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DYNAMICAL SYSTEM FOR EPITAXIAL GROWTH MODEL UNDER DIRICHLET CONDITIONS

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Abstract. This paper treats the initial-boundary value problem for a semilinear parabolic equation of forth order which has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. In the preceding papers [4, 5, 6, 7], we have already treated the problem under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$. In this paper, we want to handle the same equation but under the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$, more natural boundary conditions than before. In the previous case, the leading linear operator Δ^2 was decomposed into the product $(-\Delta)^2$, where $-\Delta$ is a negative Laplace operator equipped with the usual Neumann boundary conditions and is a positive definite self-adjoint operator of L_2 space. Such a favorable decomposition is now no longer available. We have to handle a very fourth order operator Δ^2 equipped with the homogeneous Dirichlet boundary conditions.

Our goal of this paper is to construct a dynamical system generated by the initial-boundary value problem as done in [4] for the Neumann like boundary conditions.

1 Introduction We study the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty), \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

in a two-dimensional bounded domain Ω. Here, Ω denotes a substrate domain and the unknown function $u = u(x,t)$ denotes a displacement of surface height from the standard level at position $x \in \Omega$ and time t. And $n(x)$ denotes the outer normal vector of the boundary at boundary point $x \in \partial\Omega$.

Such a nonlinear parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. They pay attentions on the two main effects. One is diffusion of adatoms on the surface caused by the difference of the chemical potential proportional to the curvature of the surface. The adatoms have tendency to migrate from the positions of large curvature to those of small one. Such a current is called the surface diffusion. According to Mullins [10], a linearized surface diffusion is described by the fourth order equation $\frac{\partial u}{\partial t} \approx -a\Delta^2 u$. The other is a uphill current of adatoms caused by step edge barriers [3, 11, 14]. The step edge barriers prevent adatoms from hopping down from the upper terraces to lower ones. As a consequence, diffusing adatoms preferably

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attach to steps from the terrace below rather than from above and non-equilibrium uphill currents are induced. Such a current is called the roughening. In the mentioned paper [8], the authors introduced as a macroscopic representation of the roughening the negative diffusion equation $\frac{\partial u}{\partial t} \approx -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$. Combining these positive and negative diffusion equations, we obtain the fourth order equation of (1.1).

In the preceding papers [4, 5, 6, 7], we used the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$. The fourth order operator Δ^2 equipped with these homogeneous boundary conditions is then decomposed into $\Delta^2 = (-\Delta)^2$, where $-\Delta$ is the negative Laplace operator equipped with the usual Neumann boundary conditions. Although mathematical treatments are easier, the boundary conditions $\frac{\partial}{\partial n}\Delta u = 0$ seem to be somewhat artificial. In this paper, we want to impose on u the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$. Physically, this means that the surface level is always controlled to $u = 0$ on the boundary $\partial\Omega$ together with the conditions $\frac{\partial u}{\partial n} = 0$ on the normal derivatives.

We first construct a global solution for any $u_0 \in H^{-2}(\Omega)$. For this purpose, we will appeal to the general theory of abstract parabolic equations in infinite-dimensional spaces, see [9, 12, 15]. The theory is available to the higher order semilinear parabolic equations, too. We secondly construct a dynamical system generated by (1.1) in the underlying space $H^{-2}(\Omega)$. Furthermore, it is shown that the dynamical system has an exponential attractor, see [1, 13, 15]. In particular, for any initial function $u_0 \in H^{-2}(\Omega)$, the trajectory starting from u_0 admits a nonempty ω-limit set.

Throughout the paper, Ω denotes a convex or $\mathcal{C}^2$, bounded domain in $\mathbb{R}^2$. For $s \geq 0$, $H^s(\Omega)$ is the complex Sobolev space with exponent s. As usual, $H^0(\Omega) = L_2(\Omega)$. For $s > 0$, $H_0^s(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ (space of infinitely differentiable functions in Ω with compact support) in the topology $H^2(\Omega)$. We shall also use the Sobolev space $H^{-s}(\Omega) = [H_0^s(\Omega)]'$ with negative exponent $-s$. The coefficients $a > 0$ and $\mu > 0$ are fixed constants.

2 Abstract formulation

In order to employ the theory of abstract parabolic equations, let us formulate (1.1) as the Cauchy problem for an abstract evolution equation. We first define a realization of the operator $a\Delta^2$ under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. For this purpose, we consider a symmetric sesquilinear form

$$a(u,v) = a\int_\Omega \Delta u \cdot \Delta\overline{v}\, dx, \qquad u,\ v \in H_0^2(\Omega),$$

defined on $H_0^2(\Omega)$. Since $\nabla u \in H_0^1(\Omega)$ if $u \in H_0^2(\Omega)$, $u \in H_0^2(\Omega)$ satisfies $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Of course, $u \in H_0^2(\Omega)$ satisfies $u = 0$ on $\partial\Omega$. Therefore, $u \in H_0^2(\Omega)$ satisfies the homogeneous Dirichlet boundary conditions. Furthermore, as Ω is convex or of class $\mathcal{C}^2$, in either case, the elliptic estimates yield that

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L_2}, \qquad u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.1}$$

This then implies the coercive estimate

$$a(u,u) \geq \delta\|u\|_{H^2}^2 \qquad \text{for all } u \in H_0^2(\Omega),$$

with some constant $\delta > 0$. As a consequence, we see that $a(u,v)$ determines a linear operator A from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$ by the formula $a(u,v) = \langle Au, v\rangle_{H^{-2}\times H_0^2}$, see [2]. Here, $H^{-2}(\Omega)$ is the dual space of $H_0^2(\Omega)$ and these spaces compose a triplet

$$H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega). \tag{2.2}$$

The operator A thus defined is considered as a realization of $a\Delta^2$ under the homogeneous Dirichlet boundary conditions which is a densely defined, closed operator in $H^{-2}(\Omega)$ whose spectrum is contained in the positive real line $(0,\infty)$. (Note that the part of A in $L_2(\Omega)$ is a positive definite self-adjoint operator of $L_2(\Omega)$.)

For $0 \le \theta \le 1$, A^θ denotes the fractional power of A of exponent θ. Of course, $A^0 = I$ (identity operator on $H^{-2}(\Omega)$) and $A^1 = A$. As a general result (cf. [15, Theorem 2.35]), it follows from (2.2) that $\mathcal{D}(A^{\frac{1}{2}}) = L_2(\Omega)$ with norm equivalence. From this fact it is further deduced that, for $\frac{1}{2} \le \theta \le 1$.

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A)]_{2\theta-1} = [L_2(\Omega), H_0^2(\Omega)]_{2\theta-1} \subset H^{4\theta-2}(\Omega). \tag{2.3}$$

As well, (2.1) can be extended for $\frac{1}{2} \le \theta \le 1$ by

$$\|u\|_{H^{4\theta-2}} \le C\|A^{\theta-\frac{1}{2}}u\|_{L^2}, \qquad u \in \mathcal{D}(A^\theta).$$

We next define a realization of the nonlinear operator $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ in the framework of (2.2). Since ∇ is a bounded operator from $L_2(\Omega)$ into $H^{-1}(\Omega)$, if $\frac{\nabla u}{1+|\nabla u|^2}$ is in $L_2(\Omega)$, then we see that $\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \in H^{-1}(\Omega) \subset H^{-2}(\Omega)$. So, it is natural to set

$$f(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right), \qquad u \in H^1(\Omega). \tag{2.4}$$

In view of (2.3), $\mathcal{D}(A^{\frac{3}{4}}) \subset H^1(\Omega)$. This shows that f is defined on the domain $\mathcal{D}(A^{\frac{3}{4}})$ and can be regarded as a subordinate operator to A.

We thus arrive at an abstract formulation of (1.1) which is written as

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{2.5}$$

in the underlying space $H^{-2}(\Omega)$. It is now possible to apply the various results of the theory of semilinear abstract parabolic equations.

3 Construction of solutions We begin with constructing local solutions to (2.5) by using [15, Theorem 4.4]. To this end, it suffices to verify a suitable Lipschitz condition for $f(u)$. In fact, for u, $v \in H^1(\Omega)$,

$$\begin{aligned} \frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla v}{1+|\nabla v|^2} &= \frac{(1+|\nabla v|^2)\nabla(u-v) - (|\nabla u|^2 - |\nabla v|^2)\nabla v}{(1+|\nabla u|^2)(1+|\nabla v|^2)} \\ &= \frac{\nabla(u-v)}{1+|\nabla u|^2} - \frac{(|\nabla u| - |\nabla v|)(|\nabla u| + |\nabla v|)\nabla v}{(1+|\nabla u|^2)(1+|\nabla v|^2)}. \end{aligned}$$

Therefore,

$$\left\|\frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla v}{1+|\nabla v|^2}\right\|_{L_2} \le C\|u-v\|_{H^1}.$$

This then yields that

$$\|f(u) - f(v)\|_{H^{-1}} \le C\|A^{\frac{3}{4}}(u-v)\|_{H^{-2}}, \qquad u, v \in \mathcal{D}(A^{\frac{3}{4}}),$$

i.e., f fulfills [15, (4.21)] with $\eta = \frac{3}{4}$.

As a direct consequence of [15, Theorem 4.4], for any $u_0 \in H^{-2}(\Omega)$, there exists a unique local solution to (2.5) in the function space:

$$u \in \mathcal{C}([0, T_{u_0}]; H^{-2}(\Omega)) \cap \mathcal{C}^1((0, T_{u_0}]; H^{-2}(\Omega)) \cap \mathcal{C}((0, T_{u_0}]; H_0^2(\Omega)). \tag{3.1}$$

The local solution u satisfies the estimate

$$t\|u(t)\|_{H^2} + t^{\frac{3}{4}}\|u(t)\|_{H^1} + \|u(t)\|_{H^{-2}} \le C_{u_0}, \qquad 0 < t \le T_{u_0}. \tag{3.2}$$

The time $T_{u_0} > 0$ and constant C_{u_0} are determined by the norm $\|u_0\|_{H^{-2}}$ alone.

For constructing global solutions, the essential thing is to establish the *a priori* estimates for local solutions, cf. [15, Corollary 4.3]. By the smoothing effect of solutions seen by (3.1) we have $u(t) \in H^2(\Omega)$ for any $t > 0$. So, in proving the *a priori* estimates (and hence constructing a global solution to (2.5)), there is no loss of generality to assume that $u_0 \in L_2(\Omega) = \mathcal{D}(A^{\frac{1}{2}})$. Under this assumption, let u denote any local solution to (2.5) in the space:

$$u \in \mathcal{C}([0, T_u]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_u]; H^{-2}(\Omega)) \cap \mathcal{C}((0, T_u]; H_0^2(\Omega)). \tag{3.3}$$

Proposition 3.1. *There exists a constant $C > 0$ such that the estimate*

$$\|u(t)\|_{L_2} \le C(\|u_0\|_{L_2} + 1), \quad 0 \le t \le T_u,$$

holds true for any local solution u lying in (3.3)*, C being independent of the interval* $[0, T_u]$.

Proof. Take a scaler product of the equation of (2.5) and $\overline{u}$. Noting that $\|u(t)\|_{L_2}^2$ is differentiable for $t > 0$ with derivative $\frac{d}{dt}\|u(t)\|_{L_2}^2 = 2\mathrm{Re}\,\langle \frac{du}{dt}(t), u(t)\rangle_{H^{-2}\times H_0^2}$ and that $\langle Au(t), u(t)\rangle_{H^{-2}\times H_0^2} = a(u(t), u(t))$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + a\int_\Omega |\Delta u|^2 dx &= \mu\int_\Omega \frac{|\nabla u|^2}{1 + |\nabla u|^2}\, dx \\ &\le \mu|\Omega|.\end{aligned}$$

By (2.1) there exists a constant $\delta > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \delta\int_\Omega |u|^2 dx \le \mu|\Omega|.$$

Solving this integral inequality, we obtain that

$$\|u(t)\|_{L^2}^2 \le e^{-2\delta t}\|u_0\|_{L^2}^2 + \mu\delta^{-1}|\Omega|, \qquad 0 \le t \le T_u.$$

□

Proposition 3.1 shows that the norm $\|u(t)\|_{L_2}$ remains uniformly bounded for any interval $[0, T_u]$. This then means that one can always extend any local solution with a uniform time interval to obtain a global solution in the space:

$$u \in \mathcal{C}([0, \infty); L_2(\Omega)) \cap \mathcal{C}^1((0, \infty); H^{-2}(\Omega)) \cap \mathcal{C}((0, \infty); H_0^2(\Omega)). \tag{3.4}$$

Of course, the global solution satisfies the similar estimate

$$\|u(t)\|_{L_2}^2 \le e^{-2\delta t}\|u_0\|_{L_2}^2 + \mu\delta^{-1}|\Omega|, \qquad 0 \le t < \infty. \tag{3.5}$$

Finally, let us remark that, if the initial function u_0 is real, then the solution $u(t)$ with $u(0) = u_0$ is also real for every time $t > 0$. In fact, we notice that the complex conjugate $\overline{u}$ of the solution u to (2.5) satisfies the same evolution equation for every t. So, $\overline{u}$ is a solution satisfying an initial condition $\overline{u(0)} = \overline{u_0}$. If u_0 is real, i.e., $u_0 = \overline{u_0}$, then uniqueness of solution implies $u(t) = \overline{u(t)}$ and $u(t)$ must be real for every t.

4 Dynamical systems The next step is to observe that the problem (2.5) generates a dynamical system. For this purpose, we can again follow the general procedure for semilinear abstract parabolic equations, see [15, Section 6.5].

For $u_0 \in H^{-2}(\Omega)$, let $u(t; u_0)$ denote the global solution of (2.5), and set

$$S(t)u_0 = u(t; u_0), \qquad 0 \le t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on $H^{-2}(\Omega)$, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \le s, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, u_0) \mapsto S(t)u_0$ is continuous from $[0, \infty) \times H^{-2}(\Omega)$ into $H^{-2}(\Omega)$. Whence, $S(t)$ defines a dynamical system in $H^{-2}(\Omega)$ which is denoted by $(S(t), H^{-2}(\Omega))$.

We can see from the dissipative estimate (3.5) that $(S(t), H^{-2}(\Omega))$ has an exponential attractor. Remember that a set $\mathcal{M}$ satisfying the following conditions is called the exponential attractor:

1. $\mathcal{M}$ is a compact subset of $H^{-2}(\Omega)$ with finite fractal dimension.
2. $\mathcal{M}$ is a positively invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for any $0 < t < \infty$.
3. There exists an exponent $k > 0$ such that, for any bounded subset B of $H^{-2}(\Omega)$, it holds true that
$$h(S(t)B, \mathcal{M}) \le C_B e^{-kt}, \qquad 0 < t < \infty,$$
with a constant $C_B > 0$.

Here, $h(B_1, B_2) = \sup_{f \in B_1} \inf_{g \in B_2} \|f - g\|_{H^{-2}}$ is a semi-distance of two bounded subsets B_1 and B_2.

As explained in [15, Section 6.4], the compact smoothing property

$$\|S(t^*)u_0 - S(t^*)v_0\|_{L_2} \le C\|u_0 - v_0\|_{H^{-2}}, \qquad u_0, v_0 \in \mathcal{B}, \tag{4.1}$$

of $S(t)$ provides existence of exponential attractors, where $\mathcal{B}$ is an attractive, positively invariant, compact subset of $H^{-2}(\Omega)$ and where $t^* > 0$ is any fixed time. But, this property is also easily verified from the known estimates (3.2) and (3.5). In fact, let B be any bounded subset of $H^{-2}(\Omega)$. Then, it follows from (3.2) that there exist a bounded ball $B_{2,B}$ of $L_2(\Omega)$ and time $t_B > 0$ both depending on B such that $S(T_B)B \subset B_{2,B}$. In addition, (3.5) yields that, for any $u_0 \in B$,

$$\|S(t)u_0\|_{L_2}^2 = \|S(t - T_B)S(T_B)u_0\|_{L_2}^2 \le e^{-2\delta(t - T_B)} R_{2,B} + \mu\delta^{-1}|\Omega|, \qquad \forall t \ge T_B,$$

where $R_{2,B}$ is the radius of $B_{2,B}$. This shows that the ball $B(0; \sqrt{1 + \mu\delta^{-1}|\Omega|})$ of $L_2(\Omega)$ is an absorbing set. Let $\mathcal{B}$ be the collection of all trajectories starting from this ball. Obviously, $\mathcal{B}$ is an absorbing and invariant set; moreover, since $\mathcal{B}$ is a bounded subset of $L_2(\Omega)$, it is a compact set of $H^{-2}(\Omega)$. Finally, the desired Lipschitz condition (4.1) can be verified by using the standard techniques described in [15, Subsection 6.5.3]. In this way, we verify that our dynamical system admits an exponential attractor.

Finally, let us notice that $S(t)$ defines a dynamical system even in the space $L_2(\Omega)$ and the restricted dynamical system denoted by $(S(t), L_2(\Omega))$ also admits an exponential attractor. In fact, as seen in (3.4), $S(t)$ maps $L_2(\Omega)$ into itself. In addition, it is proved that $S(t)$ is continuous from $L_2(\Omega)$ into itself. Therefore, (2.5) generates a dynamical system in $L_2(\Omega)$, too. Furthermore, the exponential attractor $\mathcal{M}$ in $H^{-2}(\Omega)$ constructed above is obviously a bounded subset of $\mathcal{D}(A)$ $(= H_0^2(\Omega))$, and remains to be an exponential attractor of $(S(t), L_2(\Omega))$.

5 Lyapunov function Multiply the equation of (1.1) by $-\frac{\partial \overline{u}}{\partial t}$ and integrate the product in Ω. By somewhat formal computations, its real part is given by

$$\begin{aligned}-\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx &= a\mathrm{Re}\int_\Omega \varDelta u \frac{\partial}{\partial t}\varDelta \overline{u}\, dx - \mu \mathrm{Re}\int_\Omega \left[\frac{\nabla u}{1+|\nabla u|^2}\right]\cdot \frac{\partial}{\partial t}\nabla \overline{u}\, dx\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \big[a|\varDelta u|^2 - \mu \log(1+|\nabla u|^2)\big]dx.\end{aligned}$$

These computations then suggest that the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega \big[a|\varDelta u|^2 - \mu\log(1+|\nabla u|^2)\big]dx, \qquad u\in H_0^2(\Omega), \tag{5.1}$$

becomes a Lyapunov function of the dynamical system.

In order to justify this, however, we need a higher regularity of solution u than the known (3.4). Let $u_0 \in H_0^2(\Omega)$ and assume that the solution to (2.5) belongs to

$$u \in \mathcal{C}^1((0,\infty); L_2(\Omega)) \quad \text{and} \quad \varDelta^2 u \in \mathcal{C}((0,\infty); L_2(\Omega)). \tag{5.2}$$

It is clear that

$$\begin{aligned}&\|\varDelta u(t+h)\|_{L_2}^2 - \|\varDelta u(t)\|_{L_2}^2\\ &\quad = (\varDelta[u(t+h)-u(t)], \varDelta u(t+h)) + (\varDelta u(t), \varDelta[u(t+h)-u(t)])\\ &\quad = (u(t+h)-u(t), \varDelta^2 u(t+h)) + (\varDelta^2 u(t), u(t+h)-u(t)).\end{aligned}$$

In view of (5.2), it is observed that

$$\begin{aligned}\frac{d}{dt}\|\varDelta u(t)\|_{L_2}^2 &= \left(\frac{du}{dt}(t), \varDelta^2 u(t)\right) + \left(\varDelta^2 u(t), \frac{du}{dt}(t)\right)\\ &= 2\mathrm{Re}\left(\varDelta^2 u(t), \frac{du}{dt}(t)\right).\end{aligned}$$

In the meantime, for u, $v\in H_0^2(\Omega)$, consider

$$\int_\Omega [\log(1+|\nabla v|^2) - \log(1+|\nabla u|^2)]dx.$$

For a.e. $x\in\Omega$, we have

$$\begin{aligned}\log[1+|\nabla v(x)|^2] - \log[1+|\nabla u(x)|^2] &= \int_0^1 \frac{d}{d\theta}\log\{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2\}d\theta\\ &= \int_0^1 \frac{2\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x) + 2\theta|\nabla[v(x)-u(x)]|^2}{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2}d\theta.\end{aligned}$$

Moreover, since

$$\begin{aligned}\frac{1}{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2} &= \frac{1}{1+|\nabla u(x)|^2}\\ &\quad - \frac{2\theta\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x) + \theta^2|\nabla[v(x)-u(x)]|^2}{\{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2\}(1+|\nabla u(x)|^2)},\end{aligned}$$

we have

$$\left|\log[1+|\nabla v(x)|^2]-\log[1+|\nabla u(x)|^2]-\frac{2\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x)}{1+|\nabla u(x)|^2}\right| \\ \leq C\{|\nabla[v(x)-u(x)]|^2+|\nabla[v(x)-u(x)]|^4\}.$$

Therefore, integration in Ω yields that

$$\left|\int_\Omega\left[\log(1+|\nabla v|^2]-\log(1+|\nabla u|^2)-\frac{2\mathrm{Re}\nabla[v-u]\cdot\nabla\overline{u}}{1+|\nabla u|^2}\right]dx\right| \\ \leq C\{\|\nabla(v-u)\|_{L_2}^2+\|\nabla(v-u)\|_{L_4}^4\}.$$

We here use Galiardo-Nireberg's inequality ([15, Theorem 1.37]) to obtain that

$$\|\nabla(v-u)\|_{L_4}\leq C\|\nabla(v-u)\|_{L_2}^{\frac{1}{2}}\|\nabla(v-u)\|_{H^1}^{\frac{1}{2}}\leq C\|v-u\|_{H^1}^{\frac{1}{2}}\|v-u\|_{H^2}^{\frac{1}{2}} \\ \leq C\|v-u\|_{L_2}^{\frac{1}{4}}\|v-u\|_{H^2}^{\frac{3}{4}}.$$

Then,

$$\left|\int_\Omega\left\{\log(1+|\nabla v|^2]-\log(1+|\nabla u|^2)+2\mathrm{Re}\left[\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)(\overline{v}-\overline{u})\right]\right\}dx\right| \\ \leq C\|v-u\|_{L_2}(\|v-u\|_{H^2}+\|v-u\|_{H^2}^3).$$

Let us apply this estimate with $v=u(t+h)$ and $u=u(t)$, where u is the solution mentioned above. Then, since $\|u(t+h)-u(t)\|_{H^2}\to 0$ as $h\to 0$, it is easily verified that

$$\frac{d}{dt}\int_\Omega\log[1+|\nabla u(t)|^2]dx=-2\mathrm{Re}\int_\Omega\nabla\cdot\left(\frac{\nabla u(t)}{1+|\nabla u(t)|^2}\right)\frac{d\overline{u}}{dt}(t)\,dx.$$

We have thus proved that, for any solution lying in (5.2), the function $\Phi(u(t))$ is differentiable with derivative

$$\frac{d}{dt}\Phi(u(t))=-\left\|\frac{du}{dt}(t)\right\|_{L_2}^2,\qquad 0<t<\infty. \tag{5.3}$$

6 Numerical Results We shall conclude this paper with illustrating some numerical examples. Let us consider (1.1) in the square domain $\Omega=(0,1)\times(0,1)$. The coefficient a is fixed as $a=1$ but $\mu>0$ is treated as a control parameter. The initial function is taken as

$$u_0(x_1,x_2)=0.1[\sin(2\cdot 3.14x_1)\times\sin(2\cdot 3.14x_2)],\qquad(x_1,x_2)\in\Omega,$$

which is a perturbation of the null solution $u\equiv 0$. Clearly, the null solution is a unique homogeneous stationary solution.

Set first $\mu=12$. As seen by Figure 1, the solution tends to the null solution as $t\to\infty$. The graph of Lyapunov function along this trajectory is given by Figure 2.

Take next $\mu=13$. As seen by Figure 3, the solution no longer tends to the null solution. Instead, the small perturbation grows into two columns of ridges. One can count in each column 12 ridges. The graph of Lyapunov function along the trajectory is given by Figure 4.

Finally, take a sufficiently large μ, say $\mu=40$. As seen by Figure 5, the perturbation again grows into two columns of ridges. The number of ridges in a column increases more than in the case of $\mu=13$. As before, the Lyapunov function is monotone decreasing along the trajectory, see Figure 6.

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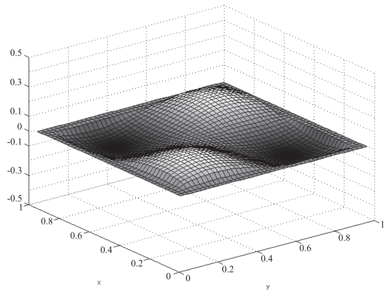
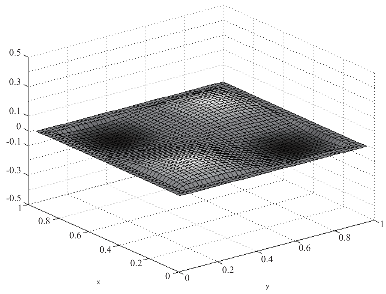
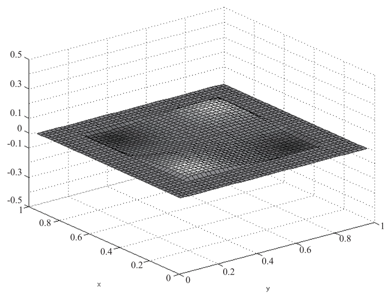


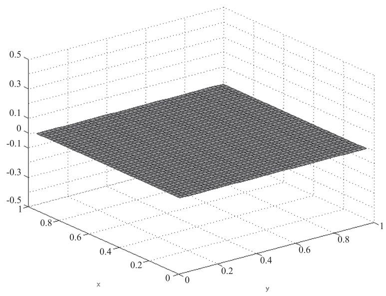
(a) t=0



(b) t=20



(c) t=40



(d) t=60

Fig. 1: Dynamics for $\mu = 12$

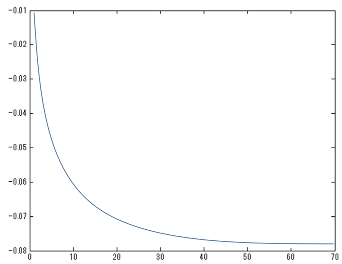
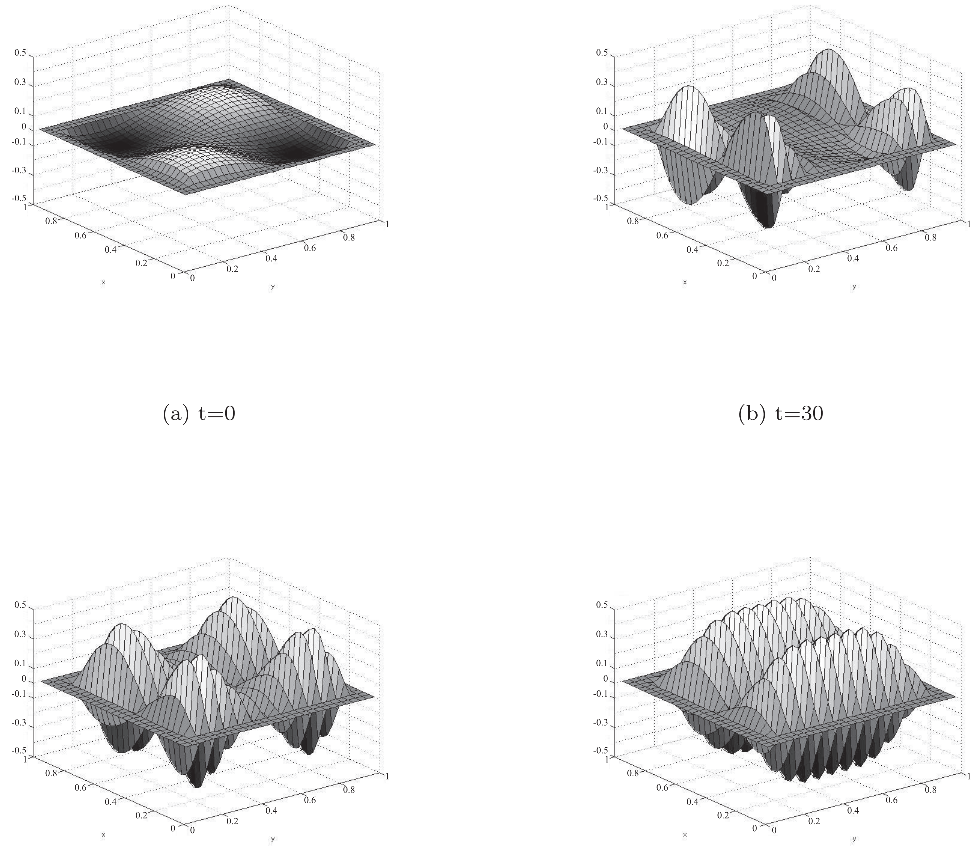


Fig. 2: Lyapunov function for $\mu = 12$



(a) t=0

(b) t=30

(c) t=60

(d) t=90

Fig. 3: Dynamics for $\mu = 13$

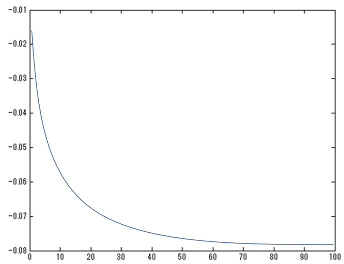
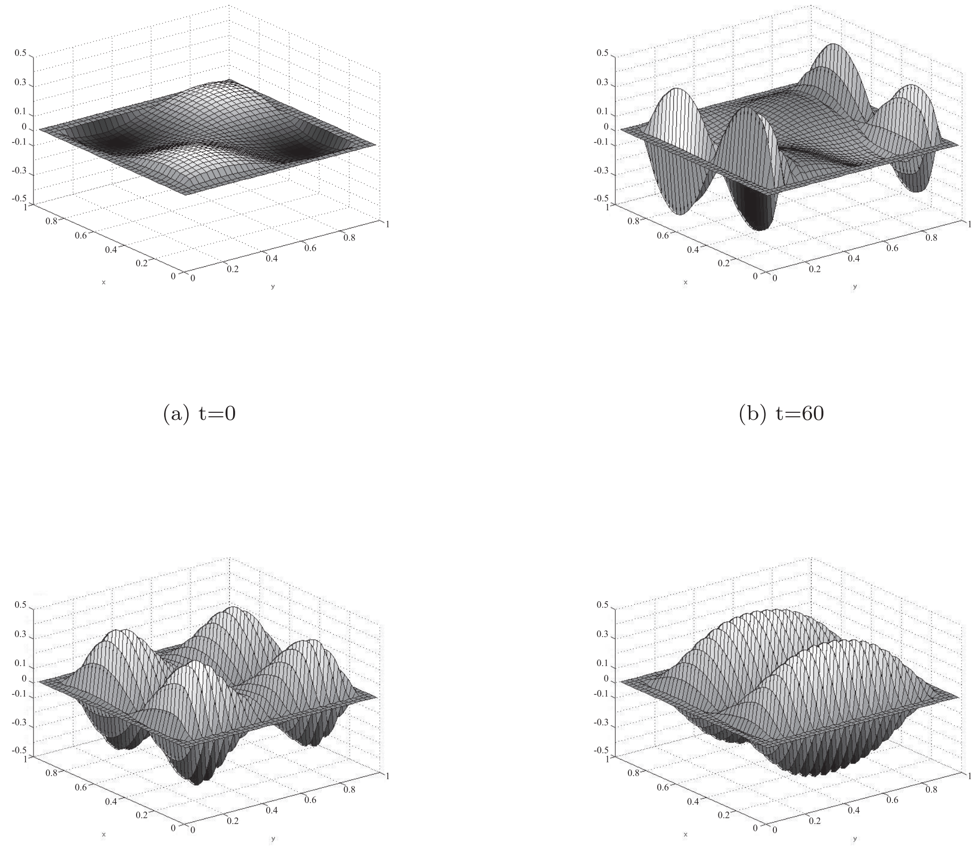


Fig. 4: Lyapunov function for $\mu = 13$



(a) t=0 (b) t=60

(c) t=120 (d) t=180

Fig. 5: Dynamics for $\mu = 40$

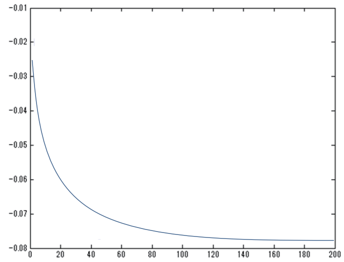


Fig. 6: Lyapunov function for $\mu = 40$

HOMOGENEOUS STATIONARY SOLUTION TO EPITAXIAL GROWTH MODEL UNDER DIRICHLET CONDITIONS

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Abstract. This paper continues a study on the initial-boundary value problem for a nonlinear parabolic equation of fourth order under the homogeneous Dirichlet boundary conditions. The parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [10] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. In the previous papers [1, 2], we constructed a dynamical system generated by the problem and showed that every trajectory converges to some stationary solution as $t \to \infty$. This paper is then devoted to investigating stability or instability of the null solution which is a unique homogeneous stationary solution. We shall also illustrate some numerical results to observe how changes the structure of stationary solutions as the roughening coefficient increases.

1 Introduction

We are concerned with the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty), \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

in a two-dimensional bounded domain Ω. Such a nonlinear parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [10] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. Here, Ω denotes a substrate domain and the unknown function $u = u(x,t)$ denotes a displacement of surface height from the standard level at position $x \in \Omega$ and time t. For detailed physical background, see [5, 12, 13, 16].

As in the preceding papers [1, 2], we will formulate (1.1) as the Cauchy problem for an abstract parabolic equation of the form (2.1) with underlying space $L_2(\Omega)$. In [1], we constructed a dynamical system $(S(t), L_2(\Omega))$ generated by (2.1), where $S(t)$ is a continuous nonlinear semigroup acting on $L_2(\Omega)$ determined by global solutions of (2.1). In addition, the dynamical system was shown to have a finite-dimensional attractor and to admit a Lyapunov function given by (2.8). In the subsequent paper [2], we succeeded in proving longtime convergence. For any $u_0 \in L_2(\Omega)$, $S(t)u_0$ was shown to converge as $t \to \infty$ to a stationary solution $\overline{u}$ of (2.1).

This paper is then concerned with stationary solutions of (2.1). Among others, we are concerned with stability and instability of the null solution $\overline{u} \equiv 0$. Clearly, the null solution is a unique homogeneous stationary solution. For this purpose, we will appeal

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to the linearized principle in infinite-dimensional spaces invented by Babin-Vishik [3] and Temam [15], see also [17, Section 6.6]. Indeed, we shall prove that, when $\mu < ad^{-2}$, where $d > 0$ is a constant determined by (3.5), the null solution is globally stable and that, when $\mu > ad^{-2}$, the null solution is unstable. The constant d can be estimated by an optimal coefficient of the Poincare inequality. In the latter case, there must exist non-null stationary solutions (remember that every trajectory converges to some stationary solution).

In the papers [6, 7, 8, 9], we handled the same fourth order parabolic equation but under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$. Among others in [8] we studied stability and instability of the homogeneous stationary solution using the fact that, under these Neumann like boundary conditions, the fourth order operator Δ^2 can be reduced into the product $(-\Delta)^2$ of the negative Laplace operator $-\Delta$ equipped with the usual Neumann boundary conditions which is a positive definite self-adjoint operator of $L_2(\Omega)$. In the present case, however, such a favorable reduction is not available and we have to handle a very fourth order elliptic operator.

Throughout the paper, Ω is a rectangular or $\mathcal{C}^4$, bounded domain in $\mathbb{R}^2$. And $n(x)$ denotes the outer normal vector of the boundary at boundary point $x \in \partial\Omega$. As noticed by [2, Proposition 2.1], for $f \in L_2(\Omega)$, the elliptic problem $-\Delta^2 u = f$ in Ω under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ admits a unique solution u such that $u \in H^4(\Omega)$. For $1 \le p \le \infty$, $L_p(\Omega)$ is the space of complex valued L_p functions in Ω. For $s \ge 0$, $H^s(\Omega)$ is the complex Sobolev space in Ω with exponent s. For $s \ge 0$, $H_0^s(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ (the space of all infinitely differentiable functions with compact support) in the topology of $H^s(\Omega)$. The coefficients $a > 0$ and $\mu > 0$ are given constants.

2 Reviews of known results In this section, let us review known results obtained in the previous papers [1, 2].

Abstract Formulation. As in [1, 2], we formulate (1.1) as the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

in the underlying space $X = L_2(\Omega)$. Here, A is an associated linear operator in the framework of a triplet $H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega)$ $(= H_0^2(\Omega)')$ with a symmetric sesquilinear form defined by

$$a(u, v) = a\int_\Omega \Delta u \cdot \Delta\overline{v}\,dx, \qquad u,\, v \in H_0^2(\Omega),$$

(cf. [4]). Then, A is a positive definite self-adjoint operator of X with domain $\mathcal{D}(A) \subset H_0^2(\Omega)$. The operator A is considered as a realization of the fourth order operator $a\Delta^2$ in X under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

As seen by [2, Proposition 2.1], our assumption on Ω yields a characterization of $\mathcal{D}(A)$ in such a way that $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$ with norm equivalence. As the sesquilinear form is symmetric, $\mathcal{D}(A^{\frac{1}{2}})$ coincides with the from domain, i.e., $\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega)$ with norm equivalence. By interpolation, we can then verify that, for $\frac{1}{2} \le \theta \le 1$,

$$\mathcal{D}(A^\theta) \subset H^{4\theta}(\Omega) \cap H_0^2(\Omega),$$

and for $0 \le \theta < \frac{1}{2}$,

$$\mathcal{D}(A^\theta) \subset H^{4\theta}(\Omega).$$

In addition, for any $0 \le \theta \le 1$, the inequality

$$\|u\|_{H^{4\theta}} \le C\|A^{\theta}u\|_X, \qquad u \in \mathcal{D}(A^{\theta}), \tag{2.2}$$

is satisfied, namely, the embedding described above is continuous.

Meanwhile, f is a nonlinear operator defined by

$$\begin{aligned} f(u) &= -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \\ &= -\mu\left[\frac{\Delta u}{1+|\nabla u|^2} - \frac{\nabla|\nabla u|^2\cdot\nabla u}{(1+|\nabla u|^2)^2}\right], \qquad u \in \mathcal{D}(A^{\frac{7}{8}}). \end{aligned} \tag{2.3}$$

Note that, since $\mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega)$ due to (2.2) and $H^{\frac{7}{2}}(\Omega) \subset \mathcal{C}^2(\overline{\Omega})$, $u \in \mathcal{D}(A^{\frac{7}{8}})$ certainly implies $f(u) \in L_2(\Omega)$. Furthermore, according to [2, (2.8)], it holds true that

$$\begin{aligned} \|f(u)-f(v)\|_X \le C\big[&\|A^{\frac{1}{2}}(u-v)\|_X \\ &+ (\|A^{\frac{7}{8}}u\|_X + \|A^{\frac{7}{8}}v\|_X)\|A^{\frac{1}{4}}(u-v)\|_X\big], \qquad u, v \in \mathcal{D}(A^{\frac{7}{8}}). \end{aligned} \tag{2.4}$$

The general result on abstract semilinear evolution equations (cf. [17, Theorem 4.1]) readily provides local existence of solutions. For any $u_0 \in \mathcal{D}(A^{\frac{1}{4}})$, (2.1) possesses a unique local solution. As a matter of fact, we can formulate (1.1) even in a larger underlying space $H^{-2}(\Omega)$ of the form (2.1). As shown in [1], for any $u_0 \in H^{-2}(\Omega)$, there exists a unique local solution. Combining these two existence results, we can claim that, for any $u_0 \in L_2(\Omega) = X$, (2.1) possesses a unique local solution in the function space:

$$u \in \mathcal{C}((0,T_{u_0}];\mathcal{D}(A)) \cap \mathcal{C}([0,T_{u_0}];X) \cap \mathcal{C}^1((0,T_{u_0}];X), \tag{2.5}$$

$T_{u_0} > 0$ being determined by the norm $\|u_0\|_X$ alone.

In the subsequent sections, we need to use differentiability of $f(u)$.

Proposition 2.1. *$f:\mathcal{D}(A^{\frac{7}{8}}) \to X$ is Fréchet differentiable with derivative*

$$f'(u)h = -\mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2} - \frac{2(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right), \qquad u, h \in \mathcal{D}(A^{\frac{7}{8}}).$$

Proof. Let $u, h \in \mathcal{D}(A^{\frac{7}{8}})$. From (2.3) it follows that

$$\begin{aligned} f(u+h)-f(u) &= -\mu\nabla\cdot\left[\left(\frac{1}{1+|\nabla(u+h)|^2} - \frac{1}{1+|\nabla u|^2}\right)\nabla(u+h)\right] \\ &\qquad - \mu\nabla\cdot\left(\frac{\nabla(u+h)-\nabla u}{1+|\nabla u|^2}\right) \\ &= -\mu\nabla\cdot\left[\frac{(-2\nabla u\cdot\nabla h - |\nabla h|^2)\nabla(u+h)}{(1+|\nabla(u+h)|^2)(1+|\nabla u|^2)}\right] - \mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2}\right). \end{aligned}$$

By the similar calculations as for (2.4),

$$\|f(u+h)-f(u)-f'(u)h\|_X \le C\|A^{\frac{7}{8}}h\|_X^2(\|A^{\frac{7}{8}}u\|_X + \|A^{\frac{7}{8}}h\|_X).$$

This means that $f:\mathcal{D}(A^{\frac{7}{8}}) \to X$ is Fréchet differentiable at u. □

Proposition 2.2. *Let* $u \in \mathcal{D}(A^{\frac{7}{8}})$ *varies in a ball* $B^{\mathcal{D}(A^{\frac{1}{2}})}(0;1)$. *Then,* $f'(u)$ *satisfies the Lipschitz condition*

$$\|[f'(u)-f'(v)]h\|_X \le C\|A^{\frac{1}{2}}(u-v)\|_X\|A^{\frac{7}{8}}h\|_X,$$
$$u,\, v \in \mathcal{D}(A^{\frac{7}{8}}) \cap B^{\mathcal{D}(A^{\frac{1}{2}})}(0;1);\ h \in \mathcal{D}(A^{\frac{7}{8}}).$$

Proof. From the formula giving $f'(u)$, we can estimate directly the difference $f'(u)-f'(v)$. □

Dynamical System. The [2, Proposition 3.1] provides *a priori* estimates for local solutions obtained above in the space (2.5). Indeed, any local solution to (2.1) on interval $[0,T_u]$ satisfies the estimate

$$\|u(t)\|_X^2 \le e^{-2\delta t}\|u_0\|_X^2 + \mu\delta^{-1}, \qquad 0 \le t \le T_u,$$

with some fixed exponent $\delta > 0$. Then, by the standard argument, we conclude that, for any $u_0 \in X$, (2.1) possesses a unique global solution u in the function space:

$$u \in \mathcal{C}((0,\infty);\mathcal{D}(A)) \cap \mathcal{C}([0,\infty);X) \cap \mathcal{C}^1((0,\infty);X). \tag{2.6}$$

Furthermore, u also satisfies the same estimate

$$\|u(t)\|_X^2 \le e^{-2\delta t}\|u_0\|_X^2 + \mu\delta^{-1}, \qquad 0 \le t < \infty, \tag{2.7}$$

which shows dissipation of u. Set a nonlinear semigroup $S(t)$, $0 \le t < \infty$, on X by $S(t)u_0 = u(t;u_0)$, using the global solution $u(t;u_0)$ to (2.1) with initial data $u_0 \in X$. Then, we obtain a dynamical system $(S(t),X)$ generated by (2.1). The dissipate estimates yield existence of a finite-dimensional attractor $\mathcal{M}$ which attracts every trajectory $S(t)u_0$ at an exponential rate. Such an attractor is called the exponential attractor. In particular, we know that every trajectory has a nonempty ω-limit set $\omega(u_0)$.

As shown by [1, Section 5], our system $(S(t),X)$ admits a Lyapunov function of the from

$$\Phi(u) = \frac{1}{2}\int_\Omega [a|\Delta u|^2 - \mu\log(1+|\nabla u|^2)]dx, \qquad u \in H_0^2(\Omega). \tag{2.8}$$

That is, the value $\Phi(S(t)u_0)$ is monotone decreasing as $t \to \infty$ along any trajectory. Furthermore, it is seen that, for $\overline{u} \in \mathcal{D}(A)$, $\Phi'(\overline{u}) = 0$ and $A\overline{u} = f(\overline{u})$ (i.e., $\overline{u}$ is a stationary solution) are equivalent. From this equivalence, we see that, if $\overline{u} \in \omega(u_0)$, then $\overline{u}$ must be a stationary solution of (2.1). The set $\omega(u_0)$ consists only of stationary solutions.

Convergence of Solutions. The objective of [2] was then to show that $\omega(u_0)$ is a singleton for every u_0. We proved that $\Phi(u)$ satisfies the Łojasiewicz-Simon inequality

$$\|\Phi'(u)\|_{H^{-2}} \ge D|\Phi(u)-\Phi(\overline{u})|^{1-\theta}$$

in a neighborhood of $\overline{u}$, where $\overline{u} \in \omega(u_0)$, with some exponent $0 < \theta \le \frac{1}{2}$. This inequality readily implies that

$$\|S(t)u_0 - \overline{u}\|_X \le C[\Phi(S(t)u_0) - \Phi(\overline{u})]^\theta. \tag{2.9}$$

As $\Phi(S(t)u_0)$ converges to $\Phi(\overline{u})$ as $t \to \infty$, we observe that $S(t)u_0$ converges to $\overline{u}$ in X with some rate of convergence.

3 Linearized Stability Let us now investigate stability and instability of the stationary solutions of (2.1). For this purpose, we will employ the general methods for abstract evolution equations, see [3, 15].

Let $\overline{u} \in \mathcal{D}(A)$ be any stationary solution to (2.1), i.e., $A\overline{u} = f(\overline{u})$. By Propositions 2.1 and 2.2, $f : \mathcal{D}(A^{\frac{7}{8}}) \to X$ is of class $\mathcal{C}^{1,1}$ in a neighborhood of $\overline{u}$, and the derivative satisfies a Lipschitz condition

$$\|[f'(u) - f'(v)]h\|_X \le C\|A^{\frac{1}{2}}(u-v)\|_X \|A^{\frac{7}{8}}h\|_X, \quad u, v \in \mathcal{D}(A^{\frac{7}{8}}) \cap \mathcal{O}(\overline{u});\ h \in \mathcal{D}(A^{\eta}),$$

$\mathcal{O}(\overline{u})$ being a neighborhood of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$. It is known that this condition in turn implies Fréchet differentiability of the semigroup. Indeed, for $0 \le t \le t^*$ where $t^* > 0$ is arbitrarily fixed time, $S(t) : \mathcal{D}(A^{\frac{1}{2}}) \to \mathcal{D}(A^{\frac{1}{2}})$ is of class $\mathcal{C}^{1,1}$ in a neighborhood $\mathcal{O}'(\overline{u})$ of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$ together with the estimate

$$\|S(t)'u - S(t)'v\|_{\mathcal{L}(\mathcal{D}(A^{\frac{1}{2}}))} \le C\|A^{\frac{1}{2}}(u-v)\|_X, \qquad u, v \in \mathcal{O}'(\overline{u});\ 0 \le t \le t^*. \tag{3.1}$$

For the detailed proof, see the proof of [17, Subsection 6.6.3].

We here assume a spectral separation condition for $\sigma(A - f'(\overline{u}))$ of the form

$$\sigma(A - f'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda = 0\} = \emptyset.$$

Then, since $S(t)'\overline{u} = e^{-t\overline{A}}$, where $\overline{A} = A - F'(\overline{u})$, we have in turn a spectral separation for $S(t)'\overline{u}$ of the from

$$\sigma(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \emptyset. \tag{3.2}$$

According to [17, Theorem 6.9], under (3.1) and (3.2), a smooth local unstable manifold $\mathcal{M}_+(\overline{u}; \mathcal{O})$ can be constructed in a neighborhood $\mathcal{O}$ of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$. When

$$\sigma(A - F'(\overline{u})) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}, \tag{3.3}$$

we have $\sigma(S(t)'\overline{u}) \subset \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$ and $\mathcal{M}_+(\overline{u}; \mathcal{O})$ reduces to a singleton $\{\overline{u}\}$. Whence, if (3.3) takes place, $\overline{u}$ is stable. In the meantime, when

$$\sigma(A - f'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < 0\} \ne \emptyset, \tag{3.4}$$

we have $\sigma(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| > 1\} \ne \emptyset$ and $\mathcal{M}_+(\overline{u}; \mathcal{O})$ is not trivial. Whence, if (3.4) takes place, $\overline{u}$ is unstable.

Let us now apply these discussions to the null solution $\overline{u} \equiv 0$. We see from Proposition 2.1 that $A - f'(0) = a\Delta^2 + \mu\Delta$. So, it is necessary to investigate the spectrum of the operator $a\Delta^2 + \mu\Delta$. To this end, we will introduce a normalization of A; indeed, when $a = 1$, we denote $A = A_1$; and, regarding a as a positive parameter, we denote in general $A = aA_1$. Of course, A_1 is a realization of the operator Δ^2 in $L_2(\Omega)$ under the homogeneous Dirichlet conditions on $\partial\Omega$, and is a positive definite self-adjoint operator of X. As verified above, we have $\mathcal{D}(A_1) = H^4(\Omega) \cap H_0^2(\Omega)$ with norm equivalence and $\mathcal{D}(A_1^{\frac{1}{2}}) = H_0^2(\Omega)$ with norm equivalence.

We here notice a fact that a mapping $u \mapsto \frac{\|\nabla u\|_X}{\|\Delta u\|_X}$ is continuous from $H_0^2(\Omega) - \{0\}$ into $\mathbb{R}$ and has a maximum on the sphere $\|A_1 u\|_X = 1$ because of compact embedding $\mathcal{D}(A_1) \subset \mathcal{D}(A_1^{\frac{1}{2}})$. Put

$$d \equiv \max_{\|A_1 u\|_X = 1} \frac{\|\nabla u\|_X}{\|\Delta u\|_X}. \tag{3.5}$$

In other words, the d is an optimal coefficient in the inequality

$$\|\nabla u\|_X \le d\|\Delta u\|_X \qquad u \in \mathcal{D}(A_1).$$

Stability of the null solution is then determined by dominance in magnitude of the two coefficients a and μ to the other but with weight d^{-2} for a.

Theorem 3.1. *If $ad^{-2} > \mu$, then the null solution is stable. If $ad^{-2} < \mu$, then the null solution is unstable.*

Proof. We notice that $a\Delta^2+\mu\Delta$ is a self-adjoint operator of X whose domain $H^4(\Omega)\cap H_0^2(\Omega)$ is compactly embedded in $L_2(\Omega)$. Therefore, the spectrum set $\sigma(a\Delta^2+\mu\Delta)$ is contained in the real axis and consists of point spectrum alone.

For any $u \in \mathcal{D}(A_1) - \{0\}$, we observe that

$$(a\Delta^2 u + \mu\Delta u, u) = a\|\Delta u\|_X^2 - \mu\|\nabla u\|_X^2 \ge (ad^{-2} - \mu)\|\nabla u\|_X^2 > 0,$$

provided $ad^{-2} > \mu$. Therefore, if μ is dominated as $\mu < ad^{-2}$, then $\sigma(a\Delta^2 + \mu\Delta) \subset (0,\infty)$ and the null solution is stable. To the contrary, if μ is large enough so that $\mu > ad^{-2}$, i.e., $d > \sqrt{\frac{a}{\mu}}$, then there exists an element $u_0 \in \mathcal{D}(A_1) - \{0\}$ such that $\|\nabla u_0\|_X > \sqrt{\frac{a}{\mu}}\|\Delta u_0\|_X$. Therefore,

$$(a\Delta^2 u_0 + \mu\Delta u_0, u_0) = a\|\Delta u_0\|_X^2 - \mu\|\nabla u_0\|_X^2 < 0.$$

This means that $\sigma(a\Delta^2 + \mu\Delta) \cap (-\infty, 0) \ne \emptyset$. Hence, the null solution is unstable. □

As a matter of fact, when $ad^{-2} > \mu$, every trajectory converges to 0, that is, the null solution is globally stable.

Theorem 3.2. *Let $ad^{-2} > \mu$. For any $u_0 \in X$, $S(t)u_0$ converges to 0 as $t \to \infty$ at an exponential rate.*

Proof. Multiply the equation of (1.1) by $\overline{u}$ and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + a\int_\Omega |\Delta u|^2 dx &= \mu\int_\Omega \frac{|\nabla u|^2}{1+|\nabla u|^2}dx \\ &\le \mu\int_\Omega |\nabla u|^2 dx.\end{aligned}$$

It then follows from (3.5) that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_X^2 dx \le -(ad^{-2}-\mu)\|\nabla u(t)\|_X^2 \le -(ad^{-2}-\mu)D^{-1}\|u(t)\|_X^2,$$

where $D > 0$ is a coefficient for the Poincare inequality given by (4.1) below. Hence, $\|u(t)\|_X \le e^{-(ad^{-2}-\mu)D^{-1}t}\|u_0\|_X$ for $t \ge 0$. □

4 Estimation of d from above. The weight constant d can be easily estimated from above from the Poincare inequality

$$\|u\|_X \le D\|\nabla u\|_X \qquad u \in H_0^1(\Omega). \tag{4.1}$$

Theorem 4.1. *Let d be the constant determined by* (3.5) *and let D be an optimal coefficient for the Poincare inequality* (4.1)*. Then, it always holds true that $d \le D$.*

Proof. Indeed,

$$\|\nabla u\|_X^2 = (-\Delta u, u) \le \|\Delta u\|_X \|u\|_X \le D\|\Delta u\|_X \|\nabla u\|_X, \qquad u \in H_0^2(\Omega).$$

Therefore, $\|\nabla u\|_X \le D\|\Delta u\|_X$ for $u \in H_0^2(\Omega)$. Of course, it holds that $\|\nabla u\|_X \le D\|\Delta u\|_X$ for $u \in \mathcal{D}(A_1)$. □

The coefficient D is usually estimated by the band width of Ω, see [4, Section 4.7].

The rest of this section is devoted to obtaining an optimal estimate of D in the specific case where

$$\Omega = \{(x_1, x_2);\ 0 < x_1 < \ell_1,\ 0 < x_2 < \ell_2\}.$$

Let Λ denote a realization of $-\Delta$ equipped with the boundary condition $u = 0$ in $L_2(\Omega)$. Then, Λ is a positive definite self-adjoint operator of $L_2(\Omega)$ with domain $\mathcal{D}(\Lambda) = H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, since its minimal eigenvalue is $\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}$ with eigenfunction $\sin \frac{\pi}{\ell_1} x_1 \cdot \sin \frac{\pi}{\ell_2} x_2$, we have $(\Lambda u, u) \ge \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right) \|u\|_X^2$ for any $u \in \mathcal{D}(\Lambda)$. It then follows that

$$\|\nabla u\|_X^2 = (-\Delta u, u) \ge \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right) \|u\|_X^2, \qquad u \in \mathcal{D}(\Lambda).$$

Since $\mathcal{D}(\Lambda)$ is dense in $\mathcal{D}(\Lambda^{\frac{1}{2}})$ and since $\mathcal{D}(\Lambda^{\frac{1}{2}})$ coincides with $H_0^1(\Omega)$, this inequality holds true for every $u \in H_0^1(\Omega)$. Hence, (4.1) takes place with $D = \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right)^{-\frac{1}{2}}$ and, in fact, this is optimal.

Theorem 4.2. *Let $\Omega = (0, \ell_1) \times (0, \ell_2)$. Then, an optimal coefficient D for the Poincare inequality (4.1) is given by $D = \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$. Consequently, the weight constant d is estimated by $d \le \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$.*

Corollary 1. *Let $\Omega = (0, \ell_1) \times (0, \ell_2)$. If $\mu < \frac{\pi^2(\ell_1^2 + \ell_2^2)a}{\ell_1^2 \ell_2^2}$, then the null solution is globally stable.*

5 Numerical Results Let us here illustrate some numerical examples which shows some agreements to Corollary 1. We consider (1.1) in one of the following rectangular domains

$$\Omega = \left(0, \tfrac{1}{\ell}\right) \times (0, \ell), \quad \text{where } \ell \text{ is } 1, 2 \text{ or } 4.$$

When $\ell = 1$, Ω is square. Otherwise, Ω is strictly rectangular. The area of Ω is constantly equal to 1. The coefficients a and μ are fixed as $a = 1$ and $\mu = 40$.

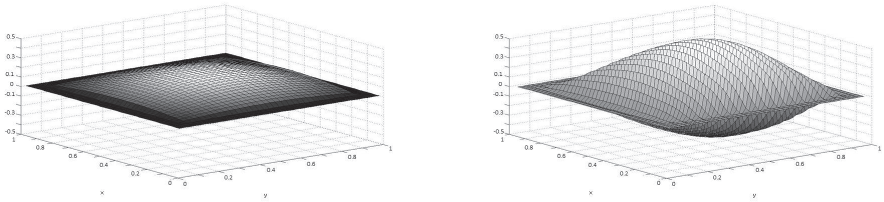
Set first $\Omega = (0, 1) \times (0, 1)$. We also set the initial function as

$$u_0(x_1, x_2) = 0.1[\sin(3.14x_1) \times \sin(3.14x_2)], \qquad (x_1, x_2) \in \Omega,$$

see Figure 1 (a). This is a small perturbation of the null solution. The solution then converges to some non-null stationary solution as $t \to \infty$. Its profile is given by Figure 1 (b). This means that the null stationary solution is unstable.

Set secondly $\Omega = \left(0, \frac{1}{2}\right) \times (0, 2)$. We accordingly replace the initial function with

$$u_0(x_1, x_2) = 0.1[\sin(2 \cdot 3.14x_1) \times \sin(3.14x_2)], \qquad (x_1, x_2) \in \Omega,$$



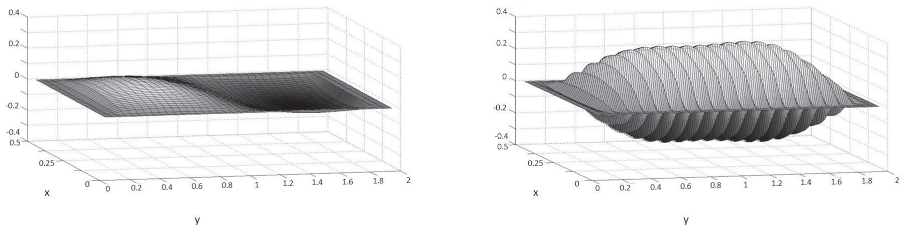
(a) t=0 (b) t=120

Fig. 1: Case where $\Omega = (0,1) \times (0,1)$

see Figure 2 (a). The solution again converges to some non-null stationary solution as $t \to \infty$ whose profile is given by Figure 2 (b). This means that the null stationary solution is still unstable.

Finally, set $\Omega = \left(0, \frac{1}{4}\right) \times (0,4)$, and replace the initial function with

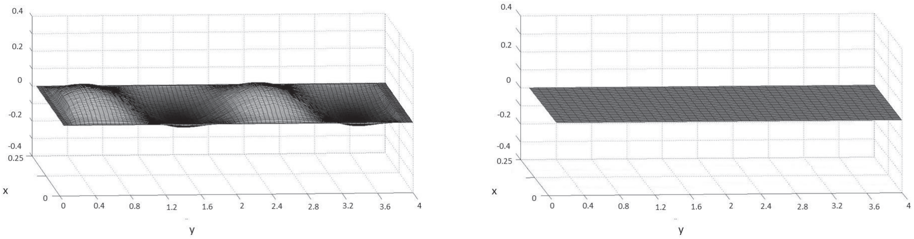
$$u_0(x_1, x_2) = 0.1[\sin(4 \cdot 3.14 x_1) \times \sin(3.14 x_2)], \qquad (x_1, x_2) \in \Omega,$$



(a) t=0 (b) t=180

Fig. 2: Case where $\Omega = (0, \frac{1}{2}) \times (0,2)$

see Figure 3 (a). As seen by Figure 3 (b), the solution now converges to the null solution. The domain Ω is slender enough to reduce the weight constant d in such a way that $d \leq \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$ (by Theorem 4.2) and to globally stabilize the null solution as ensured by Corollary 1.



(a) t=0 (b) t=240

Fig. 3: Case where $\Omega = (0, \frac{1}{4}) \times (0,4)$

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BERTRAND VERSUS COURNOT COMPETITION IN A VERTICAL DUOPOLY

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Abstract. This paper examines whether firms prefer to choose prices or quantities with a manufacturing duopoly in which each upstream firm sells its product to its own downstream firm. The degree of product differentiation plays an important role in whether firms set prices or quantities. We show that price competition performs better than quantity competition, from the upstream and downstream firms' point of view, regardless of the product differentiation. We also show that pay-offs are larger in Bertrand (price) competition than in Cournot (quantitiy) competition if both products are differentiated to a certain extent.

1 Introduction As we well know, two classical models in oligopoly theory are Cournot and Bertrand. In a non-cooperative profit maximization environment, one may wonder whether firms prefer to choose prices (Bertland) or quantities (Cournot). Singh and Vives (1984) first analyzed the issue of whether firms prefer to set prices or quantities. They show that consumer and total surplus in Bertrand competition are larger than those in Cournot competition regardless of the nature of goods.[1] They also show that Cournot equilibrium profits are higher than Bertrand equilibrium profits when the goods are substitutes, and vice versa when the goods are complements.[2]

During the past 30 years, many literatures have produced an array of extensions and generalizations of the analysis in Singh and Vives (1984). Previous literature on the issue has followed two separate streams. One stream focuses on extensions and generalizations of Singh and Vives (1984). For example, Dastidar (1997), Qiu (1997), Lambertini (1997), Häckner (2000), and Amir and Jin (2001), among others, have analyzed counter-examples based on the framework of Singh and Vives (1984) by allowing for a wider range of cost and demand asymmetries.[3] The other stream of the literature focuses on expanding the Bertrand-Cournot competition with vertically related duopoly. Correa-Lopez (2007) examines the Bertrand-Cournot profits ranking in a vertically related duopoly model focusing on substitutes and vertical product differentiation. They show that Bertrand profits may exceed Cournot profits when decentralized bargaining over the labor cost is introduced.[4]

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[1] When the goods are independent, they are equal.

[2] See Cheng (1985) for a graphical description of Singh and Vives' analysis.

[3] In particular, Zanchettin (2006) found that Singh and Vives's (1984) result that firms always make larger profits under Cournot competition than under Bertrand competition fails to hold.

[4] Symeonidis (2003, 2008) also analyzes the effects of downstream competition when there is bargaining between downstream firms and upstream agents (firms or unions).

Arya et al. (2008) explore the standard conclusions about duopoly competition when the production of key input is outsourced to a vertically integrated retail competitor with upstream market power. They show that prices and industry profits can be larger in Bertrand competition than in Cournot, while consumer and total surplus can be smaller in Bertrand than in Cournot. Mukherjee et al. (2012) compare Cournot with Bertrand competition in a vertical structure in which a monopoly upstream firm sells its product to two downstream firms, assuming there are asymmetric costs between downstream firms and homogeneous final goods. They demonstrate that the technology differences among the downstream firms and the pricing strategy (i.e., uniform pricing or price discrimination) of the upstream firm play an important role in the ranking of profit and social welfare. We revisit the profit ranking under Bertrand and Cournot competition in a vertically related duopoly in which each upstream firm sells its product to its own downstream firm. Our paper differs from the existing literature in at least two important aspects. First, previous studies consider Bertrand and Cournot competitions under wage bargaining and input prices negotiation. Our study examines them without negotiation. Second, previous ones produced the counter-results of Signs and Vives (1984) under costs and demand asymmetry. However, this paper analyzes the issue under symmetric conditions. This paper is organized as follows; in Section 2, we set up the model. Section 3 examines the Cournot competition, and then, Section 4 analyzes the Bertrand competition. Section 5 deals with comparative analysis. Finally, Section 6 contains concluding remarks.

Consider a manufacturing duopoly in which each upstream firm sells its product to its own downstream firm. There is a continuum of consumers of the same type with a utility function separable and linear in numeraire goods. Therefore, there are no income effects. The representative consumer maximizes $U(q_i, q_j) - \Sigma p_i q_i; i = 1, 2; i \neq j$, where q_i is the quantity of good i and p_i its price. U is assumed to be quadratic and strictly concave $U(q_i, q_j) = q_i + q_j - (q_i^2 + 2bq_iq_j + q_j^2)/2; i = 1, 2; i \neq j$. This utility function gives rise to a linear demand structure. Inverse demands are given by

$$p_i = 1 - q_i - bq_j,\ 0 \leq b \leq 1,\ i, j = 1, 2, i \neq j. \tag{1}$$

where p_i is the retail price for product i, and q_i and q_j are the amount of goods produced by channel i and j, respectively. Each unit of retail output requires exactly one unit of the input. The products are differentiated $(0 \leq b \leq 1)$. Upstream firms and downstream firms are risk-neutral and there are no production or retailing costs.

We posit a two-stage game. At stage one, each upstream firm sets an wholesale price. At stage two, each downstream firm also sets the retail price or quantity.

2 Cournot Competition We first consider Cournot competition in which each downstream firm sets a quantity. In this case the equilibrium concept is the sub-game perfect Nash equilibrium.

Stage Two (Quantity): At stage two, downstream firm i sets a quantity, q_i, so as to maximize its profit for a given input price, w_i. Downstream firm i's maximization problem is as follows:

$$\max \pi_i = (p_i - w_i)q_i,\ w.r.t.\ q_i.$$

where w_i is the input price. Therefore, downstream firm i sets the quantity, q_i, as the function of input prices as follows:

$$q_i(w_i, w_j) = \frac{2(1 - w_i) - b(1 - w_j)}{4 - b^2}. \tag{2}$$

Stage one (Wholesale Price): At stage one, upstream firm i sets wholesale, w_i, to maximize its profit for a given w_j. Upstream firm i's maximization problem is as follows:

$$\max \Pi_i = w_i q_i(w_i, w_j) = \frac{w_i[(2-w_i) - b(1-w_j)]}{4-b^2}, \ w.r.t. \ w_i.$$

The equilibrium wholesale price for upstream firm i is derived as follows:

$$w_i = \frac{2-b}{4-b}. \tag{3.1}$$

Substituting the wholesale price into Eq. (1) and Eq. (2), we obtain the retail price, p_i, the quantity, q_i, the upstream firm i's payoff, Π_i, and downstream firm i's payoff, π_i,

$$p_i^C = \frac{6-b^2}{(2+b)(4-b)}, \tag{3.2}$$

$$q_i^C = \frac{2}{(2+b)(4-b)}, \tag{3.3}$$

$$\Pi_i^C = \frac{2(2-b)}{(2+b)(4-b)^2}, \text{ and} \tag{3.4}$$

$$\pi_i^C = \frac{4}{(2+b)^2(4-b)^2}. \tag{3.5}$$

where superscripts C denote Cournot equilibrium.

3 Bertrand Competition We now turn to Bertrand competition in which each downstream firm sets a retail price. From Eq. (1), the following direct demand function can be derived as follows:

$$q_i = \frac{1-b-p_i+bp_j}{1-b^2}, \ 0 \le b \le 1, \ i,j = 1,2, i \ne j. \tag{4}$$

Stage Two (Retail Price): At stage two, downstream firm i sets retail price, p_i, so as to maximize its profit for a given wholesale price, w_i. Downstream firm i's maximization problem is as follows:

$$\max \pi_i = (p_i - w_i)q_i = \frac{(p_i - w_i)(1-b-p_i+bp_j)}{1-b^2}, \ w.r.t. \ p_i.$$

Therefore, downstream firm i sets the retail price, p_i, as the function of wholesale prices as follows:

$$p_i(w_i, w_j) = \frac{2(1-w_i) - b(1-w_j) - b^2}{4-b^2}. \tag{5}$$

Stage One (Wholesale Price): At stage one, upstream firm i sets a wholesale price, w_i, to maximize its profit for a given wholesale price, w_j. Upstream firm i's maximization problem is as follows:

$$\max \Pi_i = w_i q_i(w_i, w_j) = \frac{w_i[(2-b^2)(1-w_i) - b(1-w_j)]}{(4-b^2)(1-b^2)}, \ w.r.t. \ w_i.$$

The equilibrium wholesale price for upstream firm i is derived as follows:

$$w_i = \frac{2-b-b^2}{4-b-2b^2}. \tag{6.1}$$

Substituting the wholesale price into Eq. (4) and Eq. (5), we obtain the retail price, p_i, the quantity, q_i, the upstream firm i's payoff, Π_i, and downstream firm i's payoff, π_i,

$$p_i^B = \frac{2(1-b)(3-b^2)}{(2-b)(4-b-2b^2)}, \tag{6.2}$$

$$q_i^B = \frac{(2-b^2)}{(2-b)(4-b-2b^2)}, \tag{6.3}$$

$$\Pi_i^B = \frac{(1-b)(2+b)(2-b^2)}{(1+b)(2-b)(4-b-2b^2)^2}, \text{ and} \tag{6.4}$$

$$\pi_i^B = \frac{(1-b)(2-b^2)^2}{(1+b)(2-b)2(4-b-2b^2)^2}. \tag{6.5}$$

4 Comparative Analysis We turn now to compare the equilibrium under Bertrand and Cournot competition. Firstly, we compare wholesale prices between two types of contracts. From Eq. (3.1) and Eq. (6.1), we obtain the following results:

$$w_i^C - w_i^B = \frac{b^3}{(4-b)(4-b-2b^2)} \geq 0.$$

where superscripts B and C denote Bertrand and Cournot, respectively.
Lemma 1. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium wholesale prices are higher in Cournot than in Bertrand competition. If $b = 0$, both have the same wholesale prices.

Secondly, the equilibrium levels of retail prices and quantities are shown in Table 1.

Table 1: Equilibrium Levels of Retail Price and Quantity

	Retail Price	Quantity
Bertrand	$\frac{2(1-b)(3-b^2)}{(2-b)(4-b-2b^2)}$	$\frac{2-b^2}{(2-b)(4-b-2b^2)}$
Cournot	$\frac{(6-b^2)}{(2+b)(4-b)}$	$\frac{2}{(2+b)(4-b)}$

Lemma 2. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium prices for both downstream firms are higher in Cournot than in Bertrand competition. If $b = 0$, both have the same prices.
Lemma 3. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, the equilibrium outputs for both downstream firms are larger in Bertrand than in Cournot competition. If $b = 0$, both have the same input prices.

Quantities are larger and prices lower in Bertrand than in Cournot competition regardless of the nature of goods.[5] Lower prices and higher quantities are always better in welfare terms. Consumer and total surplus are decreasing as a function of prices. Therefore, in terms of consumer surplus and total surplus, the Bertrand equilibrium dominates the Cournot one. Proposition 1 summarizes the results thus far.

[5] When $b = 0$, they are equal.

Proposition 1. Under Eq. (1) and Eq. (4), if $0 < b \leq 1$, consumer surplus and total surplus are larger in Bertrand than in Cournot competition. If $b = 0$, they are equal.
For proof, see Appendix.

Thirdly, we turn to the equilibrium profits for Bertrand and Cournot competition. From Eq. (3.4) and Eq. (6.4), when $0 \leq b \leq 1$, notice that the following results are satisfied:

$$\Pi_i^B - \Pi_i^C = \frac{b^2(4+b-b^2)(16-b(2-b)(10+7b))}{(1+b)(2-b)(2+b)(4-b)^2(4-b-2b^2)^2},$$
$$\Pi_i^B > \Pi_i^C \Leftrightarrow 0 < b < 0.8868 \equiv \bar{b}.$$

Proposition 2. Under Eq. (1) and Eq. (4), if $0 < b \leq \bar{b}$, the Bertrand strategy is dominant for upstream firms. If $\bar{b} < b \leq 1$, the Cournot strategy is dominant for upstream firms. If $b = 0$, payoffs for both upstream firm are equal.

Proposition 2 can be explained as follows. If $0 < b < \bar{b}$, pay-offs in Bertrand competition are higher than those in Cournot, and vise versa. The degree of product differentiation plays an important role in equilibrium. As the degree of product differentiation decreases, the product market competition is more intense under Bertrand compared with Cournot competition. Therefore, pay-offs of Cournot competition are higher than those of Bertrand competition because of monopolistic effect. On the other hand, as the degree of product differentiation decreases, even if the wholesale price is lower in Bertrand competition than in Cournot competition, a more intense competition in the former helps to create a larger wholesale demand than in the latter. As a result, the upstream firm obtains higher pay-offs in Bertrand competition than in Cournot competition.

5 Concluding Remarks We may summarize the results derived from the model as follows:
(1) With linear demand function, if $0 < b \leq 1$, consumer and total surplus are larger in Bertrand than in Cournot competition.
(2) Pay-offs of both upstream firms are larger, equal, or smaller in Bertrand competition than in Cournot competition, according to whether $0 < b < \bar{b}$, or $\bar{b} < b \leq 1$.

We can also extend our analysis for each upstream firm and each downstream firm to make a precommitment to quantity or price contract in a vertically related market. In such a situation, we are wondering the results are the same as Singh and Vives (1984).

Appendix

Proof of Proposition 1. Consumer Surplus ranking of Bertrand and Cournot equilibria. In view of Lemma 2, consumer surplus is clearly higher under Bertrand than under Cournot competition. From the utility function, we get

$$CS = U(q_i, q_j) - (p_i q_i + p_j q_j) = q_i + q_j - \frac{(q_i^2 + 2bq_iq_j + q_j^2)}{2} - (p_i q_i + p_j q_j)$$
$$= q_i + q_j - \frac{(q_i + q_j)^2}{2} + (1-b)q_iq_j - (p_i q_i + p_j q_j) = (1-p_i)\frac{q_i}{2} + (1-p_j)\frac{q_j}{2}.$$

For $0 \leq b \leq 1$, inequality $CS^B > CS^C$ reduces to

$$CS^B - CS^C = \frac{b^2(8-3b^2)(32+8b-28b^2-4b^3+5b^4)}{(1+b)(2-b)^2(2+b)^2(4-b)^2(4-b-2b^2)^2} > 0.$$

This inequality holds for any $0 < b \leq 1$. For $b = 0$, consumer surplus is equal. From the utility function, we get

$$\begin{aligned} TS &= CS + \Pi_i + \Pi_j + \pi_i + \pi_j \\ &= U(q_i, q_j) - (p_i q_i + p_j q_j) + (w_i q_i + w_j q_j) + (p_i - w_i) q_i + (p_j - w_j) q_j \\ &= q_i + q_j - \frac{(q_i^2 + 2b q_i q_j + q_j^2)}{2} \\ &= q_i + q_j - \frac{(q_i + q_j)^2}{2} + (1-b) q_i q_j \\ &= \frac{(1-p_i) q_i}{2} + p_i q_i + \frac{(1-p_j) q_j}{2} + p_j q_j \\ &= \frac{(1+p_i) q_i}{2} + \frac{(1+p_j) q_j}{2}. \end{aligned}$$

For $0 \leq b \leq 1$, inequality $TS^B > TS^C$ reduces to

$$TS^B - TS^C = \frac{b^2(8-3b^2)(96-72b-60b^2+36b^3+9b^4-4b^5)}{(1+b)(2-b)^2(2+b)^2(4-b)^2(4-b-2b^2)^2} > 0$$

This inequality holds for any $0 < b \leq 1$. For $b = 0$, total surplus is equal.

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ON SOME MATRIX MEAN INEQUALITIES WITH KANTOROVICH CONSTANT

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Abstract. Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and σ, τ two arbitrary means between the harmonic and the arithmetic means. Put $h = \frac{M}{m}$. Then for every unital positive linear map Φ,

$$\begin{aligned} \Phi^2(A\sigma B) &\leq K^2(h)\Phi^2(A\tau B), \\ \Phi^2(A\sigma B) &\leq K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2, \\ (\Phi(A)\sigma\Phi(B))^2 &\leq K^2(h)\Phi^2(A\tau B), \\ (\Phi(A)\sigma\Phi(B))^2 &\leq K^2(h)(\Phi(A)\tau\Phi(B))^2, \end{aligned}$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ is the Kantorovich constant.

We also give a new characterization of the trace property and operator monotonicity by the squared Cauchy inequality.

Keywords: Matrix means, unital positive linear maps, Kantorovich inequality, trace, operator monotonicity.

1. Introduction

Throughout this paper, $\mathbb{M}_n$ stands for the algebra of all $n \times n$ matrices over the field of complex numbers. A continuous function f on an interval J ($\subset \mathbb{R}$) is said to be *operator monotone if*

$$A \leq B \quad \Longrightarrow \quad f(A) \leq f(B) \tag{1}$$

for any pair of self-adjoint bounded operators A, B on a separable infinite dimensional Hilbert space H with spectra $\sigma(A), \sigma(B) \subset J$. We

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know that the function f is operator monotone if and only if the inequality (1) holds for every self-adjoint matrices A, B of order n for every $n \in \mathbb{N}$.

We reserve M, m for scalars and I (in $\mathbb{M}_n$ or in $B(H)$) for the identity operator. The axiomatic theory for connections and means for pairs of positive operators on H have been studied by Kubo and Ando [6]. A binary operation σ defined on the cone of positive operators is called a connection if

(i) $A \le C, B \le D$ implies $A\sigma B \le B\sigma D$;
(ii) $C^*(A\sigma B)C \le (C^*AC)\sigma(C^*BC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then σ is called a mean. This definition can also be defined for positive operators on a finite dimensional Hilbert space. Operators on an n-dimensional space are identified with complex matrices of order n, hence we usually call connections/means in this case *matrix connections/means* of order n (see [1]). The fact is that an operator connection/mean is a matrix connection/mean of every order. However, throughout this paper operator means/connections will be used even the main theorem (and some other consequences) still hold for matrix means/connections.

Many authors study matrix inequalities containing means and unital positive linear maps on the matrix algebras. Such inequalities are interesting by themselves and have many applications in quantum information theory. One of the most important inequalities is the noncommutative AM-GM inequality which states that, for positive semidefinite matrices A, B,

$$A\nabla B = \frac{A+B}{2} \ge A\sharp B, \tag{2}$$

where, for positive definite matrices A, B,

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}$$

and, for positive semidefinite matrices A, B,

$$A\sharp B = \lim_{\varepsilon\to 0^+}(A+\varepsilon I)\sharp(B+\varepsilon I).$$

Moreover, by [11, Lemma 4.2] the left hand side and the right hand side of Cauchy inequality cover all ordered pairs of positive semidefinite matrices $X \geq Y$. However, this property does not hold for squares because the following inequality fails in general

$$(\frac{A+B}{2})^2 \geq (A\sharp B)^2. \tag{3}$$

Indeed, take the following matrices

$$X = \begin{pmatrix} 5/6 & 2 \\ 2 & 5 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}. \tag{4}$$

By help of Matlab, we can see that

$$det((X+Y)^2/4 - (X\sharp Y)^2) = -1.2301 < 0.$$

In [2], Lin proved the following Theorem.

Theorem 1.1 ([2])**.** *Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and put $h = \frac{M}{m}$. Then for every unital positive linear map Φ on $\mathbb{M}_n$,*

$$\Phi^2(A\nabla B) \leq K^2(h)\Phi^2(A\sharp B), \tag{5}$$

and

$$\Phi^2(A\nabla B) \leq K^2(h)(\Phi(A)\sharp\Phi(B))^2, \tag{6}$$

where $K(h) = \frac{(h+1)^2}{4h}$ is the Kantorovich constant.

It is well-known that the arithmetic mean ∇ is the biggest one among symmetric means (see [6]). A natural question is that: *Is the theorem above still true if we replace the biggest mean by a smaller one?* In this paper, we consider such inequalities for two different means with Kantorovich constant. In applications, we give an analogous result of Uchiyama and Yamazaki in [9] and the reverse of Minkovskii type inequality in [10].

It is well-known that for a monotone increasing function f on $\mathbb{R}^+$: $Tr(f(A)) \leq Tr(f(B))$ whenever $0 \leq A \leq B$. Consequently, for any two positive semidefinite matrices A, B and $p \geq 0$ we have $Tr((A\sharp B)^p) \leq Tr((A\nabla B)^p)$ even the inequality (3) does not hold.

Also, we can characterize positive linear functionals φ on $\mathbb{M}_n$ and operator monotone functions satisfying the following inequality: $\varphi(f(A\sharp B)) \leq \varphi(f(A\nabla B))$.

2. Main results

Lemma 2.1. *Let A and B be positive definite matrices with $0 < m \leq A, B \leq M$ for some scalar $0 < m \leq M$, and σ, τ two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map Φ,*

$$\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \leq M + m, \tag{7}$$

and

$$\Phi(A)\sigma\Phi(B) + Mm\Phi^{-1}(A\tau B) \leq M + m. \tag{8}$$

Proof. It is easy to see that

$$(M - A)(m - A)A^{-1} \leq 0,$$

or

$$mMA^{-1} + A \leq M + m.$$

Consequently,

$$\Phi(A) + mM\Phi(A^{-1}) \leq M + m.$$

Similarly,

$$\Phi(B) + mM\Phi(B^{-1}) \leq M + m.$$

Summing up two above inequalities, we get

$$\Phi(A \bigtriangledown B) + mM\Phi((A!B)^{-1}) \leq M + m.$$

Besides, by the hypothesis $\bigtriangledown \geq \sigma$ and $\tau \geq !$, we get

$$\begin{aligned}
\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B) &\leq \Phi(A\sigma B) + mM\Phi((A\tau B)^{-1}) \\
&\leq \Phi(A \bigtriangledown B) + mM\Phi((A!B)^{-1}) \\
&\leq M + m.
\end{aligned}$$

By a similar argument, we can get the inequality (8) using the fact that

$$\Phi(A)\sigma\Phi(B) \leq \Phi(A) \bigtriangledown \Phi(B) = \Phi(A \bigtriangledown B).$$

□

The following main theorem of this paper is a generalization of Lin's result (Theorem 1.1).

Theorem 2.1. *Let* A *and* B *be positive definite matrices with* $0 < m \leq A, B \leq M$ *for some scalar* $0 < m \leq M$*, and* σ, τ *two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map* Φ*,*

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{9}$$

$$\Phi^2(A\sigma B) \leq K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2, \tag{10}$$

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)\Phi^2(A\tau B), \tag{11}$$

and

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)(\Phi(A)\tau\Phi(B))^2, \tag{12}$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$.

Proof. We prove the inequality (9). The inequality (9) is equivalent to the following

$$\Phi^{-1}(A\tau B)\Phi^2(A\sigma B)\Phi^{-1}(A\tau B) \leq K^2(h),$$

or

$$||\Phi(A\sigma B)\Phi^{-1}(A\tau B)|| \leq K(h).$$

On the other hand, it is well known that [7, Theorem 1] for $A, B \geq 0$,

$$||AB|| \leq \frac{1}{4}||A+B||^2.$$

So, it is necessary to prove that

$$\frac{1}{4mM}||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)||^2 \leq \frac{(M+m)^2}{4Mm},$$

or,

$$||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)|| \leq M + m.$$

The last inequality follows from Lemma 2.1.

The remain inequalities in this theorem can be proved analogously. □

From the operator monotonicity of the function $f(t) = t^{1/2}$ on $[0, \infty)$ it obviously implies the following proposition.

Proposition 2.1. *Let $0 < m \leq A, B \leq M$ and σ, τ are two arbitrary means between the harmonic and the arithmetic means. Then for every unital positive linear map Φ,*

$$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$$

$$\Phi(A\sigma B) \leq K(h)\left(\Phi(A)\tau\Phi(B)\right),$$

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B)$$

and

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ with $h = \dfrac{M}{m}$.

Remark 1. From the following well-known fact:

$$A\tau B \leq A\nabla B \leq K(h)A!B \leq K(h)A\sigma B,$$

it implies immediately Proposition 2.1. On the other hand, it is well-known in [5] that if σ is a symmetric mean, then

$$\frac{m\sigma M}{m\nabla M} A\nabla B \leq A\sigma B.$$

Therefore, we have

$$\frac{m\sigma M}{m\nabla M} A\tau B \leq A\sigma B. \tag{13}$$

Also, Theorem 13 in [5] says that if $0 \leq m \leq A, B \leq M$, then

$$\frac{2\sqrt{mM}}{m+M} A\nabla B \leq A\sharp B \leq \frac{M+m}{2\sqrt{mM}} A!B. \tag{14}$$

Now we will show that the inequality (13) could not be squared when $\sigma = \sharp, \tau = \nabla$. Indeed, let's take $m = 0.02, M = 11$ and the matrices X, Y as in the equation (4). It is obvious that $m \leq X, Y \leq M$. With a help of Matlab we get

$$det(K(h)(X\sharp Y)^2 - (X\nabla Y)^2) = -4.1122,$$

and

$$det(K(h)(X!Y)^2 - (X\sharp Y)^2) = -1.7545.$$

Hence, the following inequalities

$$K(h)(X\sharp Y)^2 \geq (X\nabla Y)^2, \quad (X\sharp Y)^2 \geq K(h)(X!Y)^2$$

do not hold.

Corollary 2.1. *Let f, g be symmetric operator monotone functions on $[0, \infty)$. Then for any pair $0 < m < M$,*

$$max\{\frac{f(t)}{g(t)}, \frac{g(t)}{f(t)}\} \leq \frac{(m+M)^2}{4mM}, \quad t \in [m, M]. \tag{15}$$

Proof. It is necessary to apply [6, Theorem 3.2] and the above Proposition 2.1 for the symmetric means σ and τ corresponding to the functions f and g, and definition of means via their representation functions. □

The inequality (15) is interesting by itself, and the authors do not know any elementary proof even in the case when $f(t) = \sqrt{t}$.

As an application, now we give a similar result as in [9]. Uchiyama and Yamazaki showed that for an operator monotone function f on $[0, \infty)$ if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leq I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$. By applying Proposition 2.1, we get a similar result for any symmetric mean.

Corollary 2.2. *Let f be an operator monotone function on $[0, \infty)$ with $f(1) = 1$ and σ an arbitrary mean between the harmonic and the arithmetic ones. Let $0 < m < 1 < M$ and A, B positive definite matrices such that $0 < m \leq A, B \leq M$. If for all sufficiently small $\lambda > 0$*

$$f(\lambda B + I)^{-1} \sigma f(\lambda A + I) \leq K^{-1} \quad (where\ K = \frac{(m+M)^2}{4Mm}), \tag{16}$$

then

$$f(\lambda A + I) \leq f(\lambda B + I) \quad and \quad A \leq B.$$

Proof. From the continuity of the function f and assumptions, it follows that for all sufficiently small $\lambda > 0$

$$m \leq f(\lambda B + I)^{-1}, f(\lambda A + I) \leq M.$$

On account of Proposition 2.1 and (16), we obtain

$$\begin{aligned} f(\lambda B + I)^{-1} \sharp f(\lambda A + I) &\leq K(f(\lambda B + I)^{-1} \sigma f(\lambda A + I)) \\ &\leq I. \end{aligned}$$

By [9, Theorem 1], we get

$$f(\lambda A + I) \leq f(\lambda B + I) \quad \text{and} \quad A \leq B.$$

□

The famous Minkovskii determinantal inequality is

$$det^{1/n}(A+B) \geq det^{1/n}A + det^{1/n}B,$$

for any positive semidefinite matrices of order n A, B. In [10] Bourin and Hiai obtained the Minkovskii type inequality as follows: Let σ be an operator mean whose representing function f is geometrically convex, i.e., $f(\sqrt{xy}) \geq \sqrt{f(x)f(y)}$. Then, for every positive semidefinite matrices of order n A, B,

$$det^{1/n}(A\sigma B) \geq (det^{1/n}A)\sigma(det^{1/n}B),$$

and the reverse inequality holds if the representing function is geometrically concave.

Combining the reverse inequalities in Proposition 2.1, we give the lower bound and upper bound of the value $det(A\sigma B)$ for any operator mean σ between the arithmetic and the harmonic ones.

Corollary 2.3. *Let σ be a symmetric mean. If A, B are positive definite matrices such that $0 < m \leq A, B \leq M$, then*

$$K^{-1}(det^{1/n}(A)\nabla det^{1/n}(B)) \leq det^{1/n}(A\sigma B) \leq K(det^{1/n}(A)!det^{1/n}(B)),$$

where $K = \frac{(M+m)^2}{4Mm}$.

Proof. By Proposition 2.1, we have

$$A\nabla B \leq KA\sigma B.$$

We also know that $det^{1/n}$ preserves the order of matrices by the famous Minkovskii determinantal inequality. Consequently,

$$\begin{aligned} K^{-1}det^{1/n}(A)\nabla det^{1/n}(B) &\leq K^{-1}det^{1/n}(A\nabla B) \\ &\leq det^{1/n}(A\sigma B). \end{aligned}$$

Now we prove the second inequality of the corollary. By Proposition 2.1, we have $A\sigma B \leq KA!B$. Hence,

$$det^{1/n}(A\sigma B) \leq K\det^{1/n}(A!B).$$

Moreover, the function $f(t) = 1!t = \frac{2t}{1+t}$ corresponding the harmonic mean is geometrically concave, by the result in [10] mentioned above, we have

$$det^{1/n}(A!B) \leq det^{1/n}(A)!det^{1/n}(B)$$

and then the second inequality is obtained.

□

Now let us consider the problem of characterization of the trace property which is closed to the characterization of the operator monotonicity. It is well-known that, for a monotone increasing function f on $\mathbb{R}^+$, from the assumption $0 < A \leq B$ it follows that

$$Tr(f(A)) \leq Tr(f(B)).$$

Consequently, for any two positive definite matrices A, B we have

$$Tr(f(A\sharp B)) \leq Tr(f(A\nabla B)).$$

In [12] O. E. Tikhonov and A. M. Bikchentaev showed that for a positive linear functional φ on $\mathbb{M}_n$ and any given $p > 1$ if the inequality

$$\varphi(A^p) \leq \varphi(B^p)$$

holds for any pair of positive definite matrices $A \leq B$, then φ should be a scalar of the canonical trace. Note that the function $f(t) = t^p$ for $p > 1$ is not operator monotone on $[0, \infty)$. In the following proposition, replacing A, B by the geometric and the arithmetic means we can get the characterization of the trace.

Proposition 2.2. *For a positive linear functional φ on $\mathbb{M}_n$ and a given $p > 1$. If the following inequality*

$$\varphi((A\nabla B)^p) \geq \varphi((A\sharp B)^p)$$

holds whenever positive definite matrices $A \leq B$, then φ is a scalar of the canonical trace.

Proof. It is well-know that for arbitrary $0 < A \leq B$ we can find positive definite matrices X, Y such that $B = X\nabla Y, A = X\sharp Y$ (see [11, Lemma 4.2]). In fact, put $X = B - B\sharp(B - AB^{-1}A)$ and $Y = B + B\sharp(B - AB^{-1}A)$. By the assumption,

$$\varphi((X\nabla Y)^p) \geq \varphi((X\sharp Y)^p)$$

or

$$\varphi(A^p) \leq \varphi(B^p).$$

By the characterization mentioned above, φ is a scalar of the trace. □

The following corollary is just an immediate consequence of [13, Theorem 1]. However, we can give here a direct proof.

Corollary 2.4. *Let H be an infinite dimensional, separable Hilbert space and φ a normal state on $B(H)$ such that its corresponding density operator is not finite rank. Let f be a function defined on $(0,\infty)$. Then the following statements are equivalent:*

(i) *f is operator monotone;*
(ii) *the following inequality*

$$\varphi(f(A\nabla B)) \geq \varphi(f(A\sharp B))$$

holds for any positive operators $A, B \in B(H)$ satisfying the condition $\sigma(A\nabla B), \sigma(A\sharp B) \subset (0,\infty)$.

Proof. $(i) \Rightarrow (ii)$ is obvious. Conversely, for arbitrary positive invertible operators $X \leq Y$, we can find $A, B \geq 0$ such that $Y = A\nabla B, X = A\sharp B$ (see [11, Lemma 4.2]). In fact, put $A = Y - Y\sharp(Y - XY^{-1}X)$ and $B = Y + Y\sharp(Y - XY^{-1}X)$. By the assumption,

$$\varphi(f(Y)) = \varphi(f(A\nabla B)) \geq \varphi(f(A\sharp B)) = \varphi(f(X)).$$

By [13, Theorem 1], it follows that the function f is operator monotone. □

3. Some comments on the Kantorovich inequality

The Kantorovich inequality [14] states that for any $0 < m \leq A \leq M$. and any unital positive linear map Φ,

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm}\Phi(A)^{-1}. \tag{17}$$

From the Kantorovich inequality it is easy to get the following:

Corollary 3.1 ([15]). *Let $0 < m \leq A \leq M$. Then for every unital positive linear map Φ,*

$$\Phi(A^{-1})\sharp\Phi(A) \leq \frac{M+m}{2\sqrt{Mm}}. \tag{18}$$

Question: If we replace the geometric mean by an arbitrary mean, does the inequality (18) hold?

Counterexample: Let σ be an operator mean corresponding to the function $f(t) = t$. Assume that the inequality (18) holds for any $0 < m \le A \le M$, then it should hold for $A = M$. That means,

$$M^{-1}\sigma M \le \frac{M+m}{2\sqrt{Mm}},$$

or

$$M \le \frac{M+m}{2\sqrt{Mm}}.$$

Substitute $M = 2$ and $m = 1$ into the latter inequality, we get a contradiction. Even for symmetric means, for example, the arithmetic mean, the inequality (18) does not hold.

However, it is easy to see that for a unital positive linear map Φ and any mean σ,

$$\Phi(A^{-1})\sigma\Phi(B) \le m^{-1}\sigma M,$$

for $0 < m \le A, B \le M$, and the latter inequality can be squared.

Back to the above question, if we restrict our attention to the class of symmetric means, the inequality (18) holds true as follows.

Proposition 3.1. *Let $0 < m \le A \le M$. Then for any symmetric mean σ,*

$$A^{-1}\sigma A \le \sqrt{K(h')}, \tag{19}$$

where $K(h') = \frac{(M'+m')^2}{4M'm'}$ and M' and m' are the maximum and the minimum of the set $\{M, m, 1/m, 1/M\}$.

Proof. Let f be the symmetric operator monotone function corresponding to σ. Then the function $g(t) = \frac{t}{f(t)}$ is symmetric. From Corollary 2.1 and by direct calculation we get

$$\begin{aligned} A^{-1}\sigma A &= A^{-1}f(A^2) \\ &\le K(h')A^{-1}g(A^2) \\ &= K(h')Af^{-1}(A^2) \\ &= K(h')(A^{-1}\sigma A)^{-1}. \end{aligned}$$

Then we obtain (19). □

Let $0 < m \leq A \leq M$. Then for any symmetric mean σ and any unital positive linear map Φ,

$$\Phi^{-1}(A)\sigma\Phi(A) \leq \sqrt{K(h)}.$$

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GEOMETRICAL EXPANSION OF AN OPERATOR EQUATION

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Abstract. Lawson-Lim [9] had given a generalization of Karcher equation and the equations determining power means. We formulate these operator equations by simpler forms, which are geometrically meaningful.

Let $\mathbb{A}(A_1, \cdots, A_n)$ be positive operators and $\omega = \{\omega_1, \cdots, \omega_n\}$ be a weight. Then the operator equation

$$0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$$

has a unique positive solution for each $r \in [-1,\ 1]$, where $T_r(X|A) = \dfrac{X \natural_r A - X}{r}$ is the Tsallis relative operator entropy and $A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}$ for $r \in \mathbf{R}$. We show the exact form to the unique solution of the above operator equation in the case $n = 2$.

1 Introduction. Let $\mathbb{P}^+$ be the set of all positive invertible operators acting on a Hilbert space. Lawson-Lim [9] showed the Karcher equation for $A_1, \cdots, A_n\ X \in \mathbb{P}^+$ and a weight $\{\omega_1, \cdots, \omega_n\}$

$$\textbf{(KE)} \qquad 0 = \sum_{i=1}^{n} \omega_i \log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}})$$

has a unique positive invertible solution $X = G_K(\omega_i; A_i)$ which is called (*weighted n-variable*) Karcher mean. In [2], we introduced the relative operator entropy; for $A, B \in \mathbb{P}^+$,

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = \lim_{t \to 0} \frac{A \sharp_t B - A}{t},$$

that is, $\dfrac{d}{dt} A \sharp_t B|_{t=0} = S(A|B)$, where $A \sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}},\ t \in [0,\ 1]$. From this viewpoint, the equation **(KE)** is represented by multiplying $X^{\frac{1}{2}}$ on both sides as follows: For $A_1, \cdots, A_n, X \in \mathbb{P}^+$

$$(*) \qquad 0 = \sum_{i=1}^{n} \omega_i S(X|A_i).$$

The path $A \sharp_t B$ $(t \in [0,\ 1])$ is geodesic in $\mathbb{P}^+$ under the Thompson metric, i.e., $d(A, B) = \| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \|$, and $S(A|B) = \dfrac{d}{dt} A \sharp_t B\ |_{t=0}$. Therefore geometrical meaning of $(*)$ is clarified, see Figure 1.

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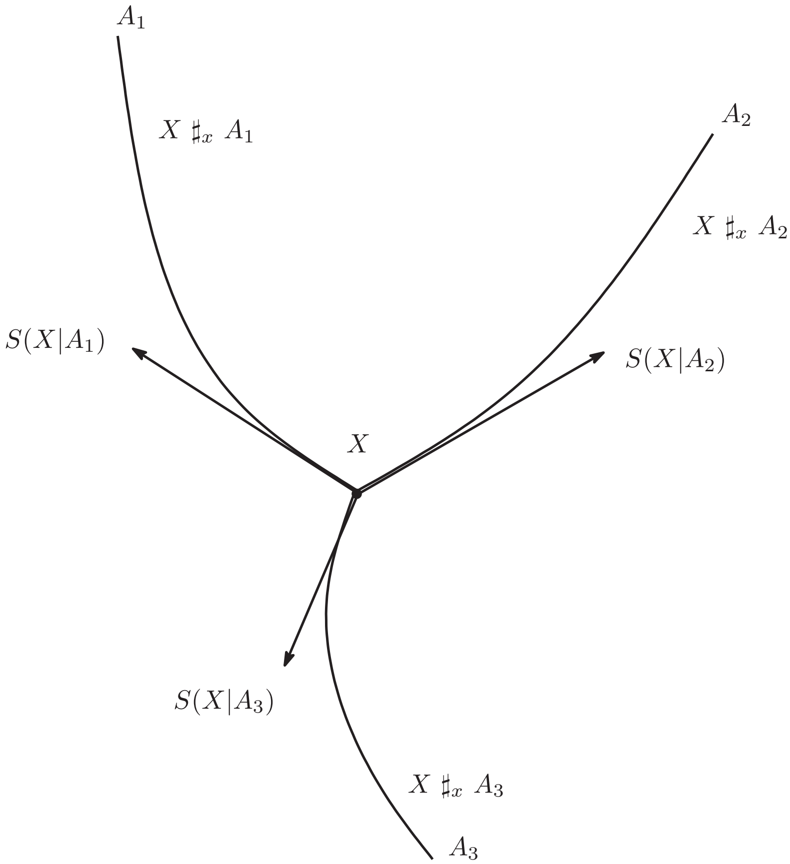


$$\omega_1 S(X|A_1) + \omega_2 S(X|A_2) + \omega_3 S(X|A_3) = 0$$

Figure 1

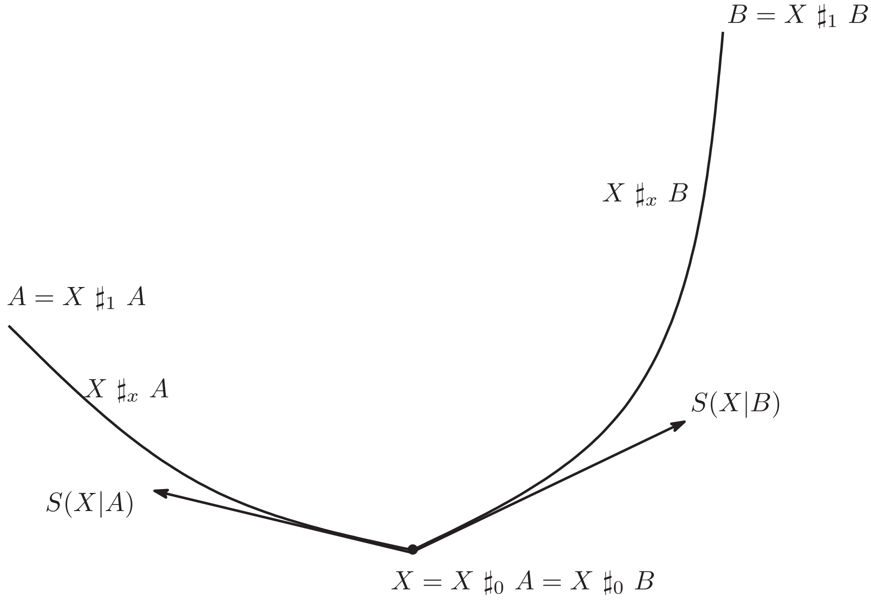
In the case $n = 2$, the equation $(*)$ is

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0, \quad \alpha \in [0,\ 1]$$

for $A,\ B,\ X \in \mathbb{P}^+$. It has the unique solution $X = A \sharp_\alpha B$, this is given in section 3 as Corollary 3 (cf. [6],[9]). Figure 2 shows if a path combining A and B is not geodesic, then X can't be a solution of the operator equation $(*)$. The Figure 3 gives an image that any X on the geodesic path combining A and B can be the solution of the operator equation $(*)$, where the notations $A \sharp_t B \supset X \sharp_y B$ and $X \sharp_x A \subset A \sharp_t B$ in Figure 3 mean that the path $A \sharp_t B$ contains the paths $X \sharp_y B$ and $X \sharp_x A$ as subdivisions of itself.
In the final section, we show that the above equation can be more generalized as

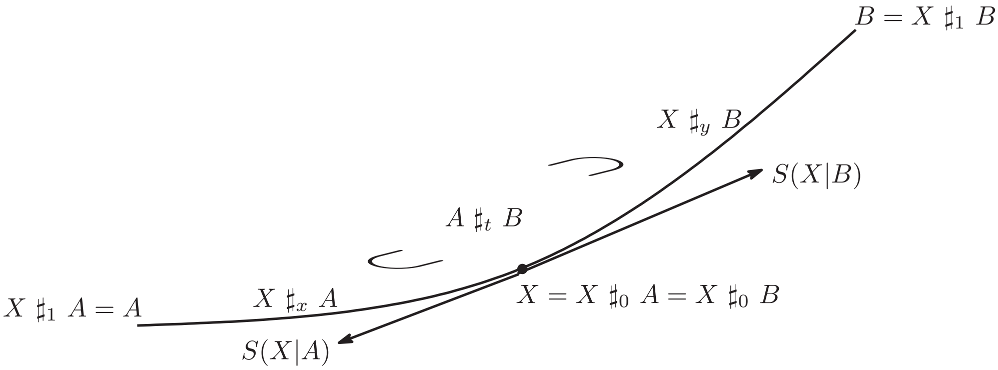
$$\alpha S(X|A) + \beta S(X|B) = 0, \quad \alpha,\ \beta \in \mathbf{R}$$

and this has the unique solution $X = A \natural_{\frac{\beta}{\alpha+\beta}} B$ also under some reasonable conditions.



$$(1-\alpha)S(X|A) + \alpha S(X|B) \neq 0$$

Figure 2



$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0$$

Figure 3

In [9], Lawson-Lim defined the ω-weighted power mean $P_t(\omega : \mathbb{A})$ of order t of $\mathbb{A} = (A_1, \cdots, A_n)$ as the solution of

$$(\textbf{LLE}) \qquad X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i), \quad \text{for} \quad t \in (0,\ 1],$$

for the weighted geometric operator mean [8] and $\omega = \{\omega_1, \cdots, \omega_n\}$ is a weight (cf.[10]). We show this equation has a geometrical structure.

As an approximation to the relative operator entropy, the Tsallis relative operator entropy is defined in [11] by

$$T_r(X|A) = \frac{X \natural_r A - X}{r}, \quad \text{for a fixed} \ \ r \neq 0,$$

where $X \natural_r A = X^{\frac{1}{2}}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r X^{\frac{1}{2}}$, which is not a Kubo-Ando mean but still the geodesic in the CPR-geometry [1]. We note that $S(X|A) = \lim_{r\to 0} T_r(X|A)$ (cf.[3],[4],[5]).

In section 3, we show in the following: For $\alpha \in [0,\ 1]$ and $r \in \mathbf{R}$, the equation

$$X = (1-\alpha)X \natural_r A + \alpha X \natural_r B$$

has the unique solution $X = A \sharp_{\alpha,r} B = A^{\frac{1}{2}}\{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}}$. We propose the following operator equation:

$$(**) \qquad 0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i), \quad r \in \mathbf{R},$$

especially if $r = 0$, $(**)$ is understood as $(*)$. For the 2 variable equations of $(**)$ and $(*)$, we can give the solutions.

2 Expanded LL equation.

For $t \in (0,\ 1]$, the operator equation **(LLE)** is understood as

$$(***) \qquad 0 = \sum_{i=1}^{n} \omega_i T_t(X|A_i),$$

whose unique solution is just $P_t(\omega;\mathbb{A})$, the ω-weighted power mean of order t of $\mathbb{A}$.
On the other hand, for $t \in [-1,\ 0)$, we show the power mean is also the solution of $(***)$. It has a parallelism with the Karcher equation $(*)$ by the use of the relative operator entropy.

For $t \in [-1,\ 0)$, Lawson-Lim [9] give $P_t(\omega;\mathbb{A})$ as the unique positive definite solution of the operator equation

$$(\mathbf{LLE}*) \qquad X = \Big(\sum_{i=1}^{n} \omega_i (X \sharp_{-t} A_i)^{-1}\Big)^{-1} \quad \text{or} \quad X^{-1} = \sum_{i=1}^{n} \omega_i (X^{-1} \sharp_{-t} A_i^{-1}),$$

and $P_t(\omega;\mathbb{A}) = P_{-t}(\omega;\mathbb{A}^{-1})^{-1}$ where $\mathbb{A}^{-1} = (A_1^{-1},\cdots,A_n^{-1})$. It is rewritten with no use of $\sharp_{-t}$ as follows: $(\mathbf{LLE}*)$ is equivalent to

$$I = \sum_{i=1}^{n} \omega_i (X^{\frac{1}{2}}A_i^{-1}X^{\frac{1}{2}})^{-t} = \sum_{i=1}^{n} \omega_i (X^{-\frac{1}{2}}A_iX^{-\frac{1}{2}})^{t}.$$

Furthermore it is given with the use of the Tsallis relative operator entropy; for n positive operators $\mathbb{A} = (A_1,\cdots,A_n)$ and a weight $\{\omega_1,\cdots,\omega_n\}$, we can propose the operator equation $0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$, for $r \in \mathbb{R}$. Lawson-Lim [9] teaches us this equation has a unique positive solution if $r \in [-1,\ 1]$.

Theorem 1. *Let $\mathbb{A} = (A_1,\cdots,A_n)$ be positive operators and $\{\omega_1,\cdots,\omega_n\}$ be a weight. Then for each $r \in [-1,\ 1]$, the operator equation*

$$(**) \qquad 0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$$

has a unique positive solution.

3 On 2 variable expanded Karcher equation. Let A and B be positive invertible operators on a Hilbert space. The operator power mean we use in this note is the following:

$$A \sharp_{\alpha,r} B = A^{\frac{1}{2}}\{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}} = A \natural_{\frac{1}{r}} \{A \nabla_\alpha (A \natural_r B)\},$$

where $0 \leq \alpha \leq 1$ and $r \in \mathbf{R}$. We regard this as a path combining A and B for each $r \in \mathbf{R}$, and $A \sharp_{\alpha,1} B = A \nabla_\alpha B = (1-\alpha)A + \alpha B$, weighted arithmetic operator mean, $A \sharp_{\alpha,0} B = A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$, weighted geometric operator mean and $A \sharp_{\alpha,-1} B = A \Delta_\alpha B = 2(A^{-1}+B^{-1})^{-1}$, weighted harmonic operator mean.
Related to the representing function $1 \sharp_{\alpha,r} x$, we consider for a fixed $x > 0$ and $r \in \mathbb{R}$, the function $\psi(\alpha) = (1-\alpha+\alpha x^r)^{\frac{1}{r}}$, $\alpha \in [0,\ 1]$, and

$$\frac{d}{d\alpha}\psi(\alpha) = \big(1-\alpha+\alpha x^r\big)^{\frac{1}{r}-1}\frac{x^r-1}{r} = (1-\alpha+\alpha x^r)^{\frac{1}{r}}(1-\alpha+\alpha x^r)^{-1}\frac{x^r-1}{r}.$$

So we gave in [4] the relative operator entropy along this path as follows:

$$(\heartsuit) \qquad S_{\alpha,r}(A|B) = \big(A \sharp_{\alpha,r} B\big)\big(A \nabla_\alpha (A \natural_r B)\big)^{-1}T_r(A|B),$$

especially $S_{0,r}(A|B) = T_r(A|B)$.

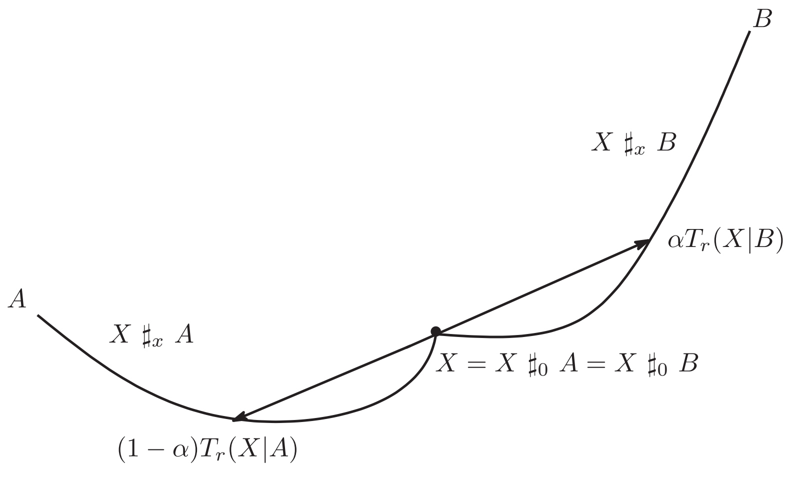
$T_r(A|B)$ has a property that for $t \in [0,\ 1]$, $tT_r(A|B) = T_r(A|A \sharp_{t,r} B)$ ([4],[5],[7]).

Theorem 2. *For $A,\ B,\ X \in \mathbb{P}^+$ and $\alpha \in [0,\ 1]$, $r \in \mathbf{R}$, the operator equation*

$$(1-\alpha)T_r(X|A) + \alpha T_r(X|B) = 0$$

has a unique solution $X = A \sharp_{\alpha,r} B$.

We give the follwing Figure 4 as an image of this theorem.



$$(1-\alpha)T_r(X|A) + \alpha T_r(X|B) = 0$$

Figure 4

Proof of Theorem 2. The following equivalent relations lead us to the conclusion.

$$
\begin{aligned}
& (1-\alpha)T_r(X|A)+\alpha T_r(X|B)=0 \\
\Longleftrightarrow\quad & (1-\alpha)(X \natural_r A - X)+\alpha(X \natural_r B - X)=0 \\
\Longleftrightarrow\quad & (1-\alpha)(X \natural_r A)+\alpha(X \natural_r B)=X \\
\Longleftrightarrow\quad & (1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r+\alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r=I \\
\Longleftrightarrow\quad & \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r=I-(1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & B=\alpha^{-\frac{1}{r}}X^{\frac{1}{2}}\big(I-(1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r\big)^{\frac{1}{r}}X^{\frac{1}{2}} \\
\Longleftrightarrow\quad & A^{-\frac{1}{2}}BA^{-\frac{1}{2}}=\alpha^{-\frac{1}{r}}A^{-\frac{1}{2}}X^{\frac{1}{2}}\big(I-(1-\alpha)(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}X^{\frac{1}{2}}A^{-\frac{1}{2}} \\
& \qquad\qquad\quad \overset{(*)}{=} \alpha^{-\frac{1}{r}}\big(I-(1-\alpha)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\
\Longleftrightarrow\quad & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r=\alpha^{-1}\big(I-(1-\alpha)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r=(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r-(1-\alpha)I \\
\Longleftrightarrow\quad & (1-\alpha)I+\alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r=(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & A^{-\frac{1}{2}}XA^{-\frac{1}{2}}=\{(1-\alpha)I+\alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}} \\
\Longleftrightarrow\quad & X=A^{\frac{1}{2}}\{(1-\alpha)I+\alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}}=A \sharp_{\alpha,r} B. \ \square
\end{aligned}
$$

The equation $\overset{(*)}{=}$ holds because $Yf(Y^*Y)=f(YY^*)Y$ for a continuous function f on an interval containing spectra of Y^*Y and YY^*.

Since $T_0(A|B)=\lim_{r\to 0}T_r(A|B)=S(A|B)$ and $A \sharp_{\alpha,0} B=\lim_{r\to 0}A \sharp_{\alpha,r} B=A \sharp_\alpha B$, we have the following if $r=0$ (cf.[6],[9]):

Corollary 3. *For $A,B,X\in\mathbb{P}^+$ and $\alpha\in[0,\ 1]$, the operator equation*

$$(1-\alpha)S(X|A)+\alpha S(X|B)=0$$

has a unique solution $X=A \sharp_\alpha B$.

4 Generalizations of Theorem 2 and Corollary 3.

In this section, we point out that Theorem 2 has more general form.

Theorem 4. *Let $A,\ B,\ X\in\mathbb{P}^+$ and $\alpha,\ \beta\in\mathbf{R}$ such that $\alpha+\beta\neq 0$ and $\alpha\beta\neq 0$. If the operator equation*

$$\beta T_r(X|A)+\alpha T_r(X|B)=0$$

holds, then $\frac{\beta}{\alpha+\beta}I+\frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\geq 0$ and this equation has the unique solution

$$X=A^{\frac{1}{2}}\Big\{\frac{\beta}{\alpha+\beta}I+\frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\Big\}^{\frac{1}{r}}A^{\frac{1}{2}}.$$

Proof. The following equation $\overset{(*)}{=}$ is led by the same reason in the proof of Theorem 2.

$$
\begin{aligned}
& \beta T_r(X|A) + \alpha T_r(X|B) = 0 \\
\iff\quad & \beta(X \natural_r A - X) + \alpha(X \natural_r B - X) = 0 \\
\iff\quad & \beta(X \natural_r A) + \alpha(X \natural_r B) = (\alpha+\beta)X \\
\iff\quad & \beta(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r + \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = (\alpha+\beta)I \\
\iff\quad & \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = (\alpha+\beta)I - \beta(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \\
\iff\quad & (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = \frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \geq 0 \\
\iff\quad & B = X^{\frac{1}{2}}\big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r\big)^{\frac{1}{r}}X^{\frac{1}{2}} \\
\iff\quad & A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = A^{-\frac{1}{2}}X^{\frac{1}{2}}\big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}X^{\frac{1}{2}}A^{-\frac{1}{2}} \\
& \qquad\qquad\quad \overset{(*)}{=} \big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\
\iff\quad & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = \big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\iff\quad & \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (\alpha+\beta)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r - \beta I, \\
\iff\quad & \beta I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (\alpha+\beta)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r, \\
\iff\quad & \frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \geq 0 \\
\iff\quad & A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = \{\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}} \\
\iff\quad & X = A^{\frac{1}{2}}\Big\{\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\Big\}^{\frac{1}{r}}A^{\frac{1}{2}}
\end{aligned}
$$

The next theorem is a modification of Corollary 3, which is given as the case $r = 0$ in Theorem 4. But we here pose an independent proof of Theorem 4.

Theorem 5. *For $\alpha,\ \beta \in \mathbf{R}$ such that $\alpha+\beta \neq 0$ and $\alpha\beta \neq 0$, the operator equation*

$$\beta S(X|A) + \alpha S(X|B) = 0$$

has the unique solution $X = B \natural_{\frac{\beta}{\alpha+\beta}} A = A \natural_{\frac{\alpha}{\alpha+\beta}} B$.

We recall $rS(A|B) = S(A|A \natural_r B)$ for $r \in \mathbf{R}$, (cf.[4],[5],[7]), and prepare the next lemma.

Lemma 6. (cf.[9]) *Let $A,\ B,\ X \in \mathbb{P}^+$, then the following hold:*

$$S(X|A) + S(X|B) = 0 \quad \text{if and only if} \quad X = A \sharp B$$

Proof of Theorem 5. Since

$$\beta S(X|A) + \alpha S(X|B) = S(X|X \natural_\beta A) + S(X|X \natural_\alpha B) = 0, \quad \alpha,\beta \in \mathbf{R},$$

we have $X = (X \natural_\beta A) \sharp (X \natural_\alpha B)$ by the above lemma, which is equivalent to

$$I = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^\beta \sharp (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^\alpha.$$

Since $C \sharp D = I \iff C = D^{-1}$ for $C,\ D \in \mathbb{P}^+$, we have

$$(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^\alpha = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{-\beta},$$

that is,

$$A = X^{\frac{1}{2}}(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{-\frac{\alpha}{\beta}}X^{\frac{1}{2}} = X \natural_{-\frac{\alpha}{\beta}} B = B \natural_{\frac{\alpha+\beta}{\beta}} X.$$

Hence $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^{\frac{\alpha+\beta}{\beta}}$, we have $(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{\beta}{\alpha+\beta}} = B^{-\frac{1}{2}}XB^{-\frac{1}{2}}$, and

$$X = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{\beta}{\alpha+\beta}}B^{\frac{1}{2}} = B \natural_{\frac{\beta}{\alpha+\beta}} A = A \natural_{\frac{\alpha}{\alpha+\beta}} B.$$

So we have

$$\beta S(A \natural_{\frac{\alpha}{\alpha+\beta}} B|A) + \alpha S(B \natural_{\frac{\beta}{\alpha+\beta}} A|B) = 0. \qquad \square$$

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SOME OPERATOR DIVERGENCES BASED ON PETZ-BREGMAN DIVERGENCE

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Abstract. Let A and B be strictly positive operators on a Hilbert space. For relative operator entropies $S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, $T_\alpha(A|B) \equiv \frac{1}{\alpha}(A \sharp_\alpha B - A)$ and $S_\alpha(A|B) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, we showed

$$(*) \qquad S_1(A|B) \geq -T_{1-\alpha}(B|A) \geq S_\alpha(A|B) \geq T_\alpha(A|B) \geq S(A|B) \ \text{ for } \alpha \in (0,1).$$

Petz gave an operator divergence $D_0(A|B) = B - A - S(A|B)$ which we call Petz-Bregman divergence. Petz also defined Bregman divergence $D_\Psi(X,Y)$ for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space.

In this paper, firstly, we define new operator divergences as the differences between two terms in $(*)$ and represent them by using $D_0(A|B)$. Secondly, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B)$ for $\Psi(t) = A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1,0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Bregman divergences $D_\Psi(1,0)$ for several functions Ψ which relate to the operator divergences defined above.

1 Introduction. Throughout this paper, a bounded linear operator T on a Hilbert space H is positive (denoted by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in H$, and T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Fujii and Kamei [2] defined the following relative operator entropy for strictly positive operators A and B:

$$S(A|B) \equiv A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Furuta [7] defined generalized relative operator entropy as follows:

$$S_\alpha(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}, \ \alpha \in \mathbb{R},$$

We know immediately $S_0(A|B) = S(A|B)$. Yanagi, Kuriyama and Furuichi [15] introduced Tsallis relative operator entropy as follows:

$$T_\alpha(A|B) \equiv \frac{A \sharp_\alpha B - A}{\alpha}, \ \alpha \in (0,1],$$

where $A \sharp_\alpha B \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$ is the weighted geometric operator mean (cf. [12]). Since $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log a$ holds for $a > 0$, we have $T_0(A|B) \equiv \lim\limits_{\alpha \to 0} T_\alpha(A|B) =$

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$S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha \in \mathbb{R}$. In [8], we had given the following relations among these relative operator entropies:

$$(*) \qquad S_1(A|B) \geq -T_{1-\alpha}(B|A) \geq S_\alpha(A|B) \geq T_\alpha(A|B) \geq S(A|B),\ \alpha \in (0,1).$$

A path $A \natural_x B$ passing through A and B is given as follows ([3, 4, 11] etc.):

$$A \natural_x B \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^x A^{\frac{1}{2}},\ x \in \mathbb{R}.$$

We remark that $A \natural_x B = B \natural_{1-x} A$ holds for $x \in \mathbb{R}$. If $x \in [0,1]$, the path $A \natural_x B$ coincides with $A \sharp_x B$. We can regard $S_\alpha(A|B)$ as the slope of the tangent line of the path $A \natural_x B$ at $x = \alpha$ and $T_\alpha(A|B)$ as the slope of the line passing through A and $A \sharp_\alpha B$ on the path.

Fujii [1] defined an operator valued α-divergence $D_\alpha(A|B)$ for $\alpha \in (0,1)$ as follows:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)},$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean. The operator valued α-divergence has the following relations at end points for interval $(0,1)$.

Theorem A ([5, 6])**.** *For strictly positive operators A and B, the following hold*:

$$\begin{aligned} D_0(A|B) &\equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B), \\ D_1(A|B) &\equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = A - B - S(B|A). \end{aligned}$$

Petz [14] introduced the right hand side in the first equation in Theorem A as an operator divergence, so we call $D_0(A|B)$ *Petz-Bregman divergence*. We remark that $D_1(A|B) = D_0(B|A)$ holds. Figure 1 shows our interpretation of $D_0(A|B)$.

In [9], we represented $D_\alpha(A|B)$ as follows:

$$D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B),\ \alpha \in (0,1),$$

which is a difference between two of five terms in $(*)$. Moreover, $D_0(A|B)$ can be also represented as $D_0(A|B) = T_1(A|B) - S(A|B)$. From these facts, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. In section 2, we represent these operator divergences by using Petz-Bregman divergence.

On the other hand, for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space, Petz [14] defined a divergence $D_\Psi(X,Y)$ as follows:

$$D_\Psi(X,Y) \equiv \Psi(X) - \Psi(Y) - \lim_{\alpha \to +0} \frac{\Psi(Y + \alpha(X-Y)) - \Psi(Y)}{\alpha}.$$

We call $D_\Psi(X,Y)$ Ψ*-Bregman divergence* of Y and X in this paper. Petz gave some examples for invertible density matrices X and Y. If $\Psi(X) = \eta(X) \equiv X \log X$ and X commutes with Y, then $D_\Psi(X,Y) = Y - X + X(\log X - \log Y)$, and if $\Psi(X) = tr\ \eta(X)$, then $D_\Psi(X,Y) = tr\ X(\log X - \log Y)$, which is the usual quantum relative entropy.

In section 3, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B)$ for $\Psi(t) = A \natural_t B$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1,0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Ψ-Bregman divergences $D_\Psi(1,0)$ for several functions Ψ which relate to the operator divergences given in section 2.

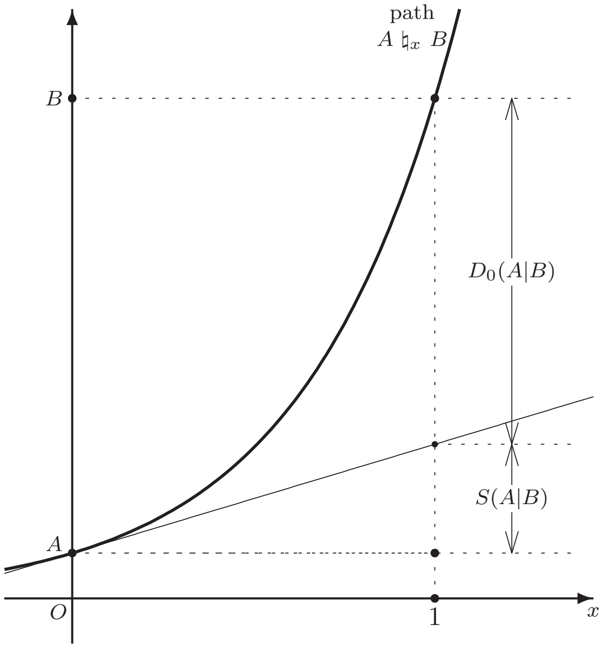

Figure 1: An interpretation of $D_0(A|B)$.

2 Divergences given by the differences among relative operator entropies. As we mentioned in section 1, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. There are 10 such divergences. For convenience, we use symbols Δ_i for them as follows:

$$\begin{aligned} &\Delta_1 = T_\alpha(A|B) - S(A|B), && \Delta_2 = S_\alpha(A|B) - T_\alpha(A|B), \\ &\Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B), && \Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A), \\ &\Delta_5 = S_\alpha(A|B) - S(A|B), && \Delta_6 = -T_{1-\alpha}(B|A) - T_\alpha(A|B) = D_\alpha(A|B), \\ &\Delta_7 = S_1(A|B) - S_\alpha(A|B), && \Delta_8 = -T_{1-\alpha}(B|A) - S(A|B), \\ &\Delta_9 = S_1(A|B) - T_\alpha(A|B), && \Delta_{10} = S_1(A|B) - S(A|B). \end{aligned}$$

In this section, we consider a relation between each of $\Delta_1, \cdots, \Delta_{10}$ and the Petz-Bregman divergence $D_0(A|B)$. It is sufficient to consider Δ_1, Δ_2, Δ_3 and Δ_4 since the following relations hold:

$$\begin{aligned} &\Delta_5 = \Delta_1 + \Delta_2, && \Delta_6 = \Delta_2 + \Delta_3, && \Delta_7 = \Delta_3 + \Delta_4, \\ &\Delta_8 = \Delta_1 + \Delta_2 + \Delta_3, && \Delta_9 = \Delta_2 + \Delta_3 + \Delta_4, && \Delta_{10} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4. \end{aligned}$$

The order of the differences among $\Delta_1, \cdots, \Delta_{10}$ are given as in Table 1.

The next lemma is essential tools in our discussion.

Lemma 2.1 ([8, 9])**.** *For strictly positive operators A and B, the following hold for $s, t \in \mathbb{R}$:*

(1) $$S_t(A|A \natural_s B) = sS_{st}(A|B),$$

(2) $$S_t(A|B) = -S_{1-t}(B|A).$$

Table 1

$$
\begin{array}{lllll}
S_1(A|B)-S(A|B) & \geq S_1(A|B)-T_\alpha(A|B) & \geq S_1(A|B)-S_\alpha(A|B) & \geq S_1(A|B)+T_{1-\alpha}(B|A) & \geq 0 \\
\quad\vee| & \quad\vee| & \quad\vee| & & \\
-T_{1-\alpha}(B|A)-S(A|B) & \geq -T_{1-\alpha}(B|A)-T_\alpha(A|B) & \geq -T_{1-\alpha}(B|A)-S_\alpha(A|B) & \geq 0 & \\
\quad\vee| & \quad\vee| & & & \\
S_\alpha(A|B)-S(A|B) & \geq S_\alpha(A|B)-T_\alpha(A|B) & & & \\
\quad\vee| & \quad\vee| & & & \\
T_\alpha(A|B)-S(A|B) & \quad 0 & & & \\
\quad\vee| & & & & \\
\quad 0 & & & &
\end{array}
$$

The following are results on Δ_1, Δ_2, Δ_3 and Δ_4.

Theorem 2.2. *For strictly positive operators A and B, the following hold*:

(1) $$\Delta_1 = T_\alpha(A|B) - S(A|B) = \frac{1}{\alpha}D_0(A|A \sharp_\alpha B) \text{ for } \alpha \in (0,1],$$

(2) $$\Delta_2 = S_\alpha(A|B) - T_\alpha(A|B) = \frac{1}{\alpha}D_0(A \sharp_\alpha B|A) \text{ for } \alpha \in (0,1],$$

(3) $$\Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = \frac{1}{1-\alpha}D_0(A \sharp_\alpha B|B) \text{ for } \alpha \in [0,1),$$

(4) $$\Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B|A \sharp_\alpha B) \text{ for } \alpha \in [0,1).$$

Proof. (1) By (1) in Lemma 2.1, we have

$$
\begin{aligned}
T_\alpha(A|B) - S(A|B) &= \frac{A \sharp_\alpha B - A}{\alpha} - S(A|B) = \frac{1}{\alpha}\big(A \sharp_\alpha B - A - \alpha S(A|B)\big) \\
&= \frac{1}{\alpha}\big(A \sharp_\alpha B - A - S(A|A \sharp_\alpha B)\big) = \frac{1}{\alpha}D_0(A|A \sharp_\alpha B).
\end{aligned}
$$

(2) By Lemma 2.1, we have

$$
\begin{aligned}
S_\alpha(A|B) - T_\alpha(A|B) &= \frac{A - A \sharp_\alpha B}{\alpha} + S_\alpha(A|B) = \frac{1}{\alpha}\big(A - A \sharp_\alpha B + \alpha S_\alpha(A|B)\big) \\
&= \frac{1}{\alpha}\big(A - A \sharp_\alpha B + S_1(A|A \sharp_\alpha B)\big) \\
&= \frac{1}{\alpha}\big(A - A \sharp_\alpha B - S(A \sharp_\alpha B|A)\big) = \frac{1}{\alpha}D_0(A \sharp_\alpha B|A).
\end{aligned}
$$

(3) By Lemma 2.1 and (2) in this theorem, we have

$$
\begin{aligned}
-T_{1-\alpha}(B|A) - S_\alpha(A|B) &= -T_{1-\alpha}(B|A) + S_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B \sharp_{1-\alpha} A|B) \\
&= \frac{1}{1-\alpha}D_0(A \sharp_\alpha B|B).
\end{aligned}
$$

(4) By (2) in Lemma 2.1 and (1) in this theorem, we have

$$\begin{aligned} T_{1-\alpha}(B|A) + S_1(A|B) &= T_{1-\alpha}(B|A) - S(B|A) \\ &= \frac{1}{1-\alpha} D_0(B|B \sharp_{1-\alpha} A) = \frac{1}{1-\alpha} D_0(B|A \sharp_\alpha B). \end{aligned}$$

□

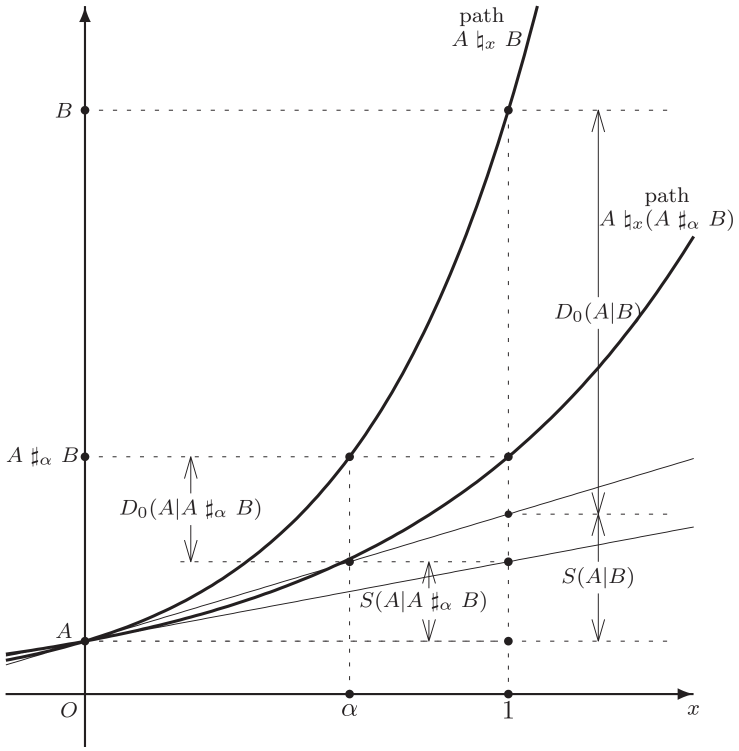

Figure 2: An interpretation of $D_0(A|A \sharp_\alpha B) = A \sharp_\alpha B - A - S(A|A \sharp_\alpha B)$.

Figure 2 shows an interpretation of $D_0(A|A \sharp_\alpha B)$ appeared in (1) in Theorem 2.2, and in Figure 3 we illustrate an interpretation of (1) and (2) in Theorem 2.2.

Theorem 2.2 leads the next theorem.

Theorem 2.3. *For strictly positive operators A and B, the following hold*:

$$D_\alpha(A|B) = \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A) \quad \text{for } \alpha \in (0,1).$$

Proof. By (2) and (3) in Theorem 2.2, we have

$$\begin{aligned} D_\alpha(A|B) &= -T_{1-\alpha}(B|A) - T_\alpha(A|B) \\ &= \big(-T_{1-\alpha}(B|A) - S_\alpha(A|B)\big) + \big(S_\alpha(A|B) - T_\alpha(A|B)\big) \\ &= \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A). \end{aligned}$$

□

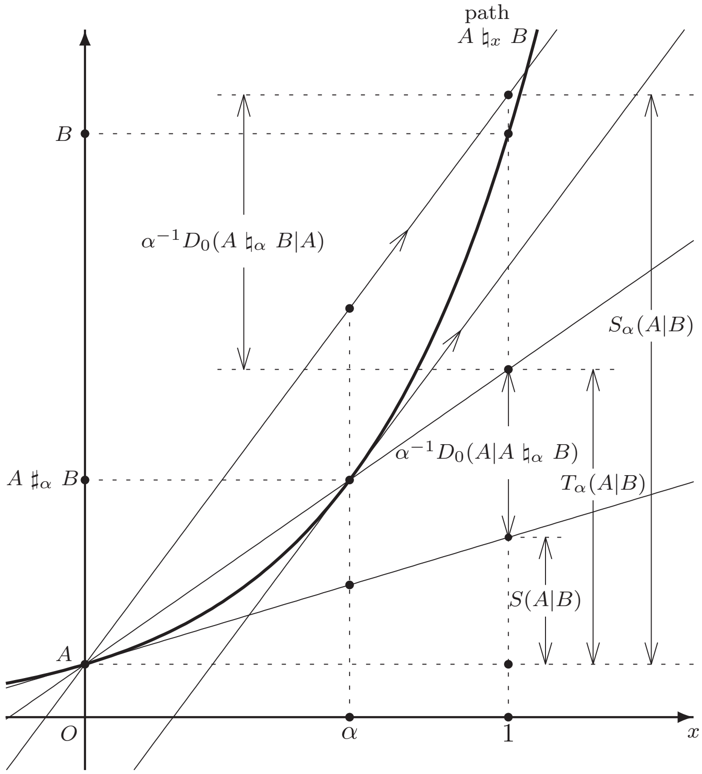

Figure 3: An interpretation of $S_\alpha(A|B) - T_\alpha(A|B) = \alpha^{-1}D_0(A \natural_\alpha B|A)$ and $T_\alpha(A|B) - S(A|B) = \alpha^{-1}D_0(A|A \natural_\alpha B)$.

By Theorem 2.3, we have

$$\begin{aligned}\alpha(1-\alpha)D_\alpha(A|B) &= \alpha D_0(A \sharp_\alpha B|B) + (1-\alpha)D_0(A \sharp_\alpha B|A) \\ &= \alpha\big(B - A \sharp_\alpha B - S(A \sharp_\alpha B|B)\big) + (1-\alpha)\big(A - A \sharp_\alpha B - S(A \sharp_\alpha B|A)\big) \\ &= A \nabla_\alpha B - A \sharp_\alpha B - \big((1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B)\big),\end{aligned}$$

and then

$$(1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B) = 0,$$

since $D_\alpha(A|B) = \dfrac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)}$. This means that $A \sharp_\alpha B$ is a solution of $(1-\alpha)S(X|A) + \alpha S(X|B) = 0$ which is called Karcher equation. For 2-variable cases, we can rewrite the result of Lawson-Lim [13] as follows:

Theorem 2.4 ([13]). *For strictly positive operators A, B and X, and for $\alpha \in [0,1]$,*

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0 \text{ if and only if } X = A \sharp_\alpha B.$$

3 Ψ-Bregman divergences on the differences of relative operator entropies. In this section, we consider Ψ-Bregman divergence in the case $\mathbf{C} = \mathbb{R}$ as follows: For an operator valued smooth function $\Psi : \mathbb{R} \to B(H)$ and $x, y \in \mathbb{R}$,

$$D_\Psi(x,y) \equiv \Psi(x) - \Psi(y) - \lim_{\alpha \to +0} \frac{\Psi(y + \alpha(x-y)) - \Psi(y)}{\alpha}.$$

From the following proposition, it is natural that we consider $D_\Psi(1,0)$ as a divergence of operators A and B.

Proposition 3.1. *Let $\Psi(t) = A \natural_t B$ for strictly positive operators A and B. Then for $x, y \in \mathbb{R}$,*

$$D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B).$$

In particular, $D_\Psi(1,0) = D_0(A|B)$.

Proof.

$$\begin{aligned} D_\Psi(x,y) &= A \natural_x B - A \natural_y B - \lim_{\alpha\to+0} \frac{A \natural_{y+\alpha(x-y)} B - A \natural_y B}{\alpha} \\ &= A \natural_x B - A \natural_y B - \lim_{\alpha\to+0} \frac{(A \natural_y B) \natural_\alpha (A \natural_x B) - A \natural_y B}{\alpha} \quad \text{by [10, Lemma 2.2]} \\ &= A \natural_x B - A \natural_y B - S(A \natural_y B | A \natural_x B) = D_0(A \natural_y B | A \natural_x B). \end{aligned}$$

□

In the rest of this section, we obtain $D_\Psi(1,0)$ for functions Ψ which relate to the operator divergences Δ_1, Δ_2, Δ_5 and Δ_6 in section 2.

Theorem 3.2. *For strictly positive operators A and B, the following hold:*

(1) If $\Psi(t) = T_t(A|B) - S(A|B)$, then

$$D_\Psi(1,0) = D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

(2) If $\Psi(t) = S_t(A|B) - S(A|B)$, then

$$D_\Psi(1,0) = D_0(A|B) + D_0(B|A) - S(A|B)A^{-1}S(A|B).$$

(3) If $\Psi(t) = S_t(A|B) - T_t(A|B)$, then

$$D_\Psi(1,0) = D_0(B|A) - \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

Proof. (1) For $a > 0$, we have

$$\lim_{\alpha\to+0} \frac{a^\alpha - 1 - \alpha \log a}{\alpha^2} = \frac{1}{2}(\log a)^2.$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned} \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} &= \lim_{\alpha\to+0} \frac{A^{\frac{1}{2}}\big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha - I - \alpha\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)A^{\frac{1}{2}}}{\alpha^2} \\ &= \frac{1}{2}A^{\frac{1}{2}}\big(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)^2 A^{\frac{1}{2}} = \frac{1}{2}S(A|B)A^{-1}S(A|B), \end{aligned}$$

then

$$\begin{aligned} D_\Psi(1,0) &= T_1(A|B) - S(A|B) - \big(T_0(A|B) - S(A|B)\big) \\ &\quad - \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B) - \big(T_0(A|B) - S(A|B)\big)}{\alpha} \\ &= T_1(A|B) - S(A|B) - \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} \\ &= D_0(A|B) - \frac{1}{2}S(A|B)A^{-1}S(A|B). \end{aligned}$$

(2) For $a > 0$, we have

$$\lim_{\alpha\to+0} \frac{a^\alpha \log a - \log a}{\alpha} = (\log a)^2.$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned}
&\lim_{\alpha\to+0} \frac{S_\alpha(A|B) - S(A|B)}{\alpha} \\
&\quad = \lim_{\alpha\to+0} \frac{A^{\frac{1}{2}}\big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)A^{\frac{1}{2}}}{\alpha} \\
&\quad = A^{\frac{1}{2}}\big(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)^2 A^{\frac{1}{2}} = S(A|B)A^{-1}S(A|B),
\end{aligned}$$

then by (2) in Lemma 2.1,

$$\begin{aligned}
D_\Psi(1,0) &= S_1(A|B) - S(A|B) - \big(S_0(A|B) - S(A|B)\big) \\
&\quad - \lim_{\alpha\to+0} \frac{S_\alpha(A|B) - S(A|B) - \big(S_0(A|B) - S(A|B)\big)}{\alpha} \\
&= S_1(A|B) - S(A|B) - \lim_{\alpha\to+0} \frac{S_\alpha(A|B) - S(A|B)}{\alpha} \\
&= \big(B - A - S(A|B)\big) + \big(A - B - S(B|A)\big) - S(A|B)A^{-1}S(A|B) \\
&= D_0(A|B) + D_0(B|A) - S(A|B)A^{-1}S(A|B).
\end{aligned}$$

(3) This relation is obtained from (1) and (2) immediately.

□

Theorem 3.3. *Let* $\Psi(t) = D_t(A|B)$ *for* $t \in [0,1]$ *and strictly positive operators* A *and* B*. Then*

$$D_\Psi(1,0) = D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B).$$

Proof. For $a > 0$, we have

$$\begin{aligned}
&\lim_{\alpha\to+0} \frac{1 - \alpha + \alpha a - a^\alpha - \alpha(1-\alpha)(a - 1 - \log a)}{\alpha^2(1-\alpha)} \\
&\quad = \lim_{\alpha\to+0} \frac{-1 + a - a^\alpha \log a - (1-2\alpha)(a - 1 - \log a)}{2\alpha - 3\alpha^2} \\
&\quad = \lim_{\alpha\to+0} \frac{-a^\alpha(\log a)^2 + 2(a - 1 - \log a)}{2 - 6\alpha} \\
&\quad = -\frac{1}{2}(\log a)^2 + a - 1 - \log a.
\end{aligned}$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned}
\lim_{\alpha\to+0} \frac{D_\alpha(A|B) - D_0(A|B)}{\alpha} &= \lim_{\alpha\to+0} \frac{\frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} - \big(B - A - S(A|B)\big)}{\alpha} \\
&= \lim_{\alpha\to+0} \frac{A \nabla_\alpha B - A \sharp_\alpha B - \alpha(1-\alpha)\big(B - A - S(A|B)\big)}{\alpha^2(1-\alpha)} \\
&= -\frac{1}{2}S(A|B)A^{-1}S(A|B) + D_0(A|B),
\end{aligned}$$

then

$$\begin{aligned} D_\Psi(1,0) &= D_1(A|B) - D_0(A|B) - \lim_{\alpha\to+0}\frac{D_\alpha(A|B)-D_0(A|B)}{\alpha} \\ &= D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B). \end{aligned}$$

□

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ON DUCCI MATRIX SEQUENCES

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Abstract. For each irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, there is a unique Ducci matrix sequence $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots$ associated with it. We first consider the function j that maps each $\alpha \in (0,1) \setminus \mathbb{Q}$ to the sequence $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ of indexes of its Ducci matrix sequence expansion. While continuity of j and j^{-1} is easily checked, we show that j^{-1} is moreover uniformly continuous. We then study the distribution of Ducci matrices in the Ducci matrix sequence expansion of a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ by considering the following three conditions on the sequence $j(\alpha)$:

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$$

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\};$$

$$\lim_{n\to\infty} \sqrt[p]{\frac{\sum_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

We prove that the top implies the middle and the middle implies the bottom. We also give examples witnessing that the converse to these two implications are not true in general. In addition, various equivalent statements to the first condition will be presented. Furthermore, we shall give measure theoretic treatment of the subject: We prove that for almost every α, each Ducci matrix appears in the Ducci matrix sequence expansion of α infinitely often. We then ask if the second (and also the third) condition above holds almost everywhere. Related questions as well as several partial results will be presented.

1 Introduction.

A Ducci sequence is a sequence of vectors generated by iterating the following *Ducci map* D to a starting vector:

$$(v_1, v_2, \ldots, v_n) \overset{D}{\longmapsto} (|v_1 - v_2|, |v_2 - v_3|, \ldots, |v_n - v_1|)$$

Ciamberlini and Marengoni attributed a question about the limiting behavior of such sequences to E. Ducci in their paper [4]. Since then, a substantial amount of literature on various generalizations as well as the dynamics of the Ducci map has appeared ([2] provides a large list of references.)

Due to the simplicity of the definition, one can consider the Ducci map on various domains. While more works can be found on the Ducci map on $\mathbb{Z}^n$, there are several important results in the real setting, i.e. $\mathbb{R}^n$. For $n = 4$, though every vector in $\mathbb{Z}^4$ is known to converge to the zero vector in finite time [1, 4], Lotan [8] constructed vectors in $\mathbb{R}^4$ whose Ducci sequence never reach the zero vector. However, not many vectors exhibit such asymptotic behavior — A vector does not reach the zero vector if and only if it reaches a trivial transformation of the vector $(1, q, q^2, q^3)$ after finite time, where $1 < q < 2$ is the unique positive solution of the equation $x^3 - x^2 - x - 1 = 0$ [8]. For $n = 3$, Brockman and Zerr [2] proved that if a starting vector $\boldsymbol{v}$ is *heterogeneous*, i.e. $\lambda(\boldsymbol{v} + (x, x, x)) \notin \mathbb{Q}^3$ holds for all $\lambda, x \in \mathbb{R}$ with $\lambda \neq 0$, then

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its Ducci sequence will never become periodic and approaches the zero vector asymptotically. For a non-heterogeneous starting vector, it was also proved in [2] that its Ducci sequence is eventually periodic. (An alternative proof can be found in [5].) For a general n, it is known [3] that any starting vector in $\mathbb{R}^n$ converges asymptotically to a periodic sequence, but not necessarily to the sequence of zero vectors. Indeed, a vector which converges asymptotically to a non-trivial periodic sequence is constructed in [3] for $n = 7$.

Hogenson et al. [6] made a new approach to the subject by introducing the concept *Ducci matrix sequences.* For each vector in $\mathbb{R}^n$, one can find an $n \times n$ matrix whose application to the vector is equivalent to the application of the Ducci map. This matrix depends, of course, on the chosen vector. Thus, one may associate with a vector $\boldsymbol{v}$ not a single matrix but a sequence $M_{j_1}, M_{j_2}, \ldots$ of matrices such that the matrix M_{j_n} implements the n-th application of the Ducci map to $\boldsymbol{v}$. By considering those starting vectors in $\mathbb{R}^3$ that lead to unique Ducci matrix sequences, Hogenson et al. [6] established a connection between the Ducci map, the process of forming *mediants* of rational numbers and the *Stern-Brocot tree.*

In this paper, we focus on the Ducci map on $\mathbb{R}^3$. After presenting necessary concepts and their properties in Section 2, we consider in Section 3 the function j that maps each $\alpha \in (0,1) \setminus \mathbb{Q}$ to the sequence $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ of indexes of its Ducci matrix sequence expansion. While continuity of j and j^{-1} is easily checked, we show that j^{-1} is moreover uniformly continuous. We then study the distribution of Ducci matrices in the Ducci matrix sequence expansion of a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ by considering the following three conditions on the sequence $j(\alpha)$:

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\};$$
$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

In Section 4, we prove that the top implies the middle and the middle implies the bottom. We also give examples witnessing that the converse to these two implications are not true in general. In addition, various equivalent statements to the first condition will be presented. In the final section, we shall provide measure theoretic treatment of the subject: We prove that for almost every α, each Ducci matrix appears in the Ducci matrix sequence expansion of α infinitely often. We then ask if the second (and the third) condition above holds almost everywhere. We have not succeeded in solving these questions; We will however see the following partial result:

$$\limsup_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} \ge \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\} \text{ holds a.e.}$$

Related questions as well as some other partial results will be presented.

2 Preliminaries. In this preparatory section, we recapitulate materials presented in [5]. For more details, we refer the reader to [5].

2.1 Ducci map and continued fractions. Let us start by fixing certain terminology on continued fractions (as taken from Khinchin's book [7]). We write $[a_0; a_1, a_2, \ldots]$ and $[a_0; a_1, \ldots, a_l]$ for the following infinite and finite continued fraction, respectively:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \qquad \text{and} \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_l}}}.$$

We assume that a_0 is an integer and $a_1, a_2, \dots$ are positive integers. We call $a_0, a_1, \dots$ the *elements* of a continued fraction. For an infinite continued fraction $\alpha = [a_0; a_1, a_2, \dots]$, we call $s_k := [a_0; a_1, \dots, a_k]$ and $r_k := [a_k; a_{k+1}, \dots]$ a *segment* and a *remainder* of α, respectively. Obviously, remainders satisfy the relation $r_k = r_{k+1}^{-1} + a_k$. For finite continued fractions, segments and remainders are defined analogously.

Another important concept in the theory of continued fractions is that of *convergent*. For a given $\alpha = [a_0; a_1, a_2, \dots]$, we write p_k/q_k for the k-th order convergent, i.e. p_k and q_k are non-negative relatively prime integers such that $p_k/q_k = s_k$. It is customary to set $p_{-1} := 1$ and $q_{-1} := 0$. A folklore theorem gives us the rule for the formation of the convergents: For any $k \geq 1$, it holds that $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$.

It is well-known that continued fraction can be used as an apparatus for representing real numbers (A proof of the next theorem can be found in, e.g. [7, Theorem 14]):

Theorem 1. *Assume that the last element of any finite continued fraction is greater than* 1*. Then, to every real number α, there corresponds a unique continued fraction with value equal to α. This fraction is finite if α is rational, and is infinite if α is irrational.* □

Using continued fraction expansion, one can completely describe the orbit of $(0, \alpha, 1)$ under the Ducci map D for irrational $\alpha > 0$ as follows. Observe that $\alpha > 0$ implies that the first element a_0 of α's continued fraction expansion is non-negative.

Theorem 2 ([5])**.** *Let $\alpha = [a_0; a_1, a_2, \dots] > 0$. For a given positive integer $n \geq 1$, let $k = k(n)$ be the least integer satisfying the relation $n \leq \Sigma_{i=0}^{k} a_i$. Then*

$$D^n(0, \alpha, 1) = \frac{\alpha}{r_0 \cdots r_k} \tau_{n,k} \cdot \begin{pmatrix} 1 \\ r_{k+1}^{-1} + \Sigma_{i=0}^{k} a_i - n \\ r_{k+1}^{-1} + \Sigma_{i=0}^{k} a_i - n + 1 \end{pmatrix}^{\mathrm{T}},$$

where $\tau_{n,k} \in \mathfrak{S}_3$ is a permutation that depends only on n if $k = 0$, and n and a segment s_{k-1} if $k > 0$. □

(We put $\tau \cdot \boldsymbol{v} := (v_{\tau(1)}, v_{\tau(2)}, v_{\tau(3)})$ for a permutation $\tau \in \mathfrak{S}_3$ and a vector $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.) It is not hard to check the validity of this theorem also for finite continued fractions. More precisely, for a finite continued fraction $\alpha = [a_0; a_1, \dots, a_l]$, the formula is correct for $n = 1, 2, \dots, \Sigma_{i=0}^{l-1} a_i$. For n with $\Sigma_{i=0}^{l-1} a_i < n \leq \Sigma_{i=0}^{l} a_i$, we obtain a correct formula by deleting all the occurrences of the term r_{l+1}^{-1} in the entries of the vector. Specifically, we have $D^n(0, \alpha, 1) = (\alpha / r_0 \cdots r_l)\, \tau_{n,l} \cdot (1, \Sigma_{i=0}^{l} a_i - n, \Sigma_{i=0}^{l} a_i - n + 1)$ in this case.

For convenience, let us introduce one more concept here:

Definition 1. *We say that a real vector $\boldsymbol{v} \in \mathbb{R}^3$ is of*

- type 1 *if it is of the form $\boldsymbol{v}_1\langle c; x; n\rangle := c(1, x+n, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 2 *if it is of the form $\boldsymbol{v}_2\langle c; x; n\rangle := c(x+n, 1, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*

- type 3 *if it is of the form* $\boldsymbol{v}_3\langle c; x; n\rangle := c(x+n, x+n+1, 1)$ *for some* $c > 0, 0 < x < 1$ *and a natural number* $n \geq 1$;
- type 4 *if it is of the form* $\boldsymbol{v}_4\langle c; x; n\rangle := c(1, x+n+1, x+n)$ *for some* $c > 0, 0 < x < 1$ *and a natural number* $n \geq 1$;
- type 5 *if it is of the form* $\boldsymbol{v}_5\langle c; x; n\rangle := c(x+n+1, 1, x+n)$ *for some* $c > 0, 0 < x < 1$ *and a natural number* $n \geq 1$;
- type 6 *if it is of the form* $\boldsymbol{v}_6\langle c; x; n\rangle := c(x+n+1, x+n, 1)$ *for some* $c > 0, 0 < x < 1$ *and a natural number* $n \geq 1$.

In any of these cases, we call n the *integer part* of the vector $\boldsymbol{v}_i\langle c; x; n\rangle$.

Let an irrational number α be given. Observe that its reminders r_i satisfy $0 < r_i^{-1} < 1$, and that we have

$$\frac{\alpha}{r_0 \cdots r_k} \tau_{n,k} \cdot \begin{pmatrix} 1 \\ r_{k+1}^{-1} \\ r_{k+1}^{-1} + 1 \end{pmatrix}^{\mathrm{T}} = \frac{\alpha}{r_0 \cdots r_k r_{k+1}} \tau_{n,k} \cdot \begin{pmatrix} r_{k+2}^{-1} + a_{k+1} \\ 1 \\ r_{k+2}^{-1} + a_{k+1} + 1 \end{pmatrix}^{\mathrm{T}}$$

with $a_{k+1} \geq 1$. It is then not hard to see from Theorem 2 that for every $n \geq 1$, the vector $D^n(0, \alpha, 1)$ is of some type.

An easy computation shows the following

Proposition 1 ([5]). *Let* $\alpha = [a_0; a_1, a_2, \ldots] > 0$ *be irrational. Then for any positive real number* $c > 0$ *and a natural number* $n > 1$, *it holds that* $D(\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle) = \boldsymbol{v}_{i+1}\langle c; r_k^{-1}; n-1\rangle$ *for every* $k \geq 0$ *and* $i = 1, 2, \ldots, 6$, *where any subscript greater than* 6 *is to be understood by modulo* 6.

If the integer part of $\boldsymbol{v}_i$ *is* 1, *then we have the following:*

- $D(\boldsymbol{v}_1\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_1\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_2\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_4\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_3\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_3\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_4\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_6\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_5\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_5\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_6\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_2\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$. □

Therefore, an application of the Ducci map D to a vector of the form $\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle$ with $n \geq 1$ yields the increment of the type by 1 (modulo 6) if and only if the integer part n is greater than 1. This property will play a key role later on.

The above proposition will bring the reader clearer understanding of the computation of the permutation $\tau_{n,k(n)}$.

2.2 Ducci matrix sequence.

Definition 2. *The regions* $\mathcal{R}_1, \ldots, \mathcal{R}_6 \subset \mathbb{R}^3$ *are defined as follows:*

- $\mathcal{R}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$;
- $\mathcal{R}_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq x_1 \leq x_3\}$;

- $\mathcal{R}_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \le x_1 \le x_2\}$;
- $\mathcal{R}_4 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \le x_3 \le x_2\}$;
- $\mathcal{R}_5 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \le x_3 \le x_1\}$;
- $\mathcal{R}_6 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \le x_2 \le x_1\}$.

We say that a matrix M *implements* the action of the Ducci map D on $\boldsymbol{v} \in \mathbb{R}^3$ if $D\boldsymbol{v} = \boldsymbol{v}M$ holds. Matrices $M_1, \dots, M_6$ are defined so that M_i implements the application of the Ducci map to any vector in the region $\mathcal{R}_i$ uniformly, i.e. $D\boldsymbol{v} = \boldsymbol{v}M_i$ holds for every $\boldsymbol{v} \in \mathcal{R}_i$. For instance,

$$M_1 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

Observe that two distinct regions can overlap each other. For example, $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x_1, x_2, x_3) \mid x_1 = x_2 \le x_3\} \ne \emptyset$. Consequently, either M_1 or M_2 serves as an implementation of an application of the Ducci map to any vector $\boldsymbol{v} \in \mathcal{R}_1 \cap \mathcal{R}_2$. It is also easy to observe that if all entries of a vector $\boldsymbol{v}$ are pairwise distinct, then $\boldsymbol{v}$ belongs to a unique region, and hence has only one implementation.

It would be interesting to consider a sequence of implementations of applications of the Ducci map to a given starting vector. To make this precise, let us introduce one more piece of terminology.

Definition 3 ([6])**.** *For a given vector $\boldsymbol{v} \in \mathbb{R}^3$, a* Ducci matrix sequence associated with *$\boldsymbol{v}$ is a sequence $M_{j_1}, M_{j_2}, \dots$ of matrices with $j_1, j_2, \dots \in \{1, 2, \dots, 6\}$ such that $D^n\boldsymbol{v} = \boldsymbol{v}M_{j_1} \cdots M_{j_n}$ holds for all $n \ge 1$.*

For a real number $\alpha \in \mathbb{R}$, we define a Ducci matrix sequence associated with α *to be a Ducci matrix sequence associated with the vector $(0, \alpha, 1)$.*

One may naturally ask which α have a unique Ducci matrix sequence. This question has been answered in [6] as follows (A different proof can be found in [5].):

Theorem 3 ([6])**.** *α is irrational if and only if there is only one Ducci matrix sequence associated with α.* □

Thus, for a given α, we call the unique Ducci matrix sequence associated with it the *Ducci matrix sequence expansion* of α.

3 Uniform continuity.

At the end of the last section, we mentioned the result that α is irrational if and only if there is only one Ducci matrix sequence associated with α. This gives us a function j that sends an irrational number $\alpha \in (0, 1)$ to the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle \in \{1, 2, \dots, 6\}^\omega$ of indexes of the Ducci matrix sequence expansion $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \dots$ of α. In this section, we shall study (uniform) continuity of j and its inverse j^{-1}.

Before proceeding any further, it will be useful to summarize the relationship among relevant concepts defined so far:

Proposition 2. *For irrational $\alpha > 0$ and $n \ge 1$, we have the following relations:*

$$D^n(0, \alpha, 1) \text{ is of type } t \iff D^n(0, \alpha, 1) \in \mathcal{R}_t \iff j_\alpha(n+1) = t. \qquad \square$$

Remark 1. *Actually, the second equivalence in the above proposition holds for $n = 0$. We have formulated the above proposition in this way because the type of vector $D^0(0, \alpha, 1)$ is undefined.*

The uniqueness of the continued fraction expansion (Theorem 1) and Theorem 2 entail the injectivity of j. Thus, considered as a function from $(0,1) \setminus \mathbb{Q}$ to $j((0,1) \setminus \mathbb{Q})$, j is bijective.

In order to see that j is continuous, the following result plays a key role:

Theorem 4 ([5])**.** *Let two distinct positive irrational numbers $\alpha, \alpha' > 0$ be given, and consider their infinite continued fraction expansions: $[a_0; a_1, a_2, \dots]$ and $[a'_0; a'_1, a'_2, \dots]$. If we have $a_l < a'_l$ for $l = \min\{ l \mid a_l \neq a'_l \}$, then the length of the maximal common initial segment of Ducci matrix sequence expansions of α and α' is $\Sigma_{i=0}^{l} a_i$.* □

For a given irrational $\alpha \in (0,1)$, take an irrational α' sufficiently close to α so that the first n elements of their continued fraction expansions coincide. Then, by virtue of this theorem, the sequences $j(\alpha)$ and $j(\alpha')$ are identical up to the first $\min\{\Sigma_{i=0}^{n} a_i, \Sigma_{i=0}^{n} a'_i\} \geq n$ segments. Given the definition of the standard metric on $\{1, 2, \dots, 6\}^\omega$, i.e. the distance of two sequences is set to be 2^{-m}, where m is the first place at which the sequences differ, it is easily observed that continuity of j follows from this argument.

Uniform continuity however does not hold for j, as witnessed by the example below:

Example 1. *Take a positive irrational number $\varepsilon < 1/6$ and consider two irrational numbers $1/2 - \varepsilon$ and $1/2 + \varepsilon$. The first element of the continued fraction expansion of the former is 2, while the latter is 1. This means that, no matter how small ε is, and consequently no matter how close the numbers $1/2 - \varepsilon$ and $1/2 + \varepsilon$ are, the second coordinates of $j(1/2 - \varepsilon)$ and $j(1/2 + \varepsilon)$ are different.*

When it comes to the inverse $j^{-1} : j((0,1) \setminus \mathbb{Q}) \to (0,1) \setminus \mathbb{Q}$, not only continuity but also uniform continuity holds. Let us prove this assertion.

Theorem 5. *$j^{-1} : j((0,1) \setminus \mathbb{Q}) \to (0,1) \setminus \mathbb{Q}$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. Since $j((0,1) \setminus \mathbb{Q})$ is considered as a subspace of $\{1, 2, \dots, 6\}^\omega$, in view of the definition of the standard metric on $\{1, 2, \dots, 6\}^\omega$, it suffices to show that there exists an n such that $|\alpha - \alpha'| < \varepsilon$ holds for any $\alpha, \alpha' \in (0,1) \setminus \mathbb{Q}$ whenever $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle$ and $j(\alpha') = \langle j_{\alpha'}(1), j_{\alpha'}(2), \dots \rangle$ agree up to first n elements.

We claim that any natural number n greater than $1/\varepsilon$ has the desired property. To see this, take $\alpha, \alpha' \in (0,1) \setminus \mathbb{Q}$ so that the initial segments of $j(\alpha)$ and $j(\alpha')$ are identical up to n. Let $k = k(n)$ and $k' = k'(n)$ be as in the statement of Theorem 2, i.e. $k \geq 1$ (resp. $k' \geq 1$) is the least integer satisfying that $n \leq \Sigma_{i=0}^{k} a_i$ (resp. $n \leq \Sigma_{i=0}^{k} a'_i$). It is not difficult to see from Theorem 4 that k and k' are equal and that we have $a_0 = a'_0, \dots, a_{k-1} = a'_{k-1}$.

Now let p_l/q_l and p'_l/q'_l denote the k-th order convergent of α and α', respectively: $p_l/q_l = s_l$ and $p'_l/q'_l = s'_l$. It is known [7, pp. 8] that α and α' can be expressed in terms of their convergents and remainders as follows:

$$\alpha = \frac{p_{l-1} r_l + p_{l-2}}{q_{l-1} r_l + q_{l-2}} \quad \text{and} \quad \alpha' = \frac{p'_{l-1} r'_l + p'_{l-2}}{q'_{l-1} r'_l + q'_{l-2}} \quad \text{for every } l \geq 1.$$

Note that $a_0 = a'_0, \dots, a_{k-1} = a'_{k-1}$ imply $p_l = p'_l$ and $q_l = q'_l$ for $l \leq k$. Therefore, by writing $f(x) = (p_{k-1}x + p_{k-2})/(q_{k-1}x + q_{k-2})$, we have $\alpha = f(r_k)$ and $\alpha' = f(r'_k)$.

Since $q_{k-1} \geq 1$ and $q_{k-2} \geq 0$, the function $f(x)$ is monotone on $(0, \infty)$. As we have $r_k \geq \lfloor r_k \rfloor = a_k \geq n - \Sigma_{i=0}^{k-1} a_i > 0$ and similarly $r'_k \geq n - \Sigma_{i=0}^{k-1} a_i > 0$, this monotonicity of

$f(x)$ proves that $\alpha = f(r_k)$ and $\alpha' = f(r'_k)$ lie between $f(n - \Sigma_{i=0}^{k-1} a_i)$ and $\lim_{x\to\infty} f(x) = p_{k-1}/q_{k-1}$. Hence it holds that

$$
\begin{aligned}
|\alpha - \alpha'| &\leq \left| f(n - \Sigma_{i=0}^{k-1} a_i) - \lim_{x\to\infty} f(x) \right| \\
&= \left| \frac{p_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + p_{k-2}}{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} \right| \\
&= \frac{|p_{k-2}q_{k-1} - p_{k-1}q_{k-2}|}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\
&= \frac{1}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}}.
\end{aligned}
$$

(The well-known identity $p_{k-2}q_{k-1} - p_{k-1}q_{k-2} = (-1)^{k-1}$ was used in the last step.)

Since n is greater than $1/\varepsilon$, if $k = 1$, then $q_0 = 1$ and $q_{-1} = 0$ proves that $|\alpha - \alpha'| \leq 1/n < \varepsilon$. In order to deal with the case that $k \geq 2$, we need the following lemma, which is easily proved via induction.

Lemma 1. $q_{k-1} \geq \Sigma_{i=0}^{k-1} a_i$. □

Using this lemma, we resume the evaluation of the difference $|\alpha - \alpha'|$:

$$
\begin{aligned}
|\alpha - \alpha'| &\leq \frac{1}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\
&\leq \frac{1}{\Sigma_{i=0}^{k-1} a_i\{\Sigma_{i=0}^{k-1} a_i(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\
&\leq \frac{1}{\Sigma_{i=0}^{k-1} a_i(n - \Sigma_{i=0}^{k-1} a_i) + 1} \\
&\leq \frac{1}{n} < \varepsilon.
\end{aligned}
$$

This makes the proof of the theorem complete. □

Remark 2. *If the domain of a continuous function is compact, then uniform continuity follows automatically from continuity. This time, however, we had to prove the above theorem directly because the domain $j((0,1)\setminus\mathbb{Q}) \subset \{1, 2, \ldots, 6\}^\omega$ of the continuous function j^{-1} is not compact. Indeed, there is a Cauchy sequence $\{j([0; n, 1, 1, 1, \ldots])\}_{n\geq 1}$ in $j((0,1)\setminus\mathbb{Q})$ with its limit $\langle 1\rangle^\frown\langle 1, 2, \ldots, 6\rangle^\omega \in \{1, 2, \ldots, 6\}^\omega$ outside $j((0,1)\setminus\mathbb{Q})$. (Here and in what follows, we use the symbol $^\frown$ for the concatenation of two sequences: $\langle x_1, x_2, \ldots, x_n\rangle^\frown\langle y_1, y_2, \ldots\rangle := \langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots\rangle$.)*

4 Distribution of Ducci matrices. For a given irrational number $\alpha \in (0,1)\setminus\mathbb{Q}$, we are interested in the distribution of indexes in the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \ldots\rangle$. In this section, we consider several statements regarding the distribution of Ducci matrices in a given sequence $j(\alpha)$ and examine their relationships. Note that, since we shall deal with irrational numbers only from $(0,1)\setminus\mathbb{Q}$, we always have $a_0 = 0$ from now on. Also, the index i of the sequence $\{j_\alpha(i)\}_i$ starts from 1. For notational convenience, we thus make the following convention: Throughout this and the next section, any index and subscript start from 1 unless otherwise stated.

To begin with, let us prove the following

Lemma 2. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $l \geq 1$, we have*

$$| \{ i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6 \} | \geq n - l \cdot | \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} | - \frac{l(l-1)}{2}.$$

Equality holds if $l = 1$.

Proof. It is not hard to check that for any $i \geq 2$, we have

$$j_\alpha(i) + 1 \not\equiv j_\alpha(i+1) \pmod 6 \overset{(1)}{\Longleftrightarrow} i \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1$$
$$\overset{(2)}{\Longleftrightarrow} \text{The integer part of } D^{i-1}(0, \alpha, 1) \text{ is } 1.$$

(Actually, Equivalence (1) holds for $i = 1$; We wrote $i \geq 2$ above because the integer part of $D^n(0, \alpha, 1)$ for $n = 0$ is undefined. See Remark 1.)

The assertion for $l = 1$ follows at once from Equivalence (1), which holds for any $i \geq 1$.

In order to deal with a general $l > 1$, let us put

$$\hat{I}_p^n := \{ i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+p-1) + l - p + 1 \not\equiv j_\alpha(i+p) + l - p \pmod 6 \} \text{ and}$$
$$I_p^n := \{ i \leq n \mid \text{The integer part of } D^i(0, \alpha, 1) \text{ is } p \}$$

for $1 \leq p \leq l$.

If $p = 1$, then (1) implies that $| \hat{I}_1^n | = | \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} |$.

If $p > 1$, then (1), (2) and Proposition 1 tell us that $\hat{I}_p^n = I_{p-1}^n$. Now we evaluate the size $| I_{p-1}^n |$ of I_{p-1}^n. If $p > 2$, then $i \in I_{p-1}^n$ implies $i + 1 \in I_{p-2}^n$ for any $i < n$. Note however that the converse is in general not true. (For example, it can happen that an $i \in I_1^n$ satisfies $i + 1 \in I_{p-2}^n$.) Therefore, we have $| I_{p-1}^n | \leq | I_{p-2}^n | + 1$. Applying this inequality repeatedly, we obtain $| I_{p-1}^n | \leq | I_1^n | + p - 2$ for any $p > 1$. Using (2), we thus see that

$$\begin{aligned}
| \hat{I}_p^n | &= | I_{p-1}^n | \\
&\leq | \{ i \leq n \mid i + 1 \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1 \} | + p - 2 \\
&\leq | \{ i \leq n \mid i \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1 \} | + p - 1 \\
&= | \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} | + p - 1,
\end{aligned}$$

for any $p > 1$.

Putting these arguments together, we get

$$\begin{aligned}
&| \{ i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6 \} | \\
&= n - \Sigma_{p=1}^l | \hat{I}_p^n | \\
&\geq n - | \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} | - \Sigma_{p=2}^l \left(| \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} | + p - 1 \right) \\
&= n - l \cdot | \{ m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n \} | - \frac{l(l-1)}{2},
\end{aligned}$$

as desired. □

We are interested in the relation $j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6$, especially in the frequency that this happens in a given sequence $j(\alpha)$. In some situations, this relation between $j_\alpha(i)$ and $j_\alpha(i+1)$ can be equivalently expressed using only elements a_i of α. Specifically, we have

Theorem 6. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following are equivalent:*

1. $\displaystyle\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$

2. $\displaystyle\lim_{n\to\infty} \frac{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \le n\}|}{n} = 0;$

3. $\displaystyle\frac{\Sigma_{i=1}^n a_i}{n}$ *diverges.*

Proof. 1 ⇔ 2: This follows easily from Lemma 2 (for the case $l = 1$).
2 ⇒ 3: Let us temporarily put $A(n) := |\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \le n\}|$. It is then clear that $n = A(\Sigma_{i=1}^n a_i)$. Since $\Sigma_{i=1}^n a_i \to \infty$ as $n \to \infty$, 2 implies $A(\Sigma_{i=1}^n a_i)/\Sigma_{i=1}^n a_i$ converges to 0. Hence $\Sigma_{i=1}^n a_i/n = \Sigma_{i=1}^n a_i/A(\Sigma_{i=1}^n a_i)$ diverges.
3 ⇒ 2: From the definition of $A(n)$, it follows that $\Sigma_{s=1}^{A(n)} a_s \le n$. Hence we have $0 \le A(n)/n \le A(n)/\Sigma_{s=1}^{A(n)} a_s$. Since $A(n)$ diverges as $n \to \infty$, 3 implies $A(n)/\Sigma_{s=1}^{A(n)} a_s$ converges to 0. Therefore $A(n)/n$ also converges to 0. □

Corollary 1. *No $\alpha \in (0,1) \setminus \mathbb{Q}$ with bounded elements satisfies* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$.

Proof. Let M be such that $a_i \le M$ holds for all $i \in \mathbb{Z}_{>0}$. This implies that $\Sigma_{i=1}^n a_i/n \le M$, in particular, $\Sigma_{i=1}^n a_i/n$ is not divergent. Theorem 6 now proves our assertion. □

Corollary 2. *The set of all α satisfying* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ *is dense.*

Proof. The set of all α with $\Sigma_{i=1}^n a_i/n$ divergent is dense. The assertion again follows from Theorem 6. □

Our condition above concerns the relationship between only two indexes. Here, the following question arises naturally: does it give rise to any difference if we require more than two indexes, say $j_\alpha(i), j_\alpha(i+1), \ldots, j_\alpha(i+l)$ with $l > 1$, to be consecutive by modulo 6? The next theorem answers this question negatively.

Theorem 7. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $l \ge 1$, we have*

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1 \iff$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} = 1.$$

Proof. Evidently, the first condition follows from the second. So let us prove the other direction:

Assume that the first condition is true for a given α. From Theorem 6, it then follows that $\Sigma_{i=1}^m a_i/m$ is divergent. For a given $\varepsilon > 0$, let $M \in \mathbb{Z}_{>0}$ be such that every $M' \ge M$ satisfies $\Sigma_{i=1}^{M'} a_i/M' > 2l/\varepsilon$. Take an $n \ge \max\{\Sigma_{i=1}^M a_i, l(l-1)/\varepsilon\}$. Then, since we have $|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}| \ge M$, it holds that

$$\frac{n}{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} \ge \frac{\Sigma_{i=1}^{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} a_i}{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} > \frac{2l}{\varepsilon}.$$

Using Lemma 2, we thus obtain

$$\begin{aligned}1 &\geq \frac{|\{i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} \\ &\geq 1 - \frac{l \cdot |\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^{m} a_i \leq n\}|}{n} - \frac{l(l-1)}{2n} \\ &> 1 - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this completes the proof. □

Now we turn to another condition on the distribution of indexes in the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle$. If the sequence $j(\alpha)$ is distributed uniformly, any $\jmath \in \{1, 2, \dots, 6\}$ will occur with the same probability. As there are only six possible values of $\jmath$, the probability should then be 1/6. Hence the following statement is to be seen as a necessary condition for a sequence $j(\alpha)$ to be uniformly distributed:

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ holds for every } \jmath \in \{1, 2, \dots, 6\}.$$

The next theorem shows that this condition follows from our first condition.

Theorem 8. *Let an $\alpha \in (0,1) \setminus \mathbb{Q}$ be given. If $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ holds, then $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds for every $\jmath \in \{1, 2, \dots, 6\}$.*

Proof. In view of Theorem 6, it suffices to prove the consequent assuming that $\Sigma_{i=1}^{n} a_i/n$ is divergent.

Fix a $\jmath \in \{1, 2, \dots, 6\}$ and choose an $\varepsilon > 0$ arbitrarily. Since $\Sigma_{i=1}^{n} a_i/n$ diverges, there exists a positive integer $K > 1$ such that $\Sigma_{i=1}^{l} a_i/l > 3/\varepsilon$ holds for every $l \geq K$. For this K, we claim that the difference between $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ and 1/6 is less than ε whenever n is greater than $\Sigma_{i=1}^{K} a_i$. For this purpose, fix an $n > \Sigma_{i=1}^{K} a_i$ and let $k = k(n)$ be as in the statement of Theorem 2.

Using the sequence $j_\alpha(1), j_\alpha(2), \dots, j_\alpha(n)$, we shall construct a new sequence w (of finite length) as follows: Consider two numbers $j_\alpha(1+\Sigma_{i=1}^{p} a_i)$ and $j_\alpha(\Sigma_{i=1}^{p} a_i)$ for $p = 1, \dots, k-1$, and iterate the next process from $p = 1$ to $p = k-1$. If $j_\alpha(1 + \Sigma_{i=1}^{p} a_i) = j_\alpha(\Sigma_{i=1}^{p} a_i)$, then remove $j_\alpha(\Sigma_{i=1}^{p} a_i)$ from the initial sequence. If, on the other hand, we have $j_\alpha(1+\Sigma_{i=1}^{p} a_i) \equiv j_\alpha(\Sigma_{i=1}^{p} a_i) + 2 \pmod 6$, then insert the number $j_\alpha(\Sigma_{i=1}^{p} a_i) + 1 \pmod 6$ between these two. Call the resulting sequence w. It is then immediate from Propositions 1 and 2 that the new finite sequence w is eventually periodic: $w = \langle 1, 1, 2, \dots, 6, 1, 2, \dots, 6, \dots \rangle$. (Periodic part starts from the second coordinate.)

Since we have removed or inserted $k-1$ numbers, the length $\mathrm{lh}(w)$ of w is at most $n+k-1$ and at least $n - k + 1$. Eventual periodicity of w implies that at most $\lfloor(\mathrm{lh}(w) - 1)/6\rfloor + 2$ and at least $\lfloor(\mathrm{lh}(w) - 1)/6\rfloor$ coordinates of w are equal to $\jmath$. It might be the case that all removed numbers are equal to $\jmath$; it might be the case that all inserted numbers are equal to $\jmath$. Taking these worst case scenarios into account, one obtains the following estimate:

$$\frac{\lfloor\frac{n-k}{6}\rfloor - (k-1)}{n} \leq \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \leq \frac{\lfloor\frac{n+k-2}{6}\rfloor + k + 1}{n}.$$

As we have $\Sigma_{i=1}^{k} a_i \geq n > \Sigma_{i=1}^{K} a_i$ for $k = k(n)$, it follows that $k - 1 \geq K > 1$. In view of the definition of K, we thus see that

$$\frac{1}{n} < \frac{k-1}{n} < \frac{k-1}{\Sigma_{i=1}^{k-1} a_i} < \frac{\varepsilon}{3}.$$

Putting these arguments together, we compute the difference as follows:

$$\begin{aligned}\left|\frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} - \frac{1}{6}\right| &\le \frac{7k+4}{6n} \\ &= \frac{7(k-1)}{6n} + \frac{11}{6n} \\ &< \frac{7\varepsilon}{18} + \frac{11\varepsilon}{18} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ and $\jmath \in \{1, 2, \dots, 6\}$ were chosen arbitrarily, this proves that the limit exists and is equal to $1/6$, as desired. □

We note that the converse to this theorem is not true. Here is a witness:

Example 2. *Consider the following eventually periodic sequence of Ducci matrices:*

$$\langle M_1 \rangle^\frown \langle M_1, M_2, M_3, M_4, M_5, M_6, M_1, M_1, M_2, M_3, M_4, M_5, M_6, M_2, M_3, M_4, M_5, M_6 \rangle^\omega.$$

It is not hard to check that this is the Ducci matrix sequence expansion of an irrational number $\alpha := [0; 8, 6, 12, 6, 12, 6, 12, 6, \dots] \in (0,1) \setminus \mathbb{Q}$.

Since the elements of α *is bounded by 12, it is clear that* $\Sigma_{i=1}^n a_i/n$ *is not divergent, and accordingly, we do not have* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$.

We need to check that this α *satisfies* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *for every* $\jmath \in \{1, 2, \dots, 6\}$. *To see this, choose an* $\varepsilon > 0$ *and a* $\jmath \in \{1, 2, \dots, 6\}$ *arbitrarily and pick a natural number* $N > 4/\varepsilon$. *Take an arbitrary natural number* $n \ge N$ *and express it as* $n = 1 + 18a + b$ *with non-negative integers* $a \ge 0$ *and* $0 \le b < 18$. *Note that in the Ducci matrix sequence expansion of* α, *the number of occurrences of the Ducci matrix* $M_\jmath$ *from the* $18m + 2$*nd matrix to the* $18(m+1) + 1$*st matrix is three for every* $m \ge 0$. *Therefore,*

$$\frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} \le \frac{3a+4}{n} \le \frac{1}{6} + \frac{4}{n} < \frac{1}{6} + \varepsilon$$

and

$$\frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} \ge \frac{3a}{n} = \frac{1}{6} - \frac{1+b}{6n} > \frac{1}{6} - \frac{4}{n} > \frac{1}{6} - \varepsilon.$$

Since $\varepsilon > 0$ *was chosen arbitrarily, this proves that* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$. *As we took* $\jmath$ *also arbitrarily, this proves our claim.*

In studying distribution, one natural attempt is to take the average. The uniformity of the distribution of indexes in the sequence $\langle j_\alpha(1), j_\alpha(2), \dots \rangle$ can also be captured using the notion of average. Specifically, we consider the following formula to be a plausible formulation of uniformity: $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$. Let us investigate the relationship of this condition to the preceding one.

Proposition 3. *Let* $p \ge 1$ *be a positive integer. For any* $\alpha \in (0,1) \setminus \mathbb{Q}$, *if* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *holds for every* $\jmath \in \{1, 2, \dots, 6\}$, *then we have*

$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

Proof. Let $\varepsilon > 0$ be given. Then there exists an $N \in \mathbb{Z}_{>0}$ such that $|\,|\{i \le n \mid j_\alpha(i) = \jmath\}|/n - 1/6\,| < \varepsilon$ holds for every $\jmath$ and $n \ge N$. This means that

$$\left(\frac{1}{6} - \varepsilon\right) n < |\{i \le n \mid j_\alpha(i) = \jmath\}| < \left(\frac{1}{6} + \varepsilon\right) n$$

holds for every $\jmath$. By multiplying by $\jmath^p$ and taking the sum over all $\jmath$, we get

$$(1^p+2^p+\cdots+6^p)\left(\frac{1}{6}-\varepsilon\right)n < \Sigma_\jmath\left(\jmath^p|\{i \le n \mid j_\alpha(i)=\jmath\}|\right) < (1^p+2^p+\cdots+6^p)\left(\frac{1}{6}+\varepsilon\right)n.$$

Since $\Sigma_\jmath\left(\jmath^p|\{i \le n \mid j_\alpha(i)=\jmath\}|\right)$ is simply $\Sigma_{i=1}^n j_\alpha(i)^p$, this easily entails the assertion. □

The reader may wonder if the converse to the above implication is true. In order to answer this question, let us introduce the following example:

Example 3. *For any given $p \ge 1$, we define an infinite sequence $\mathbf{M}_p$ of Ducci matrices by putting $\mathbf{M}_p := \langle M_1\rangle^\frown(\vec{M}_1{}^\frown\vec{M}_2{}^\frown\cdots{}^\frown\vec{M}_{5^p-1})^\omega$, where a finite sequence $\vec{M}_i$ $(1 \le i \le 5^p-1)$ is given by*

$$\vec{M}_i := \begin{cases} \langle M_1, M_1, M_2, M_4, M_5, M_5, M_6\rangle & (i \le 3^p-1) \\ \langle M_1, M_1, M_2, M_4, M_5, M_6\rangle & (3^p-1 < i \le 5^p-3^p) \\ \langle M_1, M_2, M_4, M_5, M_6\rangle & (5^p-3^p < i \le 5^p-1) \end{cases}.$$

With the help of Proposition 1, one can check that for each $p \ge 1$, the (eventually periodic) sequence $\mathbf{M}_p$ is realized as the Ducci matrix sequence expansion of some $\alpha_p \in (0,1)\setminus\mathbb{Q}$. For example, we have

$$\begin{aligned}\mathbf{M}_1 &= \langle M_1\rangle^\frown\langle M_1, M_1, M_2, M_4, M_5, M_5, M_6, M_1, M_1, M_2, M_4, M_5, M_5, M_6,\\ &\qquad M_1, M_2, M_4, M_5, M_6, M_1, M_2, M_4, M_5, M_6\rangle^\omega,\\ \alpha_1 &= [0; 2, \underline{2,2,3,2,2,4,5,4}, \underline{2,2,3,2,2,4,5,4}, \underline{2,2,3,2,2,4,5,4}, \ldots].\end{aligned}$$

By construction, M_3 does not appear in the Ducci matrix sequence expansion $\mathbf{M}_p$ of α_p. Therefore, we have $\lim_{n\to\infty}|\{i \le n \mid j_{\alpha_p}(i)=3\}|/n = 0 \ne 1/6$.

One can easily check that the number of occurrences of M_{2i} $(i=1,2,3)$ in the finite sequence $\vec{M}_1{}^\frown\vec{M}_2{}^\frown\cdots{}^\frown\vec{M}_{5^p-1}$ is 5^p-1. In this finite sequence, M_1 appears $2\cdot5^p-3^p-1$ times and M_5 appears 5^p+3^p-2 times. By the definition of $\mathbf{M}_p$, it is thus clear that

$$\begin{aligned}\Sigma_{i=6m(5^p-1)+2}^{6(m+1)(5^p-1)+1} j_{\alpha_p}(i)^p &= (2\cdot5^p-3^p-1)\cdot1^p+(5^p-1)\cdot2^p+(5^p-1)\cdot4^p\\ &\quad+(5^p+3^p-2)\cdot5^p+(5^p-1)\cdot6^p\\ &=(5^p-1)\cdot(1^p+2^p+3^p+4^p+5^p+6^p)\\ &=\Sigma_{i=6m(5^p-1)+2}^{6(m+1)(5^p-1)+1} i^p,\end{aligned}$$

for every $m \in \mathbb{Z}_{\ge0}$. From this, it is not hard to conclude $\lim_{n\to\infty}\sqrt[p]{\Sigma_{i=1}^n j_{\alpha_p}(i)^p/n} = \sqrt[p]{(1^p+2^p+\cdots+6^p)/6}$ for this α_p.

For any given positive integer q, the above argument also proves that

$$\begin{aligned}&\lim_{n\to\infty}\sqrt[q]{\frac{\Sigma_{i=1}^n j_{\alpha_p}(i)^q}{n}}\\ &=\sqrt[q]{\frac{(2\cdot5^p-3^p-1)\cdot1^q+(5^p-1)\cdot2^q+(5^p-1)\cdot4^q+(5^p+3^p-2)\cdot5^q+(5^p-1)\cdot6^q}{6(5^p-1)}}\\ &=\sqrt[q]{\frac{(5^p-1)\cdot(1^q+2^q+4^q+5^q+6^q)+5^p-3^p+3^p\cdot5^q-5^q}{6(5^p-1)}}.\end{aligned}$$

In order for this value to be equal to

$$\sqrt[q]{\frac{1^q + 2^q + \cdots + 6^q}{6}} = \sqrt[q]{\frac{(5^p - 1) \cdot (1^q + 2^q + 3^q + 4^q + 5^q + 6^q)}{6(5^p - 1)}},$$

$q \geq 1$ *has to satisfy* $5^p - 3^p + 3^p \cdot 5^q - 5^q = 5^p \cdot 3^q - 3^q$*. An elementary computation shows that this happens only when* q *is equal to* p*. Hence for any* q *different from* p*, we have* $\lim_{n\to\infty} \sqrt[q]{\Sigma_{i=1}^n j_{\alpha_p}(i)^q/n} \neq \sqrt[q]{(1^q + 2^q + \cdots + 6^q)/6}$.

From this example, we can conclude as follows:

Theorem 9. *For every positive integer* $p \geq 1$*, the condition "*$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$*" is weaker than "*$\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *for every* $\jmath$*".*

Moreover, the statements "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$*" for* $p = p_1$ *and for* $p = p_2$ *are independent from each other whenever* p_1 *and* p_2 *are distinct.* □

Before ending this section, let us present variants to the preceding results. As in the proof of Theorem 6, one can prove

Theorem 10. *For any* $\alpha \in (0,1) \setminus \mathbb{Q}$*, the following are equivalent:*

1. $\limsup_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$

2. $\liminf_{n\to\infty} \frac{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \leq n\}|}{n} = 0;$

3. $\frac{\Sigma_{i=1}^n a_i}{n}$ *is unbounded.* □

Also, one can show as in the proof of Theorem 8 that

Theorem 11. *If* $\limsup_{n\to\infty} |\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ *holds for a given* $\alpha \in (0,1) \setminus \mathbb{Q}$*, then this* α *satisfies* $\limsup_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n \geq 1/6$ *for every* $\jmath \in \{1, 2, \ldots, 6\}$*.* □

Observe that Example 2 witnesses the failure of the converse to Theorem 8 but also to Theorem 11.

5 Measure theory. Given an irrational number $\alpha > 0$, how often for a fixed index $\jmath$, does the matrix $M_\jmath$ appear in its Ducci matrix sequence expansion $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots$? We shall present several measure theoretic approaches around this problem. In this section, measure refers to the Lebesgue measure on $\mathbb{R}$.

Our first result is the next

Theorem 12. *The following set is of measure zero:*

$$\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists \jmath\, (j_\alpha(n) = \jmath \text{ holds for only finitely many } n)\}.$$

Before proving this theorem, let us remind the reader of the following result (For a proof, see, e.g. [7, Theorem 29]):

Theorem 13. *The set of all numbers in the interval* $(0,1)$ *with bounded elements is of measure zero.* □

Therefore, it is sufficient to prove that every element α from the set that we are concerning has bounded elements.

Proof of Theorem 12. Let α and $\jmath$ be such that only finitely many n satisfy $j_\alpha(n) = \jmath$. Then there exists an N such that $j_\alpha(n) \neq \jmath$ holds for all $n \geq \Sigma_{i=1}^N a_i$. In particular, we have $j_\alpha(1 + \Sigma_{i=1}^N a_i) \neq \jmath$. Since the integer part of the vector $D^{\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is a_{N+1}, if $a_{N+1} \geq 2$, then Proposition 1 implies that the type of $D^{1+\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is 1 plus the type of $D^{\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ modulo 6. In view of Proposition 2, this yields that $j_\alpha(2 + \Sigma_{i=1}^N a_i) \equiv j_\alpha(1 + \Sigma_{i=1}^N a_i) + 1 \pmod 6$. Now, since the integer part of $D^{1+\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is $a_{N+1} - 1$, if $a_{N+1} - 1 \geq 2$, the same reasoning proves $j_\alpha(3 + \Sigma_{i=1}^N a_i) \equiv j_\alpha(2 + \Sigma_{i=1}^N a_i) + 1 \pmod 6$. Repeating in this manner, we see that, for $m = 1, 2, \ldots, a_{N+1} - 1$

$$j_\alpha(1 + m + \Sigma_{i=1}^N a_i) \equiv j_\alpha(1 + \Sigma_{i=1}^N a_i) + m \pmod 6.$$

These arguments, together with the assumption that $j_\alpha(n) \neq \jmath$ holds for all $n \geq \Sigma_{i=1}^N a_i$, proves that $j_\alpha(1 + \Sigma_{i=1}^N a_i) + m \not\equiv \jmath \pmod 6$ for $0 \leq m \leq a_{N+1} - 1$. This clearly entails that $a_{N+1} < 6$.

Continuing this way, we reach the conclusion that $a_{N+l} < 6$ holds for all $l \geq 1$. As every element a_i satisfies $a_i \leq \max\{a_0, a_1, \ldots, a_N, 6\}$, this finishes the proof. □

In the last section, we considered the condition "$\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$". One can of course study this condition from the viewpoint of measure theory:

Question 1. *Does the following property hold for a.e. $\alpha \in (0, 1) \setminus \mathbb{Q}$?*

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath = 1, 2, \ldots, 6.$$

We do not know the answer to this question. Note however that Theorems 10 and 11, together with the fact that $\Sigma_{i=1}^n a_i/n$ is unbounded almost everywhere [7, pp. 94], shows that

Theorem 14. *Almost every $\alpha \in (0, 1) \setminus \mathbb{Q}$ has the following property:*

$$\limsup_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \geq \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\}. \qquad \square$$

Similarly, one will be able to conclude that $\liminf_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n \leq 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$.

Here is another partial result:

Theorem 15. *For almost every $\alpha \in (0, 1) \setminus \mathbb{Q}$, the sequence $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ has $1/6$ as an accumulation point for every $\jmath \in \{1, 2, \ldots, 6\}$.*

Proof. The following theorem plays an important role in our proof:

Theorem 16 ([7, Theorem 30])**.** *Let $\varphi(i)$ be an arbitrary positive function with natural argument n. If the series $\Sigma_{i=1}^\infty 1/\varphi(i)$ diverges, then for almost every α, infinitely many i satisfy the inequality $a_i \geq \varphi(i)$.* □

Applying this theorem for $\varphi(i) := Ki$ ($K \in \mathbb{Z}_{>0}$) and taking the countable intersection for all positive integers K, one sees that almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ has the following property: For any positive real number x, infinitely many i satisfy the inequality $a_i \geq xi$. We claim that $1/6$ is an accumulation point of the sequence $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ for every $\jmath \in \{1,2,\dots,6\}$ whenever $\alpha \in (0,1) \setminus \mathbb{Q}$ has the above property. To prove our claim, choose an α having the above property and take $\jmath$ from $\{1,2,\dots,6\}$ arbitrarily.

Let $\varepsilon > 0$ and $N \in \mathbb{Z}_{>0}$ be given. What we have to show is that there exists an $n' \geq N$ such that $||\{i \leq n' \mid j_\alpha(i) = \jmath\}|/n' - 1/6| < \varepsilon$ holds. By the assumption on α, there exists an $n > \max\{N, 5/3\varepsilon\}$ satisfying the inequality $a_n > 2n/\varepsilon$. We claim that $a_1 + \cdots + a_n$ ($\geq n > N$) works as n'.

For $m \geq 2$, Proposition 1 tells us that the sequence $j_\alpha((\Sigma_{s=1}^{m-1} a_s) + 1), j_\alpha((\Sigma_{s=1}^{m-1} a_s) + 2), \dots, j_\alpha((\Sigma_{s=1}^{m-1} a_s) + a_m) = j_\alpha(\Sigma_{s=1}^{m} a_s)$ is periodic. This periodicity enables us to estimate the number of occurrences of $\jmath$ in this sequence:

$$\left\lfloor \frac{a_m}{6} \right\rfloor < |\{\Sigma_{s=1}^{m-1} a_s < i \leq \Sigma_{s=1}^{m} a_s \mid j_\alpha(i) = \jmath\}| < \left\lfloor \frac{a_m}{6} \right\rfloor + 1.$$

When $m = 1$, periodic part is $j_\alpha(2), j_\alpha(3), \dots, j_\alpha(a_1)$. Hence we have

$$\left\lfloor \frac{a_1 - 1}{6} \right\rfloor < |\{i \leq a_1 \mid j_\alpha(i) = \jmath\}| < \left\lfloor \frac{a_1 - 1}{6} \right\rfloor + 2.$$

These arguments, together with inequalities $2n/\varepsilon < a_n \leq \Sigma_{s=0}^{n} a_s$ and $5/3\varepsilon < n \leq \Sigma_{s=1}^{n} a_s$, proves that

$$\begin{aligned} \frac{|\{i \leq \Sigma_{s=1}^{n} a_s \mid j_\alpha(i) = \jmath\}|}{\Sigma_{s=1}^{n} a_s} &\leq \frac{\frac{\Sigma_{s=1}^{n} a_s}{6} + n + \frac{5}{6}}{\Sigma_{s=1}^{n} a_s} \\ &< \frac{1}{6} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \frac{1}{6} + \varepsilon. \end{aligned}$$

Similarly, one can prove

$$\frac{|\{i \leq \Sigma_{s=1}^{n} a_s \mid j_\alpha(i) = \jmath\}|}{\Sigma_{s=1}^{n} a_s} > \frac{1}{6} - \varepsilon.$$

This completes the proof. □

Note that Theorem 14 follows also from this result as a corollary.

The rest of this paper studies another possible question concerning Question 1. Specifically, we ask if it gives rise to any difference to replace "for *every* $\jmath$" in Question 1 with "for *some* $\jmath$". The next results shows that the notion of parity plays a crucial part in this investigation.

Theorem 17. *Let $\jmath_1, \jmath_2 \in \{1,2,\dots,6\}$ be two distinct numbers with the same parity. If* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ *holds a.e., then so does* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$.

Proof. We give the proof only for the case $\jmath_1 = 1$ and $\jmath_2 = 3$; the other cases are left to the reader.

For $\jmath \in \{1,2,\dots,6\}$, put

$$N_\jmath := \left\{ \alpha \in (0,1) \setminus \mathbb{Q} \;\middle|\; \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \text{ does not converge to } \frac{1}{6} \right\}.$$

What we need to prove is that if N_1 is of measure zero, then so is N_3.

Define a function $g : (0,1) \to (0,1/3)$ by $g(x) = x/(2x+1)$. Clearly, g is a homeomorphism on $(0,1)$. An elementary computation shows that g is invertible and that both g and g^{-1} are *bi-Lipschitz*, in particular both send measure zero sets to measure zero sets. Since we trivially have $N_3 = \{\alpha \in N_3 \mid a_1 \geq 3\} \cup \{\alpha \in N_3 \mid a_1 = 1 \text{ or } 2\}$, in order to prove that N_3 is of measure zero, it suffices to show the next two identities:

$$\{\alpha \in N_3 \mid a_1 \geq 3\} = g(N_1);$$
$$\{\alpha \in N_3 \mid a_1 = 1 \text{ or } 2\} = g^{-2}(\{\alpha \in N_1 \mid a_1 = 5 \text{ or } 6\}).$$

Observe that g maps $[0; a_1, a_2, a_3, \dots]$ to $[0; a_1+2, a_2, a_3, \dots]$, i.e. adds 2 to the first element of the continued fraction expansion. This observation, together with Proposition 1, leads to the next

Lemma 3. *For every $\alpha \in (0,1)$ and $n \geq 1$, the type of $D^{n+2}(0, g(\alpha), 1)$ is the type of $D^n(0, \alpha, 1)$ plus 2 (modulo 6).* □

This lemma implies that α satisfies $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = 1\}|/n = 1/6$ if and only if $g(\alpha)$ satisfies $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = 3\}|/n = 1/6$. In other words, $\alpha \in N_1 \iff g(\alpha) \in N_3$. The desired two identities follows from this easily. □

Remark 3. *In the above proof, the value $1/6$ did not play any role. Indeed, the statement remains valid even when $1/6$ is replaced by any other real number in $(0,1)$.*

In the proof of the preceding theorem, Lemma 3 played an important role. A similar statement when the parity of $\jmath_1$ is different from that of $\jmath_2$ is no longer true. This makes it difficult to prove an analogous statement to Theorem 17 for a pair of different parity.

Question 2. *Let two distinct numbers $\jmath_1, \jmath_2 \in \{1, 2, \dots, 6\}$ with different parity be given. If $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ holds a.e., does $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$ also hold a.e. ?*

If this (technical) question has a positive answer, then we can actually replace "for *every* $\jmath$" with (seemingly weaker) "for *some* $\jmath$" in the statement of Question 1. Indeed, suppose there exists some $\jmath \in \{1, 2, \dots, 6\}$ such that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds almost everywhere. Then, we know from Theorem 17 and the positive answer to Question 2 that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ holds almost everywhere for every $\jmath'$. Hence we see that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ for every $\jmath' = 1, 2, \dots, 6$ at almost everywhere.

Note that an affirmative answer to the following question solves Question 2 positively:

Question 3. *Are the following two conditions equivalent for every $\jmath \in \{1, 2, \dots, 6\}$?*

1. $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) \text{ and } \jmath \text{ have the same parity}\}|/n = 1/2$ *holds a.e.;*
2. $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *holds a.e.*

One can easily deduce the first condition from the second one. Indeed, 2 and Theorem 17 imply that we have $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ a.e. for every $\jmath'$ having the same parity as $\jmath$. As the conjunction of finitely many properties that hold almost everywhere again holds almost everywhere, 1 now follows.

We now deduce Question 2 assuming that 2 follows from 1: Let us suppose $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ holds a.e. for a given $\jmath_1 \in \{1, 2, \dots, 6\}$. Take a $\jmath_2 \in \{1, 2, \dots, 6\}$

having different parity from that of $\jmath_1$. We wish to prove that $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$ holds a.e.

Since 2 implies 1, $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ and $\jmath_1$ have the same parity$\}|/n = 1/2$ holds almost everywhere. Now observe that for an arbitrary $\alpha \in (0,1) \setminus \mathbb{Q}$, $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ is even$\}|/n = 1/2$ holds if and only if $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ is odd$\}|/n = 1/2$ holds. Therefore, we have

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) \text{ is even}\}|}{n} = \frac{1}{2} \; a.e. \iff \lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) \text{ is odd}\}|}{n} = \frac{1}{2} \; a.e.$$

As $\jmath_1$ and $\jmath_2$ have different parity, it follows that $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ and $\jmath_2$ have the same parity$\}|/n = 1/2$ holds a.e. By applying our assumption that the condition 2 follows from 1, we get the desired result.

One can also ask a

Question 4. *Do we have* $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ *almost everywhere?*

We also do not know the answer to this question. What we can certainly say is that, in view of Proposition 3, Question 4 is at least as likely to be true as Question 1. Although we know from Theorem 9 that the condition "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$" for $p = p_1$ and for $p = p_2$ are independent from each other whenever p_1 and p_2 are distinct, there might be some relationship between statements "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ holds almost everywhere" for $p = p_1$ and for $p = p_2$ even when $p_1 \ne p_2$; It is interesting to see how the strength of the above statement changes as the value of p increases.

Our final remark is on "mod 2". Instead of modulo 6, one can consider indexes of Ducci matrices by modulo 2 and formulate statements for them, e.g. $j_\alpha(i) \equiv \jmath \pmod 2$ in place of $j_\alpha(i) = \jmath$. Even if we do so, the results in this paper remain valid (with trivial modifications).

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Note Added in Proof. At the time of submission of the manuscript, the author was not aware of the following fact: Almost every $\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q}$ satisfies $\lim_{n\to\infty} \Sigma_{i=1}^n a_i/n = \infty$. This appears to be a standard result in ergodic theory.

On the other hand, Theorem 6 states that for each $\alpha \in (0,1) \setminus \mathbb{Q}$, $\Sigma_{i=1}^n a_i/n$ diverges if and only if $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ holds. Combining these two results, we thus get a

Corollary 3. $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$. □

In view of Theorem 8 and Proposition 3, this in turn gives us a

Corollary 4. *1.* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *for every* $\jmath = 1, 2, \ldots, 6$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$;

2. For every positive integer p, $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$. □

1 and 2 positively answer Questions 1 and 4, respectively. Questions 2 and 3 are vacuously true.

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ON A NONLOCAL BIHARMONIC MEMS EQUATION WITH THE NAVIER BOUNDARY CONDITION

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Abstract. We study a biharmonic nonlocal MEMS equation. It arises in the Micro-Electro Mechanical System(MEMS) devices. First we establish the local solution and extend it globally in time by the use of the energy. Next, we consider the dynamical properties. The dynamical system has an absorbing set and a global attractor. Finally we prove the convergence of the global solution to a stationary solution.

1 Introduction We consider the following biharmonic nonlocal MEMS equation:

$$(1)\qquad \begin{cases} u_{tt}+u_t+\Delta^2 u = G\left(\beta,\gamma,\nabla u\right)\Delta u + \frac{\lambda}{(1-u)^\sigma} I\left(\sigma,\chi,u\right) & x\in\Omega,\ t>0, \\ u=\Delta u = 0 & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x) & x\in\Omega, \\ u_t(x,0)=u_1(x) & x\in\Omega, \end{cases}$$

where $\lambda>0$, $\beta>0$, $\gamma>0$, $\chi>0$, $\sigma\geq 2$, $\Omega\subset\mathbb{R}^n$ for $n\in\mathbb{N}$ is a bounded domain with smooth boundary $\partial\Omega$,

$$G\left(\beta,\gamma,\nabla u\right)=\beta\int_\Omega |\nabla u|^2\,dx+\gamma,$$

$$I\left(\sigma,\chi,u\right)=\frac{1}{\left(H\left(\sigma,\chi,u\right)\right)^\sigma}\qquad\text{and}\qquad H\left(\sigma,\chi,u\right)=1+\chi\int_\Omega\frac{dx}{(1-u)^{\sigma-1}}.$$

If the solution $u(x,t)$ of (1) reaches 1 at some point in Ω in finite time $t=T_q$, the right-hand side of (1) becomes infinite, which leads to the singularity. In this case, the solution $u(x,t)$ is said to quench in finite time $t=T_q$ and T_q is called the quenching time of the solution. This equation has been considered in [2, 4] and is a natural extension of MEMS equation [7, 8, 20, 23]. The MEMS (Micro-Electro Mechanical System) equation arises in the study of the MEMS devices which are often utilized to combine electronics with micro-size mechanical devices. They can be modelled as the dynamic deflection of an elastic membrane inside this system and arise in the accelerometers for airbag deployment in automobiles, in the ink jet printer heads, in the optical switches, in the chemical sensors and so on.

In [2], the authors establish the stationary solution with Steklov and Dirichlet boundary condition by the implicit function theorem [25]. They construct the stationary solution $u\in H^4(\Omega)\cap H^1_0(\Omega)$ of (1) provided that the diameter of Ω is sufficiently small. In [4], the authors consider the periodic solution of (1) by [25]. In the limiting case $\chi=0$, there is supposed to be no capacitor in the circuit, which is studied in [13] with $\beta=0$ and $\sigma=2$. The author derives the results of existence, convergence to the stationary solution and exponential decay of the global solution. On the other hand, he deals with the quenching of the solution. The aim of this paper is to investigate the dynamical properties to the biharmonic nonlocal

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problem (1). For the second order nonlocal equation, see [10, 11, 12, 16, 17, 19, 21, 22, 24]. First, we obtain the theorem concerned with the local existence of the solution. Throughout this paper, the definition of the function spaces and their norms is presented in Section 2.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n = 1, 2, 3$. We denote $X \equiv H^2(\Omega) \cap H_0^1(\Omega)$, $D \equiv X \times L^2(\Omega)$ and $H \equiv L^2(\Omega) \times H^{-2}(\Omega)$. For any $\lambda > 0$, $\beta > 0$, $\gamma > 0$, $\chi > 0$, $\sigma \geq 2$ and $\phi_0 \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D$ with*

$$\|u_0\|_C < 1 - \delta$$

for some $\delta \in (0, 1)$, there exists a unique solution of (1) *with*

$$\phi \equiv \begin{pmatrix} u \\ u_t \end{pmatrix} \in C([0, T); D) \cap C^1([0, T); H)$$

for sufficiently small $T > 0$, where T depends only on λ, β, γ, χ, σ, Ω, (u_0, u_1) and δ. The solution u can be continued as long as $\|u(\cdot, t)\|_C < 1$. Here, $\|\cdot\|_C$ denotes the standard $C(\overline{\Omega})$ norm defined in Section 2.

To establish the global solution, we define the energies by

$$\mathcal{E}(\phi(t)) = \frac{1}{2}\int_\Omega u_t^2\, dx + \frac{1}{2}\int_\Omega (\Delta u)^2\, dx + \frac{\beta}{4}\left(\int_\Omega |\nabla u|^2\, dx\right)^2 + \frac{\gamma}{2}\int_\Omega |\nabla u|^2\, dx$$

and

$$\mathcal{E}_0 \equiv \frac{1}{2}\int_\Omega u_1^2\, dx + \frac{1}{2}\int_\Omega (\Delta u_0)^2\, dx + \frac{\beta}{4}\left(\int_\Omega |\nabla u_0|^2\, dx\right)^2 + \frac{\gamma}{2}\int_\Omega |\nabla u_0|^2\, dx,$$

respectively. Then

$$E(\phi(t)) \equiv \mathcal{E}(\phi(t)) + \frac{\lambda}{(\sigma - 1)^2 \chi}\left(H(\sigma, \chi, u)\right)^{1-\sigma}$$

is a Lyapunov function for (1), which plays an important role in proving the global existence and dynamical properties of the solution. We impose the smallness condition on the parameter $\lambda > 0$ and initial energy $\mathcal{E}_0$. To state the condition, we define

$$\lambda^* \equiv \frac{(\sigma - 1)^2 \chi a}{2}\left(\frac{2^{\sigma-1} + \chi|\Omega|}{2^{\sigma-1}}\right)^{\sigma-1}$$

for any fixed $\gamma > 0$, $\chi > 0$, $\sigma \geq 2$ and $\Omega \subset \mathbb{R}^n$, where $a > 0$ depends only on γ and Ω and is defined in Section 4. Moreover we define

$$\mathcal{E}_0^* \equiv \frac{a}{2} - \frac{\lambda}{(\sigma - 1)^2 \chi}\left(\frac{2^{\sigma-1}}{2^{\sigma-1} + \chi|\Omega|}\right)^{\sigma-1} > 0$$

for these fixed constants and any $0 < \lambda < \lambda^*$. Then we have the next theorem on the global existence of the solution.

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n = 1, 2, 3$. For any $\beta > 0$, $\gamma > 0$, $\chi > 0$ and $\sigma \geq 2$, let $\lambda < \lambda^*$ be fixed arbitrarily. For all $\kappa \in (0, \mathcal{E}_0^*)$, there exists $\delta_0 \in (0, 1)$ such that for any $\phi_0 \in D$ with*

$$\|u_0\|_C < 1 - \delta_0$$

and

$$\mathcal{E}_0 < \mathcal{E}_0^* - \kappa,$$

(1) *has a global solution satisfying*

$$\phi \in C\left([0,\infty); D\right) \cap C^1\left([0,\infty); H\right)$$

and

$$\|u(\,\cdot\,,t)\|_C < 1 - \delta_0$$

for all $t \geq 0$. *Here,* δ_0 *depends only on* λ, γ, χ, σ, κ *and* Ω.

Next theorem is on the regularity of the solution obtained in Theorem 2. To state the theorem, we define Y by

$$Y = \left\{u \in H^4(\Omega) \cap H_0^1(\Omega) \mid \Delta u = 0 \text{ on } \partial\Omega\right\}.$$

If $n \leq 3$, the Sobolev embedding $H^2(\Omega) \subset C(\overline{\Omega})$ holds. Hence we note that $\Delta u = 0$ on $\partial\Omega$ makes sense in this paper. Now we denote $E \equiv Y \times X$. Then we have the following:

Theorem 3 *Under the same hypotheses as Theorem 2, for any* $\phi_0 \in E$, *there exists a unique global solution of* (1) *with*

$$\phi \in C\left([0,\infty); E\right) \cap C^1\left([0,\infty); D\right) \cap C^2\left([0,\infty); H\right).$$

We define

$$Z_{\delta_0} \equiv \left\{ \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in D \mid \left\|u^1\right\|_C < 1 - \delta_0 \right\}$$

and consider the nonlinear semigroup $S(t) : Z_{\delta_0} \to Z_{\delta_0}$ by

$$S(t)\phi_0 = \phi(t).$$

In the fourth theorem, we establish an absorbing set to show that $S(t)$ has a global attractor in Z_{δ_0}.

Theorem 4 *In addition to the same hypotheses as Theorem 2, let*

$$\mathcal{E}_0 + \frac{\lambda}{(\sigma-1)^2\chi} < \frac{\gamma K_1}{2\beta K_2} \tag{2}$$

hold, where $K_1 > 0$ *and* $K_2 > 0$ *depend only on* γ *and* Ω *and are defined in Lemma 4. Then the dynamical system* $S(t) : Z_{\delta_0} \to Z_{\delta_0}$ *possesses an absorbing set* $\mathcal{B} \subset Z_{\delta_0}$. *The omega limit set* $\mathcal{A} = \omega(\mathcal{B})$ *of* $\mathcal{B}$ *is a global attractor in* Z_{δ_0}.

To argue the behaviour as $t \to +\infty$, we introduce the set $\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma}$ of stationary solution by

$$\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} = \left\{\eta \in Z_{\delta_0} \mid \eta = \eta(x) \text{ is a stationary solution for (1)}\right\}.$$

In [2], they construct the stationary solution by the implicit function theorem for the small domain. We also find the stationary solution without imposing any smallness condition on Ω. We derive the following theorem on dynamical properties of $S(t)$.

Theorem 5 *Under the same hypotheses as Theorem 4, the omega limit set* $\omega(\phi_0)$ *is invariant, non-empty, compact and connected in* Z_{δ_0}. *Moreover* $\omega(\phi_0) \subset \mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \times \{0\}$. *In particular,*

$$\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \neq \emptyset$$

for $\lambda \in (0, \underline{\lambda}^*)$, *where*

$$\underline{\lambda}^* \equiv \min\left(\lambda^*, \frac{\gamma\chi\,(\sigma-1)^2\,K_1}{2\beta K_2}\right).$$

We prove that the omega limit set is composed of a single point in Z_{δ_0}.

Theorem 6 *Under the same hypotheses as Theorem 4, there exists* $\eta \in \mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma}$ *such that the omega limit set is composed of a single point in* Z_{δ_0} *with*

$$\omega(\phi_0) = (\eta, 0)$$

and

$$\lim_{t\to+\infty}\left(\|u(\,\cdot\,,t)-\eta\|_X + \|u_t(\,\cdot\,,t)\|_2\right) = 0. \tag{3}$$

This paper is organized as follows: In Section 2, we recall the facts about Sobolev space and dynamical system. We introduce the existence theorem [2] of stationary solution. In Section 3, we establish the local solution by the contraction mapping theorem. In Section 4, we extend the local solution to the global one for small parameters and initial values. Moreover we study the regularity of the global solution. In Section 5, we consider the dynamical properties. By the existence of the Lyapunov function, we can treat the omega limit set and global attractor. In Section 6, by the Lojasiewicz-Simon inequality, we show that the omega limit set is composed of a single point. In an appendix, we prove the Lojasiewicz-Simon inequality. This kind of inequalities is proven in many situations. In this paper, we treat the case with nonlocal term.

2 Preliminaries First, we introduce the notations of function spaces and the Sobolev embedding theorems. In this paper, $C(\overline{\Omega})$ denotes the space of all continuous functions in $\overline{\Omega}$ with the norm

$$\|u\|_C = \sup_{x\in\overline{\Omega}} |u(x)|$$

for $u \in C(\overline{\Omega})$. For $1 \leq p \leq +\infty$, we denote the usual Sobolev space in Ω by $W^{s,p}(\Omega)$ and in particular write $W^{s,2}(\Omega) = H^s(\Omega)$. $H^s_0(\Omega)$ is defined as the closure of the set $\mathcal{D}(\Omega)$ in the space $H^s(\Omega)$, where we denote by $\mathcal{D}(\Omega)$ the space of all infinitely differentiable functions on Ω with compact supports. $H^{-s}(\Omega)$ is defined as the dual space of $H^s_0(\Omega)$ equipped with the norm

$$\|u\|_{H^{-s}} = \sup_{w\in H^s_0(\Omega),\|w\|_{H^s_0}\leq 1}\left|\int_\Omega uw\,dx\right|.$$

$(\ ,\ )$ and $(\ ,\ )_{H^{-s}}$ denote the inner product in $L^2(\Omega)$ and $H^{-s}(\Omega)$, respectively. According to [1, 3] , we adopt the norm in $H^1_0(\Omega)$, X and Y as

$$\|u\|_{H^1_0} = \|\nabla u\|_2\,, \quad \|u\|_X = \|\Delta u\|_2 \quad \text{and} \quad \|u\|_Y = \left\|\Delta^2 u\right\|_2,$$

respectively. Here, $\|\cdot\|_p$ denotes the standard L^p norm in Ω with $p \in [1,\infty]$. We define

$$\|\phi\|_D = \left(\left\|u^1\right\|_X^2 + \left\|u^2\right\|_2^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|\phi\|_E = \left(\left\|u^1\right\|_Y^2 + \left\|u^2\right\|_X^2\right)^{\frac{1}{2}}$$

for $\phi = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$. We introduce the embedding inequalities [1].

Lemma 1 *Let $n = 1$. For $u \in H_0^1(\Omega)$, we have*

$$\|u\|_C \leq C_S \|u\|_{H_0^1},$$

where $C_S > 0$ depends only on Ω.

Lemma 2 *Let $n = 2, 3$. For $u \in X$, we have*

$$\|u\|_C \leq C_S \|u\|_X,$$

where $C_S > 0$ depends only on Ω.

Lemma 3 *For $u \in H_0^1(\Omega)$, we have*

$$\|u\|_2 \leq C_P \|u\|_{H_0^1},$$

where $C_P > 0$ depends only on Ω.

Henceforth we shall adopt universal notations C_S and C_P to denote these constants for the case $n = 1, 2, 3$.
We introduce the theorem of existence of the global attractor. For other basic notions and results, see [26, 28]. Let Z be Banach space and $S(t)$ be a continuous semigroup on Z. The semigroup $S(t)$ is said to be uniformly compact if for every bounded set $B \subset Z$, there exists t_0 such that $\cup_{t \geq t_0} S(t)B$ is relatively compact in Z.

Theorem 7 (Theorem 1.1 in [26]) *Let $S(t)$ be a continuous semigroup on Banach space Z. We assume that it can be decomposed into $S(t) = S_1(t) + S_2(t)$, where $S_1(t)$ is uniformly compact for large $t > 0$ and $S_2(t)$ is continuous from Z to Z satisfying the following condition: For any bounded set $B \subset Z$,*

$$\sup_{\phi_0 \in B} \|S_2(t)\phi_0\|_Z \to 0$$

as $t \to \infty$. We also assume that there exist an open set $\mathcal{U}$ and absorbing set $\mathcal{B} \subset \mathcal{U}$. Then the omega limit set $\mathcal{A} = \omega(\mathcal{B})$ of $\mathcal{B}$ is a global attractor in $\mathcal{U}$ for $S(t)$.

Finally we mention the existence of stationary solution. We consider the corresponding elliptic equation

$$\text{(4)} \qquad \begin{cases} \Delta^2 \eta = G(\beta, \gamma, \nabla\eta)\,\Delta\eta + \frac{\lambda}{(1-\eta)^\sigma} I(\sigma, \chi, \eta) & x \in \Omega, \\ \eta = \Delta\eta - d\frac{\partial\eta}{\partial\nu} = 0 & x \in \partial\Omega, \end{cases}$$

where $d \in [0, +\infty]$, ν is the outer unit normal vector and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then the existence of the stationary solution is guaranteed by the implicit function theorem [25].

Theorem 8 (Theorem 1 in [2]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n \leq 7$. For any $\lambda > 0$, $\beta > 0$, $\gamma > 0$, $\chi > 0$ and $\sigma \geq 2$, there exist $\overline{\lambda} > 0$ and $d_0 > 0$ such that* (4) *possesses a solution $\eta \in H^4(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \overline{\lambda})$ provided that one of the following holds*:
Steklov boundary condition: $0 \leq d < d_0$
or
Dirichlet boundary condition: $d = +\infty$ *and Ω is a ball*
and the diameter of Ω is sufficiently small.

The relation between $\overline{\lambda}$ and $\underline{\lambda}^*$ in Theorem 5 is not clear.

3 Local existence We consider the linear wave equation

$$
(5) \qquad \begin{cases} w_{tt} + w_t + Aw = 0 & x \in \Omega,\ t > 0, \\ w = \Delta w = 0 & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x) & x \in \Omega, \\ w_t(x,0) = w_1(x) & x \in \Omega, \end{cases}
$$

where

$$
Aw = \Delta^2 w - \gamma \Delta w
$$

and derive the decay estimate of the solution. Next we construct the time local solution of (1) by the contraction mapping theorem. We omit the detail of the computations. See [13, 18, 21].

Lemma 4 (Proposition 4.3.4 in [15] and (3.5) in [13]) *For any $\psi_0 \equiv \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in D$, there exists a unique solution*

$$
\psi \equiv \begin{pmatrix} w \\ w_t \end{pmatrix} \in C\left([0,\infty); D\right) \cap C^1\left([0,\infty); H\right)
$$

of (5). *Moreover, we have*

$$
\|\psi\|_D \leq K_2 \|\psi_0\|_D \, e^{-K_1 t},
$$

where $K_1 > 0$ and $K_2 > 0$ depend only on γ and Ω.

Proof of Theorem 1. To deal with the nonlinear term with the singularity, we modify $1/(1-u)$ and $I(\sigma, \chi, u)$ by

$$
F_\delta(u) = \begin{cases} \frac{1}{1-u} & u \leq 1 - \frac{\delta}{2}, \\ \frac{4}{\delta} & u \geq 1 - \frac{\delta}{4} \end{cases}
$$

and

$$
I_\delta(\sigma, \chi, u) = \frac{1}{\left(H_\delta(\sigma, \chi, u)\right)^\sigma} \quad \text{with} \quad H_\delta(\sigma, \chi, u) = 1 + \chi \int_\Omega F_\delta(u(x))^{\sigma - 1}\, dx,
$$

where we continue $F_\delta(u)$ suitably in the range $(1 - \delta/2, 1 - \delta/4)$ so that we assume that F_δ is positive, bounded and sufficiently smooth. Under the abstract setting

$$
\phi = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i_d \\ A & i_d \end{pmatrix}
$$

and

$$J_\delta(u) = \begin{pmatrix} 0 \\ \beta\left(\int_\Omega |\nabla u|^2 \, dx\right)\Delta u + \lambda F_\delta(u)^\sigma I_\delta(\sigma, \chi, u) \end{pmatrix},$$

we transform (1) into the modified equation

$$\phi_t + B\phi = J_\delta(u)$$

and consider the corresponding integral equation

$$\phi = e^{-Bt}\phi_0 + \int_0^t e^{-B(t-s)} J_\delta(u(s)) \, ds. \tag{6}$$

We construct the local solution by the contraction mapping theorem. Taking $l = \|\phi_0\|_D$, we set

$$X_T \equiv \left\{\phi \in C([0,T]; D) \mid \|\phi\|_{X_T} \le 2K_2 l\right\},$$

where T is a positive constant to be determined later. Here in the space X_T, the norm is equipped with

$$\|\phi\|_{X_T} = \sup_{t\in[0,T]} \|\phi(\,\cdot\,, t)\|_D .$$

For $\phi \in X_T$, we define the mapping $V(t)$ on D by the right-hand side of (6), that is,

$$V(t)\phi = e^{-Bt}\phi_0 + \int_0^t e^{-B(t-s)} J_\delta(u(s)) \, ds.$$

Then we can show that V is a contraction mapping from X_T into itself for small $T > 0$.

Lemma 5 (Cf. Lemmas 2 and 3 in [21]) *If $T < \tau$, then V is a contraction mapping from X_T into X_T, where $\tau > 0$ is a constant determined only by λ, β γ, χ, σ, Ω, l and δ.*

By Lemma 5, (6) has a unique time local solution $\phi \in C([0,T]; D) \cap C^1([0,T]; H)$. If the solution of (6) begins with $\|u_0\|_C < 1 - \delta$ and satisfies $\|u(\,\cdot\,, t)\|_C \le 1 - \delta/2$ for all $t > 0$, then u is a solution of (1). Otherwise there is a finite time $T_0 > 0$ at which $\max_{x\in\overline{\Omega}} u(x, T_0) = 1 - \delta/2$. We choose $\delta_1 \in (0, \delta)$ and apply the contraction mapping theorem to (6) with δ replaced by δ_1. We may extend $u(x,t)$ uniquely to an interval $(0, T_0')$ with $T_0 < T_0'$ such that $\|u(\,\cdot\,, t)\|_C \le 1 - \delta_1/2$ for all $t \in [0, T_0')$. Since we can take $\delta_1 \in (0, \delta)$ arbitrarily small, $u(x,t)$ is a solution of (1) on $\overline{\Omega} \times [0, T_0')$ as long as $\|u(\,\cdot\,, t)\|_C < 1$. □

4 Global existence In this section, we shall show that the local solution can be continued up to $t = +\infty$. We introduce the Lyapunov function to obtain the necessary estimates and extend it globally in time. The idea is from [17]. At first, in order to introduce the lemma, we set

$$a = \begin{cases} \frac{\gamma}{C_S^2} & \text{for } n = 1, \\ \frac{1}{C_S^2} & \text{for } n = 2, 3, \end{cases} \qquad b = \frac{2\lambda}{(\sigma - 1)^2 \chi} \quad \text{and} \quad c = \chi |\Omega|$$

and define

$$g(x) = ax^2 + b\left\{\frac{(1-x)^{\sigma-1}}{(1-x)^{\sigma-1} + c}\right\}^{\sigma-1}$$

for $-1 \le x \le 1$, where C_S is the constant defined in Lemmas 1 and 2. Let

$$G(x) = g(x) - 2\mathcal{E}_0^* - g(-1) + g(1) + 2\kappa$$

for $0 \le x \le 1$.

Lemma 6 *Under the same hypotheses as Theorem 2, there exists a zero $x_0 \in (0,1)$ of $G(x)$, where x_0 depends only on λ, γ, χ, σ, κ and Ω.*

Proof. Since we have

$$g(-1) = a + b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1}, \quad g(0) = b\left(\frac{1}{1+c}\right)^{\sigma-1} \quad \text{and} \quad g(1) = a$$

and

$$h(x) = \left(\frac{x}{x+c}\right)^{\sigma-1}$$

is increasing for $x \geq 0$, a simple computation yields

$$G(0) = 2\left(\kappa - \mathcal{E}_0^*\right) + b\left(h(1) - h(2^{\sigma-1})\right) < 0$$

by the hypotheses $0 < \kappa < \mathcal{E}_0^*$ and $\sigma \geq 2$. On the other hand, we have

$$G(1) = a - b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1} - 2\mathcal{E}_0^* + 2\kappa = 2\kappa > 0.$$

Thus the intermediate theorem guarantees at least one zero in $(0,1)$. Henceforth, we denote the least zero by

$$x_0 = 1 - \delta_0$$

with $\delta_0 \in (0,1)$. □

Proof of Theorem 2. For (1), we have the Lyapunov function

$$E(\phi(t)) \equiv \mathcal{E}(\phi(t)) + \frac{\lambda}{(\sigma-1)^2\chi}\left(H\left(\sigma,\chi,u\right)\right)^{1-\sigma}$$

for $t \in [0,T)$ and set

$$E_0 \equiv E(\phi_0) = \mathcal{E}_0 + \frac{\lambda}{(\sigma-1)^2\chi}\left(H\left(\sigma,\chi,u_0\right)\right)^{1-\sigma},$$

where T is the maximal existence time of the solution determined in Section 3. In fact, we obtain

$$\frac{d}{dt}E(\phi(t)) = -\int_\Omega u_t^2\,dx \leq 0,$$

which implies that

$$E(\phi(t)) \leq E(\phi(t)) + \int_0^t\int_\Omega u_t^2\,dx\,ds = E_0. \tag{7}$$

Now we estimate $E(\phi(t))$ and E_0 as follows:

$$2E(\phi(t)) \geq \|u\|_X^2 + \gamma\|u\|_{H_0^1}^2 + \frac{b}{\left(1 + c\left(\frac{1}{1-\|u\|_C}\right)^{\sigma-1}\right)^{\sigma-1}} \geq g(\|u\|_C)$$

by Lemmas 1 and 2 and

$$2E_0 < 2\mathcal{E}_0 + b\left(H\left(\sigma,\chi,-1\right)\right)^{1-\sigma} \leq 2\mathcal{E}_0^* + b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1} - 2\kappa$$

due to $-1 < u_0$. Then the energy inequality (7) yields

$$G(\|u(t)\|_C) = g(\|u(t)\|_C) - 2\mathcal{E}_0^* - g(-1) + g(1) + 2\kappa < 0$$

for all $t \in [0, T)$. By Lemma 6 and $\|u_0\|_C < 1 - \delta_0$,

$$\|u(t)\|_C < 1 - \delta_0 \tag{8}$$

holds for all $t \in [0, T)$. Owing to the energy (7), we have

$$\gamma \|u(t)\|_{H_0^1}^2 + \|u(t)\|_X^2 + \|u_t(t)\|_2^2 \leq 2E_0 \tag{9}$$

for all $t \in [0, T)$. We note that

$$E_0 < \mathcal{E}_0^* + \frac{\lambda^*}{(\sigma - 1)^2 \chi} - \kappa < \frac{a}{2} + \frac{a}{2}\left(\frac{2^{\sigma-1} + \chi |\Omega|}{2^{\sigma-1}}\right)^{\sigma-1}. \tag{10}$$

Hence the right-hand side depends only on γ, χ, σ and Ω and is independent of $\|\phi_0\|_D$ and T. Finally (8) and (9) are valid for all $t \geq 0$, which ensures that the solution exists globally in time. Since $L^2(\Omega) \subset H^{-2}(\Omega)$, we have

$$u_{tt} = -u_t - \Delta^2 u + G(\beta, \gamma, \nabla u)\Delta u + \frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u) \in H^{-2}(\Omega)$$

and $\phi \in C([0,\infty); D) \cap C^1([0,\infty); H)$. □

Proof of Theorem 3. In this proof, by L we denote the universal positive constants which depend only on λ, β, γ, χ, σ, κ, Ω and $\|\phi_0\|_E$. We define $(u_1)_t$ by

$$(u_1)_t = -u_1 - \Delta^2 u_0 + G(\beta, \gamma, \nabla u_0)\Delta u_0 + \frac{\lambda}{(1-u_0)^\sigma} I(\sigma, \chi, u_0) \in L^2(\Omega).$$

First by differentiating $\|u_{tt}\|_2^2$ with respect to t, we obtain

$$\frac{d}{dt}\int_\Omega u_{tt}^2\, dx = -2\int_\Omega u_{tt}^2\, dx - \frac{d}{dt}\int_\Omega (\Delta u_t)^2\, dx + 2I_1 + 2I_2. \tag{11}$$

The definition and estimation of I_1 and I_2 are as follows. (8), (9), the Hölder and Young inequalities yield

$$\begin{aligned} I_1 &\equiv \int_\Omega u_{tt}\Big(G(\beta,\gamma,\nabla u)\Delta u\Big)_t\, dx \\ &= -2\beta\int_\Omega \Delta u u_t\, dx \int_\Omega u_{tt}\Delta u\, dx - G(\beta,\gamma,\nabla u)\int_\Omega \nabla u_{tt}\cdot\nabla u_t\, dx \\ &\leq \frac{1}{4}\int_\Omega u_{tt}^2\, dx + L\int_\Omega u_t^2\, dx - \frac{1}{2}G(\beta,\gamma,\nabla u)\frac{d}{dt}\int_\Omega |\nabla u_t|^2\, dx \end{aligned} \tag{12}$$

and

$$I_2 \equiv \int_\Omega u_{tt}\left(\frac{\lambda}{(1-u)^\sigma} I(\sigma,\chi,u)\right)_t dx \leq \frac{1}{4}\int_\Omega u_{tt}^2\, dx + L\int_\Omega u_t^2\, dx, \tag{13}$$

respectively. Eventually integrating (11) over $(0,t)$ with respect to t together with (12) and (13), we obtain

$$\begin{aligned}&\int_0^t\int_\Omega u_{tt}^2\,dx\,ds+\int_\Omega\Big(u_{tt}^2+(\Delta u_t)^2\Big)\,dx+G(\beta,\gamma,\nabla u)\int_\Omega|\nabla u_t|^2\,dx\\ &\le\ \|(u_1)_t\|_2^2+\|u_1\|_X^2+G(\beta,\gamma,\nabla u_0)\,\|u_1\|_{H_0^1}^2\\ &\qquad+\int_0^t\frac{d}{dt}\Big(G(\beta,\gamma,\nabla u)\Big)\left(\int_\Omega|\nabla u_t|^2\,dx\right)ds+L\int_0^t\int_\Omega u_t^2\,dx\,ds.\end{aligned}$$

Since the fourth integral term in the right-hand side yields

$$\begin{aligned}\frac{d}{dt}\Big(G(\beta,\gamma,\nabla u)\Big)\int_\Omega|\nabla u_t|^2\,dx\ &=\ 2\beta\int_\Omega\Delta u u_t\,dx\int_\Omega\Delta u_t u_t\,dx\\ &\le\ L\int_\Omega u_t^2\,dx+\beta\int_\Omega u_t^2\,dx\int_\Omega(\Delta u_t)^2\,dx,\end{aligned}$$

we have

$$\begin{aligned}&\int_0^t\int_\Omega u_{tt}^2\,dx\,ds+\int_\Omega\Big(u_{tt}^2+(\Delta u_t)^2\Big)\,dx+G(\beta,\gamma,\nabla u)\int_\Omega|\nabla u_t|^2\,dx\\ &\le\ L+\beta\int_0^t\left(\int_\Omega u_t^2\,dx\right)\left(\int_\Omega(\Delta u_t)^2\,dx\right)ds\end{aligned}$$

and in particular

$$\int_\Omega(\Delta u_t)^2\,dx\le L+\beta\int_0^t\left(\int_\Omega u_t^2\,dx\right)\left(\int_\Omega(\Delta u_t)^2\,dx\right)ds.$$

We apply the Gronwall inequality (Lemma 2.1.1 in [15]) to derive

$$\int_\Omega(\Delta u_t)^2\,dx\le L\exp\left(\beta\int_0^{+\infty}\int_\Omega u_t^2\,dx\,ds\right)\le L,$$

which implies that $u\in Y$, $u_t\in X$ and $u_{tt}\in L^2(\Omega)$ due to

$$\Delta^2u=-u_{tt}-u_t+G(\beta,\gamma,\nabla u)\,\Delta u+\frac{\lambda}{(1-u)^\sigma}I(\sigma,\chi,u)\in L^2(\Omega).$$

□

5 Global attractor First, we show that the orbit $\cup_{t\ge0}\phi(t)$ is contained in some absorbing set in Z_{δ_0}. Hence this fact leads us to the existence of a global attractor by Theorem 7 in Section 2. Next, we consider the properties of $\omega(\phi_0)$. We show that $(\eta,0)\in\omega(\phi_0)$ for some $\eta\in\mathcal{S}^\lambda_{\beta,\gamma,\chi,\sigma}$. In other words, there exist $\eta\in\mathcal{S}^\lambda_{\beta,\gamma,\chi,\sigma}$ and $t_n\to+\infty$ such that

$$\lim_{n\to+\infty}\Big(\|u(\,\cdot\,,t_n)-\eta\|_X+\|u_t(\,\cdot\,,t_n)\|_2\Big)=0. \tag{14}$$

In [2], the authors establish the stationary solution $\eta\in Y$. However they impose the smallness condition on Ω. See Theorem 8 in this paper. In this section, the Lyapunov

function plays an important role in the argument.
For a solution u of (1) obtained in Theorem 2, we denote by v a solution of

$$(15)\qquad \begin{cases} v_{tt}+v_t+Av=\beta\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta v & x\in\Omega,\ t>0,\\ v=\Delta v=0 & x\in\partial\Omega,\ t>0,\\ v(x,0)=u_0(x) & x\in\Omega,\\ v_t(x,0)=u_1(x) & x\in\Omega. \end{cases}$$

Let $\psi=\begin{pmatrix} v\\ v_t\end{pmatrix}$ and $S_2(t)\phi_0=\psi(t)$. From now on, we show that the semigroup S_2 has a decaying property. First, ψ satisfies

$$\psi=e^{-Bt}\phi_0+\int_0^t e^{-B(t-s)}P\left(u(s),v(s)\right)\,ds,$$

where

$$P\left(u(t),v(t)\right)=\begin{pmatrix} 0\\ \beta\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta v\end{pmatrix}.$$

We set

$$K_3\equiv K_1-\frac{2\beta E_0K_2}{\gamma},$$

where $K_1>0$ and $K_2>0$ are constants defined in Lemma 4 and derive

$$K_3>K_1-\frac{2\beta K_2}{\gamma}\left(\mathcal{E}_0+\frac{\lambda}{(\sigma-1)^2\chi}\right)>0$$

provided that (2) holds.

Lemma 7 *Under the same hypotheses as Theorem 4, for any $\phi_0\in D$, there exists a unique solution*

$$\psi\in C\left([0,\infty);D\right)\cap C^1\left([0,\infty);H\right)$$

of (15). *Moreover, we have*

$$\|\psi(t)\|_D\le K_2e^{-K_3t}\|\phi_0\|_D\,.$$

Proof. Thanks to Lemma 4, we have

$$\|\psi\|_D\le K_2\|\phi_0\|_De^{-K_1t}+\frac{2\beta E_0K_2}{\gamma}\int_0^t e^{-K_1(t-s)}\|\psi\|_D\,ds$$

by the use of (9) and

$$e^{K_1t}\|\psi(t)\|_D\le K_2\|\phi_0\|_D+(K_1-K_3)\int_0^t e^{K_1s}\|\psi(s)\|_D\,ds.$$

The Gronwall inequality yields

$$\|\psi(t)\|_D\le K_2e^{-K_3t}\|\phi_0\|_D\,.$$

□

Next, in order to prove uniformly compactness, we introduce the lemmas. Their proofs are similar to that of Theorem 3. Henceforth we shall adopt universal notations $M>0$ to denote the various constants which depend only on λ, β, γ, χ, σ, κ, Ω and $\|\phi_0\|_D$.

Lemma 8 *The solution ψ obtained in Lemma 7 satisfies*

$$\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2 + 2\int_0^t \|v_t\|_2^2 \, ds \le M.$$

Proof. We have

$$\left(\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2\right)_t + 2\|v_t\|_2^2 = -\beta \|u\|_{H_0^1}^2 \left(\|v\|_{H_0^1}^2\right)_t$$

and integrate this equation with respect to t to get

$$\begin{aligned}
&\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2 + 2\int_0^t \|v_t\|_2^2 \, ds \\
\le\ & M\|\phi_0\|_D^2 + M\|\phi_0\|_D^4 + \beta\int_0^t \left(\|u\|_{H_0^1}^2\right)_t \|v\|_{H_0^1}^2 \, ds \\
\le\ & M + M\int_0^t \|u\|_X \|u_t\|_2 \|\psi\|_D^2 \, ds \\
\le\ & M + M\|\phi_0\|_D^2 \int_0^t e^{-2K_3 s} \, ds \\
\le\ & M
\end{aligned}$$

by (9). □

Next, let w be a solution of

$$(16) \qquad \begin{cases} w_{tt} + w_t + Aw = \beta\left(\int_\Omega |\nabla u|^2 \, dx\right)\Delta w + \frac{\lambda}{(1-u)^\sigma} I(\sigma,\chi,u) & x \in \Omega,\ t > 0, \\ w = \Delta w = 0 & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_t(x,0) = 0 & x \in \Omega. \end{cases}$$

We set $\xi = \begin{pmatrix} w \\ w_t \end{pmatrix}$ and $S_1(t)\phi_0 = \xi(t)$. Then we have

$$\phi(t) = \xi(t) + \psi(t) \quad \text{and} \quad S(t) = S_1(t) + S_2(t).$$

Lemma 9 (16) *possesses a unique solution*

$$\xi \in C([0,\infty); E) \cap C^1([0,\infty); D) \cap C^2([0,\infty); H).$$

Proof. First of all, we note that

$$\|w\|_X = \|u - v\|_X \le \|u\|_X + \|v\|_X \le M$$

by (9) and Lemma 8 and that

$$\int_0^\infty \|w_t\|_2^2 \, ds = \int_0^\infty \|u_t - v_t\|_2^2 \, ds \le 2\int_0^\infty \|u_t\|_2^2 \, ds + 2\int_0^\infty \|v_t\|_2^2 \, ds \le M$$

by (7) and Lemma 8. In the same computations as (12) and (13) in the proof of Theorem 3, we have

$$\frac{d}{dt}\int_\Omega w_{tt}^2\,dx = -2\int_\Omega w_{tt}^2\,dx - \frac{d}{dt}\int_\Omega (\Delta w_t)^2\,dx + 2I_3 + 2I_4,$$

where I_3 and I_4 are defined and computed similarly as follows:

$$\begin{aligned} I_3 &\equiv \int_\Omega w_{tt}\Big(G(\beta,\gamma,\nabla u)\,\Delta w\Big)_t\,dx \\ &\leq \frac{1}{4}\int_\Omega w_{tt}^2\,dx + M\int_\Omega u_t^2\,dx - \frac{1}{2}G(\beta,\gamma,\nabla u)\frac{d}{dt}\int_\Omega |\nabla w_t|^2\,dx \end{aligned}$$

and

$$I_4 \equiv \int_\Omega w_{tt}\left(\frac{\lambda}{(1-u)^\sigma}I(\sigma,\chi,u)\right)_t\,dx \leq \frac{1}{4}\int_\Omega w_{tt}^2\,dx + M\int_\Omega u_t^2\,dx.$$

Hence we obtain

$$\begin{aligned} &\int_\Omega w_{tt}^2\,dx + \frac{d}{dt}\int_\Omega \Big(w_{tt}^2 + (\Delta w_t)^2\Big)\,dx + G(\beta,\gamma,\nabla u)\frac{d}{dt}\int_\Omega |\nabla w_t|^2\,dx \\ &\leq M\int_\Omega u_t^2\,dx. \end{aligned}$$

We integrate this inequality to derive

$$\begin{aligned} &\int_0^t\int_\Omega w_{tt}^2\,dx\,ds + \int_\Omega \Big(w_{tt}^2 + (\Delta w_t)^2\Big)\,dx + G(\beta,\gamma,\nabla u)\int_\Omega |\nabla w_t|^2\,dx \\ &\leq M + 2\beta\int_0^t\left(\int_\Omega \Delta u u_t\,dx\right)\left(\int_\Omega \Delta w_t w_t\,dx\right)ds \\ &\leq M + M\int_0^t\left(\int_\Omega w_t^2\,dx\right)\left(\int_\Omega (\Delta w_t)^2\,dx\right)ds \end{aligned}$$

and apply the Gronwall inequality to obtain

$$\int_\Omega (\Delta w_t)^2\,dx \leq M\exp\left(M\int_0^t\int_\Omega w_t^2\,dx\,ds\right) \leq M,$$

which yields $w \in Y$, $w_t \in X$ and $w_{tt} \in L^2(\Omega)$. □

Proof of Theorem 4. Since λ and $\mathcal{E}_0$ are restricted to the hypotheses in Theorem 2 and (2), it is obvious that $S(t)$ has an absorbing set owing to (10) and

$$\|\phi\|_D^2 \leq 2E(\phi(t)) \leq 2E_0.$$

S_2 has a decaying property from Lemma 7. Lemma 9 implies that $\|\xi(t)\|_E$ is bounded for all $t > 0$. The inclusion $E \subset D$ is compactly embedded. Hence S_1 is uniformly compact. Thus we apply Theorem 7 to $S(t)$ to complete the proof. □

Proof of Theorem 5. Following the argument of Lemma 7.6.2 in [15] and originally [27] along with the proof of Theorem 4, we can prove that the orbit $\cup_{t\geq 0}\phi(t)$ is relatively compact in Z_{δ_0}. Hence, the omega limit set $\omega(\phi_0)$ is invariant, non-empty, compact and connected in Z_{δ_0} by Theorem 5.1.8 in [15]. Moreover by Theorem 7.6.1 in [15] together with both the existence of Lyapunov function $E(\phi)$ and the precompactness of the orbit, we have

$$\lim_{t\to+\infty} \|u_t(\,\cdot\,,t)\|_2 = 0.$$

Finally we reach

$$\omega(\phi_0) = \left\{(\eta,0) \mid \text{there exist } \eta \in \mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \text{ and } t_n \to \infty \text{ such that (14) holds}\right\}.$$

$\square$

6 Omega limit set We discuss the convergence of the global solution $u(\,\cdot\,,t)$ to the stationary solution η in the norm of X. We conclude that the omega limit set is composed of a stationary solution η in Z_{δ_0} with (3). For $z = u - \eta$, the method in [14, 15] is valid by existence of Lyapunov function and the precompactness of the orbit in X as proven in Section 5. In the limiting case $\chi = 0$ and $\beta = 0$, the same conclusion is obtained in [13].

Proof of Theorem 6. In this proof, by N_i for $i \in \mathbb{N}$, we denote the positive constant which depends only on the constants λ, β, γ, χ, σ, κ, Ω and $\|\eta\|_X$. Changing variable $z = u - \eta$, we consider

$$\begin{cases} z_{tt} + z_t + Az = f(\beta,z,\eta) + g(\lambda,\sigma,\chi,z,\eta) & x\in\Omega,\ t>0,\\ z = \Delta z = 0 & x\in\partial\Omega,\ t>0,\\ z(x,0) = u_0(x) - \eta(x) & x\in\Omega,\\ z_t(x,0) = u_1(x) & x\in\Omega \end{cases}$$

and obtain

$$\lim_{n\to+\infty}\Big(\|z(\,\cdot\,,t_n)\|_X + \|z_t(\,\cdot\,,t_n)\|_2\Big) = 0$$

instead of (1) and (14), where

$$f(\beta,z,\eta) = \beta\left(\int_\Omega |\nabla(z+\eta)|^2\,dx\right)\Delta(z+\eta) - \beta\left(\int_\Omega |\nabla\eta|^2\,dx\right)\Delta\eta$$

and

$$g(\lambda,\sigma,\chi,z,\eta) = \frac{\lambda}{(1-(z+\eta))^\sigma} I(\sigma,\chi,z+\eta) - \frac{\lambda}{(1-\eta)^\sigma} I(\sigma,\chi,\eta),$$

respectively. For the sake of simplicity, we shall write

$$f = f(z,\eta) = f(\beta,z,\eta) \quad \text{and} \quad g = g(z,\eta) = g(\lambda,\sigma,\chi,z,\eta).$$

Defining

$$\begin{aligned} F(z) \;=\; & \frac{1}{2}\int_\Omega \Big((\Delta z)^2 + \gamma|\nabla z|^2\Big)\,dx + \frac{\beta}{4}\left(\int_\Omega |\nabla(z+\eta)|^2\,dx\right)^2 \\ & -\frac{\beta}{4}\left(\int_\Omega |\nabla\eta|^2\,dx\right)^2 + \beta\left(\int_\Omega |\nabla\eta|^2\,dx\right)\left(\int_\Omega z\Delta\eta\,dx\right) \\ & +\lambda\int_\Omega \frac{z}{(1-\eta)^\sigma}\,dx\, I(\sigma,\chi,\eta) \\ & +\frac{\lambda}{(\sigma-1)^2\chi}\Big(H(\sigma,\chi,z+\eta)^{1-\sigma} - H(\sigma,\chi,\eta)^{1-\sigma}\Big) \end{aligned}$$

and

$$G(t)=\frac{1}{2}\int_{\Omega}z_t^2\,dx+F(z(t))+\varepsilon\Big(Az-f\left(z,\eta\right)-g\left(z,\eta\right),z_t\Big)_{H^{-2}},$$

where $\varepsilon>0$ is a small constant to be determined later, we have

$$\frac{d}{dt}F(z(t))=-\frac{1}{2}\frac{d}{dt}\int_{\Omega}z_t^2\,dx-\int_{\Omega}z_t^2\,dx$$

and

$$\begin{aligned}G'(t)=&-\int_{\Omega}z_t^2\,dx-\varepsilon\Big(Az-f\left(z,\eta\right)-g\left(z,\eta\right),z_t\Big)_{H^{-2}}\\&-\varepsilon\left\|Az-f\left(z,\eta\right)-g\left(z,\eta\right)\right\|_{H^{-2}}^2\\&+\varepsilon\Big(Az_t-f_z\left(z,\eta\right)z_t-g_z\left(z,\eta\right)z_t,z_t\Big)_{H^{-2}},\end{aligned}$$

where f_z and g_z are linearized operators from $L^2(\Omega)$ to $H^{-2}(\Omega)$ given by

$$f_z\left(z,\eta\right)w=2\beta\left(\int_{\Omega}\nabla(z+\eta)\cdot\nabla w\,dx\right)\Delta(z+\eta)+\beta\left(\int_{\Omega}\left|\nabla(z+\eta)\right|^2\,dx\right)\Delta w$$

and

$$\begin{aligned}g_z\left(z,\eta\right)w=&\frac{\lambda\sigma w}{\left(1-(z+\eta)\right)^{\sigma+1}}I\left(\sigma,\chi,z+\eta\right)\\&-\frac{\lambda\sigma\left(\sigma-1\right)\chi}{\left(1-(z+\eta)\right)^{\sigma}}H\left(\sigma,\chi,z+\eta\right)^{-\sigma-1}\int_{\Omega}\frac{w}{\left(1-(z+\eta)\right)^{\sigma}}\,dx,\end{aligned}$$

respectively. Then the Young and Hölder inequalities yield

$$\begin{aligned}G'(t)\ \le\ &-\left\|z_t\right\|_2^2-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2+\frac{\varepsilon}{2}\left\|z_t\right\|_{H^{-2}}^2\\&+\varepsilon\left\|z_t\right\|_2^2+\varepsilon\gamma\left\|z_t\right\|_2\left\|z_t\right\|_{H^{-2}}-\varepsilon\Big(f_zz_t+g_zz_t,z_t\Big)_{H^{-2}}\\\le\ &\left(\varepsilon N_1-1\right)\left\|z_t\right\|_2^2-\varepsilon\Big(f_zz_t+g_zz_t,z_t\Big)_{H^{-2}}-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2.\end{aligned}$$

Since we estimate the linearized operators as

$$\left|\Big(f_z\left(z,\eta\right)w,w\Big)_{H^{-2}}\right|\le\beta\Big(2\left\|z+\eta\right\|_X\left\|z+\eta\right\|_2+\left\|z+\eta\right\|_{H_0^1}^2\Big)\left\|w\right\|_2\left\|w\right\|_{H^{-2}}$$

and

$$\left|\Big(g_z\left(z,\eta\right)w,w\Big)_{H^{-2}}\right|\le\frac{N_2\left\|w\right\|_2^2}{\left(1-\left\|z+\eta\right\|_C\right)^{\sigma+1}}+\frac{N_2\left\|w\right\|_2^2}{\left(1-\left\|z+\eta\right\|_C\right)^{2\sigma}},$$

respectively, thanks to (8) and (9), we can take sufficiently small $\varepsilon>0$ so that the following estimate is valid:

$$\begin{aligned}G'(t)\ \le\ &\left(\varepsilon N_3-1\right)\left\|z_t\right\|_2^2-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2\\\le\ &-2N_4\Big(\left\|z_t\right\|_2^2+\left\|Az-f-g\right\|_{H^{-2}}^2\Big)\\\le\ &-N_4\Big(\left\|z_t\right\|_2+\left\|Az-f-g\right\|_{H^{-2}}\Big)^2\qquad(17)\end{aligned}$$

for $t\ge0$. Hence since $G(t)$ is non-increasing in $t\ge0$ and $(0,0)\in\omega(u_0-\eta,u_1)$, we have $G(t)\to0$ as $t\to+\infty$ and $G(t)>0$ for $t\ge0$. As in [13, 14], we can prove the following type of Lojasiewicz-Simon inequality:

Lemma 10 (Theorems 2.2 in [14] and 11.2.7 in [15]) *There exist $\theta \in (0, \frac{1}{2})$ and $\rho > 0$ such that for all $z \in X$ with $\|z\|_X < \rho$, we have*

$$|F(t)|^{1-\theta} \le \|Az - f(z,\eta) - g(z,\eta)\|_{H^{-2}}.$$

This inequality is proven in the same argument as Theorems 2.2 in [14] and 11.2.7 in [15]. For the sake of completeness, we give a sketch of a proof in an appendix. The proof of Theorem 6 is also similar to that of Theorem 1.2 in [14]. For all $t \ge 0$, we have

$$\begin{aligned} -\frac{d}{dt}(G(t))^{\theta} &= -\theta (G(t))^{\theta-1} G'(t) \\ &\ge \theta N_4 (G(t))^{\theta-1} \Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big)^2 \end{aligned} \tag{18}$$

by (17). Now that $\lim_{t\to+\infty} \|z_t\|_2 = 0$ holds, there exists sufficiently large $T > 0$ such that we may suppose that $\|z_t\|_2 \le 1$ as long as $t \ge T$. Noting that $1/2 < 1-\theta < 1$ and that $1 < (1-\theta)/\theta$ for $\theta \in (0, 1/2)$, we have

$$\begin{aligned} (G(t))^{1-\theta} &\le \frac{1}{2^{1-\theta}} \|z_t\|_2^{2(1-\theta)} + |F(t)|^{1-\theta} + \varepsilon^{1-\theta} \|Az - f - g\|_{H^{-2}}^{1-\theta} \|z_t\|_{H^{-2}}^{1-\theta} \\ &\le \frac{1}{2^{1-\theta}} \|z_t\|_2^{2(1-\theta)} + |F(t)|^{1-\theta} \\ &\quad + \varepsilon^{1-\theta}(1-\theta) \|Az - f - g\|_{H^{-2}} + \varepsilon^{1-\theta}\theta \|z_t\|_{H^{-2}}^{\frac{1-\theta}{\theta}} \\ &\le N_5 \Big(\|z_t\|_2 + |F(t)|^{1-\theta} + \|Az - f - g\|_{H^{-2}} \Big) \end{aligned} \tag{19}$$

for $t \ge T$. For any $0 < \xi < \rho$, there exists $N \in \mathbb{N}$ such that $t_n > T$ satisfying

$$\|z(\,\cdot\,, t_n)\|_2 < \frac{1}{2}\xi, \quad (G(t_n))^{\theta} < \frac{\theta N_4}{4N_5}\xi \quad \text{and} \quad \|z(\,\cdot\,, t_N)\|_X < \frac{1}{2}\xi$$

for all $n \ge N$. Let

$$\bar{t} = \sup\{t \ge t_N \mid \|z(\,\cdot\,, s)\|_X < \rho \text{ for all } s \in [t_N, t]\}.$$

Hence for all $t \in [t_N, \bar{t}]$, (18) becomes

$$-\frac{d}{dt}(G(t))^{\theta} \ge \frac{\theta N_4}{2N_5}\Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big) \ge \frac{\theta N_4}{2N_5} \|z_t\|_2$$

owing to

$$(G(t))^{1-\theta} \le 2N_5 \Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big)$$

by (19) together with Lemma 10. By integrating this inequality over $[t_N, \bar{t}]$, we obtain

$$\int_{t_N}^{\bar{t}} \|z_t(\,\cdot\,, s)\|_2 \, ds \le \frac{2N_5}{\theta N_4} (G(t_N))^{\theta} < \frac{1}{2}\xi.$$

Claim 1

$$\bar{t} = +\infty$$

holds.

Proof. If $\bar{t} < +\infty$, we have

$$\begin{aligned}
\left\|z(\,\cdot\,,\bar{t})\right\|_2 &= \int_{t_N}^{\bar{t}} \left(\frac{d}{dt} \|z(\,\cdot\,,s)\|_2 \right) ds + \|z(\,\cdot\,,t_N)\|_2 \\
&\leq \int_{t_N}^{\bar{t}} \left(\|z(\,\cdot\,,s)\|_2^{-1} \int_\Omega |z(x,s)|\,|z_t(x,s)|\; dx \right) ds + \|z(\,\cdot\,,t_N)\|_2 \\
&\leq \int_{t_N}^{\bar{t}} \|z_t(\,\cdot\,,s)\|_2\; ds + \|z(\,\cdot\,,t_N)\|_2 \\
&< \xi.
\end{aligned}$$

From the compactness of $z(t)$ in X, we can choose $\xi > 0$ sufficiently small to obtain $\left\|z(\,\cdot\,,\bar{t})\right\|_X < \rho$, which contradicts the definition of $\bar{t}$. □

Since the claim is shown,

$$\lim_{t\to+\infty} \|z(\,\cdot\,,t)\|_2 \leq \int_{t_N}^{+\infty} \|z_t(\cdot\,,s)\|_2\; ds + \|z(\,\cdot\,,t_N)\|_2 < \xi,$$

which implies the convergence of z in X. □

A Proof of Lemma 10 In this section, we sketch the proof of the Lojasiewicz-Simon inequality. If we establish Lemma 11, we can follow the argument in [14, 15]. As in our problem, the Lojasiewicz-Simon inequality can be applicable to the convergence problem in infinite dimensions We remark that the lemma is also proven by [5, 6] and applied to [9].

Sketch of the proof of Lemma 10. Let

$$\mathcal{M}z = Az - f(z,\eta) - g(z,\eta).$$

We prepare an orthogonal projection. We denote the i-th eigenpair of A by (μ_i, φ_i), where $\{\varphi_i\}_{i\in\mathbb{N}}$ is a set of orthonormal eigenfunctions in $L^2(\Omega)$. We define by W_k the vector space spanned by $\varphi_1, \varphi_2, \ldots, \varphi_k$. Let

$$Q_k : L^2(\Omega) \to W_k$$

be the orthogonal projection onto W_k. For all $k \in \mathbb{N}$, we have

$$\begin{aligned}
(Az + \mu_k Q_k z, z) &= \frac{1}{2}(Az,z) + \frac{1}{2}(Az,z) + \mu_k (Q_k z, z) \\
&\geq \frac{1}{2}\|z\|_X^2 + \frac{\gamma}{2}\|z\|_{H_0^1}^2 + \frac{\mu_k}{2}\|z - Q_k z\|_2^2 + \mu_k \|Q_k z\|_2^2 \\
&\geq \frac{1}{2}\|z\|_X^2 + \frac{\gamma}{2}\|z\|_{H_0^1}^2 + \frac{\mu_k}{2}\left(\|z - Q_k z\|_2^2 + \|Q_k z\|_2^2\right) \\
&\geq \frac{1}{2}\|z\|_X^2 + \frac{\gamma}{2}\|z\|_{H_0^1}^2 + \frac{\mu_k}{4}\|z\|_2^2.
\end{aligned}$$

Putting

$$\mathcal{L} = A - f_z(0,\eta) - g_z(0,\eta) + \mu_k Q_k,$$

we obtain

$$\begin{aligned}(\mathcal{L}z,z) \geq\ & \frac{1}{2}\|z\|_X^2+\frac{\gamma}{2}\|z\|_{H_0^1}^2+\frac{\mu_k}{4}\|z\|_2^2+\beta\|\eta\|_{H_0^1}^2\|z\|_{H_0^1}^2\\ &+2\beta\left(\int_\Omega \nabla z\cdot\nabla\eta\,dx\right)^2-\lambda\sigma I(\sigma,\chi,\eta)\int_\Omega\frac{1}{(1-\eta)^{\sigma+1}}z^2\,dx\\ &+\lambda\sigma(\sigma-1)\chi H(\sigma,\chi,\eta)^{-\sigma-1}\left(\int_\Omega\frac{z}{(1-\eta)^\sigma}\,dx\right)^2\\ \geq\ & \frac{1}{2}\|z\|_X^2+\left(\frac{\mu_k}{4}-\frac{\lambda\sigma}{(1-\|\eta\|_C)^{\sigma+1}}\right)\|z\|_2^2.\end{aligned}$$

Now we take k so large that the inequality

$$\mu_k>\frac{4\lambda\sigma}{(1-\|\eta\|_C)^{\sigma+1}}$$

is satisfied. Hence $\mathcal{L}$ is coercive and bijective from X to $H^{-2}(\Omega)$. Let

$$\mathcal{N}=\mu_k Q_k+\mathcal{M}.$$

$\mathcal{N}$ is a C^1 diffeomorphism in the neighbourhood of $0\in X$ to $H^{-2}(\Omega)$ and its derivative at 0 is $\mathcal{L}$. The inversion theorem implies the following lemma:

Lemma 11 (Lemmas 2.6 in [14] and 11.2.8 in [15]) *There exist a neighbourhood V_1 of 0 in X, V_2 of 0 in $H^{-2}(\Omega)$, $C_1>0$ and $C_2>0$ such that*

$$\|\mathcal{M}z-\mathcal{M}w\|_{H^{-2}}\leq C_1\|z-w\|_X$$

for all z, $w\in V_1$ and

$$\left\|\mathcal{N}^{-1}(f)-\mathcal{N}^{-1}(g)\right\|_X\leq C_2\|f-g\|_{H^{-2}}$$

for all f, $g\in V_2$.

Owing to Lemma 11, the same estimates as in [14, 15] hold, which yields the conclusion. □

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(73 − age) × ¥3,000

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