Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 80, No. 1 Whole Number 285

June 2017

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

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SIG-DIMENSION OF $K_{2,2}$−FREE GRAPHS

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Received June 26, 2016

Abstract. This paper introduces an algorithmic approach to investigate into the SIG-dimension of graphs, under the sup-norm. We provide an upper bound for the SIG-dimension of graphs, without isolated vertices, which do not contain an induced subgraph isomorphic to $K_{2,2}$.

1 Introduction The sphere-of-influence graph (SIG) on a set of points, each with an open ball centered about it of radius equal to the distance between that point and its nearest neighbor, is defined to be the intersection graph of these balls.

The notion of the sphere of influence graphs was introduced by Toussaint to model situations in pattern recognition and computer vision. These are used to help separate objects or otherwise capture perceptual relevance, see [6, 7, 8].

Toussaint has used the SIGs under L_2-norm to capture low-level perceptual information in certain dot patterns. The SIGs in general metric spaces are considered in [3]. It is known that the SIGs under the L_∞-norm perform better for this purpose, see [4]. Below we provide the construction of SIGs in this case.

Let d be a natural number and $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. For any $z \in \mathbb{R}^d$, let $z[j]$ denotes the j^{th} component of z. The distance between any $x, y \in \mathbb{R}^d$ under the L_∞-metric, denoted by $\rho(x, y)$, is defined as,

$$\rho(x, y) := \max\{|x[j] - y[j]| : j = 1, 2, \ldots, d\}.$$

Let $P \subset \mathbb{R}^d$ be a finite set having atleast two points. For a point $v \in P$, let r_v denotes the distance of v to its *nearest neighbor,* that is

$$r_v = \min\{\rho(u, v) : u \in P \setminus \{v\}\}.$$

The open ball $B_v := \{u \in \mathbb{R}^d : \rho(u, v) < r_v\}$ is known as the sphere of influence at v. The sphere of influence graph of P, denoted by $SIG^d_\infty(P)$, is the graph with vertex set P and edges corresponding to the pairs of intersecting spheres of influence. That is, the edge set of $SIG^d_\infty(P)$ is

$$\{uv : B_u \cap B_v \neq \emptyset; u, v \in P\}.$$

Throughout this paper, $E(G)$ and $V(G)$ will denote the vertex set and the edge set of a graph G. Note that for $G = SIG^d_\infty(P)$ and $u, v \in P$,

$$uv \in E(G) \iff \rho(u, v) < r_u + r_v.$$

A graph G is said to be *realizable* in $\mathbb{R}^d$ if there exists a finite set $P \subset \mathbb{R}^d$ such that G is isomorphic to $SIG^d_\infty(P)$. Note that if G is realized in $\mathbb{R}^d$, then it is realizable in $\mathbb{R}^{d+e}$ for every $e \in \mathbb{N}$. This can be observed by appending e zero coordinates to each point in the vertex set. The smallest such d is called the SIG-dimension of a graph G, denoted by $SIG(G)$. That is,

$$SIG(G) = \min\{d : G \text{ is realizable in } \mathbb{R}^d\}.$$

2000 *Mathematics Subject Classification.* 05C62, 05C75, 68R10.
Key words and phrases. Sphere-of-influence, SIG-dimension, Sup-norm.

It is trivial to see that if a graph with at least two vertices is realizable in some $\mathbb{R}^d$, then it can not have isolated vertices. Also, each graph G with atleast two vertices and no isolated vertices can be realized in $\mathbb{R}^d$, for some $d \in \mathbb{N}$. This can be seen as the rows of the matrix $2I + A$ realize G, where A is the adjacency matrix for G and I is the identity matrix, for more details see [4, Theorem 1].

Recently in [9], Taussaint has surveyed the theory and applications of sphere of influence graphs. In [4], several open problems on SIG-dimension have been discussed, the one regarding SIG-dimension of trees has already been solved, for details see [2]. In [5], we have proved the SIG-dimension conjecture for graphs having a perfect matching. A few partial results regarding the SIG-dimension for some particular graphs are proved in [1, 4].

It is easy to see that if G is path of size n, then $SIG(G) = 1$. Also it is known that if G is a graph of size n with no isolated vertex, then $SIG(G) \leq n - 1$, for details see [4].

In this paper, we consider the graphs which do not contain an induced subgraph isomorphic to $K_{2,2}$. We call them $K_{2,2}$-*free* graphs. We prove that if G is a $K_{2,2}$-free graph of order n which has no isolated vertex, then

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$

2 Definitions and Notations To establish our main result for $K_{2,2}$-free graphs, we will map our graph to a suitably required finite dimensional Euclidean space. But before that, we simply categorize the vertices in terms of triplets and pairs as per the following algorithm.

We start with a $K_{2,2}$-free graph, of size n, without an isolated vertex. The fact that G is $K_{2,2}$-free will be used later in our constructions, not for the following algorithm.

Algorithm 1_Step I. Let G be a $K_{2,2}$-free graph, of size n, without an isolated vertex._

Step II. Take an edge $pq \in E(G)$. There are two possible cases:

Case 1. There is a vertex $s \in V(G)$ such that exactly one of ps or qs is an edge. That is,

$$\text{(1)} \qquad \textit{either } 'ps \in E(G) \ \ \& \ \ qs \notin E(G)' \textit{ or } 'ps \notin E(G) \ \ \& \ \ qs \in E(G)'.$$

Define $n(p) = n(q) = n(s) = 0$. The set $\{p, q, s\}$ will be called a root of G.

Case 2. There is no vertex $s \in V(G)$ satisfying (1). That is, for all $s \in V(G)$,

$$ps \in E(G) \iff qs \in E(G).$$

Define $n'(p) = n'(q) = 0$. The set $\{p, q\}$ will be called a root of G.

Step III. Let $G_1 = G \setminus R$, where $R \neq \emptyset$ is a root of G and $r \in R$. Let

$$k = \begin{cases} n'(r) + 1, & \textit{if } |R| = 2 \\ n(r) + 1, & \textit{if } |R| = 3 \end{cases}$$

Case 1. $E(G_1) \neq \emptyset$. As above, let R_1 be a root of G_1.

_If $|R_1| = 2$, define $n'(u) = k$ and if $|R_1| = 3$, define $n(u) = k$, for all $u \in R_1$._

Set $G = G_1$ and repeat Step 2.

Case 2. $E(G_1) = \emptyset$. For all $v \in V(G_1)$, define $n''(v) = k$.

Note that, the vertices v for which $n''(v)$ is defined, form an independent set. Therefore, the vertices of our graph are divided into triplets, pairs and the remaining independent set.

In order to facilitate our argument, we now fix up few notations. Note that for any $v \in V(G)$, exactly one of $n(v)$, $n'(v)$ and $n''(v)$ is defined.

Notations 2. 1. For any $v \in V(G)$, the *index* of v, denoted by $m(v)$, is defined as follows:

$$m(v) := \begin{cases} n(v) & \text{if } n(v) \text{ is defined} \\ n'(v) & \text{if } n'(v) \text{ is defined} \\ n''(v) & \text{if } n''(v) \text{ is defined.} \end{cases}$$

2. Let α denotes the maximum value of $m(v); v \in V(G)$.

3. If v is a vertex such that $n''(v)$ is defined, choose a vertex u such that $uv \in E(G)$ and call it $N(v)$. That is, $N(v) = u$.

   Comment: There can be more than one such vertices u, which have an edge with v. In that case we fix up any one of these and call it $N(v)$.

4. Let $r > 0$ be any real and $\delta := \frac{r}{n+2}$.

5. For $0 \leq k \leq \alpha$ and for $v \in V(G)$, let $r(v) := r + \delta m(v)$.

As a common practice in most analytic proofs, the purpose of the above particular choice of $\delta > 0$ will be cleared later, in our proofs.

Remark 3. For any triplet $\{p, q, s\}$, $r(p) = r(q) = r(s) = r + \delta m(p)$. Similarly, it is same on every pair and on the residual independent set.

3 Mapping the graph to a Euclidean space In this section, we map the vertices of our given graph to a Euclidean space. This mapping will be done in a way that the corresponding SIG becomes isomorphic to the given graph. The bijection will be proved in the next section.

Each triplet, as per the previus section, will determine two dimensions of the Euclidean space, while the pairs will determine a single dimension. The final independent set will be considered in a separate manner later, while assigning new dimensions to the vertices.

Below we present the detailed algorithm to ensure the same.

Algorithm 4. *Step 1. Let G be a $K_{2,2}$-free graph, of size $n(\geq 2)$, without an isolated vertex.*

Step 2. Apply Algorithm 1 on G to categorize its vertices into triplets, pairs and an independent set.

Step 3. Repeat this Step, for each $k = 0, 1, \ldots \alpha$. Find $v \in V(G)$ with $m(v) = k$.

Case 1. There is a triplet $\{p, q, s\}$ such that $m(p) = m(q) = m(s) = k$ and $n(p)$ is defined. Without loss of generality, we assume that $qs \notin E(G)$. We define $c_{1(k)}$ and $c_{2(k)}$ on vertices of G as follows. Let $v \in V(G)$.

Case 1.1. If $m(v) < k$, then we define

$$c_{1(k)}(v) := c_{2(k)}(v) := \frac{3}{2}r(p).$$

Case 1.2. *If* $m(v)=k$, *then* $v\in\{p,q,s\}$. *Define*

$$c_{1(k)}(v):=\begin{cases}0 & \text{if } v=q\\ r(p) & \text{if } v=p\\ 2r(p) & \text{if } v=s\end{cases} \quad \text{and } c_{2(k)}(v):=\begin{cases}0 & \text{if } v=s\\ r(p) & \text{if } v=p\\ 2r(p) & \text{if } v=q.\end{cases}$$

Case 1.3. *If* $k<m(v)<\alpha$, *then we define*

$$c_{1(k)}(v):=\begin{cases}2r(p)+r(v) & \text{if } vp\notin E(G), vq\notin E(G) \text{ and } vs\notin E(G)\\ 2r(p)+r(v) & \text{if } vp\notin E(G), vq\notin E(G) \text{ and } vs\in E(G)\\ r(p)+r(v) & \text{if } vp\in E(G), vq\notin E(G) \text{ and } vs\notin E(G)\\ r(p)+r(v) & \text{if } vp\in E(G), vq\notin E(G) \text{ and } vs\in E(G)\\ r(p)+r(v)-\delta & \text{if } vp\in E(G), vq\in E(G) \text{ and } vs\in E(G)\\ r(p)+r(v)-\delta & \text{if } vp\notin E(G), vq\in E(G) \text{ and } vs\notin E(G)\\ r(p)+r(v)-\delta & \text{if } vp\in E(G), vq\in E(G) \text{ and } vs\notin E(G)\end{cases}$$

and

$$c_{2(k)}(v):=\begin{cases}2r(p)+r(v) & \text{if } vp\notin E(G), vq\notin E(G) \text{ and } vs\notin E(G)\\ r(p)+r(v)-\delta & \text{if } vp\notin E(G), vq\notin E(G) \text{ and } vs\in E(G)\\ r(p)+r(v) & \text{if } vp\in E(G), vq\notin E(G) \text{ and } vs\notin E(G)\\ r(p)+r(v)-\delta & \text{if } vp\in E(G), vq\notin E(G) \text{ and } vs\in E(G).\\ r(p)+r(v)-\delta & \text{if } vp\in E(G), vq\in E(G) \text{ and } vs\in E(G)\\ 2r(p)+r(v) & \text{if } vp\notin E(G), vq\in E(G) \text{ and } vs\notin E(G)\\ r(p)+r(v) & \text{if } vp\in E(G), vq\in E(G) \text{ and } vs\notin E(G)\end{cases}$$

Note that if $vp\notin E(G), vq\in E(G)$ *and* $vs\in E(G)$, *then the induced subgraph of* G *on the vertices* p,q,s *and* v *is isomorphic to* $K_{\{2,2\}}$, *which is not possible.*

Case 1.4. *If* $m(v)=\alpha$, *define*

$$c_{1(k)}(v):=c_{2(k)}(v):=\begin{cases}r(p) & \text{if } N(v)\in\{p,q,s\}\\ 2r(p) & \text{if } N(v)\notin\{p,q,s\}.\end{cases}$$

Case 2. *There is a pair* $\{p,q\}$ *such that* $n'(p)$ *is defined and* $m(p)=m(q)=k$. *We define* $c_{1(k)}$ *on vertices* $v\in V(G)$ *as follows:*

Case 2.1. *If* $m(v)<k$, *define* $c_{1(k)}(v):=\frac{3}{2}r(p)$.

Case 2.2. *If* $m(v)=k$, *then* $v\in\{p,q\}$. *Define*

$$c_{1(k)}(v):=\begin{cases}0 & \text{if } v=p\\ r(p) & \text{if } v=q.\end{cases}$$

Case 2.3. *If* $k<m(v)<\alpha$, *then we define*

$$c_{1(k)}(v):=\begin{cases}2r(p)+r(v) & \text{if } vp\notin E(G) \text{ and } vq\notin E(G)\\ r(p)+r(v)-\delta & \text{if } vp\in E(G) \text{ and } vq\in E(G).\end{cases}$$

Case 2.4. *If* $m(v)=\alpha$, *we define*

$$c_{1(k)}(v):=\begin{cases}r(p) & \text{if } N(v)\in\{p,q\}\\ 2r(p) & \text{if } N(v)\notin\{p,q\}.\end{cases}$$

Case 3. $k=\alpha$. *Assume that there are exactly* n_0 *vertices* $v_1,\ldots,v_{n_0}$ *such that* $m(v_1)=\cdots=m(v_{n_0})=\alpha$. *For each* $l=1,\ldots,n_0$, *we define*

$$c_{v_l(k)}(v):=\begin{cases}0 & \text{if } v=v_l\\ r(v_l) & \text{if } vv_l\in E(G)\\ r(v_l)+r(v) & \text{if } vv_l\notin E(G)\end{cases}$$

Step 4. Define $c_{1'}$ and $c_{2'}$ on vertices $u \in V(G)$ as follows:

Case 1. $n(u)$ is defined. Then there exist two other vertices v_1 and v_2 such that $m(v_1) = m(v_2) = m(u)$. Define

$$\begin{aligned} & (c_{1'}(u), c_{2'}(u)) && := (0, r(u)), \\ & (c_{1'}(v_1), c_{2'}(v_1)) && := (r(u), 0) \\ \textit{and} \quad & (c_{1'}(v_2), c_{2'}(v_2)) && := (r(u), r(u)). \end{aligned}$$

Case 2. $n'(u)$ is defined. Then there exists only one other vertex v such that $m(v) = m(u)$. Define

$$(c_{1'}(u), c_{2'}(u)) := (0, r(u)) \textit{ and } (c_{1'}(v), c_{2'}(v)) := (r(u), 0).$$

Case 3. $n''(u)$ is defined. Define $(c_{1'}(u), c_{2'}(u)) := (0, 0)$.

Step 5. Let $d_0 := \lceil \log_2 \alpha \rceil$ and $P := \{p : p[j] = 1 \text{ or } -1\} \subset \mathbb{R}^{d_0}$. In this step we choose a point in $\mathbb{R}^{d_0}$, corresponding to every triplet and pair.

For each $k = 0, 1, \ldots, \alpha - 1$, pick a different point from P, say $p'_k \in P$ and let $p_k = (r - \delta)p'_k$. Also let

$$S_k = \{v : m(v) < \alpha \;\; \& \;\; m(v) = k\} \cup \{v : m(v) = \alpha \;\; \& \;\; m(N(v)) = k \;\; \}.$$

Now append p_k to each $s \in S_k$.

Remark 5. Note that $S_k \cap S_{k'} = \emptyset$, for all $k \neq k'$. Therefore the last step above, won't add more than $\lceil \log_2 \alpha \rceil$ dimensions to our mapping.

Remark 6. Further, every vertex is appended with $\lceil \log_2 \alpha \rceil$ coordinates in the mapping, as for every vertex v such that $n''(v)$ is defined, there exists at least one u such that $u = N(v)$ and $n''(u)$ is not defined. Otherwise, v has to be an isolated vertex in our graph, which is not the case.

4 The mapping is an isomorphism In the previous section, we mapped the vertex set on a Euclidean space by assigning the coordinates with respect to each triplet, pair and the independent set. For convenience, we will use the same symbol v for the image of v, under this mapping.

In this sense, the vertex set $V(G)$ is now projected in a Euclidean space endowed with sup metric. We now prove that the SIG of this mapped vertex set is isomorphic to our given graph. We prove our main result through a series of claims.

In the sequel, for $u, v \in G$ we will use the notation $|c_k(u) - c_k(v)|$, even when c_k represents a pair of Euclidean dimensions. In that case, as an abuse of notation, it will represent the sup-norm in those two dimensions.

Lemma 7. For all $v \in V(G)$, we have $r_v \leq r(v)$.

Proof. Let $u \in V(G)$. We have the following cases:

Case 1. $n(u)$ is defined. Then there exist v_1 and v_2 such that $n(v_1) = n(v_2) = n(u)$.

Case 1.1. $v_1 v_2 \notin E(G)$. Then $uv_1 \in E(G)$ & $uv_2 \in E(G)$. Note that it is enough to prove that $\rho(u, v_1) \leq r(u)$. Therefore it is enough to prove that

$$|u[j] - v_1[j]| \leq r(u), \text{ for each } j = 1, 2, \ldots \tag{2}$$

We verify (2), for each co-ordinate separately. First let $k = \{0, 1, \ldots, \alpha\}$.

Case 1.1.1. $k = m(u)$. Then the only possibilities for $c_{1(k)}$ and $c_{2(k)}$ are

$$c_{1(k)}(u) = r(u) = c_{2(k)}(u), c_{1(k)}(v_1) \in \{0, 2r(u)\} \text{ and } c_{2(k)}(v_1) \in \{0, 2r(u)\}.$$

Thus (2) is verified for $c_{1(k)}$ and $c_{2(k)}$, as we have

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| = r(u) = |c_{2(k)}(u) - c_{2(k)}(v_1)|.$$

Case 1.1.2. $k < m(u)$. Let $v_3 \in V(G)$ be such that $m(v_3) = k$.

Case 1.1.2.1. $n(v_3)$ is defined. In this case, we have
'$c_{1(k)}(u) = 2r(v_3) + r(u)$ or $r(v_3) + r(u)$ or $r(v_3) + r(u) - \delta$'
'$c_{1(k)}(v_1) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$'
Hence we see that

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| \le r(v_3) + \delta = r + \delta k + \delta \le r + \delta m(u) = r(u).$$

The second inequality above holds, as we have $m(u) \ge k + 1$. Similarly, we obtain ,

$$|c_{2(k)}(u) - c_{2(k)}(v_1)| \le r(u).$$

Case 1.1.2.2. $n'(v_3)$ is defined. In this case, we have
'$c_{1(k)}(u) = 2r(v_3) + r(u)$ or $r(v_3) + r(u) - \delta$'
'$c_{1(k)}(v_1) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$'
Hence. as earlier, we see that

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| \le r(v_3) + \delta \le r(u).$$

Case 1.1.3. $k > m(u)$. Let $v_3 \in V(G)$ be such that $m(v_3) = k$.

Case 1.1.3.1. $n(v_3)$ is defined. Then
$c_{1(k)}(u) = \frac{3}{2}r(v_3)$ and $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$.
Therefore $|c_{1(k)}(u) - c_{1(k)}(v_1)| = 0$.
Similarly, $|c_{2(k)}(u) - c_{2(k)}(v_1)| = 0$. Similarly we deal with the case when $n'(v_3)$ is defined.

Case 1.1.3.2. $n''(v_3)$ is defined. For each $l = 1, 2, \ldots, i$, we have
$c_{v_l(k)}(u) = r(v_3)$ or $r(v_3) + r(u)$
and $c_{v_l(k)}(v_1) = r(v_3)$ or $r(v_3) + r(v_1)$
Therefore $|c_{v_l(k)}(u) - c_{v_l(k)}(v_1)| \le r(u)$.

Also note that $\max\{|c_{1'}(u) - c_{1'}(v_1)|, |c_{2'}(u) - c_{2'}(v_1)|\} = r(u)$ and $\max\{|p_{m(u)}[j] - p_{m(v_1)}[j]| : j = 1, 2, \ldots\} = 0$.

This verifies (2) and hence, in this case $r_u \le r(u)$.

Case 1.2. Either $uv_1 \notin E(G)$ or $uv_2 \notin E(G)$. Let $uv_1 \notin E(G)$. Then we have $uv_2 \in E(G)$ and $v_1v_2 \in E(G)$. This case is similar to Case 1.1.

Case 2. $n'(u)$ is defined. This case is analogous to Case 1.

Case 3. $n''(u)$ is defined. Then there is v such that $N(u) = v$. Therefore $uv \in E(G)$. Let $k \in \{0, 1, \ldots, \alpha\}$.

Case 3.1. $k = m(u)$. For $l = 1, 2, \ldots, n_0$, $c_{v_l(k)}(u) = 0$ or $2r(u)$.
If $c_{v_l(k)}(u) = 0$, we have $c_{v_l(k)}(v) = r(u)$.
If $c_{v_l(k)}(u) = 2r(u)$, we have $c_{v_l(k)}(v) = r(u)$ or $r(u) + r(v)$.
In both cases, we have $|c_{v_l(k)}(u) - c_{v_l(k)}(v)| \le r(u)$.

Case 3.2. $k < m(u)$. Let $w \in V(G)$ be such that $m(w) = k$.

Case 3.2.1. $n(w)$ is defined.

Case 3.2.1.1. $m(v) = k$. Then we have $c_{1(k)}(u) = r(v)$ and $c_{1(k)}(v) = 0, r(v)$ or $2r(v)$. Then

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) = r + \delta k < r + \delta m(u) = r(u).$$

Similarly, we have $|c_{2(k)}(u) - c_{2(k)}(v)| < r(u)$.

Case 3.2.1.2. $m(v) \neq k$. Then we have $c_{1(k)}(u) = 2r(v)$ and $c_{1(k)}(v) = 2r(w) + r(v), r(w) + r(v), r(w) + r(v) - \delta$ or $\frac{3}{2}r(w)$. Again, we have

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Case 3.2.2. $n'(w)$ is defined.

Case 3.2.2.1. $m(v) = k$. Then we have $c_{1(k)}(u) = r(v)$ and $c_{1(k)}(v) = 0$ or $r(v)$. Then we see that

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Case 3.2.2.2. $m(v) \neq k$. Then we have $c_{1(k)}(u) = 2r(v)$ and $c_{1(k)}(v) = 2r(w) + r(v), r(w) + r(v) - \delta$ or $\frac{3}{2}r(w)$. Hence

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Also, as earlier, we have

$$\max\{|c_{1'}(u) - c_{1'}(v)|, |c_{2'}(u) - c_{2'}(v)|\} = r(v) < r(u) \text{ and } \max\{|p_{m(u)}[j] - p_{m(v)}[j]| : j = 1, 2, \ldots\} = 0.$$

This implies that $\rho(u, v) = r(u)$. Therefore $r_u \leq r(u)$.

Hence the result. □

Lemma 8. For all $v \in V(G)$, we have $r_v \geq r(v)$.

Proof. Let $v_1, v_2 \in V(G)$.

Case 1. There is some $k < \alpha$ such that $v_1, v_2 \in S_k$.

Case 1.1. Either $m(v_1) < \alpha$ or $m(v_2) < \alpha$. Then we have

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} = r(v_1).$$

Case 1.2. $m(v_1) = m(v_2) = \alpha$. Then $v_1 v_2 \notin E(G)$ and we have $c_{v_1(n''(v_1))}(v_1) = 0$ and $c_{v_1(n''(v_1))}(v_2) = r(v_1) + r(v_2)$. Therefore

$$|c_{v_1(n''(v_1))}(v_1) - c_{v_1(n''(v_1))}(v_2)| = r(v_1) + r(v_2) > r(v_1).$$

Case 2. $v_1 \in S_{k_1}$ and $v_2 \in S_{k_2}$, where $k_1 \neq k_2$. Then, by our construction

$$\begin{aligned}
&\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \ldots\} \\
&= 2(r - \delta) \qquad = 2\left(r - \tfrac{r}{n+2}\right) \\
&= 2r\left(\tfrac{n+1}{n+2}\right) \qquad = r\left(\tfrac{2n+2}{n+2}\right) \\
&\geq r\left(\tfrac{n+k_1+2}{n+2}\right) \qquad = r\left(1 + \tfrac{k_1}{n+2}\right) \\
&= r + k_1\left(\tfrac{r}{n+2}\right) \qquad = r + k_1\delta \\
&= r(v_1).
\end{aligned}$$

This implies $\rho(v_1, v_2) \geq r(v_1) \Rightarrow r_{v_1} \geq r(v_1)$, which establishes our lemma. □

The following is immediate from Lemma 7 and Lemma 8.

Proposition 9. *For all $v \in V(G)$, we have $r_v = r(v)$.*

Lemma 10. *If $v_1, v_2 \in V(G)$ are such that $v_1v_2 \notin E(G)$, then $\rho(v_1, v_2) \geq r_{v_1} + r_{v_2}$.*

Proof. Case 1. Either $n''(v_1)$ or $n''(v_2)$ or both $n''(v_1)$ and $n''(v_2)$ are defined. Without loss of generality, let $n''(v_1)$ is defined. Then $c_{v_1(n''(v_1))}(v_1) = 0$ and $c_{v_1(n''(v_1))}(v_2) = r(v_1) + r(v_2)$. Hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{v_1(n''(v_1))}(v_1) - c_{v_1(n''(v_1))}(v_2)| \\ &= r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2. Both $n''(v_1)$ are $n''(v_2)$ not defined.

Case 2.1. $m(v_1) = m(v_2)$. Clearly by our construction, the case that both $n'(v_1)$ and $n'(v_2)$ are defined fails, as in that case $v_1v_2 \in E(G)$. Therefore both $n(v_1)$ and $n(v_2)$ must be defined and $n(v_1) = n(v_2)$. Then, we have $c_{1(n(v_1))}(v_1) = 0$ or $2r(v_1)$. Also $c_{1(n(v_1))}(v_1) = 0$ implies $c_{1(n(v_1))}(v_2) = 2r(v_1)$
and $c_{1(n(v_1))}(v_1) = 2r(v_1)$ implies $c_{1(n(v_1))}(v_2) = 0$.
Therefore, $|c_{1(n(v_1))}(v_1) - c_{1(n(v_1))}(v_2)| = 2r(v_1) = r_{v_1} + r_{v_2}$ and hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{v_1(n(v_1))}(v_1) - c_{v_1(n(v_1))}(v_2)| \\ &= r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2.2. $m(v_1) \neq m(v_2)$. Let $m(v_1) = k_1$ and $m(v_2) = k_2$. Without loss of generality, assume that $k_1 < k_2$.

Case 2.2.1. $n(v_1)$ is defined. Then $c_{1(m(v_1))}(v_1) = 0$ or $r(v_1)$ or $2r(v_1)$. In each of the following arguments, we look at the possibilities from our construction.
If $c_{1(m(v_1))}(v_1) = 0$ then

$$c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2).$$

If $c_{1(m(v_1))}(v_1) = r(v_1)$ then

$$c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Incase $c_{1(m(v_1))}(v_2) = r(v_1) + r(v_2) - \delta$, we have

$$c_{2(m(v_1))}(v_2) = 2r(v_1) + r(v_2). \text{ Already } c_{2(m(v_1))}(v_1) = r(v_1).$$

If $c_{1(m(v_1))}(v_1) = 2r(v_1)$ then $c_{2(m(v_1))}(v_1) = 0$ and

$$c_{1(m(v_1))}(v_2) = r(v_1) + r(v_2), 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Therefore

$$c_{2(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2).$$

Hence we observe that

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq \max\{|c_{i(m(v_1))}(v_1) - c_{i(m(v_1))}(v_2)| : i = 1, 2\} \\ &\geq r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2.2.2. $n'(v_1)$ is defined. Then we have $c_{1(m(v_1))}(v_1) = 0$ or $r(v_1)$ and $c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2)$. Hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{1(m(v_1))}(v_1) - c_{1(m(v_1))}(v_2)| \\ &\geq r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

This proves our lemma. □

Lemma 11. *If $v_1, v_2 \in V(G)$ are such that $v_1v_2 \in E(G)$, then $\rho(v_1, v_2) < r_{v_1} + r_{v_2}$.*

Proof. Pick $v_1, v_2 \in V(G)$ with $v_1v_2 \in E(G)$ and let $k_1 = m(v_1)$ and $k_2 = m(v_2)$.

Case 1. $k_1 = k_2$.

Case 1.1. $n''(v_1)$ is defined. Then $n''(v_2)$ is defined. This implies $v_1v_2 \notin E(G)$, which is not the case.

Case 1.2. $n'(v_1)$ is defined. Then $n'(v_2)$ is defined. Repeat the following steps for $k = 0$ to α.

Case 1.2.1. $k = k_1$. Then $c_{1(k)}(v_1) = 0$ or $r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1)$ and if $c_{1(k)}(v_1) = r(v_1)$ then $c_{1(k)}(v_2) = 0$. Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = r(v_1) = r_{v_1}.$$

Case 1.2.2. $k > k_1$.

Case 1.2.2.1. k denotes the index of vertices in the independent set (left at the end of our algorithm), if any.
Then $c_{v_l(k)}(v_1) = r(v_l)$ or $r(v_l) + r(v_1)$. Also $c_{v_l(k)}(v_2) = r(v_l)$ or $r(v_l) + r(v_1)$ and therefore

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| \leq r(v_1) = r_{v_1}.$$

Case 1.2.2.2. Otherwise, $c_{1(k)}(v_1) = \frac{3}{2}r(v_0)$, with $v_0 \in V(G)$ is such that $m(v_0) = k$. Also $c_{1(k)}(v_2) = \frac{3}{2}r(v_0)$ and therefore

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0.$$

If $n(v_0)$ is defined, then $c_{2(k)}$ is defined and we have $c_{2(k)}(v_1) = \frac{3}{2}r(v_0)$ and $c_{2(k)}(v_2) = \frac{3}{2}r(v_0)$. Therefore

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| = 0.$$

Case 1.2.3. $k < k_1$.

Case 1.2.3.1. There exists a vertex $v_3 \in V(G)$ such that $n'(v_3)$ is defined with $k = n'(v_3)$. Then

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

Also $c_{1(k)}(v_2) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$. Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| \leq r(v_3) + \delta \leq r(v_1) = r_{v_1}.$$

Case 1.2.3.2. There exists a vertex $v_3 \in V(G)$ such that $n(v_3)$ is defined with $k = n(v_3)$. Then both $c_{1(k)}(v_1)$ and $c_{1(k)}(v_2)$ are either

$$2r(v_3) + r(v_1), r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

Therefore, we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| \leq r(v_3) + \delta \leq r(v_1) = r_{v_1}.$$

Similarly, $|c_{2(k)}(v_1) - c_{2(k)}(v_2)| \leq r_{v_1}$.

Case 1.2.4. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1))$ or $(r(v_1), 0)$.
If $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1))$, then $(c_{1'}(v_2), c_{2'}(v_2)) = (r(v_1), 0)$.
If $(c_{1'}(v_1), c_{2'}(v_1)) = (r(v_1), 0)$, then $(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_1))$. Hence $\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} = r(v_1) = r_{v_1}$.

Case 1.2.5. $\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \dots\} = 0$.

Therefore, if $n'(v_1)$ and $n'(v_2)$ are defined and $n'(v_1) = n'(v_2)$, then

$$\rho(v_1, v_2) \leq r_{v_1} < r_{v_1} + r_{v_2}.$$

Case 1.3. $n(v_1)$ is defined. Then $n(v_2)$ is also defined.

Case 1.3.1. $k = k_1$. Then $c_{1(k)}(v_1) = 0, r(v_1)$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1)$.
If $c_{1(k)}(v_1) = r(v_1)$ then $c_{1(k)}(v_2) = 0$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 2r(v_1)$ then $c_{1(k)}(v_2) = r(v_1)$.
Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = r(v_1) = r_{v_1}.$$

Case 1.3.2. $k > k_1$. This case is same as Case 1.2.2.

Case 1.3.3. $k < k_1$. This case is same as Case 1.2.3.

Case 1.3.4. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$.
$(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$. Hence

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} \leq r(v_1) = r_{v_1}.$$

Case 1.3.5. $\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \dots\} = 0$.

Therefore, if $n(v_1)$ and $n(v_2)$ are defined such that $n(v_1) = n(v_2)$ and $v_1v_2 \in E(G)$, then we have

$$\rho(v_1, v_2) \leq r_{v_1} < r_{v_1} + r_{v_2}.$$

This proves the result for the case $m(v_1) = m(v_2)$.

Case 2. $k_1 \neq k_2$. Without loss of generality, assume that $k_1 < k_2$. Repeat the following for $k = 0$ to α.

Case 2.1. $k < k_1$.

Case 2.1.1. There exists some $v_3 \in V(G)$ such that $n(v_3) = k$. Therefore

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1), r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta,$$

$$c_{1(k)}(v_2) = 2r(v_3) + r(v_2), r(v_3) + r(v_2), r(v_3) + r(v_2) - \delta, 2r(v_3) \text{ or } r(v_3).$$

Hence we obtain

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Similarly, $|c_{2(k)}(v_1) - c_{2(k)}(v_2)| \leq r_{v_1} + r_{v_2}$.

Case 2.1.2. There exists $v_3 \in V(G)$ be such that $n'(v_3)$ is defined and $n'(v_3) = k$. Then we see that

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

$$c_{1(k)}(v_2) = 2r(v_3) + r(v_2), r(v_3) + r(v_2) - \delta, 2r(v_3) \text{ or } r(v_3).$$

Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.2. $k = k_1$.

Case 2.2.1. $n(v_1)$ is defined. Then we have $c_{1(k)}(v_1) = 0, r(v_1)$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1), 2r(v_1)$ or $r(v_1) + r(v_2) - \delta$. Hence we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

If $c_{1(k)}(v_1) = r(v_1)$ then we have

$$c_{1(k)}(v_2) = r(v_1), 2r(v_1), r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Hence, as earlier

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

If $c_{1(k)}(v_1) = 2r(v_1)$ then

$$c_{1(k)}(v_2) = r(v_1), 2r(v_1), 2r(v_1) + r(v_2), r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Therefore $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.
Similarly, we obtain

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| < r_{v_1} + r_{v_2}.$$

Case 2.2.2. $n'(v_1)$ is defined. Then we have
$c_{1(k)}(v_1) = 0$ or $r(v_1)$ and $c_{1(k)}(v_2) = r(v_1), 2r(v_1)$ or $r(v_1)+r(v_2)-\delta$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.3. $k_1 < k < k_2$. Then there exists $v_3 \in V(G)$ such that $m(v_3) = k$.

Case 2.3.1. $n(v_3)$ is defined. Then we have $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and

$$c_{1(k)}(v_2) = r(v_3), 2r(v_3), 2r(v_3) + r(v_2), r(v_3) + r(v_2) \text{ or } r(v_3) + r(v_2) - \delta.$$

Case 2.3.2. $n'(v_3)$ is defined. Then we have $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and
$c_{1(k)}(v_2) = r(v_3), 2r(v_3), 2r(v_3) + r(v_2)$ or $r(v_3) + r(v_2) - \delta$.
Therefore, in both of the above cases, we observe that

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.4. $k = k_2$.

Case 2.4.1. $n(v_2)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_2)$ and $c_{1(k)}(v_2) = 0, r(v_2)$ or $2r(v_2)$.
Therefore, we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Similarly, we obtain

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| < r_{v_1} + r_{v_2}.$$

Case 2.4.2. $n'(v_2)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_2)$ and $c_{1(k)}(v_2) = 0$ or $r(v_2)$.

Case 2.4.3. $n''(v_2)$ is defined. Then $c_{v_2(k)}(v_2) = 0$ and $c_{v_2(k)}(v_1) = r(v_2)$. Also, for $v_l \neq v_2$ such that $n''(v_l)$ is defined, we have $c_{v_l(k)}(v_2) = r(v_l) + r(v_2) = 2r(v_2)$ and $c_{v_l(k)}(v_1) = r(v_2)$ or $r(v_1) + r(v_2)$. Hence

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.5. $k > k_2$. Then there exists $v_3 \in V(G)$ such that $m(v_3) = k$.

Case 2.5.1. $n(v_3)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and $c_{1(k)}(v_2) = \frac{3}{2}r(v_3)$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0 < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.5.2. $n'(v_3)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_3), c_{1(k)}(v_2) = \frac{3}{2}r(v_3)$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0 < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.5.3. $n''(v_3)$ is defined. Then we have $c_{v_l(k)}(v_1) = r(v_l)$ or $r(v_l) + r(v_1)$ and $c_{v_l(k)}(v_2) = r(v_l)$ or $r(v_l) + r(v_2)$. Hence

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.6. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$.
$(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_2)), (r(v_2), 0), (r(v_2), r(v_2))$ or $(0, 0)$.
Therefore $|c_{1'}(v_1) - c_{1'}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.
Also $|c_{2'}(v_1) - c_{2'}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$. Hence

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} < r_{v_1} + r_{v_2}.$$

Case 2.7. Let $p_{k_{1'}}$ be associated with v_1 and $p_{k_{2'}}$ be associated with v_2.
Then, either $|p_{k_{1'}} - p_{k_{1'}}| = 0$ or

$$|p_{k_{1'}} - p_{k_{1'}}| = 2(r - \delta) < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

This proves the result for the case $m(v_1) \neq m(v_2)$. Hence the result. □

The previous two lemmas essentially prove the following theorem.

Theorem 12. *For $v_1, v_2 \in V(G)$, we have*

$$v_1v_2 \in E(G) \text{ if and only if } \rho(v_1, v_2) < r_{v_1} + r_{v_2}.$$

Therefore the SIG of our mapping of $V(G)$ on the Euclidean space is isomorphic to G. In other words, G is realizable in a Euclidean space, whose dimension is fixed according to our algorithm. Next we will count the dimension of this Euclidean space.

5 The Main Result We need the following result from [1, Corollary 9].

Lemma 13. *If G is a graph of order n with no isolated vertex. If G has an independent set of size $t > 1$, then*

$$SIG(G) \leq n - 1 - t + \lceil \log_2 t \rceil.$$

Remark 14. In Step 3 of our construction, we attach $\lceil \log_2 \alpha \rceil$ co-ordinates to each vertex. As $\alpha \leq n/2$, we attach maximum $\lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil$ co-ordinates. Since

$$\left\lceil \log_2 \left\lfloor \frac{n}{2} \right\rfloor \right\rceil \leq \left\lceil \log_2 \frac{n}{2} \right\rceil = \lceil \log_2 n - \log_2 2 \rceil = \lceil \log_2 n \rceil - 1,$$

we attach maximum $\lceil \log_2 n \rceil - 1$ co-ordinates.

Now we prove the main result of this paper.

Theorem 15. *Let G be a $K_{2,2}$-free graph with $n(\geq 2)$ vertices. If G has no isolated vertex, then*

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$

Proof. Let $S := \{v : v \in V(G) \text{ and } n''(v) \text{ is defined}\}$. Let $|S| = \beta$. Using our construction in Section 3 along with Remark 14, we obtain

$$SIG(G) \leq \frac{2}{3}(n-\beta) + \beta + (\lceil \log_2 n \rceil - 1) + 2 = \frac{2}{3}n + \frac{1}{3}\beta + \lceil \log_2 n \rceil + 1.$$

If $\beta = \frac{n}{4}$, then

$$SIG(G) \leq \frac{2n}{3} + \frac{n}{12} + \lceil \log_2 n \rceil + 1 = \frac{3n}{4} + \lceil \log_2 n \rceil + 1.$$

If $\beta < \frac{n}{4}$, then $\beta = \frac{n}{4} - k$, for some $k > 0$ and we have

$$SIG(G) \leq \frac{2n}{3} + \frac{n}{12} - \frac{k}{3} + \lceil \log_2 n \rceil + 1 < \frac{3n}{4} + \lceil \log_2 n \rceil + 1.$$

If $\beta > \frac{n}{4}$, then $\beta = \frac{n}{4} + k$, for some $k > 0$ and then the maximum independent set has cardinality greater than or equal to $\frac{n}{4} + k$. Let t be the cardinality of the maximum independent set of G. Then $t \geq \frac{n}{4} + k$. Also, as in Lemma 13, we have

$$SIG(G) \leq n - 1 - t + \lceil \log_2 t \rceil.$$

Therefore,

$$SIG(G) \leq \frac{3n}{4} - k - 1 + \lceil \log_2 t \rceil < \frac{3n}{4} + \lceil \log_2 n \rceil.$$

Hence, we have proved that $SIG(G) \leq \frac{3n}{4} + \lceil \log_2 n \rceil + 2$. Hence

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$

□

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Communicated by *Hiroaki Ishii*

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ON GENERALIZED DIGITAL LINES *

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Received June 21, 2014 ; revised July 17, 2016,January 16,2017,January 18,2017

Abstract. In the present paper, we introduce and study the concept of *generalized digital lines*, say $(\mathbb{Z}, \kappa(q,n))$, where q and n are positive integers with $2 \leq q < n$ and $n \not\equiv 0 \pmod q$; especially, for $q = 2$ and $n = 3$, $(\mathbb{Z}, \kappa(2,3))$ is identical with the digital line $(\mathbb{Z}, \kappa)$ (=the Khalimsky line due to E.D. Khalimsky).

1 Introduction and preliminaries *The Khalimsky line* or so called *the digital line* is the set $\mathbb{Z}$ of integers equipped with the topology κ having $\mathcal{G}_\kappa := \{\{2m-1, 2m, 2m+1\} \mid m \in \mathbb{Z}\}$ as a subbase ([25]: e.g. [26], [27, p.905, p.906], [28, Definition 2, p.175], [10, Example 4.6, p.23], [8, p.50], [13, p.164], [14, p.31], [44, p.601], [43, p.46], [18, p.926], [37, Example 2.4], [19, p.1034, p.1035], [36, Section 3(I)]). In 1970, the concept of the digital line was published by Khalimsky [25] above from Russia. In 1990, Khalimsky, Kopperman and Meyer [26] investigated the concepts of *connected ordered topological spaces*, *digital planes* and a proof of digital Jordan closed curve theorem using purely digital topological methods (cf. the references of [26], [27]). The digital line is denoted by $(\mathbb{Z}, \kappa)$. Roughly speaking, $(\mathbb{Z}, \kappa)$ has a covering $\mathcal{G}_\kappa$ by infinitely many open subsets which are three points subset $\{2m-1, 2m, 2m+1\}$, where $m \in \mathbb{Z}$, and two adjacent open sets $\{2m-1, 2m, 2m+1\}$ and $\{2m+1, 2m+2, 2m+3\}$ are connected with a singleton $\{2m+1\}$ as their intersection of two such open subsets. For any integer m, the singleton $\{2m+1\}$ is open in $(\mathbb{Z}, \kappa)$ and $\{2m\}$ is closed in $(\mathbb{Z}, \kappa)$. From a point of view in general topology approaches, the digital line $(\mathbb{Z}, \kappa)$ is a typical and geometrical example of a topological space which satisfies a $T_{1/2}$ separation axiom. In 1970, Levine [31] published, from Italy, the concept of $T_{1/2}$-spaces by introducing the concept of *generalized closed subsets* [31, Definition 2.1] of a topological space; a topological space is called $T_{1/2}$ [31, Definition 5.1] if every generalized closed set is closed. The class of $T_{1/2}$-spaces is properly placed between the classes of T_0- and T_1-spaces [31, Corollary 5.6]. In 1977, Dunham [11, Theorem 2.5] proved that a topological space (X, τ) is $T_{1/2}$ if and only if each singleton $\{x\}$ is open or closed in (X, τ), where $x \in X$. Therefore, we know that $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ (cf. [26, p.7], [10, Example 4.6]). In 1996, Dontchev and Ganster [10] investigated the class of $T_{3/4}$-spaces which is properly placed between the classes of T_1- and $T_{1/2}$-spaces; and the authors proved that $(\mathbb{Z}, \kappa)$ is $T_{3/4}$ [10, Example 4.6].

The purpose of the present paper is to construct *generalized digital lines*, say $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2 below) and investigate its fundamental properties (cf. Theorem A below and related properties).

Throughout the present paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed unless otherwise mentioned and $P(X)$ denotes the power set of X.

Theorem A *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line in the sense of* Definition 2.2,*where the integers* q *and* n *satisfy the following conditions:* $2 \leq q < n$ *and* $n \not\equiv 0 \pmod q$, *say* $n \equiv r \pmod q$ $(1 \leq r \leq q-1)$. *Then, we have the following fundamental properties.*

(i) $\kappa(q,n) \neq P(\mathbb{Z})$ *holds*;

*2010 *Math. Subject classification.* 54B99, 54D10, 54F65; Secondary: 68U05.
Key words and phrases. Digital lines; Preopen sets; Semi-open sets; Lower separation axioms; Semi-T_2-spaces; Semi-kernels;Connected Spaces, Semi-connected spaces; Preconnected spaces.

(ii) (ii-1) *if* $2 \leq r$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is pre-*T_2; (ii-2) *if* $r = 1$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is semi-*T_2; *especially if* $q = 2$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is* $T_{3/4}$;
(iii) $(\mathbb{Z}, \kappa(q,n))$ *is connected.*

The proof of Theorem A(i) (resp. (ii), (iii)) is shown in Section 5 (resp. Section 6, Section 7). When $q = 2$ and $n = 3$, then we see $(\mathbb{Z}, \kappa(2,3)) = (\mathbb{Z}, \kappa)$ (cf. Remark 2.3).

In the present paper, sometimes, we use the following notation:

Notation. For integers $a, b \in \mathbb{Z}$ with $a \leq b$, $[a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ (by [6], this set is called a *digital interval* if $a \lneq b$). For a set A, we denote by $|A|$ the cardinality of A (e.g. Lemma 2.8, Proof of Theorem 5.1(ii)).

2 Open sets and classifications of generalized digital lines

Definition 2.1 Let n and q be given two positive integers. Let $\mathcal{G}(q,n) := \{B_k(q,n) \mid k \in \mathbb{Z}\}$ be the family of subsets $B_k(q,n)$ of Z, where $k \in \mathbb{Z}$ and $B_k(q,n) := \{kq+i \in \mathbb{Z} \mid 1 \leq i \leq n\}$.

Definition 2.2 (the generalized digital line) Suppose that the following conditions: $2 \leq q < n$ and $n \equiv r \pmod{q}$ $(1 \leq r \leq q-1)$ hold for the integers q and n in Definition 2.1 above. Then, a *generalized digital line* is the set of the integers, $\mathbb{Z}$, equipped with the topology $\kappa(q,n)$ having $\mathcal{G}(q,n)$ as a subbase. It is denoted by $(\mathbb{Z}, \kappa(q,n))$.

Remark 2.3 In Definition 2.2 above, let $q = 2$ and $n = 3$. Then, for each $k \in \mathbb{Z}, B_k(2,3) = \{2(k+1)-1, 2(k+1), 2(k+1)+1\}$ and the space $(\mathbb{Z}, \kappa(2,3))$ coincides with the digital line $(\mathbb{Z}, \kappa)$ (cf. [26], e.g. [10], Section 1 above).

We investigate the smallest open set (resp. closed set) containing a point of $(\mathbb{Z}, \kappa(q,n))$.

Definition 2.4 For a subset A of a topological space (X, τ),
(i) $Ker(A) := \bigcap\{U \mid A \subset U, U \in \tau\}$, (e.g. in [35, Definition 2.1], $Ker(A)$ is denoted by A^{Λ});
(ii) $Cl(A) := \bigcap\{F \mid A \subset F, F$ is closed in $(X,\tau)\}$.

Definition 2.5 Let (X, τ) be a topological space, A and B subsets of (X, τ) and $x \in X$.
(i) A is called *the smallest open set containing* x if $x \in A, A \in \tau$ and $G = A$ holds for any open set G such that $x \in G$ and $G \subset A$. The uniqueness of the smallest open sets is assured by Remark 2.6(i) below.
(ii) B is called *the smallest closed set containing* x, if $x \in B, X \setminus B \in \tau$ and $F = B$ holds for any closed set F such that $x \in F$ and $F \subset B$.

Remark 2.6 (i) If subsets A and B are the smallest open subsets containing $x \in X$, then $A = B$.
(ii) For an open subset A of X and a point $x \in A$, the following properties are equivalent:
(1) A is the smallest open set containing x;
(2) for any open set U containing $x, A \subset U$ holds.

Lemma 2.7 *Let* (X, τ) *be a topological space and* $A \subset X, x \in X$.
(i) *If* A *is the smallest open set containing* x, *then* $Ker(\{x\}) = A$ *holds.*
(ii) *If* $Ker(\{x\}) = A$ *and* $A \in \tau$, *then* A *is the smallest open set containing* x.
(iii) A *is the smallest closed set containing* x *if and only if* $Cl(\{x\}) = A$ *holds.* □

Lemma 2.8 *Let* X *be a set and* $\mathcal{G} = \{V_i \mid i \in \mathcal{A}\}$ *be a collection of subsets of* X. *Let* (X, τ) *be a topological space, where* τ *is the topology having* $\mathcal{G}$ *as subbase. Suppose that, for each point* $w \in X$, *the collection* $\{V \mid V \in \mathcal{G}, w \in V\} := \mathcal{G}_w$ *is a finite subcollection of* $\mathcal{G}$, *i.e.,* $|\mathcal{G}_w| < \infty$. *Then, for a point* $x \in X$ *and a subset* $A \subset X$, *the following properties on* $Ker(\{x\}), Cl(\{x\})$ *and* $Cl(A)$ *hold.*

(i) $Ker(\{x\}) = \bigcap\{V \mid V \in \mathcal{G}, x \in V\}(= \bigcap\{V \mid V \in \mathcal{G}_x\})$ *and it is the smallest open set containing* x.

(ii) *Moreover, suppose that* $Ker(\{x\}) \cap Ker(\{y\}) = \emptyset$ *or* $Ker(\{x\}) = Ker(\{y\})$ *hold for any distinct points* x, y *of* X.

Then, $Cl(\{x\}) = Ker(\{x\})$.

(iii) $Cl(A) = X \setminus U_A$, *where* $U_A = \{y \in X \mid Ker(\{y\}) \cap A = \emptyset\}$.

Proof. **(i)** We claim that $Ker(\{x\}) \supset \bigcap\{V \mid V \in \mathcal{G}_x\}$ holds. For each open set G containing x, we are able to set $G = \bigcup\{B_i \mid i \in I\}$, where the subset B_i is a finite intersection of some elements of $\mathcal{G}$ and I is an index set. For each open set G, there exists an element $i_0 \in I$ such that $x \in B_{i_0}$ and $B_{i_0} = \bigcap\{V_j \mid V_j \in \mathcal{G}_x, j \in J\}$ for some finite set $J \subset \mathcal{A}$. Then, we have $G \supset B_{i_0} \supset \bigcap\{V \mid V \in \mathcal{G}_x\} \ni x$ and so $Ker(\{x\}) \supset \bigcap\{V \mid V \in \mathcal{G}_x\}$. Conversely, the implication $Ker(\{x\}) \subset \bigcap\{V \mid V \in \mathcal{G}_x\}$ is easily proved. Thus we have that $Ker(\{x\}) = \bigcap\{V \mid V \in \mathcal{G}_x\}$ holds and it is open. By Lemma 2.7 (ii), the set $Ker(\{x\})$ is the smallest open set containing x.

(ii) For a given point $x \in X$, let $F := X \setminus U$, where $U := \bigcup\{Ker(\{y\}) \mid y \notin Ker(\{x\})\}$. Then, by the assumption in (ii), $F = Ker(\{x\})$ holds. Indeed, first we show that $U \subset X \setminus Ker(\{x\})$. Let $z \in U$. Then, there exists a point $y \in X$ such that $y \notin Ker(\{x\})$ and $z \in Ker(\{y\})$. It is shown that $Ker(\{y\}) \cap Ker(\{x\}) = \emptyset$ holds; and so $z \notin Ker(\{x\})$. Thus, we have the property that $U \subset X \setminus Ker(\{x\})$. Finally, we show that $U \supset X \setminus Ker(\{x\})$, because $U := \bigcup\{Ker(\{y\}) \mid y \notin Ker(\{x\})\} \supset \bigcup\{\{y\} \mid y \notin Ker(\{x\})\} = X \setminus Ker(\{x\})$. Therefore, $U = X \setminus Ker(\{x\})$ holds, i.e., $F = Ker(\{x\})$ holds. Since $Ker(\{y\})$ is open by (i), $F := X \setminus U$ is a closed subset containing x and so $Cl(\{x\}) \subset F = Ker(\{x\})$. Conversely, we claim that $Ker(\{x\}) \subset Cl(\{x\})$. Let y be a point such that $y \notin Cl(\{x\})$. Then, there exists an open subset V_y containing y such that $V_y \cap \{x\} = \emptyset$. Since $Ker(\{y\}) \subset V_y$, we have $Ker(\{y\}) \cap \{x\} = \emptyset$ and so $Ker(\{x\}) \neq Ker(\{y\})$. Using assumption we have $Ker(\{x\}) \cap Ker(\{y\}) = \emptyset$ and hence $y \notin Ker(\{x\})$ for any $y \notin Cl(\{x\})$. Thus we conclude that $Cl(x) = Ker(\{x\})$ holds.

(iii) It is shown that $Cl(A) \subset X \setminus U_A$. Indeed, let $a \notin X \setminus U_A$. Then, $Ker(\{a\}) \cap A = \emptyset$ and so $a \notin Cl(A)$ (cf. (i) above). Conversely, let $b \notin Cl(A)$. Then, there exists an open set V containing the point b such that $V \cap A = \emptyset$. Thus, we have that $Ker(\{b\}) \cap A = \emptyset$ and so $b \notin X \setminus U_A$. This shows that $X \setminus U_A \subset Cl(A)$ holds. □

Remark 2.9 (i) The following example shows that even if A is the smallest open set containing a point x there exists a proper open subset G such that $G \subset A$. Let $(\mathbb{Z}, \kappa)$ be the digital line, $x := 0$ and $A =: \{-1, 0, 1\}$ be the smallest open set containing x. Then, $Ker(\{x\}) = A$; however, subsets $G := \{1\}, G' := \{-1\}$ are open proper subsets of A. Note that $x \notin G$ and $x \notin G'$.

(ii) The following example shows that the converse of Lemma 2.7 (i) is not true in general. Let $(\mathbb{R}, \tau)$ be the Euclidian line. A subset $A := \{0\}$ is not open; $Ker(\{0\}) = \{0\}$ holds.

Lemma 2.10 *Assume that* $2 \leq q < n$ *and* $n = sq + r$, *where* $r, s \in \mathbb{N}$ *with* $1 \leq r \leq q - 1$. *Then, a subset* $\{y \in \mathbb{Z} \mid kq + 1 \leq y \leq (k + t)q + r\}$ *is open in* $(\mathbb{Z}, \kappa(q, n))$, *where* $k \in \mathbb{Z}$ *and* $t \in \mathbb{Z}$ *with* $1 \leq t \leq s$.

Proof. Using notation above (cf. the end of Section 1), we show that $[kq+1, kq+n]_{\mathbb{Z}} \cap [(k-(s-t))q + 1, (k-(s-t))q + n]_{\mathbb{Z}} = [kq + 1, (k + t)q + r]_{\mathbb{Z}}$ holds, because $kq - (s-t)q + 1 \leq kq+1 \leq (k-(s-t))q+n \leq kq+n$. Since $[kq+1, kq+n] \in \mathcal{G}(q, n)$ and $[(k-(s-t))q+1, (k-(s-t))q + n]_{\mathbb{Z}} \in \mathcal{G}(q, n)$ (cf. Definition 2.1), we show that $[kq + 1, (k + t)q + r]_{\mathbb{Z}} \in \kappa(q, n)$ (cf. Defintion 2.2). □

Lemma 2.11 *Suppose that* $2 \leq q < n$ *for the integers* q *and* n *of the sets* $B_k(q, n) \subset \mathbb{Z}(k \in \mathbb{Z})$ *and the family* $\mathcal{G}(q, n) \subset P(\mathbb{Z})$ *in* Definition 2.1. *Let* $n = sq + r(s, r \in \mathbb{Z}$ *with*

$0 \leq r \leq q-1$). *For a point* $x \in \mathbb{Z}$ *and* $B_{k'}(q,n) \in \mathcal{G}(q,n)$, *where* $k' \in \mathbb{Z}$ (cf. Definition 2.1), *the following properties hold.*

(i) *Assume that* $n \equiv 0 \pmod q$. *For a point* $x = kq+i$, *where* $k, i \in \mathbb{Z}$ *with* $1 \leq i \leq q, x \in B_{k'}(q,n)$ *if and only if* $k' \in \{y \in \mathbb{Z} \mid k-(s-1) \leq y \leq k\}$.

(ii) *Assume that* $n \equiv r \pmod q$, *where* $0 < r \leq q-1$.

(b1) *For a point* $x = kq+i$, *where* $k, i \in \mathbb{Z}$ *with* $1 \leq i \leq r, x \in B_{k'}(q,n)$ *if and only if* $k' \in \{y \in \mathbb{Z} \mid k-s \leq y \leq k\}$.

(b2) *For a point* $x = kq+j$, *where* $k, j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q, x \in B_{k'}(q,n)$ *if and only if* $k' \in \{y \in \mathbb{Z} \mid k-s+1 \leq y \leq k\}$.

Proof. First we recall that $B_{k'}(q,n) = [k'q+1, k'q+n]_{\mathbb{Z}}$ for $k' \in \mathbb{Z}$ (cf. Definition 2.1).

(i) Suppose that $x = kq+i \in B_{k'}(q,n)$ $(1 \leq i \leq q)$ and $n = sq$, where $s \in \mathbb{Z}$. Then, $k'q+1 \leq kq+i \leq k'q+sq$ and so $kq-sq < kq-sq+i \leq k'q \leq kq+i-1 \leq kq+q-1 < kq+q$. Thus we have $k-s < k' < k+1$,i.e., $k' \in [k-s+1, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s+1, k]_{\mathbb{Z}}$, then $kq-sq+i \leq kq-sq+q \leq k'q \leq kq \leq kq+i-1$ and so $kq+i \leq k'q+sq$ and $k'q+1 \leq kq+i$. Thus, we have $x = kq+i \in [k'q+1, k'q+sq]_{\mathbb{Z}} = [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$.

(ii)(b1) Suppose that $n = sq+r$ $(0 < r \leq q-1)$ and $x = kq+i \in B_{k'}(q,n)$ $(1 \leq i \leq r)$. Then, $k'q+1 \leq kq+i \leq k'q+sq+r$ and so $kq-sq+i-r \leq k'q \leq kq+i-1$. Then, we have $kq-sq+i-(q-1) \leq kq-sq+i-r \leq k'q \leq kq+i-1$ and so $kq-sq-q < kq-sq+1-(q-1) \leq kq-sq+i-(q-1) \leq k'q \leq kq+r-1 \leq kq+(q-2) < kq+q$. Thus, we have $k' \in [k-s, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s, k]_{\mathbb{Z}}$, then $kq-sq \leq k'q \leq kq$ and so $kq-sq+i-r \leq k'q \leq kq+i-1$. Thus, we show that $k'q+1 \leq kq+i \leq k'q+sq+r = k'q+n$ and so $x \in [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$.

(b2) Suppose that $n = sq+r$ $(0 < r \leq q-1)$ and $x = kq+j \in B_{k'}(q,n)$ $(r+1 \leq j \leq q)$. Then, $k'q+1 \leq kq+j \leq k'q+sq+r$ and so $kq-sq+j-r \leq k'q \leq kq+j-1$. Thus we have $kq-sq < kq-sq+j-r \leq k'q \leq kq+j-1$ and so $kq-sq < k'q < kq+q$. Namely, we have $k' \in [k-s+1, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s+1, k]_{\mathbb{Z}}$, then $kq-sq+q \leq k'q \leq kq$ and so $kq-sq-r+j < kq-sq+j \leq kq-sq+q \leq k'q < kq+j-1$. Thus, we show that $k'q+1 < kq+j < k'q+sq+r = k'q+n$ and so $x \in [k'q+2, k'q+n-1]_{\mathbb{Z}} \subset [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$. □

Remark 2.12 For the generalized digital line $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2), its topology $\kappa(q,n)$ satisfies the assumptions in Lemma 2.8. Indeed, for each point $x \in \mathbb{Z}$, by Lemma 2.11, it is shown that $\mathcal{G}_x = \{B_{k'}(q,n) |\ x \in B_{k'}(q,n)\}$ is a finite subcollection of $\mathcal{G}(q,n)$. Namely, $\{k' |\ x \in B_{k'}(q,n)\}$ is a finite set for each point $x \in \mathbb{Z}$. Thus, for each point $x \in \mathbb{Z}$, we can get $Ker(\{x\}) = \bigcap\{B_{k'}(q,n) |\ x \in B_{k'}(q,n)\}$. We note that $Ker(\{x\})$ is the smallest open set containng x in $(\mathbb{Z}, \kappa(q,n))$.

We are able to determine the structure of $Ker(\{x\})$ for a point x in $(\mathbb{Z}, \kappa(q,n))$, where $q < n$, using Lemma 2.8 (i) and Remark 2.12 and also $Cl(\{x\})$ using Lemma 2.8 (iii), cf. Theorem 2.13 below.

Theorem 2.13 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Assume that* $n \equiv r \pmod q$, *where* $1 \leq r \leq q-1$. *The following properties hold:*

(b1) *For a point* $x = kq+i$, *where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$, $Ker(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$ *and it is the smallest open set containing* x.

(b2) *For a point* $x = kq+j$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q, Ker(\{x\}) = \{y \in Z \mid kq+1 \leq y \leq (k+1)q+r\}$ *and it is the smallest open set containing* x.

(b1)$'$ *For a point* $x = kq+i$, *where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$, $Cl(\{x\}) = \{y \in \mathbb{Z} \mid (k-1)q+r+1 \leq y \leq kq+q\}$ *holds;*

(b2)$'$ *For a point* $x = kq+j$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$, $Cl(\{x\}) = \{y \in \mathbb{Z} \mid kq+r+1 \leq y \leq kq+q\}$ *holds.*

Proof. We recall that $2 \leq q < n, n = sq + r$ ($s, r \in \mathbb{Z}$ with $1 \leq r \leq q-1$) and the family $\mathcal{G}(q,n) := \{B_{k'}(q,n)| \ k' \in \mathbb{Z}\}$ generates the topology $\kappa(q,n)$ on $\mathbb{Z}$ and $B_{k'}(q,n) = \{y \in \mathbb{Z} \mid k'q+1 \leq y \leq k'q+n\}$ is open in $(\mathbb{Z}, \kappa(q,n))$, where $k' \in \mathbb{Z}$.

(b1) Let $x = kq + i \in \mathbb{Z}$ be a point with $1 \leq i \leq r$. We have the following property (cf. Lemma 2.11 (ii) (b1)):
$(*2)$ $x = kq+i \in [k'q+1, k'q+sq+r]_{\mathbb{Z}}$ $(1 \leq i \leq r)$ if and only if $k' \in [k-s, k]_{\mathbb{Z}}$.
Using $(*2)$ and Lemma 2.8 (i) (cf. Remark 2.12), we show that $Ker(\{x\}) = \bigcap\{B_{k'}(n,q) \mid k' \in [k-s,k]_{\mathbb{Z}}\} = \bigcap\{[(k-a)q+1, (k-a)q+sq+r]_{\mathbb{Z}} \mid a \in [0,s]_{\mathbb{Z}}\} = [kq+1, kq+r]_{\mathbb{Z}}$ and $Ker(\{x\})$ is the smallest open set containing x.

(b2) Let $x = kq + j \in \mathbb{Z}$ be a point with $r+1 \leq j \leq q$. We have the followng property (cf. Lemma 2.11 (ii)(b2)):

$(*3)$ $x = kq+j \in [k'q+1, k'q+sq+r]_{\mathbb{Z}}$ $(r+1 \leq j \leq q)$ if and only if $k' \in [k-s+1, k]_{\mathbb{Z}}$.

Using $(*3)$ and Lemma 2.8 (i) (cf. Remark 2.12), we show that $Ker(\{x\})=\bigcap\{B_{k'}(q,n) \mid k' \in [k-s+1,k]_{\mathbb{Z}}\} =\bigcap\{[(k-a)q+1, (k-a)q+sq+r]_{\mathbb{Z}} \mid a \in [0,s-1]_{\mathbb{Z}}\} =[kq+1,(k+1)q+r]_{\mathbb{Z}}$ and $Ker(\{x\})$ is the smallest open set containing x.

(b1)$'$ We prove (b1)$'$ using Lemma 2.8 (iii). Let $U_{\{x\}} := \{y \in \mathbb{Z} \mid Ker(\{y\}) \cap \{x\} = \emptyset\}$ for given point x. For $x = kq + i$ with $1 \leq i \leq r$, we claim that

$(*)$ $U_{\{x\}} = [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$, where $[d, +\infty)_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid d \leq z\}$ and $(-\infty, e]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid z \leq e\}$ for some integers $d, e \in \mathbb{Z}$.

First we show that

$(*)^1$ $[(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}} \subset U_{\{x\}}$ holds.

Let $y \in [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$.

Case 1. $y \in [(k+1)q+1, +\infty)_{\mathbb{Z}}$: if $y = tq+i$ $(1 \leq i \leq r$ and $t \in \mathbb{Z}$ with $k+1 \leq t)$, then $Ker(\{y\}) = [tq+1, tq+r]_{\mathbb{Z}}$; it is shown by replacing the point y for the point x in the result of (b1) above. If $y = tq+j$ $(r+1 \leq j \leq q$ and $t \in \mathbb{Z}$ with $k+1 \leq t)$, then $Ker(\{y\}) = [tq+1, (t+1)q+r]_{\mathbb{Z}}$; it is obtained by replacing the point y for x in the result of (b2) above. Thus, we show that $x = kq+i \notin Ker(\{y\})$ $(1 \leq i \leq r)$ for this case and so $y \in U_{\{x\}}$.

Case 2. $y \in (-\infty, (k-1)q+r]_{\mathbb{Z}}$: if $y = tq+i$ $(1 \leq i \leq r$ and $t \in \mathbb{Z}$ with $t \leq k-1)$, then $Ker(\{y\}) = [tq+1, tq+r]_{\mathbb{Z}}$ (cf. the result of (b1) above). If $y = tq+j$ $(r+1 \leq j \leq q$ and $t \in \mathbb{Z}$ with $t \leq k-2)$, then $Ker(\{y\}) = [tq+1, (t+1)q+r]_{\mathbb{Z}}$ (cf. the result of (b2) above). For this case, we have $x = kq+i \notin Ker(\{y\})$ $(1 \leq i \leq r)$ and so $y \in U_{\{x\}}$.

Finally, we show the converse implication:

$(*)^2$ $U_{\{x\}} \subset [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$.

Let $y \in [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$ be any point. By the result of (b2) above, it is shown that $Ker(\{y\}) = [(k-1)q+1, kq+r]_{\mathbb{Z}}$ if $y \in [(k-1)q+r+1, kq]_{\mathbb{Z}}$. By the result of (b1) above, it is shown that $Ker(\{y\}) = [kq+1, kq+r]_{\mathbb{Z}}$ if $y \in [kq+1, kq+r]_{\mathbb{Z}}$. Moreover, if $y \in [kq+r+1, kq+q]_{\mathbb{Z}}$, we have that $Ker(\{y\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ holds (cf. the result of (b2) above). Thus, we show that, for these points y above, $x = kq+i \in Ker(\{y\})$ and so $y \notin U_{\{x\}}$, where $1 \leq i \leq r$. This concludes that $(*)^2$ above holds.

Using $(*)^1$ and $(*)^2$ above, we have done the proof of the claim $(*)$ above. Therefore, by Lemma 2.8 (iii) (cf. Remark 2.12), it is obtained that $Cl(\{x\}) = X \setminus U_{\{x\}} = [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$.

(b2)$'$ We claim that, for a given point $x = kq+j$ $(r+1 \leq j \leq q)$,

$(**)$ $U_{\{x\}} = [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, kq+r]_{\mathbb{Z}}$ holds, where $U_{\{x\}}$ is defined in the top of the proof of (b1)$'$ above. The property $(**)$ is proved by argument similar to that in the proof of $(*)$ in (b1)$'$ above. By Lemma 2.8 (iii) (cf. Remark 2.12), it is obtained that $Cl(\{x\}) = X \setminus U_{\{x\}} = [kq+r+1, (k+1)q]_{\mathbb{Z}}$. □

In the end of the present section, the following Corollary 2.14 shows the classification of families of topologies: • $\{\kappa(q,n)|n \in \mathbb{Z}$ with $2 \leq q < n$ and $n \not\equiv 0 \pmod q\}$, for a given positive integer $q \in \mathbb{Z}$ with $2 \leq q$. Throughout the proof of Corollary 2.14, the kernel of a singleton $\{x\}$ in a topological space (X, τ) also denoted by τ-$Ker(\{x\})$.

Corollary 2.14 *Let n, n' and q be positive integers such that $2 \leq q < n$, $2 \leq q < n'$, $n \not\equiv 0$ (mod q) and $n' \not\equiv 0$ (mod q). Then, $\kappa(q, n) = \kappa(q, n')$ if and only if $n \equiv n'$ (mod q).*

Proof. We denote shortly the kernel of a singleton $\{x\}$ in $(\mathbb{Z}, \kappa(q, n))$ (resp. $(\mathbb{Z}, \kappa(q, n'))$) by κ-$Ker(\{x\})$ (resp. κ'-$Ker(\{x\})$).

(Necessity) It follows from assumption that κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds for each point $x \in \mathbb{Z}$. Let $n \equiv r$ (mod q) and $n' \equiv r'$ (mod q) for some integer r and r' with $1 \leq r \leq q-1$ and $1 \leq r' \leq q-1$. We shall show $r = r'$. First we suppose $r \leq r'$. Take a point $x := kq+i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$; then we have κ-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ (cf. Theorem 2.13 (b1)). Since $x = kq+i$ $(1 \leq i \leq r')$, by Theorem 2.13 (b1) for the singleton $\{x\}$ in $(\mathbb{Z}, \kappa(q, n'))$ it is shown κ'-$Ker(\{x\}) = [kq+1, kq+r']_{\mathbb{Z}}$. Thus we have $r = r'$ for this first case, because κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$. Finally, we suppose $r' \leq r$. By the similar fashion to above first case, it is obtained that $r' = r$ for this case. Therefore, we show $r = r'$; and so we conclude that $n \equiv n'$ (mod q).

(Sufficiency) In oder to prove the sufficiency, we claim the following properties (1) and (2) of topological spaces; (2) is proved by (1).
Claim: Let (X, τ) and (X, τ') be two topological spaces.

(1) If U is an open set in (X, τ), then $U = \bigcup\{\tau\text{-}Ker(\{x\}) |\ x \in U\}$ holds.

(2) If τ-$Ker(\{x\}) \in \tau$, τ'-$Ker(\{x\}) \in \tau'$ and τ-$Ker(\{x\}) = \tau'$-$Ker(\{x\})$ hold for each point $x \in X$, then $\tau = \tau'$ and so $(X, \tau) = (X, \tau')$.

We prove the sufficiency of the present Corollary 2.14. Let $(\mathbb{Z}, \kappa(q, n))$ and $(\mathbb{Z}, \kappa(q, n'))$ be two generalized digital lines. We suppose $n \equiv r$ (mod q) and $n' \equiv r$ (mod q) for an integer r with $1 \leq r \leq q-1$. Let $x \in \mathbb{Z}$ and $x = kq+i$ for some $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq q-1$. We consider the following Case 1 and Case 2 on the point x.
Case 1. $x = kq+i$, where $1 \leq i \leq r$: by Theorem 2.13 (b1) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n))$, it is obtained that κ-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$; and by Theorem 2.13(b1) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n'))$, it is obtained that κ'-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$. Thus, for the point $x = kq+i$ $(1 \leq i \leq r)$, κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds.
Case 2. $x = kq+j$, where $r+1 \leq j \leq r$: by Theorem 2.13 (b2) for the point $x = kq+j$ in $(\mathbb{Z}, \kappa(q, n))$, it is obtained that κ-$Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$; and, by Theorem 2.13 (b2) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n'))$, it is obtained that κ'-$Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$. Thus, for the point $x = kq+j$ $(r+1 \leq j \leq q)$, κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds.

Therefore, for both cases above we see κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ for any point x. By using Theorem 2.13 (b1), (b2) and the claim (2) above, we have $\kappa(q, n) = \kappa(q, n')$. □

Remark 2.15 Kojima [29] investigated the classification of a family $\{\tau(3, m) | m \in \mathbb{Z}\}$ of the natural fuzzy topologies on $\mathbb{Z}$.

3 Semi-open sets in generalized digital lines In the first of the present section, we recall some notation with definitions and some properties (3.1) - (3.11) on familes of generalized open sets of a topological space (X, τ) (i.e., semi-open sets, preopen sets, α-open sets, β-open sets, semi-preopen sets, b-open sets):

(3.1) $SO(X, \tau) := \{A \mid A$ is semi-open in $(X, \tau)\} = \{A | A \subset Cl(Int(A))\} = \{A \mid$ there exists a subset $U \in \tau$ such that $U \subset A \subset Cl(U)\}$ [30],

(3.2) $PO(X, \tau) := \{A \mid A$ is preopen in $(X, \tau)\} = \{A \mid A \subset Int(Cl((A))\} = \{A|$ there exists a subset $V \in \tau$ such that $A \subset V \subset Cl(A)\}$ [34],

(3.3) $\tau^{\alpha} := \{A \mid A$ is α-open in $(X, \tau)\} = \{A \mid A \subset Int(Cl(Int(A)))\}$ [38].

(3.4) For every topological space (X, τ), $PO(X, \tau) \cap SO(X, \tau) = \tau^{\alpha}$ holds [42] and τ^{α} is a topology on X [38] (e.g., [40]);

(3.5) $\beta O(X, \tau) := \{A \mid A$ is β-open in $(X, \tau)\} = \{A \mid A \subset Cl(Int(Cl(A)))\}$ [1], (3.6) $SPO(X,\tau) := \{A \mid A$ is semi-preopen in $(X,\tau)\} = \{A \mid$ there exists a preopen set U such that $U \subset A \subset Cl(U)\}$ [4].

(3.7) For every topological space (X,τ),$SPO(X,\tau) = \beta O(X,\tau)$ holds [4, Theorem 2.4].

(3.8) $BO(X,\tau) := \{A \mid A$ is b-open in $(X,\tau)\} = \{A \mid A \subset Int(Cl(A)) \cup Cl(Int(A))\}$ [5].

(3.9) For every topological space (X,τ),

$\tau \subset PO(X,\tau) \cap SO(X,\tau) \subset PO(X,\tau) \cup SO(X,\tau) \subset BO(X,\tau) \subset \beta O(X,\tau) = SPO(X,\tau)$ hold [4, Theorem 2.2], [5, p.60] (e.g., [17, Proposition 1.1]).

(3.10) The following properties are well known and important ones:

if $V_i \in SO(X,\tau)$ (resp. $PO(X,\tau), SPO(X,\tau), BO(X,\tau)), i \in \Gamma$, then $\bigcup\{V_i \mid i \in \Gamma\} \in SO(X,\tau)$ (resp. $PO(X,\tau), SPO(X,\tau), BO(X,\tau)$), where the index set Γ is not necessarily finite.

(3.11) The complement of a semi-open set (resp. preopen set, α-open set, β-open set, pre-semi-open set, b-open set) is called a semi-close set (resp. preclosed set, α-closed set, β-closed set, pre-semi-closed set, b-closed set).

In the present section, we investigate mainly the semi-closure and the semi-kernel of a singleton of $(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 3.2). We note that [39, Lemma 2] if A is a nonempty semi-open set of (X,τ), then $Int(A) \neq \emptyset$.

Lemma 3.1 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2) *and* $A \in SO(\mathbb{Z}, \kappa(n,q))$ *with a point* $x \in A$. *Assume that* $n \equiv r \pmod q$, *where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q-1$.

(b1) *If* $x = kq + i \in A$, *where* $k \in \mathbb{Z}$, *and* $i \in Z$ *with* $1 \leq i \leq r$, *then there exists a subset* $U_1(x) \in \kappa(q,n)$ *such that* $x \in U_1(x) \subset A$ *and* $U_1(x)$ *is the smallest open set containing* x, *where* $U_1(x) := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$.

(b2) *If* $x = kq + j \in A$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$, *then there exist a point* $kq+h$ $(1 \leq h \leq q+r)$ *such that* $kq+h \in Int(A)$ *and an open set* V *such that* $V \subset A$, *where* V *is defined as follows:*

$V := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$ *if* $1 \leq h \leq r$; $V := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq (k+1)q+r\}$ *if* $r+1 \leq h \leq q$; $V := \{y \in \mathbb{Z} \mid (k+1)q+1 \leq y \leq (k+1)q+r\}$ *if* $q+1 \leq h \leq q+r$.

Proof. **(b1)** Suppose that $x = kq + i$ $(1 \leq i \leq r), x \in A$ and $A \in SO(\mathbb{Z}, \kappa(q,n))$. Since $x \in Cl(Int(A))$ holds, by using Theorem 2.13 (b1) for the point x, there exists the smallest open set $Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ containing x, say $U_1(x)$, such that $U_1(x) \cap Int(A) \neq \emptyset$. Take a point $y_x \in \mathbb{Z}$ such that $y_x \in U_1(x) \cap Int(A)$, say $y_x = kq+h$ $(1 \leq h \leq r)$. Then, using Theorem 2.13 (b1) for the point $y_x = kq+h$ $(1 \leq h \leq r)$, the set $Ker(\{y_x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ is the smallest open set containing y_x and so $y_x \in [kq+1, kq+r]_{\mathbb{Z}} \subset Int(A) \subset A$. Thus, it is obtained that $U_1(x) = [kq+1, kq+r]_{\mathbb{Z}}$ is the smallest open set containing x such that $U_1(x) \subset A$.

(b2) By using Theorem 2.13 (b2) for the point x, there exists the smallest open set $Ker(\{x\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ containing x. Since $x \in A$ and $A \subset Cl(Int(A))$ hold, we have $[kq+1, (k+1)q+r]_{\mathbb{Z}} \cap Int(A) \neq \emptyset$ and so there exists a point $kq+h \in Int(A)$ with $1 \leq h \leq q+r$. Thus we investigate the following Case 1, Case 2 and Case 3.

Case 1. $kq+h \in Int(A)$, where $1 \leq h \leq r$; Case 2. $kq+h \in Int(A)$, where $r+1 \leq h \leq q$; Case 3. $kq+h \in Int(A)$, where $q+1 \leq h \leq q+r$.

For Case 1, by using Theorem 2.13 (b1) for the point $kq+h$ and the definition of V, it is shown that $Ker(\{kq+h\}) = [kq+1, kq+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \notin V$ for this case. For Case 2, by using Theorem 2.13 (b2) for the point $kq+h$ and the definition of V, it is shown that $Ker(\{kq+h\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \in V$ for this case. For Case 3, by using Theorem 2.13 (b1) for the point $kq+h = (h+1)q+h'$, where $h' \in \mathbb{Z}$ with $1 \leq h' \leq r$, and the definition of V, it is shown that $Ker(\{kq+h\}) = [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \notin V$ for this case. □

For the digital line $(\mathbb{Z}, \kappa), \kappa(2,3) = \kappa$, i.e., $q = 2, n = 3$ and so $r = 1$, it is known that $SO(\mathbb{Z}, \kappa(2,3)) \neq \kappa(2,3)$ and $\kappa(2,3) \subsetneq SO(\mathbb{Z}, \kappa(2,3))$. For example, a subset $\{q+r, q+q\} = \{3,4\}$ is a semi-open set, where $q = 2$ and $r = 1$; it is not open in $(\mathbb{Z}, \kappa(2,3))$.

We recall the following definitions: for a subset B of a topological space (X, τ), $sKer(B) = \bigcap\{U \mid U \in SO(X,\tau), B \subset U\}$; $sCl(B) = \bigcap\{F \mid X \setminus F \in SO(X,\tau), B \subset F\}$. It is well nown that [4, Theorem 2.1 (a)] $sCl(A) = A \cup Int(Cl(A))$ holds for any subset A of (X, τ).

Theorem 3.2 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2) *and a point* $x \in \mathbb{Z}$. *Assume that* $n \equiv r \pmod q$, *where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q-1$. *The following properties hold:*
(b1) *Let* $x = kq + i$, *where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$. *Then,*
(b1-1) *there exists a subset* $U_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ *such that* $x \in U_1(x)$, *where* $U_1(x) := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$;
(b1-2) *if there exists a semi-open set* A_1 *containing the point* x *such that* $A_1 \subset U_1(x)$, *then* $A_1 = U_1(x)$ *and* $x \in U_1(x)$ *hold, where* $U_1(x)$ *is defined in* (b1-1) *above;*
(b1-3) $sKer(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\} \in SO(\mathbb{Z}, \kappa(q,n))$ *and* $sKer(\{x\})$ *is semi-open in* $(\mathbb{Z}, \kappa(q,n))$.
(b2) *Let* $x = kq + j \in \mathbb{Z}$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$. *Then,*
(b2-1) *there exist two subsets* $V_i(x) \in SO(\mathbb{Z}, \kappa(q,n)), i \in \{1,2\}$, *such that* $\{x\} = V_1(x) \cap V_2(x)$, *where* $V_1(x) := \{x\} \cup \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$ *and* $V_2(x) := \{x\} \cup \{y \in \mathbb{Z} \mid (k+1)q+1 \leq y \leq (k+1)q+r\}$;
(b2-2) $sKer(\{x\}) = \{x\}$ *and* $\{x\}$ *is not semi-open in* $(\mathbb{Z}, \kappa(q,n))$;
(b2-3) *if there exists a semi-open set* G_1 (resp. *a semi-open set* G_2) *such that* $x \in G_1 \subset V_1(x)$ (resp. $x \in G_2 \subset V_2(x)$), *then* $G_1 = V_1(x)$ (resp. $G_2 = V_2(x)$), *where* $V_1(x)$ *and* $V_2(x)$ *are defined in* (b2-1) *above.*
(b1)$'$ *For a point* $x = kq + i$, *where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$,
$sCl(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\} = sKer(\{x\})$ *hold.*
(b2)$'$ *For a point* $x = kq + j$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$,
$sCl(\{x\}) = \{x\} = sKer(\{x\})$ *hold.*

Proof. **(b1)** **(b1-1)** Let $x = kq + i$ $(1 \leq i \leq r)$. By using Lemma 3.1 (b1) for the semi-open set $\mathbb{Z}$ of $(\mathbb{Z}, \kappa(q,n))$ and the point $x \in \mathbb{Z}$ and a fact that $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$, there exists a subset $U_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ such that $x \in U_1(x)$, where $U_1(x) = [kq+1, kq+r]_{\mathbb{Z}}$.
(b1-2) Suppose that there exists a semi-open set A_1 such that $x \in A_1 \subset U_1(x)$. Then, by Lemma 3.1 for A_1 and x, it is shown that $x \in U_1(x) \subset A_1$ and so $A_1 = U_1(x)$.
(b1-3) By (b1-2) above, it is obtained that $sKer(\{x\}) = U_1(x)$ holds and $sKer(\{x\})$ is semi-open in $(\mathbb{Z}, \kappa(q,n))$.

(b2) Throughout (b2) we recall that $x = kq + j$ $(r+1 \leq j \leq q)$.
(b2-1) First we claim that $V_1(x) := \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}}$ is a semi-open set containing x. Put $V_1 := [kq+1, kq+r]_{\mathbb{Z}}$. Using Theorem 2.13 (b1) for a point $y \in V_1$, $Ker(\{y\}) = V_1$ is the smallest open set containing y. It is shown that $V_1(x) \subset Cl(V_1)$. Indeed, by Theorem 2.13 (b1)$'$, $Cl(V_1) = \bigcup\{Cl(\{kq+h\}) \mid h \in [1,r]_{\mathbb{Z}}\} = [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$ and so $V_1(x) \subset Cl(V_1)$. Thus, there exists an open set V_1 such that $V_1 \subset V_1(x) \subset Cl(V_1)$. Namely, $V_1(x)$ is a semi-open set containing x. Finally, we can prove that $V_2(x) := \{x\} \cup [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$ is a semi-open set containing x. Put $V_2 := [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$. Using Theorem 2.13 (b2) for a point $z \in V_2$, $Ker(\{z\}) = V_2$ is the smallest open set containing z. By Theorem 2.13 (b1)$'$, $Cl(V_2) = \bigcup\{Cl(\{(k+1)q+h\}) \mid h \in [1,r]_{\mathbb{Z}}\} = [kq+r+1, (k+1)q+q]_{\mathbb{Z}}$ and $x \in Cl(V_2)$. Thus, there exists an open set V_2 such that $V_2 \subset V_2(x) \subset Cl(V_2)$. Namely, $V_2(x)$ is a semi-open set containing x. Obviously, we have $\{x\} = V_1(x) \cap V_2(x)$.
(b2-2) It follows from (b2-1) above that $\{x\} \subset sKer(\{x\}) \subset V_1(x) \cap V_2(x) = \{x\}$ and so $sKer(\{x\}) = \{x\}$. By Theorem 2.13 (b2), it is obtained that $Int(\{x\}) = \emptyset$ and so

$\{x\} \not\subset Cl(Int((\{x\}))=\emptyset$, i.e., {x} is not semi-open in $(\mathbb{Z},\kappa(q,n))$.
(b2-3) Let $\xi := \{[kq{+}1,kq{+}r]_{\mathbb{Z}},[kq{+}1,(k{+}1)q{+}r]_{\mathbb{Z}},[(k{+}1)q{+}1,(k{+}1)q{+}r]_{\mathbb{Z}}\}$ throughout the present proof. First, we claim that $V_1(x)=G_1$. Indeed, using Lemma 3.1 (b2) for G_1 and the point x, there exists an open set V such that $V\subset G_1$; by Lemma 3.1 (b2), it is shown explicitly that $V\in\xi$. Because of $V\subset G_1\subset V_1(x)=\{kq{+}j\}\cup[kq{+}1,kq{+}r]_{\mathbb{Z}}$, where $r{+}1\le j\le q$, we have $V=[kq{+}1,kq{+}r]_{\mathbb{Z}}$. Thus, $V_1(x)=\{x\}\cup V\subset\{x\}\cup G_1=G_1\subset V_1(x)$ and hence $V_1(x)=G_1$. Finally, we prove that $V_2(x)=G_2$. Using Lemma 3.1 (b2) for the semi-open set G_2 and the point x, there exists an open set V such that $V\subset G_2$; explicitly that $V\in\xi$. Because of $V\subset G_2\subset V_2(x)=\{kq+j\}\cup[(k+1)q+1,(k+1)q+r]_{\mathbb{Z}}$, where $r+1\le j\le q$, we conclude that $V=[(k+1)q+1,(k+1)q+r]_{\mathbb{Z}}$. Thus, we obtain $V_2(x)=\{x\}\cup V\subset\{x\}\cup G_2=G_2\subset V_2(x)$ and hence $V_2(x)=G_2$.

(b1)′ By Theorem 2.13 (b1)′, (b1) and (b2), for a point $x=kq+i$ $(1\le i\le r)$, it is shown that $Int(Cl(\{x\}))=Int([(k-1)q+r+1,kq+q]_{\mathbb{Z}})=[kq+1,kq+r]_{\mathbb{Z}}$. Then, $sCl(\{x\})=\{x\}\cup Int(Cl(\{x\}))=[kq+1,kq+r]_{\mathbb{Z}}$ hold. We have $sCl(\{x\})=sKer(\{x\})$ (cf. (b1) above).

(b2)′ Let $x=kq{+}j$ $(r{+}1\le j\le q)$. By Theorem 2.13 (b2′), $Cl(\{x\})=[kq{+}r{+}1,kq{+}q]_{\mathbb{Z}}$. By Theorem 2.13 (b2), it is obtained that $Int(Cl(\{x\}))=Int([kq+r+1,kq+q]_{\mathbb{Z}})=\emptyset$ and so $sCl(\{x\})=\{x\}$. It is noted that $sCl(\{x\})=sKer(\{x\})$ (cf. (b2-2) above). □

Remark 3.3 It is shown that $sKer(\{x\})$ is not necessarily semi-open (cf. Theorem 3.2 (b2-2)).

4 Preopen sets of generalized digital lines In the present section, we investigate prekernels and preclosures of singletons in $(\mathbb{Z},\kappa(q,n))$. We recall the following definitions: for a subset A of a topological space (X,τ), $pKer(A):=\bigcap\{U\mid A\subset U,U\in PO(X,\tau)\}$ [21]; $pCl(A):=\bigcap\{F\mid A\subset F,X\setminus F\in PO(X,\tau)\}$ [12]. It is well known that [4, Theorem 1.5 (e)] $pCl(A)=A\cup Cl(Int(A))$ holds for any subset A of (X,τ).

Lemma 4.1 *Let* $(\mathbb{Z},\kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Assume that* $n\equiv r \pmod q$, *where* $r\in\mathbb{Z}$ *with* $1\le r\le q-1$. *Let* $x=kq+j\in Z$, *where* $k\in\mathbb{Z}$ *and* $j\in\mathbb{Z}$ *with* $r+1\le j\le q$. *If* $A\in PO(\mathbb{Z},\kappa(q,n))$ *and* $x\in A$, *then there exist two points* $kq+a$ *and* $kq+q+b$ *such that* $\{kq+a,kq+q+b\}\subset A$ *for some integers* a *and* b *with* $1\le a\le r$ *and* $1\le b\le r$.

Proof. There exists a subset $W\in\kappa(q,n)$ such that $x\in W\subset Cl(A)$, because $x\in A\subset Int(Cl(A))$. Since $Ker(\{x\})\subset W$, by Theorem 2.13 (b2), $[kq{+}1,kq{+}q{+}r]_{\mathbb{Z}}\subset Cl(A)$ holds. Thus, we have $kq+1\in Cl(A)$ and $kq+q+r\in Cl(A)$. By using Theorem 2.13 (b1) for the above two points, it is obtained that $[kq{+}1,kq{+}r]_{\mathbb{Z}}\cap A\ne\emptyset$ and $[kq{+}q{+}1,kq{+}q{+}r]_{\mathbb{Z}}\cap A\ne\emptyset$, respectively. Then there exist two points $kq+a\in A$ and $kq+q+b\in A$ for some integers a,b with $1\le a\le r$ and $1\le b\le r$. □

Theorem 4.2 *Let* $(\mathbb{Z},\kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Assume that* $n\equiv r \pmod q$, *where* $r\in\mathbb{Z}$ *with* $1\le r\le q-1$.
(b1) *For a point* $x=kq+i\in Z$, *where* $k\in\mathbb{Z}$ *and* $i\in\mathbb{Z}$ *with* $1\le i\le r$, *the following properties hold.*

(b1-1) $pKer(\{x\})=\{x\}$ *and* $\{x\}$ *is preopen.*

(b1-1)′ *If* $r\ge 2$, *then* $pCl(\{x\})=\{x\}$, *i.e.,* $\{x\}$ *is preclosed.*

If $r=1$, *then* $x=kq+1$ *and* $pCl(\{x\})=\{y\in\mathbb{Z}\mid (k-1)q+2\le y\le kq+q\}$.

(b2) *For a point* $x=kq+j\in\mathbb{Z}$, *where* $k\in\mathbb{Z}$ *and* $j\in\mathbb{Z}$ *with* $r+1\le j\le q$, *the following properties* (b2-1) - (b2-4) *and* (b2-3)′ *hold. Let* $V_{h,h'}(x):=\{kq+h,x,kq+q+h'\}$, *where* $h,h'\in\mathbb{Z}$ *with* $1\le h\le r$ *and* $1\le h'\le r$.

(b2-1) $V_{h,h'}(x)\in PO(\mathbb{Z},\kappa(q,n))$ *and* $pKer(\{x\})\subset V_{h,h'}(x)$ *for each integers* h *and* h' *with* $1\le h\le r, 1\le h'\le r$.

(b2-2) *Suppose that* $r = 1$. *If there exists a preopen set* G *containing the point* x, *then* $x \in V_{1,1}(x) \subset G$.

(b2-3) $pKer(\{x\}) = V_{1,1}(x)$ *if* $r = 1$; $pKer(\{x\}) = \{x\}$ *if* $r \geq 2$; *for the singleton* $\{x\}, \{x\} \notin PO(\mathbb{Z}, \kappa(q, n))$.

(b2-4) *If there exists a subset* $G \in PO(\mathbb{Z}, \kappa(q, n))$ *such that* $x \in G \subset V_{h,h'}(x)$, *then* $G = V_{h,h'}(x)$.

(b2-3)$'$ $pCl(\{x\}) = \{x\}$, *i.e.,* $\{x\}$ *is preclosed.*

Proof. **(b1) (b1-1)** For the point $x = kq + i$ $(1 \leq i \leq r)$, by using Theorem 2.13 (b1)$'$, (b1) and (b2), it is shown that $Int(Cl(\{x\})) = Int([(k-1)q+r+1, kq+q]_{\mathbb{Z}}) = [kq+1, kq+r]_{\mathbb{Z}} \supset \{x\}$ and so $\{x\} \in PO(\mathbb{Z}, \kappa(q, n))$. This implies $pKer(\{x\}) = \{x\}$.
(b1-1)$'$ By Theorem 2.13 (b1), it is shown that, for the case where $r \geq 2$, $Int(\{x\}) = \emptyset$ and so $pCl(\{x\}) = \{x\} \cup Cl(Int(\{x\})) = \{x\}$. For the case where $r = 1$, $x = kq+1$ holds. And, by Theorem 2.13 (b1) and (b1)$'$, it is shown that $Cl(Int(\{x\})) = Cl(\{x\}) = [(k-1)q+2, kq+q]_{\mathbb{Z}}$ and so $pCl(\{kq + 1\}) = [(k-1)q + 2, kq + q]_{\mathbb{Z}}$.

(b2) (b2-1) Put $V_{h,h'}(x) := \{x, kq+h, kq+q+h'\}$ for a point $x = kq+j$ $(r+1 \leq j \leq q)$ and each integers h and h' with $1 \leq h \leq r$ and $1 \leq h' \leq r$. Then, by Theorem 2.13, it is shown that $Int(Cl(V_{h,h'}(x)))=Int([kq+r+1, kq+q]_{\mathbb{Z}} \cup [(k-1)q+r+1, kq+q]_{\mathbb{Z}} \cup [kq+r+1, (k+1)q+q]_{\mathbb{Z}})=Int([(k-1)q+r+1, (k+1)q+q]_{\mathbb{Z}})=[kq+1, (k+1)q+r]_{\mathbb{Z}} \supset V_{h,h'}(x)$ and so $V_{h,h'}(x) \in PO(\mathbb{Z}, \kappa(q, n))$. Thus, we show that $pKer(\{x\}) \subset V_{h,h'}(x)$ for each integers h and h' with $1 \leq h \leq r$ and $1 \leq h' \leq r$.
(b2-2) If $r = 1$, then $V_{1,1}(x) = \{kq+1, x, kq+q+1\} \subset G$ for any preopen set G containing x (cf. Lemma 4.1).
(b2-3) Using (b2-1) and (b2-2) above, we have that $pKer(\{x\}) = V_{1,1}(x)$ if $r = 1$. If $r \geq 2$, then there exist two preopen sets $V_{1,1}(x)$ and $V_{2,2}(x)$ such that $V_{1,1}(x) \cap V_{2,2}(x) = \{x\}$. Thus we have that $pKer(\{x\}) = \{x\}$ if $r \geq 2$. By Theorem 2.13 (b2)$'$ and (b2), it is shown that $\{x\} \not\subset Int(Cl(\{x\})) = \emptyset$ and so $\{x\} \notin PO(\mathbb{Z}, \kappa(q, n))$.
(b2-4) Let $G \in PO(\mathbb{Z}, \kappa(q, n))$ such that $G \subset V_{h,h'}(x)$ and $x \in G$. We claim that $G = V_{h,h'}(x)$ holds. Indeed, by Lemma 4.1, $\{kq+a, kq+q+b\} \subset G \subset V_{h,h'}(x)$, for some $a, b \in \mathbb{Z}$ with $1 \leq a \leq r$ and $1 \leq b \leq r$. Thus, we have $a = h, b = h'$ and so $G = V_{h,h'}(x)$, because $x \in G$.
(b2-3)$'$ By Theorem 2.13 (b2), $pCl(\{x\}) = \{x\} \cup Cl(Int(\{x\})) = \{x\} \cup Cl(\emptyset) = \{x\}$. Thus $\{x\}$ is preclosed. □

5 Proof of Theorem A(i) and related properties In the present section, the proof of Theorem A(i) (cf. Section 1) shall be given (cf. Theorem 5.1 (i) or (ii) below); moreover we investigate some related properties on structures of $SO(\mathbb{Z}, \kappa(q, n))$ and $PO(\mathbb{Z}, \kappa(q, n))$ (cf. Theorems 5.1 and 5.2 below).

For a topological space (X, τ), we recall that (X, τ) is said to be *extremally disconnected* if the closure of every open set is open; by [23, Proposition 4.1], [22], it is well known that a topological space (X, τ) is extremally disconnected if and only if $SO(X, \tau) \subset PO(X, \tau)$ holds. A topological space (X, τ) is said to be a *PS-space* [2] if $PO(X, \tau) \subset SO(X, \tau)$ holds. It is well known that the following properties are equivalent to each others: (X, τ) is a PS-space; $SO(X, \tau) = SPO(X, \tau); \tau^{\alpha} = PO(X, \tau)$; (X, τ^{α}) is submaximal; (X, τ) is quasi-submaximal (cf. [15, Theorem 4], [16, Proposition 8]; [2, Theorem 2.1]; [3, Theorem 3.4], e.g. [43, Theorem 3.4]).

Theorem 5.1 *Let* $(\mathbb{Z}, \kappa(q, n))$ *be a generalized digital line* (cf. Definition 2.2)*. Assume that* $n \equiv r \pmod q$, *where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q-1$. *Then, the following properties hold.*

(i) *A singleton* $\{kq + j\}$ *is not preopen in* $(\mathbb{Z}, \kappa(q, n))$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r + 1 \leq j \leq q$. *Namely,* $PO(\mathbb{Z}, \kappa(q, n)) \neq P(\mathbb{Z})$ *holds.*

(ii) *A singleton* $\{kq+j\}$ *is not semi-open in* $(\mathbb{Z}, \kappa(q,n))$*, where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$*. Namely,* $SO(\mathbb{Z}, \kappa(q,n)) \neq P(\mathbb{Z})$ *holds.*

(iii) *Especially, assume that* $2 \leq r$*. For a singleton* $\{kq+i\}$*, where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$*, we have* $\{kq+i\} \in PO(\mathbb{Z}, \kappa(q,n))$ *and* $\{kq+i\} \notin SO(\mathbb{Z}, \kappa(q,n))$*.*

(iv) *There exists a subset* V *such that* $V \notin PO(\mathbb{Z}, \kappa(q,n))$ *and* $V \in SO(\mathbb{Z}, \kappa(q,n))$*.*

(v) (e.g., [13, Theorem 2.1 (i)(b)]) *Especially, if* $q = 2, n = 3$ *and* $r = 1$*, then* $PO(\mathbb{Z}, \kappa(2,3)) \subset SO(\mathbb{Z}, \kappa(2,3))$ *and* $\kappa(2,3)^\alpha = \kappa(2,3)$ *hold.*

Proof. **(i)** By using Theorem 4.2 (b2)(b2-3) for the point $x := kq+j$ $(r+1 \leq j \leq q)$, it is obtained that $\{kq+j\} \notin PO(\mathbb{Z}, \kappa(q,n))$ and so $PO(\mathbb{Z}, \kappa(q,n)) \subsetneq P(\mathbb{Z})$.

(ii) We claim that the singleton $\{kq+j\}$ is not semi-open in $(\mathbb{Z}, \kappa(q,n))$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q$. Suppose that $\{kq+j\}$ is semi-open in $(\mathbb{Z}, \kappa(q,n))$. By Theorem 3.2 (b2)(b2-1), there exists a semi-open set $V_1(kq+j) = \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}}$. Then, by using Theorem 3.2 (b2)(b2-3) for the point $x := kq+j$ and the semi-open set $G_1 := \{kq+j\}$, it is shown that $\{kq+j\} = V_1(kq+j)$ holds. Thus, we have $|\{kq+j\}| = 1 = |V_1(kq+j)| = r+1$ and so $r = 0$; thus this contradicts to the assumption. Thus, $\{kq+j\} \notin SO(\mathbb{Z}, \kappa(q,n))$ and so $SO(\mathbb{Z}, \kappa(q,n)) \subsetneq P(\mathbb{Z})$.

(iii) By using Theorem 4.2 (b1)(b1-1) for the point $x := kq+i$ $(1 \leq i \leq r)$, the singleton $\{kq+i\}$ is preopen in $(\mathbb{Z}, \kappa(q,n))$. Since $2 \leq r$, the singleton $\{kq+i\}$ is not semi-open in $(\mathbb{Z}, \kappa(q,n))$, because $sKer(\{kq+i\})$=$[kq+1, kq+r]_{\mathbb{Z}} \supsetneq \{kq+i\}$ and $sKer(\{kq+i\})$ is the intersection of all semi-open sets containing the point $kq+i$ (cf. Theorem 3.2 (b1)(b1-3)).

(iv) By using Theorem 3.2 (b2)(b2-1) for the point $x := kq+j$ $(r+1 \leq j \leq q)$, there exists a semi-open set $V_1(kq+j) := \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}}$. We put $V := V_1(kq+j)$ and so $V \in SO(\mathbb{Z}, \kappa(q,n))$. We claim that $V \not\subset Int(Cl(V))$. Indeed, by using Theorem 2.13 (b2)$'$ and (b1$'$) for the point $kq+j$ and points $kq+i$ $(1 \leq i \leq r)$, respectively, it is shown that $Cl(V) = Cl(\{kq+j\}) \cup (\bigcup_{i=1}^{r} Cl(\{kq+i\})) = [(k-1)q+r+1, kq+q]_{\mathbb{Z}}$. Using Theorem 2.13 (b1) and (b2), we have $Int(Cl(V)) = [kq+1, kq+r]_{\mathbb{Z}}$ and hence $V := V_1(kq+j) = \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}} \not\subset [kq+1, kq+r]_{\mathbb{Z}} = Int(Cl(V))$. Therefore, we have $V \notin PO(\mathbb{Z}, \kappa(q,n))$ and $V \in SO(\mathbb{Z}, \kappa(q,n))$. □

Proof of Theorem A(i) The proof is shown by using Theorem 5.1 (i) or (ii) above, because $\kappa(q,n) \subset PO(\mathbb{Z}, \kappa(q,n))$ or $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$ hold in general. □

Theorem 5.1 (iii) and (v) (resp. (iv)) suggest the property of Theorem 5.2 (i) (resp. (ii)) below.

Theorem 5.2 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2)*. Assume that* $n \equiv r \pmod{q}$*, where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q-1$*.*

(i) $PO(\mathbb{Z}, \kappa(q,n)) \subset SO(\mathbb{Z}, \kappa(q,n))$ *holds if and only if* $n \equiv 1 \pmod{q}$*.*

(ii) *A non-implicaton* $SO(\mathbb{Z}, \kappa(q,n)) \not\subset PO(\mathbb{Z}, \kappa(q,n))$ *holds.*

(iii) *The topology* $\kappa(q,n)$ *is a proper subfamily of* $SO(\mathbb{Z}, \kappa(q,n))$*. And, if* $q+r > 3$ *then* $\kappa(q,n)$ *is a proper subfamily of* $PO(\mathbb{Z}, \kappa(q,n))$*.*

Proof. **(i) (Necessity)** By Theorem 4.2 (b1)(b1-1) for a point $x := kq+i$ $(1 \leq i \leq r)$, it is shown that $\{kq+i\} = pKer(\{kq+i\}) \in PO(\mathbb{Z}, \kappa(q,n))$. It follows our assumption that $\{kq+i\} \in SO(\mathbb{Z}, \kappa(q,n))$; by definition, $sKer(\{kq+i\}) = \{kq+i\}$ holds. Using Theorem 3.2 (b1)(b1-3) for the point $kq+i$, we have $sKer(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}}$ and so $|[kq+1, kq+r]_{\mathbb{Z}}| = 1$; therefore $r = 1$.

(**Sufficiency**) Suppose that $r = 1$. Let $V \in PO(\mathbb{Z}, \kappa(q,n))$. The set V has a decomposition $V = A_V \cup B_V$, where $A_V := \bigcup\{V \cap \{kq+1\} | \ k \in \mathbb{Z}\}$ and $B_V := \bigcup\{V \cap [kq+2, kq+q]_{\mathbb{Z}} | \ k \in \mathbb{Z}\}$.

First, we show that: (∗1) $A_V \in SO(\mathbb{Z}, \kappa(q,n))$. Indeed, we have that $V \cap \{kq+1\} = \{kq+1\}$ or $\emptyset$ and $sKer(\{kq+1\}) = [kq+1, kq+r]_{\mathbb{Z}} = \{q+1\}$ hold and $\{kq+1\} \in SO(\mathbb{Z}, \kappa(q,n))$ by Theorem 3.2 (b1)(b1-3); thus $A_V \in SO(\mathbb{Z}, \kappa(q,n))$.

Secondly, we show that: $(*2)$ for a point $x \in B_V$, there exist a preopen set $V_{1,1}(x) := \{kq+1, x, kq+q+1\}$ such that $x \in V_{1,1}(x)$ and $V_{1,1}(x) \subset V$. Indeed, the point $x \in B_V$, there exist integers k and j with $r+1 = 2 \leq j \leq q$ such that $x = kq+j$. Since $x \in [kq+2, kq+q]_{\mathbb{Z}}$, $x \in V$ and $V \in PO(\mathbb{Z}, \kappa(q,n))$, we use Theorem 4.2 (b2)(b2-1) and (b2-2) for the point $x = kq+j$, the preopen set V, $r = 1$ and $h = h' = 1$. Then, there exist a preopen set $V_{1,1}(x)$ such that $x \in V_{1,1}(x)$ and $V_{1,1}(x) \subset V$, where $V_{1,1}(x) := \{kq+1, x, kq+q+1\} \subset V$.

Thus, by using $(*2)$, it is obtained that: $(*2')$ $B_V \subset \bigcup\{V_{1,1}(x) | \ x \in B_V\} \subset V$ hold.

Thirdly, we show that: $(*3)$ $V_{1,1}(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for the point $x = kq+j \in B_V$. Indeed, using Theorem 3.2 (b2)(b2-1) for the point $x = kq+j$ and $r = 1$, fortunately, we have two semi-open sets $V_1(x) = \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}} = \{x, kq+1\}$ and $V_2(x) = \{x\} \cup [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}} = \{x, kq+q+1\}$. Since $V_1(x) \cup V_2(x) = \{kq+1, x, kq+q+1\}$ and $V_i(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for each $i \in \{1,2\}$, we have $V_1(x) \cup V_2(x) = V_{1,1}(x)$ and $V_{1,1}(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for the point $x = kq+j \in B_V$.

Finally, by the properties $(*1)$, $(*2')$ and $(*3)$ above, it is shown that $V = A_V \cup B_V \subset A_V \cup (\bigcup\{V_{1,1}(x) | \ x \in B_V\}) \subset V$ and so $V = A_V \cup (\bigcup\{V_{1,1}(x) | \ x \in B_V\})$ and hence $V \in SO(\mathbb{Z}, \kappa(q,n))$ (cf. (3.10) in Section 3). Therefore, $PO(\mathbb{Z}, \kappa(q,n)) \subset SO(\mathbb{Z}, \kappa(q,n))$ holds if $q < n$ and $n \equiv 1 \pmod{q}$.

(ii) By Theorem 5.1 (iv), there exists a semi-open set, say V, such that $V \notin PO(\mathbb{Z}, \kappa(q,n))$; this shows $SO(\mathbb{Z}, \kappa(q,n)) \not\subset PO(\mathbb{Z}, \kappa(q,n))$.

(iii) First, let $V_1(x) := \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}}$ be the semi-open set in Theorem 3.2 (b2) (b2-1), where $x := kq+j$ $(r+1 \leq j \leq q, k \in \mathbb{Z})$. The semi-open set $V_1(x)$ is not open because $V_1(x) \subsetneq Ker(\{x\})$ and $Ker(\{x\})$ is the smallest open set containing x (cf. Theorem 2.13 (b2), $Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$). Thus, we have that $V_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ and $V_1(x) \notin \kappa(q,n)$ (i.e., $\kappa(q,n)$ is a proper subfamily of $SO(\mathbb{Z}, \kappa(q,n))$, because $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$ holds in general). Finally, let $V_{h,h'}(x) := \{kq+h, x, kq+q+h'\}$ be the preopen set containing x in Theorem 4.2 (b2), where $x := kq+j (r+1 \leq j \leq q, \ k \in \mathbb{Z})$ and $h, h' \in [1, r]_{\mathbb{Z}}$ (cf. (b2-1)). However, the preopen set $V_{h,h'}(x)$ is not open in $(\mathbb{Z}, \kappa(q,n))$ if $q+r > 3$. Indeed, $Ker(\{x\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ is the smallest open set containing the point $x := kq+j$ (cf. Theorem 2.13 (b2)), $|Ker(\{x\})| = q+r$ and $|V_{h,h'}(x)| = 3$ hold; and so the point x is not an interior point of $V_{h,h'}(x)$. Thus, we have that $V_{h,h'}(x) \in PO(\mathbb{Z}, \kappa(q,n))$ and if $q+r > 3$ then $V_{h,h'}(x) \notin \kappa(q,n)$ (i.e., $\kappa(q,n)$ is a proper subfamily of $PO(\mathbb{Z}, \kappa(q,n))$, because $\kappa(q,n) \subset PO(\mathbb{Z}, \kappa(q,n))$ holds in general). □

6 Some separation axioms of generalized digital lines and proof of Theorem A(ii)

The purpose of the present section is to investigate some separation axioms of generalized digital lines (cf. Theorem A(ii) in Section 1; and Theorem 6.2, Tables 1 and 2 below). The proof of Theorem A(ii) shall be given by quoting some results in Theorem 6.2 below.

We first recall the following properties (6.1) - (6.6) for a topological space (X, τ).

(6.1) (X, τ) is $T_{1/2}$ if and only if every singleton $\{x\}, x \in X$, is open or closed in (X, τ) ([11, Theorem 2.5]).

(6.2) (X, τ) is $T_{3/4}$ if and only if every singleton $\{x\}$ of (X, τ) is δ-open or closed (equivalently, regular open or closed) in (X, τ) ([10, Theorem 4.3, Example 4.6]).

(6.3) (X, τ) is semi-pre-$T_{1/2}$ if and only if every singleton $\{x\}$ of (X, τ) is semi-preopen or closed (=preopen or closed) in (X, τ) ([9, Theorem 4.1]).

(6.4) For each integer $i \in \{2, 1, 0\}$, the semi-T_i axiom [32] (resp. pre-T_i axiom [24], β-T_i axiom [33]) is defined by using as ordinary T_i axiom except each open set replaced by semi-open set (resp. preopen sets, β-open set(=semi-preopen sets)).

(6.5) (X, τ) is semi-T_1 (resp. pre-T_1, β-T_1) if and only if every singleton $\{x\}, x \in X$, is semi-closed (resp. preclosed, β-closed) in (X, τ).

(6.6) The following implications of separation axioms above are well known:

$\cdot$ $T_2 \Rightarrow T_1 \Rightarrow T_{3/4} \Rightarrow T_{1/2} \Rightarrow T_0$,

· $T_2 \Rightarrow$ semi-$T_2 \Rightarrow$ semi-$T_1 \Rightarrow$ semi-$T_{1/2} \Rightarrow$ semi-T_0,
· $T_2 \Rightarrow$ pre-$T_2 \Rightarrow$ pre-$T_1 \Rightarrow$ pre-$T_{1/2} \Rightarrow$ pre-T_0,
· $T_2 \Rightarrow \beta$-$T_2 \Rightarrow \beta$-$T_1 \Rightarrow \beta$-$T_{1/2} \Rightarrow \beta$-$T_0$,
· for each $i \in \{2, 1, 1/2, 0\}$, $T_i \Rightarrow$ semi-$T_i \Rightarrow \beta$-T_i,
· for each $i \in \{2, 1, 1/2, 0\}$, $T_i \Rightarrow$ pre-$T_i \Rightarrow \beta$-T_i.

In order to investigate some separation axioms of the generalized digital line, we need the following theorem on topological properties of singletons $\{x\}$ of $(\mathbb{Z}, \kappa(q, n))$ (cf. Defintion 2.2).

Theorem 6.1 *For a generalized digital line* $(\mathbb{Z}, \kappa(q, n))$ (cf. Definition 2.2) *and a point* $x \in \mathbb{Z}$, *the following properties hold. Assume that* $n \equiv r \pmod q$, *where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q - 1$.

(b1) *For a point* $x := kq + i$, *where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \leq i \leq r$, $\{x\}$ *is semi-preopen* (=β-*open*). *Especially, if* $2 \leq r$, *then* $\{x\}$ *is semi-preclosed* (=β-*closed*).

(b2) *For a point* $x := kq+j$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$, $\{x\}$ *is semi-closed and so semi-preclosed* (=β-*closed*).

Proof. **(b1)** By using Theorem 2.13 for the point $x = kq+i$ $(k \in \mathbb{Z}, 1 \leq i \leq r)$, it is obtained that $Cl(Int(Cl(\{kq + i\}))) = Cl(Int([(k - 1)q + r + 1, kq + q]_{\mathbb{Z}})) = Cl([kq + 1, kq + r]_{\mathbb{Z}}) = [(k - 1)q + r + 1, kq + q]_{\mathbb{Z}} \supset \{kq + i\}$; so $\{x\}$ is semi-preopen (cf. (3.7), (3.5) in Section 3). We shall show that if $2 \leq r$ then the singleton $\{kq + i\}$ is semi-preclosed, where $1 \leq i \leq r$. Since $Ker(\{kq + i\}) = [kq + 1, kq + r]_{\mathbb{Z}}$ (cf. Theorem 2.13 (b1)), we have that if $2 \leq r$ then $Int(\{kq + i\}) = \emptyset$ and so $Int(Cl(Int(\{kq + i\}))) = \emptyset \subset \{kq + i\}$; therefore, $\{x\}$ is semi-preclosed (cf. (3.11) in Section 3).

(b2) Using Theorem 2.13 (b2)′ for the point $x = kq + j$ $(k \in \mathbb{Z}, r + 1 \leq j \leq q)$, we have $Cl(\{kq+j\}) = [kq+r+1, kq+q]_{\mathbb{Z}}$. Moreover, by using Theorem 2.13 (b2), it is shown that $Int([kq+r+1, kq+q]_{\mathbb{Z}}) = \emptyset$ and hence $Int(Cl(\{x\})) = \emptyset \subset \{x\}$. Namely, the singleton $\{x\}$ is semi-closed; it is semi-peclosed (cf. (3.7), (3.5) and (3.11) in Section 3). □

Theorem 6.2 *Let* $(\mathbb{Z}, \kappa(q, n))$ *be a generalized digital line* (cf. Definition 2.2). *Assume that* $n \equiv r \pmod q$ *and* $1 \leq r \leq q - 1$.

(1) (T_i*-axioms, where* $i \in \{2, 1, 3/4, 1/2, 0\}$; cf. (6.1),(6.2)).

(1-1) *If* $2 \leq r \leq q - 1$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is not a* T_0*-space.*

(1-2) *If* $r = 1$ *and* $q = 2$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is a* $T_{3/4}$*-space and so it is a* $T_{1/2}$*-space; it is not a* T_1*-space* (cf. [10, Definition 4, Example 4.6]).

(1-3) *If* $r = 1$ *and* $3 \leq q$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is not a* T_0*-space.*

(2) (*Semi-*T_i*-separation axioms, where* $i \in \{2, 1, 1/2, 0\}$; cf. (6.4), (6.5).

(2-1) *If* $r = 1$ *and* $2 \leq q$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is a semi-*T_2*-space.*

(2-2) *If* $2 \leq r \leq q - 1$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is not a semi-*T_0*-space.*

(3) (*Pre-*T_i*-separation axioms, where* $i \in \{2, 1\}$; cf. (6.4), (6.5)).

(3-1) *If* $r = 1$ *and* $2 \leq q$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is not a pre-*T_1*-space.*

(3-2) *If* $2 \leq r \leq q - 1$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is a pre-*T_2*-space.*

(4) (β*-*T_i*-separation axioms, where* $i \in \{2, 1, 1/2\}$; cf. (6.4), (6.5)).

$(\mathbb{Z}, \kappa(q, n))$ *is a* β*-*T_2*-space.*

(5) (*Semi-pre-*$T_{1/2}$*-space*; cf. (6.3))

(5-1) *If* $1 \leq r \leq q - 2$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is not semi-pre-*$T_{1/2}$.

(5-2) *If* $1 \leq r = q - 1$, *then* $(\mathbb{Z}, \kappa(q, n))$ *is semi-pre-*$T_{1/2}$.

Proof. **(1) (1-1)** Assume that $n \equiv r \pmod q$, where $2 \leq r$ and $r \leq q-1$. Let $x := kq+1 \in \mathbb{Z}$ and $y := kq + r \in \mathbb{Z}$ for some integer k. We have $x \neq y$ because of $r \neq 1$. By Theorem 2.13 (b1) for the point x (resp. y), $Ker(\{x\})$ (resp. $Ker(\{y\})$) is the smallest open set containing x (resp. y). And, since $Ker(\{x\}) = [kq + 1, kq + r]_{\mathbb{Z}} = Ker(\{y\})$ hold, $y \in Ker(\{x\})$

and $x \in Ker(\{y\})$; and hence $(\mathbb{Z}, \kappa(q,n))$ is not a T_0 space, where $n \equiv r \pmod{q}$ and $2 \leq r \leq q-1$.

(1-2) We assume that $q = 2$; and we claim that $(\mathbb{Z}, \kappa(2,n))$ is a $T_{3/4}$-space and it is not T_1, where $q = 2 < n$ and $n \equiv 1 \pmod{2}$. First, by using Corollary 2.14 for $q = 2, 2 < n$ and $n' = 3$, it is shown that $\kappa(2,n) = \kappa(2,3)$ holds, since $n \equiv 3 \pmod{2}$, $q = 2 < 3$ and $q = 2 < n$. Thus, $(\mathbb{Z}, \kappa(2,n))$ is $T_{3/4}$ and it is not T_1, since it is well known that the digital line $(\mathbb{Z}, \kappa) = (\mathbb{Z}, \kappa(2,3))$ is $T_{3/4}$ (cf. [10, Example 4.6]) and it is not T_1. Finally, we note that an alternative proof is given by using Theorem 2.13; we can claim that every singleton $\{x\}$ is closed or regular open (cf. (6.2) above, [10, Theorem 4.3]) and some singleton is not closed. Indeed, by Theorem 2.13 (b2)$'$ for $j = 2 = r+1$ and assumptions that $q = 2 = r+1$, it is shown that a singleton $\{k2+2\}$ is closed, where $k \in \mathbb{Z}$. For a singleton $\{k2+1\}$, it is regular open, where $k \in \mathbb{Z}$; its proof is as follows. By using Theorem 2.13 (b1) (b2) (resp. (b1)$'$) and assumption that $q = 2 = r+1$, it is shown that $Ker(\{k2\}) = [(k-1)2+1, k2+1]_{\mathbb{Z}}, Ker(\{k2+1\}) = \{k2+1\}$ and $Ker(\{k2+2\}) = [k2+1, k2+3]_{\mathbb{Z}}$ (resp. $Cl(\{k2+1\}) = [k2, k2+2]_{\mathbb{Z}}$) hold; and so $Int([k2, k2+2]_{\mathbb{Z}}) = \{k2+1\}$. Thus, we have that $Int(Cl(\{k2+1\})) = \{k2+1\}$; and hence the singleton $\{k2+1\}$ is regular open. And, the above singleton $\{k2+1\}$ is not closed.

(1-3) We assume that $3 \leq q$ and $r = 1$. Let $x := kq+j \in \mathbb{Z}$, where $2 \leq j \leq q$ and $y := kq+j' \in \mathbb{Z}$, where $2 \leq j' \leq q$ and $j \neq j'$ for some integer k. We have $x \neq y$, because of $3 \leq q$ and $j \neq j'$. By Theorem 2.13 (b2) for $r = 1, Ker(\{x\}) = Ker(\{y\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ is the smallest open set containing x and also it is the smallest open set containing y. Thus, $(\mathbb{Z}, \kappa(q,n))$ is not a T_0-space, where $n \equiv 1 \pmod{q}$, $q < n$ and $3 \leq q$.

(2) (2-1) We first use Theorem 3.2 (b1) and (b2) for $r = 1$. For each ordered pair (x, y) of distinct points x and y, we take disjoint semi-open sets U_x and U_y containing x and y, respectively, as follows: let k, k', j and j' be integers such that $2 \leq j \leq q$ and $2 \leq j' \leq q$.

Case 1. $x = kq+1, y = kq+j$, where $2 \leq j \leq q$: $U_x := \{x\}, U_y := V_2(y) = \{y\} \cup \{(k+1)q+1\}$ (cf. Theorem 3.2 (b1), (b2)(b2-1)).

Case 2. $x = kq+1, y = k'q+1$, where $k \neq k'$: $U_x := \{x\}, U_y := \{y\}$ (cf. Theorem 3.2 (b1)).

Case 3. $x = kq+1, y = k'q+j$, where $2 \leq j \leq q, k \neq k'$: $U_x := \{x\}, U_y := V_1(y) = \{y\} \cup \{k'q+1\}$ (cf. Theorem 3.2 (b1), (b2)(b2-1)).

Case 4. $x = kq+j, y = kq+j'$, where $2 \leq j \leq q, 2 \leq j' \leq q$ and $j \neq j'$: $U_x := V_1(x) = \{x\} \cup \{kq+1\}, U_y := V_2(y) = \{y\} \cup \{(k+1)q+1\}$ (cf. Theorem 3.2 (b2)(b2-1)). Notice: for $q = 2, x = y$; Case 4 above is removed from the proof for $q = 2$.

Case 5. $x = kq+j, y = k'q+j'$, where $2 \leq j \leq q, 2 \leq j' \leq q$ and $k \neq k'$: $U_x := V_1(x) = \{x\} \cup \{kq+1\}, U_y := V_1(y) = \{y\} \cup \{k'q+1\}$ (cf. Theorem 3.2 (b2)(b2-1)).

These properties above conclude that $(\mathbb{Z}, \kappa(q,n))$ is a semi-T_2-space, where $q < n, n \equiv 1$ (mod q) and $q \geq 2$.

(2-2) Under assumption that $2 \leq r \leq q-1$, we can take two singletons $\{x\} := \{kq+1\}$ and $\{y\} := \{kq+r\}$, where $k \in \mathbb{Z}$, such that $x, y \in sKer(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}} \in SO(\mathbb{Z}, \kappa(q,n))$, where $i \in \mathbb{Z}$ with $1 \leq i \leq r$ (cf. Theorem 3.2 (b1)). Then, for every semi-open sets U_x and U_y containing x and y respectively, we have that $x \in [kq+1, kq+r]_{\mathbb{Z}} = sKer(\{y\}) \subset U_y$ and $y \in U_x$ hold. Thus, $(\mathbb{Z}, \kappa(q,n))$ is not semi-T_0.

(3) (3-1) We show that $(\mathbb{Z}, \kappa(q,n))$ is not a pre-T_1-space if $r = 1$ and $2 \leq q$. We use Theorem 4.2 (b1-1)$'$ for $r = 1$; $pCl(\{kq+1\}) = [(k-1)q+2, kq+q]_{\mathbb{Z}}$ holds and so there exists a point $kq+1$ such that $\{kq+1\}$ is not preclosed. Namely, $(\mathbb{Z}, \kappa(q,n))$ is not pre-T_1, where $q < n$ and $n \equiv 1 \pmod{q}$ (cf. (6.5)).

(3-2) We shall prove that $(\mathbb{Z}, \kappa(q,n))$ is pre-T_2 if $2 \leq r \leq q-1$. We recall that for a point $kq+j \in \mathbb{Z}$, $V_{h,h'}(kq+j) := \{kq+j\} \cup \{kq+h, kq+q+h'\}$ is a preopen set containing the point $kq+j$, where $k \in \mathbb{Z}, r+1 \leq j \leq q, 1 \leq h \leq r$ and $1 \leq h' \leq r'$ (cf. Theorem 4.2 (b2)(b2-1)); moreover, for a point $kq+i \in \mathbb{Z}, \{kq+i\}$ is a preopen set, where $1 \leq i \leq r$ (cf. Theorem 4.2 (b1)(b1-1)). Under the assumption that $2 \leq r \leq q-1$, we have that

$kq+1 \neq kq+r$ and

$(*)$ $V_{1,1}(kq+j) \cap V_{r,r}(k'q+j') = \emptyset$ for two distinct points $kq+j$ and $k'q+j'$ with $r+1 \leq j \leq q$ and $r+1 \leq j' \leq q$ (we assume $j \neq j'$ if $k = k'$).
We claim that any two distinct points, say x and y, are separated by preopen sets containing the points respectively.

Case 1. $x = kq+j$ and $y = k'q+j'$, where $j, j' \in [r+1, q]_{\mathbb{Z}}$ and $j \neq j'$ if $k = k'$: for these points x and y, we put $U_x := V_{1,1}(kq+j)$ and $U_y := V_{r,r}(k'q+j')$. Then, by $(*)$ above, it is shown that $U_x \cap U_y = \emptyset$.

Case 2. $x = kq+i$ and $y = k'q+j'$, where $i \in [1,r]_{\mathbb{Z}}$ and $j' \in [r+1,q]_{\mathbb{Z}}$: for these points x and y, we put $U_x := \{kq+i\} \in PO(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 4.2 (b1)(b1-1)) and $U_y := V_{r,r}(k'q+j')$ if $i = 1$ and $U_y := V_{1,1}(k'q+j')$ if $i \neq 1$. Then, it is directly shown that $kq+i \notin U_y$ and so $U_x \cap U_y = \emptyset$.

Case 3. $x = kq+i$ and $y = k'q+i'$, where $i, i' \in [1,r]_{\mathbb{Z}}$ and $i \neq i'$ if $k = k'$: for these points x and y, we put $U_x := \{kq+i\} \in PO(\mathbb{Z}, \kappa(q,n))$ and $U_y := \{k'q+i'\} \in PO(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 4.2 (b1)(b1-1)). Then, it is obvious that $U_x \cap U_y = \emptyset$.

Therefore, for each case it is shown that $x \in U_x, y \in U_y, U_x \cap U_y = \emptyset$ and U_x and U_y are preopen in $(\mathbb{Z}, \kappa(q,n))$ and so $(\mathbb{Z}, \kappa(q,n))$ is pre-T_2.

(4) By (2)(2-1) above, $(\mathbb{Z}, \kappa(q,n))$ is semi-T_2 if $r = 1$ and $2 \leq q$; and so it is β-T_2 (cf. (6.6)). By (3)(3-2) above, $(\mathbb{Z}, \kappa(q,n))$ is pre-T_2 if $2 \leq r \leq q-1$; and so it is β-T_2 (cf. (6.6)).

(5)(5-1) Under assumption that $1 \leq r \leq q-2$, a singleton $\{kq+j\}$ is not closed, where $r+1 \leq j \leq q$. Indeed, $Cl(\{kq+j\}) = [kq+r+1, kq+q]_{\mathbb{Z}} \neq \{kq+j\}$, because $r+1 < q$ (cf. Theorem 2.13 (b2)$'$). And, the singleton $\{kq+j\}$ is not preopen, where $r+1 \leq j \leq q$ (cf. Theorem 5.1 (i)). Thus, there exists a singleton which is neither closed nor preopen and so this generalized digital line $(\mathbb{Z}, \kappa(q,n))$ is not semi-pre-$T_{1/2}$ (cf. (6.3), i.e. [9, Theorem 4.1]).

(5-2) Let x be a point of $\mathbb{Z}$. If $x = kq+j$, where $r+1 = j = q$, then $Cl(\{kq+j\}) = \{kq+j\}$ (cf. Theorem 2.13 (b2)$'$); if $x = kq+i$, where $1 \leq i \leq r = q-1$, then $\{x\}$ is preopen (cf. Theorem 4.2 (b1)(b1-1)). Thus, this generalized digital line $(\mathbb{Z}, \kappa(q,n))$ is semi-pre-$T_{1/2}$ (cf. (6.3), i.e., [9, Theorem 4.1]). □

Proof of Theorem A(ii) The result (ii-1) is obtained by Theorem 6.2 (3)(3-2) above; the result (ii-2) is obtained by Theorem 6.2 (2)(2-1) and (1)(1-2) above. □

Let us present the tables of separation axioms of $(\mathbb{Z}, \kappa(q,n))$(cf. Definition 2.2).

Table 1. Separation axioms of $(\mathbb{Z}, \kappa(q,n))$ for the case where $q < n$ and $n \equiv r \pmod q$ $(1 \leq r \leq q-1)$

r, q	T_i-axioms	semi-T_i-axioms/pre-T_i-axioms	β-T_i-axioms
$r = 1, q = 2$	$T_{3/4}$, Non T_1	semi-T_2 / Non pre-T_1	β-T_2
$r = 1, q \geq 3$	Non T_0	semi-T_2 / Non pre-T_1	β-T_2
$2 \leq r \leq q-1$	Non T_0	Non semi-T_0 / pre-T_2	β-T_2

Table 2. Semi-pre-$T_{1/2}$ separation axioms of $(\mathbb{Z}, \kappa(q,n))$ for the case where $q < n$ and $n \equiv r \pmod q$ $(1 \leq r \leq q-1)$

r, q	semi-pre-$T_{1/2}$-axiom
$r = 1, q = 2$	semi-pre-$T_{1/2}$
$r = 1, q \geq 3$	Non semi-pre-$T_{1/2}$
$2 \leq r \leq q-2$	Non semi-pre-$T_{1/2}$
$2 \leq r = q-1$	semi-pre-$T_{1/2}$

7 The connectedness of generalized digital lines and Proof of Theorem A(iii)

We recall the following: a topological space (X, τ) is said to be *semi-connected* ([7]) (resp.

preconnected ([41])), if it cannot be represented as the disjoint union of two nonempty semi-open (resp. preopen) subsets. The class of *semi-connected* (resp. *preconnected*) topological spaces was introduced by Phullenda Das [7] (resp. Popa [41]) in 1974 (resp. 1987).

Theorem 7.1 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Suppose that* $n \equiv r \pmod q$, *where* $1 \le r \le q-1$. *Then,*
(i) $(\mathbb{Z}, \kappa(q,n))$ *is connected;*
(ii) $(\mathbb{Z}, \kappa(q,n))$ *is not semi-connected;*
(iii) *if* $2 \le r$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is not preconnected;*
(iv) *if* $r = 1$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is preconnected.*

Proof. **(i)** Suppose that $(\mathbb{Z}, \kappa(q,n))$ is not connected; i.e., there exists a nonempty open and closed subset U such that $U \ne \mathbb{Z}$. We shall show a contradiction (cf. $(*5)$, $(*6)$ below). Since $U \ne \emptyset$, we pick a point x of $\mathbb{Z}$ such that
$\cdot(*1)$ $x \in U$; let $x := kq + s$, where $k \in \mathbb{Z}$ and $s \in \mathbb{Z}$ with $1 \le s \le q$.

First, using above integer "k" of $x := kq + s$ $(1 \le s \le q)$, we construct the following sequences of points, $\{x_a\}_{a\in\mathbb{N}}$ and $\{x_a^-\}_{a\in\mathbb{N}}$ defined by:
$\cdot(*2)$ $x_a := (k+a)q$ and $x_a^- := (k-a+1)q$ for each $a \in \mathbb{N}$. Then, it is easily shown that: for each $a \in \mathbb{N}$,
$\cdot(*3)$ $x_a < x_{a+1},\ x_{a+1}^- < x_a^-$ and $x < x_a$ (if $a \ge 2$), $x \le x_1,\ x_a^- < x$.

Secondly, we claim that: for each $a \in \mathbb{N}$,
$\cdot(*4)^a$ $[x, x_a]_{\mathbb{Z}} \subset U$ and $\cdot(**4)^a$ $[x_a^-, x]_{\mathbb{Z}} \subset U$.

Proof of $(*4)^a$. The proof is done by induction on $a \in \mathbb{N}$. For $a = 1$, we show $(*4)^1$. Indeed, by Theorem 2.13 (b1)$'$ (resp. (b2)$'$), it is shown that if the point x has a form $x = kq + i$ $(1 \le i \le r)$ (resp. $x = kq + j$ $(r+1 \le j \le q)$) then $[x, x_1]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq+i\}) \subset U$ (resp. $[x, x_1]_{\mathbb{Z}} \subset [kq + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq + j\}) \subset U$) hold, because $x \in U$ and U is closed.

We suppose that $(*4)^t$ is true for an integer $t \in \mathbb{N}$ with $t \ge 2$, i.e., $[x, x_t]_{\mathbb{Z}} \subset U$, where $x_t = (k+t)q$ (cf. $(*2)$ above) and $t \ge 2$. We use Theorem 2.13 (b2) for the point $x_t = (k+t-1)q + j$, where $j = q$, and the assumption of induction, we have $Ker(\{x_t\}) = [(k+t-1)q + 1, (k+t)q + r]_{\mathbb{Z}} \subset U$ because $x_t \in U$ and U is open; and so $(k+t)q + r \in U$. By using Theorem 2.13 (b1)$'$ for the above point $(k+t)q + r \in U$, it is shown that $Cl(\{(k+t)q + r\}) = [(k+t-1)q + r + 1, (k+t)q + q]_{\mathbb{Z}} \subset U$, because U is a closed subset such that $(k+t)q + r \in U$. Thus, we prove that $(k+t+1)q \in U$ (i.e., $x_{t+1} \in U$) and $[x_t, x_{t+1}]_{\mathbb{Z}} \subset [(k+t-1)q + r + 1, (k+t+1)q]_{\mathbb{Z}} = Cl(\{(k+t)q + r\}) \subset U$. Since $[x, x_{t+1}]_{\mathbb{Z}} = [x, x_t]_{\mathbb{Z}} \cup [x_t, x_{t+1}]_{\mathbb{Z}}$, we have that $[x, x_{t+1}]_{\mathbb{Z}} \subset U$ holds. Namely, we have the required property $(*4)^a$ for $a = t+1$. Thus, for any integer $a \in \mathbb{N}$, we have $(*4)^a$. ◇

Proof of $(**4)^a$. The proof is also done by induction on $a \in \mathbb{N}$ as follows. For $a = 1$, the property $(**4)^1$ is true. Indeed, if $x = kq + i$ $(1 \le i \le r)$, then $[x_1^-, x]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq+q]_{\mathbb{Z}} = Cl(\{kq+i\}) = Cl(\{x\}) \subset U$ hold (cf. Theorem 2.13 (b1)$'$); and so $[x_1^-, x]_{\mathbb{Z}} \subset U$. If $x = kq + j$ $(r+1 \le j \le q)$, then $Ker(\{x\}) = [kq+1, (k+1)q + r]_{\mathbb{Z}} \subset U$ (cf. Theorem 2.13 (b2)); and so $kq + 1 \in U$. By using Theorem 2.13 (b1)$'$ for the point $kq+1$ above, it is shown that $x_1^- = kq \in [x_1^-, x]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq+1\}) \subset U$; and so $[x_1^-, x]_{\mathbb{Z}} \subset U$ hold.

We suppose that $(**4)^t$ is true for an integer $t \in \mathbb{N}$ with $t \ge 2$, i.e., $[x_t^-, x]_{\mathbb{Z}} \subset U$, where $x_t^- = (k-t+1)q$ (cf. $(*2)$ above) and $t \ge 2$. We see $(k-t)q + 1 \in U$. Indeed, using Theorem 2.13(b2) for the point $x_t^- = (k-t)q + j'$ with $j' = q$ and the assumption of induction, we have $(k-t)q + 1 \in [(k-t)q+1, (k-t+1)q + r]_{\mathbb{Z}} = Ker(\{(k-t)q + q) = Ker(\{x_t^-\}) \subset U$ and so $(k-t)q + 1 \in U$. Now, by using Theorem 2.13 (b1)$'$ for the above point $(k-t)q+1$, it is shown that $Cl(\{(k-t)q + 1\}) = [(k-t-1)q + r + 1, (k-t)q + q]_{\mathbb{Z}} \subset U$. Thus, for the point $x_{t+1}^- := (k-t)q$, we prove that $[x_{t+1}^-, x_t^-]_{\mathbb{Z}} \subset [(k-t-1)q + r + 1, (k-t+1)q]_{\mathbb{Z}} \subset U$ hold. Since $[x_{t+1}^-, x]_{\mathbb{Z}} = [x_{t+1}^-, x_t^-]_{\mathbb{Z}} \cup [x_t^-, x]_{\mathbb{Z}}$, we have that $[x_{t+1}^-, x]_{\mathbb{Z}} \subset U$ holds. Namely, we have the required property $(**4)^a$ for $a = t+1$. Thus, for any integer $a \in \mathbb{N}$, we have that $(**4)^a$ is true. $\diamond$

Finally, we proceed the proof as follows: take a point $y \in \mathbb{Z}$ such that
$\cdot(*5)$ $y \notin U$, because $U \neq \mathbb{Z}$; and let $y = s_0 q + i_0$, where $s_0 \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}$ with $1 \leq i_o \leq q$. Then, we consider the following two cases.

Case 1. $x < y$: for this case, using the sequence of points $\{x_a\}_{a\in\mathbb{N}}$ investigated by $(*2), (*3)$ and $(*4)$, we can pick a point $x_{t(0)}$ with $t(0) \in \mathbb{N}$ such that $y \leq x_{t(0)}$. Indeed, we take the integer $t(0)$ as $t(0) := s_0 - k + 1$ (cf. the integer k is given in $(*1)$ above); then $t(0) \geq 1$ and $y = s_0 q + i_0 \leq (s_0+1)q = (t(0)+k-1+1)q = (k+t(0))q = x_{t(0)}$ (cf. $(*2)$ above); and so $x < y < x_{t(0)}$. By $(*4)^a$ above, it is shown that $y \in [x, x_{t(0)}]_{\mathbb{Z}} \subset U$; and so $y \in U$.

Case 2. $y < x$: for this case, using the sequence of points $\{x_a^-\}_{a\in\mathbb{N}}$ investigated by $(*2), (*3)$ and $(*4)$, we can pick a point $x_{t(1)}^-$ with $t(1) \in \mathbb{N}$, such that $x_{t(1)}^- \leq y$. Indeed, we take the integer $t(1)$ as $t(1) := k - s_0 + 1$; then $t(1) \geq 1$ and $y = s_0 q + i_0 > s_0 q = (k - t(1) + 1)q = x_{t(1)}^-$ (cf. $(*2)$ above); and so $x_{t(1)}^- < y < x$. By $(**4)^a$ above, it is shown that $y \in [x_{t(1)}^-, x]_{\mathbb{Z}} \subset U$; and so $y \in U$.

By both cases above, it is obtained that:
$\cdot(*6)$ $y \in U$ holds for the point $y \notin U$ (cf. $(*5)$ above).
This shows a contradiction; therefore, $(\mathbb{Z}, \kappa(q,n))$ is a connected topological space, where $n \equiv r \pmod{q}$ with $1 \leq r \leq q-1$.

(ii) For $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2) and a point $x := kq + i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$, it is known that $sKer(\{x\}) = sCl(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ and $sKer(\{x\})$ is a nonempty semi-open proper subset of $(\mathbb{Z}, \kappa(q,n))$ and $sCl(\{x\})$ is semi-closed in $(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 3.2 (b1) and (b1)$'$). Therefore, $(\mathbb{Z}, \kappa(q,n))$ is not semi-connected.

(iii) For $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2) and a point $x := kq + i$ ($k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$), $pKer(\{x\}) = \{x\}$ holds and it is preopen (cf. Theorem 4.2 (b1)(b1-1)); if $2 \leq r$, then $\{x\}$ is preclosed (cf. Theorem 4.2 (b1)(b1-1)$'$). Thus, the singleton $\{x\}$ is a preopen and preclosed in $(\mathbb{Z}, \kappa(q,n))$ if $2 \leq r$; and so $(\mathbb{Z}, \kappa(q,n))$ is not preconnected if $2 \leq r$.

(iv) We assume that $n \equiv r \pmod{q}$ and $r = 1$. In order to prove that $(\mathbb{Z}, \kappa(q,n))$ is preconnected, we suppose that there exists a preopen and preclosed subset V such that $V \neq \emptyset$ and $V \neq \mathbb{Z}$. Since $V \neq \emptyset$, we pick a point $x \in \mathbb{Z}$ such that
$\cdot(*7)$ $x \in V$; let $x := kq + s$, where $k \in \mathbb{Z}$ and $s \in \mathbb{Z}$ with $1 \leq s \leq q$.
Using the above integer "k" of $x := kq + s$ ($1 \leq s \leq q$), let $\{x_a\}_{a\in\mathbb{N}}$ and $\{x_a^-\}_{a\in\mathbb{N}}$ be the similar sequences of points (cf. $(*2)$ in the proof of (i) above) defined by:
$\cdot(*8)$ $x_a := (k+a)q$ and $x_a^- := (k-a+1)q$ for each $a \in \mathbb{N}$. And, they have the following same properties:
$\cdot(*9)$ $x_a < x_{a+1},\ x_{a+1}^- < x_a^-$ and $x < x_a$ (if $a \geq 2$), $x \leq x_1,\ x_a^- < x$ hold.

We first claim that: under the assumption that $x := kq+s \in V$ for some s with $1 \leq s \leq q$,
$\cdot(*10)$ $kq + 1 \in V$ holds; and
$\cdot(*11)$ $[x, x_1]_{\mathbb{Z}} \subset V$ and $[x_1^-, x]_{\mathbb{Z}} \subset V$ hold.

Proof of $(*10)$. If $x = kq+s$, where $s = 1$, then $kq+1 \in V$ (cf. $(*7)$ above). If $x = kq+s \in V$, where $2 \leq s \leq q$, we use Theorem 4.2 (b2)(b2-3) for the point $kq + j$, where $j = s$ and $2 \leq j \leq q$; and so we have $pKer(\{kq+s\}) = V_{1,1}(kq+s) = \{kq+1, kq+s, (k+1)q+1\} \subset V$, because V is preopen and $x := kq + s \in V$; thus $kq + 1 \in V$. $\diamond$

Proof of $(*11)$. Using Theorem 4.2 (b1)(b1-1)$'$ for the point $kq + 1$, we have $[x, x_1]_{\mathbb{Z}} \subset [(k-1)q+2, (k+1)q]_{\mathbb{Z}} = pCl(\{kq+1\}) \subset V$, because V is preclosed and $kq + 1 \in V$ (cf. $(*10)$ above). For the points $x_1^- = kq$ and $x = kq + s$ ($1 \leq s \leq q$), we see that $[x_1^-, x]_{\mathbb{Z}} \subset pCl(\{kq+1\}) \subset V$. $\diamond$

Secondly, we claim that: for each $a \in \mathbb{N}$,
$\cdot(*12)^a$ $[x, x_a]_{\mathbb{Z}} \subset V$ and $\cdot(**12)^a$ $[x_a^-, x]_{\mathbb{Z}} \subset V$ hold.

Proof of $(*12)^a$. We shall use induction on a. The former part of $(*11)$ above shows that the case where $a = 1$ is true. We suppose the statement $(*12)^a$ for the case where $a = t > 1$ is true; then $[x, x_t]_{\mathbb{Z}} \subset V$. By Theorem 4.2 (b2)(b2-1) and (b2-3) for the point $x_t = (k+t-1)q+j \in V$, where $j = q$, it is shown that $pKer(\{x_t\}) = V_{1,1}((k+t-1)q+q) = \{(k+t-1)q+1, x_t, (k+t-1)q+q+1\}$; and so $(k+t)q+1 \in V$ holds, because $pKer(\{x_t\}) \subset V$. For the point $(k+t)q+1 \in V$, we use Theorem 4.2 (b1)(b1-1)$'$; then, we have $[x_t, x_{t+1}]_{\mathbb{Z}} = [(k+t)q, (k+t+1)q]_{\mathbb{Z}} \subset [(k+t-1)q+2, (k+t+1)q]_{\mathbb{Z}} = pCl(\{(k+t)q+1\}) \subset V$; and so $[x_t, x_{t+1}]_{\mathbb{Z}} \subset V$ hold. Since $[x, x_{t+1}]_{\mathbb{Z}} = [x, x_t]_{\mathbb{Z}} \cup [x_t, x_{t+1}]_{\mathbb{Z}}$, we show that $[x, x_{t+1}]_{\mathbb{Z}} \subset V$ holds. Therefore, by induction on a, the statement $(*12)^a$ is proved. ◇

The property $(**12)^a$ is proved by argument similar to that in the proof of $(*12)^a$ above; and so it is omitted. ◇

Finally, we shall find the following contradiction (cf. $(*14)$ bellow). There exists a point $y \in \mathbb{Z}$ such that:
$\cdot(*13)$ $y \notin V$, because $V \neq \mathbb{Z}$; and let $y = s_0q+i_0$, where $s_0 \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}$ with $1 \leq i_0 \leq q$. Since $x \neq y$, we have the following two cases:

Case 1. $x < y$: for this case, we pick the following point x_b such that $x_b \geq y$, where $b := s_0 - k + 1$. Indeed, we have that $b \geq 1$ and $x_b = (k+b)q = s_0q + q \geq y$ hold. By $(*12)^a$ for $a = b$, it is shown that $y \in [x, x_b]_{\mathbb{Z}} \subset V$; and so $y \in V$.

Case 2. $y < x$: for this case, we pick the following point x_d^- such that $x_d^- < y$, where $d := k - s_0 + 1$. Indeed, we have that $d \geq 1$ and $x_d^- = (k-d+1)q = s_0q < y$ hold, because $1 \leq i_0 \leq q$. By $(**12)^a$ for $a = d$, it is shown that $y \in [x_d^-, x]_{\mathbb{Z}} \subset V$; and so $y \in V$.

By the both cases above, it is obtained that:
$\cdot(*14)$ $y \in V$ holds for the point $y \notin V$(cf. $(*13)$ above). This $(*14)$ shows a contradiction; therefore, $(\mathbb{Z}, \kappa(q,n))$ is preconnected, where $n \equiv 1 \pmod q$ (i.e. $r = 1$). □

Proof of Theorem A(iii) The proof is shown by Theorem 7.1 (i) above. □

We present the table of connectedness of $(\mathbb{Z}, \kappa(q,n))$ from Theorem 7.1.

Table. The connectedness of $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2)

n, q		connectedness; semi-connectedness; preconnectedness
$n \equiv r \pmod q (1 \leq r \leq q-1)$	$\Rightarrow$	connected; non semi-connected
$n \equiv r \pmod q \ (2 \leq r \leq q-1)$	$\Rightarrow$	connected; non preconnected
$n \equiv 1 \pmod q$	$\Rightarrow$	preconnected

Acknowledgements. The authors would like express their gratitude to the referee for his/her helpful comments and useful suggestions. And, the authors would like to express their gratitude to Professor T. Fukutake for his kind discussion with the authors on Lemma 2.8 and also to Professor S.E. Han for his deep observations on the concept of generalized digital lines and for his further discussions with the authors on its applications (cf. [20]).

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Communicated by *Yasunao Hattori*

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JOINT TOPOLOGICAL DIVISORS AND NONREMOVABLE IDEALS IN A COMMUTATIVE REAL BANACH ALGEBRA

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Received August 6, 2015

Abstract.

The concept of joint topological zero divisors (JTZD) in a real Banach algebra was discussed in [4]. In this paper we study the concepts of cortex, Šilov boundary and non-removable ideals and relating them with ideals consisting of JTZD.

1 Introduction and Preliminaries The concepts of ideals consisting of JTZD, cortex and non removable ideals for a complex Banach algebra are studied in detail [5, 6, 7, 8]. Here we extend some of these results for a real Banach algebra. We have modified certain concepts and used the complexification technique to prove some results which was applied effectively in [3].

Throughout the paper, A denotes a real commutative Banach algebra with identity, $Car\,(A)$ and $\mathfrak{M}\,(A)$ denote the space of all nonzero (real) homomorphisms from A to $\mathbb{C}$ called the carrier space and the space of all maximal ideals of A respectively. We refer to [5] and [3] for the basic definitions.

Definition 1.1. Let A be a real Banach algebra with identity 1 and $cxA = \{(a, b) : a, b \in A\}$. Then with the following operations, cxA becomes a complex algebra with identity $(1, 0)$.

$$\left.\begin{array}{l}(a,b)+(c,d)=(a+c,b+d)\\ (\alpha+i\beta)\,(a,b)=(\alpha a-\beta b,\alpha b+\beta a)\\ (a,b)\,(c,d)=(ac-bd,ad+bc)\end{array}\right\}\begin{array}{c} for\;\; all\;\; a,b,c,d\in A\\ \alpha,\beta\in\mathbb{R}\end{array}$$

It is called the *complixification* of A. Further, there exists a norm $\|\cdot\|_{cxA}$ on cxA [3], making cxA a Banach algebra and satisfying,

(i) $\max\left(\|a\|, \|b\|\right) \leq \|(a, b)\|_{\mathrm{cx}A} \leq 2\max\left(\|a\|, \|b\|\right)$ *for all* $a, b \in A$.

(ii) $\|(a, 0)\|_{\mathrm{cx}A} = \|a\|$ *for all* $a \in A$.

2010 *Mathematics Subject Classification.* 46J10, 46J20 .
Key words and phrases. Joint TZD, Cortex, Non-removable ideals .

Note that $a \to (a, 0)$ embeds A into cxA isometrically. Now onwards we use $\|(a, b)\|$ instead of $\|(a, b)\|_{\mathrm{cx}A}$.

We associate $Car\,(cxA)$ and $\mathfrak{M}\,(cxA)$ with A. The following diagram (Figure 1) shows their interrelations.

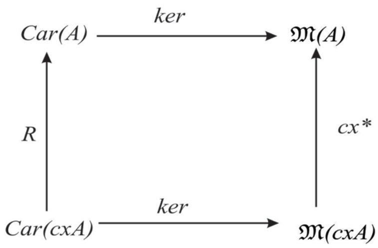

Figure 1:

We list the properties of the maps shown in the diagram.

(i) $R : Car\,(cxA) \to Car\,(A)$ defined as $R\,(\psi) = \psi_{/A}$, is a one-to-one, onto, continuous and open mapping.

(ii) $cx^* : \mathfrak{M}\,(cxA) \to \mathfrak{M}\,(A)$ defined by $cx^*\,(M) = M \bigcap A$ is a two to one, onto continuous and open mapping. Also, $cx^*\,(\Gamma\,(cxA)) = \Gamma\,(A)$ where, $\Gamma\,(A)$ denote the Šilov boundary of A [3].

(iii) $\ker : Car\,(A) \to \mathfrak{M}\,(A)$ defined by $\psi \mapsto \ker \psi$ is a two to one, onto, continuous mapping [3].

(iv) If A is a complex Banach algebra, then the map ker is a one to one mapping.

Further, we define, $\sigma : cxA \to cxA$ by $\sigma\,(f, g) = (f, -g)$. Then σ is a linear map which is also isometry.

We shall need the next proposition to prove the main result.

Proposition 1.2. *If N is a closed ideal in A, then N_{cxA} is a closed ideal in cxA where, $N_{cxA} = \{(x, y) : x, y \in N\}$. Further if N is maximal, then N_{cxA} is contained in exactly two maximal ideals of cxA namely $\ker \psi$ and $\ker\left(\bar{\psi} \circ \sigma\right)$, where $\psi = R^{-1}\,(\phi)$, $\overline{\psi}(x) = \overline{\psi(x)}$ and $N = \ker \phi$.*

Proof. It is easy to verify that N_{cxA} is a closed ideal in cxA. Let $N \in \mathfrak{M}\,(A)$. Then, $N = \ker \phi$ for some $\phi \in Car\,(A)$. Note that $\ker \phi = \ker \bar{\phi}$ and if $R^{-1}\,(\phi) = \psi$, then

$R^{-1}\left(\bar{\phi}\right) = \bar{\psi} \circ \sigma$.

Claim 1: $N_{cxA} = \ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right)$.

Let $(x, y) \in N_{cxA}$ with $x, y \in N$. Then $\phi(x) = \phi(y) = 0 = \bar{\phi}(x) = \bar{\phi}(y)$, which implies $\psi(x, y) = \phi(x) + i\phi(y) = 0$ and $\left(\bar{\psi}\circ\sigma\right)(x, y) = \bar{\phi}(x) + i\bar{\phi}(y) = 0$. Hence, $(x, y) \in \ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right)$. Thus, $N_{cxA} \subset \ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right)$.

Conversely, if $(x, y) \in \ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right)$, then $0 = \psi(x, y) = \phi(x) + i\phi(y)$ and $0 = \left(\bar{\psi}\circ\sigma\right)(x, y) = \bar{\phi}(x) + i\bar{\phi}(y)$. So, $\phi(x) - i\phi(y) = 0$. Therefore, $\phi(x) = 0 = \phi(y)$. Hence, $x, y \in N$ and so, $(x, y) \in N_{cxA}$. Therefore, $\ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right) \subset N_{cxA}$. Hence, $N_{cxA} = \ker\psi \bigcap \ker\left(\bar{\psi}\circ\sigma\right)$.

Claim 2: N_{cxA} is contained in only two maximal ideals namely $\ker\psi$ and $\ker\left(\bar{\psi}\circ\sigma\right)$.

Suppose $N_{cxA} \subset M'$, where $M' \in \mathfrak{M}(cxA)$, then $M' = \ker\psi'$ for some $\psi' \in Car(cxA)$. Let $\phi' = \psi'_{|A} = R(\psi')$. Then, we show that $\ker\phi = \ker\phi'$.

Let $x \in \ker\phi = N$. Then $(x, x) \in N_{cxA} \subset M'$. So, $\psi'(x, x) = 0$, i.e., $\phi'(x) + i\phi'(x) = 0$. Hence, $\phi'(x) = 0$. Thus, $x \in \ker\phi'$. Hence, $\ker\phi \subset \ker\phi'$. Therefore, $\ker\phi = \ker\phi'$ as both of them are maximal ideals in A. So, $\phi = \phi'$ or $\bar{\phi} = \phi'$. Hence, $\psi = \psi'$ or $\bar{\psi}\circ\sigma = \psi'$. □

2 Joint topological zero divisor In this section, we have defined joint topological zero divisor for a real Banach algebra. Also, we have proved some results similar to that of complex Banach algebras [6].

Definition 2.1. Let A be a real commutative Banach algebra. A subset S of A is said to be consisting of *joint topological zero divisors* (JTZD) if for every finite subset $\{x_1, ..., x_n\}$ of S

$$d(x_1, ..., x_n) = \inf\left\{\sum_{i=1}^{n} \|x_i z\| : z \in A, \|z\| = 1\right\} = 0.$$

Equivalently, there exists a net (z_α) in A with $\|z_\alpha\| = 1$ such that $\lim_{\alpha} x z_\alpha = 0$ for each $x \in S$ [4]. In particular, if S is an ideal, then it is called an ideal consisting of JTZD. Note that if $S = \{x\}$, then the above definition coincides with topological zero divisor.

Theorem 2.2. *If A is a real commutative Banach algebra and $I \subset A$ is a nonzero ideal consisting of JTZD, then there exists a maximal ideal N in A consisting of JTZD and $I \subset N$.*

To prove the above result we need the following lemmas.

Lemma 2.3. *If I is an ideal in A consisting of JTZD, then $I_{cxA} = \{(x, y) : x, y \in I\}$ is an ideal in cxA consisting of JTZD.*

Proof. As we have noted in Proposition 1.2, I_{cxA} is an ideal in cxA. To show that I_{cxA} consists of JTZD, let $(x, y) \in I_{cxA}$. Then $x, y \in I$. Since, I consists of JTZD, there exists a net (x_α) in A with $\|x_\alpha\| = 1$ such that $\|xx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_x$ and $\|yx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_y$. Let $\alpha_\varepsilon \geq \alpha_x$ and $\alpha_\varepsilon \geq \alpha_y$. Then $\|xx_\alpha\| < \frac{\varepsilon}{2}$ and $\|yx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_\varepsilon$.

Consider $z_\alpha = (x_\alpha, 0)$. Then, (z_α) is a net in cxA. Also, $\|z_\alpha\| = \|(x_\alpha, 0)\| = \|x_\alpha\| = 1$ and $\|z_\alpha (x, y)\| = \|(x_\alpha x, x_\alpha y)\| \leq 2 \max (\|x_\alpha x\|, \|x_\alpha y\|) < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\lim\limits_\alpha z_\alpha (x, y) = 0$ for each $(x, y) \in I_{cxA}$. Hence, I_{cxA} consists of JTZD. □

Lemma 2.4. *If J is an ideal in cxA consisting of JTZD, then $J \bigcap A$ is an ideal in A consisting of JTZD.*

Proof. Clearly, $I = J \bigcap A$ is an ideal in A. Let $x \in I$. Then, $(x, 0) \in J$. Therefore, there exists a net $(z_\alpha)_{\alpha\in\Lambda}$ in cxA with $\|z_\alpha\| = 1$ such that $\|z_\alpha (x, 0)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$.

Let $z_\alpha = (x_\alpha, y_\alpha)$. Then $\|(x_\alpha, y_\alpha)(x, 0)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Therefore, $\|(x_\alpha x, y_\alpha x)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\max (\|x_\alpha x\|, \|y_\alpha x\|) \leq \|(x_\alpha x, y_\alpha x)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Hence, $\|x_\alpha x\| < \varepsilon$ and $\|y_\alpha x\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\lim\limits_\alpha x_\alpha x = 0$ and $\lim\limits_\alpha y_\alpha x = 0$.

Now, $\max (\|x_\alpha\|, \|y_\alpha\|) \leq \|z_\alpha\| = 1 \leq 2 \max (\|x_\alpha\|, \|y_\alpha\|)$ for each α. Therefore, $\frac{1}{2} \leq \max (\|x_\alpha\|, \|y_\alpha\|) \leq 1$ for each $\alpha \in \Lambda$.

Let

$$z_{\alpha'} = \begin{cases} \frac{x_\alpha}{\|x_\alpha\|}, if \ \|x_\alpha\| \geq \frac{1}{2} \\ \frac{y_\alpha}{\|y_\alpha\|}, if \ \|x_\alpha\| < \frac{1}{2} \end{cases}$$

It is clear that $\{z_{\alpha'}\}$ is a net of norm one and $\lim\limits_\alpha z_{\alpha'} x = 0$. Hence, I consists of JTZD. □

Proof. (Theorem 2.2) Let I consist of JTZD. Then by Lemma 2.3, I_{cxA} consists of JTZD. Hence, there exists a maximal ideal M in cxA consisting of JTZD such that $I_{cxA} \subset M$ [6]. Let $N = M \bigcap A$. Then by Lemma 2.4, N is in A and it consists of JTZD, and $I \subset N$. This N is the required maximal ideal. □

3 Cortex The concept of cortex for a complex Banach algebra has been studied in [5]. The cortex for a complex Banach algebra A is defined as a subset of $Car(A)$. Here, we define the cortex slightly in a different manner.

Definition 3.1. Let A be a real commutative Banach algebra with identity. The set $\{M \in \mathfrak{M}(A) : M \text{ consists of JTZD}\}$ is called the *cortex* of A and is denoted by $Cor\,(A)$.

Note that for a complex Banach algebra A, $Cor\,(A)$ can also be looked upon as a subset of $Car\,(A)$ as $Car\,(A) \cong \mathfrak{M}(A)$. Here we have considered cortex of a complex Banach algebra A as a subset of $\mathfrak{M}(A)$. The following result for a real Banach algebra A follows immediately from the result of §2.

Theorem 3.2. *$cx^*\,(Cor\,(cxA)) = Cor\,(A)$. Consequently $Cor\,(A)$ is a nonempty compact subset of $\mathfrak{M}(A)$.*

Corollary 3.3. *$\Gamma(A) \subset Cor\,(A)$.*

Proof. $\Gamma(A) = cx^*\,(\Gamma(cxA))$[3]$\subset cx^*\,(Cor\,(cxA))$[5]$= Cor\,(A)$. □

Lemma 3.4. *Let $\psi \in Car\,(cxA)$. Then $\ker\psi \in Cor\,(cxA)$ if and only if $\ker\left(\bar{\psi}\circ\sigma\right) \in Cor\,(cxA)$.*

Proof. Let $(f,g) \in cxA$. Then,$(f,g) \in \ker\psi \Leftrightarrow \psi(f,g) = 0 \Leftrightarrow \bar{\psi}(f,g) = 0$
$\Leftrightarrow \left(\bar{\psi}\circ\sigma\right)(f,-g) = 0 \Leftrightarrow (f,-g) \in \ker\left(\bar{\psi}\circ\sigma\right)$.

Let $\ker\psi \in Cor\,(cxA)$. To show that $\ker\left(\bar{\psi}\circ\sigma\right) \in Cor\,(cxA)$, let $(f_i,g_i) \in \ker\left(\bar{\psi}\circ\sigma\right)$ for $i = 1,...,n$. Therefore, $(f_i,-g_i) \in \ker\psi$ for $i = 1,...,n$. But $\ker\psi$ consists of JTZD. Hence, for given $\varepsilon > 0$ there exists $(x,y) \in cxA$ with $\|(x,y)\| = 1$ such that

$$\sum_{k=1}^{n} \|(f_k,-g_k)(x,y)\| < \varepsilon.$$

Now, $\|(f_k,-g_k)(x,y)\| = \|(f_k,g_k)(x,-y)\|$ as $\sigma(f,g) = (f,-g)$ is an isometry. So, $\sum_{k=1}^{n}\|(f_k,g_k)(x,-y)\| < \varepsilon$. Hence, $\ker\left(\bar{\psi}\circ\sigma\right) \in Cor\,(cxA)$.

The converse follows from the fact $\overline{\bar{\psi}\circ\sigma}\circ\sigma = \psi$. □

Remark 3.5. *If we consider $F = \ker^{-1}(\Gamma(A))$ and $K = \ker^{-1}(Cor\,(A))$, then it is clear from the definition of $\Gamma(A)$ that $\ker|_F$ is also two to one onto $\Gamma(A)$. The following result shows that $\ker|_K$ is also two to one onto $Cor\,(A)$.*

Proposition 3.6. $R\left(\ker^{-1}\left(Cor\left(cxA\right)\right)\right) = \ker^{-1}\left(Cor\left(A\right)\right)$

Proof. Let $\psi \in \ker^{-1}\left(Cor\left(cxA\right)\right)$. Then $\ker\psi \in Cor\left(cxA\right)$. Now, $R\left(\psi\right) = \psi_{|A} = \phi$. To prove $\phi \in \ker^{-1}\left(Cor\left(A\right)\right)$, we have to show that $\ker\phi \in Cor\left(A\right)$. Now, $\ker\phi = \ker\psi \bigcap A$. Therefore, by Lemma 2.4, $\ker\phi$ consists of JTZD. Hence, $\phi \in \ker^{-1}\left(Cor\left(A\right)\right)$.

Conversely, let $\phi \in \ker^{-1}\left(Cor\left(A\right)\right)$. Then $\ker\phi = N \in Cor\left(A\right)$. Then, by Lemma 2.3, N_{cxA} consists of JTZD. Hence, there exists a maximal ideal $M \in Cor\left(cxA\right)$ such that $N_{cxA} \subset M$. But N_{cxA} is contained in only two maximal ideals, $\ker\psi$ and $\ker\left(\bar{\psi}\circ\sigma\right)$. Therefore, either $\ker\psi$ or $\ker\left(\bar{\psi}\circ\sigma\right)$ consists of JTZD. So, by Lemma 3.4 in any case, $\ker\psi \in Cor\left(cxA\right)$. Therefore, $R\left(\ker^{-1}\left(Cor\left(cxA\right)\right)\right) = \ker^{-1}\left(Cor\left(A\right)\right)$. □

4 Extension and Non-removable ideals In this section, we characterize the cortex of a real Banach algebra. For this, we define the concepts of extensions and non-removable ideals for a real Banach algebra. Also, we have shown that the smallest complex extension for a real Banach algebra is its complexification.

Definition 4.1. Let A and B be Banach algebras. We say that B is an *extension* of A if there exists an isometrical into isomorphism $\rho : A \to B$. In this case, we write $A \subset B$.

Theorem 4.2. *Let A be a real commutative Banach algebra.*
(i) If B is a real extension of A, then cxB is an extension of cxA with an equivalent norm.
(ii) If B is a complex extension of A, then B is also an extension of cxA, i.e., cxA is the smallest complex extension of A.

Proof. (i) Let B be a real extension of A. Then there exists a real into isometrical isomorphism $\rho : A \to B$. Define $\rho' : cxA \to cxB$ by $\rho'\left(a,b\right) = \left(\rho\left(a\right), \rho\left(b\right)\right)$. Then it is easy to check that ρ' is an algebra homomorphism. Further,$\left\|\rho'\left(a,b\right)\right\| = \left\|\left(\rho\left(a\right), \rho\left(b\right)\right)\right\|$
$\leq 2\max\left(\left\|\rho\left(a\right)\right\|, \left\|\rho\left(b\right)\right\|\right) = 2\max\left(\left\|a\right\|, \left\|b\right\|\right) \leq 2\left(\left\|\left(a,b\right)\right\|\right)$ and $\left\|\left(a,b\right)\right\| \leq 2\max\left(\left\|a\right\|, \left\|b\right\|\right)$
$= 2\max\left(\left\|\rho\left(a\right)\right\|, \left\|\rho\left(b\right)\right\|\right) \leq 2\left(\left\|\left(\rho\left(a\right), \rho\left(b\right)\right)\right\|\right) = 2\left\|\rho'\left(a,b\right)\right\|$.

Hence, $\frac{1}{2}\left\|\left(a,b\right)\right\| \leq \left\|\rho'\left(a,b\right)\right\| \leq 2\left(\left\|\left(a,b\right)\right\|\right)$. Therefore, there exists an algebra norm $\left|\left|\left|\cdot\right|\right|\right|$ on cxB equivalent to the above norm on cxB such that $\left|\left|\left|\rho'\left(a,b\right)\right|\right|\right| = \left\|\left(a,b\right)\right\|$ for every $\left(a,b\right) \in cxA$ [5]. Hence, cxB is an extension of cxA.

(ii) Let B be a complex extension of A. Then $cxB \cong B$. So, as in part (1), we get B is an extension of cxA. □

Definition 4.3. An ideal I in a commutative Banach algebra A is called *non-removable*, if in every commutative Banach algebra $B \supset A$, there exists a proper ideal J of B such that $J \supset I$.

We shall need the following lemma.

Lemma 4.4. *If I is non-removable in A, then I_{cxA} is non-removable in cxA.*

Proof. Let I be a non-removable ideal in A. To show that I_{cxA} is non-removable in cxA, let B be an extension of cxA. Then B is also an extension of A. Therefore, there exists a proper ideal $J \subset B$ such that $I \subset J$. So, $I_{cxA} \subset J$. Hence, I_{cxA} is non-removable in cxA. □

Theorem 4.5. *An ideal in a real commutative Banach algebra is non-removable if and only if it consists of JTZD.*

Proof. Let A be a real commutative Banach algebra and I be an ideal consisting of JTZD. Then there exists a net (z_α) in A with $\|z_\alpha\| = 1$ and $\lim_\alpha xz_\alpha = 0$ for every $x \in I$.

Let $B \supset A$ be a commutative extension of A and let

$$J = \{x_1b_1 + ... + x_nb_n : x_1, ..., x_n \in I, b_1, ..., b_n \in B, n \in \mathbb{N}\}$$

be the smallest ideal in B containing I. Suppose J is not proper. Then $1 \in J$. Therefore, there exists $x_1, ..., x_n \in I$ and $b_1, ..., b_n \in B$ such that $\sum_{k=1}^{n} x_kb_k = 1$.

Then, $1 = \|z_\alpha\| = \|\sum_{k=1}^{n} z_\alpha x_k b_k\| \leq \sum_{k=1}^{n} \|z_\alpha x_k\| \, \|b_k\| \to 0$ a contradiction. Hence, J is proper and so I is non-removable.

Conversely, let I be a non-removable ideal in A. Then I_{cxA} is non-removable ideal in cxA by the above Lemma. Therefore, I_{cxA} consists of JTZD [5]. Hence, $I = I_{cxA} \bigcap A$ also consists of JTZD by Lemma 2.4. □

The next theorem gives characterization of $Cor(A)$.

Theorem 4.6. *Let A be a real commutative Banach algebra and $\phi \in Car\,(A)$. Then the following statements are equivalent:*

(i) $\ker \phi \in Cor\,(A)$.

(ii) For every commutative Banach algebra $B \supset A$, there exists a multiplicative linear functional $\psi \in Car\,(B)$ such that $\phi = \psi_{|A}$.

(iii) For every commutative Banach algebra $B \supset A$, there exists a multiplicative linear functional ψ such that $\ker \psi \in Cor\,(B)$ *and $\phi = \psi_{|A}$.*

Proof. First we prove $(i) \Rightarrow (iii)$. Let $\ker \phi \in Cor\,(A)$. Then, there exists a net (z_α) in A with $\|z_\alpha\| = 1$ and $\lim\limits_{\alpha} x z_\alpha = 0$ for every $x \in \ker \phi$. Let B be a commutative Banach algebra with $B \supset A$ and $I = \{y \in B : y z_\alpha \to 0\}$. Then $I \supset \ker \phi$ and I consists of JTZD in B, so by Theorem 2.2, there exists a maximal ideal J consists of JTZD in B such that $I \subset J$. Let $J = \ker \psi$. Then $\psi_{|A} = \phi$.

$(iii) \Rightarrow (ii)$ is clear.

Finally, we prove $(ii) \Rightarrow (i)$ If $B \supset A$ and $\psi \in Car\,(B)$ extends ϕ, then $\ker \phi \subset \ker \psi$. Hence, $\ker \phi$ is a non-removable ideal in A. Hence, by Theorem 4.5, $\ker \phi$ consists of JTZD. Therefore, $\ker \phi \in Cor\,(A)$. $\square$

Acknowledgements: This research is supported by the SAP programme to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar by UGC.

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Communicated by *Anthony To-Ming Lau*

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CRITERIA FOR THE $\tilde{C}$-INTEGRAL

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Received March 3, 2015 ; revised May 20, 2015

Abstract. The C-integral was introduced by B. Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. B. Bongiorno, Di Piazza and Preiss gave some criteria for the C-integrability. The $\tilde{C}$-integral was introduced by D. Bongiorno as a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the improper Newton integral. She gave some criteria for the $\tilde{C}$-integrability. On the other hand, Nakanishi gave some criteria for the restricted Denjoy integrability. Kawasaki and Suzuki gave new criteria for the C-integrability in the style of Nakanishi. In this paper we will give new criteria for the $\tilde{C}$-integrability in the style of Nakanishi.

1 Introduction Throughout this paper we denote by $(\mathbf{L})(S)$ and $(\mathbf{D}^*)(S)$ the class of all Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a,b]$ into $(0,\infty)$ and a δ-fine McShane partition is a collection $\{(I_k,x_k) \mid k=1,\dots,k_0\}$ of non-overlapping intervals $I_k \subset [a,b]$ satisfying $I_k \subset (x_k-\delta(x_k), x_k+\delta(x_k))$ and $\sum_{k=1}^{k_0}|I_k| = b-a$. If $\sum_{k=1}^{k_0}|I_k| \le b-a$, then we say that the collection is a δ-fine partial McShane partition. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a,b]$ into $\mathbb{R}$ such that $F'=f$ on $[a,b]$. We denote by $(\mathbf{N})([a,b])$ the class of all Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [3] B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the Newton integral. Furthermore in [1–3] B. Bongiorno et al. gave some criteria for the C-integrability. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is improper Newton integrable if there exist a countable subset $N \subset [a,b]$ and a function F from $[a,b]$ into $\mathbb{R}$ such that $F'=f$ on $[a,b]\setminus N$. We denote by $(\mathbf{N}^*)([a,b])$ the class of all improper Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [4] D. Bongiorno gave a minimal constructive integration process of Riemann type which contains the Lebesgue integral and the improper Newton integral. It is given as follows:

Definition 1.1. A function f from an inteval $[a,b]$ into $\mathbb{R}$ is said to be $\tilde{C}$-integrable if there exist a countable subset $N \subset [a,b]$ and a number A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k,x_k) \mid k=1,\dots,k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$$

2010 *Mathematics Subject Classification.* 26A36, 26A39.

Key words and phrases. $\tilde{C}$-integral, C-integral, Denjoy integral, Lebesgue integral, improper Newton integral, Newton integral.

and $x_k \in N$ implies $x_k \in I_k$. The constant A is denoted by

$$A = (\tilde{C}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{C})([a,b])$ the class of all $\tilde{C}$-integrable functions from $[a,b]$ into $\mathbb{R}$.

Furthermore in [4] D. Bongiorno gave some criteria for the $\tilde{C}$-integrability. On the other hand, in [9, 12] Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, new criteria were considered in [8] for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$. In this paper, motivated by the results above, we will give new criteria for the $\tilde{C}$-integrability in the style of Nakanishi.

2 Preliminaries In [4] D. Bongiorno gave a criterion for the $\tilde{C}$-integrability. By the theorem $(\tilde{C})([a,b])$ is the minimal class which contains $(\mathbf{L})([a,b])$ and $(\mathbf{N}^*)([a,b])$. Moreover it is contained in $(\mathbf{D}^*)([a,b])$. In this paper we refer to the following theorems given by D. Bongiorno [4].

Theorem 2.1. *Let $f \in (\tilde{C})([a,b])$. Then there exists a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (\tilde{C}) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$$

and $x_k \in N$ implies $x_k \in I_k$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a,b]$ is an interval function on $[a,b]$. If an interval function F on $[a,b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a,b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive.

Definition 2.1. Let F be an interval function on $[a,b]$. Then F is said to be $\tilde{C}$-absolutely continuous on $E \subset [a,b]$ if for any positive number ε there exist a countable subset $N \subset E$, a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in N$ implies $x_k \in I_k$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_{\tilde{C}}(E)$ the class of all $\tilde{C}$-absolutely continuous interval functions on E. Moreover F is said to be $\tilde{C}$-generalized absolutely continuous on $[a,b]$ if there exists a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a,b]$ and $F \in \mathbf{AC}_{\tilde{C}}(E_m)$ for any m. We denote by $\mathbf{ACG}_{\tilde{C}}([a,b])$ the class of all $\tilde{C}$-generalized absolutely continuous interval functions on $[a,b]$.

Theorem 2.2. *For any $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in (\tilde{C})([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (\tilde{C}) \int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in (\tilde{C})([a,b])$ satisfies $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$.

On the other hand, in [9, 12] Nakanishi gave the following criteria for the restricted Denjoy integrability. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in (\mathbf{L})(F_n)$ for any n;

(3) $\left| \sum_{k=1}^{k_0} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in (\mathbf{L})(F_n)$ for any n;

(4) $\left| \sum_{k=1}^{k_0} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in (\mathbf{L})(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a, b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

Next Nakanishi gave the following theorem for the restricted Denjoy integrability.

Theorem 2.3. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a, b]$ such that the pair of f and F satisfies one of* (A), (B), (C) *and* (D). *Moreover, if the pair of f and F satisfies one of* (A), (B), (C) *and* (D), *then*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Motivated by the results of Nakanishi, in [8] Kawasaki and Suzuki gave similar criteria and theorem as Theorem 2.3 for the C-integrability.

3 Main results Firstly we consider the following four criteria for the pair of a function f from $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$.

$(A)_{\tilde{C}}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0+1, \ldots, k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in N$ implies $x_k \in I_k$.

$(B)_{\tilde{C}}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap M_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying

(4.1) $x_k \in M_n$ for any $k = k_0+1, \ldots, k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(4.3) $x_k \in N$ implies $x_k \in I_k$.

$(C)_{\tilde{C}}$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a countable subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a,b]$ satisfying

(3.1) $x_k \in F_n$ for any k;

(3.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3.3) $x_k \in N$ implies $x_k \in I_k$;

(3.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

$(\mathrm{D})_{\tilde{C}}$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a countable subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a,b]$ satisfying

(4.1) $x_k \in M_n$ for any k;

(4.2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(4.3) $x_k \in N$ implies $x_k \in I_k$;

(4.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(A)_{\tilde{C}}$ implies $(B)_{\tilde{C}}$ and $(C)_{\tilde{C}}$ implies $(D)_{\tilde{C}}$. Now we give the following theorems for the $\tilde{C}$-integrability.

Theorem 3.1. *Let $f \in (\tilde{C})([a,b])$ and let F be the additive interval function on $[a,b]$ defined by*

$$F(I) = (\tilde{C}) \int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies $(A)_{\tilde{C}}$.

Proof. Since $f \in (\tilde{C})([a,b])$, we obtain $f \in (\mathbf{D}^*)([a,b])$. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Since by Theorem 2.3 the pair of f and F satisfies (A), for $\{\frac{\varepsilon_n}{2}\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$. By Theorem 2.1 there exists a countable subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}(f(x_k)|I_k| - F(I_k))\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.2) and (3.3). Since $f\chi_{F_n} \in (\mathbf{L})([a,b])$, by the Saks-Henstock lemma for the McShane integral, for instance see [6, Lemma 10.6], for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$. Since $f = f\chi_{F_n}$ on F_n, for any n there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1}(F(I_k) - f(x_k)|I_k|)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.1), (3.2) and (3.3). Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad \le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n
\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \ne \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying (3.1), (3.2) and (3.3), that is, (3) holds. □

Theorem 3.2. *If the pair of a function f from an interval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies $(\mathrm{A})_{\tilde{C}}$, then the pair of f and F satisfies $(\mathrm{C})_{\tilde{C}}$. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies $(\mathrm{B})_{\tilde{C}}$, then the pair of f and F satisfies $(\mathrm{D})_{\tilde{C}}$.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_{\tilde{C}}$ hold. We show (3) of $(\mathrm{C})_{\tilde{C}}$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n,\varepsilon)$ such that, if $|E| < \rho(n,\varepsilon)$, then

$$\left|(L)\int_{E\cap F_n} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n,\varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n,\varepsilon) \ge n$, and put $\eta = \rho(m(n,\varepsilon),\varepsilon)$. By (3) of $(\mathrm{A})_{\tilde{C}}$ there exists a subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for $m(n,\varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a,b]$ satisfying (3.1), (3.2), (3.3) and (3.4) of $(\mathrm{C})_{\tilde{C}}$. Then we obtain

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0} |I_k| < \eta = \rho(m(n,\varepsilon),\varepsilon)$, we obtain

$$\left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} F(I_k)\right| &\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| + \left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Next we show (4) of $(\mathrm{C})_{\tilde{C}}$. Let I be a subinterval of $[a,b]$. In the case of $I \cap F_n = \emptyset$ (4) of $(\mathrm{C})_{\tilde{C}}$ is clear. Consider the case of $I \cap F_n \ne \emptyset$. Let $\{J_p \mid p = 1, 2, \ldots\}$ be the sequence of all connected components of $I^i \setminus F_n$. Since $I \cap F_m \ne \emptyset$ holds for any $m \ge n$, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| < \varepsilon_m.$$

Since $\overline{J_p} \cap F_m \ne \emptyset$ holds for any p, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left|\sum_{p=1}^{\infty}\left(F(\overline{J_p}) - (L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \le \varepsilon_m$$

for any $m \geq n$. On the other hand, we obtain

$$(L)\int_{I\cap F_m} f(x)dx = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx$$

for any $m \geq n$. Therefore we obtain

$$\begin{aligned}
&\left|F(I) - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p})\right)\right| \\
&\quad \leq \left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| \\
&\qquad + \left|(L)\int_{I\cap F_m} f(x)dx - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \\
&\qquad + \left|-\sum_{p=1}^{\infty} F(\overline{J_p}) + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right| \\
&\quad < \varepsilon_m + 0 + \varepsilon_m = 2\varepsilon_m
\end{aligned}$$

for any $m \geq n$ and hence

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}).$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_{\tilde{C}}$, then the pair of f and F satisfies $(\mathrm{D})_{\tilde{C}}$. □

Theorem 3.3. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies $(\mathrm{D})_{\tilde{C}}$, then $f \in (\tilde{C})([a,b])$ and*

$$F(I) = (\tilde{C})\int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

Proof. By (1) and (4) there exist a countable subset $N \subset [a,b]$ and a increasing sequence $\{M_n\}$ of non-empty measurable sets such that $\bigcup_{n=1}^{\infty} M_n = [a,b]$ and for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a,b]$ satisfying (4.1), (4.2), (4.3) and (4.4). Therefore we obtain

$$\begin{aligned}
\sum_{k=1}^{k_0} |F(I_k)| &= \left|\sum_{F(x_k)>0} F(I_k)\right| + \left|\sum_{F(x_k)<0} F(I_k)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

and hence $F \in \mathbf{ACG}_{\tilde{C}}([a,b])$. By Theorem 2.2 there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $g \in (\tilde{C})([a,b])$ such that

$$F(I) = (\tilde{C}) \int_I g(x)dx$$

for any interval $I \subset [a,b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [14, Theorem (5.1)] $g_n \in (\mathbf{D}^*)(I)$ for any interval $I \subset [a,b]$ and by (3)

$$\begin{aligned} (D^*) \int_I g_n(x)dx &= (D^*) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (D^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (\tilde{C}) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}), \end{aligned}$$

where $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*) \int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a,x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a,x]) = f(x)$ for almost every $x \in [a,b]$. □

By Theorems 3.1, 3.2 and 3.3 we obtain the following new criteria for the $\tilde{C}$-integrability.

Theorem 3.4. *A function f from an interval $[a,b]$ into $\mathbb{R}$ is $\tilde{C}$-integrable if and only if there exists an additive interval function F on $[a,b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_{\tilde{C}}$, $(\mathrm{B})_{\tilde{C}}$, $(\mathrm{C})_{\tilde{C}}$ *and* $(\mathrm{D})_{\tilde{C}}$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_{\tilde{C}}$, $(\mathrm{B})_{\tilde{C}}$, $(\mathrm{C})_{\tilde{C}}$ *and* $(\mathrm{D})_{\tilde{C}}$, *then*

$$F(I) = (\tilde{C}) \int_I f(x)dx$$

holds for any interval $I \subset [a,b]$.

Acknowledgements. The author would like to thank the referee for suggestions and comments.

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Communicated by *Jun Kawabe*

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A SEARCH GAME WITH INCOMPLETE INFORMATION ABOUT TARGET'S ENERGY

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Received June 10, 2015 ; revised August 21, 2015

Abstract. This paper deals with a two-person zero-sum search game called a *search allocation game*, in which a searcher distributes search resource to detect a target and the target moves to evade the searcher. When the searcher starts his search operation for the target, the target happens to stay at some position and have some energy for movement. The target knows the initial state consisting of its initial position and initial energy but the searcher does not. The problem is the game with private information about the target's initial state including initial energy. The payoff of the game is the detection probability of the target. We use a convex programming formulation and a linear programming one for the derivation of an equilibrium, which consists of the value of the game, an optimal distribution of searching resource by the searcher and an optimal movement strategy of the target. By some numerical examples, we analyze players' optimal strategies and evaluate the value of information about the target initial state.

1 Introduction Search theory originates in military operations. Koopman [32], who is a founder of search theory, summarized theoretical results of anti-submarine warfare conducted by US Navy in WW2. He [33] mainly researched one-sided problems, in which only the searcher designs a search plan by estimating the target movement. De Guenin [12] studied an optimal distribution of search efforts by adopting general function as a detection probability of target. Kadane [28] and Onaga [36] considered the criterion of searching cost and Iida et al. [26] researched the search problem of a stationary target based on risk criterion. There are other research focused on stationary targets, such as Gittins [11], who considered the optimal strategies of a stationary hider and a searcher in two regions, and Kress et al. [34], who took account of false detection occurrence in the search. Pollock [37], Schweitzer [40] and Dobbie [7] studied moving target problems in two cells and Saretsalo [39] extended their studies to the problem in a multi-dimensional Euclidean space. Iida [24], Brown [4] and Washburn [43] also studied the moving target problems and devised computational algorithms to derive a searcher's optimal strategy in a general way.

Subsequently, research of search theory progressed to search game including not only a searcher but also a target as a decision maker. Game theory is usually categorized into cooperative game and non-cooperative game. Non-cooperative search game has two kinds of models: search-evasion game (SEG) and search allocation game (SAG). In both models, the target uses the moving strategy in the search space but the searcher's strategies are different. In the SEG, the searcher has the moving strategy as well as the target, but in the SAG [17], he distributes searching resource in the search space to detect the target.

We list up the past research on the SEG as follows. Danskin [6] formulated the search game between an anti-submarine-helicopter and a submarine as a datum search game and

2010 *Mathematics Subject Classification.* 90B40, 91A95, 90C25.
Key words and phrases. Search theory, Game theory, Mathematical programming.

derived an optimal dipping position of the helicopter's sonar. Washburn [42] discussed a multi-stage game, in which the searcher makes a decision of the next position to go after knowing the location of a target at every time. Kan [29] took the searcher's search cost as a payoff in a differential game. Nakai [35] studied an optimal target motion on a line with a safety zone. Kikuta [31] investigated a SEG with the criterion of the traveling cost of the searcher. There are other SEG models such as Eagle [8], Eagle and Washburn [9], Hohzaki and Iida [19], Isler et al. [27], Zora et al. [44], Bhattachary [3] and Stipanovic et al. [41].

For the research on the SAG, Iida et al. [25] handled a two-person-zero-sum search game, in which a mobile target chooses a path and a searcher distributes a limited amount of search efforts. Hohzaki et al. [20] and Hohzaki [13] clarified the relationship between two SAGs defined in a continuous search space and in a discrete space. Hohzaki and Washburn [23] applied the SAG to a datum search in a continuous time. Ruckle [38] and Baston and Kikuta [2] dealt with a kind of the SAG called the ambush game, where player I chooses a crossing point on the border of a lattice space and player II puts obstacles to intercept player I's crossing. Dambreville and Le Carde [5] and Hohzaki [16] considered the SAG taking account of some attributes of searching resource. Hohzaki [14] and Kekka and Hohzaki [30] considered the search game with false contacts by the searcher. Hohzaki and Ikeda [21] extended the target strategy to the movement with energy supply policy in their SAG.

There are other types of search games, such as Baston and Garnaev [1], Gal and Howard [10] and Hohzaki [15, 18]. Baston and Garnaev [1] discussed a non-zero-sum game with a protector, who protects the target not to be detected by the searcher. Gal and Howard [10] discussed a zero-sum game under the situation that the searcher does not know whether the target wants to be searched or evade. Hohzaki [15] and Hohzaki [18] modeled the SAG with many cooperative searchers and the SAG with two competitive searchers against the target into a cooperative game and a nonzero-sum game, respectively.

Almost all SAGs mentioned above assume that the searcher knows the target initial position and energy. Unlike the past models of the SAG, Hohzaki and Joo [22] first studied a search game with target private information of its initial position. As well as the initial position, target's movement energy would be considered to become a private information of the target in realistic search operations. The importance of the target's energy can be seen in various situations of military and non-military operations. When the artillery fires at a retreating enemy in the long-distance area, a precise inference about residual fuel of retreating vehicles makes the firing effective or successful. In maritime operations and air-defense operations, a precise estimation on the mobility of suspicious boats or aircrafts would bring a good result to the search operation for them following the report of their invasion. In search and rescue operations, a rescue team is required to have a good estimation on the mobility and the possession of foods and fuel of missing persons or vehicles in addition to their missing point and missing time. As mentioned above, it is extremely important to consider the energy or the mobility of moving targets in search games. Of course, the target knows his initial position and energy at the beginning of the search but the searcher would not. By those reasons, this paper aims to develop a searching game model taking account of the uncertainty of the target's initial position and initial energy on the searcher's side and to derive players' rational decision making.

In the next section, we model a search game with two players, a searcher and a target, in which the target initial state consisting of its initial position and initial energy is the target's private information but is unknown to the searcher. We formulate the problem into a two-person game with incomplete information of the target initial state. In Section 3, we derive an equilibrium point, which consists of the value of the game, an optimal distribution strategy of searching resource and an optimal movement strategy of the target, by enumerating all target paths. We can easily imagine the combinatorial explosion for generating all target paths. To cope with the problem, we propose a methodology for another type of equilibrium point by a Markov movement of the target in Section 4. In Section 5, we do some sensitivity analyses on equilibria with respect to some system parameters involved in the model and then we evaluate the value of the information about the target initial state, which indicates to what extent the searcher increases the detection probability of the target by acquiring the information.

2 A One-Shot Game with Uncertainty of Target's Initial State In a competitive search game between a searcher and a target, the searcher starts a search operation if he senses the existence of the target to some extent in many cases. In the cases, the search happens to start for the target and therefore, in the beginning of the search, an initial state of the target, such as his initial position and initial possession of moving energy, has some randomness such that it seems to be given by nature. The target knows his initial state, of course, and the searcher anticipates the state in a probabilistic way based on information from his sensors. Under these situations, we consider a two-person zero-sum (TPZS) search game between the target and the searcher with detection probability of the target as payoff.

(A1) A search space consists of a finite discrete geographic space and a finite discrete time. The geographic space is represented by a set of cells $\boldsymbol{K} = \{1, \cdots, K\}$ and the time space by a set of time points $\boldsymbol{T} = \{1, \cdots, T\}$.

(A2) In the beginning of the search, nature determines an initial state of the target according to a probability law. An initial position $s \in S_0 \subset \boldsymbol{K}$ and an initial energy e_0 of the target have probability distribution $\{f(s), s \in S_0\}$ ($\sum_{s \in S_0} f(s) = 1$) and $\{g(e_0), e_0 \in E_0\}$ ($\sum_{e_0 \in E_0} g(e_0) = 1$), respectively, which are known to both players. S_0 and E_0 are finite countable sets and the random variables given by $f(s)$ and $g(e_0)$ are independent of each other.

(A3) The target moves from its initial position s as time goes by. Its movement has the following constraints. He is allowed to move from cell i at time t to a limited area of cells $N(i, t)$. He consumes energy $\mu(i, j)$ to move from cell i to j. If he exhausts his initial energy e_0, he cannot move anywhere expect for staying at his current cell.

Let us denote all target paths starting from the initial position s until using up energy e_0 by P_{se_0}. The target chooses a path among them and goes along it. If he takes a path $\omega \in P_{se_0}$, he is in cell $\omega(t) \in \boldsymbol{K}$ at time $t \in \boldsymbol{T}$.

(A4) The searcher anticipates the initial cell s and energy e_0 of the target by the probability distribution $\{f(s), s \in S_0\}$ and $\{g(e_0), e_0 \in E_0\}$, respectively, and starts a search operation.

After he is allowed to start the search at time τ, he distributes his searching resource in the space $\boldsymbol{K}$ to detect the target. We denote a time period of search after τ by $\widehat{\boldsymbol{T}} = \{\tau, \tau + 1, \cdots, T\}$. $\Phi(t)$ resources are available and infinitely divisible. Let us denote a distribution plan of resource by $\varphi = \{\varphi(i, t),\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}$, where $\varphi(i, t)$ is the amount of the resource distributed in cell i at time t.

(A5) If the searcher scatters x resources in cell i and the target is there, the searcher detects the target with probability

$$1 - \exp(-\alpha_i x). \tag{1}$$

Parameter α_i indicates the efficiency of detection per unit resource in the cell i.

If the searcher detects the target, the game ends and the searcher obtains reward 1 but the target incurs the same amount of loss.

¿From Assumption (A5), the search game is a TPZS game with detection probability as payoff.

Let us begin with enumerating conditions of a feasible path $\omega \in P_{se_0}$ for the target with its initial position s and energy e_0. We call the initial state (s, e_0) the type of the target. Using notation $e(t)$ which indicates residual energy of the target at time t, we express the feasibility conditions of $\omega \in P_{se_0}$, as follows:

(i) Condition of initial position: $\omega(1) = s$

(ii) Condition of reachable cells: $\omega(t+1) \in N(\omega(t), t),\ t = 1, \cdots, T-1$

(iii) Condition of initial energy: $e(1) = e_0$

(iv) Condition of energy conservation: $e(t+1) = e(t) - \mu(\omega(t), \omega(t+1)),\ t = 1, \cdots, T-1$

(v) Condition of movement energy: $\mu(\omega(t), \omega(t+1)) \leq e(t),\ t = 1, \cdots, T-1$

We generate P_{se_0} for the target of type (s, e_0) by enumerating path ω satisfying the conditions above. Reversely, we can calculate $e(t)$ by $e(t) = e_0 - \sum_{\xi=1}^{t-1} \mu(\omega(\xi), \omega(\xi+1))$ for a specific path ω.

We have a feasible region Ψ for a searcher's strategy φ from Assumption (A4), as follows.

$$\Psi \equiv \left\{ \varphi \;\middle|\; \sum_{i \in \boldsymbol{K}} \varphi(i,t) \leq \Phi(t),\ \varphi(i,t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}} \right\}. \tag{2}$$

Now we are going to formulate the payoff function of the game by using the players' pure strategies φ and $\omega \in P_{se_0}$ of the (s, e_0)-type target. At time t, the target is at cell $\omega(t)$ and $\varphi(\omega(t), t)$ searching resources are distributed there. Therefore, from the expression (1), we have the following function as the payoff.

$$R_{se_0}(\varphi, \omega) = 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right).$$

We denote a mixed strategy of the (s, e_0)-type target by $\pi_{se_0} \equiv \{\pi_{se_0}(\omega),\ \omega \in P_{se_0}\}$, where $\pi_{se_0}(\omega)$ is the probability of taking path ω. The feasible region for the mixed strategy π_{se_0} is

$$\Pi_{se_0} \equiv \left\{ \pi_{se_0} \;\middle|\; \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) = 1,\ \pi_{se_0}(\omega) \geq 0, \omega \in P_{se_0} \right\}. \tag{3}$$

The expected payoff, i.e. the detection probability of target, by a pure strategy φ and a mixed strategy π_{se_0} is given by

$$\begin{aligned} R_{se_0}(\varphi, \pi_{se_0}) &= \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) R_{se_0}(\varphi, \omega) \\ &= \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \left\{ 1 - \exp\left(-\sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\} \end{aligned} \tag{4}$$

$$= 1 - \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right).$$

The (s, e_0)-type target aims to minimize the payoff. The searcher does not know the type of the target with certainty and he has to evaluate his payoff taking account of all strategies of all types of target, $\pi \equiv \{\pi_{se_0}, s \in S_0, e_0 \in E_0\}$, based on the probability distribution $\{f(s), s \in S_0\}$ and $\{g(e_0), e_0 \in E_0\}$, as follows.

$$R(\varphi, \pi) = \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) R_{se_0}(\varphi, \pi_{se_0}) \tag{5}$$

$$= \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\}$$

$$= 1 - \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \sum_{\omega \in P_{se_0}} \pi_{se_0}(\omega) \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right)$$

The searcher wants to maximize the payoff. In the next section, let us discuss the game with the payoff, which looks different at first glance from each side of the searcher and the target, and derive its equilibrium point.

3 Derivation of Equilibrium Point As seen from Eqs. (4) and (5), all types of targets, each of which aims for the minimization of its own payoff $R_{se_0}(\varphi, \pi_{se_0})$, also minimize the comprehensive payoff $R(\varphi, \pi)$ in the aggregate. Therefore, an optimal strategy of the searcher is given by the maximin optimization of $R(\varphi, \pi)$. Let us begin the maximin optimization. We can carry out the minimization of $R(\varphi, \pi)$ with respect to π as follows:

$$\min_{\pi} R(\varphi, \pi) = \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \min_{\omega \in P_{se_0}} R_{se_0}(\varphi, \omega) \tag{6}$$

$$= \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \min_{\omega \in P_{se_0}} \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \right\}$$

by making $\pi_{se_0}(\omega) = 0$ for any path $\omega \notin \Omega^{+se_0} \equiv \{\omega \in P_{se_0} | R_{se_0}(\varphi, \omega) = \min_{p \in P_{se_0}} R_{se_0}(\varphi, p)\}$. Furthermore, the maximization of the above minimized value with respect φ gives us a formulation

$$(P_S^0) \qquad \max_{\varphi, \{\nu_{se_0}\}} \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \nu_{se_0}$$

$$s.t. \qquad 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \right) \geq \nu_{se_0}, \ \omega \in P_{se_0}, \ s \in S_0, e_0 \in E_0,$$

$$\varphi \in \Psi.$$

By the replacement of ν_{se_0} with $\eta_{se_0} \equiv -\log(1 - \nu_{se_0})$, i.e., $\nu_{se_0} \equiv 1 - \exp(-\eta_{se_0})$, and noting $\sum_s f(s) = 1$ and $\sum_{e_0} g(e_0) = 1$, we can equivalently transform the above formulation to

$$(P_S) \qquad \max_{\varphi, \eta} \left\{ 1 - \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \exp(-\eta_{se_0}) \right\}$$

$$\text{(7)} \qquad s.t. \quad \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi(\omega(t), t) \geq \eta_{se_0}, \ \omega \in P_{se_0}, s \in S_0, e_0 \in E_0,$$

$$\text{(8)} \qquad \sum_{i \in \boldsymbol{K}} \varphi(i,t) = \Phi(t), \ t \in \widehat{\boldsymbol{T}},$$

$$\varphi(i,t) \geq 0, \ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}.$$

Because the feasible region of the above problem is a convex set and the objective function is concave, the problem is a convex programming problem. It is easily solved by any general commercial software package of numerical optimization.

Next let us derive an optimal strategy of the (s, e_0)-type target. The target estimates φ^* by solving problem (P_S) first and is going to take an optimal strategy π_{se_0} to minimize his payoff $R_{se_0}(\varphi^*, \pi_{se_0})$ as follows:

$$\begin{aligned}
\min_{\pi_{se_0}} R_{se_0}(\varphi^*, \pi_{se_0}) &= \min_{\omega \in P_{se_0}} R_{se_0}(\varphi^*, \omega) \\
&= \min_{\omega \in P_{se_0}} \left\{ 1 - \exp\left(- \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi^*(\omega(t), t) \right) \right\} \\
&= 1 - \exp\left(- \min_{\omega \in P_{se_0}} \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi^*(\omega(t), t) \right) = 1 - \exp(-v^*_{se_0}),
\end{aligned}$$

where $v^*_{se_0}$ is given by

$$v^*_{se_0} = \min_{\omega \in P_{se_0}} \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi^*(\omega(t), t).$$

Comparing the above equation with Eq. (7), we can see that $v^*_{se_0}$ equals an optimal value $\eta^*_{se_0}$ and $1 - \exp(-\eta^*_{se_0})$ is the minimum detection probability of the (s, e_0)-type target. Using $\eta^*_{se_0}$, we redefine $\Omega^{+se_0} \equiv \{\omega \in P_{se_0} | \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \varphi^*(\omega(t), t) = \eta^*_{se_0}\}$.

Hereafter, we want to carry out the minimax optimization of $R(\varphi, \pi)$ to derive an optimal strategy of the target. However a direct approach to the optimization would be difficult. Instead, we consider the conditions of the target strategy π to which the optimal searcher's strategy φ^* given by (P_S) becomes an optimal response. On the other hand, an optimal response π to φ^* is given by minimizing a linear function $R(\varphi^*, \pi)$ of variable π or equivalently by setting $\pi_{se_0}(\omega) = 0$ for any $\omega \notin \Omega^{+se_0}$, as seen in the transformation (6).

We derive the necessary and sufficient conditions of the optimal response φ^* to π by the so-called Karush-Kuhn-Tucker (KKT) conditions of $\max_\varphi R(\varphi, \pi)$ with respect to $\varphi \in \Psi$. After defining a Lagrange function

$$L(\varphi; \lambda, \eta) \equiv R(\varphi, \pi) + \sum_{t \in \widehat{\boldsymbol{T}}} \lambda(t) \left(\Phi(t) - \sum_{i \in K} \varphi(i,t) \right) + \sum_{(i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}} \eta(i,t) \varphi(i,t)$$

with Lagrangian multipliers $\{\lambda(t), \ t \in \widehat{\boldsymbol{T}}\}$ and $\{\eta(i,t) \geq 0, \ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\}$, we have the KKT conditions as follows:

$$\text{(9)} \qquad \frac{\partial L}{\partial \varphi(i,t)} = \frac{\partial R(\varphi, \pi)}{\partial \varphi(i,t)} - \lambda(t) + \eta(i,t) = \alpha_i \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) \sum_{\omega \in \Omega^{se_0}_{it}} \pi_{se_0}(\omega)$$

$$\times \exp\left(-\sum_{t'\in\widehat{\boldsymbol{T}}} \alpha_{\omega(t')}\varphi(\omega(t'),t')\right) - \lambda(t) + \eta(i,t) = 0,\ (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}},$$

$$\varphi(i,t)\geq 0,\ (i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}},$$

$$\sum_{i\in K}\varphi(i,t) = \Phi(t),\ t\in\widehat{\boldsymbol{T}},$$

$$\eta(i,t)\geq 0,\ (i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}}, \tag{10}$$

$$\eta(i,t)\varphi(i,t) = 0,\ (i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}}. \tag{11}$$

In the conditions, we use notation $\Omega_{it}^{se_0} \equiv \{\omega\in P_{se_0}|\omega(t)=i\}$. From the previous discussion about optimal target strategy and $v^*_{se_0}$, optimal π_{se_0} should be $\pi_{se_0}(\omega)=0$ for $\omega\notin\Omega^{+se_0}$ and the condition $\sum_{t'}\alpha_{\omega(t')}\varphi^*(\omega(t'),t') = \eta^*_{se_0}$ holds for any path $\omega\in\Omega^{+se_0}$ with positive selection probability $\pi_{se_0}(\omega)>0$. Using these expressions, we can rewrite Eq. (9) into

$$\alpha_i\sum_{e_0\in E_0} g(e_0)\sum_{s\in S_0} f(s)\exp(-\eta^*_{se_0})\sum_{\omega\in\Omega_{it}^{+se_0}}\pi_{se_0}(\omega) - \lambda(t)+\eta(i,t) = 0, \tag{12}$$

$$(i,t)\in\boldsymbol{K}\times\widehat{\boldsymbol{T}},$$

where

$$\Omega_{it}^{+se_0} \equiv \{\omega\in P_{se_0}|\ \omega(t)=i,\ \sum_{t'\in\widehat{\boldsymbol{T}}}\alpha_{\omega(t')}\varphi^*(\omega(t'),t') = \eta^*_{se_0}\}.$$

Let us simplify the conditions (9), (10) and (11), as follows. If $\varphi^*(i,t)>0$, we have $\eta(i,t)=0$ from Eq. (11) and then

$$\alpha_i\sum_{e_0\in E_0} g(e_0)\sum_{s\in S_0} f(s)\exp(-\eta^*_{se_0})\sum_{\omega\in\Omega_{it}^{+se_0}}\pi_{se_0}(\omega) = \lambda(t) \tag{13}$$

from Eq. (12). If $\varphi^*(i,t)=0$, from Eq. (10), we have

$$\alpha_i\sum_{e_0\in E_0} g(e_0)\sum_{s\in S_0} f(s)\exp(-\eta^*_{se_0})\sum_{\omega\in\Omega_{it}^{+se_0}}\pi_{se_0}(\omega) \leq \lambda(t). \tag{14}$$

Reversely, if the above two conditions hold, we can easily generate optimal multipliers $\{\lambda^*(t)\}$ and $\{\eta^*(i,t)\}$ so as to satisfy the KKT conditions. Anyway, the original feasibility conditions of π_{se_0} are given by Π_{se_0} of Eq. (3).

We have discussed the conditions of an optimal target strategy $\pi=\{\pi_{se_0}, s\in S_0, e_0\in E_0\}$ so far. If π satisfies all the conditions derived so far, the optimal searcher's strategy φ^* given by problem (P_S) is optimal to π and, at the same time, π is optimal to φ^*. The two-sided optimality validates that π is in an equilibrium of the game. Summing up the discussion so far, we have a linear programming problem to derive an optimal target strategy π, as follows.

$$(P_T)\qquad \min_{\pi,\lambda}\sum_{e_0\in E_0} g(e_0)\sum_{s\in S_0} f(s)\sum_{\omega\in P_{se_0}}\pi_{se_0}(\omega)\left\{1-\exp\left(-\sum_{t\in\widehat{\boldsymbol{T}}}\alpha_{\omega(t)}\varphi^*(\omega(t),t)\right)\right\}$$

s.t.

$$\alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s)\exp(-\eta^*_{se_0}) \sum_{\omega\in\Omega^{+se_0}_{it}} \pi_{se_0}(\omega) = \lambda(t)$$
$$\textit{for } (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}} \textit{ of } \varphi^*(i,t)>0,$$
$$\alpha_i \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s)\exp(-\eta^*_{se_0}) \sum_{\omega\in\Omega^{+se_0}_{it}} \pi_{se_0}(\omega) \le \lambda(t)$$
$$\textit{for } (i,t)\in \boldsymbol{K}\times\widehat{\boldsymbol{T}} \textit{ of } \varphi^*(i,t)=0,$$
$$\sum_{\omega\in P_{se_0}} \pi_{se_0}(\omega)=1,\ s\in S_0,\ e_0\in E_0,$$
$$\pi_{se_0}(\omega)\ge 0,\ \omega\in P_{se_0},\ s\in S_0,\ e_0\in E_0.$$

4 Markov Movement Strategy of Target In Section 3, we enumerate all target paths taking account of the target movement constraints in Assumption (A3) in Section 2 and use the path set $\{P_{se_0}, s\in S_0, e_0\in E_0\}$ to derive an equilibrium point of the game. The proposed formulation is inconvenient for the game with a lot of time points because the number of target paths would increase at an exponential rate of the number of time points. To cope with the exponential expansion of the number of paths, we define a strategy of the target by Markov movement strategy, which was first introduced into search games by Eagle and Washburn [9] and sophisticated by Hohzaki et al. [20]. We represent a state of target by (s,e_0,i,t,e), where (s,e_0) is a target type, and i, t and e are the current cell, the present time and the residual energy, respectively, or a state of the (s,e_0)-type target by (i,t,e). By the Markov strategy, the target specifies the movement from the state (s,e_0,i,t,e) to a cell at the next time $t+1$ in a probabilistic manner. The Markov strategy at time t depends on not the past tracks of path before t but just a state at the present time t. We will show the equivalence between the path selection strategy discussed in the previous section and the Markov movement strategy later in the process of deriving an equilibrium point for the Markov strategy. Anyway, we have to discriminate between the beginning point of time t and the ending point of the time. Because the search operation is conducted between the two points, the target lies at different levels at two time points from the survival point of view even if the target is in the same state. We refer to the former time point as the BP of time t and the latter one as the EP of t.

Let us denote all energy states of the (s,e_0)-type target by $\boldsymbol{F}_{e_0}\equiv\{0,1,\cdots,e_0\}$. To make use of dynamic programming, we define an optimized value $z_{se_0}(i,t,e)$ as the maximized non-detection probability of the (s,e_0)-type target who is in a state (i,t,e) at the BP of time t and takes his optimal movement strategy since then until the end of the game. In this section, we adopt the non-detection probability as the payoff for the sake of formulation. Variable $v_{se_0}(i,j,t,e)$ represents a Markov strategy of the (s,e_0)-type target and indicates the probability that the target in the state (i,t,e) moves to cell j at the next time $t+1$. Let us denote a set of cells to which the target can move at time $t+1$ from (i,t,e) by $N(i,t,e)\equiv\{j\in\boldsymbol{K}|j\in N(i,t),\ \mu(i,j)\le e\}$ and a set of cells at the previous time $t-1$, from which the target can transition to the state (i,t,e) at time t, by $N^*_{e_0}(i,t,e)\equiv\{j\in\boldsymbol{K}|i\in N(j,t-1),\ e+\mu(j,i)\le e_0\}$.

The feasibility conditions of the Markov strategy v_{se_0} are given by

$$(15)\quad V_{se_0}\equiv\Big\{\{v_{se_0}(i,j,t,e), i,j\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{F}_{e_0}\}\ \Big|\ \sum_{j\in N(i,t,e)} v_{se_0}(i,j,t,e)=1,\ v_{se_0}(i,j,t,e)=0(j\notin N(i,t,e)),\ v_{se_0}(i,j,t,e)\ge 0\Big\}.$$

Before the main discussion of deriving an equilibrium, we prove the equivalency between the path selection strategy $\{\pi_{se_0}(\omega)\}$ and the Markov strategy $\{v_{se_0}(i,j,t,e)\}$ of the (s,e_0)-type target. We accomplish the proof by showing that one expression is transformable from the other one as follows, using notation $e(\omega, n) \equiv e_0 - \sum_{k=1}^{n-1} \mu(\omega(k), \omega(k+1))$:

$$\pi_{se_0}(\omega) = \prod_{t=1}^{T-1} v_{se_0}(\omega(t), \omega(t+1), t, e(\omega, t)) \ for\ \omega \in P_{se_0},$$

$$v_{se_0}(i, j, t, e) = \frac{\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e, \omega(t+1)=j\}} \pi_{se_0}(\omega)}{\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e\}} \pi_{se_0}(\omega)}.$$

If $\sum_{\{\omega \in \Omega_{it}^{se_0} | e(\omega,t)=e\}} \pi_{se_0}(\omega)$ becomes zero in the denominator, the state (s, e_0, i, t, e) is not reachable and any Markov strategy $v_{se_0}(i, j, t, e)$ is allowable.

We denote a strategy of the searcher by a distribution of searching resource $\{\varphi(i, t)\}$, as same as in Section 3. Considering the transition that the target in state (i, t, e) remains surviving from the search operation at time t and goes to cell j at the next time $t + 1$, the optimized value $z_{se_0}(i, t, e)$ has the following recursive equation at any time $t \in \widehat{\boldsymbol{T}}$:

$$z_{se_0}(i, t, e) = \max_{\{v_{se_0}(i,j,t,e), j \in N(i,t,e)\}} e^{-\alpha_i \varphi(i,t)} \sum_{j \in N(i,t,e)} v_{se_0}(i, j, t, e) z_{se_0}(j, t+1, e - \mu(i, j)).$$

Taking account of the feasibility condition V_{se_0} of Eq. (15), we further transform the above expression to

$$\begin{aligned} z_{se_0}(i, t, e) &= \max_{j \in N(i,t,e)} e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)) \\ &\geq e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)). \end{aligned} \tag{16}$$

In a similar manner, we have the following equation during a time period $\boldsymbol{T} \backslash \widehat{\boldsymbol{T}}$ with no search operation:

$$\begin{aligned} z_{se_0}(i, t, e) &= \max_{\{v_{se_0}(i,j,t,e), j \in N(i,t,e)\}} \sum_{j \in N(i,t,e)} v_{se_0}(i, j, t, e) z_{se_0}(j, t+1, e - \mu(i, j)) \\ &= \max_{j \in N(i,t,e)} z_{se_0}(j, t+1, e - \mu(i, j)) \geq z_{se_0}(j, t+1, e - \mu(i, j)). \end{aligned} \tag{17}$$

An equation $z_{se_0}(i, T, e) = \exp(-\alpha_i \varphi(i, T))$ holds at the last time T. Because the maximized non-detection probability of the (s, e_0)-type target over an entire time points is given by $z_{se_0}(s, 1, e_0)$ from its definition, the searcher wants to minimize its expectation $\sum_{e_0} \sum_s g(e_0) f(s) z_{se_0}(s, 1, e_0)$ to obtain a minimax value (a maximin value for the original payoff of the detection probability of target). From the discussion so far, we formulate the minimax optimization into the following problem.

$$\begin{aligned} (P_S^{M0}) \quad & \min_{\varphi, z} \sum_{e_0 \in E_0} g(e_0) \sum_{s \in S_0} f(s) z_{se_0}(s, 1, e_0) \\ s.t. \quad & z_{se_0}(i, t, e) \geq z_{se_0}(j, t+1, e - \mu(i, j)), \\ & \qquad j \in N(i, t, e), i \in \boldsymbol{K}, t \in \boldsymbol{T} \backslash \widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & z_{se_0}(i, t, e) \geq e^{-\alpha_i \varphi(i,t)} z_{se_0}(j, t+1, e - \mu(i, j)), \\ & \qquad j \in N(i, t, e), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}} \backslash \{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & z_{se_0}(i, T, e) = e^{-\alpha_i \varphi(i,T)}, \ i \in \boldsymbol{K}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0, \\ & \sum_{i \in K} \varphi(i, t) = \Phi(t), \ t \in \widehat{\boldsymbol{T}}, \\ & \varphi(i, t) \geq 0, \ i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}. \end{aligned}$$

We substitute $w_{se_0}(i,t,e) \equiv -\log z_{se_0}(i,t,e)$ for $z_{se_0}(i,t,e)$ to have a formulation

$$
\begin{aligned}
(P_S^M) \qquad & \min_{\varphi,w} \sum_{e_0\in E_0} g(e_0) \sum_{s\in S_0} f(s)\exp(-w_{se_0}(s,1,e_0)) \\
s.t. \qquad & w_{se_0}(i,t,e) \le w_{se_0}(j,t+1,e-\mu(i,j)), \\
& \qquad j\in N(i,t,e), i\in \boldsymbol{K}, t\in \boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in \boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& w_{se_0}(i,t,e) \le \alpha_i\varphi(i,t) + w_{se_0}(j,t+1,e-\mu(i,j)), \\
& \qquad j\in N(i,t,e), i\in \boldsymbol{K}, t\in \widehat{\boldsymbol{T}}\backslash\{T\}, e\in \boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& w_{se_0}(i,T,e) = \alpha_i\varphi(i,T),\ i\in \boldsymbol{K}, e\in \boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& \sum_{i\in K}\varphi(i,t) = \Phi(t),\ t\in\widehat{\boldsymbol{T}}, \\
& \varphi(i,t)\ge 0,\ i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}.
\end{aligned}
$$

Because $z_{se_0}(i,t,e)$ lies in $0 < z_{se_0}(i,t,e) \le 1$ from its definition, $w_{se_0}(i,t,e)$ is nonnegative. The formulation (P_S^M) is a convex minimization problem. In the formulation, there are some variables with no effect on the optimal value, such as $\{w_{se_0}(i,1,e), i\ne s, e\ne e_0\}$. It might be good to set these variables zeros. The setting corresponds to making variables z_{se_0} 1s in the formulation (P_S^{M0}). The variable setting also does not generate any constraint in the problem and therefore they do not have any effect on the optimal value of (P_S^{M0}) at all.

Hereafter, we are going to derive an optimal Markov strategy of the target by using optimal solutions φ^* and $w^*_{se_0}$ already obtained from problem (P_S^M) and $z^*_{se_0}$ from problem (P_S^{M0}). From the definition of $z^*_{se_0}(i,t,e)$, $\widehat{z}^*_{se_0}(i,t,e) \equiv z^*_{se_0}(i,t,e)\exp(\alpha_i\varphi^*(i,t))$ is the maximum non-detection probability after time t given by an optimal movement of the target conditioned that the (s,e_0)-type target is surviving in state (i,t,e) at the EP of the time t. As the Markov movement strategy, we temporarily adopt variables $\{\widehat{v}_{se_0}(i,j,t,e),\ i,j\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{F}_{e_0}\}$ other than variable v_{se_0} for the expressional sake. $\widehat{v}_{se_0}(i,j,t,e)$ indicates the probability that the (s,e_0)-type target has not been detected since the beginning, is in state (i,t,e) at the EP of time t and moves to cell j at the next time $t+1$. The movement strategy indirectly affects the following probabilities. $q_{se_0}(i,t,e)$ is the probability that the (s,e_0)-type target reaches state (i,t,e) at the BP of t with no detection. $q'_{se_0}(i,t,e)$ is the probability that the (s,e_0)-type target reaches state (i,t,e) at the EP of t with no detection.

Considering the state transition of the (s,e_0)-type target, we have the following equations.

$$
\begin{aligned}
& q_{se_0}(s,1,e_0) = 1, s\in S_0, e_0\in E_0, \\
& \sum_{i\in K}\sum_{e\in F_{e_0}} q_{se_0}(i,1,e) = 1, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = q_{se_0}(i,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = q_{se_0}(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i\in\boldsymbol{K}, t\in\widehat{\boldsymbol{T}}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q_{se_0}(i,t,e) = \sum_{j\in N^*_{e_0}(i,t,e)} \widehat{v}_{se_0}(j,i,t-1,e+\mu(j,i)), \\
& \qquad\qquad i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{1\}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0, \\
& q'_{se_0}(i,t,e) = \sum_{j\in N(i,t,e)} \widehat{v}_{se_0}(i,j,t,e), i\in\boldsymbol{K}, t\in\boldsymbol{T}\backslash\{T\}, e\in\boldsymbol{F}_{e_0}, s\in S_0, e_0\in E_0.
\end{aligned}
$$

Focusing a distribution of searching effort, $\{\varphi(i,t), i\in\boldsymbol{K}\}$, at time t, we have an expression for the non-detection probability.

$$h_t(\varphi) \equiv \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{i \in K} \sum_{e \in F_{e_0}} g(e_0) f(s) q_{se_0}(i,t,e) \exp(-\alpha_i \varphi(i,t)) \widehat{z}^*_{se_0}(i,t,e)$$

The optimal distribution $\{\varphi^*(i,t), i \in \boldsymbol{K}\}$ at time t must be an optimal solution of the minimization problem of the above objective under constraints of $\sum_i \varphi(i,t) = \Phi(t)$ and $\varphi(i,t) \geq 0$ $(i \in \boldsymbol{K})$. After defining a Lagrange function by

$$\begin{aligned} L(\varphi; \lambda, \mu) \quad \equiv \quad & \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{i \in K} \sum_{e \in F_{e_0}} g(e_0) f(s) q_{se_0}(i,t,e) \exp(-\alpha_i \varphi(i,t)) \widehat{z}^*_{se_0}(i,t,e) \\ & + \lambda(t) \left(\sum_{i \in K} \varphi(i,t) - \Phi(t) \right) - \sum_{i \in K} \mu(i,t) \varphi(i,t), \end{aligned}$$

we find KKT conditions as follows.

$$\text{(18)} \qquad \begin{aligned} \frac{\partial L}{\partial \varphi(i,t)} = -\alpha_i \exp(-\alpha_i \varphi(i,t)) & \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{e \in F_{e_0}} g(e_0) f(s) q_{se_0}(i,t,e) \widehat{z}^*_{se_0}(i,t,e) \\ & + \lambda(t) - \mu(i,t) = 0, \ i \in \boldsymbol{K}, \end{aligned}$$

$$\text{(19)} \qquad \mu(i,t) \geq 0, \ i \in \boldsymbol{K},$$

$$\text{(20)} \qquad \mu(i,t)\varphi(i,t) = 0, \ i \in \boldsymbol{K},$$

$$\text{(21)} \qquad \sum_{i \in K} \varphi(i,t) = \Phi(t),$$

$$\text{(22)} \qquad \varphi(i,t) \geq 0, \ i \in \boldsymbol{K}.$$

We reconstruct conditions (18)∼(20) into equivalent conditions:

(i) If $\varphi(i,t) > 0$,

$$\text{(23)} \qquad \alpha_i \exp(-\alpha_i \varphi(i,t)) \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{e \in F_{e_0}} g(e_0) f(s) q_{se_0}(i,t,e) \widehat{z}^*_{se_0}(i,t,e) = \lambda(t).$$

(ii) If $\varphi(i,t) = 0$,

$$\text{(24)} \qquad \alpha_i \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{e \in F_{e_0}} g(e_0) f(s) q_{se_0}(i,t,e) \widehat{z}^*_{se_0}(i,t,e) \leq \lambda(t).$$

The total non-detection probability is expressed by $\sum_{e_0,s,i,e} g(e_0) f(s) q'_{se_0}(i,T,e)$ as well as $h_t(\varphi)$. Now let us confirm the followings. First, an optimal Markov movement strategy $\widehat{v}^*$ maximizes the total non-dection probability. Secondly, if the conditions (23) and (24) are valid for arbitrary $i \in \boldsymbol{K}$ and $t \in \widehat{\boldsymbol{T}}$, φ becomes an optimal response to the Markov strategy $\widehat{v}$. The discussion so far helps us formulate a linear programming problem to derive an optimal Markov strategy $\widehat{v}^*$ by using already-obtained φ^* and $\widehat{z}^*$.

$$\begin{aligned} (P_T^M) \qquad & \max_{\widehat{v}, q, q', \lambda} \sum_{e_0 \in E_0} \sum_{s \in S_0} \sum_{i \in \boldsymbol{K}} \sum_{e \in F_{e_0}} g(e_0) f(s) q'_{se_0}(i,T,e), \\ s.t. \qquad & q_{se_0}(s,1,e_0) = 1, s \in S_0, e_0 \in E_0, \\ & \sum_{i \in K} \sum_{e \in F_{e_0}} q_{se_0}(i,1,e) = 1, s \in S_0, e_0 \in E_0, \end{aligned}$$

$$q'_{se_0}(i,t,e) = q_{se_0}(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q'_{se_0}(i,t,e) = q_{se_0}(i,t,e)\exp(-\alpha_i\varphi^*(i,t)), i \in \boldsymbol{K}, t \in \widehat{\boldsymbol{T}}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q_{se_0}(i,t,e) = \sum_{j \in N^*_{e_0}(i,t,e)} \widehat{v}_{se_0}(j,i,t-1,e+\mu(j,i)),$$
$$i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{1\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$q'_{se_0}(i,t,e) = \sum_{j \in N(i,t,e)} \widehat{v}_{se_0}(i,j,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$\alpha_i \exp(-\alpha_i\varphi^*(i,t)) \sum_{e_0 \in E_0}\sum_{s \in S_0}\sum_{e \in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e) = \lambda(t)$$
$$for\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t) > 0,$$
$$\alpha_i \sum_{e_0 \in E_0}\sum_{s \in S_0}\sum_{e \in F_{e_0}} g(e_0)f(s)q_{se_0}(i,t,e)\widehat{z}^*_{se_0}(i,t,e) \le \lambda(t)$$
$$for\ (i,t) \in \boldsymbol{K} \times \widehat{\boldsymbol{T}}\ of\ \varphi^*(i,t) = 0,$$
$$\widehat{v}_{se_0}(i,j,t,e) \ge 0,\ i,j \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0,$$
$$\widehat{v}_{se_0}(i,j,t,e) = 0,\ j \notin N(i,t,e), i \in \boldsymbol{K}, t \in \boldsymbol{T}\backslash\{T\}, e \in \boldsymbol{F}_{e_0}, s \in S_0, e_0 \in E_0.$$

Using the optimal solution $\widehat{v}^*_{se_0}(i,j,t,e)$ of the problem (P^M_T), we can reconstruct an optimal form of the original Markov strategy $v^*_{se_0}(i,j,t,e)$, as follows:

$$v^*_{se_0}(i,j,t,e) = \frac{\widehat{v}^*_{se_0}(i,j,t,e)}{\sum_{j \in N(i,t,e)} \widehat{v}^*_{se_0}(i,j,t,e)}. \tag{25}$$

$\widehat{v}^*$ includes the reachability of the target with no detection and there could be some cases that the denominator of the formula is zero for some state (s,e_0,i,t,e). The cases indicate the impossibility of the state (s,e_0,i,t,e) for the target in an optimal movement. For the state (s,e_0,i,t,e), we do not have to specify $v_{se_0}(i,j,t,e)$ or any Markov strategy $v_{se_0}(i,j,t,e)$ is allowed.

5 Numerical Example In this section, we apply our methodology proposed in previous sections to some numerical examples to analyze the features of optimal player's strategy.

We consider a discrete cell space $\boldsymbol{K} = \{1,\ldots,19\}$, shown by Fig. 1.

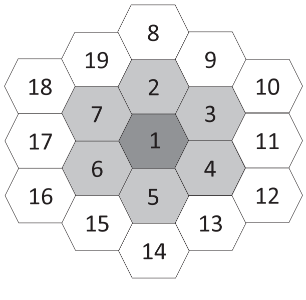

Figure 1: A search space

We set a discrete time space by $\boldsymbol{T} = \{1,2,3\}$ and a searching period by $\hat{\boldsymbol{T}} = \{2,3\}$. A searcher uses available searching resource $\Phi(2) = \Phi(3) = 1$ at each time point of $\hat{\boldsymbol{T}}$. The efficiency of detection of cell i, α_i, is set as follows : $\alpha_1 = 0.5,\ \alpha_2 = \cdots = \alpha_7 = 0.6,\ \alpha_8 =$

$\cdots = \alpha_{19} = 0.7$. Cell 1 in the center has the smallest efficiency and α_i becomes larger as the cell i is located farther from the center. The efficiency is also represented by the gradation of black color in the figure. Darker cell has the smaller efficiency. The target's initial position is assumed to be cell 1 or 2, $S_0 = \{1, 2\}$, and its initial energy 1 or 4, $E_0 = \{1, 4\}$. The searcher infers the target's initial position based on probabilities $f(1)$ and $f(2)$ $\big(f(1)+f(2) = 1\big)$ and the initial energy by $g(1)$ and $g(4)$ $\big(g(1)+g(4) = 1\big)$. The target can move from a present cell i to its neighboring cell j by consuming energy $\mu(i, j) = 1$ and move to its 2nd-neighboring cell j by consuming $\mu(i, j) = 4$ while staying at the same cell needs no energy, $\mu(i, i) = 0$. From now on, we are going to analyze the player's optimal strategy. We compute an optimal searcher's strategy φ^* and an optimal target strategy π^* by solving problem (P_S) and (P_T) in Section 3, respectively.

5.1 Features of optimal strategies Fig. 2 shows the value of game in the case of $f(1) = f(2) = 0.5$ while changing $g(1)$ from 0 through 1 at intervals of 0.1. ¿From now on, we analyze the curve of Fig. 2 by looking at searcher's and target's optimal strategies in detail.

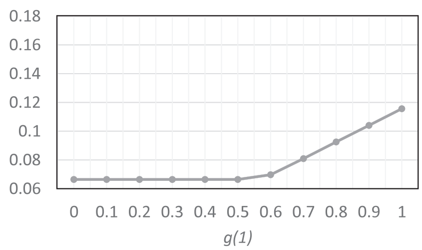

Figure 2: Value of the game in the case of $f(1) = f(2) = 0.5$

(1) Features of optimal searcher's strategy

Changing $g(1)$ from 0 through 1 at intervals of 0.1, we derive the searcher's optimal distribution of his searching resource, φ^*, at $t = 2, 3$ and illustrate it in Fig. 3 just for $g(1) = 0, 0.1, \ldots, 0.5$. For other cases of $g(1) = 0.6, 0.7, \ldots, 1.0$, we show the results in Table 1 without using space-consuming figures. A blank with no number indicates zero.

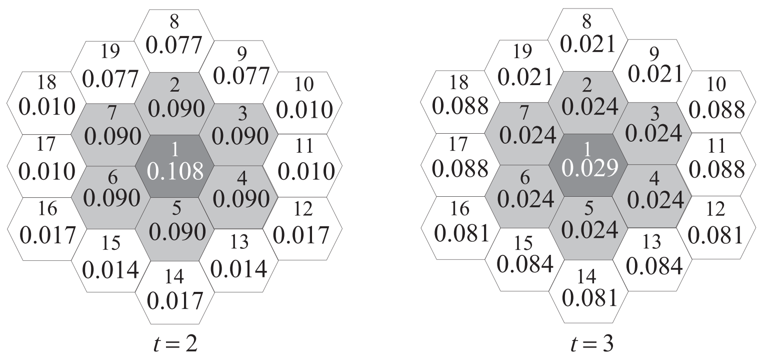

Figure 3: Optimal resource distribution in the case of $g(1) = 0 \sim 0.5$

In Fig. 3, the optimal distribution of resource keeps same even though $g(1)$ changes. Because in the case of low $g(1)$ the target has high energy 4 with high probability and it can move quickly to peripheral cells $8 \sim 19$ at the early time $t = 2$, the searcher has to widely spread its searching resource to all cells. The unchangingness of the optimal resource distribution corresponds to an optimal distribution of target's existence which also spreads to all cells at early time for $g(1) = 0 \sim 0.5$. We check it later, though. As seen in Fig. 3, the searcher distributes larger amount of searching resource in smaller numbered cells at $t = 2$ but in larger numbered cells at $t = 3$. This result explains that the searcher gradually widens its focal area of search considering the target movement. Since cell $s = 2$ is an initial target position as well as cell 1, larger amount of searching resource are distributed in its neighboring cells $8, 9$ and 19 at $t = 2$ compared to the other peripheral cells $10, \cdots, 18$.

Table 1: Optimal resource distribution in the case of $g(1) = 0.6 \sim 1.0$

Cell \ $g(1)$	$g(1) = 0.6$		$g(1) = 0.7$		$g(1) = 0.8$		$g(1) = 0.9$		$g(1) = 1.0$	
	t=2	t=3	t=2	t=3	t=2	t=3	t=2	t=3	t=2	t=3
1	.177	.031	.219	.027	.221	.025	.221	.025	.225	.021
2	.148	.026	.182	.022	.184	.021	.184	.021	.187	.017
3	.105	.069	.110	.094	.111	.094	.111	.094	.060	.144
4	.063	.110	.083	.122	.081	.124	.081	.124	.086	.119
5	.072	.102	.086	.119	.084	.120	.085	.120	.086	.119
6	.063	.110	.083	.122	.081	.124	.081	.124	.086	.119
7	.105	.069	.110	.094	.111	.094	.111	.094	.060	.144
8	.071	.077	.033	.143	.033	.143	.033	.142	.070	.106
9	.075	.074	.048	.128	.047	.128	.047	.128	.070	.106
10	.011	.065								
11	.012	.064								
12										
13										
14										
15										
16										
17	.012	.064								
18	.011	.065								
19	.075	.074	.048	.128	.047	.128	.047	.128	.070	.106
Total	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

As seen from Table 1, as $g(1)$ becomes larger and the target more likely has smaller energy, the searcher distributes more resource in smaller areas in the vicinity of target's initial positions. There is no searching resource distributed in cells $12, \cdots, 16$ in the case of $g(1) = 0.6$ and in cells $10, \cdots, 18$ in the case of $g(1) \geq 0.7$. As $g(1)$ becomes larger, the possible area of target is getting smaller and the searcher carries out more effective search by concentrating his resource on the smaller area.

(2) Features of optimal target's strategy

Here we analyze optimal target strategy. Fig. 4 shows the probability distribution of target's existence weighted over all types of (s, e_0) at time 2 and 3. The probability of target's existence in cell i at time t is calculated by $\sum_{s \in S_0} \sum_{e_0 \in E_0} f(s) g(e_0) \sum_{\omega \in \Omega_{it}^{+se_0}} \pi_{se_0}^*(\omega)$. ¿From Fig. 4, we can pick up main features of the optimal target strategy: "diffusiveness", "uniformity" and "preference to ineffective cell of search".

(i) **Diffusiveness**: The target possible area with positive probability quickly spreads to all cells even at the early time $t = 2$ and the target distribution is kept stable although there is a small bias based on efficiency parameter α_i of each cell i. The quick diffusion and spread of the target distribution over wider area is preferable for the target because it intervenes

the searcher's effective search of concentrating searching resource in small area.
(ii) **Uniformity**: From the Fig. 4, the target seems to move intentionally keeping the target distribution uniform all through the searching period and all over the cells. The target tries to deteriorate the detection probability by inducing the dispersion of searching resource by the uniform target distribution.
(iii) **Preference to ineffective cell of search**: The target changes his probability distribution such that more probabilities are allocated in the cells with smaller efficiency α_i. It would be natural that the target keeps off more detectable cells with larger α_i.

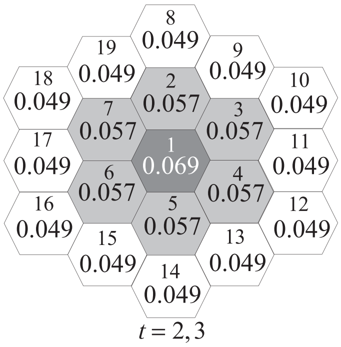

Figure 4: Target's Existence Probability in the case of $0 \leq g(1) \leq 0.5$

By the detail analysis on the optimal player's strategies mentioned above, we can explain the change of the detection probability for varying $g(1)$ in Fig. 2, as follows. In the case of $0 \leq g(1) \leq 0.5$, in which the possibility of high-mobile target is high, the detection probability remains low because the target distribution diffuses widely and uniformly all over the whole space and prevents the searcher from doing an effective search by the concentration of searching resource. On the other hand, in case of $0.6 \leq g(1) \leq 1.0$, the target's possible area is restricted by his poor mobility and the searcher distributes his resource intensively in the restricted area to increase the detection probability.

5.2 Sensitivity analysis on the value of the game and valuation of information

Here we analyze the expected detection probability of the target and the value of information. We change $f(1)$ and $g(1)$ from 0 through 1.0 at intervals of 0.1 and illustrate the value of the game or the detection probability in Fig. 5. Taking $g(1)$ on the x-axis and $f(1)$ on the y-axis, we depict the value of game on the z-axis in Fig. 5. Larger $g(1)$ indicates higher probability of target's having initial energy 1 and larger $f(1)$ higher probability of target's being in initial cell 1. Three curves of Fig. 6 are two-dimensional versions of Fig. 5, given by fixing $g(1)$ to $0, 0.5$, and 1.0, respectively. Figure 7 shows the detection probability with respect to $g(1)$ for each of fixed $f(1) = 0, 0.5$, and 1.0, respectively.

All curves of Fig. 6 have bathtub curves because an ambiguous position of target around $f(1) = 0.5$ gives the searcher some disadvantage that the searcher has to take account of a variety of target paths starting from two initial positions. On the other hand, knowing certainly the initial position near $f(1) = 0$ or 1 helps the searcher concentrate searching resource on the paths starting from the identified position for an effective search. When the target has smaller initial energy, he has less options of paths. That is why the curve of the detection probability takes higher position as $g(1)$ increases, as seen by the comparison among three cases of $g(1) = 0$, 0.5 and 1.0.

We have already analyzed the case of $f(1) = 0.5$ of Fig. 7 in section 5.1. Our analysis still works for other curves of Fig. 7. The detection probability monotonously increases as

$g(1)$ gets larger and it increases sharply over around the center of $g(1) = 0.5$. From Fig. 6, we have found the bathtub form in the change of the detection probability with respect to $f(1)$. By the 'bathtub' effect, the curve of $f(1) = 0.5$ takes a lower position than in the case $f(1) = 0$ but the curve of $f(1) = 1.0$ is located in a little higher position than one of $f(1) = 0.5$.

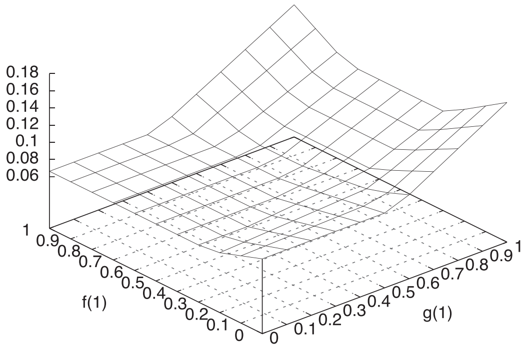

Figure 5: Value of the game for varying $f(1)$ and $g(1)$

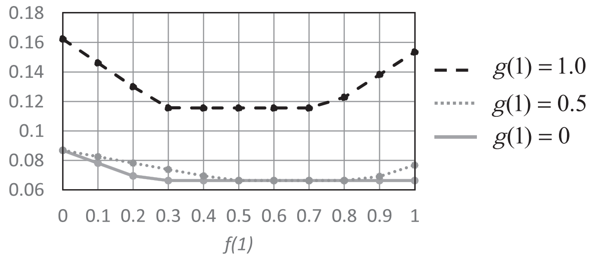

Figure 6: Value of the game for varying $f(1)$

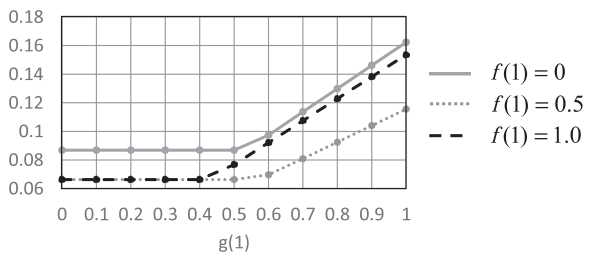

Figure 7: Value of the game for varying $g(1)$

As the last analysis, we evaluate the value of the information about the target's initial state. The value of information can be estimated from the difference between two values of the games, which are the games with private information or common knowledge about the initial state. The value of the game with common knowledge is computed by Hohzaki et al. [20]. The difference indicates how much detection probability the searcher can increase by knowing the target's initial state with certainty. Figure 8 shows the value of information for varying $g(1)$ and $f(1)$. The value reaches its maximum 0.0557 around the point of $f(1) = 0.5$ and $g(1) = 0.6$. Since the target's initial position and energy are the vaguest for the searcher around the point, the information has the maximum value if obtained.

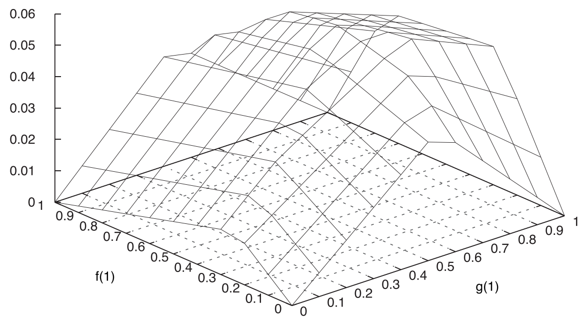

Figure 8: Value of information about target initial state

6 Conclusions In this study, we have discussed a SAG considering the targets' initial state consisting of its initial position and initial energy as a private information of the target which the searcher doesn't know. In rescue operations or military operations, however, the searcher does not start his search without any information of target's state, being concerned about the waste of search efforts in a wide operational area. The information about the target's initial state is very important for the searcher to restrict the target's possible area and make an effective use of searching resource in the restricted area.

Under the above background, we aimed to quantify the importance of the information about the target's initial state by deriving searcher's and target optimal strategies in the search game. We proposed two methods to derive an equilibrium point for the game. We also analyzed the equilibrium by some numerical examples.

¿From our analysis, we found that the target's initial energy has a great impact on the efficiency of the search. When a high-energy target is predicted to appear, the searcher has to distribute his searching resource widely, which makes the search less efficient. On the other hand, when the target is predicted to have low energy, the searcher would be able to concentrate his search efforts into comparatively small areas to make the search more efficient and bring large detection probability.

We also found three features of the targets' optimal strategy: diffusiveness, uniformity and preference to ineffective cell of search. Although the target tends to go to ineffective cells of search, he does not stay at those cells but he keeps going to expand his existence area while maintaining his probability distribution as uniform as possible over the areas with the same parameter of detection efficiency.

As for the value of the information about the target's initial state, we clarified that the value reaches its maximum when the searcher's anticipation about the target state is around the vaguest from the quantitative point of view. These results we have from our analysis are compatible with our common sense and could become precepts for practical searches.

Lastly we would like to mention our future works. In this study, we modeled our search game into a one-shot SAG. That is why the players are assumed to acquire some information about their opponents just at the beginning of the game but not to obtain any information in the middle of playing the game. In many realistic operations, however, the searcher makes efforts to update the information time by time. To examine such a situation, we need to extend our model to a multi-stage game with the change of the strategy by updated information. In some search operations such as anti-submarine warfare, it is conceivable that an evading target refuels its energy to maintain its mobility like a submarine. To investigate this problem, we require a model with an additional target strategy of energy refueling.

Acknowledgment. The authors would like to thank an anonymous referee. His comments were useful for us to improve our paper. This work is supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) Grant Number 20546048.

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Communicated by *Kensaku Kikuta*

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Scientiae Mathematicae Japonicae 80, No.1 (2017) (77-83)

SEMI-G-STABLE IN DITOPOLOGICAL TEXTURE SPACES

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Received September 18, 2014 ; revised December 14, 2015

Abstract. In this paper, the author introduce and study new notions of continuity, compactness and stability in ditopological texture spaces based on the notions of semi-g-open and semi-g-closed sets and some of their characterizations are obtained.

1 Introduction Textures and ditopological texture spaces were first introduced by L. M. Brown as a point-based setting for the study fuzzy topology. The study of compactness and stability in ditopological texture spaces was started to begin in [6]. In this paper, we introduce and study the concepts of semi-g-bicontinuity, semi-g-bi-irresolute, semi-g-compactness and semi-g-stability in ditopological texture spaces.

2 Preliminaries The following are some basic definitions of textures we will need later on.

Texture space: [6] Let S be a set. Then $\varphi \subseteq P(S)$ is called a texturing of S, and S is said to be textured by φ if

1. $(\varphi, \subseteq)$ is a complete lattice containing S and ϕ and for any index set I and $A_i \in \varphi$, $i \in I$, the meet $\bigwedge_{i\in I} A_i$ and the join $\bigvee_{i\in I} A_i$ in φ are related with the intersection and union in $P(S)$ by the equalities

   $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$

   for all I, while

   $\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$

   for all finite I.

2. φ is completely distributive.

3. φ separates the points of S. That is, given $s_1 \neq s_2$ in S we have $L \in \varphi$ with $s_1 \in L$, $s_2 \notin L$, or $L \in \varphi$ with $s_2 \in L$, $s_1 \notin L$.

If S is textured by φ then (S, φ) is called a texture space, or simply a texture.

Complementation: [6] A mapping $\sigma : \varphi \to \varphi$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \varphi$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \varphi$ is called a complementation on (S, φ) and (S, φ, σ) is then said to be a complemented texture.

For a texture (S, φ), most properties are conveniently dened in terms of the p-sets

$$P_s = \bigcap\{A \in \varphi : s \in A\}$$

and the q-sets,

$$Q_s = \bigvee\{A \in \varphi : s \notin A\}.$$

Ditopology: [6] A dichotomous topology on a texture (S, φ), or ditopology for short, is a pair (τ, k) of subsets of φ, where the set of open sets τ satisfies

2010 *Mathematics Subject Classification.* Primary: 54A05, 54A10; Secondary: 54C05.
Key words and phrases. Texture, difunction, semi-g-bi-irresolute, semi-g-stability.

1. $S, \phi \in \tau$,
2. $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and
3. $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of closed sets k satisfies

1. $S, \phi \in k$,
2. $K_1, K_2 \in k \Rightarrow K_1 \cup K_2 \in k$, and
3. $K_i \in k, i \in I \Rightarrow \bigcap K_i \in k$.

Hence a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.
For $A \in \varphi$ we define the closure $[A]$ and the interior $]A[$ of A under (τ, k) by the equalities
$[A] = \bigcap\{K \in k : A \subseteq K\}$ and $]A[= \bigvee\{G \in \tau : G \subseteq A\}$
We refer to τ as the topology and k as the cotopology of (τ, k).

If (τ, k) is a ditopology on a complemented texture (S, φ, σ), then we say that (τ, k) is complemented if the equality $k = \sigma(\tau)$ is satisfied. In this study, a complemented ditopological texture space is denoted by $(S, \varphi, \tau, k, \sigma)$.
In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)]$.

We denote by $O(S, \varphi, \tau, k)$, or when there can be no confusion by $O(S)$, the set of open sets in φ. Likewise, $C(S, \varphi, \tau, k)$, $C(S)$ will denote the set of closed sets.

Let (S_1, φ_1) and (S_2, φ_2) be textures. In the following definition we consider the product texture [3] $P(S_1) \otimes \varphi_2$, and denote by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$, respectively the p-sets and q-sets for the product texture $(S_1 \times S_2, P(S_1) \otimes \varphi_2)$.
Direlation: [5] Let (S_1, φ_1) and (S_2, φ_2) be textures. Then

1. $r \in P(S_1) \otimes \varphi_2$ is called a relation from (S_1, φ_1) to (S_2, φ_2) if it satisfies
   R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}_{(s',t)}$.
   R2 $r \not\subseteq \overline{Q}_{(s,t)} \Rightarrow \exists s' \in S_1$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
2. $R \in P(S_1) \otimes \varphi_2$ is called a corelation from (S_1, φ_1) to (S_2, φ_2) if it satisfies
   CR1 $\overline{P}_{(s,t)} \not\subseteq R$, $P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}_{(s',t)} \not\subseteq R$.
   CR2 $\overline{P}_{(s,t)} \not\subseteq R \Rightarrow \exists s' \in S_1$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
3. A pair (r, R), where r is a relation and R a corelation from (S_1, φ_1) to (S_2, φ_2) is called a direlation from (S_1, φ_1) to (S_2, φ_2).

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special type of direlation.
Difunctions: [5] Let (f, F) be a direlation from (S_1, φ_1) to (S_2, φ_2). Then (f, F) is called a difunction from (S_1, φ_1) to (S_2, φ_2) if it satisfies the following two conditions.
DF1 For $s, s' \in S_1$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists t \in S_2$ such that $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
DF2 For $t, t' \in S_2$ and $s \in S_1$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \Rightarrow P_{t'} \not\subseteq Q_t$.
Image and Inverse Image: [5] Let $(f, F) : (S_1, \varphi_1) \to (S_2, \varphi_2)$ be a difunction.

1. For $A \in \varphi_1$, the image $f^{\to}A$ and the co-image $F^{\to}A$ are defined by

$$f^{\to}A = \bigcap\{Q_t : \forall s, f \not\subseteq \overline{Q}_{(s,t)} \Rightarrow A \subseteq Q_s\},$$
$$F^{\to}A = \bigvee\{P_t : \forall s, \overline{P}_{(s,t)} \not\subseteq F \Rightarrow P_s \subseteq A\}.$$

2. For $B \in \varphi_2$, the inverse image $f^{\leftarrow}B$ and the inverse co-image $F^{\leftarrow}B$ are defined by

$$f^{\leftarrow}B = \bigvee\{P_s : \forall t, f \not\subseteq \overline{Q}_{(s,t)} \Rightarrow P_t \subseteq B\},$$
$$F^{\leftarrow}B = \bigcap\{Q_s : \forall t, \overline{P}_{(s,t)} \not\subseteq F \Rightarrow B \subseteq Q_t\}.$$

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are usually not.
Bicontinuity: [4] The difunction $(f, F) : (S_1, \varphi_1, \tau_1, k_1) \to (S_2, \varphi_2, \tau_2, k_2)$ is called continuous if $B \in \tau_2 \Rightarrow F^{\leftarrow}B \in \tau_1$, cocontinuous if $B \in k_2 \Rightarrow f^{\leftarrow}B \in k_1$, and bicontinuous if it is both continuous and cocontinuous.
Surjective difunction: [5] Let $(f, F) : (S_1, \varphi_1) \to (S_2, \varphi_2)$ be a difunction. Then (f, F) is called surjective if it satisfies the condition
SUR. For $t, t' \in S_2$, $P_t \not\subseteq Q_{t'} \Rightarrow \exists s \in S_1$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

If (f, F) is surjective then $F^{\to}(f^{\leftarrow}B) = B = f^{\to}(F^{\leftarrow}B)$ for all $B \in \varphi_2$ [[5], Corollary 2.33]

[5] Let (f, F) be a difunction between the complemented textures $(S_1, \varphi_1, \sigma_1)$ and $(S_2, \varphi_2, \sigma_2)$. The complement $(f, F)' = (F', f')$ of the difunction (f, F) is a difunction, where $f' = \bigcap\{\overline{Q}_{(s,t)} | \exists u, v$ with $f \not\subseteq \overline{Q}_{u,v}, \sigma_1(Q_s) \not\subseteq Q_u$ and $P_v \not\subseteq \sigma_2(P_t)\}$ and $F' = \bigvee\{\overline{P}_{(s,t)} | \exists u, v$ with $\overline{P}_{u,v} \not\subseteq F, P_u \not\subseteq \sigma_1(P_s)$ and $\sigma_2(Q_t) \not\subseteq Q_v\}$. If $(f, F) = (f, F)'$ then the difunction (f, F) is called complemented.

[7] Let (S, φ, τ, k) be a ditopological texture space. A set $A \in \varphi$ is called semi-open (semi-closed) if $A \subseteq []A[]$ $(][A][\subseteq A)$. We denote by $SO(S, \varphi, \tau, k)$, or when there can be no confusion by $SO(S)$, the set of semi-open sets in φ. Likewise, $SC(S, \varphi, \tau, k)$, or $SC(S)$ will denote the set of semi-closed sets. [2] Let (S, φ, τ, k) be a ditopological texture space. A subset A of a texture φ is said to be generalized closed (g-closed for short) if $A \subseteq G \in \tau$ then $[A] \subseteq G$. [2] Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. A subset A of a texture φ is said to be generalized open (g-open for short) if $\sigma(A)$ is g-closed. We denote by $gc(S, \varphi, \tau, k)$, or when there can be no confusion by $gc(S)$, the set of g-closed sets in φ. Likewise, $go(S, \varphi, \tau, k, \sigma)$, or $go(S)$ will denote the set of g-open sets.

[1] Let (S, φ, τ, k) be a ditopological texture space. A subset A of a texture φ is said to be semi-g-closed if $A \subseteq G \in SO(S)$ then $[A] \subseteq G$.

We denote by $semigc(S, \varphi, \tau, k)$, or when there can be no confusion by $semigc(S)$, the set of semi-g-closed sets in φ. [1] Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. A subset A of a texture φ is called semi-g-open if $\sigma(A)$ is semi-g-closed.

We denote by $semigo(S, \varphi, \tau, k, \sigma)$, or when there can be no confusion by $semigo(S)$, the set of semi-g-open sets in φ. [1] Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. For $A \in \varphi$, we define the semi-g-closure $[A]_{semi-g}$ and the semi-g-interior $]A[_{semi-g}$ of A under (τ, k) by the equalities

$$[A]_{semi-g} = \bigcap\{K \in semigc(S) : A \subseteq K\} \text{ and }]A[_{semi-g} = \bigcup\{G \in semigo(S) : G \subseteq A\}.$$

3 semi-g-bicontinuous, semi-g-bi-irresolute, semi-g-compact and semi-g-stable

The difunction $(f, F) : (S_1, \varphi_1, \tau_1, k_1, \sigma_1) \to (S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ is called:

1. semi-g-continuous (semi-g-irresolute), if $F^{\leftarrow}(G) \in semigo(S_1)$, for every $G \in O(S_2)$ $(G \in semigo(S_2))$.
2. semi-g-cocontinuous (semi-g-co-irresolute), if $f^{\leftarrow}(G) \in semigc(S_1)$, for every $G \in k_2$ $(G \in semigc(S_2))$.
3. semi-g-bicontinuous, if it is semi-g-continuous and semi-g-cocontinuous.
4. semi-g-bi-irresolute, if it is semi-g-irresolute and semi-g-co-irresolute.

Let $(f,F):(S_1,\varphi_1,\tau_1,k_1,\sigma_1)\to(S_2,\varphi_2,\tau_2,k_2,\sigma_2)$ be a difunction. Then:

1. Every continuous is semi-g-continuous.
2. Every cocontinuous is semi-g-cocontinuous.
3. Every semi-g-irresolute is semi-g-continuous.
4. Every semi-g-co-irresolute is semi-g-cocontinuous.

Clear. Let $(f,F):(S_1,\varphi_1,\tau_1,k_1,\sigma_1)\to(S_2,\varphi_2,\tau_2,k_2,\sigma_2)$ be a difunction. Then:

1. The following are equivalent:
   (a) (f,F) is semi-g-continuous.
   (b) $]F^{\to}A[^{S_2}\subseteq F^{\to}]A[^{S_1}_{semi-g}$, $\forall A\in\varphi_1$.
   (c) $f^{\leftarrow}]B[^{S_2}\subseteq]f^{\leftarrow}B[^{S_1}_{semi-g}$, $\forall B\in\varphi_2$.
2. The following are equivalent:
   (a) (f,F) is semi-g-cocontinuous.
   (b) $f^{\to}[A]^{S_1}_{semi-g}\subseteq[f^{\to}A]^{S_2}$, $\forall A\in\varphi_1$.
   (c) $[F^{\leftarrow}B]^{S_1}_{semi-g}\subseteq F^{\leftarrow}[B]^{S_2}$, $\forall B\in\varphi_2$.

We prove (1), leaving the dual proof of (2) to the interested reader.
$(a)\Rightarrow(b)$. Let $A\in\varphi_1$. From [[5], Theorem 2.24 (2 a)] and the definition of interior,

$$f^{\leftarrow}]F^{\to}(A)[^{S_2}\subseteq f^{\leftarrow}(F^{\to}(A))\subseteq A.$$

Since inverse image and co-image under a difunction is equal, $f^{\leftarrow}]F^{\to}(A)[^{S_2}=F^{\leftarrow}]F^{\to}(A)[^{S_2}$. Thus, $f^{\leftarrow}]F^{\to}(A)[^{S_2}\in semigo(S_1)$, by semi-g-continuity. Hence

$$f^{\leftarrow}]F^{\to}(A)[^{S_2}\subseteq]A[^{S_1}_{semi-g}$$

and applying [[5], Theorem 2.24 (2 b)] gives

$$]F^{\to}(A)[^{S_2}\subseteq F^{\to}(f^{\leftarrow}(]F^{\to}(A)[^{S_2})\subseteq F^{\to}]A[^{S_1}_{semi-g},$$

which is the required inclusion.
$(b)\Rightarrow(c)$. Take $B\in\varphi_2$. Applying inclusion (b) to $A=f^{\leftarrow}(B)$ and using [[5], Theorem 2.24 (2 b)] gives

$$]B[^{S_2}\subseteq]F^{\to}f^{\leftarrow}(B)[^{S_2}\subseteq F^{\to}]f^{\leftarrow}(B)[^{S_1}_{semi-g}.$$

Hence, we have $f^{\leftarrow}]B[^{S_2}\subseteq f^{\leftarrow}F^{\to}]f^{\leftarrow}(B)[^{S_1}_{semi-g}\subseteq]f^{\leftarrow}(B)[^{S_1}_{semi-g}$ by [[5], Theorem 2.24 (2 a)].
$(c)\Rightarrow(a)$. Applying (c) for $B\in O(S_2)$ gives

$$f^{\leftarrow}(B)=f^{\leftarrow}]B[^{S_2}\subseteq]f^{\leftarrow}(B)[^{S_1}_{semi-g},$$

so $F^{\leftarrow}(B)=f^{\leftarrow}(B)=]f^{\leftarrow}(B)[^{S_1}_{semi-g}\in semigo(S_1)$. Hence, (f,F) is semi-g-continuous.
Let $(f,F):(S_1,\varphi_1,\tau_1,k_1,\sigma_1)\to(S_2,\varphi_2,\tau_2,k_2,\sigma_2)$ be a difunction. Then:

1. The following are equivalent:

(a) (f, F) is semi-g-irresolute.

(b) $]F^{\rightarrow}A[^{S_2}_{semi-g} \subseteq F^{\rightarrow}]A[^{S_1}_{semi-g}$, $\forall A \in \varphi_1$.

(c) $f^{\leftarrow}]B[^{S_2}_{semi-g} \subseteq]f^{\leftarrow}B[^{S_1}_{semi-g}$, $\forall B \in \varphi_2$.

2. The following are equivalent:

(a) (f, F) is semi-g-co-irresolute.

(b) $f^{\rightarrow}[A]^{S_1}_{semi-g} \subseteq [f^{\rightarrow}A]^{S_2}_{semi-g}$, $\forall A \in \varphi_1$.

(c) $[F^{\leftarrow}B]^{S_1}_{semi-g} \subseteq F^{\leftarrow}[B]^{S_2}_{semi-g}$, $\forall B \in \varphi_2$.

We prove (1), leaving the dual proof of (2) to the interested reader.
$(a) \Rightarrow (b)$. Take $A \in \varphi_1$. Then

$$f^{\leftarrow}]F^{\rightarrow}A[^{S_2}_{semi-g} \subseteq f^{\leftarrow}(F^{\rightarrow}A) \subseteq A$$

by [[5], Theorem 2.24 (2 a)]. Now $f^{\leftarrow}]F^{\rightarrow}A[^{S_2}_{semi-g} = F^{\leftarrow}]F^{\rightarrow}A[^{S_2}_{semi-g} \in semigo(S_1)$ by semi-g-irresolute, so $f^{\leftarrow}]F^{\rightarrow}A[^{S_2}_{semi-g} \subseteq]A[^{S_1}_{semi-g}$ and applying [[5], Theorem 2.24 (2 b)] gives

$$]F^{\rightarrow}A[^{S_2}_{semi-g} \subseteq F^{\rightarrow}(f^{\leftarrow}]F^{\rightarrow}A[^{S_2}_{semi-g} \subseteq F^{\rightarrow}]A[^{S_1}_{semi-g},$$

which is the required inclusion.
$(b) \Rightarrow (c)$. Take $B \in \varphi_2$. Applying inclusion (b) to $A = f^{\leftarrow}B$ and using [[5], Theorem 2.24 (2 b)] gives

$$]B[^{S_2}_{semi-g} \subseteq]F^{\rightarrow}(f^{\leftarrow}B)[^{S_2}_{semi-g} \subseteq F^{\rightarrow}]f^{\leftarrow}B[^{S_1}_{semi-g}.$$

Hence, $f^{\leftarrow}]B[^{S_2}_{semi-g} \subseteq f^{\leftarrow}F^{\rightarrow}]f^{\leftarrow}B[^{S_1}_{semi-g} \subseteq]f^{\leftarrow}B[^{S_2}_{semi-g}$ by [[5], Theorem 2.24 (2 a)].
$(c) \Rightarrow (a)$. Applying (c) for $B \in semigo(S_2)$ gives

$$f^{\leftarrow}B = f^{\leftarrow}]B[^{S_2}_{semi-g} \subseteq]f^{\leftarrow}B[^{S_1}_{semi-g},$$

so $F^{\leftarrow}B = f^{\leftarrow}B =]f^{\leftarrow}B[^{S_1}_{semi-g} \in semigo(S_1)$. Hence, (f, F) is semi-g-irresolute.

Let $(S_j, \varphi_j, \tau_j, k_j, \sigma_j)$, for $j \in \{1, 2\}$, be complemented ditopology and $(f, F) : (S_1, \varphi_1) \rightarrow (S_2, \varphi_2)$ be complemented difunction. If (f, F) is semi-g-continuous then (f, F) is semi-g-cocontinuous. Since (f, F) is complemented, $(F^{'}, f^{'}) = (f, F)$. From [[5], Lemma 2.20], $\sigma_1((f^{'})^{\leftarrow}(B)) = f^{\leftarrow}(\sigma_2(B))$ and $\sigma_1((F^{'})^{\leftarrow}(B)) = F^{\leftarrow}(\sigma_2(B))$ for all $B \in \varphi_2$. The proof is clear from these equalities.

Let $(S_j, \varphi_j, \tau_j, k_j, \sigma_j)$, $j = 1, 2$, complemented ditopology and $(f, F) : (S_1, \varphi_1) \rightarrow (S_2, \varphi_2)$ be complemented difunction. If (f, F) is semi-g-irresolute then (f, F) is semi-g-co-irresolute. Clear. A complemented ditopological texture space $(S, \varphi, \tau, k, \sigma)$ is called semi-g-compact if every cover of S by semi-g-open has a finite subcover. Here we recall that $C = \{A_j : j \in J\}$, $A_j \in \varphi$ is a cover of S if $\bigvee C = S$.

Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. Then:

1. Every semi-g-compact is compact.

2. Every g-compact is semi-g-compact.

Clear. If $(S, \varphi, \tau, k, \sigma)$ is semi-g-compact and $L = \{F_j : j \in J\}$ is a family of semi-g-closed sets with $\cap L = \phi$, then $\cap\{F_j : j \in J^{'}\} = \phi$ for $J^{'} \subseteq J$ finite. Suppose that $(S, \varphi, \tau, k, \sigma)$ is semi-g-compact and let $L = \{F_j : j \in J\}$ be a family of semi-g-closed sets with $\cap L = \phi$. Clearly $C = \{\sigma(F_j) : j \in J\}$ is a family of semi-g-open sets. Moreover,

$$\bigvee C = \bigvee\{\sigma(F_j) : j \in J\} = \sigma(\cap\{F_j : j \in J\}) = \sigma(\phi) = S,$$

and so we have $J' \subseteq J$ finite with $\bigvee\{\sigma(F_j) : j \in J'\} = S$. Hence $\cap\{F_j : j \in J'\} = \phi$. Let $(f, F) : (S_1, \varphi_1, \tau_1, k_1, \sigma_1) \to (S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ be an semi-g-irresolute difunction. If $A \in \varphi_1$ is semi-g-compact then $f^{\to}A \in \varphi_2$ is semi-g-compact. Take $f^{\to}A \subseteq \bigvee_{j\in J} G_j$, where $G_j \in semigo(S_2)$, $j \in J$. Now by [[5], Theorem 2.24 (2 a) and Corollary 2.12 (2)] we have

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\textstyle\bigvee_{j\in J} G_j) = \bigvee_{j\in J} F^{\leftarrow}G_j.$$

Also, $F^{\leftarrow}G_j \in semigo(S_1)$ because (f, F) is semi-g-irresolute. So by the semi-g-compactness of A there exists $J' \subseteq J$ finite such that $A \subseteq \cup_{j\in J'} F^{\leftarrow}G_j$. Hence

$$f^{\to}A \subseteq f^{\to}(\cup_{j\in J'} F^{\leftarrow}G_j) = \cup_{j\in J'} f^{\to}(F^{\leftarrow}G_j) \subseteq \cup_{j\in J'} G_j$$

by [[5], Corollary 2.12 (2) and Theorem 2.24 (2 b)]. This establishes that $f^{\to}A$ is semi-g-compact.

Let $(f, F) : (S_1, \varphi_1, \tau_1, k_1, \sigma_1) \to (S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ be a surjective semi-g-irresolute difunction. Then, if $(S_1, \varphi_1, \tau_1, k_1, \sigma_1)$ is semi-g-compact so is $(S_2, \varphi_2, \tau_2, k_2, \sigma_2)$. This follows by taking $A = S_1$ in Theorem 3 and noting that $f^{\to}S_1 = f^{\to}(F^{\leftarrow}S_2) = S_2$ by [[5], Proposition 2.28 (1 c) and Corollary 2.33 (1)].

A complemented ditopological texture space $(S, \varphi, \tau, k, \sigma)$ is called semi-g-stable if every semi-g-closed set $F \in \varphi\backslash\{S\}$ is semi-g-compact in S. Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. Then:

1. Every semi-g-stable is stable.

2. Every g-stable is semi-g-stable.

Clear. Let $(S, \varphi, \tau, k, \sigma)$ be semi-g-stable. If G is an semi-g-open set with $G \neq \phi$ and $D = \{F_j : j \in J\}$ is a family of semi-g-closed sets with $\cap_{j\in J} F_j \subseteq G$ then $\cap_{j\in J'} F_j \subseteq G$ for a finite subsets J' of J. Let $(S, \varphi, \tau, k, \sigma)$ be semi-g-stable, let G be an semi-g-open set with $G \neq \phi$ and $D = \{F_j : j \in J\}$ be a family of semi-g-closed sets with $\cap_{j\in J} F_j \subseteq G$. Set $K = \sigma(G)$. Then K is semi-g-closed and satisfies $K \neq S$. Hence K is semi-g-compact. Let $C = \{\sigma(F) | F \in D\}$. Since $\cap D \subseteq G$ we have $K \subseteq \bigvee C$, that is C is an semi-g-open cover of K. Hence there exists $F_1, F_2, ..., F_n \in D$ so that

$$K \subseteq \sigma(F_1) \cup \sigma(F_2) \cup ... \cup \sigma(F_n) = \sigma(F_1 \cap F_2 \cap ... \cap F_n).$$

This gives $F_1 \cap F_2 \cap ... \cap F_n \subseteq \sigma(K) = G$, so $\cap_{j\in J'} F_j \subseteq G$ for a finite subsets $J' = \{1, 2, ..., n\}$ of J. Let $(S_1, \varphi_1, \tau_1, k_1, \sigma_1)$, $(S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ be two complemented ditopological texture spaces with $(S_1, \varphi_1, \tau_1, k_1, \sigma_1)$ is semi-g-stable, and $(f, F) : (S_1, \varphi_1, \tau_1, k_1, \sigma_1) \to (S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ be an semi-g-bi-irresolute surjective difunction. Then $(S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ is semi-g-stable. Take $K \in semigc(S_2)$ with $K \neq S_2$. Since (f, F) is semi-g-co-irresolute, so $f^{\leftarrow}K \in semigc(S_1)$. Let us prove that $f^{\leftarrow}K \neq S_1$. Assume the contrary. Since $f^{\leftarrow}S_2 = S_1$, by [[5], Lemma 2.28 (1 c)] we have $f^{\leftarrow}S_2 \subseteq f^{\leftarrow}K$, whence $S_2 \subseteq K$ by [[5], Corollary 2.33 (1 ii)] as (f, F) is surjective. This is a contradiction, so $f^{\leftarrow}K \neq S_1$. Hence $f^{\leftarrow}(K)$ is semi-g-compact in $(S_1, \varphi_1, \tau_1, k_1, \sigma_1)$ by semi-g-stability. As (f, F) is semi-g-irresolute, $f^{\to}(f^{\leftarrow}K)$ is semi-g-compact for the ditopology (τ_2, k_2) by Theorem 3, and by [[5], Corollary 2.33 (1)] this set is equal to K. This establishes that $(S_2, \varphi_2, \tau_2, k_2, \sigma_2)$ is semi-g-stable.

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ORDER PRESERVING PROPERTY FOR FUZZY VECTORS

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Received January 20, 2015

Abstract. In the present paper, the order preserving property for fuzzy vectors is investigated, and some classes of fuzzy vectors, which have the order preserving property and seem to be useful for applications, are constructed and proposed.

1 Introduction and preliminaries The concept of fuzzy vectors is an extension of the concept of fuzzy numbers, and it is useful for representing uncertain multidimensional quantities. Some properties of fuzzy vectors are investigated in [8]. Fuzzy linear programming problems involving oblique fuzzy vectors and fuzzy mathematical programming problems involving fuzzy vectors are considered in [2] and [7], respectively. When an ordering between any two fuzzy vectors is defined, the order preserving property for fuzzy vectors make fuzzy mathematical programming problems involving fuzzy vectors easy to solve. The order preserving property for fuzzy vectors is considered in the present paper. In the following, some basic notations and definitions are given.

For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. In addition, we set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \leq \boldsymbol{0}\}$. Let $\mathbb{N}$ be the set of all natural numbers. For $S \subset \mathbb{R}^n$, we denote the closure, interior, and complement of S by $\mathrm{cl}(S)$, $\mathrm{int}(S)$, and S^c, respectively.

A fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ is identified with its membership function $\widetilde{s} : \mathbb{R}^n \to [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$. Let $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. For $\alpha \in]0, 1]$, the set $[\widetilde{s}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) \geq \alpha\}$ is called the α-level set of $\widetilde{s}$. The 0-level set of $\widetilde{s}$ is defined as $[\widetilde{s}]_0 = \mathrm{cl}(\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\})$, and $[\widetilde{s}]_0$ is called the support of $\widetilde{s}$. The fuzzy set $\widetilde{s}$ is said to be closed if $\widetilde{s}$ is upper semicontinuous on $\mathbb{R}^n$. The fuzzy set $\widetilde{s}$ is closed if and only if $[\widetilde{s}]_\alpha$ is closed for any $\alpha \in]0, 1]$. The fuzzy set $\widetilde{s}$ is said to be convex if $\widetilde{s}(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \min\{\widetilde{s}(\boldsymbol{x}), \widetilde{s}(\boldsymbol{y})\}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, that is, $\widetilde{s}$ is quasiconcave on $\mathbb{R}^n$. The fuzzy set $\widetilde{s}$ is convex if and only if $[\widetilde{s}]_\alpha$ is convex for any $\alpha \in]0, 1]$.

We define fuzzy vectors.

Definition 1 (See [7]). A fuzzy set $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ is called a fuzzy vector on $\mathbb{R}^n$ if $\widetilde{s}$ satisfies the following conditions:

(i) there exists a unique vector $\boldsymbol{c} \in \mathbb{R}^n$, called the center of $\widetilde{s}$, such that $\widetilde{s}(\boldsymbol{c}) = 1$,

(ii) $\widetilde{s}$ is a closed fuzzy set, that is, $\widetilde{s}$ is upper semicontinuous on $\mathbb{R}^n$,

(iii) $\widetilde{s}$ is a convex fuzzy set, that is, $\widetilde{s}$ is quasiconcave on $\mathbb{R}^n$,

(iv) $[\widetilde{s}]_0$ is bounded.

Let $\mathcal{FV}(\mathbb{R}^n)$ be the set of all fuzzy vectors on $\mathbb{R}^n$. In [7], a fuzzy mathematical programming problem with a fuzzy vector-valued objective function is considered. Assume

2010 *Mathematics Subject Classification.* Primary 03E72; Secondary 90C70.
Key words and phrases. order preserving property, fuzzy vector, fuzzy max order.

that an ordering between any two fuzzy vectors is defined based on an ordering between two α-level sets of the fuzzy vectors for any $\alpha \in [0,1]$. Then, the fuzzy mathematical programming problem is equivalent to a mathematical programming problem with infinite many set-valued objective functions. If the fuzzy vector-valued objective function has the order preserving property, then the fuzzy mathematical programming problem is equivalent to a mathematical programming problem with finite many set-valued objective functions. Therefore, the order preserving property of the fuzzy vector-valued objective function make the fuzzy mathematical programmig problem easy to solve. The order preserving property of a fuzzy vector-valued function is equivalent to the order preserving property of a class of fuzzy vectors.

In the present paper, the order preserving property for fuzzy vectors is investigated, and some classes of fuzzy vectors, which have the order preserving property and seem to be useful for applications, are constructed and proposed.

For a crisp set $S \subset \mathbb{R}^n$, the function $c_S : \mathbb{R}^n \to \{0,1\}$ defined as

$$c_S(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in S, \\ 0 & \text{if } \boldsymbol{x} \notin S \end{cases}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$ is called the indicator function of S. A fuzzy set $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{s} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{s}]_\alpha}, \tag{1}$$

which is known as the decomposition theoren; see, for example, [1]. In order to construct fuzzy sets from classes of crisp sets, we set

$$\mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in]0,1], \text{ and } S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in]0,1] \text{ with } \beta < \gamma\},$$

and define a mapping $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M(\{S_\alpha\}_{\alpha \in]0,1]}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha} \tag{2}$$

for each $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. When $\widetilde{s} = M(\{S_\alpha\}_{\alpha \in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{s}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha \in]0,1]}$, and $\{S_\alpha\}_{\alpha \in]0,1]}$ is called the generator of $\widetilde{s}$. For $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$M(\{S_\alpha\}_{\alpha \in]0,1]})(\boldsymbol{x}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\},$$

where $\sup \emptyset = 0$. Based on the mapping M defined by (2), the decomposition theorem (1) can be represented as $\widetilde{s} = M(\{[\widetilde{s}]_\alpha\}_{\alpha \in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$.

The following proposition shows a relationship between level sets of a fuzzy set and the generator of the fuzzy set.

Proposition 1 (See [3]). Let $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{s} = M(\{S_\alpha\}_{\alpha \in]0,1]})$. Then, $[\widetilde{s}]_\alpha = \bigcap_{\beta \in]0,\alpha[} S_\beta$ for any $\alpha \in]0,1]$.

The remainder of the present paper is organized as follows. In Section 2, orderings of fuzzy sets are defined, and their properties are investigated. In Section 3, the concept of the order preserving property for fuzzy vectors is introduced. Then, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of orderings of crisp sets are investigated when the crisp sets vary parametrically. In Section 4, some classes of fuzzy vectors which have the order preserving property are constructed and proposed. Finally, conclusions are presented in Section 5.

2 Ordering of fuzzy sets In this section, orderings of fuzzy sets are defined, and their properties are investigated.

In order to define orderings of fuzzy sets based on level sets of the fuzzy sets, orderings of crisp sets are defined as follows.

Definition 2 (See [5, 6, 7]). Let $A, B \subset \mathbb{R}^n$.

(i) We write $A \leq_S B$ or $B \geq_S A$ if $B \subset A + \mathbb{R}^n_+$ and $A \subset B + \mathbb{R}^n_-$.

(ii) We write $A <_S B$ or $B >_S A$ if $B \subset A + \mathrm{int}(\mathbb{R}^n_+)$ and $A \subset B + \mathrm{int}(\mathbb{R}^n_-)$.

The binary relation $\leq_S$ in Definition 2 is a pseudo order on the set of all subsets of $\mathbb{R}^n$.
The following proposition shows fundamental properties of $\leq_S$ and $<_S$ in Definition 2.

Proposition 2 (See [4]). Let $A, B \subset \mathbb{R}^n$.

(i) The relation $A \leq_S B$ holds if and only if the following two conditions (i-1) and (i-2) are satisfied: (i-1) for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} \leq \boldsymbol{y}$; (i-2) for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} \leq \boldsymbol{y}$.

(ii) The relation $A <_S B$ holds if and only if the following two conditions (ii-1) and (ii-2) are satisfied: (ii-1) for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} < \boldsymbol{y}$; (ii-2) for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} < \boldsymbol{y}$.

(iii) $A \leq_S A$.

(iv) If $A <_S B$, then $A \leq_S B$.

(v) It does not always hold that $A <_S B$ even if $A \leq_S B$.

(vi) If $A = \emptyset$ and $B \neq \emptyset$, then $A \not\leq_S B$, $B \not\leq_S A$, $A \not<_S B$, and $B \not<_S A$.

(vii) $A <_S A$ and $A \not<_S A$ are both possible.

(viii) $\emptyset \leq_S \emptyset$, $\emptyset <_S \emptyset$, $\mathbb{R}^n \leq_S \mathbb{R}^n$, $\mathbb{R}^n <_S \mathbb{R}^n$.

Based on the orderings of crisp sets given in Definition 2 and level sets of fuzzy sets, orderings of fuzzy sets are defined as follows.

Definition 3 (See [4]). Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$.

(i) We write $\widetilde{a} \preceq \widetilde{b}$ or $\widetilde{b} \succeq \widetilde{a}$ if $[\widetilde{a}]_\alpha \leq_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0, 1]$.

(ii) We write $\widetilde{a} \prec \widetilde{b}$ or $\widetilde{b} \succ \widetilde{a}$ if $[\widetilde{a}]_\alpha <_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0, 1]$.

The binary relation $\preceq$ in Definition 3 is a pseudo order on $\mathcal{F}(\mathbb{R}^n)$, and $\preceq$ is called the fuzzy max order. In [7], for $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$, $\preceq_M$ and $\prec_M$ are defined as follows:

- we write $\widetilde{a} \preceq_M \widetilde{b}$ or $\widetilde{b} \succeq_M \widetilde{a}$ if $\inf([\widetilde{b}]_\alpha) \subset \inf([\widetilde{a}]_\alpha) + \mathbb{R}^n_+$ and $\sup([\widetilde{a}]_\alpha) \subset \sup([\widetilde{b}]_\alpha) + \mathbb{R}^n_-$ for any $\alpha \in [0, 1]$,

- we write $\widetilde{a} \prec_M \widetilde{b}$ or $\widetilde{b} \succ_M \widetilde{a}$ if $\inf([\widetilde{b}]_\alpha) \subset \inf([\widetilde{a}]_\alpha) + \mathrm{int}(\mathbb{R}^n_+)$ and $\sup([\widetilde{a}]_\alpha) \subset \sup([\widetilde{b}]_\alpha) + \mathrm{int}(\mathbb{R}^n_-)$ for any $\alpha \in [0, 1]$, where $\inf(S) = \{\boldsymbol{x} \in S : \text{there does not exist } \boldsymbol{y} \in S \text{ such that } \boldsymbol{y} \leq \boldsymbol{x} \text{ and } \boldsymbol{y} \neq \boldsymbol{x}\}$ and $\sup(S) = \{\boldsymbol{x} \in S : \text{there does not exist } \boldsymbol{y} \in S \text{ such that } \boldsymbol{y} \geq \boldsymbol{x} \text{ and } \boldsymbol{y} \neq \boldsymbol{x}\}$ for $S \subset \mathbb{R}^n$. The binary relation $\preceq_M$ is an extension of the fuzzy max order for fuzzy numbers given in [9].

The following proposition shows that $\preceq$ and $\prec$ in Definition 3 coincide with $\preceq_M$ and $\prec_M$ on $\mathcal{FV}(\mathbb{R}^n)$, respectively. Therefore, $\preceq$ and $\prec$ are extensions of $\preceq_M$ and $\prec_M$, respectively.

Proposition 3. Let $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$.

(i) $\widetilde{a} \preceq \widetilde{b}$ if and only if $\widetilde{a} \preceq_M \widetilde{b}$.

(ii) $\widetilde{a} \prec \widetilde{b}$ if and only if $\widetilde{a} \prec_M \widetilde{b}$.

Proof. Let $\alpha \in [0, 1]$. We set $A = [\widetilde{a}]_\alpha$ and $B = [\widetilde{b}]_\alpha$. Since A and B are nonempty compact convex sets, it follows that $\inf(A) \neq \emptyset$, $\sup(A) \neq \emptyset$, $\inf(B) \neq \emptyset$, and $\sup(B) \neq \emptyset$. In order to show (i) and (ii), it is sufficient to show that (i-1) $B \subset A + \mathbb{R}^n_+$ if and only if $\inf(B) \subset \inf(A) + \mathbb{R}^n_+$, (i-2) $A \subset B + \mathbb{R}^n_-$ if and only if $\sup(A) \subset \sup(B) + \mathbb{R}^n_-$, (ii-1) $B \subset A + \text{int}(\mathbb{R}^n_+)$ if and only if $\inf(B) \subset \inf(A) + \text{int}(\mathbb{R}^n_+)$, and (ii-2) $A \subset B + \text{int}(\mathbb{R}^n_-)$ if and only if $\sup(A) \subset \sup(B) + \text{int}(\mathbb{R}^n_-)$. We show only (i-1). (i-2), (ii-1), and (ii-2) can be shown in the similar way to (i-1). If $B \subset A + \mathbb{R}^n_+$, then $\inf(B) \subset B \subset A + \mathbb{R}^n_+ \subset \inf(A) + \mathbb{R}^n_+ + \mathbb{R}^n_+ = \inf(A) + \mathbb{R}^n_+$. If $\inf(B) \subset \inf(A) + \mathbb{R}^n_+$, then $B \subset \inf(B) + \mathbb{R}^n_+ \subset \inf(A) + \mathbb{R}^n_+ + \mathbb{R}^n_+ = \inf(A) + \mathbb{R}^n_+ \subset A + \mathbb{R}^n_+$. □

3 Order preserving property In this section, the concept of the order preserving property for fuzzy vectors is introduced. Then, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of the orderings of crisp sets are investigated when the crisp sets vary parametrically.

The orderings of two fuzzy sets in Definition 3 are defined by infinite many orderings of level sets of the fuzzy sets. If finite many orderings of level sets of two fuzzy sets imply the orderings of the fuzzy sets, then it makes the orderings of fuzzy sets easy to deal with for applications. Such property is called the order preserving property, and defined for fuzzy vectors as follows.

Definition 4. (i) Fuzzy vectors $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$ are said to be order preserving on $\mathbb{R}^n$ if $[\widetilde{a}]_0 \leq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \leq_S [\widetilde{b}]_1$ imply $\widetilde{a} \preceq \widetilde{b}$, or if $[\widetilde{a}]_0 \geq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \geq_S [\widetilde{b}]_1$ imply $\widetilde{a} \succeq \widetilde{b}$.

(ii) A class of fuzzy vectors, $\mathcal{G} \subset \mathcal{FV}(\mathbb{R}^n)$, is said to be order preserving on $\mathbb{R}^n$ if any $\widetilde{a}, \widetilde{b} \in \mathcal{G}$ are order preserving on $\mathbb{R}^n$.

Definition 5. (i) Fuzzy vectors $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$ are said to be strictly order preserving on $\mathbb{R}^n$ if $[\widetilde{a}]_0 <_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 <_S [\widetilde{b}]_1$ imply $\widetilde{a} \prec \widetilde{b}$, or if $[\widetilde{a}]_0 >_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 >_S [\widetilde{b}]_1$ imply $\widetilde{a} \succ \widetilde{b}$.

(ii) A class of fuzzy vectors, $\mathcal{G} \subset \mathcal{FV}(\mathbb{R}^n)$, is said to be strictly order preserving on $\mathbb{R}^n$ if any $\widetilde{a}, \widetilde{b} \in \mathcal{G}$ are strictly order preserving on $\mathbb{R}^n$.

In the following, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of the orderings of crisp sets are investigated when the crisp sets vary parametrically.

The following proposition shows properties of the orderings of crisp sets when the crisp sets vary parametrically.

Proposition 4. Let $A, B \subset \mathbb{R}^n$, and let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. In addition, let $r : [0, 1] \to [0, 1]$ be a monotone decreasing function. Assume that $r(0) = 1$ and $r(1) = 0$. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ and $G(\alpha) = r(\alpha)B + \boldsymbol{b}$ for each $\alpha \in [0, 1]$.

(i) If $F(0) \leq_S G(0)$ and $F(1) \leq_S G(1)$, then $F(\alpha) \leq_S G(\alpha)$ for any $\alpha \in [0, 1]$.

(ii) If $F(0) <_S G(0)$ and $F(1) <_S G(1)$, then $F(\alpha) <_S G(\alpha)$ for any $\alpha \in [0, 1]$.

Proof. We show only (i). (ii) can be shown in the similar way to (i). ¿From Proposition 2, if $A = \emptyset$ or $B = \emptyset$, then the conclusion is obtained. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\alpha \in [0, 1]$. Since $F(0) \leq_S G(0)$ and $F(1) \leq_S G(1)$, it follows that $B + \boldsymbol{b} \subset A + \boldsymbol{a} + \mathbb{R}^n_+$, $A + \boldsymbol{a} \subset B + \boldsymbol{b} + \mathbb{R}^n_-$, and $\boldsymbol{a} \leq \boldsymbol{b}$. Though it needs to show that (i-1) $r(\alpha)B + \boldsymbol{b} \subset r(\alpha)A + \boldsymbol{a} + \mathbb{R}^n_+$ and (i-2) $r(\alpha)A + \boldsymbol{a} \subset r(\alpha)B + \boldsymbol{b} + \mathbb{R}^n_-$, we show only (i-1). (i-2) can be shown in the similar way to (i-1). Let $\boldsymbol{x} \in r(\alpha)B + \boldsymbol{b}$. Then, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} = r(\alpha)\boldsymbol{y} + \boldsymbol{b}$. Since $\boldsymbol{a} \leq \boldsymbol{b}$, there exists $\boldsymbol{d}_1 \in \mathbb{R}^n_+$ such that $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{d}_1$. Since $B + \boldsymbol{b} \subset A + \boldsymbol{a} + \mathbb{R}^n_+$, there exist $\boldsymbol{z} \in A$ and $\boldsymbol{d}_2 \in \mathbb{R}^n_+$ such that $\boldsymbol{y} + \boldsymbol{b} = \boldsymbol{z} + \boldsymbol{a} + \boldsymbol{d}_2$. Therefore, we have $\boldsymbol{x} = r(\alpha)\boldsymbol{y} + \boldsymbol{b} = r(\alpha)(\boldsymbol{y} + \boldsymbol{b}) + (1 - r(\alpha))\boldsymbol{b} = r(\alpha)(\boldsymbol{z} + \boldsymbol{a} + \boldsymbol{d}_2) + (1 - r(\alpha))(\boldsymbol{a} + \boldsymbol{d}_1) = r(\alpha)\boldsymbol{z} + \boldsymbol{a} + (r(\alpha)\boldsymbol{d}_2 + (1 - r(\alpha))\boldsymbol{d}_1) \in r(\alpha)A + \boldsymbol{a} + \mathbb{R}^n_+$. □

The following proposition shows sufficient conditions for generated fuzzy sets by the mapping M defined by (2) to be fuzzy vectors.

Proposition 5. Let $A \subset \mathbb{R}^n$ be a convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0, 1] \to [0, 1]$ be a monotone decreasing function. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for each $\alpha \in [0, 1]$, and $\widetilde{s} = M(\{F(\alpha)\}_{\alpha \in]0,1]})$.

(i) $\widetilde{s}$ is a convex fuzzy set.

(ii) If A is a closed set, then $\widetilde{s}$ is a closed fuzzy set.

(iii) If A is a compact set, $r(1) = 0$, and r is left-continuous at 1, then $\widetilde{s} \in \mathcal{FV}(\mathbb{R}^n)$.

Proof. (i) and (ii) follow from Proposition 1. We show (iii). Since A is a closed convex set, $\widetilde{s}$ is a closed convex fuzzy set from (i) and (ii).

We show that $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$ is bounded. For $\boldsymbol{x} \in \mathbb{R}^n \setminus (A + \boldsymbol{a})$, since $F(\alpha) \subset F(0) = r(0)A + \boldsymbol{a} \subset A + \boldsymbol{a}$ for any $\alpha \in [0, 1]$, it follows that $\widetilde{s}(\boldsymbol{x}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{x}) = 0$. Since $(A + \boldsymbol{a})^c \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) = 0\}$, it follows that $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\} \subset A + \boldsymbol{a}$. Therefore, $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$ is bounded.

For $\boldsymbol{x} \in \mathbb{R}^n$, we show that $\widetilde{s}(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} = \boldsymbol{a}$. Since $\boldsymbol{a} = r(\alpha)\boldsymbol{0} + \boldsymbol{a} \in r(\alpha)A + \boldsymbol{a} = F(\alpha)$ for any $\alpha \in [0, 1]$, it follows that $\widetilde{s}(\boldsymbol{a}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{a}) = 1$. For $\boldsymbol{b} \in \mathbb{R}^n$, we show that $\boldsymbol{b} \neq \boldsymbol{a}$ implies $\widetilde{s}(\boldsymbol{b}) < 1$. Since A is bounded, there exists $L > 0$ such that $A \subset B_L = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq L\}$, where $\| \cdot \|$ is the Euclidean norm. Since $r(\alpha)A \subset r(\alpha)B_L$ for any $\alpha \in [0, 1]$, it follows that $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset r(\alpha)B_L + \boldsymbol{a}$ for any $\alpha \in [0, 1]$. We set $\beta = \|\boldsymbol{b} - \boldsymbol{a}\|$. Since $r(1) = 0$ and r is left-continuous at 1, there exists $\delta > 0$ such that $|\alpha - 1| < \delta$ and $\alpha \in [0, 1]$ imply $r(\alpha) < \frac{\beta}{L}$. Thus, there exists $\alpha_0 \in]0, 1[$ such that $\alpha \in [\alpha_0, 1]$ implies $\boldsymbol{b} \notin r(\alpha)B_L + \boldsymbol{a}$, and then $\boldsymbol{b} \notin F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for any $\alpha \in [\alpha_0, 1]$. Therefore, we have $\widetilde{s}(\boldsymbol{b}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{b}) \leq \alpha_0 < 1$. □

The following proposition shows a property of the ordering of crisp sets decreasing parametrically.

Proposition 6. Let $F(\beta)$, $G(\beta) \subset \mathbb{R}^n$, $\beta \in]0, 1]$ be closed sets. Assume that $F(\gamma) \supset F(\delta)$ and $G(\gamma) \supset G(\delta)$ for $\gamma, \delta \in]0, 1]$ with $\gamma \leq \delta$, and that $\cup_{\beta \in]0,1]} F(\beta)$ and $\cup_{\beta \in]0,1]} G(\beta)$ are bounded. Let $\alpha \in]0, 1]$. Assume that $\cap_{\beta \in]0,\alpha[} F(\beta) \neq \emptyset$ and $\cap_{\beta \in]0,\alpha[} G(\beta) \neq \emptyset$. If $F(\beta) \leq_S G(\beta)$ for any $\beta \in]0, \alpha[$, then $\cap_{\beta \in]0,\alpha[} F(\beta) \leq_S \cap_{\beta \in]0,\alpha[} G(\beta)$.

Proof. For any $\beta \in]0, \alpha[$, since $F(\beta) \leq_S G(\beta)$, it follows that $G(\beta) \subset F(\beta) + \mathbb{R}^n_+$ and $F(\beta) \subset G(\beta) + \mathbb{R}^n_-$. Though it needs to show that (i) $\cap_{\beta \in]0,\alpha[} G(\beta) \subset \cap_{\beta \in]0,\alpha[} F(\beta) + \mathbb{R}^n_+$ and (ii) $\cap_{\beta \in]0,\alpha[} F(\beta) \subset \cap_{\beta \in]0,\alpha[} G(\beta) + \mathbb{R}^n_-$, we show only (i). (ii) can be shown in the similar way to (i). Since $G(\beta) \subset F(\beta) + \mathbb{R}^n_+$ for any $\beta \in]0, \alpha[$, it follows that $\cap_{\beta \in]0,\alpha[} G(\beta) \subset \cap_{\beta \in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right)$. Thus, it is sufficient to show that $\cap_{\beta \in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right) \subset \cap_{\beta \in]0,\alpha[} F(\beta) + \mathbb{R}^n_+$. Let $\boldsymbol{x} \in \cap_{\beta \in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right)$. For each $\beta \in]0, \alpha[$, there exist $\boldsymbol{y}_\beta \in F(\beta)$ and $\boldsymbol{d}_\beta \in \mathbb{R}^n_+$ such that $\boldsymbol{x} = \boldsymbol{y}_\beta + \boldsymbol{d}_\beta$. Fix any $\{\beta_k\} \subset]0, \alpha[$ with $\beta_k \to \alpha$. Since $\{\boldsymbol{y}_{\beta_k}\} \subset \cup_{\beta \in]0,1]} F(\beta)$ is bounded, without loss of generality, suppose that $\boldsymbol{y}_{\beta_k} \to \boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \mathbb{R}^n$. Then, it follows that $\boldsymbol{d}_{\beta_k} = \boldsymbol{x} - \boldsymbol{y}_{\beta_k} \to \boldsymbol{x} - \boldsymbol{y}_0 \in \mathbb{R}^n_+$. For any $\beta \in]0, \alpha[$, there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $\beta_k \in]\beta, \alpha[$, and it follows that $\{\boldsymbol{y}_{\beta_k}\}_{k \geq k_0} \subset F(\beta)$, and that $\boldsymbol{y}_{\beta_k} \to \boldsymbol{y}_0 \in F(\beta)$ since $F(\beta)$ is a closed set. Since $\boldsymbol{y}_0 \in F(\beta)$ for any $\beta \in]0, \alpha[$, we have $\boldsymbol{x} = \boldsymbol{y}_0 + (\boldsymbol{x} - \boldsymbol{y}_0) \in \cap_{\beta \in]0,\alpha[} F(\beta) + \mathbb{R}^n_+$. □

The following proposition shows a property of 0-level sets of generated fuzzy sets by the mapping M defined by (2).

Proposition 7. Let $A \subset \mathbb{R}^n$ be a compact convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0, 1] \to [0, 1]$ be a monotone decreasing function. Assume that $r(0) = 1$, and that r is right-continuous at 0. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for each $\alpha \in [0, 1]$, and $\widetilde{s} = M\left(\{F(\alpha)\}_{\alpha \in]0,1]}\right)$. Then, $A + \boldsymbol{a} = [\widetilde{s}]_0$.

Proof. Since $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset F(0) = A + \boldsymbol{a}$ for any $\alpha \in [0, 1]$, it follows that $\widetilde{s}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus (A + \boldsymbol{a})$. Since $(A + \boldsymbol{a})^c \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) = 0\}$, it follows that $A + \boldsymbol{a} \supset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$. Therefore, we have $A + \boldsymbol{a} \supset [\widetilde{s}]_0$ since $A + \boldsymbol{a}$ is a closed set.

Let $\boldsymbol{x}_0 \in A + \boldsymbol{a}$. Then, there exists $\boldsymbol{y}_0 \in A$ such that $\boldsymbol{x}_0 = \boldsymbol{y}_0 + \boldsymbol{a}$. If $\boldsymbol{y}_0 = \boldsymbol{0}$, then $\boldsymbol{x}_0 = \boldsymbol{a} \in [\widetilde{s}]_0$. Thus, suppose that $\boldsymbol{y}_0 \neq \boldsymbol{0}$. We set $\lambda_0 = \max\{\lambda \geq 0 : \lambda \boldsymbol{y}_0 \in A\} \geq 1$.

Suppose that $\lambda_0 > 1$, and fix any sufficiently small $\delta > 0$. Since r is right-continuous at 0, $\alpha \in [0, \delta[$ implies $1 - r(\alpha) < 1 - \frac{1}{\lambda_0}$. For any $\alpha \in [0, \delta[$, it follows that $0 < \frac{1}{r(\alpha)\lambda_0} < 1$ and $r(\alpha)\lambda_0 \boldsymbol{y}_0 \in r(\alpha)A$, and that $\boldsymbol{y}_0 = \frac{1}{r(\alpha)\lambda_0} \cdot r(\alpha)\lambda_0 \boldsymbol{y}_0 \in r(\alpha)A$, and that $\boldsymbol{x}_0 = \boldsymbol{y}_0 + \boldsymbol{a} \in r(\alpha)A + \boldsymbol{a} = F(\alpha)$, and that $c_{F(\alpha)}(\boldsymbol{x}_0) = 1$. Therefore, since $\widetilde{s}(\boldsymbol{x}_0) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{x}_0) \geq \delta > 0$, we have $\boldsymbol{x}_0 \in [\widetilde{s}]_0$.

Suppose that $\lambda_0 = 1$. By the same arguments as in the case $\lambda_0 > 1$, it can be seen that $\widetilde{s}(\lambda \boldsymbol{y}_0 + \boldsymbol{a}) > 0$ for any $\lambda \in]0, 1[$. Choose any $\{\lambda_k\} \subset]0, 1[$ with $\lambda_k \to 1$. Since $\{\lambda_k \boldsymbol{y}_0 + \boldsymbol{a}\} \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$, we have $\lambda_k \boldsymbol{y}_0 + \boldsymbol{a} \to \boldsymbol{y}_0 + \boldsymbol{a} = \boldsymbol{x}_0 \in [\widetilde{s}]_0$. □

The following proposition shows a property of crisp sets decreasing parametrically.

Proposition 8. Let $A \subset \mathbb{R}^n$ be a compact convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0, 1] \to [0, 1]$ be a monotone decreasing function. We set $F(\beta) = r(\beta)A + \boldsymbol{a}$ for each $\beta \in [0, 1]$. If r is left-continuous at $\alpha \in]0, 1]$, then $F(\alpha) = \cap_{\beta \in]0,\alpha[} F(\beta)$.

Proof. It follows that $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset \cap_{\beta \in]0,\alpha[} (r(\beta)A + \boldsymbol{a}) = \cap_{\beta \in]0,\alpha[} F(\beta)$. In order to show that $r(\alpha)A + \boldsymbol{a} \supset \cap_{\beta \in]0,\alpha[} (r(\beta)A + \boldsymbol{a})$, suppose that $\boldsymbol{x}_0 \in \cap_{\beta \in]0,\alpha[} (r(\beta)A + \boldsymbol{a})$ and $\boldsymbol{x}_0 \notin r(\alpha)A + \boldsymbol{a}$. Since $\boldsymbol{x}_0 \in \cap_{\beta \in]0,\alpha[} (r(\beta)A + \boldsymbol{a}) \subset A + \boldsymbol{a}$, it follows that $\boldsymbol{x}_0 - \boldsymbol{a} \in A$. Since $\boldsymbol{x}_0 \notin r(\alpha)A + \boldsymbol{a}$, it follows that $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\alpha)A$. Thus, it follows that $r(\alpha) < 1$ and $\boldsymbol{x}_0 - \boldsymbol{a} \neq \boldsymbol{0}$. We set $\lambda_0 = \max\{\lambda \geq 0 : \lambda(\boldsymbol{x}_0 - \boldsymbol{a}) \in A\} \geq 1$. Then, since $\boldsymbol{x}_0 - \boldsymbol{a} \in \frac{1}{\lambda_0} A$ and $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\alpha)A$, it follows that $\frac{1}{\lambda_0} > r(\alpha)$. Fix any sufficiently small $\delta > 0$. Since r is left-continuous at α, $\beta \in]\alpha - \delta, \alpha]$ implies $r(\beta) - r(\alpha) < \frac{1}{\lambda_0} - r(\alpha)$. Fix any $\beta_0 \in]\alpha - \delta, \alpha[$. Then, it follows that $r(\beta_0) < \frac{1}{\lambda_0}$. If $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\beta_0)A$, then it follows that $\boldsymbol{x}_0 \notin r(\beta_0)A + \boldsymbol{a}$, and that $\boldsymbol{x}_0 \notin \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$, wihch is a contradiction. Thus, in order to show that $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\beta_0)A$, suppose that $\boldsymbol{x}_0 - \boldsymbol{a} \in r(\beta_0)A$. Then, since $\lambda_0(\boldsymbol{x}_0 - \boldsymbol{a}) \in \lambda_0 r(\beta_0)A$ and $\lambda_0 r(\beta_0) < 1$, for sufficiently small $\varepsilon > 0$, it follows that $(1+\varepsilon)\lambda_0(\boldsymbol{x}_0 - \boldsymbol{a}) \in (1+\varepsilon)\lambda_0 r(\beta_0)A \subset A$ and $\lambda_0 < (1+\varepsilon)\lambda_0$, which contradict the definition of λ_0. □

4 Main results In this section, based on the mapping M defined by (2), some classes of fuzzy vectors which have the order preserving property are constructed and proposed.

The following proposition shows sufficient conditions for generated fuzzy vectors by the mapping M defined by (2) to have the order preserving property.

Proposition 9. Let $A, B \subset \mathbb{R}^n$ be compact convex sets containing the origin, and let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. In addition, let $r : [0,1] \to [0,1]$ be a monotone decreasing function. Assume that $r(0) = 1$ and $r(1) = 0$, and that r is right-continuous at 0 and left-continuous at 1. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ and $G(\alpha) = r(\alpha)B + \boldsymbol{b}$ for each $\alpha \in [0,1]$, and $\widetilde{a} = M\left(\{F(\alpha)\}_{\alpha\in]0,1]}\right)$ and $\widetilde{b} = M\left(\{G(\alpha)\}_{\alpha\in]0,1]}\right)$.

(i) If $[\widetilde{a}]_0 \leq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \leq_S [\widetilde{b}]_1$, then $\widetilde{a} \preceq \widetilde{b}$.

(ii) Assume that r is left-continuous. If $[\widetilde{a}]_0 <_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 <_S [\widetilde{b}]_1$, then $\widetilde{a} \prec \widetilde{b}$.

Proof. (i) It follows that $A + \boldsymbol{a} \leq_S B + \boldsymbol{b}$ from Proposition 7, and that $\boldsymbol{a} \leq \boldsymbol{b}$ from Proposition 5. ¿From Proposition 4, it follows that $r(\alpha)A + \boldsymbol{a} \leq_S r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in [0,1]$. Since $[\widetilde{a}]_\alpha = \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$ and $[\widetilde{b}]_\alpha = \cap_{\beta\in]0,\alpha[}(r(\beta)B + \boldsymbol{b})$ for any $\alpha \in]0,1]$ from Proposition 1, it follows that $[\widetilde{a}]_\alpha \leq_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0,1]$ from Proposition 6. Therefore, we have $\widetilde{a} \preceq \widetilde{b}$.

(ii) It follows that $A + \boldsymbol{a} <_S B + \boldsymbol{b}$ from Proposition 7, and that $\boldsymbol{a} < \boldsymbol{b}$ from Proposition 5. ¿From Proposition 4, it follows that $r(\alpha)A + \boldsymbol{a} <_S r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in [0,1]$. Since $[\widetilde{a}]_\alpha = \bigcap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a}) = r(\alpha)A + \boldsymbol{a}$ and $[\widetilde{b}]_\alpha = \bigcap_{\beta\in]0,\alpha[}(r(\beta)B + \boldsymbol{b}) = r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in]0,1]$ from Propositions 1 and 8, it follows that $[\widetilde{a}]_\alpha <_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0,1]$. Therefore, we have $\widetilde{a} \prec \widetilde{b}$. □

In the following, some classes of fuzzy vectors which have the order preserving property are constructed based on the obtained results. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$ containing the origin, and let $\mathcal{R}$ be the set of all monotone decreasing functions from $[0,1]$ to $[0,1]$. We set

$$\begin{aligned}
\mathcal{R}_1 &= \{r \in \mathcal{R} : r(0) = 1, r(1) = 0, \\
&\qquad \text{and } r \text{ is right-continuous at 0 and left-continuous at 1 }\}, \\
\mathcal{R}_2 &= \{r \in \mathcal{R}_1 : \ r \text{ is left-continuous }\}.
\end{aligned}$$

In addition, we set

$$\begin{aligned}
\mathcal{S}^r(\mathbb{R}^n) &= \{\{r(\alpha)A + \boldsymbol{a}\}_{\alpha\in]0,1]} : A \in \mathcal{C}(\mathbb{R}^n), \boldsymbol{a} \in \mathbb{R}^n\}, \\
\mathcal{FV}_1^r(\mathbb{R}^n) &= \{M(\{S_\alpha\}_{\alpha\in]0,1]}) : \{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}^r(\mathbb{R}^n)\} = M(\mathcal{S}^r(\mathbb{R}^n))
\end{aligned}$$

for each $r \in \mathcal{R}_1$, and

$$\mathcal{FV}_2^r(\mathbb{R}^n) = \{M(\{S_\alpha\}_{\alpha\in]0,1]}) : \{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}^r(\mathbb{R}^n)\} = M(\mathcal{S}^r(\mathbb{R}^n))$$

for each $r \in \mathcal{R}_2$.

The following proposition shows that the classes of fuzzy sets constructed in the above are classes of fuzzy vectors which have the order preserving property.

Proposition 10. (i) $\mathcal{FV}_1^r(\mathbb{R}^n) \subset \mathcal{FV}(\mathbb{R}^n)$ for any $r \in \mathcal{R}_1$, and $\mathcal{FV}_2^r(\mathbb{R}^n) \subset \mathcal{FV}(\mathbb{R}^n)$ for any $r \in \mathcal{R}_2$.

(ii) $\mathcal{FV}_1^r(\mathbb{R}^n)$ is order preserving for any $r \in \mathcal{R}_1$.

(iii) $\mathcal{FV}_2^r(\mathbb{R}^n)$ is strictly order preserving for any $r \in \mathcal{R}_1$.

Proof. (i) follows from Proposition 5. (ii) and (iii) follow from Proposition 9. □

5 Conclusions In the present paper, we dealt with orderings of fuzzy vectors. When orderings of two fuzzy vectors were defined based on orderings of level sets of the fuzzy vectors, it needed to consider infinite many orderings of level sets of the fuzzy vectors. If finite many orderings of level sets of two fuzzy vectors imply the orderings of the fuzzy vectors, then it makes the orderings of fuzzy vectors easy to deal with for applications. Such property was defined as the order preserving property, and the order preserving property for fuzzy vectors was investigated. Based on classes of crisp sets decreasing parametrically, some classes of fuzzy vectors, which had the order preserving property and seemed to be useful for applications, were constructed and proposed.

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Communicated by *Junzo Watada*

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A CONDITION FOR REDUCING EXPANSIVE VARIATIONS OF OPTIMAL POLICY IN RESTAURANT REVENUE MANAGEMENT

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Received May 31, 2015; revised January 19, 2016

Abstract. An industry which is recently applied to revenue management is restaurant. The revenue management for restaurant is called *restaurant revenue management*. The restaurant revenue management has a problem by which state space enormously expands because of multi-dimensional resources and customers. This problem gives rise to some practical difficulty: computation complexity increases, required data size for optimal policy becomes larger and etc.. This paper presents a sufficient condition for substantially reducing data size of optimal policy.

1 Introduction There are many scenes at which a business manager controls the limited resources for variable demand to aim to maximize his(her) company's benefit. For companies with fixed capacity, dealing with perishable products and large fixed cost, how to manage the demand (e.g. setting variable terms and prices for each product and etc.) significantly affects their benefit. This management is widely known as *revenue management* or *yield management*. Traditional applications of the revenue management are airline, hotel and car rental industries.

In theory of the revenue management, there is a problem in which threshold price is solved by using dynamic programming. This problem is used to decide whether a revenue manager should accept for a request of reservation in a certain period to maximize revenue. This control by using the threshold price is called *bid price control*. Lee and Hersh(1993) suggested a bid price control model for airline industry with single resource, multiple booking classes and multiple seat booking. Further, they indicated monotonicity of threshold price for their model. However, the model did not include assumptions of cancellation and overbooking. Subramanian et al.(1999) considered a model with cancellation and overbooking, and added some assumptions to declare monotonicity of threshold price. Researches, problems, traditional models, and a glossary of revenue management for airline can be found in McGill and Ryzin(1999).

Recently, for non-traditional industries, the bid price control models have been widely researched. Chiang, Chen and Xu(2007) reviewed recent application and techniques of revenue management. One of the non-traditional industries which is applicable to the theory of revenue management is restaurant industry. The revenue management for restaurant is called *restaurant revenue management*. The bid price control model for the restaurant revenue management additionally need to decide which table a party should be allocated if the party should be accepted. The policy is called *seating policy* in Guerriero et al.(2014). There are not many researches which deals with the seating policy. Bertsimas and Shioda(2003) presented some models: an integer programming, a stochastic programming, and an approximate dynamic programming model. Guerriero et al.(2014) suggested a dynamic programming model with no waiting line, reservation, and meal duration by using the

2010 *Mathematics Subject Classification.* Primary 90B50 ; Secondary 90B80.
Key words and phrases. Revenue management, Network revenue management, Restaurant revenue management, Bid price control, Dynamic programming, Monotonicity.

techniques of *network revenue management.* These studies have focused on making models and algorithms for solving expected total revenue because the bid price control approach in restaurant revenue management is difficult for solving. The difficulty is due largely to *the curse of dimensionality.*

1.1 The curse of dimensionality in restaurant revenue management The bid price control model in restaurant revenue management is referred as a model in *network revenue management* because restaurants have multi-dimensional capacity which is the different size of tables. It is known that a model in the network revenue management is more complex than a model with single-resource. A part of reasons for the complexity is that state space enormously expands. Furthermore, in restaurant revenue management, state space of a bid price control model needs to enlarge more than ordinary models(seeing as an example in Sec.3.2 of Talluri and Ryzin(2005)) in network revenue management. Because the bid price control model in restaurant revenue management must include departure process of parties which implies cancellation process in the airline or hotel industry. Fig.1 shows states for cases with no-cancellation and with cancellation. The case without cancellation process is Case 1 and the another case with cancellation process is Case 2 in Fig.1.

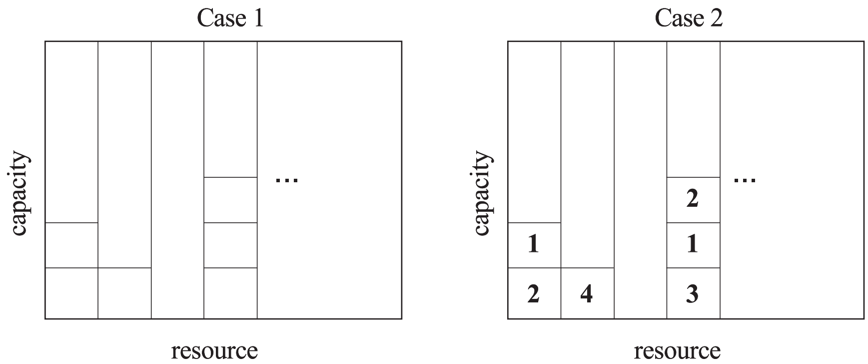

Figure 1: States in the cases without cancellation process(Case 1) and with cancellation process(Case 2).

In revenue management, the departure process commonly depends on a customer class. (See p.500 in Talluri and Ryzin(2005).) It is actually intuitive that the departure process depends on the customer class which implies size of party in restaurant revenue management. The state of the Case 1 in Fig.1 does not need to preserve the customer classes which have arrived until a certain period because of an assumption of the no-departure process. Hence, the state in the Case 1 is shown as a vector for capacity. In contrast, the state in the Case 2 needs to preserve the customer classes that have arrived until a certain period. This means that each resource in the Case 2 have a vector for the customer classes. Thus, the state space of Case 2 is much larger than the one of the Case 1. Additionally, If meal duration for each customer which is stated in Kimes et al.(1999)(2002), Guerriero et al.(2014) and etc. is considered, then an information about how long each customer has been in the state must add to the state and solving the seating problem as exact dynamic programming approach is practically impossible.

To broach this argument, in section 2, this paper presents an exact dynamic programming model for seating policy, given some conditions to simplify. Furthermore, some monotonicities are indicated by setting some realistic assumptions. From the monotonicities, this paper shows a sufficient condition for reducing varieties of optimal policy, and its structural property. In section 3, the structural property is confirmed by numerical examples.

2 A model and its property

2.1 Conditions and notation To simplify a model, some conditions are given to parties and tables. The conditions are that a composition of the tables can not be modified to suit the arriving party, size of the parties can not be divided to suit the tables, and the size of the parties does not exceed a maximum of the tables in the restaurant. Further, tables of the same size and seats are not distinguished. Suppose sets $P = \{1, \cdots, \overline{P}\}$ and $I = \{1, \cdots, \overline{I}\}$ for notations. The notations about the party and the table are shown as

- $\overline{P}$: the number of different party sizes,
- $\overline{I}$: the number of different table sizes,
- g_p: the party size for $p \in P$,
- t_i: the table size for $i \in I$,
- m_i: the number of the table for $i \in I$.

To simplify, we regard $p \in P$ as a party with party size g_p, and $i \in I$ as a table with table size t_i, respectively. Throughout this paper, a party p and a table i are indexed as $g_1 < g_2 < \cdots < g_p$ and $t_1 < t_2 < \cdots < t_i$, respectively. In addition, subsets for $p \in P$ and $i \in I$ are indicated as

- $P_i = \{p \in P : g_p \leq t_i\}, i \in I$: the party set which is able to be allocated to a table $i \in I$ with the number of the different party sizes $\overline{P}_i$,
- $I_p = \{i \in I : g_p \leq t_i\}, p \in P$: the table set to which a party $p \in P$ is able to be allocated with the number of the different table sizes $\overline{I}_p$.

The opening horizon is sufficiently divided into the $N + 1$ periods $n = 0, 1, \cdots, N$. One event of the customer's arrival or departure occurs in the period n. A period N corresponds to opening of the restaurant and a period 0 corresponds to closing of the restaurant. Parties arrive according to time-dependent Poisson process while the restaurant is opening. All of them are walk-in customers, without reservation. Departure process of the parties depends on not their length of staying time, but the state of restaurant and the period. Notations about the state space, the arrival and departure rate, and expected revenue are shown as

- $\overline{X}_i = \{\boldsymbol{x}_i = (x_p^i) : x_p^i \geq 0, p \in P_i\,; \sum_p x_p \leq m_i\}, i \in I$: state space for a table $i \in I$ where x_p^i is the number of parties who are sitting in a table $i \in I_p$,
- $X_n = \{X = (\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_{\overline{I}}) : \boldsymbol{x}_i \in \overline{X}_i, i \in I; \sum_i \sum_p x_p^i \leq N - n\}, n = 0, \cdots N$: state space for a restaurant with a submatrix $\boldsymbol{x_i}$ in a period n,
- r_p^n: the expected revenue for a party $p \in P$ in a period n,
- $\lambda_p^n(X)$: the arrival rate for a party $p \in P$ and a state $X \in X_n$ in a period n, where $\lambda_p^n(X) > 0$,
- $q_{ip}^n(X)$: the departure rate for a party $p \in P_i$ where $i \in I$, and a state $X \in X_n$ in a period n,

- λ_0^n: a probability of a null event in period n.

Suppose that $|X_n|$ corresponds to the number of elements of the state space X_n for n. Referring p.15 in Stanley(1997), we can obtain a maximum of $|X_n|$ for n: $\chi = \max_n\{|X_n|\}$ as

$$\chi = \prod_{i=1}^{\overline{I}} \binom{m_i + \overline{P}_i}{\overline{P}_i}. \tag{1}$$

The eq.(1) is helpful to roughly estimate size of state space for a restaurant. From the assumption of the arrival and the departure process in a period n, the equation

$$\sum_{p=1}^{\overline{P}} \lambda_p^n(X) + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) + \lambda_0^n(X) = 1 \tag{2}$$

is obtained.

2.2 A formulation of model Let $U_n(X)$ be the maximal expected revenue from operating over periods n to 0. Firstly, Suppose the maximal expected revenue in a general form as follows.

$$\begin{aligned} U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n(X) \left\{ \left(r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) U_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left(1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) \right) U_{n-1}(X), \\ X \in X_n, n \geq 1, \end{aligned} \tag{3}$$

where $\boldsymbol{e}_p^i = (\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_{\overline{I}})$ in which $x_p^i = 1$ and otherwise 0, $(a)^+ = \max\{a, 0\}$, and $\Delta_p^i U_n(X) = U_n(X) - U_n(X + \boldsymbol{e}_p^i)$. Boundary conditions are that $U_n(X) = -\infty$ for $X \notin X_n$, and $U_0(X) = 0$ for $X \in X_0$. The $\min_{i \in I_p} \Delta_p^i U_n(X)$ means a threshold price for a party $p \in P_i$, such that the party p who arrives for the state X in n is acceptable if r_p^n exceeds the threshold price $\min_{i \in I_p} \Delta_p^i U_n(X)$ and not acceptable if r_p^n is less than the threshold price $\min_{i \in I_p} \Delta_p^i U_n(X)$(See pp.31-32 in Talluri and Ryzin(2005).). $\Delta_p^i U_n(X)$ is an opportunity cost of accepting the party p for the table $i \in I_p$ in $n + 1$. Note that $\lambda_0^n(X) = 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X)$ from eq.(2). The first member of the right hand in (3) indicates a expected value in a case where a party arrives at a restaurant in a period n. If a p is accepted in the table $i \in I_p$, then a expected value for the case is $r_p^n - \Delta_p^i U_{n-1}(X)$ in n. The second member indicates a expected value in a case where a party sitting in a restaurant leaves in a period n. The third member is for a case where no event occurs in a period n. From eq.(3), optimal policy is indicated as below.

Optimal policy: An optimal policy for a party $p \in P$ and a state $X \in X_n$ is that if

$r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) \geq 0$, then a party p is accepted in a table $\arg\min_{i\in I_p} \Delta_p^i U_{n-1}(X)$, and if $r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) < 0$, then a party p is denied.

Then, Some assumptions are supposed to simplify the eq.(3).

Assumption 1. assume $\lambda_p^n(X) = \lambda_p^n$ for $p \in P$ and $X \in X_n$ in $n = 0, \cdots, N$.

Assumption 2. assume $q_{ip}^n(X) = x_p^i q_{ip}^n$ for $p \in P_i$ where $i \in I$ and $X \in X_n$ in $n = 0, \cdots, N$.

The Assumption 1 indicates that arrival rates do not depend on states, which means that congestion level of a restaurant does not affect the arrival rates. The Assumption 2 indicates that a party p in a table i and a period n departs independently of other parties sitting in other table, which implies that a party leaves from a restaurant according to exponential distribution. Let t be $\Delta_N + \Delta_{N-1} + \cdots + \Delta_{n+1}$ where Δ_n is the length of the nth period. Suppose $t = 0$ for Nth period. λ_p^n indicates $f_p(t)\Delta_n$ where $f_p(t), 0 \leq t \leq \Delta_N + \Delta_{N-1} + \cdots + \Delta_1$ is a mean of time-dependent Poisson distribution for a p. q_{ip}^n indicates $\mu_{ip}(t)\Delta_n$ where $\mu_{ip}(t), 0 \leq t \leq \Delta_N + \Delta_{N-1} + \cdots + \Delta_1$ is a parameter of exponential distribution at time t for a p sitting in a table $i \in I_p$. For detail of this method, Subramanian et al.(1999) explained in Appendex A.

Under these assumptions, the eq.(3) can be rewritten as the equation

$$
\begin{aligned}
U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n \left\{ \left(r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\
+ \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} x_p^i q_{ip}^n U_{n-1}(X - \boldsymbol{e}_p^i) \\
+ \left(1 - \sum_{p=1}^{\overline{P}} \lambda_p^n - \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} x_p^i q_{ip}^n \right) U_{n-1}(X), \\
X \in X_n, n \geq 1.
\end{aligned}
\tag{4}
$$

Boundary conditions are not modified. The eq.(4) is close to a equation which is extended by cancellation process for the model with upgrades which is suggested as eq.(1) in Steinhardt and Gönsch(2012). However, state space of the model in Steinhardt and Gönsch(2012) is different from the one which is defined in this paper as previously shown in Sec.1.1. Note that the first member of eq.(4) is a case of the one of eq.(1) in Steinhardt and Gönsch(2012) because of physical bundles between parties and tables, and the condition on which composition of the tables and size of the parties are fixed. For proofs as following sections, policy vector $\boldsymbol{d}$ is defining.

Let the policy vector be $\boldsymbol{d} = (d_p)$ where $p \in P$. An element of the policy vector d_p is a table $i \in I_p$ ($d_p = i$) if a party p is accepted into the table, or 0 ($d_p = 0$) if a party p is denied. Assume that if there are some acceptable tables, then the smallest i is selected. As the result, a set of policy vector is defined as

$$D_n(X) = \{\boldsymbol{d} = (d_p) : (d_p = 0) \vee \left((X + \boldsymbol{e}_p^{d_p} \in X_n) \wedge (d_p \in I_p) \right), p \in P\}, X \in X_n, n = 1, \cdots, N.$$

2.3 Property of $\Delta_p^i U_n(X)$ and the optimal policy Supposing the Assumption 3 as below, a monotonicity which is similar to the monotonicity suggested as Proposition 1 in Steinhardt and Gönsch(2012) is obtained for $\Delta_p^i U_n(X)$ in eq.(4).

Assumption 3. assume $q_{\delta p}^n = q_{\delta' p}^n$ for $p \in P$ and $\delta, \delta' \in I_p$ in $n = 0, \cdots, N$ where $\overline{I}_p \geq 2$ and $\delta \neq \delta'$.

Lemma 1. Under assumuption 1 to 3, for a given $p \in P$ and $X \in X_n$ in $n = 0, \cdots, N$,

$$\Delta_p^{\delta} U_n(X) \leq \Delta_p^{\delta'} U_n(X) \tag{5}$$

where δ, $\delta' \in I_p$, $t_\delta < t_{\delta'}$, $\sum_p x_p^\delta < m_\delta$, and $\sum_p x_p^{\delta'} < m_{\delta'}$.

Proof. $U_n(X + \boldsymbol{e}_p^\delta) \geq U_n(X + \boldsymbol{e}_p^{\delta'})$ should be indicated by induction for $\Delta_p^\delta U_n(X) \leq \Delta_p^{\delta'} U_n(X)$. For $n = 0$, It is obvious that $U_0(X + \boldsymbol{e}_p^\delta) = U_0(X + \boldsymbol{e}_p^{\delta'}) = 0$. Then, assume that $U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$. Let the first member, the second member, and the third member of the equation (4) call *arrival part, departure part*, and *null part*, respectively. In the following, we are indicating the orderings of each part.

Firstly, an order of the arrival part is indicated. The arrival part of eq.(4) is rewritten using the optimal vector as

$$\max_{\boldsymbol{d} \in D_n(X)} \left\{ \sum_{p|d_p \neq 0} \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{d_p})) + \sum_{p|d_p = 0} \lambda_p^n U_{n-1}(X) \right\}.$$

Let optimal policy vectors for $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$ be $\boldsymbol{d}^{(\delta)*}$ and $\boldsymbol{d}^{(\delta')*}$, respectively.

For a given $p \in P$, there are four cases for $d_p^{(\delta)*}$ and $d_p^{(\delta')*}$ as follows.

i) In the case: $d_p^{(\delta)*} \neq 0$ and $d_p^{(\delta')*} \neq 0$, we should make a comparison between $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta)*}})$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta')*}})$ for the arrival parts of $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$. Further, this case is divided into two cases for ordering between $d_p^{(\delta)*}$ and $d_p^{(\delta')*}$.

i-1)In the case: $d_p^{(\delta)*} \leq d_p^{(\delta')*}$, from the inductive hypothesis, $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta)*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta)*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta')*}})$ is obtained

i-2)In the case: $d_p^{(\delta)*} > d_p^{(\delta')*}$, from the inductive hypothesis and number of capacities of tables, $d_p^{(\delta)*} \leq \delta'$ and $d_p^{(\delta')*} = \delta$ is obtained. Thus, $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta)*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^\delta) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta')*}})$.

ii)In the case: $d_p^{(\delta)*} = 0$ and $d_p^{(\delta')*} \neq 0$, we should make a comparison between $U_{n-1}(X + \boldsymbol{e}_p^\delta)$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta')*}})$. From the inductive hypothesis and $d_p^{(\delta)*} = 0$, $U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta')*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{d_p^{(\delta')*}})$.

iii)In the case: $d_p^{(\delta)*} \neq 0$ and $d_p^{(\delta')*} = 0$, we should make a comparison between $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta)*}})$ and $U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$. From the inductive hypothesis and $d_p^{(\delta)*} \neq 0$, $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{d_p^{(\delta)*}}) \geq U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$.

iv)In the case: $d_p^{(\delta)*} = d_p^{(\delta')*} = 0$, from the inductive hypothesis, It is obvious that $U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$.

Next, we consider the departure parts. To simplify the notation, suppose that $q_{ip}^n = q_p^n$. For the p, the departure parts of $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$ are

$$\sum_{i \in I_p} (x_p^i + e_p^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \tag{6}$$

and

$$\sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{7}$$

respectively, where $e_p^{ki} = 1$ if $i = k$ and otherwise $e_p^{ki} = 0$. The eq.(6) and (7) can stand for

$$\begin{aligned} q_p^n \left\{ \cdots + (x_p^\delta + 1)U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^\delta) + \cdots + x_p^{\delta'} U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^{\delta'}) + \cdots \right\} \\ = q_p^n \left\{ U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^1) + \cdots \right\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} q_p^n \left\{ \cdots + x_p^\delta U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^\delta) + \cdots + (x_p^{\delta'} + 1)U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^{\delta'}) + \cdots \right\} \\ = q_p^n \left\{ U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^1) + \cdots \right\}, \end{aligned} \tag{9}$$

respectively. Therefore, from the inductive hypothesis, $\sum_{i\in I_p}(x_p^i + e_p^{\delta i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \geq \sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i)$ is obtained.

Finally, we consider the null parts. For the p, the null parts of $U_n(X+\boldsymbol{e}_p^\delta)$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$ are

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta i})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^\delta) \tag{10}$$

and

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^{\delta'}), \tag{11}$$

respectively. In these equations, the coefficients of the $U_{n-1}(X + \boldsymbol{e}_p^\delta)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$ are the same. Thus,

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^\delta)q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq \left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta'})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$$

is obtained from the inductive hypothesis.

From these ordering of the arrival parts, the departure parts, and the null parts of $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$, the eq.(5) is indicated. □

The Assumption 3 means that departure rate depends on only a period and a party size. Thus, q_{ip}^n stands for q_p^n to simplify in the following. For this assumption, Kimes et al.(2004) suggested that meal duration which relates to the departure rate did not depend on position, configuration, and size of tables while it depended on the size of a party. Therefore, the Assumption 3 can be considered as realistic one.

For the submatrix $\boldsymbol{x}_i$ of $X \in X_n$, suppose $\sum_p x_p^i := x^i$. Furthermore, let $X \in X_n$ and $\hat{X} \in X_n$ be the states with submatrices $\boldsymbol{x}_i$ and $\hat{\boldsymbol{x}}_i$, respectively, where $X \neq \hat{X}$ and $x^i = \hat{x}^i$ for $i \in I$. This assumption for X and $\hat{X}$ is used in the following this section.

The Claim 1 is obtained from the Lemma 1.

Claim 1. If optimal policy vectors $\boldsymbol{d}^*$ and $\hat{\boldsymbol{d}}^*$ for the states X and $\hat{X}$, respectively, are $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, then $d_p^* = \hat{d}_p^*$.

Proof. From $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, arrival parts of $U_n(X)$ and $U_n(\hat{X})$ are

$$\lambda_p^n(r_p^n + U_{n-1}(X + \boldsymbol{e}^{d_p^*})) \tag{12}$$

and

$$\lambda_p^n(r_p^n + U_{n-1}(\hat{X} + \boldsymbol{e}^{\hat{d}_p^*})), \tag{13}$$

respectively. From $x^i = \hat{x}^i$, the table sets which are able to be d_p^* and $\hat{d}_p^*$ for $p \in P$ are the same. Then, $d_p^* = \hat{d}_p^*$ is obtained. □

Suppose an assumption for the ordering of departure process of parties $p \in P$, and a proposition about a monotonicity of $\Delta_p^i U_n(X)$ for $p \in P$ as below.

Assumption 4. For $\psi \in P$ and $\psi' \in P$ where $\psi < \psi'$, assume $q_\psi^n \geq q_{\psi'}^n$ in $n = 0, \cdots, N$.

Proposition 1. Under the Assumption 1 to 4, for a given $\sigma \in I$ at which $\overline{P}_\sigma \geq 2$,

$$\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X), \tag{14}$$

where $\psi, \psi' \in P_\sigma$ and $\psi < \psi'$, in $n = 0, \cdots, N$.

Proof. It is obtained by induction. $U_n(X + \boldsymbol{e}_\psi^\delta) \geq U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ should be indicated for $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. In the case $n = 0$, $U_0(X + \boldsymbol{e}_\psi^\delta) = U_0(X + \boldsymbol{e}_{\psi'}^\delta)$ is clear. Then, assume that $\Delta_\psi^\delta U_{n-1}(X) \leq \Delta_{\psi'}^\delta U_{n-1}(X)$.

Firstly, we consider about the arrival parts. Let the optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ be $\boldsymbol{d}^{(\psi)^*}$ and $\boldsymbol{d}^{(\psi')^*}$, respectively.

i)In the case: $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. The optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ are $d_p^{(\psi)^*} = d_p^{(\psi')^*}$ from the Claim 1 because capacities of the states are the same. Hence, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is indicated.

ii)In the case: $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} = 0$, we compare $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ to $U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$. From the inductive hypothesis and $d_p^{(\psi')^*} = 0$, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq U_{n-1}(X + \boldsymbol{e}_\psi^\delta) \geq U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$ is obtained.

iii)In the case: $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $U_{n-1}(X + \boldsymbol{e}_\psi^\delta)$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. From the inductive hypothesis and $d_p^{(\psi)^*} = 0$, $U_{n-1}(X + \boldsymbol{e}_\psi^\delta) > r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is obtained.

iv)In the case: $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} = 0$, it is obvious.

Then, we consider the departure parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ which are

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \tag{15}$$

and

$$\sum_{p=1}^{\overline{P}}\sum_{i\in I_p}(x_p^i + e_{\psi' p}^{\delta i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{16}$$

respectively, where $e_{lp}^{ki} = 1$ if $i = k$ and $p = l$, otherwise $e_{lp}^{ki} = 0$.

We should consider only the cases $p = \psi$, $i = \sigma$ and $p = \psi'$, $i = \delta$ for the eq.(15) and eq.(16) as

$$\begin{aligned} &\cdots + q_\psi^n U_{n-1}(X) + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ &\qquad \cdots + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\qquad \cdots + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ &\cdots + q_{\psi'}^n U_{n-1}(X) + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots . \end{aligned} \tag{18}$$

From the inductive hypothesis and the Assumption 4, it is indicated that

$$\sum_{p=1}^{\overline{P}}\sum_{i\in I_p}(x_p^i + e_{\psi p}^{\delta i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta} - \boldsymbol{e}_p^i) \geq \sum_{p=1}^{\overline{P}}\sum_{i\in I_p}(x_p^i + e_{\psi' p}^{\delta i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i).$$

Finally, we consider the null parts. It is clear that coefficients of the null parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ are the same.

From the ordering of the each part, we obtain that $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. □

The Assumption 4 means that a party stochastically stays longer than the smaller one. Thompson(2009) applied this assumption to his simulation study. Furthermore, the researches in Kimes et al.(2003) and Bell and Pliner(2004) showed that a correlation between the size of a party and meal duration is significantly positive for real restaurants. Therefore, the Assumption 4 is considered as realistic one.

The Remark 1 for the Proposition 1 is indicated as follows.

Remark 1. Note that the monotonicity of the Proposition 1 does not depend on the expected revenue r_p^n, which is same to the Lemma 1. Seeing the proof for the Proposition1, we can recognize that the Assumption 4 is used in only the members of $U_{n-1}(X)$ in the eq.(17) and (18). Further, the orderings for the each part expect the the members of $U_{n-1}(X)$ in eq.(17) and (18) is conditioned by the inductive hypothesis and facts of the cases. Thus, the ordering of the Proposition 1 is conditioned by only the ordering of departure rates between the parties.

Thus, from the Proposition 1 and its character, a difference between the maximal expected revenues $U_n(X)$ and $U_n(\hat{X})$ stems from differences for departure rates among parties. If there are differences for departure rates among parties, then they are affected by all factors; arrival rates, rewards, and etc. as a matter of course. However, If there are not the differences for departure rates among parties, then there is not the difference between $U_n(X)$ and $U_n(\hat{X})$, nevertheless the parties have difference parameters each other.

From the monotonicities which is indicated in this paper, a sufficient condition which is able to reduce variations of optimal policies can be obtained. The sufficient condition is shown as Theorem 1. For given a party $p \in P$, let $\overline{d}_p^*$ be the minimum $i \in I_p$ where the $m_i - \sum_{k\in P_i} x_k^i > 0$.

Theorem 1. If the condition

$$(19)\quad r_p^n \notin \left[\min(\Delta_p^{\overline{d}_p^*}U_{n-1}(X), \Delta_p^{\overline{d}_p^*}U_{n-1}(\hat{X})), \max(\Delta_p^{\overline{d}_p^*}U_{n-1}(X), \Delta_p^{\overline{d}_p^*}U_{n-1}(\hat{X}))\right)$$

is satisfied for a given $p \in P$ and n, then the optimal vectors $\boldsymbol{d}^*$ and $\hat{\boldsymbol{d}}^*$ for the states X and $\hat{X}$, respectively is that $d_p^* = \hat{d}_p^*$ for the $p \in P$ in the period n.

Proof. The d_p^* and $\hat{d}_p^*$ are divided in four cases.
i)In the case: $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, from the Claim 1, $d_p^* = \hat{d}_p^*$.
ii)In the case: $d_p^* = 0$ and $\hat{d}_p^* \neq 0$, from $d_p^* = 0$, we obtain that

$$(20)\qquad \lambda_p^n U_{n-1}(X) > \max_{d_p | d_p \neq 0} \left\{ \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{d_p})) \right\}.$$

In addition, the eq.(20) can be rewritten to

$$(21)\qquad \lambda_p^n U_{n-1}(X) > \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\hat{d}_p^*}))$$

from the condition $x^i = \hat{x}^i$. We also obtain that

$$(22)\qquad \lambda_p^n U_{n-1}(\hat{X}) \leq \lambda_p^n (r_p^n + U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\hat{d}_p^*}))$$

because of $\hat{d}_p^* \neq 0$. From the eq.(21) and (22), we indicate

$$(23)\qquad \Delta_p^{\hat{d}_p^*} U_{n-1}(\hat{X}) \leq r_p^n < \Delta_p^{\hat{d}_p^*} U_{n-1}(X)$$

as a condition for $d_p^* = 0$ and $\hat{d}_p^* \neq 0$.
iii)In the case: $d_p^* \neq 0$ and $\hat{d}_p^* = 0$, calculating this case similar to the case ii), we can obtain

$$(24)\qquad \Delta_p^{d_p^*} U_{n-1}(X) \leq r_p^n < \Delta_p^{d_p^*} U_{n-1}(\hat{X})$$

as a condition for $d_p^* \neq 0$ and $\hat{d}_p^* = 0$.
iv)In the case: $d_p^* = 0$ and $\hat{d}_p^* = 0$, it is clearly.

Then, the relation between $\hat{d}_p^*$ and d_p^* in eq.(23) and (24) is $\hat{d}_p^* = d_p^* = \overline{d}_p^*$ due to $x^i = \hat{x}^i$, Lemma 1, and $\hat{d}_p^*, d_p^* \neq 0$. Therefore, if a range which does not include the ranges (23) and (24):

$$(25)\quad r_p^n \notin \left[\min(\Delta_p^{\overline{d}_p^*}U_{n-1}(X), \Delta_p^{\overline{d}_p^*}U_{n-1}(\hat{X})), \max(\Delta_p^{\overline{d}_p^*}U_{n-1}(X), \Delta_p^{\overline{d}_p^*}U_{n-1}(\hat{X}))\right)$$

is satisfied for a $p \in P$ and n, then $d_p^* = \hat{d}_p^*$. □

The remark of the Theorem 1 is below.

Remark 2. The range(19) indicates a sufficient condition which makes the same optimal policy for the state X and $\hat{X}$. The width of the range $|\Delta_p^{\overline{d}_p^*}U_{n-1}(X) - \Delta_p^{\overline{d}_p^*}U_{n-1}(\hat{X})|$ stands for difficulty of reducing variety of the optimal policies. If the width becomes narrower, then it is more difficult to insert the expected revenue r_p^n into the range and optimal policy goes to depend only capacities for tables.

The width of the range $|\Delta_p^{\overline{d}_p^*} U_{n-1}(X) - \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})|$ can be rewritten $|U_{n-1}(\hat{X}) - U_{n-1}(X) + U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})|$ where the Proposition 1 is applicable to $U_{n-1}(\hat{X}) - U_{n-1}(X)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})$. If there are not differences in departure rates among parties, then the width is effected by nothing because the width is zero, regardless of existing differences in arrival rates or expected revenues among the parties. As a consequence of this property, existing the differences in departure rates among parties is an only trigger for expanding varieties of optimal policy.

3 Numerical Examples In this section, we confirm the feature which is stated in the Remark 2. Numerical examples are computed using an equation which is applied the Assumptions 1 to 4 to the eq.(4). Configurations for tables and parties are which $\overline{P} = 2$, $\overline{I} = 2$, $g_1 = 1, g_2 = 2$, $t_1 = 1, t_2 = 2$, $m_1 = 2$, and $m_2 = 2$. From this parameters sets, χ is 18 by using eq.(1). Arrival rates, departure rates, and expected revenues for each party $p \in P$ in a period n are shown in Table 1.

The parameters set in Table 1 is named Sample 1. The Sample 1 has a single peak for the arrival rates, departure rates, and expected revenues. The peak time is likely lunch time. The expected revenues in the Sample 1 are set to increase as they get closer to the peak time since a restaurant which is considered for this section also serves as a cafe except in lunch time. Optimal policies for $p = 1$ which is computed from the Sample 1 are shown in Table 2. The values in cells of the Table 2 stand for policy vectors.

Seeing optimal policies for states (2|1,0) and (2|0,1), we can find that the optimal policies in $n = 16$ and 17 are difference between the states; nevertheless capacities for the states are the same. Let the states (2|1,0) and (2|0,1) be X and $\hat{X}$, respectively. To confirm the Theorem 1 for the states, $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ where $\overline{d}_1^* = 2$, are shown in Table 3 which also includes the expected revenue r_1^n to make a comparison easily.

Table 1: Arrival rates, departure rates, and expected revenues of Sample 1.

	Arrival Rate		Departure Rate		Reward	
n	λ_1^n	λ_2^n	q_1^n	q_2^n	r_1^n	r_2^n
0-5	.021	.014	.018	.014	3	6
6-7	.105	.070	.088	.070	4	8
8-11	.150	.100	.125	.100	5	10
12-13	.105	.070	.088	.070	4	8
14-20	.021	.014	.018	.014	3	6

r_1^{16} and r_1^{17} are put in the ranges between $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ for $n = 16$ and $n = 17$. Then, for a case where there is not difference in departure rates between parties, we have computed the range. Sample 2 is the case in which the departure rates for $p = 2$ become the same to the ones for $p = 1$ for the Sample 1. $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$, $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$, and the width of the range are shown as Table 4.

Table 2: Optimal policies for $p = 1$ in period n.

$n\backslash X$	2∣0,2	2∣2,0	2∣0,0	2∣1,1	2∣0,1	2∣1,0	0∣0,2	0∣2,0	0∣0,0	0∣1,1	0∣0,1	0∣1,0	1∣0,2	1∣2,0	1∣0,0	1∣1,1	1∣0,1	1∣1,0
0	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
1	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
4	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
5	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
6	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
7	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
8	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
9	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
10	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
11	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
12	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
13	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
14	0	0	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
15	0	0	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
16	0	0	2	0	0	2	1	1	1	1	1	1	1	1	1	1	1	1
17	–	–	2	–	0	2	1	1	1	1	1	1	1	1	1	1	1	1
18	–	–	2	–	–	–	1	1	1	1	1	1	–	–	1	–	1	1
19	–	–	–	–	–	–	–	–	1	–	1	1	–	–	1	–	–	–
20	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–

Table 3: The range in Theorem 1 for the states X and $\hat{X}$.

n	r_1^n	$\Delta_1^{\overline{d_1^*}} U_{n-1}(X)$	$\Delta_1^{\overline{d_1^*}} U_{n-1}(\hat{X})$
0	3	–	–
1	3	0.000	0.000
2	3	0.147	0.147
3	3	0.282	0.282
4	3	0.405	0.406
5	3	0.518	0.521
6	4	0.622	0.626
7	4	1.348	1.360
8	5	1.785	1.814
9	5	2.551	2.601
10	5	2.932	3.006
11	5	3.174	3.262
12	4	3.337	3.434
13	4	3.172	3.272
14	3	3.095	3.193
15	3	3.040	3.140
16	3	2.989	3.090
17	3	2.941	3.043

Table 4: The range and the difference for Sample 2.

n	$\Delta_1^{\overline{d_1^*}} U_{n-1}(\hat{X})$	$\Delta_1^{\overline{d_1^*}} U_{n-1}(X)$	Dif.
0	–	–	–
1	0.000	0.000	0.000
2	0.147	0.147	0.000
3	0.282	0.282	0.000
4	0.405	0.405	0.000
5	0.518	0.518	0.000
6	0.622	0.622	0.000
7	1.348	1.348	0.000
8	1.786	1.786	0.000
9	2.555	2.555	0.000
10	2.940	2.940	0.000
11	3.189	3.189	0.000
12	3.359	3.359	0.000
13	3.199	3.199	0.000
14	3.128	3.128	0.000
15	3.075	3.075	0.000
16	3.026	3.026	0.000
17	2.980	2.980	0.000

We can confirm that the width of eq.(19) is zero since there is not difference in the departure rates between the parties. Remember that there is difference in the arrival rates and the expected revenues between the parties. Additionally, how the range has influence on the difference for departure rates is indicated. Let additional datasets in where the departure rate for $p = 2$ is multiplied by 0.75, 0.5, and 0.25 for the Sample 1 be Sample 3, 4, and 5, respectively. The widths of the ranges for the states X and $\hat{X}$ which are computed from the Sample 1 to 5 are shown in Table 5. We can recognize that the widths enlarge for all n if the differences for the departure rates enlarge. Thus, what increasing difference for the departure rates enlarges the width of the range is suggested.

Table 5: The widths of the ranges for the samples.

n	Sample2	Sample1	Sample3	Sample4	Sample5
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.001	0.001	0.002	0.002
4	0.000	0.001	0.003	0.004	0.006
5	0.000	0.003	0.005	0.008	0.011
6	0.000	0.004	0.008	0.013	0.017
7	0.000	0.013	0.025	0.038	0.052
8	0.000	0.029	0.058	0.088	0.119
9	0.000	0.050	0.101	0.154	0.209
10	0.000	0.074	0.150	0.229	0.310
11	0.000	0.088	0.180	0.275	0.374
12	0.000	0.097	0.198	0.302	0.410
13	0.000	0.101	0.204	0.311	0.421
14	0.000	0.098	0.199	0.302	0.407
15	0.000	0.100	0.202	0.306	0.411
16	0.000	0.101	0.204	0.308	0.411
17	0.000	0.101	0.204	0.308	0.410

4 Conclusion This study has presented the formulation which is modeled seating problem as bid price control by dynamic programming(Markov decision process). Further, the sufficient condition which makes variations of optimal policy reduce and its property have been indicated. It is meaningful to investigate the sufficient condition because reducing variations of optimal policies leads requisite data capacity to reduce.

This paper's result indicates that we should pay attention to difference for departure rates among parties. Specially, if there is not difference in departure rates among the parties for big scale problem, then results of the Theorem 1 and Proposition 1 have significance. If parameters sets are that $\overline{P} = 4$, $\overline{I} = 2$, $g_p = p$, $t_1 = 2, t_2 = 4$, $m_1 = 6$, and $m_2 = 7$ where $p \in P$, then $\chi = 9240$. However, if there is not difference in departure rates for the case, a maximum of the variations of optimal policies is reduced to 56.

However, This study's result is based on the assumption which is that departure rates depend on exponential distribution. It is mystery that what kinds of restaurant; first-food restaurant, traditional restaurant, cafeteria restaurant, cafe restaurant, and etc.. can be approximately applied to this assumption. This question is a big future issue for this study.

Although this study's model has the problem for a restaurant, the model also corresponds to a upgrade model with departure of parties where resources are rooms or tables. Some other future issues are mentioned that for example, considering meal duration as probability distribution, investigating effect of elements; arrival rate, reward, and etc. for the width of the sufficient condition, and making a relation between this results and heuristic calculation method of existing researches clear.

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