfy the cancellation condition* (1)*. Then*

(i) *if $\Omega \in L(\log L)(S^{n-1})$, $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2Z_2Q_3Q_4Z_4$ when $1 < \gamma < 2$ and $Q_1Q_2S_2Q_3Q_4S_4$ when $2 \le \gamma \le \infty$, where $Q_1 = (0, 0)$, $Q_2 = (\frac{1}{\gamma'}, 0)$, $Q_3 = (1, 1)$, $Q_4 = (\frac{1}{\gamma}, 1)$, $S_2 = (1, \frac{1}{\gamma})$, $S_4 = (\frac{1}{\gamma}, 0)$, $Z_2 = (1, \frac{1}{2})$, and $Z_4 = (\frac{1}{2}, 0)$.*

(ii) *if $\Omega \in \cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})$, $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$*

(iii) *if $\Omega \in L(\log L)(S^{n-1}) \cup \big(\cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})\big)$, $T_{\Omega,h,\phi}$ is bounded on $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$.*

See the following Figures 1-3 and 1-4 for the conclusion of Theorem 2(i).

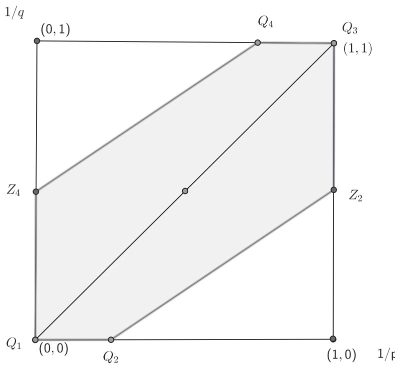

Figure 1 − 3 $(1 < \gamma < 2)$

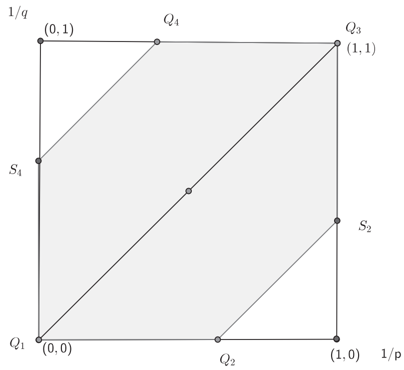

Figure 1 − 4 $(2 \le \gamma \le \infty)$

As a corresponding result to the case Ω belongs to $W\mathcal{F}_\alpha$, we have the following:

Theorem 3. *Let ϕ be a nonnegative (or nonpositive) and monotonic function on $(0, \infty)$ satisfying the same condition as in Theorem 1. Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Suppose $\Omega \in W\mathcal{F}_\beta = W\mathcal{F}_\beta(S^{n-1})$ for some $\beta > \max(\gamma', 2)$, and satisfies the cancellation condition* (1). *Then*

(i) *the singular integral operator $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{S}_2\mathcal{Q}_3\mathcal{Q}_4\mathcal{S}_4$, where $\mathcal{Q}_1 = \big(\frac{\max(\gamma',2)}{2\beta}, \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_2 = \big(\frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'}), \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_3 = \big(1 - \frac{\max(\gamma',2)}{2\beta}, 1 - \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{Q}_4 = \big(\frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2}), 1 - \frac{\max(\gamma',2)}{2\beta}\big)$, $\mathcal{S}_2 = \big(, 1 - \frac{\max(\gamma',2)}{2\beta}, \frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2})\big)$, and $\mathcal{S}_4 = \big(\frac{\max(\gamma',2)}{2\beta}, \frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'})\big)$.*

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$, $\frac{2\beta}{2\beta - \max(\gamma',2)} < p < \frac{2\beta}{\max(\gamma',2)}$ and $1 < q < \infty$.*

See the Figure 1-5 for the conclusion (i) of Theorem 3.

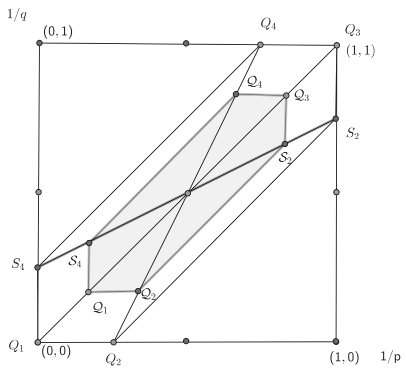

Figure 1 − 5

Remark 1. In [10] we have shown these theorems under the stronger assumption on ϕ, i.e, when ϕ is a positive increasing function on $(0, \infty)$ satisfying the doubling condition $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$ besides (12). Note also that we improve Theorems 1.2 and 1.3 in [10] even in the case $\phi(t) = t$.

Example 1. As typical examples of ϕ satisfying the condition (12), we list the following four: $t^\alpha e^t$ $(\alpha > 0)$, $t^\alpha \log^\beta(1+t)$ $(\alpha > 0,\ \beta \ge 0)$, $(2t^2 - 2t + 1)t^{1+\alpha}$ $(\alpha \ge 0)$, and $\phi(t) = 2t^2 + t$ $(0 < t < \frac{\pi}{2})$, $= 2t^2 + t\sin t$ $(t \ge \frac{\pi}{2})$. Note that linear combinations with positive coefficients of functions ϕ's satisfying the above two conditions also satisfies them. Note that the first example satisfies (12), but does not satisfy the doubling condition.

2 Proofs of Theorems.

One can prove these theorems by a change of variable and the corresponding theorems in case $\phi(t) = t$ in [10], like in [2] or [5].

To prove the theorems, we prepare the following three lemmas: Lemma 1, Lemma 2 and Lemma 4. The first one is Lemma 2.2 in [2], and the second one is Lemma 2.3 in [2].

Lemma 1. *Let ϕ and φ be the same as in Theorem 1. If $b \in \Delta_\gamma$ for some $\gamma \ge 1$, then*

$$\frac{1}{R}\int_0^R |b(|\Phi|^{-1}(t))\varphi(|\Phi|^{-1}(t))|^\gamma\,dt \le C_\gamma(\|\varphi\|_\infty^{\gamma-1} + \|\varphi\|_\infty^\gamma), \quad R > 0, \tag{13}$$

that is, $b(|\Phi|^{-1})\varphi(|\Phi|^{-1}) \in \Delta_\gamma$.

Lemma 2. *Let ϕ and φ be the same as in Theorem 1. Then*

$$T_{\Omega,\phi,h}f(x) = \begin{cases} T_{\Omega,\varphi(\phi^{-1})h(\phi^{-1})}f(x), & \text{if } \phi \text{ is nonnegative and increasing,} \\ -T_{\Omega,\varphi(\phi^{-1})h(\phi^{-1})}f(x), & \text{if } \phi \text{ is nonnegative and decreasing,} \\ T_{\tilde{\Omega},\varphi(\phi^{-1}(-\cdot))h(\phi^{-1}(-\cdot))}f(x), & \text{if } \phi \text{ is nonpositive and decreasing,} \\ -T_{\tilde{\Omega},\varphi(\phi^{-1}(-\cdot))h(\phi^{-1}(-\cdot))}f(x), & \text{if } \phi \text{ is nonpositive and increasing,} \end{cases} \tag{14}$$

where $\tilde{\Omega}(y) = \Omega(-y)$.

To state the third one we prepare some definitions and a lemma. For $\Omega \in L^1(S^{n-1})$, $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, a suitable function ϕ on $\mathbb{R}_+$, and $k \in \mathbb{Z}$, we define the measures $\sigma_{\Omega,h,\phi,k}$ on $\mathbb{R}^n$ and the maximal operator $\sigma^*_{\Omega,h,\phi}f(x)$ by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_{\Omega,h,\phi,k}(x) = \int_{\mathbb{R}^n} f(\phi(|x|x') \frac{\Omega(x')h(|x|)}{|x|^n} \chi_{\{2^{k-1}<|x|\le 2^k\}}(x)\, dx, \tag{15}$$

$$\sigma^*_{\Omega,h,\phi}f(x) = \sup_{k\in\mathbb{Z}} \big| |\sigma_{\Omega,h,\phi,k}| * f(x) \big|, \tag{16}$$

where $|\sigma_{\Omega,h,\phi,k}|$ is defined in the same way as $\sigma_{\Omega,h,\phi,k}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

we also define the maximal functions $M_{\Omega,h,\phi}$ by

$$M_{\Omega,h,\phi}f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{2^{kn}} \int_{\{2^{k-1}<|y|\le 2^k\}} |\Omega(y')h(|y|)f(x-\phi(|y|)y')|\, dy. \tag{17}$$

We see easily that $M_{\Omega,h,\phi}$ is equivalent to $\sigma^*_{\Omega,h,\phi}(|f|)$.

In [10], we have shown the following auxiliary lemma.

Lemma 3. *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, and $\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0,\infty)$. Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Then, for $\gamma' < p, q < \infty$ we have*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |M_{\Omega,h,\phi} f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)} \le C \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)}. \tag{18}$$

Using this we get our third lemma.

Lemma 4. *Let ϕ be the same as above, and $\ell(j) \in \mathbb{Z}$ for $j \in \mathbb{Z}$. Then, if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$, we have*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,\ell(j)} * f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)} \le C \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^n)}, \tag{19}$$

where $Q_1 = (0,0)$, $Q_2 = (\frac{1}{\gamma'}, 0)$, $Q_3 = (1,1)$, $Q_4 = (\frac{1}{\gamma}, 1)$, $S_2 = (1, \frac{1}{\gamma})$, and $S_4 = (\frac{1}{\gamma}, 0)$.

Proof. By Lemma 3, we see that

$$\left\|\Big(\sum_{j\in\mathbb{Z}}|\sigma_{\Omega,h,\phi,\ell(j)}*f_j|^q\Big)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\Big(\sum_{j\in\mathbb{Z}}|M_{\Omega,h,\phi}f_j|^q\Big)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)},$$

if $\gamma' < p, q < \infty$. By duality, we see that the estimate (19) holds if $1 < p, q < \gamma$. Interpolating these two cases, we see that the estimate (19) holds, if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$. □

Now we can prove our theorems.

Using Lemmas 1 and 2 and applying Theorem 1.1 in [10] for $\phi(t) = t$, we get our Theorem 1.

Next, using Lemma 4 in place of Lemma 2.4(ii) in [10], we modify the proof of the inequality (3.4) in [10], and obtain that estimate if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_3Q_3Q_4S_4$. Thus we get our Theorem 3(i) under the additional assumption $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, in particular when $\phi(t) = t$. Similarly, we get our Theorem 2(ii) under the additional assumption $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$. So, using Lemmas 1 and 2 and applying Theorems 2(ii) and 3(i) for $\phi(t) = t$, we get our Theorems 2(ii) and 3(i), respectively.

Next, we consider Theorem 2(i) i.e. the case $\Omega \in L(\log L)(S^{n-1})$. Similarly to the case Ω belonging to block spaces, we see that $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$.

On the other hand, by Theorem 1.3 in [1] we know that $T_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n) = \dot{F}^0_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, if $\Omega \in L(\log L)(S^{n-1})$ and $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. So, using Lemmas 1 and 2, we see that $T_{\Omega,h,\phi}$ is bounded on $L^p(\mathbb{R}^n) = \dot{F}^0_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$.

Hence, interpolating between this case and the case $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the hexagon $Q_1Q_2S_2Q_3Q_4S_4$, we see that $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the quadrilateral $Q_1Q_2Z_2Z_4$ or $Q_3Q_4Z_4Z_2$. Interpolating between the cases $Q_1Q_2Z_2Z_4$ and $Q_3Q_4Z_4Z_2$, we have the desired conclusion of Theorem 2(i).

Theorems 2(iii) and 3(ii) follow by using the property (f) of Triebel-Lizorkin spaces and interpolating the cases $\dot{F}^{\alpha+1}_{p,p}(\mathbb{R}^n)$ and $\dot{F}^{\alpha-1}_{p,p}(\mathbb{R}^n)$. This completes the proofs of our theorems. □

References

[1] A. Al-Salman and Y. Pan, Singular integrals with rough kernels in $L\log L(S^{n-1})$, J. London Math. Soc. (2), **66** (2002), no. 1, 153–174.

[2] Y. Ding, Q. Xue and K. Yabuta, On singular integral operators with rough kernel along surface, Integral Equations Operator Theory **68** (2010), no. 2, 151–161.

[3] D. Fan and S. Sato, A note on singular integrals associated with a variable surface of revolution, Math. Inequal. Appl. **12**(2) (2009), 441–454.

[4] L. Grafakos and A. Stefanov, L^p bounds for singular integrals and maximal singular integrals with rough kernels, Indiana Univ. Math. J. **47** (1998), 455–469.

[5] F. Liu and H. Wu, Rough singular integrals supported by submanifolds in Triebel-Lizorkin spaces and Besov spaces, Taiwanese J. Math. **18** (2014), no. 1, 127–146.

[6] F. Liu and H. Wu, Rough singular integrals associated with surfaces of Van der Corput type, Appl. Math. J. Chinese Univ. Ser. B **29** (2014), no. 1, 86–100.

[7] W. Li, Z. Si and K. Yabuta, Boundedness of singular integrals associated to surfaces of revolution on Triebel-Lizorkin spaces, Forum Mathematicum, Published Online:08/14/2014.

[8] E. M. Stein, Note on the class $L \log L$, Studia Math. **32** (1969), 305–310.

[9] H. Triebel, *Theory of Function Spaces*, Monogr. Math. vol. **78**, Birkhäuser, Basel, 1983.

[10] K. Yabuta, Triebel-Lizorkin space boundedness of rough singular integrals associated to surfaces, J. Inequal. Appl. (2015) 2015:107 DOI 10.1186/s13660-015-0630-7, 26 pages.

Communicated by *Eiichi Nakai*

Research Center for Mathematical Sciences
Kwansei Gakuin University
Gakuen 2-1, Sanda 669-1337, Japan
Email: kyabuta3@kwansei.ac.jp

ON THE FIRST-PASSAGE TIME OF AN INTEGRATED GAUSS-MARKOV PROCESS

Mario Abundo

Received June 3, 2015

Abstract. It is considered the integrated process $X(t) = x + \int_0^t Y(s)ds$, where $Y(t)$ is a Gauss-Markov process starting from y. The first-passage time (FPT) of X through a constant boundary and the first-exit time of X from an interval (a, b) are investigated, generalizing some results on FPT of integrated Brownian motion. An essential role is played by a useful representation of X, which allows to reduces the FPT of X to that of a time-changed Brownian motion. Some explicit examples are reported; when theoretical calculation is not available, the quantities of interest are estimated by numerical computation.

Keywords: Diffusion, Gauss-Markov process, first-passage-time
Mathematics Subject Classification: 60J60, 60H05, 60H10.

1 Introduction First-passage time (FPT) problems for integrated Markov processes arise both in theoretical and applied Probability. For instance, in certain stochastic models for the movement of a particle, its velocity, $Y(t)$, is modeled as Ornstein-Uhlenbeck (OU) process, which is indeed suitable to describe the velocity of a particle immersed in a fluid; as the friction parameter approaches zero, $Y(t)$ becomes Brownian motion B_t (BM). More generally, the particle velocity $Y(t)$ can be modeled by a diffusion. Thus, particle position turns out to be the integral of $Y(t)$, and any question about the time at which the particle first reaches a given place leads to the FPT of integrated $Y(t)$. This kind of investigation is complicated by the fact that the integral of a Markov process such as $Y(t)$, is no longer Markovian; however, the two-dimensional process $\mathcal{Y}(t) = \left(\int_0^t Y(s)ds, Y(t)\right)$ is Markovian, so the FPT of integrated $Y(t)$ can be studied by using Kolmogorov's equations approach. The first apparition in the literature of $\mathcal{Y}(t)$, with $Y(t) = B_t$, dates back to the beginning of the twentieth century (see [24]), in modeling a harmonic oscillator excited by a Gaussian white noise (see [25] and references therein).

The study of $\int_0^t Y(s)ds$ has interesting applications in Biology, in the framework of diffusion models for neural activity; if one identifies $Y(t)$ with the neuron voltage at time t, then $\frac{1}{t}\int_0^t Y(s)ds$ represents the time average of the neural voltage in the interval $[0, t]$. Moreover, integrated Brownian motion arises naturally in stochastic models for particle sedimentation in fluids (see [22]). Another application can be found in Queueing Theory, if $Y(t)$ represents the length of a queue at time t; then $\int_0^t Y(s)ds$ represents the cumulative waiting time experienced by all the "users" till the time t. Furthermore, as an application in Economy, one can suppose that $Y(t)$ represents the rate of change of a commodity's price, i.e. the current inflation rate; hence, the price of the commodity at time t is $X(t) = X(0) + \int_0^t Y(s)ds$. Finally, integrated diffusions also play an important role in connection with the so-called realized stochastic volatility in Finance (see e.g. [8], [17], [20]).

FPT problems of integrated BM (namely, when $Y(t) = B_t$) through one or two boundaries, attracted the interest of a lot of authors (see e.g. [10], [18], [22], [26], [27], [29], [35] for single boundary, and [25], [32], [33] for double boundary); the FPT of integrated Ornstein-Uhlenbeck process was studied in [10], [30]. Motivated by these works, our aim is to extend to integrated Gauss-Markov processes the literature's results concerning FPT of integrated BM.

Let $m(t),\ h_1(t),\ h_2(t)$ be C^1-functions of $t \geq 0$, such that $h_2(t) \neq 0$ and $\rho(t) = h_1(t)/h_2(t)$ is a non-negative and monotonically increasing function, with $\rho(0) = 0$.
If $B(t) = B_t$ denotes standard Brownian motion (BM), then

$$Y(t) = m(t) + h_2(t)B(\rho(t)),\ t \geq 0, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 60J60, 60H05, 60H10.
Key words and phrases. Diffusion, Gauss-Markov process, first-passage-time .

is a continuous Gauss-Markov process with mean $m(t)$ and covariance $c(s,t) = h_1(s)h_2(t)$, for $0 \le s \le t$.
Throughout the paper, Y will denote a Gauss-Markov process of the form (1.1), starting from $y = m(0)$.

Besides BM, a noteworthy case of Gauss-Markov process is the Ornstein-Uhlenbeck (OU) process, and in fact any continuous Gauss-Markov process can be represented in terms of a OU process (see e.g. [36]).
Given a continuous Gauss-Markov process Y, we consider its integrated process, starting from $X(0)$:

$$X(t) = X(0) + \int_0^t Y(s)ds. \tag{1.2}$$

For a given boundary a, we study the FPT of X through a, with the conditions that $X(0) = x < a$ and $Y(0) = y$, that is:

$$\tau_a(x,y) = \inf\{t > 0 : X(t) = a | X(0) = x, Y(0) = y\}; \tag{1.3}$$

moreover, for $b > a$ and $x \in (a,b)$, we also study the first-exit time of X from the interval (a,b), with the conditions that $X(0) = x$ and $Y(0) = y$, that is:

$$\tau_{a,b}(x,y) = \inf\{t > 0 : X(t) \notin (a,b) | X(0) = x, Y(0) = y\}. \tag{1.4}$$

In our investigation, an essential role is played by the representation of X in terms of BM, which was previously obtained by us in [1]. By using this, we avoid to address the FPT problem by Kolmogorov's equations approach, namely to study the equations associated to the two-dimensional process $(X(t), Y(t))$; many authors (see the references cited above) followed this analytical approach to study the distribution and the moments of the FPT of integrated BM, and they obtained explicit solutions, in terms of special functions. On the contrary, our approach is based on the properties of Brownian motion and continuous martingales and it has the advantage to be quite simple, since the problem is reduced to the FPT of a time-changed BM. Actually, for $Y(0) = y = 0$ we present explicit formulae for the density and the moments of the FPT of the integrated Gauss-Markov process X, both in the one-boundary and two-boundary case; in particular, in the two-boundary case, we are able to express the nth order moment of the first-exit time as a series involving only elementary functions.

2 Preliminary Results on Integrated Gauss-Markov Processes We recall from [1] the following:

Theorem 2.1 *Let Y be a Gauss-Markov process of the form* (1.1)*; then $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\gamma(\rho(t))$, where $M(t) = \int_0^t m(s)ds$, $\gamma(t) = \int_0^t (R(t) - R(s))^2 ds$ and $R(t) = \int_0^t h_2(\rho^{-1}(s))/\rho'(\rho^{-1}(s))ds$. Moreover, if $\gamma(+\infty) = +\infty$, then there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}(\widehat{\rho}(t))$, where $\widehat{\rho}(t) = \gamma(\rho(t))$. Thus, the integrated process X can be represented as a Gauss-Markov process with respect to a different BM.*

□

Remark 2.2 Notice that, if $\gamma(+\infty) = +\infty$, though X is represented as a Gauss-Markov process for a suitable BM $\widehat{B}$, X is not Markov with respect to its natural filtration $\mathcal{F}_t$ (i.e. the $\sigma-$field generated by X up to time t). In fact, a Gaussian process X enjoys this property if and only if its covariance $K(s,t) = cov(X(s), X(t))$ satisfies the condition (see e.g. [16], [31], [34]) $K(u,t) = \frac{K(u,s)K(s,t)}{K(s,s)}$, $u \le s \le t$. Really, if X is e.g. integrated BM with $y = 0$, $x = 0$ (that is, $X(t) = \int_0^t B_s ds$), one has $K(s,t) = cov\big(\int_0^s B_u du, \int_0^t B_u du\big) = \frac{s^2}{6}(3t - s)$ (see e.g. [39], pg. 654 or [23], pg. 105), and so the above condition does not hold. On the other hand, the two-dimensional process $\big(\int_0^s B_u du, \int_0^t B_u du\big)$ has not the same joint distribution as $\big(\widehat{B}(\widehat{\rho}(s)), \widehat{B}(\widehat{\rho}(t))\big)$, because $cov\big(\widehat{B}(\widehat{\rho}(s)), \widehat{B}(\widehat{\rho}(t))\big) = E[\widehat{B}(\widehat{\rho}(s)) \cdot \widehat{B}(\widehat{\rho}(t))] = \widehat{\rho}(s) = s^3/3$, for $s \le t$ (see Example 1 below), which is different from $K(s,t)$. However, the process (X, B) is Markov, and the marginal distributions of the random vector $\big(X(s), X(t)\big)$ are equal to the distributions of $\widehat{B}(\widehat{\rho}(s))$ and $\widehat{B}(\widehat{\rho}(t))$, respectively; this is enough for the FPT problems we aim to investigate.

Remark 2.3 By Theorem 2.1 the FPT problem for integrated Gauss-Markov process is reduced to the FPT problem for another suitable Gauss-Markov process. Thus, to compute the FPT densities involving one or two boundaries, one can use the methods (both analytical and numerical) developed in [15] and [37] for general Gauss-Markov processes, in which the FPT densities are obtained as solutions to non-singular second-kind Volterra integral equations.

Example 1 (integrated Brownian motion)
Let be $Y(t) = y + B_t$, then $m(t) = y,\ h_1(t) = t,\ h_2(t) = 1$ and $\rho(t) = t$. Moreover, $R(t) = \int_0^t ds = t$ and $\gamma(t) = \int_0^t (t-s)^2 ds = t^3/3$. Thus, $\widehat{\rho}(t) = t^3/3,\ \gamma(+\infty) = +\infty$, and so there exists a BM $\widehat{B}$ such that $X(t) = x + yt + \widehat{B}(t^3/3)$ (see [4]).

The following three examples are taken from Section 3 of [1]; notice that the process here denoted by Y was there indicated by X.

Example 2 (integrated OU process)
Let $Y(t)$ be the solution of the SDE (Langevin equation):

$$dY(t) = -\mu(Y(t) - \beta)dt + \sigma dB_t,\ Y(0) = y,$$

where $\mu, \sigma > 0$ and $\beta \in \mathbb{R}$. The explicit solution is (see e.g. [2]):

$$Y(t) = \beta + e^{-\mu t}[y - \beta + \widetilde{B}(\rho(t)], \tag{2.1}$$

where $\widetilde{B}$ is Brownian motion and $\rho(t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$. Thus, Y is a Gauss-Markov process with $m(t) = \beta + e^{-\mu t}(y - \beta),\ h_1(t) = \frac{\sigma^2}{2\mu}\left(e^{\mu t} - e^{-\mu t}\right),\ h_2(t) = e^{-\mu t}$ and $c(s,t) = h_1(s)h_2(t)$. The functions $M(t),\ R(t)$ and $\gamma(t)$ defined in the statement of Theorem 2.1 can be easily obtained in closed form (see Example 2 of [1] for calculations and more details). Then, by Theorem 2.1, we get that $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\widehat{\rho}(t) = \gamma(\rho(t))$. Moreover, as easily seen, $\lim_{t\to+\infty}\gamma(t) = +\infty$, so there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}\left(\widehat{\rho}(t)\right)$.

Notice that in both Example 1 and 2 it holds $\rho(+\infty) = +\infty$, so the condition $\gamma(+\infty) = +\infty$ is equivalent to $\widehat{\rho}(+\infty) = +\infty$.

Example 3 (integrated Brownian bridge)
For $T > 0$ and $\alpha, \beta \in \mathbb{R}$, let $Y(t)$ be the solution of the SDE:

$$dY(t) = \frac{\beta - Y(t)}{T - t}\,dt + dB_t,\ 0 \le t \le T,\ Y(0) = y = \alpha.$$

This is a transformed BM with fixed values at each end of the interval $[0,T],\ Y(0) = y = \alpha$ and $Y(T) = \beta$. The explicit solution is (see e.g. [38]):

$$Y(t) = \alpha\,(1 - t/T) + \beta t/T + (T - t)\int_0^t \frac{1}{T - s}dB(s)$$

$$= \alpha\,(1 - t/T) + \beta t/T + (T - t)\widetilde{B}\left(\frac{t}{T(T-t)}\right),\ 0 \le t \le T, \tag{2.2}$$

where $\widetilde{B}$ is BM. So, for $0 \le t \le T,\ Y$ is a Gauss-Markov process with:

$$m(t) = \alpha\,(1 - t/T) + \beta t/T,\ h_1(t) = t/T,\ h_2(t) = T - t,\ \rho(t) = \frac{t}{T(T-t)},\ c(s,t) = h_1(s)h_2(t).$$

Notice that now $\rho(t)$ is defined only in $[0,T)$. The functions $M(t),\ R(t)$ and $\gamma(t)$ defined in the statement of Theorem 2.1 can be easily obtained in closed form (see Example 3 of [1] for calculations and more details). Then, by Theorem 2.1, we get that $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\widehat{\rho}(t) = \gamma(\rho(t))$. As easily seen, $\lim_{t \to T^-} \rho(t) = +\infty$; moreover, by a straightforward, but boring calculation, we get that $\lim_{t \to T^-} \widehat{\rho}(t) = \gamma_1(+\infty) = +\infty$, so there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}(\widehat{\rho}(t))$, $t \in [0, T]$.

Example 4 (the integral of a generalized Gauss-Markov process)
Let us consider the diffusion $Y(t)$ which is the solution of the SDE:

$$dY(t) = m'(t)dt + \sigma(Y(t))dB_t, \ Y(0) = y,$$

where $\sigma(y) > 0$ is a smooth deterministic function. In this Example, we denote by $\rho(t)$ the quadratic variation of $Y(t)$, that is, $\rho(t) := \langle Y \rangle_t = \int_0^t \sigma^2(Y(s))ds$, and suppose that $\rho(+\infty) = +\infty$; then, it follows (see Example 4 of [1]) that $Y(t) = m(t) + \widehat{B}(\rho(t))$, $t \geq 0$ ($m(0) = y$), where $\widehat{B}$ is BM; here, $\rho(t)$ is an increasing, but not necessarily deterministic function, namely it can be a random function. For this reason, we call Y a generalized Gauss-Markov process. By using the arguments leading to the proof of Theorem 2.1, we conclude that, under certain conditions, for fixed t the law of $\int_0^t Y(s)ds$, conditional to $\rho(t)$, is normal with mean $M(t) = \int_0^t m(s)ds$ and variance $\widehat{\rho}(t)$, where $\widehat{\rho}(t)$ is increasing and bounded between two certain deterministic functions (see [1] for more details).

In the sequel, we suppose that all the assumptions of Theorem 2.1 hold, and $\gamma(+\infty) = +\infty$; we limit ourselves to consider the special case when $m(t)$ is a constant (that is, $m(t) \equiv Y(0) = y$, $\forall t$), thus $Y(t) = y + h_2(t)B(\rho(t))$ and $X(t) = x + yt + \int_0^t h_2(s)B(\rho(s))ds$. Our aim is to investigate the FPT problem of X, for one or two boundaries. One approach to the FPT problem of X consists in considering the two-dimensional process $(X(t), Y(t))$ given by:

$$\begin{cases} X(t) = x + \int_0^t Y(s)ds \\ Y(t) = y + h_2(t)B(\rho(t))dt \ , \end{cases}$$

or, in differential form:

$$\begin{cases} dX(t) = Y(t)dt \\ dY(t) = h_2'(t)B(\rho(t))dt + h_2(t)\sqrt{\rho'(t)}dB_t \ , \end{cases}$$

and to study the associated Kolmogorov's equations.
Many authors (see e.g. [19], [25], [26], [27], [29]) followed this way in the case of integrated BM, namely for $Y(t) = y + B_t$. In fact, for $\tau = \tau_a$ or $\tau = \tau_{a,b}$, the law of the couple $(\tau(x, y), B_{\tau(x,y)})$ was investigated. Let us denote by $\mathcal{G}$ the generator of (X, B), that is:

$$\mathcal{G}f(x, y) = \frac{\partial f}{\partial x} \cdot y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \ , \ f \in C^2;$$

if one considers, for instance, the one boundary case, then the Laplace transform of $\left(\tau_a(x, y), B_{\tau_a(x,y)}\right)$, defined for $x \leq a$, $y \in \mathbb{R}$, by $u(\lambda, \nu) := E\left[\exp\left(-\lambda\tau_a(x, y) - \nu B_{\tau_a(x,y)}\right)\right]$ (λ, $\nu \geq 0$), is the solution of the problem with boundary conditions:

$$\begin{cases} \mathcal{G}u(x, y) = \lambda u(x, y), \ x \leq a, \ y \in \mathbb{R} \\ u(a^-, y) = e^{-\nu y}, \ y \geq 0 \\ u(a^+, y) = e^{\nu y}, \ y < 0 \end{cases} \tag{2.3}$$

(see e.g. [27], Lemma 3, or ref. [4], [5], [7], therein). Moreover, for $n = 1, 2, \ldots$ the nth order moments $T_n(x, y) = E(\tau_a^n(x, y))$ are solutions to the equations $\mathcal{G}T_n = -nT_{n-1}$ ($T_0 \equiv 1$), subjected to certain boundary conditions; however, these boundary value problems are not well-posed (see [22], where some numerical methods to estimate T_n were also considered).

Notice that, in the case of integrated BM, explicit, rather complicated formulae for the joint distribution of $\left(\tau_a(x, y), B_{\tau_a(x,y)}\right)$ (and therefore for the density of $\tau_a(x, y)$) were found in [18],

[26], [35]). In order to avoid not convenient formulae, we propose an alternative approach, based on the representation of the integrated process X as a Gauss-Markov process, with respect to the BM $\widehat{B}$ (see Theorem 2.1); this way works very simply, almost in the case when $y = 0$. Thus, in the following, we suppose that $Y(t) = y + h_2(t)B(\rho(t))$ and $\gamma(+\infty) = +\infty$, so the integrated process is of the form $X(t) = x + yt + \widehat{B}(\widehat{\rho}(t))$, where $\widehat{\rho}(t) = \gamma(\rho(t))$ and $\widehat{B}$ is a suitable BM. Notice however, that the integrated OU process and the integrated Brownian bridge belong to this class only if $y = \beta$, and $\alpha = \beta = y$, respectively; this easily follows from the explicit expressions of $M(t)$ given in Examples 2 and 3 in [1].

3 FPT through one boundary Under the previous assumptions, let a be a fixed constant boundary; for $x < a$ and $y \in \mathbb{R}$, the FPT of X through a can be written as follows:

$$\tau_a(x, y) = \inf\{t > 0 : x + yt + \widehat{B}(\widehat{\rho}(t)) = a\}. \tag{3.1}$$

Thus, if we set $\widehat{\tau}_a(x, y) = \widehat{\rho}(\tau_a(x, y))$, we get:

$$\widehat{\tau}_a(x, y) = \inf\{t > 0 : \widehat{B}_t = h(t)\}, \tag{3.2}$$

where $h(t) = a - x - y\widehat{\rho}^{\ -1}(t)$, and so we reduce to consider the FPT of BM through a curved boundary. Since, for $x < a$ and $y \geq 0$ the function $h(t)$ is not increasing, we are able to conclude that $\tau_a(x, y)$ is finite with probability one, if $y \geq 0$. In fact, as it is well-known, the FPT of BM $\widehat{B}_t$ through the constant barrier $h(0) = a - x$, say $\bar{\tau}(x)$, is finite with probability one; then, if $y \geq 0$, from $h(t) \leq h(0)$ we get that $\widehat{\tau}_a(x, y) \leq \bar{\tau}(x)$ and therefore also $\widehat{\tau}_a(x, y)$ is finite with probability one. Finally, if $y \geq 0$, we obtain that $P(\tau_a(x, y) < +\infty) = 1$, because $\tau_a(x, y) = \widehat{\rho}^{-1}(\widehat{\tau}_a(x, y)) \leq \widehat{\rho}^{-1}(\bar{\tau}_a(x))$. Note, however, that this argument does not work for $y < 0$.
A more difficult problem is to find the distribution of $\widehat{\tau}_a(x, y)$, and then that of $\tau_a(x, y)$. However, if $h(t)$ is either convex or concave, then lower and upper bounds to the distribution of $\widehat{\tau}_a(x, y)$ can be obtained by considering a "polygonal approximation" of $h(t)$ by means of a piecewise-linear function (see e.g. [3], [6]), but in general, it is not possible to find the distribution of $\widehat{\tau}_a(x, y)$ exactly.

Remark 3.1 Actually, it is possible to find explicitly the density of the FPT of X through certain moving boundaries, by using results on the FPT-density of BM (see [7], [9], [11], [13], [14], [21], [40], [41]). Let $v(t),\ t \geq 0$, be a curved boundary for which the FPT-density $\widehat{f}_v(t|x)$ of BM through v, when starting from $x < v(0)$, is explicitly known; if $S(t) = v(\widehat{\rho}(t)) + yt$, one can easily find the density of the FPT of X through S, with the condition that $x < S(0) = v(0)$. In fact, if $\tau_S(x, y) = \inf\{t > 0 : X(t) = S(t)|X(0) = x, Y(0) = y\}$, one gets $\tau_S(x, y) = \inf\{t > 0 : x + ty + \widehat{B}(\widehat{\rho}(t)) = S(t)\}$; then, $\widehat{\tau}_v(x, y) := \widehat{\rho}(\tau_S(x, y)) = \inf\{t > 0 : x + \widehat{B}(t) = v(t)\}$ has density $\widehat{f}_v$ and so the density of $\tau_S(x, y)$ turns out to be

$$f_S(t|x) = \widehat{f}_v(\widehat{\rho}(t)|x)\widehat{\rho}'(t). \tag{3.3}$$

For instance, if X is integrated BM ($\widehat{\rho}(t) = t^3/3$), and we consider the cubic boundary $S(t) = a + ty + bt^3 \ \ (a > 0,\ b < 0)$, it results $S(t) = v(\widehat{\rho}(t)) + yt$, with $v(t) = a + 3bt$ and so, for $x < a$ we reduce to consider the FPT of BM starting from x through the linear boundary $a + 3bt$. Since this has the inverse Gaussian density i.e. $\frac{a-x}{\sqrt{2\pi}\ t^{3/2}} e^{-(3bt+a-x)^2/2t}$ (see e.g. [6]), the density of $\tau_S(x, y)$ can be easily recovered from (3.3).

Formula (3.2), with $y = 0$, allows to find the density of $\tau_a(x, 0)$ in closed form; in fact, $\widehat{\tau}_a(x, 0)$ is the FPT of BM $\widehat{B}$ through the level $a - x > 0$, and so its density is:

$$\widehat{f}_a(t|x) := \frac{d}{dt} P(\widehat{\tau}_a(x, 0) \leq t) = \frac{a - x}{\sqrt{2\pi}\ t^{3/2}} e^{-(a-x)^2/2t}, \tag{3.4}$$

from which the density of $\tau_a(x, 0) = \widehat{\rho}^{-1}(\widehat{\tau}_a(x, 0)$ follows:

$$f_a(t|x) := \frac{d}{dt} P(\tau_a(x, 0) \leq t) = \widehat{f}_a(\widehat{\rho}(t)|x)\widehat{\rho}'(t) = \frac{(a - x)\ \widehat{\rho}'(t)}{\sqrt{2\pi}\ \widehat{\rho}(t)^{3/2}} e^{-(a-x)^2/2\widehat{\rho}(t)}. \tag{3.5}$$

If X is integrated BM, we have $X(t) = x + \widehat{B}(\widehat{\rho}(t))$, with $\widehat{\rho}(t) = t^3/3$, so we get (cf. [18]):

$$f_a(t|x) = \frac{3^{3/2}(a-x)}{\sqrt{2\pi}\, t^{5/2}} e^{-3(a-x)^2/2t^3}. \tag{3.6}$$

If X is integrated OU process, the density of $\tau_a(x, 0)$ can be obtained by inserting in (3.5) the function $\widehat{\rho}(t)$ deducible from Example 2, but it takes a more complex form.

Remark 3.2 Formula (3.5) implies that the nth order moment of the FPT, $E(\tau_a^n(x, 0))$, is finite if and only if the function $t^n \widehat{\rho}'(t)/\widehat{\rho}(t)^{3/2}$ is integrable in $(0, +\infty)$.
Now, let us suppose that there exists $\alpha > 0$ such that $\widehat{\rho}(t) \sim const \cdot t^\alpha$, as $t \to +\infty$; then, in order that $E(\tau_a^n(x, 0)) < \infty$, it must be $\alpha = 2(n+\delta)$, for some $\delta > 0$. For integrated BM, we have $\alpha = 3$, then for $n = 1$ the last condition holds with $\delta = 1/2$, so we obtain the finiteness of $E(\tau_a(x, 0))$ (notice that the mean FPT of BM through a constant barrier is instead infinite). Of course, this is not always the case; in fact, if X is integrated OU process, we have $\rho(t) \sim const \cdot e^{2\mu t}$, $\gamma(t) \sim const \cdot \ln(2\mu t/\sigma^2)$, as $t \to +\infty$, and so $\widehat{\rho}(t) = \gamma(\rho(t)) \sim const \cdot t$, as $t \to +\infty$, namely $\alpha = 1$ and the condition above is not satisfied with $n = 1$; therefore $E(\tau_a(x, 0)) = +\infty$. Not even $E((\tau_a(x, 0))^{1/2})$ is finite, but $E((\tau_a(x, 0))^{1/4})$ is so. Notice that the moments of any order of the FPT of (non integrated) OU through a constant barrier are instead finite.

As for the second order moment of the FPT of integrated BM, instead, we obtain $E\left[(\tau_a(x,0))^2\right] = +\infty$, since the equality $\alpha = 2(n + \delta)$ with $\alpha = 3$ and $n = 2$ is not satisfied, for any $\delta > 0$.

From (3.4) we get that the nth order moment of $\tau_a(x, 0)$, if it exists finite, is explicitly given by:

$$E\left[(\tau_a(x,0))^n\right] = E\left[(\widehat{\rho}^{\,-1}(\widehat{\tau}_a(x,0)))^n\right]$$

$$= \int_0^{+\infty} (\widehat{\rho}^{\,-1}(t))^n \frac{a-x}{\sqrt{2\pi} t^{3/2}} e^{-(a-x)^2/2t} dt. \tag{3.7}$$

For instance, if X is integrated BM, one has:

$$E(\tau_a(x,0)) = E((3\,\widehat{\tau}_a(x,0))^{1/3}) = \int_0^{+\infty} (3t)^{1/3} \frac{a-x}{\sqrt{2\pi} t^{3/2}} e^{-(a-x)^2/2t} dt$$

$$= \frac{3^{1/3}(a-x)}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{t^{7/6}} e^{-(a-x)^2/2t} dt.$$

By the variable's change $z = 1/t$, the integral can be written as:

$$\int_0^{+\infty} \frac{1}{z^{5/6}} e^{-(a-x)^2 z/2} dz = \frac{\Gamma\left(\frac{1}{6}\right) 2^{1/6}}{(a-x)^{1/3}} \int_0^{+\infty} \left(\frac{(a-x)^2}{2}\right)^{1/6} \frac{1}{\Gamma\left(\frac{1}{6}\right)} z^{1/6-1} e^{-\frac{(a-x)^2}{2} z} dz$$

$$= \frac{\Gamma\left(\frac{1}{6}\right) 2^{1/6}}{(a-x)^{1/3}},$$

where we have used that the last integral equals one, because the integrand is a Gamma density. Thus, for integrated BM, we finally obtain:

$$E(\tau_a(x,0)) = \left(\frac{3}{2}\right)^{1/3} \Gamma\left(\frac{1}{6}\right) \frac{(a-x)^{2/3}}{\sqrt{\pi}}. \tag{3.8}$$

Notice that an asymptotic expression of $E(\tau_a(x, y))$ for large $y > 0$ was obtained in [22].

3.1 Random starting point Until now we have supposed that the starting point $X(0) < a$ is given and fixed. We can introduce a randomness in the starting point, replacing $X(0)$ with a random variable x, having density $g(u)$ whose support is the interval $(-\infty, a)$; the corresponding FPT problem is particularly relevant in contexts such as neuronal modeling, where the reset value of the membrane potential is usually unknown (see e.g. [28]). In fact, the quantity of interest becomes now the unconditional FPT through the boundary a, that is, $\inf\{t > 0 : X(t) = a | Y(0) =$

$y\}$. Notice that results on FPT problems for general Gauss-Markov processes in the presence of random initial position are available in [15].
In particular, if X is integrated BM and $y = 0$, one gets from (3.8) that the average FPT through the boundary a, over all initial positions $x < a$, is:

$$\overline{T}_a = \int_{-\infty}^{a} E(\tau_a(u,0))g(u)du = \left(\frac{3}{2}\right)^{1/3} \frac{\Gamma\left(\frac{1}{6}\right)}{\sqrt{\pi}} \int_{-\infty}^{a} (a-u)^{2/3} g(u)du. \tag{3.9}$$

For instance, suppose that $a - x$ has Gamma distribution with parameters $\alpha,\ \lambda > 0$, namely, x has density

$$g(u) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda(a-u)}(a-u)^{\alpha-1} \cdot \mathbb{I}_{(-\infty,a)}(u).$$

Then, by the change of variable $z = a - u$ one obtains that the above integral is nothing but $E\left(Z^{2/3}\right)$, where Z is a random variable with the same distribution of $a - x$; then, recalling the expressions of the moments of the Gamma distribution, one obtains $E\left(Z^{2/3}\right) = \frac{\Gamma(\alpha+\frac{2}{3})}{\lambda^{2/3}\Gamma(\alpha)}$. Finally, by inserting this quantity in (3.9), it follows that:

$$\overline{T}_a = \frac{\left(\frac{3}{2\lambda^2}\right)^{1/3}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)}.$$

Remark 3.3 For $y = Y(0) = 0$, we have considered the FPT of X through the boundary a from "below", with the condition $x = X(0) < a$; if one considers the FPT of X through the barrier a from "above", with the condition $X(0) > a$ (namely, $\inf\{t > 0 : X(t) \le a | X(0) = x, Y(0) = 0\}$), then in all formulae $a - x$ has to be replaced with $x - a$. More generally, if one considers the first hitting time of X to a (from above or below), $a - x$ must be replaced by $|a - x|$.

4 First exit time from an interval Assume, as always, that $\gamma(+\infty) = +\infty$; for $x \in (a,b)$ and $y \in \mathbb{R}$, the first-exit time of X from the interval (a,b) is:

$$\tau_{a,b}(x,y) = \inf\{t > 0 : x + yt + \widehat{B}(\widehat{\rho}(t)) \notin (a,b)\}. \tag{4.1}$$

Set $\widehat{\tau}_{a,b}(x,y) = \widehat{\rho}(\tau_{a,b}(x,y))$, then:

$$\widehat{\tau}_{a,b}(x,y) = \inf\{t > 0 : x + \widehat{B}_t \le a - y\widehat{\rho}^{-1}(t) \text{ or } x + \widehat{B}_t \ge b - y\widehat{\rho}^{-1}(t)\}. \tag{4.2}$$

If $\widehat{\tau}_{a,b}(x,y)$ is finite with probability one, also $\tau_{a,b}(x,y)$ is so. In the sequel, we will focus on the case when $y = 0$, namely we will consider $\tau_{a,b}(x,0) = \widehat{\rho}^{-1}(\widehat{\tau}_{a,b}(x,0))$, where $\widehat{\tau}_{a,b}(x,0) = \inf\{t > 0 : x + \widehat{B}_t \notin (a,b)\}$; as it is well-known, $\widehat{\tau}_{a,b}(x,0)$ is finite with probability one and its moments are solutions of Darling and Siegert's equations (see [12]).

First, we will find sufficient conditions so that the moments of $\tau_{a,b}(x,0)$ are finite; then, we will carry on explicit computations of them, in the case of integrated BM.

Proposition 4.1 *If $\widehat{\rho}$ is convex, then $E\left(\tau_{a,b}(x,0)\right) < \infty$; moreover, if there exist constants $c,\ \delta > 0$, such that $0 \le \widehat{\rho}^{-1}(t) \le c \cdot t^\delta$, then $E\left(\tau_{a,b}(x,0)\right)^n < \infty$, for any integer n.*

Proof. If $\widehat{\rho}$ is convex, then $\widehat{\rho}^{-1}$ is concave, and the finiteness of $E\left(\tau_{a,b}(x,0)\right)$ follows by Jensen's inequality written for concave functions. Next, denote by $\widehat{f}_{-\alpha,\alpha}(t|x)$ the density of the first-exit time of $x + \widehat{B}_t$ from the interval $(-\alpha,\alpha),\ \alpha > 0$; we recall from [12] that the Laplace transform of $\widehat{f}_{-\alpha,\alpha}(t|x)$, namely, $\int_0^{+\infty} e^{-\theta t}\widehat{f}_{-\alpha,\alpha}(t|x)dt$ is:

$$\mathcal{L}\left[\widehat{f}_{-\alpha,\alpha}\right](\theta|x) = \frac{\cosh(\sqrt{2\theta}x)}{\cosh(\sqrt{2\theta}\alpha)}\ ,\ -\alpha < x < \alpha,\ \theta \ge 0. \tag{4.3}$$

By inverting this Laplace transform, one obtains (see [12]):

$$\widehat{f}_{-\alpha,\alpha}(t|x) = \frac{\pi}{\alpha^2}\sum_{k=0}^{\infty}(-1)^k\left(k+\frac{1}{2}\right)\cos\left[\left(k+\frac{1}{2}\right)\frac{\pi x}{\alpha}\right]\exp\left[-\left(k+\frac{1}{2}\right)^2\frac{x^2 t}{2\alpha^2}\right]. \tag{4.4}$$

The case of an interval (a,b), $b > a$, is reduced to the previous one; in fact, as easily seen, if $\alpha = (b-a)/2$ one has:

$$\widehat{f}_{a,b}(t|x) = \widehat{f}_{-\alpha,\alpha}\left(t|x - \frac{a+b}{2}\right).$$

Of course, the density of $\tau_{a,b}(x,0)$ turns out to be $\widehat{f}_{a,b}(\widehat{\rho}(t)|x)\widehat{\rho}'(t)$. For the sake of simplicity, we take $a = -\alpha$, $b = \alpha$, $\alpha > 0$; then, for $x \in (-\alpha,\alpha)$ and an integer n :

$$E\left[(\tau_{a,b}(x,0))^n\right] = E\left[(\tau_{-\alpha,\alpha}(x,0))^n\right] = E\left[\left(\widehat{\rho}^{-1}(\widehat{\tau}_{-\alpha,\alpha}(x,0))^n\right] = \sum_{k=0}^{\infty} A_k(x), \tag{4.5}$$

where

$$A_k(x) = \frac{\pi}{\alpha^2}(-1)^k\left(k+\frac{1}{2}\right)\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{\alpha}\right)\int_0^{+\infty} e^{-(k+1/2)^2\pi^2 t/2\alpha^2}\left(\widehat{\rho}^{-1}(t)\right)^n dt. \tag{4.6}$$

The integral can be written as:

$$\frac{2\alpha^2}{\pi^2(k+1/2)^2} E\left(\widehat{\rho}^{-1}(Z_k)\right)^n,$$

where Z_k is a random variable with exponential density of parameter $\lambda_k = (k+1/2)^2\pi^2/2\alpha^2$; so:

$$A_k(x) = (-1)^k\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{\alpha}\right)\frac{2}{\pi(k+1/2)}E\left(\widehat{\rho}^{-1}(Z_k)\right)^n.$$

Recalling that $E[(Z_k)^{n\delta}] = \frac{\Gamma(1+n\delta)}{(\lambda_k)^{n\delta}}$, by the hypotheses we get $E\left((\widehat{\rho}^{-1}(Z_k))^n\right) \le c^n E[(Z_k)^{n\delta}] = const \cdot \frac{\Gamma(1+n\delta)}{(k+1/2)^{2n\delta}}$; thus:

$$|A_k(x)| \le \frac{const'}{(k+1/2)^{1+2n\delta}},$$

from which it follows that the series $\sum_k A_k(x)$ is absolutely convergent for every $x \in (-\alpha,\alpha)$, and therefore $E\left[(\tau_{-\alpha,\alpha}(x,0))^n\right] < +\infty$. The finiteness of $E\left[(\tau_{a,b}(x,0))^n\right]$ in the general case is easily obtained.

□

Remark 4.2 The condition $0 \le \widehat{\rho}^{-1}(t) \le c \cdot t^\delta$ is satisfied e.g. for integrated BM, since $\widehat{\rho}^{-1}(t) = 3^{1/3}t^{1/3}$ (see Example 1), and for integrated OU process, because from the expression of $\widehat{\rho}(t)$ deducible from Example 2, it can be shown that $c_1 t \le \widehat{\rho}(t) \le c_2 t$ for suitable $c_1, c_2 > 0$ which depend on μ and σ, and therefore $\frac{1}{c_2}t \le \widehat{\rho}^{-1}(t) \le \frac{1}{c_1}t$.

Now, we carry on explicit computations of $E\left[\tau_{a,b}(x,0)\right]$ and $E\left[(\tau_{a,b}(x,0))^2\right]$, in the case of integrated BM. Inserting $\widehat{\rho}(t) = t^3/3$, $(\widehat{\rho}^{-1}(y) = (3y)^{1/3})$, and $n = 1, 2$ in (4.5), (4.6), after some calculations we obtain:

$$E\left[\tau_{a,b}(x,0)\right] = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^{5/3}}\cos\left[\frac{\pi(2k+1)}{b-a}\left(x-\frac{a+b}{2}\right)\right]. \tag{4.7}$$

$$E\left[(\tau_{a,b}(x,0))^2\right] = \frac{12(b-a)^4}{\pi^4}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^4}\cos\left[\frac{\pi(2k+1)}{b-a}\left(x-\frac{a+b}{2}\right)\right]. \tag{4.8}$$

Notice that it is arduous enough to express the sums of the Fourier-like series above in terms of elementary functions of $x \in (a,b)$, and then to obtain the moments of $\tau_{a,b}(x,0)$ in a simple closed form; actually, by using the Kolmogorov's equations approach, in [32], [33], it was obtained a formula for $E(\tau_{a,b}(x,0))$ in terms of hypergeometric functions. This kind of difficulty does not

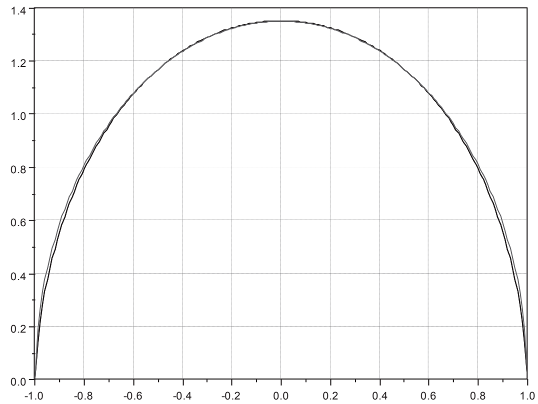

Figure 1: Plots of the mean exit time, $E(\tau_{-1,1}(x,0))$, of integrated BM from the interval $(-1,1)$ (lower curve), and of the function $z(x) = 1.35\cdot(1-x^2)^{1/2}$ (upper curve), as functions of $x \in (-1,1)$.

arise, for instance, in the case of (non-integrated) BM; in fact, by using formula (4.5) with $\widehat{\rho}(t) = t$ and $n = 1$, one obtains:

$$E\left[\tau_{-\alpha,\alpha}(x)\right] = \frac{32\alpha^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^3}\cos\left[(2k+1)\frac{\pi}{2\alpha}x\right];$$

on the other hand, the well-known formula for the mean first-exit time of BM from the interval $(-\alpha,\alpha)$, provides that the sum of the series must be $\alpha^2 - x^2$.

However, (4.7) and (4.8) turn out to be very convenient to estimate the first two moments of $\tau_{a,b}(x,0)$ for integrated BM; in fact the two series converge fast enough, so to obtain "good" estimates of the moments, it suffices to consider a few terms of them. As for $E\left[\tau_{a,b}(x,0)\right]$, it appears to be fitted very well by the square root of a quadratic function; this was obtained by least square interpolation implemented in MATLAB. In the Figure 1, for integrated BM, we compare the graphs of $E(\tau_{a,b}(x,0))$, calculated by replacing the series in (4.7) with a finite summation over the first 20 addends, and that of $C\cdot[(b-x)(x-a)]^{1/2}$, as functions of $x \in (a,b)$, for $a=-1,\ b=1$, and $C = 1.35$; the two curves appear to be almost undistinguishable.

We have also calculated the second order moment of the first-exit time of integrated BM, by summing the first 20 addends of the series in (4.8). In the Figure 2, we plot $E\left[(\tau_{a,b}(x,0))^2\right]$, $E^2\left[\tau_{a,b}(x,0)\right]$ and the variance $Var\left[\tau_{a,b}(x,0)\right]$, as a function of $x \in (-1,1)$, for $a=-1,\ b=1$; as we see, the maximum of $Var\left[\tau_{a,b}(x,0)\right]$ is about 10% times the maximum of $E(\tau_{-1,1}(x,0))$.

4.1 Random starting point As in the one boundary case, if we introduce a randomness in the starting point, replacing $X(0) \in (a,b)$ with a random variable x, having density $g(u)$ whose support is the interval (a,b), we can consider the average exit time over all initial positions $x \in (a,b)$. If $y = 0$, this quantity is:

$$\overline{T}_{a,b} = \int_a^b E(\tau_{a,b}(u,0))g(u)du.$$

Notice that results on first-exit times for general Gauss-Markov processes, in the presence of random initial position, are available in [37].

In the case of integrated BM, $\overline{T}_{a,b}$ can be calculated by using the expression of $E(\tau_{a,b}(x,0))$ given

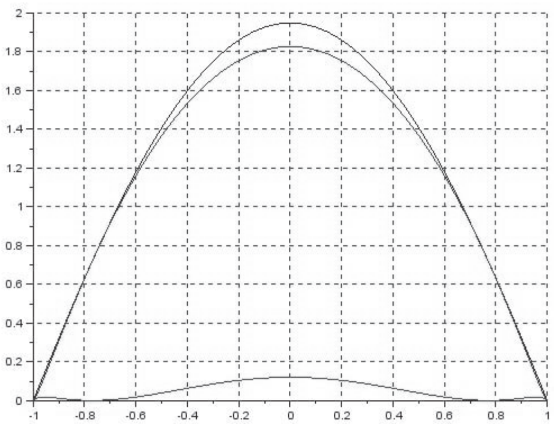

Figure 2: From top to bottom: plot of the second moment (first curve), the square of the first moment (second curve), and the variance of the first-exit time $\tau_{-1,1}(x,0)$ (third curve) of integrated BM from the interval $(-1,1)$, as functions of $x \in (-1.1)$.

by (4.7). We obtain:
(4.9)

$$\overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} \int_a^b \cos\left[\frac{\pi(2k+1)}{b-a}\left(u-\frac{a+b}{2}\right)\right] g(u)du$$

(it has been possible to exchange the integral of the sum with the sum of the integrals, thanks to the dominated convergence theorem); the integral in (4.9) equals $E(U_k)$, where
$U_k = \cos\left[\frac{\pi(2k+1)}{b-a}\left(\eta - \frac{a+b}{2}\right)\right] \leq 1$. Therefore:

$$\overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} E(U_k). \tag{4.10}$$

In the special case when g is the uniform density in the interval (a,b), we get by calculation:

$$\overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} \int_a^b \cos\left[\frac{\pi(2k+1)}{b-a}\left(x-\frac{a+b}{2}\right)\right] \frac{1}{b-a}\, dx$$

$$= \frac{3^{1/3}2^{10/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{8/3}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{8/3}}\ . \tag{4.11}$$

Thus, $\overline{T}_{a,b} = const \cdot (b-a)^{2/3}$. This confirms the result by Masoliver and Porrà (see [32], [33]), obtained by the Kolmogorov's equations approach in the case of integrated BM, with $y=0$ and uniform distribution of the $X-$ starting point, according to which, the dependence of $\overline{T}_{a,b}$ on the size $L=(b-a)$ of the interval, is $L^{2/3}$.

As far as integrated OU process is concerned, the moments of $\tau_{a,b}(x,0)$ can be found again by formula (4.5), where $\widehat{\rho}(t)$ can be deduced from Example 2; however, it is not possible to calculate explicitly the integral which appears in the expression of $A_k(x)$, so it has to be numerically computed. Since the integrand function decreases exponentially fast, it suffices to calculate the integral over the interval $(0,10)$, to obtain precise enough estimates. In the Figure 3 we have

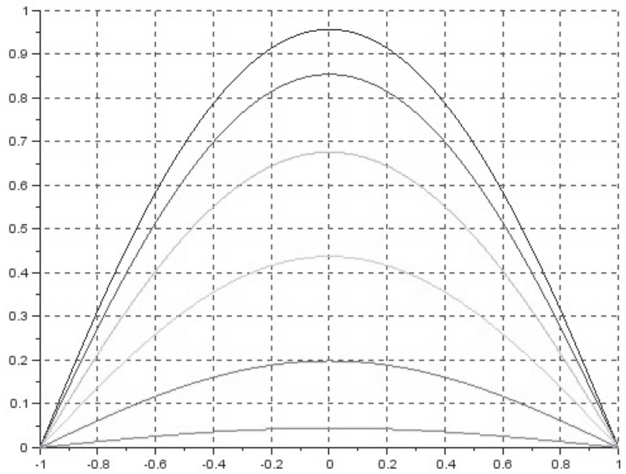

Figure 3: Plot of numerical evaluation of the mean exit time, $E\left(\tau_{-1,1}(x,0)\right)$, of integrated OU with $\beta = y = 0$, from the interval $(-1,1)$, as a function of $x \in (-1,1)$, for $\sigma = 1$ and several values of μ. From top to bottom, with respect to the peak of the curve: $\mu = 2; 1.8; 1.6; 1.4; 1.2; 1$.

plotted, for comparison, the numerical evaluation of the mean exit time of integrated OU process with $y = \beta = 0$, from the interval $(-1,1)$, as a function of $x \in (-1,1)$, for $\sigma = 1$ and several values of μ; in the Figure 4 we have plotted the numerical evaluation of $E\left[\left(\tau_{-1,1}(x,0)\right)^2\right]$, $E^2\left[\tau_{-1,1}(x,0)\right]$ and the variance $Var\left[\tau_{-1,1}(x,0)\right]$ of the first exit time of integrated OU process, for $\sigma = 1$ and $\mu = 1$. As we see, the maximum of $Var\left[\tau_{-1,1}(x,0)\right]$ is about 5% times the maximum of $E\left(\tau_{-1,1}(x,0)\right)$.

Finally, we mention the exit probabilities of the integrated Gauss-Markov process X through the ends of the interval (a,b), namely:

$$\pi_a(x,y) = P\left(\tau_a(x,y) < \tau_b(x,y)\right) = P\left(X(\tau_{a,b}(x,y)) = a\right),$$

and

$$\pi_b(x,y) = P\left(\tau_b(x,y) < \tau_a(x,y)\right) = P\left(X(\tau_{a,b}(x,y)) = b\right).$$

Recalling the well-known formulae for exit probabilities of BM, we get, for $y = 0$ and $x \in (a,b)$:

$$\pi_a(x,0) = P\left(x + \widehat{B}(\widehat{\tau}_{a,b}(x,0)) = a\right) = \frac{b-x}{b-a}, \quad \pi_b(x,0) = P\left(x + \widehat{B}(\widehat{\tau}_{a,b}(x,0)) = b\right) = \frac{x-a}{b-a}.$$

Notice that, in the case of integrated BM, several probability laws related to the couple $\left(\tau_{a,b},\ B_{\tau_{a,b}}\right)$ were evaluated in [25] (in particular, explicit formulae for $\pi_a(x,0)$ and $\pi_b(x,0)$ were obtained), but they are written in terms of special functions.

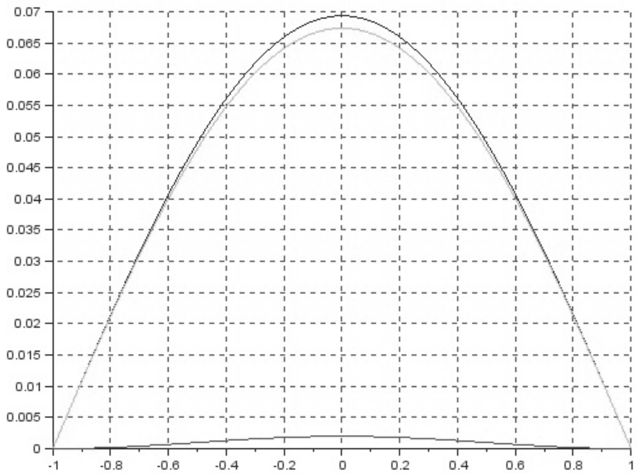

Figure 4: From top to bottom: plot of the second moment (first curve), the square of the first moment (second curve), and the variance of the first-exit time $\tau_{-1,1}(x, 0)$ (third curve) of integrated OU with $y = \beta = 0$, from the interval $(-1, 1)$, as a function of $x \in (-1, 1)$, for $\sigma = 1, \mu = 1$.

Acknowledgment I would like to express particular thanks to the anonymous referees for the constructive comments and suggestions leading to improvements of the paper.

REFERENCES

[1] Abundo, M., 2013. On the representation of an integrated Gauss-Markov process. Sci. Math. Jpn. Online e-2013, 719-723.

[2] Abundo, M., 2012. An inverse first-passage problem for one-dimensional diffusions with random starting point. Statistics and Probability Letters 82 (1), 7–14.

[3] Abundo, M., 2012. First-passage time of a stochastic integral process trough a linear boundary. International Journal of Applied Mathematics (IJAM) 25 (1), 41–49.

[4] Abundo, M., 2008. On the distribution of the time average of a jump-diffusion process. International Journal of Applied Mathematics (IJAM) 21 (3), 447–454.

[5] Abundo, M., 2002. Some conditional crossing results of Brownian motion over a piecewise-linear boundary. Statistics and Probability Letters 58 (2), 131–145.

[6] Abundo, M., 2001. Some results about boundary crossing for Brownian motion. Ricerche di Matematica L (2), 283–301.

[7] Alili, L., and Patie, P., 2010. Boundary-crossing identies for diffusions having the time-inversion property. J. Theor. Probab. 23, 65–84.

[8] Andersen, T.G., Bollerslev, T., Diebold, F.X. and Labys, P., 2001. The distribution of realized exchange rate volatility. J. Amer. Statist. Assoc. 96, 42–55.

[9] Bachelier, L., 1941. Probabilités des oscillation maxima. C. R. Acad. Sci. Paris 212, 836–838.

[10] Benedetto, E., Sacerdote, L. and Zucca, C., 2013. A first passage problem for a bivariate diffusion process: Numerical solution with an application to neuroscience when the process is GaussMarkov. J. Comput. Appl. Math. 242, 41–52.

[11] Breiman, L., 1965/1966. First exit times from a square root boundary. In: Proc. Fifth Berkeley Sympos. Math. Statis. and Probability, Berkeley, Calif. Contributions to Probability Theory. Part 2, vol II, 9–16. University of California Press, Berkeley.

[12] Darling, D. A. and Siegert, A.J.F., 1953. The first passage problem for a continuous Markov process. Ann. Math. Statis., 24, 624–639.

[13] Daniels, H. E., 1982. Sequential tests constructed from images. Ann. Statist., 10 (2), 394–400.

[14] Daniels, H. E., 1969. Minimum of a stationay Markov process superimposed on a U-shaped trend. J. Appl. Probab., 6 (2), 399–408.

[15] Di Nardo, E., Nobile A.G., Pirozzi E., Ricciardi L.M., 2001. A computational approach to first-passage-time problem for Gauss-Markov processes. Adv. Applied Probability, 33, 453–482.

[16] Doob, J.L., 1949. Heuristic approach to the Kolmogorov-Smirnov theorem. Ann. Math. Statist., 20, 393–403.

[17] Genon-Catalot, V., Jeantheau, T. and Larédo, C., 2000. Stochastic volatility models as hidden Markov models and statistical applications. Bernoulli 6, 1051–1079.

[18] Goldman, M., 1971. On the first-passage time of the integrated Wiener process. Ann. Math. Statis. 42 (6), 2150–2155.

[19] Gor'kov, Ju P., 1975. A formula for the solution of a certain boundary value problem for the stationary equation of Brownian motion. Probab. Theory. Relat. Fields 81 (1), 79–109.

[20] Gloter, A., 2000. Parameter estimation for a discrete sampling of an integrated Ornstein-Uhlenbeck process. Statistics 35, 225–243.

[21] Groeneboom, P., 1989. Brownian motion with a parabolic drift and Airy functions. Sov. Math. Dokl. 16, 904–908.

[22] Hesse, C.H., 2005. On the first-passage time of Integrated Brownian motion. Journal of Applied Mathematics and Stochastic Analysis 3, 237–246.

[23] Klebaner, F.C., 2005. Introduction to Stochastic Calculus with Applications. Second Edition. Imperial College Press, London.

[24] Kolmogorov, A., 1934 Zufällige Bewegungen (zur Theorie der Brownschen Bewegung) Ann. of Math. (2) 35, no. 1, 116-117 (German).

[25] Lachal, A., 1997. Temps de sortie d'un intervalle borné pour l'intégrale du mouvement Brownien. C.R. Acad. Sci. Paris 324, serie I, 559–564.

[26] Lachal, A., 1993. L'integrale du mouvement Brownien. J. Appl. Prob. 30, 17–27.

[27] Lachal, A., 1991. Sur le premier instant de passage de l'integrale du mouvement Brownien. Annales de l' I.H.P. B, 27 (3), 385–405.

[28] Lansky, P. and Smith, C.E., 1989. The effect of a random initial value in neural first- passage-time models. Math. Biosci. 93 (2), 191-215.

[29] Lefebvre, M., 1989. First-passage densities of a two-dimensional process. SIAM J. Appl. Math. 49 (5), 1514–1523.

[30] Lefebvre, M., 1989. Moment generating function of a first hitting place for the integrated Ornstein-Uhlenbeck process. Stoch. Proc. Appl. 32, 281–287.

[31] Marcus M.B. and Rosen, J., 2006. Markov processes, Gaussian processes, and local time. Cambridge University Press, Cambridge.

[32] Masoliver, J. and Porrà, J.M., 1996. Exact solution to the exit-time problem for an undamped free particle driven by Gaussian white noise. Phys. Rev. E 53 (3), 2243–2256.

[33] Masoliver, J. and Porrà, J.M., 1995. Exact solution to the mean exit-time problem for free inertial processes driven by Gaussian white noise. Phys. Rev. Lett. 75 (2), 189–192.

[34] Mehr, C.B. and McFadden, J.A., 1965. Certain properties of Gaussian processes and their first-passage times. J. R. Statist. Soc. B 27, 505–522.

[35] McKean, H.P. Jr., 1963. A winding problem for a resonator driven by a white noise. J. Math. Kyoto Univ. 2 (2), 227–235.

[36] Nobile, A.G., Pirozzi, E., Ricciardi, L.M. , 2008. Asymptotics and evaluations of FPT densities through varying boundaries for Gauss-Markov processes. Sci. Math. Jpn. 67, (2), 241–266.

[37] Nobile, A.G.,Pirozzi, E., Ricciardi, L.M. , 2006. On the two-boundary first-passage-time problem for Gauss-Markov processes . Sci. Math. Jpn. 64(2), 421-442.

[38] Revuz, D. and Yor, M., 1991. Continous martingales and Brownian motion. Springer-Verlag, Berlin Heidelberg.

[39] Ross, S.M., 2010. Introduction to Probability Models. Tenth Edition. Academic Press, Elsevier, Burlington.

[40] Salminen, P., 1988. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. Adv. Appl. Probab. 20 (2), 411–426.

[41] Sato, S., 1977. Evaluation of the first-passage time probability to a square root boundary for the Wiener process.. J. Appl. Probab. 14 (4), 850–856.

Communicated by *Antonio Di Crescenzo*

Mario Abundo
Dipartimento di Matematica, Università "Tor Vergata"
via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail: abundo@mat.uniroma2.it

CZECH REPUBLIC

(a) **Milan Vlach**
(b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
(b') milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
(c) Game theory, Fair division, Optimization

(a) **Vaclav Koubek**
(b) Department of Theoretical Computer Science and Mathematical Logic (KTIML), The Faculty of Mathematics and Physics (MFF), Malostranske nam. 25, 118 00 Praha 1, Czech Republic
(b') koubek@ktiml.mff.cuni.cz
(c) Theory of lattices. Theory of semigroups. Theory of graphs and their complexity. Complexity theory. Universal algebra

(a) **Vera Trnková**
(b) Mathematical Institute of Charles University, 186 75 Praha 8, Sokolovska 83, Czech Republik
(b') trnkova@karlin.mff.cz
(c) Universal algebra, Category theory, General topology

FINLAND

(a) **Arto Salomaa**
(b) Jaanintie 34 A 26, 20540 Turku, Finland
(b') asalomaa@utu.fi
(c) Formal languages, automata, computability, cryptography

(a) **Jarkko Kari**
(b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
(b') jkari@utu.fi
(c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

(a)**Klaus Denecke**
(b) 14542 Werder (Havel), Germany, Zanderweg 3
(b') klausdenecke@hotmail.com
(c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

(a) **Maria Fragoulopoulou**
(b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
(b') fragoulop@math.uoa.gr
(c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory, Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

(a) **Gyula Maksa**
(b) Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
(b') maksa@science.unideb.hu
(c) Functional equations

(a) **Kálmán Györy**
(b) University of Debrecen, Institute of Mathematics, 4010 Debrecen, Hungary
(b') gyory@science.unideb.hu
(c) Number Theory (mainly Diophantine and Algebraic Number Theory)

(a) **Pál Dömösi**
(b) Institute of Mathematics and Informatics, Nyíregyháza College, Nyíregyháza, Sóstói út 31/B, H-4400, Hungary
(b') domosi@nyf.hu
(c) Theoretical Computer Science, Algebra

ISRAEL

(a) **Dany Leviatan**
(b) School of Mathematics, Tel Aviv University, 69978 Tel Aviv, Israel
(b') leviatan@post.tau.ac.il
(c) Approximation Theory, Computer Added Geometric Design, Summability

ITALY

(a) **Angelo Favini**
(b) Dept. of Mathematics, Bologna Univ., Piazza di Porta S. Donato, 5, 40126, Bologna, Italy
(b') angero.favini@unibo.it
(c) Evolution equations and Control theory for abstract differential equations and PDE.

(a) **Antonio Di Crescenzo**
(b) Università di Salerno, Dipartimento di Matematica, Via Giovanni Paolo II, n.132, 84084 Fisciano (SA), Italy
(b') adicrescenzo@unisa.it
(c) Stochastic Processes and Applications, Reliability Theory, Queueing Systems, Stochastic Models in Biology

(a) **Tonia Ricciardi**
(b) Department of Mathematics and its Applications, Federico II University, Via Cintia, 80126 Naples, Italy.
(b') tonia.ricciardi@unina.it
(c) Nonlinear elliptic partial differential equations

NETHERLANDS

(a) **Grzegorz Rozenberg**
(b) Leiden Institute of Advanced Computer Science (LIACS) Leiden University, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
(b') rozenber@liacs.nl
(c) Natural computing, Formal languages and automata theory

POLAND

(a) **Dariusz Zagrodny**
(b) Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszynski University, Dewajtis 5, 01-815 Warsaw, Poland
(b') d.zagrodny@uksw.edu.pl
(c) Nonsmooth Analysis (this is my main field of research), Nonlinear Programming (convex, nonconvex), Maximal Monotonicity (with respect to duality)

(a)**Henryk Hudzik**
(b) Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska Street 87, 61-614 Poznań, Poland
(b') hudzik@amu.edu.pl
(c) Function spaces theory and abstract Banach spaces theory, Banach Lattices , Geometry Banach Spaces,Composition and multiplication operators between Köthe spaces

(a) **Krzysztof Szajowski**
(b) Faculty of Fundamental Problems of Technology, Wroclaw University of Technology, Wybrzeze, Wyspianskiego 27, PL-50-370 Wroclaw, Poland
(b') Krzysztof.Szajowski@pwr.wroc.pl
(c) Applied Probability; Game Theory; Operations Research

(a) **Piotr Multarzynski**
(b) Faculty of Mathematics and Information Science, Warsaw University of Technology, PL. Politechniki 1, 00-661 Warsaw, Poland
(b') multarz@mini.pw.edu.pl
(c) Algebraic analysis (calculus of right invertible operators); Algebraic approach to differential geometry; Discrete counterparts of the classical concepts in analysis and differential geometry; q-calculus, Sikorski and Froelicher differential (or smooth) spaces; 12H10, 39A12, 39A70, 47B39, Groupoids - theory and applications in physics.

(a) **Tomasz Kubiak**
(b) Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
(b') tkubiak@amu.edu.pl
(c) Many-valued topology (fuzzy topology), Pointfree topology (frames and locales)

ROMANIA

(a) **Adrian Petrusel**
(b) Babes-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Department of Mathematics, Kogalniceanu street no. 1, 400084 Cluj-Napoca, Romania
(b') petrusel@math.ubbcluj.ro
(c) Nonlinear Analysis

(a) **Ioan A. Rus**
(b) Department of Mathematics, Babes-Bolyai University, Str. Kogalniceanu No. 1, 400084 Cluj-Napoca, Romania
(b') iarus@math.ubbcluj.ro
(c) Fixed Point Theory

(a) **Vasile Berinde**
(b) Department of Mathematics and Computer Science, Faculty of Sciences North University of Baia Mare, Victoriei Nr. 76, 430122 Baia Mare, Romania
(b') vberinde@ubm.ro
(c) Fixed Point Theory, iterative approximation of fixed points

RUSSIA

(a) **Andrei Vesnin**
(b) Sobolev Institute of Mathematics, pr. ak. Koptyuga 4, Novosibirsk, 630090, Russia
(b') vesnin@math.nsc.ru

(c) Low-dimensional topology, Knot theory, Hyperbolic manifolds and orbifolds.
(a) **Semen S. Kutateladze**
(b) The Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Academician Koptyug's Avenue 4, Novosibirsk, 630090, RUSSIA
(b') sskut@member.ams.org and sskut@math.nsc.ru
(c) Functional Analysis, Operator Theory, Convex Geometry, Optimization and Programming, Nonstandard Analysis, Boolean Valued Models

(a) **Vladimir V. Mazalov**
(b) Institute of Applied Mathematical Research, Karelia Research Center of Russian Academy of Sciences Pushkinskaya str., 11, Petrozavodsk 185610, Russia
(b') vmazalov@krc.karelia.ru
(c) Optimal Stopping Theory, Game with Optimal Stopping, Stochastic Dynamic Programming, Applications in Behavioral Ecology
.

SOUTH AFRICA

(a) **Joachim Schröder**
(b) Department van Wiskunde, Universiteit van die Vrystaat, Posbus 339, Bloemfontein 9300, South Africa
(b') schroderjd@qwa.ufs.ac.za
(c) Enumerative combinatorics, Categorical methods in topology, Set theoretic topology (cardinal invariants, elementary submodels)

SPAIN

(a) **Javier Gutierrez Garcia**
(b) Departamento de Matematicas, Universidad del Pais Vasco/Euskal Herriko Unibertsitatea UPV/EHU, Apartado 644, 48080, Bilbao, Spain
(b') javier.gutierrezgarcia@ehu.es
(c) Lattice-valued topology, General topology (in particular, insertion and extension of functions), Pointfree topology

(a) **Jorge Galindo**
(b) Instituo de Matemáticas y Aplicaciones de Castellón (IMAC), Departamento de Matemáticas, Universidad Jaume I, 12071-Castellón, Spain.
(b') jgalindo@mat.uji.es
(c) Topological Algebra, Abstract Harmonic Analysis, General Topology.

(a) **Luis M. Sanchez Ruiz**
(b) ETSID-Depto. de Matematica Aplicada, Universidad Politecnica de Valencia, E-46022 Valencia, Spain
(b') LMSR@mat.upv.es
(c) Functional Analysis, Topological Vector Spaces, Barrelledness Properties, Baire-like Spaces, Continuous Function Spaces, Wavelets

(a) **Salvador Hernandez**
(b) Departamento de Matematicas, Universitat Jaume I, 12071 Castellon, Spain
(b') hernande@mat.uji.es
(c) Topological groups and semigroups, Continuous functions, Operators defined between spaces of continuous functions, General topology.

TAIWAN

(a) **Hang-Chin Lai**
(b) Department of Mathematics, National Tsing Hua University, Hsin Chu City, Taiwan
(b’) laihc@mx.nthu.edu.tw
(c) Nonlinear analysis and convex analysis, Optimization theory, Harmonic analysis

UNITED STATES OF AMERICA

(a) **Andreas Blass**
(b) Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1043, USA
(b') ablass@umich.edu
(c) Mathematical logic, set theory, category theory

(a) **John B Conway**
(b) Professor Emeritus, George Washington University, 2115 G Street NW; Room 240;
Washington, DC 20052, U.S.A
(b') Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**
(b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**
(b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**
(b) 28-21 Misasagiikezutsumicho, Yamashinaku, Kyoto 607-8433, Japan
(b') yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Haruo Maki**
(b) 2-10-13 Wakagi-dai, Fukutsu-shi, Fukuoka 811-3221, Japan
(b') makih@pop12.odn.ne.jp
(c) (Topological) digital n-spaces (n>0), Generalized closed sets (after Levine),
Operation theory in topology (in the sense of Kasahara and Ogata)

(a) **Kohzo Yamada**
(b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b') eckyama@ipc.shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**
(b) Dept. of Mathematics and Computer Sciences, Shimane Univ., Matsue, Shimane 690-8504, Japan
(b') hattori@riko.shimane-u.ac.jp
(c) General Topology

(a) **Yoshikazu Yasui**
(b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho,
Kitakaturagi-gun, Nara, 635-0832, Japan
(b') y.yasui@kio.ac.jp
(c) General Topology

(a) **Eiichi Nakai**
(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
(b') enakai@mx.ibaraki.ac.jp
(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**
(b) Department of Mathematical Sciences, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
(b') jkawabe@shinshu-u.ac.jp
(c) Measure and integration, Vector measure, Fuzzy measure

(a) **Shizu Nakanishi**
(b) 4-15 Jonanteramachi, Tennoji-Ku, Osaka 543-0017, Japan
(b') shizu.nakanishi@nifty.ne.jp
(c) measures and integrations

(a) **Jun Ichi Fujii**
(b) Department of Arts and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
(b') fujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Masaru Nagisa**
(b) Dept. of Mathematics and Informatics, Faculty of Science, Chiba Univ., Yayoi-cho, Chiba, 263-8522, Japan
(b') nagisa@math.s.chiba-u.ac.jp
(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**
(b) Noji-Higashi 1-1-1, Kusatsu,525-8577, Japan
(b') osaka@se.ritsumei.ac.jp
(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**
(b) 46 Taka, Kashiba, Nara 639-0241, Japan
(b') mfujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Moto O'uchi**
(b) Dept. of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Sakai City, Osaka 599-8531, Japan
(b') ouchi@mi.s.osakafu-u.ac.jp
(c) C*-algebras, C*-modules, groupoid C*-algebras

(a) **Wataru Takahashi**
(b) Shiratoridai 52-27, Aoba-ku, Yokohama 227-0054, Japan
(b') wataru@is.titech.ac.jp, wataru@a00.itscom.net
(c) Nonlinear Functional Analysis

(a) **Shigeo Akashi**
(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, 2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan
(b') akashi@is.noda.tus.ac.jp
(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**
(b) Department of Mathematical Sciences, Osaka Prefecture University, 1-1, Gakuen-cho, Naka-ku, Sakai, Osaka 599-8531, Japan
(b') kabeya@ms.osakafu-u.ac.jp
(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**
(b) Dept. of Applied Physics, Graduate School of Engineering, Osaka Univ., 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') yagi@ap.eng.osaka-u.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yoshimasa Nakamura**
(b) Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
(b)' ynaka@i.kyoto-u.ac.jp
(c) integrable systems, numerical linear algebra, special functions

(a) **Yasumasa Fujisaki**
(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') fujisaki@ist.osaka-u.ac.jp
(c) Control Systems Theory

(a) **Naruhiko Aizawa**
(b) Department of Mathematics and Information Sciences,Graduate School of Science,Osaka Prefecture University,Sakai,Osaka 599-8531,Japan
(b')aizawa@mi.s.osakafu-u.ac.jp
(c) representation theory

(a) **Hisao Nagao**
(b) 5-1-25-506, Senrioka, Settu, Osaka 566-0001, Japan
(b') nagao@sb.dcns.ne.jp
(c) Multivariate Analysis, Sequential Analysis, Jackknife Statistics and Bootstrap Method

(a) **Masamori Ihara**
(b) 18-8 Neayagawa, Osaka, 572-8530, Japan
(b') ihara@isc.osakac.ac.jp
(c) Structural equations modeling, Statistical Quality Managment, Factor analysis, Multivariate Analysis

(a) **Masanobu Taniguchi**
(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo,169-8555, Japan, Tel & Fax: 03-5286-8095
(b') taniguchi@waseda.jp
(c) Statistical Inference for Stochastic Processes

(a) **Masao Kondo**
(b) Dept. of Mathematics and Computer Science, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890-0065, Japan
(b') kondo@sci.kagoshima-u.ac.jp
(c) Time Series Analysis

(a) **Masao Fukushima**
(b) Dept. of Systems and Mathematical Science, Faculty of Science and Engineering, Nanzan University, Nagoya, Aichi 466-8673, Japan
(b)' fuku@nanzan-u.ac.jp
(c) Mathematical Programming, Nonlinear Optimization

(a) **Ryusuke Hohzaki**
(b) Department of Computer Science, National Defense Academy, 1-10-20, Hashirimizu,
Yokosuka, 239-8686, Japan
(b') hozaki@cc.nda.ac.jp
(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Hiroaki Ishii**
(b) Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan
(b') ishiihiroaki@kwansei.ac.jp
(c) Operations Research and Fuzzy Theory, especially Mathematical Programming (Stochastic Programming, Combinatorial Optimization, Fuzzy Programming), Scheduling Theory, Graph and Network Theory, Inventory control, Mathematical evaluation method

(a) **Junzo Watada**
(b) Graduate School of Information, Production and Systems (IPS) Waseda Univ., 2-7 Hibikino, Wakamatsuku,
Kitakyushu, Fukuoka 808-0135, Japan
(b') junzow@osb.att.ne.jp
(c) Fuzzy systems, Management Engineering

(a) **Kensaku Kikuta**
(b) School of Business Administration, University of Hyogo,
8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN
(b') kikuta@biz.u-hyogo.ac.jp
(c) Game Theory, Operations Research,

(a) **Wuyi Yue**
(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto,
Higashinada-ku , Kobe 658-8501, JAPAN
(b') yue@konan-u.ac.jp
(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov Processes, Probabilistic Methods, Systems Engineering

(a) **Hiroaki Sandoh**
(b) Graduate School of Economics, Osaka University, 1-7, Machikaneyama, Toyonaka, Osaka 560-0043, Japan
(b')sandoh@econ.osaka-u.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

(a) **Yoshio Tabata**
(b) Takigi Ogaki 21, Kyotanabe, Kyoto 610-0341, Japan
(b') tabata@econ.osaka-u.ac.jp
(c) Mathematical Finance, Sequential Decision Theory

(a) Katsunori Ano
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku
Saitama-city, 337-8570, Japan
(b)' k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

(a) **Koyu Uematsu**
(b)Dept.of Economics and Finance,Faculty of Business Administration,Osaka International University
3-50-1 Sugi Hirakata Osaka,573-0192,Japan
(b') uematsu@oiu.jp
(c)Stochastic Process and its Applications,Reliability Analysis,and Game Theory

(a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences , Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa 259-1293, Japan
(b’) yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

(a) **Shunsuke Sato**
(b)2-6-20 Hanayashiki-Soen,Takarazuka,Hyogo 665-0808, Japan
(b’)ss_22362@nifty.com
(c) Mathematical biology in general, Neural networks, application of stochastic process

(a)**Tadashi Takahashi**
(b)Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto, Higashinada, Kobe, Hyogo 658-8501, Japan
(b’) takahasi@konan-u.ac.jp
(c)Mathematics Education

(a) **Benoit Collins**
(b) Department of Mathematics, Faculty of Science, Kyoto University
(b') collins@math.kyoto-u.ac.jp
(c) Random Matrix Theory, Free Probability, Quantum Information Theory
Quantum Groups (operator algebra side), Operator Algebra

****

Managing Editor
Koyu Uematsu (Professor of Osaka International University)
International Society for Mathematical Sciences
1-5-12-202 Kaorigaoka-cho, Sakai-ku, Sakai-city, 590-0011,Japan
uematsu@jams.jp

Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names

3'. 3. in Japanese for Japanese authors

4. Corresponding author's name and postal address (affiliation)

4'. 4. in Japanese for Japanese authors

5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$225)　　☐Print ＋ Online (US$225)
University (Institution)	
Department	
Postal Address where SCMJ should be sent	
E-mail address	
Person in charge	Name: Signature:
Payment Check one of the two.	☐Bank transfer　　☐Credit Card (Visa, Master)
Name of Associate Membership	1.
	2.

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in "**Join ISMS !**" on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4's address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in "**Join ISMS !**")

Individual Membership Application Form

1. **Name**	
2. **Birth date**	
3. **Academic background**	
4. **Affiliation**	
5. **4's address**	
6. **Doctorate**	
7. **Contact address**	
8. **E-mail address**	
9. **Special fields**	
10. **Membership category**	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.
The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

****

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.
(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:
(1) Post Office Transfer Account - 00930-3-73982 or
(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.
All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting**: Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2015

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ￥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

Notices from the ISMS