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DEFORMATIONS OF THE CHEBYSHEV HYPERGROUPS

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Abstract. In the present paper we introduce q-deformations of the Chebyshev hypergroups of the first kind and of the second kind as models of q-deformations of countable discrete hypergroups. Moreover we study q-deformations of character hypergroups $\mathcal{K}(\hat{G})$ of certain compact groups G.

1 Introduction

The notion of compact quantum groups is introduced in [14] and [15] by S. L. Woronowicz. Especially, he studied the structure of $SU_q(2)$ which is obtained by a q-deformation of $SU(2)$ in the category of Hopf algebras. Compact quantum groups play an important role not only in mathematics but also in theoretical physics.

Deformations of groups and hypergroups are investigated in [16] by K. A. Ross and D. Xu and our previous paper [6] in the category of hypergroups. Many new hypergroups are produced by deforming groups and hypergroups. The notion of q-deformations of groups and hypergroups is one of the way to understand hypergroup structures.

The structure of countable discrete hypergroups arising from orthogonal polynomials has been studied by many authors (for example [7], [8] and [9]). But there is no notion of q-deformations of countable discrete hypergroups in the category of hypergroups. In the present paper, we consider q-deformations of countable discrete commutative hypergroups, mainly of the Chebyshev hypergroups $\mathcal{T}$ of the first kind and $F_d(\mathcal{U})$ of the second kind. In the present paper the q-deformation K_q of a countable discrete hypergroup K is to deform continuously structures of K by a parameter q $(0 < q \leq 1)$ and $K_1 = K$ in the category of hypergroups. A notion of dimension functions of countable discrete hypergroups and of fusion rule algebras plays an essential role in our discussions.

In section 3, we consider dimension functions of countable discrete hypergroups as well as of fusion rule algebras. In section 4, we discuss q-deformations $\mathcal{T}_q$ of the Chebyshev hypergroup $\mathcal{T}$ of the first kind. Moreover we consider q-deformations $\mathcal{K}_q(\hat{G})$ of a character hypergroup $\mathcal{K}(\hat{G})$ of the compact group $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}_2$ as an application of q-deformations of $\mathcal{T}$. In section 5, we discuss q-deformations $\mathcal{U}_q$ of the Chebyshev hypergroup $F_d(\mathcal{U})$ of the second kind which is obtained by normalization of the fusion rule algebra $\mathcal{U}$ by the dimension function d of $\mathcal{U}$. Moreover we investigate q-deformations $\mathcal{K}_q(\widehat{SU(2)})$ of a character hypergroup $\mathcal{K}(\widehat{SU(2)})$ of the compact Lie group $SU(2)$.

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2 Preliminary

For a countable discrete set $K = \{X_0, X_1, X_2, \cdots, X_n, \cdots\}$ we denote the algebraic complex linear space based on K by $\mathbb{C}K$, namely

$$\mathbb{C}K = \left\{ X = \sum_{k=0}^{\infty} a_k X_k : a_k \in \mathbb{C},\ |\mathrm{supp}(X)| < +\infty \right\},$$

where $|\mathrm{supp}(X)|$ is the cardinal number of $\mathrm{supp}(X)$ and the *support* of X is

$$\mathrm{supp}(X) := \{k : a_k \neq 0\}.$$

A *countable discrete hypergroup* $(K, \mathbb{C}K, \circ, *)$ consists of the set $K = \{X_0, X_1, \cdots, X_n, \cdots\}$ together with a product (called convolution) $\circ$ and an involution $*$ in the complex linear space $\mathbb{C}K$ satisfying the following conditions.

(1) For $X_m, X_n \in K$, the convolution $X_m \circ X_n$ belongs to $\mathbb{C}K$ and

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k,$$

where

$$S(m,n) := \mathrm{supp}(X_m \circ X_n),\ a_{mn}^k \geq 0 \text{ and } \sum_{k \in S(m,n)} a_{mn}^k = 1.$$

(2) The space $(\mathbb{C}K, \circ, *)$ is an associative $*$-algebra with unit X_0.

(3) The map $X_n \mapsto X_n^*$ is a bijection on K. Moreover for all $X_m, X_n \in K$, $X_n = X_m^*$ if and only if $0 \in \mathrm{supp}(X_m \circ X_n)$.

We denote the hypergroup $(K, \mathbb{C}K, \circ, *)$ by K. A hypergroup K is called *commutative* if the convolution $\circ$ on $\mathbb{C}K$ is commutative and be called *hermitian* if $X_n^* = X_n$.

If the given countable discrete hypergroup K is commutative, its dual $\hat{K}$ can be introduced as the set of all bounded functions $\chi \neq 0$ on $\mathbb{C}K$ satisfying

$$\chi(X_m \circ X_n) = \chi(X_m)\chi(X_n), \quad \chi(X_n^*) = \overline{\chi(X_n)}$$

for all $X_i, X_j \in K$. This set of characters $\hat{K}$ of K becomes a compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup. If $\hat{K}$ is a hypergroup, then K is called a strong hypergroup or a hypergroup of strong type.

Let G be a compact group and $\hat{G}$ the set of all equivalence classes of irreducible representations of G. Put

$$\mathcal{K}(\hat{G}) := \{ch(\pi) : \pi \in \hat{G}\},$$

where

$$ch(\pi)(g) := \frac{1}{\dim \pi} \mathrm{tr}(\pi(g)) \quad (g \in G).$$

Then $\mathcal{K}(\hat{G})$ always becomes a discrete commutative hypergroup which is called the character hypergroup of G. (Refer to [1] for details.)

Let $K = (K, \mathbb{C}K, \circ, *)$ be a countable discrete hypergroup where $K = \{X_0, X_1, \cdots, X_n, \cdots\}$. For q $(0 < q \leq 1)$, put $K_q = \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}$ a new basis in $\mathbb{C}K$. Then the convolution $X_m(q)$ and $X_n(q)$ of $\mathbb{C}K$ is defined by

$$X_m(q) \circ X_n(q) := \sum_{k=0}^{\infty} a_{mn}^k(q) X_k(q)$$

where $a_{mn}^k(q)$ is continuous with respect to q. The involution $*$ of K_q is given by

$$X_n(q)^* = X_m(q)^* \quad \text{when} \quad X_n^* = X_m.$$

For the hypergroup $K_q = (K_q, \mathbb{C}K, \circ, *)$ satisfies the following conditions

$$X_n(1) = X_n \ \text{ and } \ X_n(q) \to X_n \ \text{ as } \ q \to 1,$$

we call K_q a q-deformation of K.

A *fusion rule algebra* $(F, \mathbb{C}F, \diamond, ^-)$ consists of the set $F = \{Y_0, Y_1, \cdots, Y_n, \cdots\}$ together with a product (called convolution) $\diamond$ and an involution $^-$ in the complex linear space $\mathbb{C}F$ based on F satisfying the following conditions.

(1) For $Y_m, Y_n \in F$, the convolution $Y_m \diamond Y_n$ belongs to $\mathbb{C}F$ and

$$Y_m \diamond Y_n = \sum_{k \in S(m,n)} a_{mn}^k Y_k \quad (a_k \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}),$$

$$Y_n^- \diamond Y_n = Y_0 + \sum_{\substack{k \in S(m,n) \\ k \neq 0}} a_{mn}^k Y_k.$$

where $S(m,n) := \text{supp}(Y_m \diamond Y_n)$.

(2) The space $(\mathbb{C}F, \diamond, ^-)$ is an associative involutive algebra with unit Y_0.

We denote the fusion rule algebra $(F, \mathbb{C}F, \diamond, ^-)$ by F.

For the dual $\hat{G}$ of a compact group G, put

$$\mathcal{F}(\hat{G}) := \{Ch(\pi) : \pi \in \hat{G}\},$$

where

$$Ch(\pi)(g) := \text{tr}(\pi(g)) \quad (g \in G).$$

Then $\mathcal{F}(\hat{G})$ always becomes a fusion rule algebra.

3 A dimension function

In this section, we discuss a dimension function of a countable discrete hypergroup and a fusion rule algebra.

For a countable discrete hypergroup K, the mapping d from K to $\mathbb{R}_+^\times = \{x \in \mathbb{R} : x > 0\}$ is called a *dimension function* of K if d is a homomorphism in the sense that

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k \Rightarrow d(X_m) d(X_n) = \sum_{k \in S(m,n)} a_{mn}^k d(X_k).$$

The dimension function d of K is uniquely extendable as a linear mapping from $\mathbb{C}K$ to $\mathbb{C}$ and satisfies

$$d(X_m \circ X_n) = d(X_m)d(X_n).$$

Proposition 3.1 Let K be a countable discrete hypergroup where $K = \{X_0, X_1, \cdots, X_n, \cdots\}$. For the dimension function d of K, put

$$c_n := \frac{1}{d(X_n)} X_n \quad \text{and} \quad K_d := \{c_0, c_1, \cdots, c_n, \cdots\}.$$

Then K_d is a hypergroup.

Proof By the axiom (1) of a countable discrete hypergroup, the structure equation of K is written by

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k.$$

Hence, the structure equation of K_d is

$$\begin{aligned} c_m \circ c_n &= \frac{1}{d(X_m)d(X_n)} X_m \circ X_n \\ &= \frac{1}{d(X_m)d(X_n)} \sum_{k \in S(m,n)} a_{mn}^k X_k \\ &= \sum_{k \in S(m,n)} \frac{a_{mn}^k d(X_k)}{d(X_m)d(X_n)} c_k. \end{aligned}$$

Here we note that

$$\sum_{k \in S(m,n)} \frac{a_{mn}^k d(X_k)}{d(X_m)d(X_n)} = 1$$

by the fact

$$d(X_m)d(X_n) = \sum_{k \in S(m,n)} a_{mn}^k d(X_k).$$

It is clear that the coefficients of the convolution $c_m \circ c_n$ are non-negative. It is easy to check other conditions of axiom of a countable discrete hypergroup. Hence, K_d is a countable discrete hypergroup. □

Remark If K is a finite hypergroup, the dimension function d of K is known to be unique such that $d(X_k) = 1$ for all $X_k \in K$.

For a fusion rule algebra F, the dimension function d of F is defined in a similar way to the above.

Proposition 3.2 Let F be a fusion rule algebra where $F = \{Y_0, Y_1, \cdots, Y_n, \cdots\}$. For the dimension function d of a fusion rule algebra F, put

$$b_n := \frac{1}{d(Y_n)} Y_n \quad \text{and} \quad F_d := \{b_0, b_1, \cdots, b_n, \cdots\}.$$

Then F_d becomes a hypergroup.

Proof The desired assertion is obtained in a similar way to the proof of Proposition 3.1. □

4 Q-deformations of the Chebyshev hypergroup of the first kind

Let $T_n(x)$ be the Chebyshev polynomial of the first kind of degree n, then $T_n(x)$ $(n = 0, 1, 2, \cdots)$ satisfy the following equation.

$$T_m(x)T_n(x) = \frac{1}{2}T_{|m-n|}(x) + \frac{1}{2}T_{m+n}(x).$$

Then, for the set $\mathcal{T} = \{T_0, T_1, \cdots, T_n, \cdots\}$, $(\mathcal{T}, \mathbb{C}\mathcal{T}, \circ, *)$ is a countable discrete hypergroup by the product

$$T_m \circ T_n := T_m(x)T_n(x).$$

The hypergroup $\mathcal{T} = (\mathcal{T}, \mathbb{C}\mathcal{T}, \circ, *)$ is called the Chebyshev hypergroup of the first kind.

Next, we consider a mapping d_q from $\mathcal{T}$ to $\mathbb{R}_+^\times$. For $T_n \in \mathcal{T}$ the mapping d_q defined by

$$d_q(T_n) := T_n\left(\frac{q+q^{-1}}{2}\right) = \frac{q^n + q^{-n}}{2} \geq 1.$$

Proposition 4.1 The mapping d_q from $\mathcal{T}$ to $\mathbb{R}_+^\times$ is a dimension function of $\mathcal{T}$.

Proof Put $x(q) := \frac{q+q^{-1}}{2}$ and $d_q(T_n) = T_n(x(q))$. Then

$$\begin{aligned} d_q(T_m)d_q(T_n) &= T_m(x(q))T_n(x(q)) \\ &= \frac{1}{2}T_{|m-n|}(x(q)) + \frac{1}{2}T_{m+n}(x(q)) \\ &= \frac{1}{2}d_q(T_{|m-n|}) + \frac{1}{2}d_q(T_{m+n}) \end{aligned}$$

Hence, d_q is a dimension function of $\mathcal{T}$. □

Next, put

$$X_n(q) := \frac{1}{d_q(T_n)}T_n \quad \text{and} \quad \mathcal{T}_q := \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}.$$

Then the following theorem holds.

Theorem 4.2 $\mathcal{T}_q$ becomes a hypergroup which is a q-deformation of $\mathcal{T}$. The structure equation is

$$X_m(q) \circ X_n(q) = \sum_{k \in S(m,n)} a_{mn}^k(q) X_k(q)$$

where $S(m,n) = \{|m-n|, m+n\}$ and

$$a_{mn}^k(q) = \frac{q^k + q^{-k}}{(q^m + q^{-m})(q^n + q^{-n})}.$$

Proof By Proposition 3.1, $\mathcal{T}_q$ is a hypergroup. The convolution $X_m(q)$ and $X_n(q)$ is

$$X_m(q) \circ X_n(q) = \frac{1}{d_q(T_m)d_q(T_n)}T_m \circ T_n = \sum_{k \in S(m,n)} \frac{d_q(T_k)}{2d_q(T_m)d_q(T_n)}X_k(q)$$

where $S(m,n) = \{|m-n|, m+n\}$. Then

$$a_{mn}^k(q) = \frac{d_q(T_k)}{2d_q(T_m)d_q(T_n)} = \frac{q^k + q^{-k}}{(q^m + q^{-m})(q^n + q^{-n})}.$$

Hence, we see that $a_{mn}^k(q)$ is continuous with respect to q. The involution $*$ of $\mathcal{T}_q$ is an identity map by the fact that

$$X_n(q) = \frac{1}{d_q(T_n)} T_n \quad \text{and} \quad T_n^* = T_n.$$

When $q = 1$, it is clear that $X_n(1) = T_n$. Since $d_q(T_n) = \frac{q^n + q^{-n}}{2}$ is continuous, $X_n(q) \to T_n$ as $q \to 1$. Hence, $\mathcal{T}_q$ is a q-deformation of $\mathcal{T}$. $\square$

Let α be an action of $\mathbb{Z}_2 = \{e, g\}$ $(g^2 = e)$ on the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ defined by

$$\alpha_g(z) = \overline{z}.$$

Then we have a compact group $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}_2$ in the form of a semi-direct product. We consider q-deformations of the character hypergroup $\mathcal{K}(\hat{G})$ of $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}_2$. The dual of $\hat{\mathbb{T}}$ of $\mathbb{T}$ and $\widehat{\mathbb{Z}_2}$ of $\mathbb{Z}_2$ are given by

$$\begin{aligned} &\hat{\mathbb{T}} = \{\chi_n : n \in \mathbb{Z}\}, \text{ where } \chi_n(z) = z^n \quad \text{for } z \in \mathbb{T}, \\ &\widehat{\mathbb{Z}_2} = \{\tau_0, \tau_1\}, \text{ where } \tau_1^2 = \tau_0. \end{aligned}$$

For $\chi \in \hat{\mathbb{T}}$ and $\tau \in \widehat{\mathbb{Z}_2}$, the irreducible representations ρ_0, ρ_1 and π_n $(n = 1, 2, \cdots)$ of $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}_2$ are written by

$$\begin{aligned} &\rho_0((z,h)) = \tau_0(h) = 1, \quad \rho_1((z,h)) = \tau_1(h), \\ &\pi_n = \mathrm{ind}_{\mathbb{T}}^G \chi_n \quad (n = 1, 2, \cdots). \end{aligned}$$

Then the dual $\hat{G}$ of G is determined by $\hat{G} = \{\rho_0, \rho_1, \pi_1, \pi_2, \cdots, \pi_n, \cdots\}$ by Mackey Machine. For the irreducible representations π_n $(n = 1, 2, \cdots)$, put

$$Ch(\pi_n)(g) := \mathrm{tr}(\pi_n(g)) \quad (g \in G)$$

and

$$\mathcal{F}(\hat{G}) := \{\rho_0, \rho_1, Ch(\pi_1), Ch(\pi_2), \cdots, Ch(\pi_n), \cdots\}.$$

Then $\mathcal{F}(\hat{G})$ becomes a fusion rule algebra with unit ρ_0. The structure equations are

$$\begin{aligned} &\rho_1^2 = \rho_0, \quad \rho_1 Ch(\pi_n) = Ch(\pi_n), \\ &Ch(\pi_m)Ch(\pi_n) = Ch(\pi_{|m-n|}) + Ch(\pi_{m+n}) \quad (m \neq n), \\ &Ch(\pi_n)^2 = \rho_0 + \rho_1 + Ch(\pi_{2n}). \end{aligned}$$

Moreover, put

$$ch(\pi_n) := \frac{1}{\dim \pi_n} Ch(\pi_n) = \frac{1}{2} Ch(\pi_n)$$

and

$$\mathcal{K}(\hat{G}) = \{\rho_0, \rho_1, ch(\pi_1), ch(\pi_2), \cdots, ch(\pi_n), \cdots\}.$$

Then $\mathcal{K}(\hat{G})$ becomes a hypergroup with unit ρ_0 by Proposition 3.2. This hypergroup $\mathcal{K}(\hat{G})$ is the character hypergroup of G. The hypergroup structure of $\mathcal{K}(\hat{G})$ is the hypergroup join of $\mathbb{Z}_2$ by $\mathcal{T}$ which is written by

$$\mathcal{K}(\hat{G}) = \mathbb{Z}_2 \vee \mathcal{T}.$$

Hence, we obtain a q-deformation $\mathcal{K}_q(\hat{G})$ of the countable discrete hypergroup $\mathcal{K}(\hat{G})$ as follows.

Theorem 4.3 The hypergroup $\mathcal{K}_q(\hat{G}) = \mathbb{Z}_2 \vee \mathcal{T}_q$ is a q-deformation of $\mathcal{K}(\hat{G})$.

The hypergroups $\mathcal{T}$ and $\mathcal{K}(\hat{G})$ are strong hypergroup. Since $\hat{\mathcal{T}} = \mathcal{K}^\alpha(\mathbb{T})$ and $\widehat{\mathcal{K}(\hat{G})} = \mathcal{K}(G)$ where $\mathcal{K}^\alpha(\mathbb{T})$ is the orbital hypergroup of the action α of $\mathbb{Z}_2$ on $\mathbb{T}$ and $\mathcal{K}(G)$ is the conjugacy class hypergroup of G.

Conjecture When $q \neq 1$, the hypergroups $\mathcal{T}_q$ and $\mathcal{K}_q(\hat{G})$ are not strong.

5 Q-deformations of the Chebyshev hypergroup of the second kind

Let $U_n(x)$ be the Chebyshev polynomial of the second kind of degree n, then $U_n(x)$ $(n = 0, 1, 2, \cdots)$ satisfy the following equation.

$$U_m(x)U_n(x) = U_{|m-n|}(x) + U_{|m-n|+2}(x) + \cdots + U_{m+n}(x).$$

Hence, for the set $\mathcal{U} = \{U_0, U_1, \cdots, U_n, \cdots\}$, $(\mathcal{U}, \mathbb{C}\mathcal{U}, \diamond, ^-)$ has the structure of a fusion rule algebra by the product

$$U_m \diamond U_n := U_m(x)U_n(x).$$

The canonical dimension function d of $\mathcal{U}$ is given by

$$d(U_n) = n + 1.$$

Put

$$c_n := \frac{1}{d(U_n)} U_n \quad \text{and} \quad F_d(\mathcal{U}) := \{c_0, c_1, \cdots, c_n, \cdots\}.$$

Then $F_d(\mathcal{U})$ becomes a hypergroup by the product

$$c_m \circ c_n := \frac{1}{d(U_m)} U_m \diamond \frac{1}{d(U_n)} U_n.$$

This hypergroup is called the Chebyshev hypergroup of the second kind. The structure equation is

$$\begin{aligned} c_m \circ c_n = & \frac{|m-n|+1}{(m+1)(n+1)} c_{|m-n|} + \frac{|m-n|+3}{(m+1)(n+1)} c_{|m-n|+2} + \cdots \\ & + \frac{m+n+1}{(m+1)(n+1)} c_{m+n}, \end{aligned}$$

where c_0 is the unit element and $c_n^* = c_n$.

Next, we consider a mapping d_q from $\mathcal{U}$ to $\mathbb{R}_+^\times$. For $U_n \in \mathcal{U}$, the mapping d_q defined by

$$d_q(U_n) := U_n\left(\frac{q+q^{-1}}{2}\right) = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = q^n + q^{n-2} + \cdots + q^{-n} \quad (0 < q \leq 1).$$

Proposition 5.1 The mapping d_q from $\mathcal{U}$ to $\mathbb{R}_+^\times$ is a dimension function of $\mathcal{U}$.

Proof The proof is obtained in a similar way to Proposition 4.1. □

Put
$$X_n(q) := \frac{1}{d_q(U_n)} U_n \quad \text{and} \quad \mathcal{U}_q := \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}.$$
Then the following theorem holds.

Theorem 5.2 $\mathcal{U}_q$ becomes a hypergroup which is a q-deformation of $F_d(\mathcal{U})$. The structure equation is

$$X_m(q) \circ X_n(q) = \sum_{k \in S(m,n)} a_{mn}^k(q) X_k(q)$$

where $S(m,n) = \{|m-n|, |m-n|+2, \cdots, m+n\}$ and

$$a_{mn}^k(q) = \frac{(q-q^{-1})(q^{k+1} - q^{-(k+1)})}{(q^{m+1} - q^{-(m+1)})(q^{n+1} - q^{-(n+1)})}.$$

Proof By Proposition 3.1, $\mathcal{U}_q$ is a hypergroup. The convolution $X_m(q)$ and $X_n(q)$ is

$$X_m(q) \circ X_n(q) = \frac{1}{d_q(U_m) d_q(U_n)} U_m \diamond U_n = \sum_{k \in S(m,n)} \frac{d_q(U_k)}{d_q(U_m) d_q(U_n)} X_k(q)$$

where $S(m,n) = \{|m-n|, |m-n|+2, \cdots, m+n\}$. Then,

$$a_{mn}^k(q) = \frac{d_q(U_k)}{d_q(U_m) d_q(U_n)} = \frac{(q-q^{-1})(q^{k+1} - q^{-(k+1)})}{(q^{m+1} - q^{-(m+1)})(q^{n+1} - q^{-(n+1)})}.$$

The coefficients $a_{mn}^k(q)$ can also write

$$a_{mn}^k(q) = \frac{(q^k + q^{k-2} + \cdots + q^{-k})}{(q^m + q^{m-2} + \cdots + q^{-m})(q^n + q^{n-2} + \cdots + q^{-n})}.$$

Hence, we see that $a_{mn}^k(q)$ is continuous with respect to q. The involution of $\mathcal{U}_q$ is an identity map by the fact that

$$X_n(q) = \frac{1}{d_q(U_n)} U_n \quad \text{and} \quad U_n^* = U_n.$$

When $q = 1$, it is clear that $X_n(1) = \frac{1}{n+1} U_n$. Since $d_q(U_n) = q^n + q^{n-2} + \cdots + q^{-n}$ is continuous, $X_n(q) \to \frac{1}{n+1} U_n$ as $q \to 1$. Hence, $\mathcal{U}_q$ is a q-deformation of $F_d(\mathcal{U})$. □

Next, we consider the relation with the dual $\widehat{SU(2)} = \{\pi_0, \pi_1, \cdots, \pi_n, \cdots\}$, where

$$\dim \pi_n = n+1 \quad \text{and} \quad \pi_m \otimes \pi_n \cong \pi_{|m-n|} \oplus \pi_{|m-n|+2} \oplus \cdots \oplus \pi_{m+n}.$$

The character ρ_n of $\pi_n \in \widehat{SU(2)}$ is given by

$$\rho_n(g) = \mathrm{tr}(\pi_n(g)) \quad (g \in SU(2)).$$

Then,

$$\rho_m \rho_n = \rho_{|m-n|} + \rho_{|m-n|+2} + \cdots + \rho_{m+n}$$

as a function on $SU(2)$. Put $\mathcal{F}(\widehat{SU(2)}) = \{\rho_0, \rho_1, \cdots, \rho_n, \cdots\}$. Then $\mathcal{F}(\widehat{SU(2)})$ becomes a fusion rule algebra and isomorphic to $\mathcal{U}$.

For the representative element $g = g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SU(2)$ in the conjugacy class of $SU(2)$,

$$\rho_n(g_\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta).$$

Put

$$\chi_n = \frac{1}{d_q(U_n)}\rho_n \quad (0 < q \leq 1)$$

and

$$\mathcal{K}_q(\widehat{SU(2)}) = \{\chi_0, \chi_1, \cdots, \chi_n, \cdots\}.$$

Then, $\mathcal{K}_q(\widehat{SU(2)})$ is a hypergroup which is isomorphic to $\mathcal{U}_q$.

Remark The hypergroups $F_d(\mathcal{U})$ and $\mathcal{K}(\widehat{SU(2)})$ are strong hypergroups.

Conjecture The q-deformations $\mathcal{U}_q$ of $F_d(\mathcal{U})$ and $\mathcal{K}_q(\widehat{SU(2)})$ of $\mathcal{K}(\widehat{SU(2)})$ are not strong when $q \neq 1$.

Conjecture The character hypergroup $\mathcal{K}(\widehat{SU_q(2)})$ of the quantum group $SU_q(2)$ is well defined and

$$\mathcal{K}_q(\widehat{SU(2)}) \cong \mathcal{K}(\widehat{SU_q(2)}).$$

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A ROUTE OPTIMIZATION PROBLEM IN ELECTRICAL PCB INSPECTIONS WITH ALIGNMENT OPERATIONS

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Abstract. This paper considers a route optimization problem in advanced electrical PCB inspections. By considering the constraint that "camera-based alignment of position" needs to be conducted before electrical tests, the PCB inspection route optimization problem (PCBIRP) is modeled as a precedence-constrained traveling salesman problem (PCTSP), especially, as a pickup and delivery traveling salesman problem (PDTSP). Two of mixed 0-1 integer programming problem formulations are proposed. The computational times for the proposed formulations are compared by solving benchmark instances using some of well-known mathematical programming solvers.

1 Introduction Printed circuit boards (PCBs) have been used in almost all electric devices. There are many of previous studies on optimization techniques for PCB manufacturing processes such as assembly operations [1, 5, 11] and drilling processes [2]. On the other hand, optimization techniques for PCB inspections have not been sufficiently developed so far except for some studies on multi-chip module substrate testing [10, 14].

PCB inspections are quite important to enhance the reliability of manufactured PCBs. In addition, since the number of PCBs to be inspected has been recently increasing, the speedup of PCB inspections has become one of the most important issues in the field. In production processes of PCBs, defect generation may arise due to some trouble, which prevents PCBs from working properly. In electrical PCB inspections, all the PCBs arrayed in a plain are visited and tested by an inspection jig in some sequence or order.

Since the inspection time is dependent on the length of traveling (visiting) route of an inspection jig, it is worth finding the best inspection sequence or route in order to reduce the inspection time. On the other hand, the procedure of the camera-based "alignment" of position (hereafter we call it just an *alignment operation*) is additionally needed before electrical tests in recently-developed PCB inspection machines. However, a route optimization problem in such advanced inspections with alignment operations has not been discussed so far, and there has been no article to model the problem using mathematical programming.

This paper is organized as follows: Section 2 reviews an advanced electrical PCB inspection method involving "alignment" operations, and discusses the necessity of route optimization. In Section 3, we model the PCB inspection route optimization problem (PCBIRP) as a class of pickup and delivery traveling salesman problems (PDTSPs) [3, 15] and provide two of mixed 0-1 integer programming problem formulations. In Section 4, numerical experiments are conducted by solving benchmark instances based on real PCB wiring patterns, using some of well-known mathematical programming solvers. Finally, in Section 5, we summarize this paper and discuss future works.

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Key words and phrases. Printed circuit board (PCB), inspection route optimization, pickup and delivery traveling salesman problems, mixed 0-1 integer programming problem, exact solutions.

2 Preliminaries In this section, we review an advanced electrical PCB inspection method and explain the reason why the *alignment operations* via a camera have been recently necessary in PCB inspections. In addition, we discuss the necessity of considering route optimization problems in electrical PCB inspections with alignment operations.

2.1 Electrical test of PCB In production process of PCBs, various wiring patterns are etched on PCBs. When some trouble happens in forming wiring patterns, PCBs may include some defects such as open (disconnection) defects and/or short-circuit defects. PCBs have bulged parts, called *contact pads*, as shown in Figure 1, which are used to electrically inspect PCBs. In some cases, the number of pins is more than 1,000. The diameter of contact pads is about $100 \sim 300\ \mu m$.

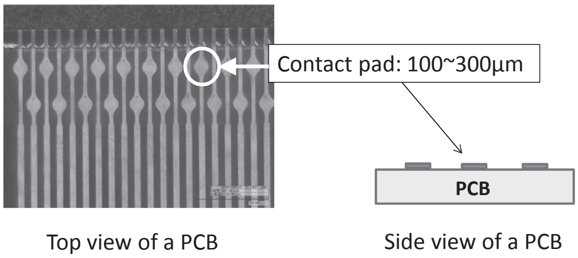

Figure 1: Contact pads in a PCB

In order to electrically test wiring patterns on PCBs, a test jig, called a *probe jig*, is used, as shown on the left-hand side of Figure 2. A probe jig contains many sharp, thin and conductive pins with a diameter of $20{\sim}130\mu m$. The number of pins is usually equal to that of contact pads, and there is a one-to-one correspondence between contact pads and pins.

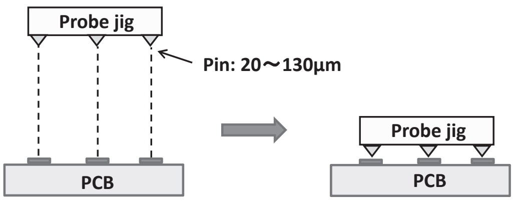

Figure 2: Electrical test via a probe jig

Electrical tests are conducted by pressing a probe jig onto a PCB and by feeding electric currents through pins into contact pads of PCB wirings, as shown on the right-hand side of Figure 2. Electric currents are fed into wiring through pins in order to check if each wiring has defects such as open or short circuits. Therefore, for valid tests, every pin of the probe jig needs to have a contact with the corresponding contact pad.

To exactly conduct the electrical continuity test, each PCB has the so-called “test position.” For valid tests, it is necessary to make the reference point of a probe jig exactly located at the test position of a PCB, as shown on the left-hand side of Figure 3.

To be more specific, when the reference point of a probe jig is precisely placed at the test position of a PCB, each pin of the probe jig has a contact with its corresponding contact pad of the PCB, as shown on the right-hand side of Figure 3.

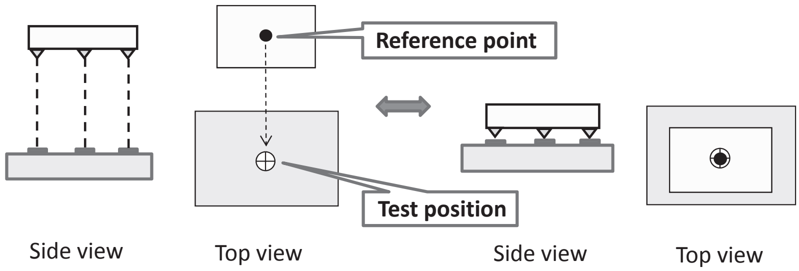

Figure 3: Proper inspection

On the other hand, as shown in Figure 4, when the reference point is not placed at the test position, inspection failures occur. In this case, because some of pins do not have contacts with their corresponding contact pads, electric currents are not fed into wiring lines through the pins, which causes the misjudgment of whether there is a defect or not.

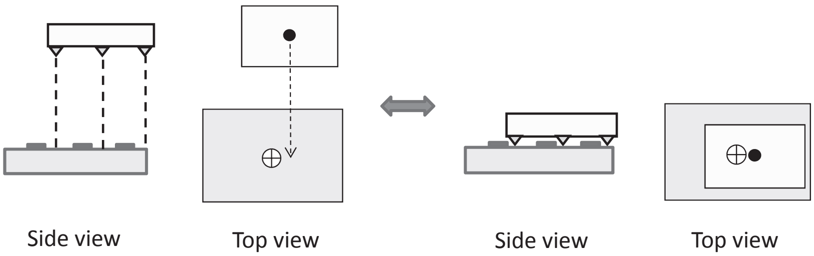

Figure 4: Failed inspection

2.2 Electrical PCB inspection with alignment operations Recently, inspection failures tend to occur more frequently than before, because the miniaturization of electronic devices makes it more difficult to exactly put the reference point of a probe jig onto the test position of a PCB. Since PCB sheets are very thin like pieces of paper, they easily undergo deflections. When there is a deflection in a PCB sheet, the position of the reference point of a probe jig is changed from the regular position, which leads to the situation that some pins of the probe jig do not have contacts with their corresponding contact pads.

In order to deal with such a change of the test position due to the deflection of PCB sheets, the "alignment" operation has been introduced in advanced PCB inspection systems. A camera is attached to the probe jig, and each PCB has one or two alignment marks, as shown in Figure 5 (in this case, the PCB has two alignment marks). We call a probe jig with a camera "a probe unit." The camera is used to capture the images of alignment marks, which acquires the information on the exact coordinate of the test position.

Thus, the advanced PCB inspection method involving alignment operations consists of two steps, *Alignment operation* and *Electrical test*, as follows:

[Procedures of electrical PCB inspections with alignment operations]

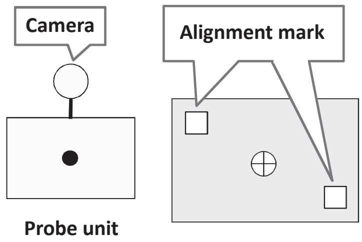

Figure 5: Alignment marks (captured by the camera of a probe unit)

Step 1 (Alignment operation):
Move the camera of the probe unit to the position of an alignment mark so that the camera can capture the image of the alignment mark (cf. the left-hand side of Figure 6).

Step 2 (Electrical test):
Move the reference point of the probe unit to the test position of the PCB so that the probe jig can properly press onto each wiring pattern of the PCB (cf. the right-hand side of Figure 6).

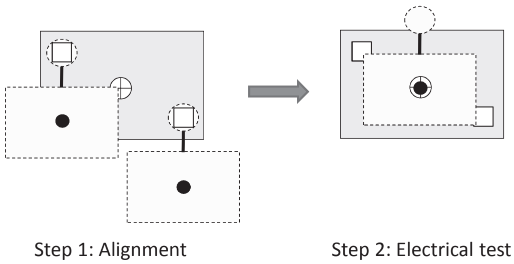

Figure 6: Electrical PCB inspection (alignment operation and electrical test)

2.3 Route optimization Route optimization in PCB inspections is considerably important. Even if the reduced inspection time is several percent, it can bring a great effect on cost reduction as well as on productive efficiency of PCBs. It is because there are a large number of PCB sheets to be inspected in the field, and the number of PCB sheets to be inspected by one machine per day is more than 1,000.

Each of PCB sheets consists of many PCBs with the same wiring patterns which are arrayed in a plane. Figure 7 is a simple example of a PCB sheet which consists of only 4 (2×2) PCBs. In general, the number of the same wiring pattern arrayed in one PCB sheet ranges from 4 to around 200.

Our interest is to find the shortest route for inspecting all the PCBs in the sheet. The route length is dependent on the visiting sequence (order) of the probe jig. Figure 8 shows a simple inspection sequence in which the test position for each PCB is visited immediately after the corresponding alignment marks are visited in order. It should be noted here that each test position does not need to be visited immediately after the corresponding alignment marks are visited.

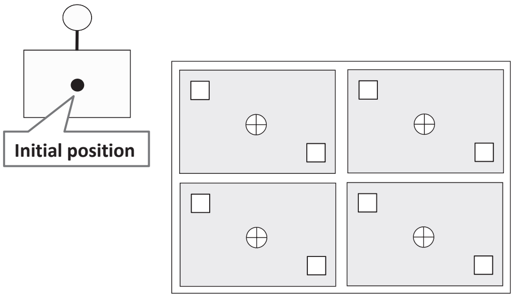

Figure 7: PCB sheet

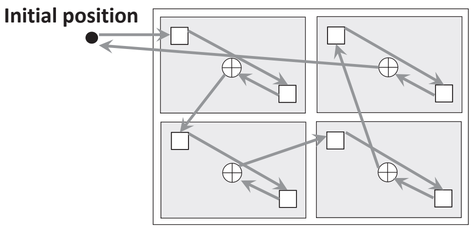

Figure 8: Simple visiting sequence (order): Example 1

Figure 9 shows another simple inspection sequence in which test positions are visited after all the alignment marks are visited. A probe unit firstly moves to the alignment mark located at the upper left, and then visits only alignment marks in order. After all the alignment marks are visited, the probe unit moves to the nearest test position from the lastly visited alignment mark, and visits only test positions in the inverse sequence (order) of alignment marks that were already visited.

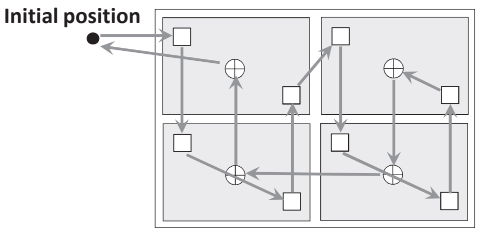

Figure 9: Simple visiting sequence (order): Example 2

Apparently, the inspection routes based on the sequence shown in Figures 8 and 9 are not optimal, and there may exist other shorter routes. This kind of optimization problem for obtaining the shortest route in electrical PCB inspections with alignment operations has not been discussed so far.

In the next section, we will address how to find an optimal inspection route in advanced electrical PCB inspections.

3 Modelling and problem formulation In this section, we show that the problem of finding an optimal inspection route, namely, an inspection route with the minimum length (an optimal route) can be modeled as a kind of PDTSPs, and that it is formulated as mixed 0-1 integer programming problems.

3.1 Pickup and delivery traveling salesman problem (PDTSP) A pickup and delivery traveling salesman problem (PDTSP) [3, 15] is a kind of TSPs in which all vertices are characterized as pickup and/or delivery vertices. PDTSPs can be roughly classified into three groups [3] such as 1) one-to-one, 2) many-to-many, 3) one-to-many-to-one. In one-to-one problems, each commodity has exactly one pickup vertex and one delivery vertex. In many-to-many PDTSPs, several origins (pickup vertices) and destinations (delivery vertices) are characterized for each commodity. In one-to-many-to-one PDTSPs, some commodities (e.g., food or drinks) are delivered from the depot to customers while other commodities (e.g., empty bottles) supplied by the customers are brought back to the depot.

As will be discussed in the next section, the PCB inspection route optimization problem (PCBIRP) can be modeled as a many-to-one or one-to-one problem in which each commodity has several (or one) pickup vertices (vertex) but only one delivery vertex.

3.2 Modelling based on a PDTSP To illuminate the readers' understanding of the ideas of this paper, we give graphical explanations with the example shown in Figure 7. In order for the camera of a probe unit to capture the images of alignment marks, the camera must be moved to alignment mark A, as shown in Figure 10. This operation is equivalent to moving the reference point of the probe unit (the center of the probe jig) to vertex A'. Thus, we consider the graph in which the alignment mark A is moved to A'.

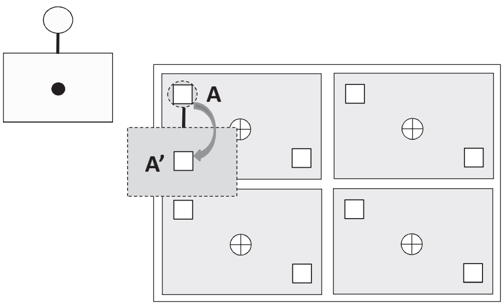

Figure 10: Image capturing of alignment mark A

In a similar manner, we transfer all the positions of alignment marks, and obtain a new graph shown in Figure 11, which represents a set of vertices to be visited by the reference point of a probe unit.

As described before, there are precedence constraints between alignment marks and their corresponding test position for each PCB. In Figure 12, dotted lines represent precedence relations, which means that for each PCB, two alignment marks must be visited before the corresponding test position is visited.

In general, when the number of alignment marks is two, the PCBIRP can be modeled as a two-to-one PDTSP; two alignment marks are regarded as pickup vertices, and the corresponding one test position is regarded as a delivery vertex. It should be noted that each vertex is characterized as *either of* pickup and delivery vertices. This is different from

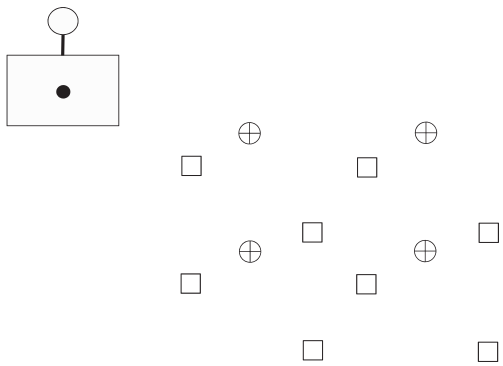

Figure 11: Vertices to be visited by the probe unit

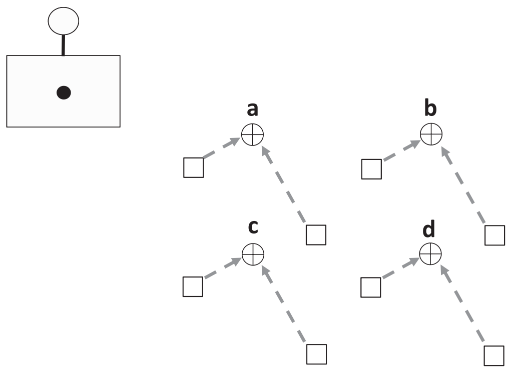

Figure 12: Precedence relationships between alignment marks and test positions

the original many-to-many PDTSPs; the original many-to-many PDTSPs allow each vertex to be *both* a pickup and a delivery vertex.

On the other hand, when the number of alignment marks is exactly one, the problem can be regarded as a one-to-one PDTSP, because there exists a one-to-one correspondence between each alignment mark and its corresponding test position.

The goal of the PCBIRP is to obtain an optimal (shortest) route. The length of the inspection route is dependent on the visiting sequence (order), and there are many feasible inspection routes. For example, Figure 13 shows the simple route constructed based on the visiting sequence (order) shown in Figure 8. In this figure, firstly, all the alignment marks are visited, and then all the test positions are visited. This is not an optimal (shortest) route.

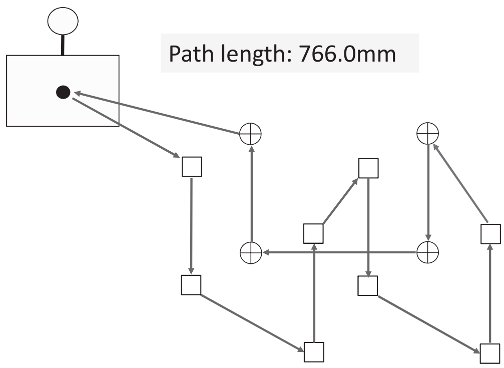

Figure 13: Simple inspection route

Figure 14 shows the optimal route, namely, the shortest route (cycle). In the next section, we shall give mathematical programming formulations in order to obtain an optimal route.

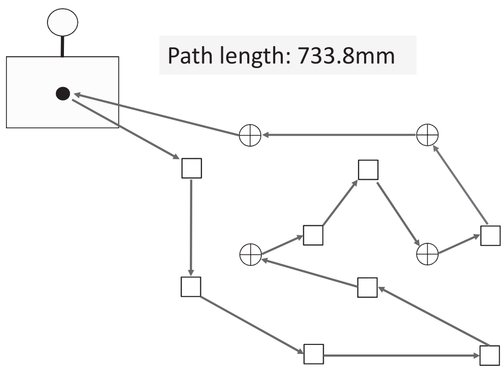

Figure 14: Optimal inspection route

3.3 Integer programming-based problem formulation Here, we formulate the PCBIRP as mixed 0-1 integer programming problems. In preparation for problem formulation, we use the following mathematical notation:

0:	Vertex corresponding to the initial point of a probe unit
B:	Set of PCBs to be tested, defined by $\{1, 2, \ldots, l\}$
A_p:	Set of vertices corresponding to alignment marks of p-th PCB ($p \in B$)
t_p:	Vertex corresponding to the test position of p-th PCB ($p \in B$)
N:	Set of all the vertices to be visited by a probe unit defined by $N = \cup_{p=1}^{l} (A_p \cup \{t_p\})$
e_{ij}:	Edge between vertices i and j ($i, j \in N \cup \{0\}$)
E:	Set of all the edges e_{ij}, $\forall i, j \in N \cup \{0\}$
c_{ij}:	Length of e_{ij} ($e_{ij} \in E$)

For notational convenience, the test position for the p-th PCB is represented as a singleton $\{t_p\}$, although there is only a single test position for each PCB. As for alignment marks, since there are one or two alignment marks, the number of elements in A_p is one or two. Figure 15 shows an example of 4 PCBs where there are 13 numbered vertices $0 \cup N = \{0, 1, \ldots, 12\}$ in which $B = \{1, 2, 3, 4\}$ ($l = 4$), $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $A_3 = \{5, 6\}$, $A_4 = \{7, 8\}$, $t_1 = 9$, $t_2 = 10$, $t_3 = 11$, $t_4 = 12$ and $N = \{1, 2, \ldots, 12\}$.

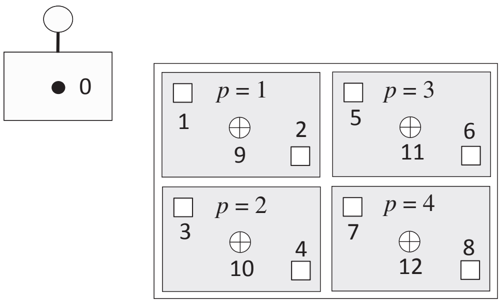

Figure 15: Number of vertices

In order to formulate PCBIRPs as mathematical programming problems, we introduce decision variables x_{ij}s as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ is visited immediately after } i \text{ is visited} \\ 0 & \text{otherwise.} \end{cases}$$

Decision variable x_{ij} is used to represent inspection routes, namely, to construct a route by connecting all edges with $x_{ij} = 1$.

In this paper, we consider the well-known "polynomial formulations" where the number of constraints and variables is a polynomial function of the number of vertices (cities). One of the most well-known polynomial formulations of TSPs is given by Miller-Tucker-Zemlin (MTZ) [13]. The PCBIRP cannot directly be modeled as the original MTZ formulation because the original MTZ formulation does not take into consideration the precedence relationship between vertices.

On the basis of MTZ formulation, we firstly provide the following new formulation, called PCBIRP-MTZ, in which the precedence constraints with respect to alignment marks and test positions are added to the original MTZ formulation:

PCBIRP-MTZ:

(1) $$\text{minimize} \quad \sum_{i\in N\cup\{0\}} \sum_{\substack{j\in N\cup\{0\}\\(j\neq i)}} c_{ij}x_{ij}$$

(2) $$\text{subject to} \quad \sum_{\substack{j\in N\cup\{0\}\\(j\neq i)}} x_{ij} = 1,\ \forall i \in N\cup\{0\}$$

(3) $$\sum_{\substack{i\in N\cup\{0\}\\(i\neq j)}} x_{ij} = 1,\ \forall j \in N\cup\{0\}$$

(4) $$u_j \geq u_i + 1 - (n-1)(1-x_{ij}),\ \forall i,j \in N,\ i\neq j$$

(5) $$1 \leq u_j \leq n-1,\ \forall j \in N$$

(6) $$u_i \leq u_j - 1,\ \forall i \in A_p,\ \forall j \in \{t_p\},\ \forall p \in B$$

(7) $$x_{ij} \in \{0,1\},\ \forall i,j \in N\cup\{0\},\ i\neq j,$$

where (1) represents the route length. Constraints (2) and (3) impose that the out-degree and in-degree of each vertex, respectively, is equal to one. Constraints (4) prevent subtours not containing node 0 and imply $u_j \geq u_i + 1$ whenever $x_{ij} = 1$, where u_i, $i \in N$ is an arbitrary real number representing the order of vertex i in the optimal tour. Together with (2) and (3), constraints (4) guarantee that subtours containing node 0 are not allowed. Constraints (6) guarantee that all the alignment marks $i \in A_p$ of the p-th PCB are visited before the corresponding test position $j \in \{t_p\}$ is visited.

We propose another formulation which is an extended version of PCBIRP-MTZ. Desrochers and Laporte [6] proposed a formulation in which the MTZ constraints were lifted into facets of the underlying TSP polytope. Along this line, we provide the following new formulation, called PCBIRP-DL, which is obtained by replacing constraints (4) and (5) in PCBIRP-MTZ with the lifted constraints (11)-(13):

PCBIRP-DL:

(8) $$\text{minimize} \quad \sum_{i\in N\cup\{0\}} \sum_{\substack{j\in N\cup\{0\}\\(j\neq i)}} c_{ij}x_{ij}$$

(9) $$\text{subject to} \quad \sum_{\substack{j\in N\cup\{0\}\\(j\neq i)}} x_{ij} = 1,\ \forall i \in N\cup\{0\}$$

(10) $$\sum_{\substack{i\in N\cup\{0\}\\(i\neq j)}} x_{ij} = 1,\ \forall j \in N\cup\{0\}$$

(11) $$u_j \geq u_i + 1 - (n-1)(1-x_{ij}) + (n-3)x_{ji},\ \forall i,j \in N,\ i\neq j$$

(12) $$1 + (1-x_{0j}) + (n-3)x_{j0} \leq u_j,\ \forall j \in N$$

(13) $$u_j \leq (n-1) - (n-3)x_{0j} - (1-x_{j0}),\ \forall j \in N,$$

(14) $$u_i \leq u_j - 1,\ \forall i \in A_p,\ \forall j \in \{t_p\},\ \forall p \in B$$

(15) $$x_{ij} \in \{0,1\},\ \forall i,j \in N\cup\{0\},\ i\neq j,$$

where constraints (11) are obtained by lifting (4), and constraints (12) and (13) are obtained by lifting (5). By introducing lifted constraints (11)-(13), PCBIRP-DL is a tighter formulation than PCBIRP-MTZ, which means that the formulation of PCBIRP-DL is expected to solve PCBIRPs faster than the formulation of PCBIRP-MTZ.

4 Numerical Experiments In order to compare the performances of the proposed two formulations, we solved benchmark instances constructed based on real PCB wiring patterns. In the benchmark instances, the numbers of wiring patterns on a PCB sheet are between 1 and 21. For all instances, there are two alignment marks. Tables 1 and 2 show the experimental results. We use a personal computer with Intel Core i5 Processor 2.5 GHz, RAM:8GB, OS:Windows 7 (64bit), and the coding was done with Python 2.7 + PuLP 1.6.0.

To solve benchmark instances, we used three mathematical programming solvers, CPLEX [4], Gurobi [8] and SCIP [16], and compared their performances (computational times). Table 1 shows the computational times for the different-size instances when two proposed formulations are solved by using three well-known solvers, CPLEX 12.6.1.0, Gurobi 6.0 and SCIP 3.1.1.1. Each figure represents the computational time for obtaining an optimal solution. Bold figures express the best one among three solvers. We started to solve the minimum-size problem (the case of $n = 1$), and increment the sizes of problems to be solved in such a manner as $n = 1$, $n = 2$, $\ldots$,. When the computational time for solving the problem excessed one hour (3,600 seconds), we stopped solving larger-size problems.

Table 1: Computational time for solving different-size benchmark instances

n	PCBIRP-MTZ			PCBIRP-DL		
	CPLEX	Gurobi	SCIP	CPLEX	Gurobi	SCIP
1	0.01	0.02	**0.001**	**0.001**	**0.001**	**0.001**
2	0.01	0.08	0.05	**0.001**	**0.001**	**0.001**
3	0.06	0.09	0.64	**0.001**	**0.001**	0.02
4	0.31	0.70	1.73	0.19	**0.03**	0.09
5	0.67	0.64	9.50	**0.09**	0.11	0.79
6	3.81	6.46	52.59	0.14	**0.11**	1.11
7	8.50	26.60	189.62	0.17	**0.16**	3.12
8	14.90	15.21	45.20	0.25	**0.16**	3.05
9	158.11	677.69	3989.33	0.98	**0.78**	9.58
10	149.14	143.06		0.69	**0.50**	7.72
11	947.63	1328.30		5.43	**2.64**	14.43
12	724.00	1079.55		**5.51**	8.01	31.76
13	4030.57	53046.92		**30.05**	40.39	119.07
14				101.24	**65.57**	1014.06
15				107.30	**37.86**	441.24
16				156.87	**155.20**	2684.08
17				326.53	**174.30**	12498.04
18				247.73	**109.28**	
19				**1687.96**	4883.33	
20				**1467.39**		
21				**6017.80**		

Figure 16 shows the results of Table 1 graphically. The vertical axis of the plot is scaled logarithmically by taking logs to base 10, namely, $log_{10}(x)$. When the formulation of PCBIRP-MTZ was used, CPLEX was the fastest among three solvers, and SCIP was the slowest one. On the other hand, when PCBIRP-DL was used, CPLEX and Gurobi were competitive. In both formulations, the computational time increases exponentially. Since the most typical number of PCBs in a sheet is from 9 to 16, the formulation of PCBIRP-DL can be considered practical when CPLEX or Gurobi is used because it takes less than 3 minutes to solve the problem with 16 PCBs in a sheet.

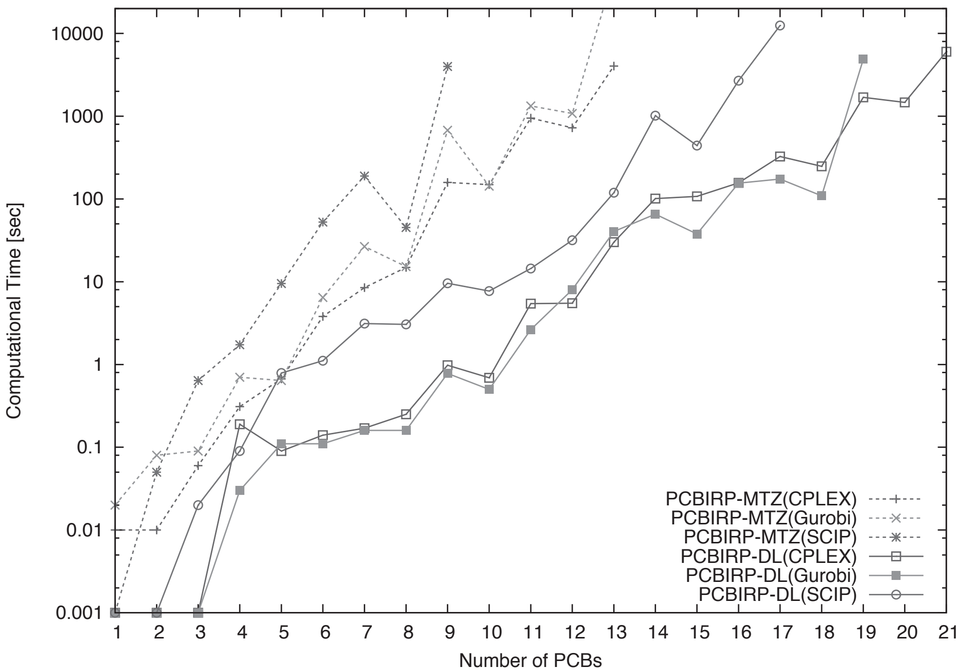

Figure 16: Computational time for solving different-size benchmark instances

Table 2 shows the route length of the optimal solution obtained using mathematical programming solvers. In this table, "Simple route" represents the simple PCB inspection route in which all the alignment marks are firstly visited and then all the test positions are visited in a simple order, like the route as shown in Figure 13. It is observed from Table 2 that the PCB inspection routes obtained by the proposed formulations are averagely around 50% shorter than simple routes that had been previously employed in the field of PCB inspections with alignment operations.

5 Conclusion In this paper, we have newly modeled a route optimization problem in advanced PCB electrical inspections, which had not been discussed so far. We have formulated the PCB inspection route optimization problem (PCBIRP) as a class of PDTSPs, and provided two types of mixed 0-1 integer programming problem formulations based on MTZ formulation and its extension. Some experiments have been conducted using bench mark instances based on real PCB wiring patterns. The proposed method can yield averagely 50% shorter inspection route than the previous method. Also, it has been shown that the procedure of "lifting" is promising for solving the PCBIRP.

As a future study, we will address other alternative polynomial formulations (formulations in which the number of constraints and variables is a polynomial function of the number of vertices) of PCBIRP using flow-based formulations [17]. In addition, it is interesting to consider branch-and-cut methods by extending the polytope of PDTSPs [7]. Since the number of PCBs is 100 to 200 in some cases, it is also important to consider some efficient heuristic algorithms. Hence, another future work is to consider novel heuristic

Table 2: The lengths of PCB inspection routes

n	Simple route	Optimal route	Improvement rate (%)
1	**647.1**	**647.1**	0.0
2	1062.2	**762.2**	28.2
3	1478.2	**895.3**	39.4
4	1894.8	**1057.4**	44.2
5	2311.6	**1198.8**	48.1
6	2640.0	**1380.4**	47.7
7	2855.4	**1459.4**	48.9
8	3072.2	**1531.2**	50.2
9	3290.9	**1693.7**	48.5
10	3512.9	**1808.0**	48.5
11	3878.2	**2050.0**	47.1
12	4277.6	**2114.4**	50.6
13	4681.7	**2225.7**	52.5
14	5089.2	**2322.7**	54.3
15	5498.9	**2383.6**	56.7
16	5850.7	**2570.7**	56.1
17	6074.9	**2644.2**	56.4
18	6302.1	**2715.9**	56.9
19	6533.2	**2883.2**	55.9
20	6769.6	**2992.8**	55.8
21	7148.0	**3241.9**	54.6

algorithms as some extensions of conventional efficient algorithms such as Lin-Kernighan method [12] and its variants [9]. These extensions will be discussed elsewhere in near future.

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CONTRA-γ-IRRESOLUTE MAPPINGS AND RELATED GROUPS *

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Abstract.The aim of the present paper is devoted to discuss some more properties of γ-irresolute mappings and contra-γ-irresolute mappings. Also, we introduce and study two new weak homeomorphisms such as contra-γr-homeomorphisms and contra-γ-homeomorphisms. Further, we investigate some groups related to the mappings above and some examples of them on digital lines.

1 Introduction and preliminaries D.Andrijević [6] (resp. A.A. EL-Atik [15] and J. Dontchev and M. Przemski [13]) introduced independently the concept of b-open sets [6] (resp. γ-open sets [15] and sp-open sets [13]). A subset A of a topological space (X, τ) is called a *γ-open set* [15] (or *b-open set* [6], *sp-open set* [13]), if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ holds in (X, τ); and the complement of a γ-open (or b-open, sp-open) set is called *γ-closed* (or *b-closed, sp-closed*). Throughout the present paper, we use the terminology due to [15] for the naming of the above set, i.e., γ-open sets, γ-closed sets. The *γ-closure* of a subset E of (X, τ) is defined by $\gamma Cl(E) := \bigcap\{F | E \subseteq F, F$ is γ-closed in $(X, \tau)\}$; and it is the smallest γ-closed set containing E (cf. Theorem 4.4(iii)); we recall some importante properties of γ-open sets in Section 4 (Theorem 4.4).

In the present paper, we use the following notations (cf. [28] [19, p.2]):
$\gamma O(X, \tau) := \{U | U$ is γ-open in $(X, \tau)\}$;
$\gamma C(X, \tau) := \{F | F$ is γ-closed in (X, τ), i.e., $Int(Cl(F)) \cap Cl(Int(F)) \subseteq F\}$.
$SO(X, \tau) := \{U | U$ is semi-open in (X, τ), i.e., $U \subseteq Cl(Int(U))\}$ [25];
$SC(X, \tau) := \{F | F$ is semi-closed in (X, τ), i.e., $Int(Cl(F)) \subseteq F\}$.
$\tau^\alpha := \{V |\ V$ is α-open in (X, τ), i.e., $V \subseteq Int(Cl(Int(V)))\}$ [27].
$\beta O(X, \tau) = SPO(X, \tau) := \{W |\ W$ is β-open (or semi-preopen) in (X, τ), i.e., $W \subseteq Cl(Int(Cl(W)))\}$ [2],[5]. It is well known that:
$\tau^\alpha \subseteq SO(X, \tau) \subseteq \gamma O(X, \tau) \subseteq \beta O(X, \tau)$ hold in general.

In Section 2, we mention some relations among γ-irresoluteness [12], pre-γ-closedness [15], contra-γ-irresoluteness ([16] [28]) and some mappings (cf. Definitions 2.1, 2.2).

In Section 3, after the work due to A.Keskin and T.Noiri [20], we study a new group, say γr-$h(X; \tau) \cup$ *contra*-γr-$h(X; \tau)$ (Theorem 3.4(i), Corollary 3.6(i)). By the article [20, Definition 4.13, Theorem 4.14(ii)], the concept of the family γr-$h(X; \tau)$ is introduced and it is proved that γr-$h(X; \tau)$ forms a group. The family *contra*-γr-$h(X; \tau)$ is one of all *contra-γ-homeomorhpisms* on (X, τ) (cf. Definition 3.2). The group γr-$h(X; \tau) \cup$ *contra*-γr-$h(X; \tau)$ is one of the group invariants of a topological space (X, τ) under a γr-homeomorphism between topological spaces (Theorem 3.5(i)). By Theorem 3.4(iii)(cf. (iv)), it is shown that the group $h(X; \tau)$ of all homeomorphisms on (X, τ) is a subgroup of the group γr-$h(X; \tau) \cup$ *contra*-γr-$h(X; \tau)$.

In Section 4, we introduce two subgroups of γr-$h(X; \tau)$ (Definition 4.1) and so we can investigate some group structure of γr-$h(H; \tau|H)$ for the subspace $(H, \tau|H)$ of (X, τ) (Theorems 4.2 and 4.9(iii)).

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In Section 5, we study some topological properties on related topics of transformations on the digital line $(\mathbb{Z},\kappa)$ (so-called Khalimsky lines [21], [22, p.7, line −6], [23, p.905, p.908]), and for a specific subset H of the digital line $(\mathbb{Z},\kappa)$, we determine the group structure (Example 5.13) of $\gamma r\text{-}h(\mathbb{Z},\mathbb{Z}\setminus H;\kappa)$, $\gamma r\text{-}h_0(\mathbb{Z},\mathbb{Z}\setminus H;\kappa)$ and $\gamma r\text{-}h(H;\kappa|H)$.

Throughout the present paper, $(X,\tau),(Y,\sigma)$ and (Z,η) (or simply X,Y and Z) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

2 Contra-γ-irresolute mappings and γ-irresolute mappings This section is devoted to discuss the relation among γ-irresolute mappings [15], contra-γ-irresolute mappings [16][28], perfectly contra-γ-irresolute mappings [16] and some mappings (cf. Definitions 2.1, 2.2).

Definition 2.1 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be:

(i) *b-continuous* [12] (or *γ-continuous* [15]), if $f^{-1}(V)$ is a b-closed (or γ-closed) set of (X,τ) for each closed set V of (Y,σ);

(ii) *perfectly continuous* [31], if $f^{-1}(V)$ is clopen in (X,τ) for each open set V of (Y,σ);

(iii) *contra-continuous* [11], if $f^{-1}(V)$ is closed in (X,τ) for each open set V of (Y,σ);

(iv) *contra-γ-continuous* [16] (or *contra-b-continuous* [28]) if $f^{-1}(V)\in\gamma C(X,\tau)$ for each open set V of (Y,σ);

(iv)$'$ *strongly contra-γ-continuous* (cf. (iv)), if f is a contra-γ-continuous mapping such that the inverse image of each open set of (Y,σ) has an interior point;

(v) *B-continuous* [34], if $f^{-1}(V)$ is a B-set of (X,τ) for each nonempty open set V of (Y,σ), where the B-set is the intersection of an open set and a semi-closed set of (X,τ) (this is defined by [34], cf. [10, Theorem 2.3]).

(v)$'$ *B^*-continuous* (cf. (v)), if $f^{-1}(V)$ contains a nonempty B-set of (X,τ) for each nonempty open set V of (Y,σ);

(vi) *pre-b-closed* [15] (or *pre-γ-closed*), if $f(G)$ is b-closed (or γ-closed) in (Y,σ) for each b-closed (or γ-closed) set G of (X,τ).

Definition 2.2 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be:

(i) *γ-irresolute* (or *b-irresolute* [15]) (resp. *irresolute* [8, Definition 1.1]), if $f^{-1}(U)\in\gamma O(X,\tau)$ (resp. $f^{-1}(U)\in SO(X,\tau)$) for every set $U\in\gamma O(Y,\sigma)$ (resp. $U\in SO(Y,\sigma)$);

(ii) *contra-γ-irresolute* [16] (or *contra-b-irresolute* [28]) (resp.- *contra-irresolute*), if $f^{-1}(U)\in\gamma C(X,\tau)$ (resp. $f^{-1}(U)\in SC(X,\tau)$) for every set $U\in\gamma O(Y,\sigma)$ (resp. $U\in SO(Y,\sigma)$);

(iii) *perfectly contra-γ-irresolute* [29] (resp. *perfectly contra-irresolute*), if $f^{-1}(V)$ is γ-clopen (resp. semi-open and semi-closed) in (X,τ) for each set $V\in\gamma O(Y,\sigma)$ (resp. $V\in SO(Y,\sigma)$).

Theorem 2.3 *A mapping $f:(X,\tau)\to(Y,\sigma)$ is B^*-continuous, if one of the following conditions is satisfied:*

(1) *f is a strongly contra-γ-continuous mapping,*

(2) *f is an onto and B-continuous mapping.* □

We have the following diagram among several mappings defined above, where p → q (resp. p' $\nleftrightarrow$ q') means that p implies q (resp. p' and q' are independent). The implications are not reversible and the independence is explained (cf. Remark 2.4 below).

contra-continuous
↙ ↘
contra-γ-continuous $\nleftrightarrow$ B-continuous ← onto and B-continuous
↑ ↓
strongly contra-γ-continuous → B^*-contiuous.

Remark 2.4 (i) Let $(\mathbb{R}, \epsilon)$ be the real line with the Euclidean topology ϵ. The following functions $f, 1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ of (i) below are seen in [12].

(i) (i-1) Let $f : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ be a mapping defined by $f(x) = [x]$, where $[x]$ is the Gaussian symbol. Then, f is contra-γ-continuous (cf. Definition 2.1(iv)). However, f is not contra-continuous, because for an open interval $(1/2, 3/2)$, $f^{-1}((1/2, 3/2)) = [1, 2)$ is not closed in $(\mathbb{R}, \epsilon)$.

(i-2) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ is B-continuous (cf. Definition 2.1(v)) but not contra-γ-continuous, since the inverse image of each singleton is not γ-open. Moreover, $1_{\mathbb{R}}$ is not contra-continuous.

(ii) The following mapping $f : (X, \tau) \to (X, \tau)$ is contra-γ-continuous; but f is not B-continuous. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a, b\}, X\}$. Then, we have $\gamma C(X, \tau) = \{\emptyset,\{a\},\{b\},\{c\}, \{b, c\},\{a, c\},X\}$ and $SC(X, \tau) = \{\emptyset,\{c\},X\}$. We define the mapping f by $f(a) := a, f(b) := c, f(c) := b$.

(iii) The converse of Theorem 2.3 under the assumption (1) is not reversible. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := b, f(b) := c, f(c) := a$. Then, since $\gamma C(X, \tau) = SC(X, \tau) = P(X)\setminus\{\{a, b\}\}$, we show f is B-continuous and onto. By Theorem 2.3 under the assumption (2), it is obtained that f is B^*-continuous. This mapping f is contra-γ-continuous; but $Int(f^{-1}(\{a\})) = Int(\{c\}) = \emptyset$ hold; and so f is not strongly contra-γ-continuous.

(iv) The converse of Theorem 2.3 under the assumption (2) is not reversible. The mapping $f : (X, \tau) \to (X, \tau)$ defined in (ii) above is not B-continuous (cf. (ii)). But, f is B^*-continuous, because $\{c\}$ and X are the nonempty B-sets.

(v) The contra-γ-continuous mapping $f : (X, \tau) \to (X, \tau)$ of (ii) above is not strongly contra-γ-continuous (cf. Definition 2.1(iv)$'$), because
$Int(f^{-1}(\{a, b\})) = \emptyset$.

Remark 2.5 (i) Let $X = \{a, b\}, \tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, X, \{b\}\}$. Then the identity mapping $1_X : (X, \tau) \to (X, \sigma)$ is a contra-γ-continuous mapping but it is not γ-continuous.

(ii) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ of Remark 2.4(i)(i-2) is γ-continuous but it is not contra-γ-continuous.

Remark 2.6 The following properties are well known. (i) [4, Theorem 3.7(i)] if $f : (X, \tau) \to (Y, \sigma)$ is contra-γ-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is γ-continuous, then $g \circ f$ is contra-γ-continuous.

(ii) Every homeomorphism is γ-irresolute.

Remark 2.7 (i) By the following examples (i-1) and (i-2), it is shown that the contra-γ-irresoluteness and γ-irresoluteness are independent notions: let $X := \{a, b, c\}$ and $\tau := \{X, \emptyset, \{a\}, \{a, b\}\}$.

(i-1) The identity mapping on (X, τ) above is γ-irresolute; but it is not contra-γ-irresolute.

(i-2) Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := b, f(b) := b, f(c) := a$. Then, f is contra-γ-irresolute; but f is not γ-irresolute.

(ii) In general, for any topological space (X, τ), the identity mapping $1_X : (X, \tau) \to (X, \tau)$ is contra-γ-irresolute if and only if $\gamma O(X, \tau) = \gamma C(X, \tau)$ holds. And, 1_X on any topological space (X, τ) is γ-irresolute.

Remark 2.8 (i) Every contra-γ-irresolute mapping is contra-γ-continuous, but it is shown that its converse is not true, by the following example. Let $X = \{a, b, c\}$, τ=$\{\emptyset$,$\{a\}$, $\{b\}$,$\{a, b\}$,$X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := c, f(b) := a, f(c) := b$.

(ii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is contra-γ-irresolute if and only if the inverse image $f^{-1}(F)$ of each γ-closed set F of (Y, σ) is γ-open in (X, τ).

Remark 2.9 (i) The following diagram of implications is well known:

· contra-irresolute $\longleftarrow$ perfectly contra-irresolute $\longrightarrow$ irresolute.

We have the following diagram of implications:

· contra-γ-irresolute $\longleftarrow$ perfectly contra-γ-irresolute $\longrightarrow$ γ-irresolute;

and they are not reversible (cf. Remark 2.7(i) above and Remark 2.10 below):

(ii) In the implications above, the irresoluteness (resp. contra-irresoluteness, perfectly contra-irresoluteness) and the γ-irresoluteness (resp. contra-γ-irresoluteness, perfectly contra-γ-irresoluteness) are independent (cf. (a), (b), (c) below).

Let $X = \{a, b, c\}$. We consider the following topologies on $X : \tau := \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1 := \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 := \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\tau_3 := \{X, \emptyset\}$. We have the following dates: $SO(X, \tau) = \gamma O(X, \tau) = P(X) \setminus \{\{c\}\}$; $SO(X, \tau_1) = \gamma O(X, \tau_1)=\{\emptyset, \{a\}, \{a, b\},\{a, c\}, X\}$; $SO(X, \tau_2) = \tau_2, \gamma O(X, \tau_2) = P(X)$; $SO(X, \tau_3) = \{\emptyset, X\}, \gamma O(X, \tau_3) = P(X)$.

(a) (a-1) Define a mapping $f : (X, \tau) \rightarrow (X, \tau_2)$ as follows: $f(a) = a, f(b) = c$ and $f(c) = b$. Then f is irresolute; f is not γ-irresolute.

(a-2) Let $f : (X, \tau_3) \rightarrow (X, \tau)$ be the identity mapping. Then f is γ-irresolute; f is not irresolute.

(b) (b-1) Let $f : (X, \tau_2) \rightarrow (X, \tau_1)$ be the identity mapping. Then f is contra-γ-irresolute; f is not contra-irresolute.

(b-2) Define a mapping $f : (X, \tau_1) \rightarrow (X, \tau_2)$ as follows: $f(a) := a, f(b) := a, f(c) := b$. Then f is contra-irresolute; f is not contra-γ-irresolute.

(c) (c-1) Let $f : (X, \tau_3) \rightarrow (X, \tau_2)$ be the identity mapping. Then f is perfectly contra-γ-irresolute; f is not perfectly contra-irresolute.

(c-2) Define a mapping $f : (X, \tau) \rightarrow (X, \tau_2)$ as follows: $f(a) := c, f(b) := a, f(c) := b$. Then f is perfectly contra-irresolute; f is not perfectly contra-γ-irresolute.

Remark 2.10 We have a decomposition of perfectly contra-γ-irresolute mappings. The following conditions (1) and (2) are equivalent: (1) $f : (X, \tau) \rightarrow (Y, \sigma)$ is perfectly contra-γ-irresolute; (2) $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-γ-irresolute and γ-irresolute.

3 Groups γr-$h(X;\tau) \cup contra$-γr-$h(X;\tau)$ and $h(X;\tau) \cup contra$-$h(X;\tau)$ We have a new homeomorphism invariant for topological spaces (cf. Theorems 3.4, 3.5, Corollary 3.6).

Definition 3.1 (i) A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

(i-1) ([20, Definiton 4.12]) a *γr-homeomorphism* if f is a γ-irresolute bijection and f^{-1} is γ-irresolute;

(i-2) a *contra-γr-homeomorphism* if f is a contra-γ-irresolute bijection and f^{-1} is contra-γ-irresolute;

(ii) (ii-1) ([20, Definition 4.12]) a *γ-homeomorphism* if f is a γ-continuous bijection and f^{-1} is γ-continuous;

(ii-2) a *contra-γ-homeomorphism* (resp. *contra-homeomorphism*) if f is a contra-γ-continuous (resp. contra-continuous) bijection and f^{-1} is contra-γ-continuous (resp. contra-continuous).

Definition 3.2 We recall and define the following families of mappings from (X, τ) onto itself.

· ([20, Definition 4.13]) γr-$h(X;\tau) := \{f | f : (X, \tau) \rightarrow (X, \tau)$ is a γr-homeomorphism$\}$ (by [20, Theorem 4.14(ii)], it is proved that γr-$h(X;\tau)$ forms a group under the composition of mappings);

· *contra*-γr-$h(X;\tau) := \{f | f : (X, \tau) \rightarrow (X, \tau)$ is a contra-γr-homeomorphism$\}$;

· $h(X;\tau) := \{f | f : (X, \tau) \rightarrow (X, \tau)$ is a homeomorphism$\}$;

· *contra*-$h(X;\tau) := \{f | f : (X, \tau) \rightarrow (X, \tau)$ is a contra-homeomorphism$\}$;

· $G_{(X,\tau)} := \gamma r$-$h(X;\tau) \cup contra$-γr-$h(X;\tau)$;

· $H_{(X,\tau)} := h(X;\tau) \cup contra$-$h(X;\tau)$.

Proposition 3.3 *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two mappings between topological spaces.*

(i) (i-1) ([20, Theorem 4.14(ii)]) *If f and g are γ-irresolute, then $g\circ f$ is γ-irresolute.*

(i-2) ([20, Theorem 4.14(ii)]) *The identity mapping $1_X:(X,\tau)\to(X,\tau)$ is γ-irresolute.*

(i-3) *If f and g are contra-γ-irresolute, then $g\circ f$ is γ-irresolute.*

(ii) (ii-1) *If f is contra-γ-irresolute and g is γ-irresolute, then $g\circ f$ is contra-γ-irresolute.*

(ii-2) *If f is γ-irresolute and g is contra-γ-irresolute, then $g\circ f$ is contra-γ-irresolute.* □

Theorem 3.4 *Let $G_{(X,\tau)}$ and $H_{(X,\tau)}$ be the families of mappings defned in* Definition 3.2.

(i) *$G_{(X,\tau)}$ forms a group under the composition of mappings.*

(ii) *γr-$h(X;\tau)$ forms a subgroup of $G_{(X,\tau)}$* (cf. [20, Theorem 4.14(ii)]).

(iii) *The group $h(X;\tau)$ is a subgroup of γr-$h(X;\tau)$* ([20, Theorem 4.14(iii)]) *and $h(X;\tau)$ is also a subgroup of $G_{(X,\tau)}$.*

(iv) *$H_{(X,\tau)}$ forms a group under the composition of mappings. The group $h(X;\tau)$ is a subgroup of $H_{(X,\tau)}$.*

(v) *If $\tau=\gamma O(X,\tau)$ holds, then $G_{(X,\tau)}=H_{(X,\tau)}$.* □

We note that the binary operation $\omega_{G(X,\tau)}:G_{(X,\tau)}\times G_{(X,\tau)}\to G_{(X,\tau)}$ is well defined by $\omega_{G(X,\tau)}(a,b):=b\circ a$, where $a,b\in G_{(X,\tau)}$ and $b\circ a$ denotes the composition of two mappings a, b defined by $(b\circ a)(x)=b(a(x))$ for any $x\in X$ (cf. Proposition 3.3). And, the restriction $\omega_{G(X,\tau)}|\gamma r\text{-}h(X;\tau)\times\gamma r\text{-}h(X;\tau)$ is denoted shortly by ω_X.

Theorem 3.5 (i) *If $f:(X,\tau)\to(Y,\sigma)$ is a γr-homeomorphism* (resp. *contra-γr-homeomorphism*), *then the mapping f induces an isomorphism $f_*:G_{(X,\tau)}\to G_{(Y,\sigma)}$, where f_* is defined by $f_*(a):=f\circ a\circ f^{-1}$ for any $a\in G_{(X,\tau)}$. Moreover,*

(a) *$(g\circ f)_*=g_*\circ f_*:G_{(X,\tau)}\to G_{(Z,\eta)}$, where $g:(Y,\sigma)\to(Z,\eta)$ is a γr-homeomorphism* (resp. *contra-γr-homeomorphism*),

(b) *$(1_X)_*=1:G_{(X,\tau)}\to G_{(X,\tau)}$ is the identity isomorphism,*

(c) *$f_*(\gamma r\text{-}h(X;\tau))=\gamma r\text{-}h(Y;\sigma)$, $f_*(h(X;\tau))\subseteq\gamma r\text{-}h(Y;\sigma)$ and $f_*(contra\text{-}\gamma r\text{-}h(X;\tau))=contra\text{-}\gamma r\text{-}h(Y;\sigma)$ hold.*

(ii) *Especially, if $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are homeomorphisms, then the induced mappings $f_*:G_{(X,\tau)}\to G_{(Y,\sigma)}$ and $g_*:G_{(Y,\sigma)}\to G_{(Z,\eta)}$ are isomorphisms* (cf. (i)). *Moreover, they have the same property of* (a),(b) *and* (c) *in* (i). *We note that, in* (c), *$f_*(h(X;\tau))=h(Y;\sigma)$ holds.* □

Corollary 3.6 (cf. Definition 3.2, Theorem 3.5) (i) *If $G_{(X,\tau)}\ncong G_{(Y,\sigma)}$* (i.e. *$G_{(X,\tau)}$ is not isomorphic to $G_{(Y,\sigma)}$ as groups), then there does not exist any γr-homeomorphism between two topological spaces (X,τ) and (Y,σ); and hence $(X,\tau)\ncong(Y,\sigma)$* (i.e., *(X,τ) is not homeomorphic to (Y,σ)*).

(ii) *If $\gamma r\text{-}h(X;\tau)\ncong\gamma r\text{-}h(Y;\sigma)$* (i.e., *$\gamma r\text{-}h(X;\tau)$ is not isomorphic to $\gamma r\text{-}h(Y;\sigma)$ as groups), then there does not exist any γr-homeomorphism between (X,τ) and (Y,σ).* □

Example 3.7 (i) In Section 5, we give a special example of group $\gamma r\text{-}h(H,\kappa|H)$, where $(H,\kappa|H)$ is a subspace of the digital line $(\mathbb{Z},\kappa)$(cf. Example 5.13).

(ii) Let (X,τ) and (Y,σ) be two topological spaces, where $X=Y:=\{a,b,c\}$, $\tau:=\{\emptyset,\{a\},\{b,c\},X\}$ and $\sigma:=\{\emptyset,\{a\},\{b\},\{a,b\},Y\}$. Then, it is shown that $G_{(X,\tau)}=\gamma r\text{-}h(X;\tau)\cong S_3$(=the symetric group of degree 3) and $G_{(Y,\sigma)}=\gamma r\text{-}h(Y;\sigma)=\{1_Y,h_c\}$, where $h_c:(Y,\sigma)\to(Y,\sigma)$ is a bijection defined by $h_c(a):=b,h_c(b):=a,h_c(c):=c$; and hence $G_{(X,\tau)}\ncong G_{(Y,\sigma)}$. Thus, using Corollary 3.6(i), we can assure that there is never exists any γr-homeomorphism between (X,τ) and (Y,σ). We note that $h(X;\tau)=\{1_X,h_a\}$ and $h(Y;\sigma)=\{1_Y,h_c\}$ hold, where $h_a:(X,\tau)\to(X,\tau)$ is a bijection defined by $h_a(a):=a,h_a(b):=c,h_a(c):=b$; and so $h(X;\tau)\cong h(Y;\sigma)$ holds.

(iii) Let (X,τ) be the topological space of (ii) above and let (Y_1,σ_1) be a topological space such that $Y_1:=\{a,b,c\}$ and $\sigma_1:=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}$

,$Y_1\}$. Then, we have that $G_{(X,\tau)} \not\cong G_{(Y_1,\sigma_1)}$ and $h(X;\tau) \not\cong h(Y_1;\sigma_1)$. Using Corollary 3.6, there is never exist any γr-homeomorphism between (X,τ) and (Y_1,σ_1).

(iv) Let (Y_1,σ_1) be the topological space of (iii) above and let (Y_2,σ_2) be a topological space such that $Y_2 := \{a,b,c\}$ and $\sigma_2 := \{\emptyset,\{a\},Y_2\}$. Then, we have that $G_{(Y_1,\sigma_1)} \cong G_{(Y_2,\sigma_2)}$, $\gamma r\text{-}h(Y_1,\sigma_1) \not\cong \gamma r\text{-}h(Y_2,\sigma_2)$ and $h(Y_1,\sigma_1) \not\cong h(Y_2,\sigma_2)$ hold. We can apply Corollary 3.6(ii) for this example (iii).

(v) For the digital line $(\mathbb{Z},\kappa)$, we have an example of a subgroup of $H_{(\mathbb{Z},\kappa)}$ (cf. Example 5.10(iv)).

4 Two subgroups of γr-$h(X;\tau)$ and their properties The purpose of the present section is to prove Theorem 4.9.

Definition 4.1 For a subset G of X, we define the following families of mappings:
(i) $\gamma r\text{-}h(X,G;\tau) := \{a|\ a \in \gamma r\text{-}h(X;\tau)$ and $a(G) = G\}$;
(ii) $\gamma r\text{-}h_0(X,G;\tau) := \{a|\ a \in \gamma r\text{-}h(X;\tau)$ and $a(x) = x$ for every point $x \in G\}$.

Theorem 4.2 *Let H be a subset of a topological space (X,τ). The families $\gamma r\text{-}h(X,X\setminus H;\tau)$ and $\gamma r\text{-}h_0(X,X\setminus H;\tau)$ form two subgroups of $\gamma r\text{-}h(X,\tau)$ and $\gamma r\text{-}h(X,X\setminus H;\tau) = \gamma r\text{-}h(X,H;\tau)$ holds.* □

For the group $\gamma r\text{-}h(X,X\setminus H;\tau)$, say A, (resp. $\gamma r\text{-}h_0(X,X\setminus H;\tau)$, say A_0,) of Theorem 4.2, we define the binary operation $\omega_{X,H} : A\times A \to A$ (resp. $\omega_{X,H_0} : A_0\times A_0 \to A_0$) by $\omega_{X,H}(a,b) := (\omega_{G(X,\tau)}|A\times A)(a,b) = b\circ a$ (resp. $\omega_{X,H_0}(a,b) := (\omega_{G(X,\tau)}|A_0\times A_0)(a,b) = b\circ a$) (cf. a few lines after Theorem 3.4).

In order to investigate precisely some structures of $\gamma r\text{-}h(H,X\setminus H;\tau|H)$ (cf. Theorem 4.9), we need the following definitions and properties.

Definition 4.3 Let H, K be subsets of X and Y, respectively. For a mapping $f : X \to Y$ satisfying a property $K = f(H)$, we define the following mapping $r_{H,K}(f) : H \to K$ by $r_{H,K}(f)(x) = f(x)$ for every $x \in H$.

Then, we have the following properties:

(4.a) $j_K \circ r_{H,K}(f) = f|H : H \to Y$, where $j_K : K \to Y$ be the inclusion defined by $j_K(y) = y$ for every $y \in K$ and $f|H : H \to Y$ is the restriction of f to H defined by $(f|H)(x) = f(x)$ for every $x \in H$.

(4.b) Especially, we consider the following case where $X = Y, H = K \subseteq X$. If $a(H) = H$ and $b(H) = H$, then $r_{H,H}(b\circ a) = r_{H,H}(b)\circ r_{H,H}(a)$ holds, where $a,b : X \to X$ are mappings.

(4.c) If a mapping $a : X \to X$ is a bijection such that $a(H) = H$, then $r_{H,H}(a) : H \to H$ is bijective and $r_{H,H}(a^{-1}) = (r_{H,H}(a))^{-1}$.

In Theorem 4.4 below, we recall well known properties on γ-open sets and they are needed later. For a subset H of (X,τ) and a subset $U \subseteq H$, $Int_H(U)$ (resp. $Cl_H(U)$) is the interior (resp. closure) of the set U in a subspace $(H,\tau|H)$. The γ-interior of a subset A of (X,τ) is defined by

· $\gamma Int(A) := \bigcup\{V|V \subseteq A, V \in \gamma O(X,\tau)\}$. It is well known that: for a set $A \subseteq X$,
· ([6, Proposition 2.5]) $\gamma Int(A) = A \cap (Int(Cl(A)) \cup Cl(Int(A)))$ and
· $\gamma Cl(A) = A \cup (Int(Cl(A)) \cap Cl(Int(A)))$ hold (e.g., [19, Lemma 2.6(iii)], [3, Lemma 3.2]).

And, by [6, Proposition 2.3(a)] (cf. Theorem 4.4(iii)), it is shown that

· $\gamma Cl(A) \in \gamma C(X,\tau)$ and $\gamma Int(A) \in \gamma O(X,\tau)$, where A is a subset of (X,τ).
· $\gamma O(H,\tau|H) := \{U \subseteq H|\ U$ is γ-open in $(H,\tau|H)\}$;
· $\gamma C(H,\tau|H) := \{F \subseteq H|\ F$ is γ-closed in $(H,\tau|H)\}$;
· $\gamma Cl_H(U) := \bigcap\{F|\ U \subseteq F, F \in \gamma C(H,\tau|H)\}$, where $U \subseteq H \subseteq X$.

Theorem 4.4 (i) ([15],e.g.,[14, Lemma 2.2];[1, Proof of Theorem 2.3(3)]). *Let $H \subseteq X$ and $A_1 \subseteq X$. If H is α-open in (X,τ) and A_1 is γ-open in (X,τ), then $A_1 \cap H$ is γ-open in $(H,\tau|H)$.*

(ii) ([15];e.g.,[14, Lemma 2.4]) *Let $A \subseteq H \subseteq X$. If A is γ-open in $(H,\tau|H)$ and H is α-open in (X,τ), then A is γ-open in (X,τ).*

(iii) ([6, Proposition 2.3(a)]) *Arbitrary union of γ-open sets of (X,τ) is γ-open in (X,τ).*

(iv) ([6, Proposition 2.4(2)]) *Let $H \subseteq X$ and $A_1 \subseteq X$. If H is α-open in (X,τ) and A_1 is γ-open in (X,τ), then $A_1 \cap H$ is γ-open in (X,τ).*

(v) *If $B \subseteq H \subseteq X$ and H is α-open in (X,τ), then $\gamma Cl(B) \cap H = \gamma Cl_H(B)$ holds.*

(vi) *Let $F \subseteq H \subseteq X$. If H is α-open and γ-closed in (X,τ) and F is γ-closed in $(H,\tau|H)$, then F is γ-closed in (X,τ).* □

Remark 4.5 It follows from the following example that one of the assumptions of Theorem 4.4(vi) is not removed. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset,\{a\},X\}$ (cf. the space (Y_2,σ_2) of Example 3.7(iv)). For a subset $H := \{a,c\}$, the set H is γ-closed in $(H,\tau|H)$ and it is α-open in (X,τ), but H is not γ-closed in (X,τ).

Proposition 4.6 (i) *If $f : (X,\tau) \to (Y,\sigma)$ is γ-irresolute and a subset H is α-open in (X,τ), then $f|H : (H,\tau|H) \to (Y,\sigma)$ is γ-irresolute.*

(ii) *Let $k : (X,\tau) \to (K,\sigma|K)$ be a mapping and $j_K : (K,\sigma|K) \to (Y,\sigma)$ be the inclusion, where $K \subseteq Y$. Then, the following properties* (1), (2) *are equivalent, under the assumption that K is α-open in (Y,σ):*

(1) *$k : (X,\tau) \to (K,\sigma|K)$ is γ-irresolute;*

(2) *$j_K \circ k : (X,\tau) \to (Y,\sigma)$ is γ-irresolute.*

(iii) *If $f : (X,\tau) \to (Y,\sigma)$ is γ-irresolute, H is α-open in (X,τ) and $f(H)$ is α-open in (Y,σ), then $r_{H,f(H)}(f) : (H,\tau|H) \to (f(H),\sigma|f(H))$ is γ-irresolute* (cf. Definition 4.3).

Proof. The properties (i) and (ii)(1)⇒(2) (resp. (ii)(2)⇒(1)) are proved by using Theorem 4.4(i) (resp. Theorem 4.4(ii)). The property (iii) is proved by (i),(ii) above and (4.a) after Definition 4.3. □

Definition 4.7 For an α-open subset H of (X,τ), the following mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are well defined as follows (cf. Proposition 4.6(iii)), respectively:

$(r_H)_*(f) := r_{H,H}(f)$ for every $f \in \gamma r\text{-}h(X,X\setminus H;\tau)$;

$(r_H)_{*,0}(g) := r_{H,H}(g)$ for every $g \in \gamma r\text{-}h_0(X,X\setminus H;\tau)$.

Lemma 4.8 (A pasting lemma for γ-irresolute mappings) *Let $X = U_1 \cup U_2$, where U_1 and U_2 are α-open sets in (X,τ), and $f_1 : (U_1,\tau|U_1) \to (Y,\sigma)$ and $f_2 : (U_2,\tau|U_2) \to (Y,\sigma)$ are γ-irresolute mappings such that $f_1(x) = f_2(x)$ for every point $x \in U_1 \cap U_2$. Then its combination $f_1\nabla f_2 : (X,\tau) \to (Y,\sigma)$ is γ-irresolute, where $(f_1\nabla f_2)(x) := f_j(x)$ for every $x \in U_j (j \in \{1,2\})$.*

Proof. Let $V \in \gamma O(Y,\sigma)$. By Theorem 4.4 (ii) and (iii), it is proved that $(f_1\nabla f_2)^{-1}(V) \in \gamma O(X,\tau)$, because $f_i^{-1}(V) \in \gamma O(U_i,\tau|U_i)$, $f_i^{-1}(V) \in$-
$\gamma O(X,\tau)$ for each $i \in \{1,2\}$ and $(f_1\nabla f_2)^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V)$ hold. □

Theorem 4.9 *Let H be a subset of a topological space (X,τ).*

(i) (i-1) *If H is α-open in (X,τ), then the mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are homomorphisms of groups* (cf. Definition 4.7). *Moreover, $(r_H)_*|B_0 = (r_H)_{*,0}$ holds, where $B_0 := \gamma r\text{-}h_0(X,X\setminus H;\tau)$.*

(i-2) *If H is α-open and α-closed in (X,τ), then the mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are onto homomorphisms of groups.*

(ii) *For an α-open subset H of (X,τ), we have the following isomorphisms of groups:*

(ii-1) $\gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong Im(r_H)_*$;

(ii-2) $\gamma r\text{-}h_0(X, X\setminus H;\tau) \cong Im(r_H)_{*,0}$, *where* $Ker(r_H)_* := \{a \in \gamma r\text{-}h(X, X\setminus H;\tau) \mid (r_H)_*(a) = 1_X\}$ *is a normal subgroup of* $\gamma r\text{-}h(X, X\setminus H;\tau)$; $Im(r_H)_* := \{(r_H)_*(a) \mid a \in \gamma r\text{-}h(X, X\setminus H;\tau)\}$ *and* $Im(r_H)_{*,0} := \{(r_H)_{*,0}(b) \mid b \in \gamma r\text{-}h_0(X, X\setminus H;\tau)\}$ *are subgroups of* $\gamma r\text{-}h(H;\tau)$.

(iii) *For an α-open and α-closed subset H of (X,τ), we have the following isomorphisms of groups:*

(iii-1) $\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_*$;

(iii-2) $\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h_0(X, X\setminus H;\tau)$.

Proof. (i) (i-1) Since H is α-open in (X,τ), the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are well defined (cf. Definition 4.7). Let $a, b \in \gamma r\text{-}h(X, X\setminus H;\tau)$ and $\omega_{X,H} : \gamma r\text{-}h(X, X\setminus H;\tau) \times \gamma r\text{-}h(X, X\setminus H;\tau)) \to \gamma r\text{-}h(X, X\setminus H;\tau)$ be the binary operation of the group $\gamma r\text{-}h(X, X\setminus H;\tau)$ (cf. a few lines after Theorem 4.2). Then, $(r_H)_*(\omega_{X,H}(a,b)) = (r_H)_*(b\circ a) = r_{H,H}(b\circ a) = (r_{H,H}(b))\circ(r_{H,H}(a)) = \omega_H((r_H)_*(a),(r_H)_*(b))$ hold, where ω_H is the binary operation of the group $\gamma r\text{-}h(H;\tau|H)$ (cf. a few lines after Theorem 3.4). Thus, $(r_H)_*$ is a homomorphism of group. Similarly, the mapping $(r_H)_{*,0} : \gamma r\text{-}h_0(X, X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ is also a homomorphism of groups. It is obviously shown that $(r_H)_*|\gamma r\text{-}h_0(X, X\setminus H;\tau) = (r_H)_{*,0}$ holds (cf. Definition 4.1, Definition 4.7).

(i-2) Let $h \in \gamma r\text{-}h(H;\tau|H)$. We consider the combination $h_1 := (j_H\circ h)\nabla(j_{X\setminus H}\circ 1_{X\setminus H}) : (X,\tau) \to (X,\tau)$. By Proposition 4.6 (ii) and the assumption of α-openness of H, it is shown that the two mappings $j_H\circ h : (H,\tau|H) \to (X,\tau)$ and $j_H\circ h^{-1} : (H,\tau|H) \to (X,\tau)$ are γ-irresolute. Moreover, under the assumption of α-openness of $X\setminus H$, $j_{X\setminus H}\circ 1_{X\setminus H} : (X\setminus H, \tau|(X\setminus H)) \to (X,\tau)$ is γ-irresolute. By using Lemma 4.8 for an α-open cover $\{H, X\setminus H\}$ of X, the combination above $h_1 : (X,\tau) \to (X,\tau)$ is γ-irresolute and h_1 is bijective and its inverse mapping $h_1^{-1} = (j_H\circ h^{-1})\nabla(j_{X\setminus H}\circ 1_{X\setminus H})$ is also γ-irresolute. Thus, we have that $h_1 \in \gamma r\text{-}h(X,\tau)$. Since $h_1(x) = x$ for every point $x \in X\setminus H$, we conclude that $h_1 \in \gamma r\text{-}h_0(X, X\setminus H;\tau)$ and so $h_1 \in \gamma r\text{-}h(X, X\setminus H;\tau)$; moreover, $(r_H)_{*,0}(h_1) = (r_H)_*(h_1) = r_{H,H}(h_1) = h$.

(ii) By (i-1) above and the first isomorphism theorem of group theory, it is shown that there are group isomorphisms below, under the assumption of the α-openness of H in (X,τ):
(4.d) $\gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong Im(r_H)_*$ and
(4.e) $\gamma r\text{-}h_0(X, X\setminus H;\tau)/Ker(r_H)_{*,0} \cong Im(r_H)_{*,0}$, where $Ker(r_H)_{*,0} := \{a \in \gamma r\text{-}h_0(X, X\setminus H;\tau) \mid (r_H)_{*,0}(a) = 1_X\}$.

It is shown that $Ker(r_H)_{*,0} = \{1_X\}$. Indeed, let $u_0 \in Ker(r_H)_{*,0} \subset \gamma r\text{-}h_0(X, X\setminus H;\tau)$; then $(r_H)_{*,0}(u_0) = 1_H$, where 1_H is the identity element of $\gamma r\text{-}h(H;\tau|H)$. By Definitions 4.7 and 4.3, we have that, for any point $x \in H$, $((r_H)_{*,0}(u_0))(x) = (r_{H,H}(u_0))(x) = u_0(x)$ and so, $u_0(x) = 1_H(x)$; and, for any point $x \in X\setminus H$, $u_0(x) = x$ (cf. Definition 4.1(ii)). Thus, we conclude that $u_0 = 1_X$; and hence $Ker(r_H)_{*,0} = \{1_X\}$. Therefore, by using the isomorphism (4.e) above, we have the isomorphism (ii-2).

(iii) By (i-2) and (ii), the isomorphisms (iii-1) and (iii-2) are obtained. □

Example 4.10 (i) In Example 5.13 of Section 5, the groups in Theorem 4.9 above are given for a special subspace $(H,\kappa|H)$ of the digital line $(\mathbb{Z},\kappa)$.

(ii) Let (X,τ) be the topological space of Example 3.7(ii) throughout the present Example 4.10(ii).
(ii-1) Let $H := \{a\}$. Since $H = \{a\}$ is α-open and α-closed in the topologica space (X,τ), then we apply Theorem 4.9(iii) to the present case; and so, we have the following result:
$\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong \gamma r\text{-}h_0(X, X\setminus H;\tau)$.
We can check directly the group isomorphisms as follows: we have the date: $\gamma r\text{-}h(X, X\setminus H;\tau) = \{1_X, h_a\}$, $Ker(r_H)_* = \{1_X, h_a\}$, $\gamma O(H,\tau|H) = \{\emptyset, H\}$, $\gamma r\text{-}h(H;\tau|H) = \{1_H\}$ and $\gamma r\text{-}h_0(X, X\setminus H;\tau) = \{1_X\}$, where $\tau|H = \{\emptyset, H\}$.
(ii-2) Let $H := \{b, c\}$. Then H is α-open and α-closed in (X,τ). Now, we apply Theorem 4.9

(iii) to the present case; and we can also check directly the group isomorphisms: we have the date as follows: γr-$h(X, X \setminus H; \tau) = \{1_X, h_a\}$, $Ker(r_H)_* = \{1_X\}$, $\gamma O(H, \tau|H) = P(H)$, γr-$h(H; \tau|H) = \{1_H, h_a|H\}$ and γr-$h_0(X, X \setminus H; \tau) = \{1_X, h_a\}$, where $\tau|H = \{\emptyset, H\}$.

Example 4.11 Even if a subset H of a topological space (X, τ) is not α-closed and it is α-open (cf. Theorem 4.9(i)(i-2)), we have some examples such that the homomorphisms $(r_H)_*$ and $(r_H)_{*,0}$ are onto.

(i) For example, let (X, τ) be a topological space and $(H, \tau|H)$ a subspace of (X, τ), where $X := \{a, b, c\}$, $H =: \{a, b\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$; and so, $\tau|H = \{\emptyset, \{a\}, \{b\}, H\}$. Then, we see that $\gamma O(X, \tau) = P(X) \setminus \{\{c\}\}$ and $\tau^\alpha = \tau$. The subset H is α-open and it is not α-closed in (X, τ). Hence by Theorem 4.9(i)(i-1), the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are homomorphisms of groups. Because of $X \setminus H = \{c\}$, we see that γr-$h_0(X, X \setminus H; \tau) = \gamma r$-$h(X, X \setminus H; \tau)$ and $(r_{H,0})_* = (r_H)_*$. And it is shown directly that γr-$h(X, X \setminus H; \tau) = \{1_X, h_c\} \cong \mathbb{Z}_2, (h_c)^2 = 1_X$, and γr-$h(H; \tau|H) = \{1_H, t_{a,b}\}$, where $h_c : (X, \tau) \to (X, \tau)$ and $t_{a,b} : (H, \tau|H) \to (H, \tau|H)$ are the bijections defined by $h_c(a) = b, h_c(b) = a, h_c(c) = c$ and $t_{a,b}(a) = b, t_{a,b}(b) = a$, respectively. Then, we prove that : $(r_H)_* : \gamma r$-$h(X, X \setminus H; \kappa) \to \gamma r$-$h(H; \tau|H)$ is onto; $Ker(r_H)_* = \{1_X\}$. By using Theorem 4.9(i)(i-1) and (ii), we have that γr-$h(H; \tau|H) \cong \gamma r$-$h(X, X \setminus H; \tau)/Ker(r_H)_* = \gamma r$-$h(X, X \setminus H; \tau)$ hold.

(ii) In Section 5, we give an example of an onto homomorphism $(r_H)_*$, where $H := \{-1, 0, 1\}$ of the digital line $(\mathbb{Z}, \kappa)$ (cf. Example 5.13(iv)).

5 Examples on the digital line $(\mathbb{Z}, \kappa)$ We recall that *the digital line* is the set of the integers, $\mathbb{Z}$, equipped with the topology κ having $\{\{2s-1, 2s, 2s+1\} |\ s \in \mathbb{Z}\}$, say $\mathbf{G}$, as a subbase (e.g., [24, p.175], [26, Section 3(I)], [23, p.905,p.908]). This topological space is denoted by $(\mathbb{Z}, \kappa)$. By the definition of topology κ, every singleton $\{2u+1\}$ is open in $(\mathbb{Z}, \kappa)$ and it is not closed in $(\mathbb{Z}, \kappa)$, where $u \in \mathbb{Z}$. Every singleton $\{2s\}$ is closed in $(\mathbb{Z}, \kappa)$ and it is not open in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. In the present paper, we denote: $U(2s) := \{2s-1, 2s, 2s+1\}$ and $U(2u+1) := \{2u+1\}$ for each point $2s$ and $2u+1$ of $(\mathbb{Z}, \kappa)$, respectively; and $U(2s)$ and $U(2u+1)$ are two typical open sets of $(\mathbb{Z}, \kappa)$. And, $U(x)$ above is called *the smallest open set containing the point* x of $(\mathbb{Z}, \kappa)$, where $x \in \mathbb{Z}$. It is well known that: for a nonempty open set U and a point x of $(\mathbb{Z}, \kappa)$, if $x \in U$, then $U(x) \subseteq U$ holds (e.g., [26, Section 3]).

(I) Characterizations of γ-open sets in the digital line $(\mathbb{Z}, \kappa)$ (cf. Theorems 5.1 and 5.5 below). First, we recall some properties on the digital line $(\mathbb{Z}, \kappa)$: $\kappa = PO(\mathbb{Z}, \kappa)$ and $PO(\mathbb{Z}, \kappa) \subsetneq SO(\mathbb{Z}, \kappa) = \gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ (cf. [9], [17], [33]). Secondly, we need some notations and properties (e.g., [18, Sections 1, 2], [26, Sections 2, 3]): let A be a nonempty subset of $(\mathbb{Z}, \kappa)$, $A_\kappa := \{x \in A |\ \{x\}$ is open in $(\mathbb{Z}, \kappa)\}$; $A_{\mathbf{F}} := \{x \in A |\ \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$. It is easily shown that:
(i) $A_\kappa = \{2s+1 \in A |\ s \in \mathbb{Z}\}$; $A_{\mathbf{F}} = \{2m \in A |\ m \in \mathbb{Z}\}$; and
(ii) $A = A_\kappa \cup A_{\mathbf{F}}$ $(A_\kappa \cap A_{\mathbf{F}} = \emptyset)$, where A is any subset of $(\mathbb{Z}, \kappa)$.

By Takigawa [32, Theorems 1, 2 and 3], some characterizations of any preopen sets, semi-open sets and semi-preopen sets in the digital n-space $(\mathbb{Z}^n, \kappa^n)$ are investigated, where $n \geq 1$. The following property is obtained by a special case of [32, Theorem 2 or Theorem 3] for the digital line (i.e., $n = 1$).

Theorem 5.1 (A special case of Takigawa [32, Theorem 2 or Theorem 3]) *A subset E is γ-open in $(\mathbb{Z}, \kappa)$ if and only if $E \subseteq Cl(E_\kappa)$ holds in $(\mathbb{Z}, \kappa)$.*

Remark 5.2 (i) If $A_\kappa = \emptyset$ for a subset A of $(\mathbb{Z}, \kappa)$, then A is closed in $(\mathbb{Z}, \kappa)$. The converse of above implication is not true; a subset $\{2s, 2s+1, 2s+2\}$ is closed in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$; and $(\{2s, 2s+1, 2s+2\})_\kappa = \{2s+1\} \neq \emptyset$.

(ii) $Cl(A) = Cl(A_\kappa) \cup A$ holds for a subset A of $(\mathbb{Z}, \kappa)$.

Definition 5.3 ([7, Definition 5.3]) Let A be a subset of $(\mathbb{Z}, \kappa)$.

(i) For a point $x \in \mathbb{Z}$, the following set $V_A(x)$ is defined: if $x+1 \in A$, then $V_A(x) := \{x, x+1\}$ (sometimes it is denoted by ${V_A}^+(x)$, or shortly $V^+(x)$); if $x+1 \notin A$, then $V_A(x) := \{x-1, x\}$ (sometimes it is denoted by ${V_A}^-(x)$, or shortly $V^-(x)$). Thus, we have that $V_A(x) = {V_A}^+(x)$ or ${V_A}^-(x)$.

(ii) $V_A := \bigcup\{V_A(x) |\ x \in A_{\mathbf{F}}\}$ if $A_{\mathbf{F}} \neq \emptyset$; $V_A := \emptyset$ if $A_{\mathbf{F}} = \emptyset$.

Example 5.4 (i) A subset $\{x, x+1\}$ of $\mathbb{Z}$ is γ-open and γ-closed in $(\mathbb{Z}, \kappa)$ for any point $x \in \mathbb{Z}$.

(ii) (cf. [7, Example 5.5]) For a point $x \in \mathbb{Z}$ and a subset $A \subseteq \mathbb{Z}$, the set $V_A(x)$ is both γ-open and γ-closed in $(\mathbb{Z}, \kappa)$ (cf. Definition 5.3).

Finally, the following characterization (Theorem 5.5) is obtained by using the equality $\gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ and [7, Theorem 5.7]. We note that we are able to have directly an alternative proof of Theorem 5.5 using the characterization of Theorem 5.1 above.

Theorem 5.5 ([7, Theorem 5.7]) *Let B be a nonempty subset of $(\mathbb{Z}, \kappa)$. Then the following statements hold.*

(i) *Assume that $B_{\mathbf{F}} \neq \emptyset$.*

(i-1) *If B is γ-open in $(\mathbb{Z}, \kappa)$, then B is expressible as the union: $B = V_B \cup B_\kappa$, where $V_B := \bigcup\{V_B(x) | x \in B_{\mathbf{F}}\}$ (cf. Definiton 5.3).*

(i-2) *If B satisfies a property that $B = V_B \cup B_\kappa$, then B is γ-open in $(\mathbb{Z}, \kappa)$.*

(ii) *Assume that $B_{\mathbf{F}} = \emptyset$. Then, $V_B = \emptyset$ and $B = B_\kappa$ hold and B is open in $(\mathbb{Z}, \kappa)$; and so B is γ-open in $(\mathbb{Z}, \kappa)$.* □

Example 5.6 Suppose that a singleton $\{x\}$ is closed in $(\mathbb{Z}, \kappa)$ (i.e., x is even in $\mathbb{Z}$) and y is any point with $y \neq x$. Then,
(i) $\{x, y\}$ is γ-closed in $(\mathbb{Z}, \kappa)$;
(ii) $\{x, y\}$ is γ-open if and only if $y = x+1$ or $y = x-1$.

(II) Some transformations on $(\mathbb{Z}, \kappa)$.

Definition 5.7 Let $t_{e+,o-} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa), t_- : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ and $f_s : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$, be the transformations defined by the following form, respectively: for every point $x \in \mathbb{Z}$,
(i) $t_{e+,o-}(x) := x+1$ if x is even and $t_{e+,o-}(x) := x-1$ if x is odd;
(ii) $t_-(x) := -x$; (iii) $f_s(x) := x+s$.

Theorem 5.8 *For any γ-open set A of $(\mathbb{Z}, \kappa)$, we have the following properties:*

(i) ${t_{e+,o-}}^{-1}(A)$ *is explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$;*

(ii) ${t_-}^{-1}(A)$ *is explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$;*

(iii)([7, Lemma 5.8(vii), Theorem 5.10(iii)]) ${f_{2m+1}}^{-1}(A)$ *and $f_{2m+1}(A)$ are explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$.*

Proof. (i) By using Definition 5.3, Example 5.6(i) and Definition 5.7, it is shown that, for any set B and any point $x \in \mathbb{Z}$, ${t_{e+,o-}}^{-1}(V_B(x))$ is γ-closed in $(\mathbb{Z}, \kappa)$ (cf. Definition 5.3(i), Example 5.6(i), Definition 5.7); ${t_{e+,o-}}^{-1}(B_\kappa) = \bigcup\{\{2s\} |\ 2s+1 \in B\}$ holds, because $B_\kappa = \bigcup\{\{2s+1\} |\ 2s+1 \in B\}$). And, so ${t_{e+,o-}}^{-1}(B_\kappa)$ is the union of the collection $\{\{2s\} |\ 2s+1 \in B\}$ of γ-closed sets. Let $A \in \gamma O(\mathbb{Z}, \kappa)$. By Theorem 5.5(i-1) and (ii), it is shown that ${t_{e+,o-}}^{-1}(A) = (\bigcup\{{t_{e+,o-}}^{-1}(V_A(x)) |\ x \in A_{\mathbf{F}}\}) \cup {t_{e+,o-}}^{-1}(A_\kappa)$ (if $A_{\mathbf{F}} \neq \emptyset$) and ${t_{e+,o-}}^{-1}(A) = {t_{e+,o-}}^{-1}(A_\kappa)$ (if $A_{\mathbf{F}} = \emptyset$); and so, by the properties above respectively, ${t_{e+,o-}}^{-1}(A)$ is the union of a collection of γ-closed sets.

(ii) By an argument similar to that in (i), the statement (ii) is proved (cf. Definition 5.3, Example 5.4).

(iii) This is shown by the property that $\gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ (cf. (I) above) and the corresponding property on β-openess version [7, Lemma 5.8(vii), Theorem 5.10(iii)]. □

Remark 5.9 Let $A_{2k} := \{2k, 2k+1\} \cup \{2(k+1)+1, 2(k+1)+2\}$. Since $Int(Cl(A_{2k})) \cap Cl(Int(A_{2k})) = \{2k+1, 2(k+1), 2(k+1)+1\} \not\subseteq A_{2k}$ hold, A_{2k} is not γ-closed. But, A_{2k} is the union of two γ-closed sets $\{2k, 2k+1\}$ and $\{2(k+1)+1, 2(k+2)\}$ of $(\mathbb{Z}, \kappa)$ (cf. Example 5.4 (i)).

Example 5.10 (i) $t_{e+,o-} \notin \gamma r\text{-}h(\mathbb{Z};\kappa)$ and $t_{e+,o-} \notin contra\text{-}\gamma r\text{-}h(\mathbb{Z};\kappa)$ hold.

(ii) $t_{-} \in h(\mathbb{Z},\kappa)$ holds and so $t_{-} \in \gamma r\text{-}h(\mathbb{Z};\kappa)$.

(iii) (iii-1) $f_{2m+1} \notin \gamma r\text{-}h(\mathbb{Z};\kappa)$ and $f_{2m+1} \notin contra\text{-}\gamma r\text{-}h(\mathbb{Z};\kappa)$;
(iii-2) $f_{2m+1} \notin h(\mathbb{Z};\kappa)$.

(iv) $f_{2m} \in h(\mathbb{Z};\kappa)$ and $f_{2m+1} \in contra\text{-}h(\mathbb{Z};\kappa)$ hold; and hence $\{f_s | s \in \mathbb{Z}\}$ forms a subgroup of $H_{(\mathbb{Z},\kappa)}$.

(III) A group structure of γr-$h(H;\kappa|H)$, where $H := \{-1, 0, 1\}$.

Lemma 5.11 *Let $s, u \in \mathbb{Z}$. If $f : (\mathbb{Z},\kappa) \to (\mathbb{Z},\kappa)$ is a γr-homeomorphism* (i.e., $f \in \gamma r$-$h(\mathbb{Z},\kappa)$)*, then*

(i) *$f(U(2s)) = U(2a)$ holds for some point $2a \in \mathbb{Z}$;*

(ii) *$f(U(2u+1)) = U(2v+1)$ holds for some point $2v+1 \in \mathbb{Z}$.* □

Notation Let H be the smallest open set containing 0, $U(0) := \{-1, 0, +1\}$, which is used in Example 5.13 below. A family of subsets of $(\mathbb{Z},\kappa)$, say $\{H_j | j \in \mathbb{Z}$ with $j \geq 1\}$, is defined by : $H_1 := H = U(0)$ and $H_i := U(-(2i-2)) \cup H_{i-1} \cup U(2i-2)$ for each integer $i \geq 2$, where $U(2s) := \{2s-1, 2s, 2s+1\}(s \in \mathbb{Z})$.

It is easily shown that $H_i = \bigcup\{U(-(2j-2)) \cup U(2j-2) |\ j \in \mathbb{Z}$ with $1 \leq j \leq i\}$ holds for each integer $i \geq 2$; and if $i \leq j$, then $H_i \subseteq H_j$ and $\bigcup\{H_j |\ j \in \mathbb{Z}$ with $j \geq 1\} = \mathbb{Z}$.

Lemma 5.12 below is proved by an argument similar to that in [30, Claim in Proof of Proposition 6.1]; we use induction on $m \in \mathbb{Z}$ and Lemma 5.11; and so we omite the proof.

Lemma 5.12 *Let $f \in \gamma r\text{-}h(\mathbb{Z}, \mathbb{Z}\backslash H;\kappa)$ and $\{H_j | j \in \mathbb{Z}$ with $j \geq 1\}$ be the family of subsets defined by Notation above, where $H = H_1 = \{-1, 0, 1\}$,* i.e., *$H = U(0)$.*

(i) *If $f|H = t_{-}|H$, then $f|H_m = t_{-}|H_m$ for any interger m with $m \geq 2$.*

(ii) *If $f|H = 1_H$, then $f|H_m = 1_{H_m}$ for any integer m with $m \geq 2$.* □

Using Lemma 5.11 and Lemma 5.12, we can examine the isomorphisms of Theorem 4.9(ii) for the following α-open set $H := U(0)$ which is not α-closed in $(\mathbb{Z},\kappa)$.

Example 5.13 Let $(H, \kappa|H)$ be a subspace of $(\mathbb{Z},\kappa)$, where $H := \{-1, 0, +1\}$ is the smallest open set containing $0 \in \mathbb{Z}$, i.e., $H = U(0)$. Then, we have the following properties: (i) γr-$h(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) = \{1_{\mathbb{Z}}, t_{-}\}$; (ii) γr-$h_0(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) = \{1_{\mathbb{Z}}\}$; (iii) γr-$h(H;\kappa|H) = \{1_H, t_{-}|H\}$; (iv) $Im(r_H)_* = \{1_H, t_{-}|H\}$ and $(r_H)_* : \gamma r\text{-}h(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) \to \gamma r\text{-}h(H, \kappa|H)$ is onto; (v) $Ker(r_H)_* = \{1_{\mathbb{Z}}\}$.

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EXISTENCE OF UNBOUNDED SOLUTIONS TO A ONE DIMENSIONAL ISENTROPIC PERIODIC FLOW OF A COMPRESSIBLE VISCOUS FLUID WITH SELF-GRAVITATION

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Abstract. We consider a one dimensional isentropic periodic flow of a compressible viscous fluid driven by a self-gravitation of the fluid. We show the existence of an unbounded solution of a system describing the flow. A sufficient condition for the unboundedness is given in terms of the initial values of an energy form.

1 Introduction Let us consider a one dimensional isentropic flow of a compressible viscous fluid in the Lagrangian mass coordinates:

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x (a v^{-\gamma}) - \mu \partial_x \left(\dfrac{\partial_x u}{v} \right) = \mathcal{G}, \end{cases} \tag{1.1}$$

where specific volume v, assumed to take positive values, and velocity u of the fluid are unknown functions of the time and space variables $t \geq 0$ and $x \in \mathbf{R}$, pressure $av^{-\gamma}$ a function of v with constants $a > 0$ and $\gamma \geq 1$, and $\mu > 0$ the viscosity constant. The second member $\mathcal{G}$ is an external force specified below. We are mainly concerned with the so-called isentropic flow, i.e., $\gamma > 1$, though, we occasionally refer to the isothermal flow, i.e. $\gamma = 1$ for the sake of comparison.

The initial or initial-boundary value problem for (1.1) with a prescribed forcing term $\mathcal{G}$ has been studied by several authors. Since the pioneering work of Kanel' [3], showing the existence of global bounded solutions to the system (1.1) on the whole line with $\mathcal{G} \equiv 0$, the boundedness is one of the crucial keys to study the asymptotic behavior of the solutions. Closely related with the present paper are the works of Matsumura and Nishida [4], and Matsumura and Yanagi [5]. In [4] it is shown that the isothermal system on a finite interval with a general bounded forcing term $\mathcal{G}$ has a unique global bounded solution for any smooth initial data. In [5] a similar result was obtained for the isentropic system but on the assumption of smallness of $\gamma - 1$ depending on the data. Both the results fail to mention whether an unbounded solution exists or not for the isentropic system with a bounded forcing term.

This paper handles the system (1.1) under the L-periodic condition:

$$v(t, x+L) = v(t, x), \quad u(t, x+L) = u(t, x) \tag{1.2}$$

with a rather special forcing term depending on the unknowns:

$$\mathcal{G}(t, x) = -\frac{4\pi G}{\bar{v}} \partial_x \int_0^L K_L(x, y)(v(t, y) - \bar{v}) dy, \tag{1.3}$$

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where $K_L(x,y)$ is the Green kernel of the operator $-\partial_x^2$ on the L-periodic functions with average 0:

$$K_L(x,y) = \sum_{n=1}^{\infty} \frac{L}{2\pi^2 n^2} \cos \frac{2\pi n}{L}(x-y),$$

or

$$K_L(x,y) = -\frac{|x-y|}{2} + \frac{(x-y)^2}{2L} + \frac{L}{12}, \quad 0 \le x, y \le L, \tag{1.4}$$

$\bar{v}$ the average of the specific volume:

$$\bar{v} = \frac{1}{L}\int_0^L v(t,x)dx,$$

and $G > 0$ the gravitational constant. This is the representation in the Lagrangian mass coordinates of a force field that takes into account only the part of Newton's gravitation corresponding to the disturbance in an infinite homogeneous fluid, and is often adopted in the classical theory of gravitational instability for the fluid. See Weinberg [7], Chapter 15. Notice that the field is consistent with static equilibria of the fluid.

Since the average $\bar{v}$ as well as that of u is a constant of motion in view of (1.1), the forcing term (1.3) is a bounded function of the variables t and x. This enables us to show the boundedness of any smooth solutions to the isothermal system just in the same manner as in [4]. As for the isentropic system, however, the situation proves to be quite different. Indeed, on the assumption $1 < \gamma < 2$ we show the existence of unbounded solutions in the sense that

$$\sup_{t,x} v(t,x) = \infty.$$

To be precise we present an initial condition for unbounded solutions in terms of the form for a state (v,u) given by

$$\mathcal{E}(v,u) = \int_0^L \frac{1}{2}u(x)^2 dx + \mathcal{E}(v) \tag{1.5}$$

with

$$\begin{aligned} \mathcal{E}(v) = &\int_0^L a\left(\frac{v(x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma}\right)dx \\ &- \frac{2\pi G}{\bar{v}}\int_0^L\int_0^L K_L(x,y)(v(x)-\bar{v})(v(y)-\bar{v})dxdy. \end{aligned} \tag{1.6}$$

This form, called the energy form associated with the system (1.1)–(1.3), is decreasing and bounded along the orbit of a solution to the system, which turns out to be the key to find out the unboundedness condition.

The paper is organized as follows. In Section 2, after a brief comment on the class of solutions concerned, we present two theorems. One refers to the structure of the whole stationary solutions. The other constitutes the main part of the paper showing an initial condition for unbounded solutions to the Cauchy problem. In Section 3 we study the large time behavior of bounded solutions and show a reason why the stationary problem is inevitably related to the unboundedness of solutions to the Cauchy problem. In Section 4 we study the structure of the whole stationary solutions with the comparison of the values of energy form at the stationary solutions. From this together with the decreasing property of energy form we formulate an initial condition for unbounded solutions. Finally in Section 5, focusing on the behavior of the energy form near the stationary solutions, we supplement the condition for unboundedness to make it meaningful.

This paper completes the preceding one [6] with details of the unboundedness of solutions to the isentropic system. By replacing (1.6) with

$$\mathcal{E}(v) = \int_0^L a\left(\frac{v(x)-\bar{v}}{\bar{v}} - \log\frac{v(x)}{\bar{v}}\right)dx \\ - \frac{2\pi G}{\bar{v}}\int_0^L\int_0^L K_L(x,y)(v(x)-\bar{v})(v(y)-\bar{v})dxdy$$

some results of the present paper are, with natural modifications, valid also for the isothermal system. See [6].

2 Notation and main results For a nonnegative integer m and a positive number L let C^m be the space of m times continuously differentiable periodic real-valued functions on $\mathbf{R}$ with period L, and H^m the Sobolev space of locally square integrable L-periodic real-valued functions on $\mathbf{R}$ equipped with scalar product

$$(h_1,h_2)_{H^m} = \sum_{j=0}^m \int_0^L \partial_x^j h_1(x)\partial_x^j h_2(x)dx$$

and norm $\|h\|_{H^m} = \sqrt{(h,h)_{H^m}}$. We write $H^0 = L^2$ as usual. Let s be a nonnegative integer and X a Banach space with norm $\|\cdot\|$. The space of s-times continuously differentiable functions on $[0,\infty)$ with values in X is denoted by $C^s([0,\infty);X)$. $H^s_{\mathrm{loc}}(0,\infty;X)$ denotes the space of X-valued strongly measurable functions on $[0,\infty)$ whose distributional derivatives up to order s are locally square integrable, i.e.,

$$\int_0^T \|\partial_t^k u(t)\|^2 dt < \infty \quad \text{for any } k=0,\dots,s \text{ and } T>0.$$

Noting that the forcing term (1.3) is a bounded function of the variables t and x on the time interval of existence for a solution, we are allowed to consider a unique global solution for the Cauchy problem of (1.1)–(1.3) having initial value $(v_0,u_0)\in H^1\times H^1$ with $v_0>0$ arbitrarily given at $t=0$, as for the initial-boundary value problem on a finite interval supplemented by solid boundary condition with a general bounded forcing term. In what follows the solution of the Cauchy problem is meant by a unique global solution having the property

$$\begin{cases} v\in C^1([0,\infty);L^2)\cap C^0([0,\infty);H^1), \quad v(t,\cdot)>0,\\ u\in H^1_{\mathrm{loc}}(0,\infty;L^2)\cap L^2_{\mathrm{loc}}(0,\infty;H^2).\end{cases}$$

Without loss of generality we may assume that the average of u vanishes, taking $(v,u-\bar{u})$ as new unknown functions, if necessary.

In order to present an initial condition for unbounded solutions we first refer to the structure of the stationary solutions to (1.1)–(1.3). Noting that the average $\bar{v}$ of v is a constant of motion, we consider the stationary solutions on the following manifold in $H^1\times H^1$ parametrized by a positive number V:

$$M_V = \{(v,u)\in H^1\times H^1;\, v>0,\ \bar{v}=V,\ \bar{u}=0\}.$$

Clearly, the trivial solution $(V, 0)$ lies in M_V. A non-trivial stationary solution, if exists, has the least period L/j for some positive integer j. Let us now introduce a function I_γ on the interval $(0, (\gamma-1)^{-1/2})$ expressed as

$$I_\gamma(\theta) = \theta \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{f_+(F_+{}^{-1}(\theta^2 y))}\, dy + \theta \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{f_-(F_-{}^{-1}(\theta^2 y))}\, dy, \tag{2.1}$$

where the functions $f_+(r)$, $F_+(r)$ on $r \ge 0$, and $f_-(r)$, $F_-(r)$ on $0 \le r < 1$ are given by

$$\begin{aligned} f_+(r) &= 1 - (1+r)^{-1/\gamma}, & F_+(r) &= \int_0^r f_+(s)\, ds, \\ f_-(r) &= -\{1 - (1-r)^{-1/\gamma}\}, & F_-(r) &= \int_0^r f_-(s)\, ds. \end{aligned}$$

As shown by **Lemma 4** below in Section 4, I_γ is a monotone increasing function with $I_\gamma(\theta) > \sqrt{2\gamma}\pi$ provided that $1 < \gamma < 2$. Moreover, $I_\gamma(\theta)$ has a finite limit as $\theta \to (\gamma-1)^{-1/2} - 0$.

Theorem 1 Assume $1 < \gamma < 2$. For $V > 0$ let $k_{\min}$ and $k_{\max}$, respectively, be the smallest and the largest integers j satisfying

$$\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < \frac{L}{j} < \frac{I_\gamma((\gamma-1)^{-1/2} - 0)}{\sqrt{2\gamma}\pi} \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}. \tag{2.2}$$

Then, for $j = k_{\min}, \ldots, k_{\max}$ there exists on M_V a stationary solution of (1.1)–(1.3) with least period L/j. The whole stationary solutions lying in M_V except for the trivial one are given by $(\tilde{v}^{(j)}(\,\cdot\, - \alpha), 0)$, $0 \le \alpha < L/j$, $j = k_{\min}, \ldots, k_{\max}$, where $(\tilde{v}^{(j)}, 0)$ is one of the stationary solutions with least period L/j.

Remark 1 When $V \le \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$, no integer satisfies the condition (2.2), and hence the stationary problem admits on M_V only the trivial solution. When $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$, (2.2) holds with $j = 1$, and hence $k_{\min} = 1$, while when $V \ge \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$, $k_{\min}$, if it makes sense, must be greater than or equal to 2.

Let us recall the energy form (1.5) with (1.6). By L-periodicity of v we have $\mathcal{E}(v(\,\cdot - \alpha)) = \mathcal{E}(v)$ for any $\alpha \in \mathbf{R}$. The following theorem gives an initial condition for unbounded solutions to the isentropic system (1.1)–(1.3) with $1 < \gamma < 2$.

Theorem 2 Assume $1 < \gamma < 2$. Let $V \ge \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ and $\tilde{v}^{(k_{\min})}$ be as in **Theorem 1**.

(i) The subset of $H^1 \times H^1$ given by

$$A_V = \left\{ (v, u) \in M_V \,\middle|\, \mathcal{E}(v, u) < \begin{cases} \mathcal{E}(\tilde{v}^{(k_{\min})}), & \text{if integers } j \text{ with (2.2) exist,} \\ 0, & \text{otherwise} \end{cases} \right\} \tag{2.3}$$

is nonempty.

(ii) Any solution of (1.1)–(1.3) with initial value from A_V is unbounded, i.e.,

$$\sup_{t,x} v(t, x) = \infty.$$

Remark 2 In view of the decreasing property of the energy form shown by **Lemma 1** in the following section the statement of **Theorem 2** suggests that $\mathcal{E}(\tilde{v}^{(k_{\min})})$ if it makes sense or else $\mathcal{E}(V)=0$ is minimal amongst the values of the energy form evaluated at the stationary solutions on M_V. This itself is true also for the case $V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ as shown by **Proposition 1** in Section 4, however, in this case we fail to refer to the existence or nonexistence of unbounded solutions for some technical reasons. See **Remark 3** in the final section.

3 Large time behavior of bounded solutions As a preliminary but vital step, we devote this section to the study of the global behavior of a solution of (1.1)–(1.3) subject to

$$\sup_{t,x} v(t,x) < \infty. \tag{3.1}$$

The results in the present section have already been given in [6] with rather detailed proofs, however, we give them for the sake of completeness.

We first show that the energy form (1.5) with (1.6) is non-increasing along the orbits of solutions. This is true for any $\gamma > 1$ regardless of the boundedness of solutions.

Lemma 1 For a solution (v,u) of (1.1)–(1.3) put

$$E(t) = \mathcal{E}(v(t,\,\cdot\,), u(t,\,\cdot\,)), \quad t \geq 0. \tag{3.2}$$

Then we have

$$\frac{dE}{dt}(t) = -\mu \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \leq 0, \quad \inf_t E(t) = E(\infty) > -\infty. \tag{3.3}$$

Proof: Taking the derivative of E and then using the symmetry of the integral kernel K_L, we obtain an expression for the derivative dE/dt as

$$\int_0^L \Big\{ u(t,x)\partial_t u(t,x) + \left(a\bar{v}^{-\gamma} - av(t,x)^{-\gamma}\right)\partial_t v(t,x) - \frac{4\pi G}{\bar{v}} \int_0^L K_L(x,y)(v(t,y)-\bar{v})dy\,\partial_t v(t,x) \Big\} dx.$$

After substituting $\partial_x u$ for $\partial_t v$, by integration by parts we get

$$\frac{dE}{dt}(t) = \int_0^L u(t,x) \left\{ \partial_t u(t,x) + \partial_x\left(av(t,x)^{-\gamma}\right) + \frac{4\pi G}{\bar{v}} \partial_x \int_0^L K_L(x,y)(v(t,y)-\bar{v})dy \right\} dx.$$

Using the second equation of (1.1), by integration by parts we obtain the desired equality for dE/dt. The boundedness of E from below follows from

$$\frac{v-\bar{v}}{\bar{v}^\gamma} - \frac{v^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma} \geq 0,$$

the positivity of v and the boundedness of the kernel K_L. □

Integrating the equality (3.3) over $(0,\infty)$, we obtain the following.

Corollary of Lemma 1 We have

$$\mu \int_0^\infty \left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \right) dt = E(0) - E(\infty) < \infty. \tag{3.4}$$

The following lemma shows that if $1 < \gamma \leq 2$, the upper bound of v controls the H^1 norm as well as the lower bound of v of a solution. Notice that the same result holds true of the isothermal system without any assumptions on a priori bounds of a solution. See Matsumura and Nishida [4].

Lemma 2 Assume $1 < \gamma \leq 2$. For a solution (v, u) of (1.1)–(1.3) with $\bar{u} = 0$, if it is bounded in the sense of (3.1), then we have

$$\sup_t \|v(t,\cdot)\|_{H^1} < \infty, \quad \sup_t \|u(t,\cdot)\|_{H^1} < \infty, \quad \inf_{t,x} v(t,x) > 0. \tag{3.5}$$

Proof: We consider the forcing term (1.3) as a bounded function of the variables t and x, and make use of the equalities

$$\begin{aligned}
&\frac{d}{dt} \int_0^L \left\{ \frac{1}{2} u(t,x)^2 + a \left(\frac{v(t,x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) \right\} dx \qquad (3.6)\\
&= -\mu \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx + \int_0^L \mathcal{G}(t,x) u(t,x) dx,\\
&\frac{d}{dt} \int_0^L \left(\frac{\mu}{2} \frac{\partial_x v(t,x)^2}{v(t,x)^2} - u(t,x) \frac{\partial_x v(t,x)}{v(t,x)} \right) dx \qquad (3.7)\\
&= -a\gamma \int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}} dx + \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx - \int_0^L \mathcal{G}(t,x) \frac{\partial_x v(t,x)}{v(t,x)} dx.
\end{aligned}$$

Combining the equalities as (3.6)+$(\mu/2)\times$(3.7), we prove that the quantity

$$\begin{aligned}
\int_0^L &\left\{ \frac{1}{2} u(t,x)^2 - \frac{\mu}{2} u(t,x) \frac{\partial_x v(t,x)}{v(t,x)} + \frac{\mu^2}{4} \frac{\partial_x v(t,x)^2}{v(t,x)^2} \right.\\
&\left. + a \left(\frac{v(t,x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) \right\} dx
\end{aligned} \tag{3.8}$$

is bounded with respect to the variable t. For this purpose we need to estimate the bounds of $\int_0^L u(t,x)^2 dx$, $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2} dx$ and $\int_0^L \left(\frac{v(t,x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) dx$ in terms of $\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx$ and $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}} dx$. Since $\bar{u} = 0$, choosing such an $x_t \in [0, L)$ as $u(t, x_t) = 0$ for every $t \geq 0$, and then using Schwarz' lemma, for $x \in [0, L]$ we have

$$\begin{aligned}
|u(t,x)| &= \left| \int_{x_t}^x \partial_y u(t,y) dy \right|\\
&\leq \int_0^L v(t,y)^{1/2} \frac{|\partial_y u(t,y)|}{v(t,y)^{1/2}} dy\\
&\leq \left(\int_0^L v(t,y) dy \right)^{1/2} \left(\int_0^L \frac{\partial_y u(t,y)^2}{v(t,y)} dy \right)^{1/2},
\end{aligned}$$

from which we obtain

$$\int_0^L u(t,x)^2 dx \leq L^2 \bar{v} \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx. \tag{3.9}$$

As for the estimate of $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2}dx$ and $\int_0^L \left(\frac{v(t,x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right)dx$, we make use of the assumption (3.1) noting that $1 < \gamma \le 2$. It is clear that

$$\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2}dx \le \left(\sup_{t,x} v(t,x)\right)^\gamma \int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}}dx.$$

For every $t \ge 0$ choosing $x_t \in [0, L)$ so that $v(t, x_t) = \bar{v}$ holds, by Schwarz' lemma we have

$$\begin{aligned}\left|\frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma}\right| &= \left|\int_{x_t}^x \frac{\partial_y v(t,y)}{v(t,y)^\gamma}dy\right| \\ &\le \left(\int_0^L v(t,y)^{2-\gamma}dy\right)^{1/2}\left(\int_0^L \frac{\partial_y v(t,y)^2}{v(t,y)^{\gamma+2}}dy\right)^{1/2} \\ &\le L^{1/2}\left(\sup_{t,y} v(t,y)\right)^{(2-\gamma)/2}\left(\int_0^L \frac{\partial_y v(t,y)^2}{v(t,y)^{\gamma+2}}dy\right)^{1/2}\end{aligned}$$

for $x \in [0, L]$, and hence,

$$\begin{aligned}&\int_0^L \left(\frac{v(t,x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right)dx \\ &= \left|\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right| \\ &\le L^{3/2}\left(\sup_{t,x} v(t,x)\right)^{(2-\gamma)/2}\left(\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}}dx\right)^{1/2}.\end{aligned}$$

We thus obtain a differential inequality for (3.8) showing its boundedness. Since the first three terms of the integrand of (3.8) constitute a positive quadratic form in two variables u and $\partial_x v/v$, the boundedness of v as in (3.5) follows from that of (3.8) immediately. Once the boundedness of v is obtained, that of u in H^1 follows just in the same manner as in [4]. We thus conclude (3.5). □

Let (v, u) be a solution of (1.1)–(1.3) with initial value (v_0, u_0). If (3.5) holds, then the orbit of the solution is a precompact set of $C^0 \times C^0$ by the Ascoli-Arzelá theorem. In particular, the ω-limit set of the orbit defined by

$$\omega(v_0, u_0) = \bigcap_{n=1}^\infty \overline{\{(v(t,\cdot), u(t,\cdot)); t \ge n\}}^{C^0\times C^0}$$

is nonempty. The following lemma shows that the large time behavior of a bounded solution is under the control of the set of stationary solutions.

Lemma 3 Assume that $1 < \gamma \le 2$. Let (v, u) be a bounded solution of (1.1)–(1.3) with initial value (v_0, u_0) and $\bar{u} = 0$. Then, for $(v_\omega, u_\omega) \in \omega(v_0, u_0)$ we have $v_\omega \in C^\infty$, $v_\omega > 0$, $\overline{v_\omega} = \overline{v_0}$, $u_\omega = 0$, and

$$\partial_x\left(a v_\omega(x)^{-\gamma}\right) = -\frac{4\pi G}{\overline{v_\omega}}\partial_x \int_0^L K_L(x,y)(v_\omega(y) - \overline{v_\omega})dy, \tag{3.10}$$

that is, (v_ω, u_ω) is a static and stationary solution of (1.1)–(1.3) having the average in common with the initial value.

Proof: By **Lemma 2** we have $\inf_{t,x} v(t,x) > 0$, and hence $v_\omega > 0$. It is clear that $\overline{v_\omega} = \overline{v_0}$.

We show that $u_\omega = 0$. Choose an increasing sequence $\{t_n; n = 1, 2, \dots,\}$ of positive numbers such that $t_n \geq n$ and $\lim_{n\to\infty}(v(t_n, \cdot), u(t_n, \cdot)) = (v_\omega, u_\omega)$ in $C^0 \times C^0$. Since E given by (3.2) is decreasing, we have

$$\lim_{t\to\infty} E(t) = \lim_{n\to\infty} E(t_n) = \mathcal{E}(v_\omega, u_\omega)$$

and

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} E(t)dt = \mathcal{E}(v_\omega, u_\omega). \tag{3.11}$$

Representing $\int_{t_n}^{t_n+1} E(t)dt$ as

$$\int_{t_n}^{t_n+1} \left(\int_0^L \frac{1}{2} u(t,x)^2 dx \right) dt + \int_{t_n}^{t_n+1} \left(\mathcal{E}(v(t, \cdot)) - \mathcal{E}(v(t_n, \cdot)) \right) dt + \mathcal{E}(v(t_n, \cdot)),$$

we take the limit in (3.11) term by term. Since

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \right) dt = 0 \tag{3.12}$$

by (3.4), it follows from (3.9) that

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L \frac{1}{2} u(t,x)^2 dx \right) dt = 0. \tag{3.13}$$

We next take the limit of the second term using

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L |v(t,x) - v(t_n,x)| dx \right) dt = 0. \tag{3.14}$$

This follows from estimating the integral with respect to the variable x in (3.14) with the use of the equality $v(t,x) - v(t_n,x) = \int_{t_n}^t \partial_x u(s,x)ds$, $t \in [t_n, t_n+1]$, due to the first equation of (1.1), as

$$\begin{aligned}
&\int_0^L |v(t,x) - v(t_n,x)| dx \\
&\leq \int_{t_n}^{t_n+1} \left(\int_0^L |\partial_x u(s,x)| dx \right) ds \\
&\leq \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L v(s,x) dx \right) ds \right\}^{1/2} \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(s,x)^2}{v(s,x)} dx \right) ds \right\}^{1/2} \\
&= \sqrt{L\bar{v}} \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(s,x)^2}{v(s,x)} dx \right) ds \right\}^{1/2},
\end{aligned}$$

and then applying (3.12). From the expression

$$v(t,x)^{1-\gamma} - v(t_n,x)^{1-\gamma} = (1-\gamma) \int_0^1 \{\xi v(t,x) + (1-\xi) v(t_n,x)\}^{-\gamma} d\xi \, (v(t,x) - v(t_n,x))$$

we have

$$\begin{aligned}&\int_{t_n}^{t_n+1}\left(\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx-\int_0^L \frac{v(t_n,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right)dt\\ &=\int_0^1\left\{\int_{t_n}^{t_n+1}\left(\int_0^L \{\xi v(t,x)+(1-\xi)v(t_n,x)\}^{-\gamma}(v(t,x)-v(t_n,x))dx\right)dt\right\}d\xi.\end{aligned}$$

Since $\{\xi v(t,x)+(1-\xi)v(t_n,x)\}^{-\gamma}\le(\inf_{t,x}v(t,x))^{-\gamma}$, $0\le\xi\le 1$, we obtain

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}\left(\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx-\int_0^L \frac{v(t_n,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right)dt=0.$$

Similarly, it follows from

$$\begin{aligned}&(v(t,x)-\bar{v})(v(t,y)-\bar{v})-(v(t_n,x)-\bar{v})(v(t_n,y)-\bar{v})\\ &=(v(t,x)-v(t_n,x))(v(t,y)-\bar{v})+(v(t_n,x)-\bar{v})(v(t,y)-v(t_n,y))\end{aligned}$$

and the boundedness of the kernel K_L that

$$\begin{aligned}\lim_{n\to\infty}\int_{t_n}^{t_n+1}&\left(\int_0^L\int_0^L K_L(x,y)(v(t,x)-\bar{v})(v(t,y)-\bar{v})dxdy\right.\\ &\left.-\int_0^L\int_0^L K_L(x,y)(v(t_n,x)-\bar{v})(v(t_n,y)-\bar{v})dxdy\right)dt=0.\end{aligned}$$

We thus obtain

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}\left(\mathcal{E}(v(t,\cdot))-\mathcal{E}(v(t_n,\cdot))\right)dt=0,$$

and hence,

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}E(t)dt=\lim_{n\to\infty}\mathcal{E}(v(t_n,\cdot))=\mathcal{E}(v_\omega).$$

Comparing this result with (3.11), we have $\int_0^L u_\omega(x)^2dx=0$, that is, $u_\omega=0$.

Finally we prove that v_ω is smooth and subject to (3.10). Let $\{t_n;n=1,2,\dots,\}$ be as above. Take a test function $\phi\in H^1$, and a smooth function θ of the real variable with support contained in the interval $(0,1)$, $\theta\ge 0$, and $\int_0^1\theta(t)dt=1$. Multiply the second equation of (1.1) by $\theta(t-t_n)\phi(x)$ and integrate the both sides of the result over $[t_n,t_n+1]\times[0,L]$. Integration by parts yields

$$\begin{aligned}&-\int_{t_n}^{t_n+1}\theta'(t-t_n)\left(\int_0^L\phi(x)u(t,x)dx\right)dt\\ &-\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)av(t,x)^{-\gamma}dx\right)dt\\ &+\mu\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)\frac{\partial_x u(t,x)}{v(t,x)}dx\right)dt\\ &=\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)\frac{4\pi G}{\bar{v}}\int_0^L K_L(x,y)(v(t,y)-\bar{v})dydx\right)dt.\end{aligned}$$

With the use of (3.14) we can handle the second term on the left-hand side and the term on the right-hand side in the same manner as shown above:

$$\begin{aligned}&\lim_{n\to\infty}\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)av(t,x)^{-\gamma}dx\right)dt\\&=\int_0^L \partial_x\phi(x)av_\omega(x)^{-\gamma}dx,\\&\lim_{n\to\infty}\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)\frac{4\pi G}{\bar{v}}\int_0^L K_L(x,y)(v(t,y)-\bar{v})dydx\right)dt\\&=\int_0^L \partial_x\phi(x)\frac{4\pi G}{\bar{v}}\int_0^L K_L(x,y)(v_\omega(y)-\bar{v})dydx.\end{aligned}$$

As for the third term on the left-hand side we have the following estimate by Schwarz' lemma:

$$\begin{aligned}&\left|\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)\frac{\partial_x u(t,x)}{v(t,x)}dx\right)dt\right|\\&\le\left\{\int_{t_n}^{t_n+1}\theta(t-t_n)^2\left(\int_0^L \frac{\partial_x\phi(x)^2}{v(t,x)}dx\right)dt\right\}^{1/2}\left\{\int_{t_n}^{t_n+1}\left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)}dx\right)dt\right\}^{1/2}\\&\le\left(\frac{1}{\inf_{t,x}v(t,x)}\int_0^1\theta(t)^2dt\int_0^L \partial_x\phi(x)^2dx\right)^{1/2}\left\{\int_{t_n}^{t_n+1}\left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)}dx\right)dt\right\}^{1/2},\end{aligned}$$

which shows that the term tends to 0 as $n\to\infty$ in view of (3.12). Similarly, the first term on the left-hand side tends to 0 as $n\to\infty$ from (3.13). Thus we obtain

$$-\int_0^L \partial_x\phi(x)av_\omega(x)^{-\gamma}dx=\int_0^L \partial_x\phi(x)\frac{4\pi G}{\overline{v_\omega}}\int_0^L K_L(x,y)(v_\omega(y)-\overline{v_\omega})dydx,$$

the equality (3.10) for v_ω in the distribution sense. Using the smoothing property of the integral operator with kernel K_L, by bootstrap argument we can derive the smoothness of v_ω from $v_\omega\in C^0$. □

4 Structure of stationary solutions From the observation of the large time behavior of bounded solutions to (1.1)–(1.3) we see that a solution is necessarily unbounded if it fails to approach the set of stationary solutions. This together with **Lemma 1**, which claims that the energy form is decreasing along the orbit of any solution, implies that, if there exists a state on M_V at which the energy form takes a value smaller than those of the energy form evaluated at the stationary solutions on M_V, then the orbit passing such a state is apart from the set of the stationary solutions and must be unbounded. This gives us the idea of providing, in terms of the energy form, an initial condition for unbounded solutions in reference to the structure of stationary solutions. Based on this idea, we first prove **Theorem 1**, and then examine at which stationary solution on M_V the energy form takes the minimal value.

Let us consider the stationary problem for (1.1)–(1.3):

$$\begin{cases}\partial_x u(x)=0,\\ \partial_x(av(x)^{-\gamma})=-\dfrac{4\pi G}{\bar{v}}\partial_x\displaystyle\int_0^L K_L(x,y)(v(y)-\bar{v})dy.\end{cases}\tag{4.1}$$

Our first task in the present section is to seek all the solutions of (4.1) lying in M_V for every $V > 0$. Clearly we have $u = 0$. By the change of unknown functions $r(x) = (v(x)/V)^{-\gamma} - 1$, we transform the problem into an equivalent one of finding L-periodic solutions to the following differential equation:

$$\partial_x^2 r(x) + \lambda f(r(x)) = 0, \quad r(x) > -1, \tag{4.2}$$

with

$$f(r) = 1 - (1+r)^{-1/\gamma}, \quad \lambda = \frac{4\pi G V^\gamma}{a}.$$

An L-periodic solution r of (4.2) has a critical point x_0, i.e., $\partial_x r(x_0) = 0$. Since both $r(x+x_0)$ and $r(-x+x_0)$ satisfy (4.2) with coincidence of the Cauchy data at $x = 0$, by the uniqueness of solutions to the Cauchy problem for (4.2) we have $r(x+x_0) = r(-x+x_0)$, and therefore both are even functions. Thus, r is given by an appropriate shift of an even solution. In view of this fact, we seek even L-periodic solutions of (4.2).

To this end we make use of the relation between the period of a solution and the first integral. The first integral of (4.2), usually called the energy of the orbit, is given by

$$\mathcal{I} = \frac{1}{2}\partial_x r(x)^2 + \lambda F(r(x))$$

with

$$F(r) = \int_0^r f(s)ds = r - \frac{\gamma}{\gamma - 1}\left\{(1+r)^{1-1/\gamma} - 1\right\}, \quad r > -1.$$

F is monotone decreasing on $(-1, 0]$ and monotone increasing on $[0, \infty)$, having the limit at either end of the half line:

$$F(-1+0) = \frac{1}{\gamma - 1}, \qquad F(\infty) = \infty.$$

We can therefore find a unique closed orbit with energy $\mathcal{I}$ if and only if

$$0 < \mathcal{I} < \frac{\lambda}{\gamma - 1}.$$

The period l of the orbit with energy $\mathcal{I}$ is given by

$$l = 2\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{2(\mathcal{I} - \lambda F(r))}},$$

where $r_{\min} < 0$ and $r_{\max} > 0$ are the minimum and the maximum of the solution r, respectively. Notice that

$$F(r_{\min}) = F(r_{\max}) = \frac{\mathcal{I}}{\lambda}. \tag{4.3}$$

Dividing the integral into two parts, one over $(r_{\min}, 0)$ and the other over $(0, r_{\max})$, and changing the variables by $y = (\lambda/\mathcal{I})F(r)$, we get

$$l = \sqrt{2/\lambda}\, I_\gamma\left(\sqrt{\mathcal{I}/\lambda}\right), \tag{4.4}$$

where I_γ is a function on $(0, (\gamma-1)^{-1/2})$ given by (2.1). The following lemma shows that the period of an orbit is a monotone increasing function of its energy provided $1 < \gamma < 2$.

Lemma 4 Assume $1 < \gamma < 2$. Then, $I'_\gamma(\theta) > 0$. Moreover we have

$$I_\gamma(+0) = \sqrt{2\gamma}\pi, \qquad I_\gamma((\gamma-1)^{-1/2} - 0) < \infty.$$

Proof: Put

$$I_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{\theta}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)} dy \tag{4.5}$$

and express $I_\gamma(\theta)$ as the sum of $I_{\gamma,+}(\theta)$ and $I_{\gamma,-}(\theta)$. Since

$$\frac{\partial}{\partial\theta}\left({F_\pm}^{-1}(\theta^2 y)\right) = \frac{2\theta y}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)},$$

we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\theta}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)}\right) &= \frac{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^2 - 2\theta^2 y f'_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^3} \\ &= \frac{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left({F_\pm}^{-1}(\theta^2 y)\right) f'_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^3}.\end{aligned}$$

Noting that

$$\lim_{z\to+0} \frac{f_\pm(z)^2 - 2F_\pm(z) f'_\pm(z)}{f_\pm(z)^3} = -\frac{1}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2},$$

we apply differentiation under the integral sign to (4.5) to obtain

$$I'_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left({F_\pm}^{-1}(\theta^2 y)\right) f'_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^3} dy$$

and

$$\lim_{\theta\to+0} I'_{\gamma,\pm}(\theta) = -\frac{1}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2}\int_0^1 \frac{dy}{\sqrt{1-y}} dy = -\frac{2}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2}. \tag{4.6}$$

Similarly, we have

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left(\frac{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left({F_\pm}^{-1}(\theta^2 y)\right) f'_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^3}\right) \\ &= \frac{2\theta y\, g\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^5} \\ &= \frac{2\sqrt{y} F_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^{1/2} g\left({F_\pm}^{-1}(\theta^2 y)\right)}{f_\pm\left({F_\pm}^{-1}(\theta^2 y)\right)^5}\end{aligned}$$

with

$$g(z) = 2F_\pm(z)\left(3f'_\pm(z)^2 - f_\pm(z) f''_\pm(z)\right) - 3f_\pm(z)^2 f'_\pm(z).$$

Since

$$\lim_{z\to+0} \frac{F_\pm(z)^{1/2} g(z)}{f_\pm(z)^5} = \lim_{z\to+0}\left(\frac{F_\pm(z)}{f_\pm(z)^2}\right)^{1/2} \frac{2F_\pm(z)\left(3f'_\pm(z)^2 - f_\pm(z) f''_\pm(z)\right) - 3f_\pm(z)^2 f'_\pm(z)}{f_\pm(z)^4}$$

$$= \frac{5f''_\pm(0)^2 - 3f'_\pm(0)f'''_\pm(0)}{12\sqrt{2}f'_\pm(0)^{7/2}},$$

again by differentiation under the integral sign the second derivative of $I_{\gamma,\pm}$ is given by

$$I''_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{2\theta y\, g\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^5} dy, \tag{4.7}$$

a continuous function on $(0, (\gamma-1)^{-1/2})$. Now we put $\zeta = (1 \pm z)^{-1/\gamma}$ and express $g(z)$ as

$$g(z) = \frac{\zeta^{1+2\gamma}}{\gamma^2} h(\zeta)$$

with

$$h(\zeta) = \frac{\gamma(1+\gamma)}{1-\gamma}\zeta^{2-\gamma} + \frac{2(1+\gamma)(2-\gamma)}{1-\gamma}\zeta^{1-\gamma} + (2-\gamma)\zeta^{-\gamma} + \frac{2(\gamma-2)}{1-\gamma}\zeta - \frac{2(1+\gamma)}{1-\gamma}.$$

Since

$$h'(\zeta) = \frac{\gamma(1+\gamma)(2-\gamma)}{1-\gamma}\zeta^{1-\gamma} + 2(1+\gamma)(2-\gamma)\zeta^{-\gamma} - \gamma(2-\gamma)\zeta^{-1-\gamma} + \frac{2(\gamma-2)}{1-\gamma},$$

and since

$$\begin{aligned} h''(\zeta) &= \gamma(1+\gamma)(2-\gamma)\zeta^{-\gamma} - 2\gamma(1+\gamma)(2-\gamma)\zeta^{-1-\gamma} + \gamma(1+\gamma)(2-\gamma)\zeta^{-2-\gamma} \\ &= \gamma(1+\gamma)(2-\gamma)\zeta^{-2-\gamma}(\zeta-1)^2 \\ &\geq 0 \end{aligned}$$

for $1 < \gamma < 2$, we have $h(1) = h'(1) = 0$ and therefore $h(\zeta) > 0$, $\zeta \neq 1$. The integrand in (4.7) is positive, and so is $I''_{\gamma,\pm}(\theta)$ for $\theta \in (0, (\gamma-1)^{-1/2})$. This implies that $I'_\gamma = I'_{\gamma,+} + I'_{\gamma,-}$ as well as $I'_{\gamma,\pm}$ is monotone increasing on $(0, (\gamma-1)^{-1/2})$. Using $f'_\pm(0) = 1/\gamma$ and $f''_\pm(0) = \mp(1+1/\gamma)/\gamma$, from (4.6) we obtain $\lim_{\theta\to+0} I'_\gamma(\theta) = 0$, and hence I'_γ is positive on $(0, (\gamma-1)^{-1/2})$, as desired.

Since

$$\frac{\theta}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)} = \frac{1}{\sqrt{y}}\left(\frac{F_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^2}\right)^{1/2},$$

and since the function $z \to F_\pm(z)/f_\pm(z)^2$ is bounded on $\left(0, F_\pm{}^{-1}\left((\gamma-1)^{-1} - 0\right)\right)$, we can take the limit of $I_{\gamma,\pm}(\theta)$ at either end of the interval $(0, (\gamma-1)^{-1/2})$ under the integral sign in (4.5). Thus we obtain

$$\lim_{\theta\to+0} I_{\gamma,\pm}(\theta) = \lim_{z\to+0}\left(\frac{F_\pm(z)}{f_\pm(z)^2}\right)^{1/2}\int_0^1 \frac{dy}{\sqrt{1-y}\sqrt{y}} = \frac{\pi}{(2f'_\pm(0))^{1/2}} = \left(\frac{\gamma}{2}\right)^{1/2}\pi$$

and

$$\lim_{\theta\to(\gamma-1)^{-1/2}-0} I_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}\sqrt{y}}\left(\frac{F_\pm\left(F_\pm{}^{-1}(y/(\gamma-1))\right)}{f_\pm\left(F_\pm{}^{-1}(y/(\gamma-1))\right)^2}\right)^{1/2} dy < \infty,$$

showing that $I_\gamma(+0) = \sqrt{2\gamma}\pi$ and $I_\gamma((\gamma-1)^{-1/2} - 0) < \infty$. □

By **Lemma 4**, from the formula (4.4) we obtain a necessary and sufficient condition for the existence and uniqueness of l-periodic orbits for (4.2):

$$\sqrt{4\gamma/\lambda}\pi < l < \sqrt{2/\lambda} I_\gamma((\gamma-1)^{-1/2}-0). \tag{4.8}$$

Recalling $\lambda = 4\pi G V^\gamma / a$, we obtain the assertion of **Theorem 1** immediately.

We proceed to another task of finding out the stationary solutions with minimal value of the energy form. Assume that the stationary problem for (1.1)–(1.3) has a non-trivial solution on M_V. For $j = k_{\min}, \ldots, k_{\max}$ choose a stationary solution $(\tilde{v}^{(j)}, 0) \in M_V$ with least period L/j as in **Theorem 1**, and put

$$S_V = \{(\tilde{v}^{(j)}, 0); j = k_{\min}, \ldots, k_{\max}\} \cup \{(V, 0)\}. \tag{4.9}$$

We compare the values $\mathcal{E}(\tilde{v}^{(j)})$, $j = k_{\min}, \ldots, k_{\max}$, and $\mathcal{E}(V) = 0$ with each other. To this end we introduce the following function with respect to the periods of stationary solutions:

$$\begin{aligned}\varepsilon(l) = &\int_0^l a\left(\frac{\tilde{v}^l(x)-V}{V^\gamma} - \frac{\tilde{v}^l(x)^{1-\gamma}-V^{1-\gamma}}{1-\gamma}\right)dx \\ &- \frac{2\pi G}{V}\int_0^l\int_0^l K_l(x,y)(\tilde{v}^l(x)-V)(\tilde{v}^l(y)-V)dxdy,\end{aligned}$$

where $(\tilde{v}^l, 0)$ is the non-trivial solution of the stationary problem (4.1) parametrized by $L = l$ with $\tilde{v}^l$ having the average V, the least period l, and the maximum at $x = 0$. In view of (4.8), $\tilde{v}^l$ as well as $\varepsilon(l)$ is well defined for l with

$$\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < l < \frac{I_\gamma((\gamma-1)^{-1/2}-0)}{\sqrt{2\gamma}\pi}\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}. \tag{4.10}$$

With this function the value $\mathcal{E}(\tilde{v}^{(j)})$ is expressed as follows.

Lemma 5 For $j = k_{\min}, \ldots, k_{\max}$ we have

$$\mathcal{E}(\tilde{v}^{(j)}) = j\varepsilon(L/j). \tag{4.11}$$

Proof: Put $l_j = L/j$. Notice that $\mathcal{E}(\tilde{v}^{(j)}) = \mathcal{E}(\tilde{v}^{l_j})$. In the expression

$$\begin{aligned}\mathcal{E}(v^{l_j}) = &\int_0^L a\left(\frac{\tilde{v}^{l_j}(x)-V}{V^\gamma} - \frac{\tilde{v}^{l_j}(x)^{1-\gamma}-V^{1-\gamma}}{1-\gamma}\right)dx \\ &- \frac{2\pi G}{V}\int_0^L\int_0^L K_L(x,y)(\tilde{v}^{l_j}(x)-V)(\tilde{v}^{l_j}(y)-V)dxdy\end{aligned}$$

we divide every integral on the interval $[0, L]$ into the integrals on the subintervals $[ml_j, (m+1)l_j]$, $m = 0, \ldots, j-1$, and rewrite every piece as an integral on $[0, l_j]$ by change of variables. By periodicity of $\tilde{v}^{l_j}$ we obtain

$$\begin{aligned}\mathcal{E}(\tilde{v}^{(j)}) = j&\int_0^{l_j} a\left(\frac{\tilde{v}^{l_j}(x)-V}{V^\gamma} - \frac{\tilde{v}^{l_j}(x)^{1-\gamma}-V^{1-\gamma}}{1-\gamma}\right)dx \\ &- \frac{2\pi G}{V}\int_0^{l_j}\int_0^{l_j}\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j)(\tilde{v}^{l_j}(x)-V)(\tilde{v}^{l_j}(y)-V)dxdy.\end{aligned} \tag{4.12}$$

Noting that $0 \le x+ml_j, y+nl_j \le L$ for $0 \le x, y \le l_j$ and $m, n = 0, \dots, j-1$, we calculate the sum $\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j)$ with the use of the expression (1.4) of K_L:

$$\begin{aligned}
&\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j) \\
&= -\frac{1}{2}\sum_{m,n=0}^{j-1} |x-y+(m-n)l_j| + \frac{1}{2L}\sum_{m,n=0}^{j-1}\{x-y+(m-n)l_j\}^2 + \frac{L}{12}j^2 \\
&= -\frac{1}{2}\left[\sum_{m=n}|x-y| + \sum_{m>n}\{x-y+(m-n)l_j\} - \sum_{m<n}\{x-y+(m-n)l_j\}\right] \\
&\quad + \frac{1}{2L}\sum_{m,n=0}^{j-1}\left\{(x-y)^2 + 2(m-n)(x-y)l_j + (m-n)^2 l_j^2\right\} + \frac{L}{12}j^2 \\
&= -\frac{1}{2}\left(j|x-y| + 2\sum_{m>n}(m-n)l_j\right) + \frac{1}{2L}\left\{j^2(x-y)^2 + 2\sum_{m>n}(m-n)^2 l_j^2\right\} + \frac{L}{12}j^2 \\
&= -j\frac{|x-y|}{2} + j\frac{(x-y)^2}{2l_j} + \sum_{m>n}\left\{\frac{(m-n)^2}{j} - (m-n)\right\} l_j + \frac{l_j}{12}j^3.
\end{aligned}$$

Here we have

$$\begin{aligned}
&\sum_{m>n}\left\{\frac{(m-n)^2}{j} - (m-n)\right\} \\
&= \sum_{m=1}^{j-1}\sum_{k=1}^{m}\left(\frac{k^2}{j} - k\right) \\
&= \sum_{m=1}^{j-1}\left(\frac{2m^3+3m^2+m}{6j} - \frac{m^2+m}{2}\right) \\
&= \frac{1}{12}\left[\left\{j(j-1)^2 + (j-1)(2j-1) + (j-1)\right\} - \left\{j(j-1)(2j-1) + 3j(j-1)\right\}\right] \\
&= \frac{1}{12}(-j^3 + j),
\end{aligned}$$

and hence,

$$\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j) = j\left\{-\frac{|x-y|}{2} + \frac{(x-y)^2}{2l_j} + \frac{l_j}{12}\right\} = jK_{l_j}(x,y).$$

This together with (4.12) gives (4.11). □

Since

$$\mathcal{E}(\tilde{v}^{(j)}) = L\frac{\varepsilon(L/j)}{L/j}$$

by (4.11), the j-dependence of $\mathcal{E}(\tilde{v}^{(j)})$ would be known from the behavior of the function $l \mapsto \varepsilon(l)/l$ on the interval (4.10). We first show the differentiability of the function. Put $r^l(x) = (\tilde{v}^l(x)/V)^{-\gamma} - 1$. Notice that r^l is a solution of (4.2) having the least period l and the minimum at $x = 0$. Let us denote the minimum by $r^l_{\min}$, negative for l with (4.10). By (4.3) the energy of the orbit of r^l is $\lambda F(r^l_{\min})$. Therefore, from (4.4) we obtain

$$l = \sqrt{2/\lambda}\, I_\gamma \left(\sqrt{F(r^l_{\min})}\right).$$

By the monotonicity of I_γ due to **Lemma 4**,

$$F\left(r^l_{\min}\right) = \left({I_\gamma}^{-1}\left(l\sqrt{\lambda/2}\right)\right)^2 \tag{4.13}$$

holds. Since I_γ is continuously differentiable, so is the function $l \mapsto r^l_{\min}$ on (4.10). By continuous dependence on initial data in the Cauchy problem for (4.2) the correspondence $l \mapsto r^l$ defines a continuously differentiable function on (4.10) with values in the space of continuous functions on $\mathbf{R}$, and so does the correspondence $l \mapsto \tilde{v}^l$. From this together with the expression (1.4) of the kernel K_L with $L = l$ the differentiability of the function $l \mapsto \varepsilon(l)$ on (4.10) easily follows.

The following lemma shows that the function under consideration is monotonic and negative.

Lemma 6 We have $(\varepsilon(l)/l)' < 0$ and $\varepsilon(l) < 0$.

Proof: Put $\tilde{v}^l_0 = \tilde{v}^l(0)$. We take the derivative of $\varepsilon(l)$ and rewrite the result using $\tilde{v}^l(l) = \tilde{v}^l_0$, $\int_0^l \partial_l \tilde{v}^l(x)dx = V - \tilde{v}^l_0$ from $\int_0^l \tilde{v}^l(x)dx = Vl$, and the symmetry of the Green kernel $K_l(x, y)$. After rearrangement of terms we obtain

$$\begin{aligned}
\varepsilon'(l) = & -a\frac{(\tilde{v}^l_0)^{1-\gamma} - V^{1-\gamma}}{1-\gamma} \\
& - \int_0^l a\tilde{v}^l(x)^{-\gamma}\partial_l\tilde{v}^l(x)dx - \frac{4\pi G}{V}\int_0^l\int_0^l K_l(x,y)(\tilde{v}^l(y) - V)dy\,\partial_l\tilde{v}^l(x)dx \\
& - \frac{4\pi G}{V}\int_0^l K_l(l,y)(\tilde{v}^l(y) - V)dy(\tilde{v}^l_0 - V) \\
& - \frac{2\pi G}{V}\int_0^l\int_0^l \partial_l K_l(x,y)(\tilde{v}^l(x) - V)(\tilde{v}^l(y) - V)dxdy.
\end{aligned}$$

Here we notice that $\tilde{v}^l$ is subject to the following equation equivalent to the second one of (4.1) with $L = l$:

$$-a\tilde{v}^l(x)^{-\gamma} + \frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx - \frac{4\pi G}{V}\int_0^l K_l(x,y)(\tilde{v}^l(y) - V)dy = 0. \tag{4.14}$$

Then the sum of the second and the third terms on the right-hand side is

$$-\frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx\int_0^l \partial_l\tilde{v}^l(x)dx = \frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx\,(\tilde{v}^l_0 - V).$$

Adding the forth term to this expression and using (4.14) with $x = l$, we see that the sum of the above three terms turns out to be $a(\tilde{v}^l_0 - V)/(\tilde{v}^l_0)^\gamma$. Since $\tilde{v}^l$ is axially symmetric with respect to $x = l/2$, the last term on the right-hand side vanishes in view of

$$\partial_l K_l(x,y) = -\frac{(x-y)^2}{2l^2} + \frac{1}{12}$$

$$= -\frac{1}{2l^2}\left\{\left(x-\frac{l}{2}\right)^2+\left(y-\frac{l}{2}\right)^2-2\left(x-\frac{l}{2}\right)\left(y-\frac{l}{2}\right)\right\}+\frac{1}{12}, \quad 0 \le x, y \le l.$$

Summing up, we obtain

$$\varepsilon'(l) = a\left\{-\frac{(\tilde{v}_0^l)^{1-\gamma}-V^{1-\gamma}}{1-\gamma}+\frac{\tilde{v}_0^l-V}{(\tilde{v}_0^l)^\gamma}\right\}.$$

From this expression the function $l \mapsto \varepsilon(l)$ is twice continuously differentiable and

$$\varepsilon''(l) = -a\gamma\frac{\partial_l\tilde{v}_0^l\,(\tilde{v}_0^l-V)}{(\tilde{v}_0^l)^{\gamma+1}}.$$

Since $\tilde{v}^l$ attains its maximum at $x = 0$, we have $\tilde{v}_0^l - V > 0$. Moreover, from $\tilde{v}_0^l = V\left(1+r_{\min}^l\right)^{-1/\gamma}$ with $r_{\min}^l$ as above, we obtain

$$\partial_l\tilde{v}_0^l = -\frac{V}{\gamma}(1+r_{\min}^l)^{-1/\gamma-1}\partial_l r_{\min}^l.$$

Thus, the sign of $\varepsilon''(l)$ coincides with that of $\partial_l r_{\min}^l$. Taking the derivatives of the both sides of (4.13), we obtain

$$f(r_{\min}^l)\partial_l r_{\min}^l = \sqrt{2\lambda}{I_\gamma}^{-1}\left(l\sqrt{\lambda/2}\right)({I_\gamma}^{-1})'\left(l\sqrt{\lambda/2}\right),$$

positive in view of **Lemma 4**. Since $r_{\min}^l$ is negative, so are $f\left(r_{\min}^l\right)$ and $\partial_l r_{\min}^l$. We thus conclude that $\varepsilon''(l) < 0$.

We next take the limit as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$ in (4.13). By **Lemma 4** we have $F(r_{\min}^l) \to ({I_\gamma}^{-1}(\sqrt{2\gamma}\pi+0))^2 = 0$, and hence $r_{\min}^l \to 0$. By continuous dependence on initial data in the Cauchy problem for (4.2) we obtain the uniform convergence of both r^l and $\tilde{v}^l$ as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$, showing $r^l(x) \to 0$ and $\tilde{v}^l(x) \to V$ on $\mathbf{R}$. Thus, $\varepsilon(l)$ as well as $\varepsilon'(l)$ tends to 0 as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$. From these in combination with $(\varepsilon(l)/l)' = (l\varepsilon'(l)-\varepsilon(l))/l^2$ and $(l\varepsilon'(l)-\varepsilon(l))' = l\varepsilon''(l) < 0$, we conclude that $(\varepsilon(l)/l)' < 0$ and $\varepsilon(l) < 0$. □

As a consequence of **Lemma 6** we obtain

Proposition 1 For $j_1, j_2 = k_{\min}, \dots, k_{\max}$ with $j_1 < j_2$, we have

$$\mathcal{E}(\tilde{v}^{(j_1)}) < \mathcal{E}(\tilde{v}^{(j_2)}) < \mathcal{E}(V) = 0.$$

In particular, $\mathcal{E}(\tilde{v}^{(k_{\min})})$ is minimal amongst the values of the energy form on S_V.

5 Initial condition for unbounded solutions

Proposition 1 claims that the subset A_V of $H^1 \times H^1$ given by (2.3) consists of the states on M_V at which the energy form takes values smaller than any values of the energy form evaluated at the stationary solutions on M_V. As proved earlier, the orbit of a solution to (1.1)–(1.3) passing through A_V is necessarily unbounded, i.e., $\sup_{t,x} v(t,x) = \infty$. In this way we obtain an initial condition for unbounded solutions as presented by **Theorem 2**. The problem to be settled is to find a condition that ensures the non-emptiness of A_V. The final section is devoted to a partial answer to the problem, proving **Theorem 2**.

Our strategy is to find an element of A_V in a small neighborhood of a stationary solution giving the minimal value of the energy form on S_V given by (4.9). In order to introduce the idea we begin by examining the behavior of the energy form near an arbitrary stationary solution. Let $(\tilde{v}, 0) \in M_V$ be a stationary solution of (1.1)–(1.3), and $(v, u) \in M_V$ a state in a neighborhood of the stationary solution. We introduce the displacement from the stationary solution as

$$\phi(x) = v(x) - \tilde{v}(x), \quad \psi(x) = u(x).$$

Suppose the displacement is small enough in amplitude. Since

$$\frac{(\tilde{v}(x) + \phi(x))^{1-\gamma} - \tilde{v}(x)^{1-\gamma}}{1-\gamma} = \tilde{v}(x)^{-\gamma}\phi(x) - \frac{1}{2}\gamma\tilde{v}(x)^{-\gamma-1}\phi(x)^2 + \mathcal{O}(|\phi(x)|^3),$$

evaluating the form (1.6) at $v = \tilde{v} + \phi$, we obtain

$$\begin{aligned} &\mathcal{E}(\tilde{v} + \phi) \\ &= \mathcal{E}(\tilde{v}) + \int_0^L \left(-a\tilde{v}(x)^{-\gamma} - \frac{4\pi G}{V} \int_0^L K_L(x, y)(\tilde{v}(y) - V)dy \right) \phi(x)dx \\ &\quad + \frac{1}{2}\int_0^L a\frac{\gamma\phi(x)^2}{\tilde{v}(x)^{\gamma+1}}dx - \frac{2\pi G}{V}\int_0^L\int_0^L K_L(x, y)\phi(x)\phi(y)dxdy + \mathcal{O}(\|\phi\|_{L^\infty})\|\phi\|_{L^2}^2 \end{aligned}$$

with $\|\phi\|_{L^\infty}$ the supremum norm of ϕ. As in (4.14), $\tilde{v}$ satisfies the equation

$$(5.1) \qquad -a\tilde{v}(x)^{-\gamma} + \frac{1}{L}\int_0^L a\tilde{v}(x)^{-\gamma}dx - \frac{4\pi G}{V}\int_0^L K_L(x, y)(\tilde{v}(y) - V)dy = 0.$$

Since the average of ϕ vanishes, this implies that

$$(5.2) \qquad \mathcal{E}(\tilde{v} + \phi) = \mathcal{E}(\tilde{v}) + \frac{1}{2}Q[\phi] + \mathcal{O}(\|\phi\|_{L^\infty})\|\phi\|_{L^2}^2,$$

where Q is the quadratic form on the Hilbert space $\mathcal{H} = \{\varphi \in L^2; \bar{\varphi} = 0\}$ defined by

$$Q[\varphi] = \int_0^L a\frac{\gamma\varphi(x)^2}{\tilde{v}(x)^{\gamma+1}}dx - \frac{4\pi G}{V}\int_0^L\int_0^L K_L(x, y)\varphi(x)\varphi(y)dxdy.$$

Now suppose the quadratic form Q admits a negative value, i.e., $Q[\varphi_0] < 0$ for some $\varphi_0 \in \mathcal{H}$. By approximation of functions we may assume that φ_0 is smooth. Evaluating the energy form (1.5) with (1.6) at $(v, u) = (\tilde{v} + \varepsilon\varphi_0, 0)$ for small $|\varepsilon|$, from (5.2) we obtain

$$\mathcal{E}(\tilde{v} + \varepsilon\varphi_0, 0) = \mathcal{E}(\tilde{v}) + \frac{1}{2}\varepsilon^2 Q[\varphi_0] + \mathcal{O}(|\varepsilon|^3).$$

This shows that the energy form takes a value smaller than its value at the stationary solution in any small neighborhood of that stationary solution.

In order to examine the sign of Q we make use of the expression

$$Q[\varphi] = \frac{a}{V^{\gamma+1}}(T\varphi, \varphi)_{L^2},$$

where T is the self-adjoint operator on $\mathcal{H}$ given by

$$(5.3) \qquad (T\varphi)(x) = \frac{\gamma\varphi(x)}{(1 + \tilde{w}(x))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1 + \tilde{w}(x))^{\gamma+1}}dx - \lambda\int_0^L K_L(x, y)\varphi(y)dy$$

with $\tilde{w} = \tilde{v}/V - 1$ and $\lambda = 4\pi G V^\gamma/a$. We are concerned with the spectrum $\sigma(T)$ of T since the lower bound of $\sigma(T)$ gives $\inf_{\|\varphi\|_{L^2}=1}(T\varphi, \varphi)_{L^2}$.

In case $\tilde{v} = V$, that is, $\tilde{w} = 0$ the spectrum of T is easily obtained from that of the Green operator of $-d^2/dx^2$ on $\mathcal{H}$. The spectrum consists of double eigenvalues $\gamma - (\lambda L^2)/(4\pi^2 j^2)$ with two independent eigenvectors $\cos(2\pi j/L)x$ and $\sin(2\pi j/L)x$, $j = 1, 2, \dots$, and the accumulation point γ of them. Thus, we obtain

$$\inf \sigma(T) = \gamma - \frac{\lambda L^2}{4\pi^2} \tag{5.4}$$

immediately.

In considering the spectrum of T corresponding to a non-trivial stationary solution $(\tilde{v}, 0)$ some preliminary observations are in order. Since $(T\varphi, \varphi)_{L^2} \le \int_0^L \frac{\gamma\varphi(x)^2}{(1+\tilde{w}(x))^{\gamma+1}} dx$ for $\varphi \in \mathcal{H}$, we have $\inf \sigma(T) \le \gamma(1 + \max \tilde{w})^{-\gamma-1}$. In the region below $\gamma(1+\max\tilde{w})^{-\gamma-1}$ the spectrum in fact consists of eigenvalues of T. This follows from rewriting an equation $T\varphi - \Lambda\varphi = \psi$ in $\mathcal{H}$ with parameter $\Lambda < \gamma(1 + \max\tilde{w})^{-\gamma-1}$ as $P_\Lambda\varphi - \lambda K_L\varphi = \psi$ with

$$(P_\Lambda\varphi)(x) = \left\{ \frac{\gamma}{(1+\tilde{w}(x))^{\gamma+1}} - \Lambda \right\} \varphi(x) - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1+\tilde{w}(x))^{\gamma+1}} dx,$$
$$(K_L\varphi)(x) = \int_0^L K_L(x,y)\varphi(y)dy,$$

noting the positivity of P_Λ and the compactness of K_L, and applying the the Riesz-Schauder theory to the compact operator ${P_\Lambda}^{-1}K_L$ on $\mathcal{H}$. We next remark that $(\tilde{v}(\,\cdot\, - \alpha), 0)$ is also a stationary solution of (1.1)–(1.3) for any $\alpha \in \mathbf{R}$, and hence

$$-\frac{1}{(1+\tilde{w}(x-\alpha))^\gamma} + \frac{1}{L}\int_0^L \frac{dx}{(1+\tilde{w}(x-\alpha))^\gamma} - \lambda\int_0^L K_L(x,y)\tilde{w}(y-\alpha)dy$$

holds by (5.1). Differentiating this relation with respect to α and evaluating the result at $\alpha = 0$, we obtain

$$\frac{\gamma\tilde{w}'(x)}{(1+\tilde{w}(x))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\tilde{w}'(x)}{(1+\tilde{w}(x))^{\gamma+1}} dx - \lambda\int_0^L K_L(x,y)\tilde{w}'(y)dy = 0,$$

that is, $T\tilde{w}' = 0$. This shows that T has a non-trivial null space with an eigenvector $\tilde{w}' \neq 0$. Let us define a self-adjoint operator on $\mathcal{H}$ corresponding to the stationary solution $(\tilde{v}(\,\cdot\, - \alpha), 0)$ by (5.3):

$$(T^\alpha\varphi)(x) = \frac{\gamma\varphi(x)}{(1+\tilde{w}(x-\alpha))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1+\tilde{w}(x-\alpha))^{\gamma+1}} dx - \lambda\int_0^L K_L(x,y)\varphi(y)dy.$$

Our last remark here is that the point spectrum of T^α coincides with that of T for any $\alpha \in \mathbf{R}$ with the correspondence of associating eigenspaces given by the shift of functions $\varphi \mapsto \varphi(\,\cdot\, - \alpha)$, for from the equation $T\varphi = \Lambda\varphi$ we have

$$\frac{\gamma\varphi(x-\alpha)}{(1+\tilde{w}(x-\alpha))^{\gamma+1}} - \frac{1}{L}\int_\alpha^{L+\alpha} \frac{\gamma\varphi(x-\alpha)}{(1+\tilde{w}(x-\alpha))^{\gamma+1}} dx$$
$$- \lambda\int_\alpha^{L+\alpha} K_L(x-\alpha, y-\alpha)\varphi(y-\alpha)dy = \Lambda\varphi(x-\alpha),$$

and hence $T^\alpha\varphi(\,\cdot\, - \alpha) = \Lambda\varphi(\,\cdot\, - \alpha)$ in view of $K_L(x-\alpha, y-\alpha) = K_L(x,y)$ and the L-periodicity of $\tilde{w}$, φ and $K_L(x,\,\cdot\,)$. To sum up, we are allowed to study the lower bound of T focusing on the nonpositive eigenvalues of T after a favorable shift of $\tilde{v}$.

With the above considerations in mind we prove the following.

Lemma 7 Let k be an integer satisfying (2.2), and $\tilde{v}^{(k)}$ as in **Theorem 1**. Let T be the self-adjoint operator on $\mathcal{H}$ that corresponds to the stationary solution $(\tilde{v}^{(k)}, 0)$ by (5.3). If $k \geq 2$, then the lower bound of T is a negative eigenvalue.

Proof: As shown just before the statement of the lemma, we may assume that $\tilde{v}^{(k)}$ is even and attains its maximum at $x = 0$. Such a stationary solution with least period L/k is unique. Rewriting (2.2) with a parameter $\lambda = 4\pi G V^\gamma / a$, we consider the stationary solution as parametrized over the interval

$$\gamma \left(\frac{2\pi k}{L}\right)^2 < \lambda < 2 \left(\frac{I_\gamma\left((\gamma-1)^{-1/2} - 0\right) k}{L}\right)^2, \tag{5.5}$$

and denote $\tilde{v}^{(k)}/V - 1$ by $\tilde{w}_\lambda$. We first notice that $\lambda \mapsto \tilde{w}_\lambda$ is a continuous function with values in the space of continuously differentiable L-periodic functions on (5.5) with uniform limit

$$\lim_{\lambda \to \gamma(2\pi k/L)^2 + 0} \tilde{w}_\lambda(x) = 0.$$

To show this put $r_\lambda = (1 + \tilde{w}_\lambda)^{-\gamma} - 1$ and notice that r_λ is a solution of (4.2) attaining its minimum $r_{\lambda,\min}$, which is negative, at $x = 0$. Since the energy of the orbit of r_λ is given by $\lambda \left(I_\gamma{}^{-1}\left((L/k)\sqrt{\lambda/2}\right)\right)^2$, we have $F(r_{\lambda,\min}) = \left(I_\gamma{}^{-1}\left((L/k)\sqrt{\lambda/2}\right)\right)^2$. See (4.3) and (4.4). This together with

$$\lim_{\lambda \to \gamma(2\pi k/L)^2 + 0} I_\gamma{}^{-1}\left((L/k)\sqrt{\lambda/2}\right) = I_\gamma{}^{-1}\left(\sqrt{2\gamma}\pi + 0\right) = 0,$$

coming from **Lemma 4**, implies that $r_{\lambda,\min}$ depends continuously on λ with $r_{\lambda,\min} \to 0$ as $\lambda \to \gamma\,(2\pi k/L)^2 + 0$. The continuity of r_λ with respect to λ as well as the uniform convergence $r_\lambda(x) \to 0$ as $\lambda \to \gamma\,(2\pi k/L)^2 + 0$ follows from continuous dependence on initial data in the Cauchy problem for (4.2). Thus, the map $\lambda \mapsto \tilde{w}_\lambda$ enjoys the continuity as desired. Now put

$$(T_\lambda \varphi)(x) = \frac{\gamma\varphi(x)}{(1+\tilde{w}_\lambda(x))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1+\tilde{w}_\lambda(x))^{\gamma+1}}dx - \lambda\int_0^L K_L(x,y)\varphi(y)dy.$$

In view of $K_L(-x, y) = K_L(x, -y)$ and the L-periodicity of $\tilde{w}_\lambda$ and $K_L(x, \cdot)$, T_λ maps an odd function into an odd one. The restriction of T_λ onto the subspace $\mathcal{H}^{(o)} = \{\varphi \in \mathcal{H}; \varphi(-x) = -\varphi(x)\}$ of $\mathcal{H}$ is denoted by $T_\lambda^{(o)}$. As shown above, $T_\lambda^{(o)}$ as well as T_λ has the eigenvalue 0 with eigenvector $\tilde{w}'_\lambda \in \mathcal{H}^{(o)}$. Moreover, the eigenvalue 0 is simple. This is because the equation $T_\lambda^{(o)}\varphi = 0$ is equivalent to the second order linear differential equation $\partial_x^2\left\{\gamma(1+\tilde{w}_\lambda(x))^{-\gamma-1}\varphi(x)\right\} + \lambda\varphi(x) = 0$ and any odd solution of the differential equation must be proportional to the solution $\tilde{w}'_\lambda$ by the uniqueness of solutions to the Cauchy problem. Noting that $\inf\sigma(T_\lambda) \leq \inf\sigma(T_\lambda^{(o)})$, we show that the lower bound of $T_\lambda^{(o)}$ is negative.

From the continuous dependence of $\tilde{w}_\lambda$ on λ we see that the correspondence $\lambda \mapsto T_\lambda^{(o)}$ is continuous on the interval (5.5) up to the left end $\gamma\,(2\pi k/L)^2$ with respect to the operator norm. The limit $T^{(o)}_{\gamma(2\pi k/L)^2+0}$ is the restriction onto $\mathcal{H}^{(o)}$ of the operator (5.3) with $\lambda = \gamma\,(2\pi k/L)^2$ and $\tilde{w} = 0$, and its lower bound is the eigenvalue $\gamma(1-k^2)$ with eigenvector

$\sin(2\pi/L)x$, as shown by (5.4). Thus, the correspondence $\lambda \mapsto \inf \sigma(T_\lambda^{(\mathrm{o})})$ gives a continuous function on (5.5) with

$$\lim_{\lambda \to \gamma(2\pi k/L)^2+0} \inf \sigma(T_\lambda^{(\mathrm{o})}) = \inf \sigma(T^{(\mathrm{o})}_{\gamma(2\pi k/L)^2+0}) = \gamma(1-k^2),$$

which is negative by the assumption $k \geq 2$. Put $c(\lambda) = \inf \sigma(T_\lambda^{(\mathrm{o})})$ and suppose the function $\lambda \mapsto c(\lambda)$ admits a nonnegative value on (5.5). In view of the continuity of the function and $c(\gamma\,(2\pi k/L)^2 + 0) < 0$ as proved above, there does exist a zero of the function. The smallest zero is denoted by λ_*. For $\gamma\,(2\pi k/L)^2 < \lambda < \lambda_*$, since $c(\lambda) < 0$, $c(\lambda)$ is proved to be an eigenvalue as the lower bound of T_λ is. Let φ_λ be an eigenvector associated with $c(\lambda)$ satisfying $\|\varphi_\lambda\|_{L^2} = 1$. By the boundedness of $\{\varphi_\lambda; \gamma\,(2\pi k/L)^2 < \lambda < \lambda_*\}$ in $\mathcal{H}^{(\mathrm{o})}$ we can choose a sequence $\{\lambda_n; n = 1, 2, \dots\}$ and an element φ_{λ_*} of $\mathcal{H}^{(\mathrm{o})}$ so that $\gamma\,(2\pi k/L)^2 < \lambda_n < \lambda_*$, $\lambda_n \to \lambda_*$ as $n \to \infty$, and the sequence $\{\varphi_{\lambda_n}; n = 1, 2, \dots\}$ converges to φ_{λ_*} weakly in $\mathcal{H}^{(\mathrm{o})}$ as $n \to \infty$. Noting that $\varphi_\lambda(1 + \tilde{w}_\lambda)^{-\gamma-1}$ is an odd function, we rewrite $T_\lambda^{(\mathrm{o})}\varphi_\lambda = c(\lambda)\varphi_\lambda$ as

$$\varphi_\lambda(x) = \frac{(1 + \tilde{w}_\lambda(x))^{\gamma+1}}{\gamma} \left(\lambda \int_0^L K_L(x, y)\varphi_\lambda(y)dy + c(\lambda)\varphi_\lambda(x) \right)$$

and then take the limit along the sequence. Since, as $\lambda \to \lambda_*$, $\tilde{w}_\lambda$ converges uniformly to $\tilde{w}_{\lambda_*}$ and $c(\lambda) \to c(\lambda_*) = 0$, and since the integral operator with kernel K_L is compact on $\mathcal{H}^{(\mathrm{o})}$, the sequence $\{\varphi_{\lambda_n}; n = 1, 2, \dots,\}$ converges strongly in L^2 and also in $\mathcal{H}^{(\mathrm{o})}$. Therefore, $\|\varphi_{\lambda_*}\|_{L^2} = 1$ and $T^{(\mathrm{o})}_{\lambda_*}\varphi_{\lambda_*} = 0$ hold. This shows that φ_{λ_*} is an eigenvector of $T^{(\mathrm{o})}_{\lambda_*}$ associated with the eigenvalue 0. Since $\tilde{w}'_\lambda$ and φ_λ are orthogonal to each other for $\gamma\,(2\pi k/L)^2 < \lambda < \lambda_*$, so are $\tilde{w}'_{\lambda_*}$ and φ_{λ_*} by passage to the limit along the sequence and the continuity of $\tilde{w}'_\lambda$ with respect to λ. In particular, $\tilde{w}'_{\lambda_*}$ and φ_{λ_*} are independent, however, this contradicts the simplicity of the eigenvalue 0.

Thus, the lower bound of $T_\lambda^{(\mathrm{o})}$ must be negative over the interval (5.5), as desired. □

We are now in position to present a condition for A_V to be nonempty. Given **Proposition 1** and **Lemma 7**, we think it reasonable to pick up the cases in which the minimal value of the energy form on S_V as in (4.9) is attained either at the stationary solution $(\tilde{v}^{(k_{\min})}, 0)$ with $k_{\min} \geq 2$ or at the trivial solution $(V, 0)$ with $\inf \sigma(T) < 0$. In view of **Remark 1** and (5.4), the condition that we propose turns out to be

$$V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2} - 0)^2}{2\pi G L^2} \right)^{1/\gamma},$$

the assumption of **Theorem 2**. Now the proof of the theorem is completed.

Remark 3 In the proof of **Lemma 7** we rely on the fact that the lower bound of the operator $T_\lambda^{(\mathrm{o})}$ is somewhere negative on the interval (5.5). Here we essentially make use of the assumption $k \geq 2$. In case $k = 1$, however, the situation is subtle, and in fact the lower bound of T corresponding to the stationary solution $(\tilde{v}^{(1)}, 0)$ proves the eigenvalue 0, which is isolated and simple. An outline of the reasoning is given by [6], where the spectrum of the restriction of T_λ onto the space of even functions are considered with the use of **Lemma 4** and the result of Crandall and Rabinowitz [2] on the perturbation of simple eigenvalues along bifurcation curves of stationary solutions. The result shows, in some sense, the stability of the set of stationary solutions having a profile in common, and in order to find an element of A_V for $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ we are forced to study the behavior of the energy form beyond a small neighborhood of the set of stationary solutions, which is a global and therefore difficult problem. The situation is quite similar in case $V \le \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$ since the trivial solution $(V, 0)$ is the unique stationary solution on M_V with stability in some sense.

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SOME INTEGRALS BETWEEN THE LEBESGUE INTEGRAL AND THE DENJOY INTEGRAL

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Abstract. B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type, called the C-integral, which contains the Lebesgue integral and the Newton integral. D. Bongiorno gave a minimal constructive integration process of Riemann type, called the $\tilde{C}$-integral, which contains the Lebesgue integral and the improper Newton integral. On the other hand, Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, Kawasaki and Suzuki gave criteria for the C-integrability, and Kawasaki gave criteria for the $\tilde{C}$-integrability. In this paper, motivated by the results above, we give new integrals between the Lebesgue integral and the restricted Denjoy integral. Moreover we give criteria for the integrability of one of them in the style of Nakanishi.

1 Introduction Throughout this paper we denote by $\mathbf{(L)}(S)$, $\mathbf{(L^*)}(S)$ and $\mathbf{(D^*)}(S)$ the class of all Lebesgue integrable functions, the class of all improper Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a, b]$ into $(0, \infty)$ and a δ-fine McShane partition of an interval $[a, b] \subset \mathbb{R}$ is a collection $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals $I_k \subset [a, b]$ and $x_k \in [a, b]$ satisfying $I_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$ and $\sum_{k=1}^{k_0} |I_k| = b - a$. If $\sum_{k=1}^{k_0} |I_k| \leq b - a$, then we say that the collection is a δ-fine partial McShane partition. Moreover, if $x_k \in I_k$ for any $k = 1, \ldots, k_0$, then a δ-fine McShane partition and a δ-fine partial McShane partition are called a δ-fine Perron partition and a δ-fine partial Perron partition, respectively. We say that a function f from an interval $[a, b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a, b]$ into $\mathbb{R}$ such that $F' = f$ on $[a, b]$. We denote by $\mathbf{(N)}([a, b])$ the class of all Newton integrable functions from $[a, b]$ into $\mathbb{R}$. In [3] B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type, called the C-integral, which contains the Lebesgue integral and the Newton integral. Furthermore in [1–3] B. Bongiorno et al. gave some criteria for the C-integrability. We denote by $\mathbf{(C)}([a, b])$ the class of all C-integrable functions from $[a, b]$ into $\mathbb{R}$. We say that a function f from an interval $[a, b]$ into $\mathbb{R}$ is improper Newton integrable if there exist a countable subset $N \subset [a, b]$ and a function F from $[a, b]$ into $\mathbb{R}$ such that $F' = f$ on $[a, b] \setminus N$. We denote by $\mathbf{(N^*)}([a, b])$ the class of all improper Newton integrable functions from $[a, b]$ into $\mathbb{R}$. In [4] D. Bongiorno gave a minimal constructive integration process of Riemann type, called the $\tilde{C}$-integral, which contains the Lebesgue integral and the improper Newton integral. Furthermore in [4] D. Bongiorno gave some criteria for the $\tilde{C}$-integrability. We denote by $(\tilde{C})([a, b])$ the class of all $\tilde{C}$-integrable functions from $[a, b]$ into $\mathbb{R}$. The improper Lebesgue integral, the C-integral and the $\tilde{C}$-integral are between the Lebesgue integral and the restricted Denjoy integral.

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On the other hand, in [11, 14] Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, in [10] Kawasaki and Suzuki gave criteria for the C-integrability, and in [9] Kawasaki gave criteria for the $\widetilde{C}$-integrability.

In this paper, motivated by the results above, we give new integrals between the Lebesgue integral and the restricted Denjoy integral. Moreover we give criteria for the integrability of one of them in the style of Nakanishi.

2 Preliminaries We know that the Lebesgue integral and the restricted Denjoy integral are equivalent to the McShane integral and the Henstock-Kurzweil integral, respectively. The McShane integral and the Henstock-Kurzweil integral are Riemann type integrals and these definitions are as follows.

Definition 2.1. A function f from an interval $[a, b]$ into $\mathbb{R}$ is McShane integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$. The constant A is the value of the McShane integral of f and we denote by

$$A = (MS)\int_{[a,b]} f(x)dx = (L)\int_{[a,b]} f(x)dx.$$

We denote by **(MS)**$([a, b])$ the class of all McShane integrable functions from $[a, b]$ into $\mathbb{R}$.

Definition 2.2. A function f from an interval $[a, b]$ into $\mathbb{R}$ is Henstock-Kurzweil integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ with $x_k \in I_k$, that is, δ-fine Perron partition. The constant A is the value of the Henstock-Kurzweil integral of f and we denote by

$$A = (HK)\int_{[a,b]} f(x)dx = (D^*)\int_{[a,b]} f(x)dx.$$

We denote by **(HK)**$([a, b])$ the class of all Henstock-Kurzweil integrable functions from $[a, b]$ into $\mathbb{R}$.

In [5] D. Bongiorno showed a criterion for the improper Lebesgue integral as follows.

Theorem 2.1. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is improper Lebesgue integrable if and only if there exist a constant A and a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Moreover

$$A = (L^*) \int_{[a,b]} f(x)dx.$$

The theorem above gives a Riemann type definition for the improper Lebesgue integral. In [1], see also [2,3], B. Bongiorno gave the C-integral, which is also a Riemann type integral, as follows.

Definition 2.3. A function f from an interval $[a, b]$ into $\mathbb{R}$ is C-integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$, where $d(I, x) = \inf_{y \in I} |y - x|$. The constant A is the value of the C-integral of f and we denote by

$$A = (C) \int_{[a,b]} f(x)dx.$$

We denote by $\mathbf{(C)}([a, b])$ the class of all C-integrable functions from $[a, b]$ into $\mathbb{R}$.

In [4] D. Bongiorno gave the $\tilde{C}$-integral, which is also a Riemann type integral, as follows.

Definition 2.4. A function f from an interval $[a, b]$ into $\mathbb{R}$ is $\tilde{C}$-integrable if there exist a constant A and a countable subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the $\tilde{C}$-integral of f and we denote by

$$A = (\tilde{C}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{\mathbf{C}})([a, b])$ the class of all $\tilde{C}$-integrable functions from $[a, b]$ into $\mathbb{R}$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a, b]$ is an interval function on $[a, b]$. If an interval function F on $[a, b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a, b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive. In [11, 14] Nakanishi gave the following criteria for the restricted Denjoy integrability. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) $\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \dots, k_0\}$ of non-overlapping intervals in $[a, b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

Next Nakanishi gave the following theorem for the restricted Denjoy integrability.

Theorem 2.2. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a, b]$ such that the pair of f and F satisfies one of* (A), (B), (C) *and* (D). *Moreover, if the pair of f and F satisfies one of* (A), (B), (C) *and* (D), *then*

$$F(I) = (D^*) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Motivated by the results of Nakanishi, in [10] Kawasaki and Suzuki gave similar criteria and theorems for the C-integrability, and in [9] Kawasaki give similar criteria and theorems for the $\tilde{C}$-integrability.

3 Definitions of new integrals In this section firstly we define new integrals. By observing the definitions of the McShane, the improper Lebesgue in the sense of Theorem 2.1, the Henstock-Kurzweil integrals, C-integral and $\tilde{C}$-integral, we become aware of the following two integrals.

Definition 3.1. A function f from an interval $[a, b]$ into $\mathbb{R}$ is C*-integrable if there exist a constant A and a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the C*-integral of f and we denote by

$$A = (C^*) \int_{[a,b]} f(x)dx.$$

We denote by $(\mathbf{C}^*)([a,b])$ the class of all C*-integrable functions from $[a,b]$ into $\mathbb{R}$.

Definition 3.2. A function f from an interval $[a,b]$ into $\mathbb{R}$ is $\tilde{L}$-integrable if there exist a constant A and a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. The constant A is the value of the $\tilde{L}$-integral of f and we denote by

$$A = (\tilde{L}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{L})([a,b])$ the class of all $\tilde{L}$-integrable functions from $[a,b]$ into $\mathbb{R}$.

By the definitions of these integrals we obtain the following relations.

$$\begin{array}{ccccccc}
(\mathbf{N}) & & \subset & & (\mathbf{N}^*) & & (\mathbf{D}^*) \\
\cap & & & & \cap & & \\
(\mathbf{C}) & \subset & (\mathbf{C}^*) & \subset & (\tilde{C}) & & \| \\
\cup & & & & & \subset & \\
(\mathbf{MS}) & & \cup & & \cup & & (\mathbf{HK}) \\
\| & & & & & & \\
(\mathbf{L}) & \subset & (\mathbf{L}^*) & \subset & (\tilde{L}) & &
\end{array}$$

The above relations of inclusion are proper. We give some examples to check these. To show these, we provide the Saks-Henstock type lemmas. The following is the Saks-Henstock type lemma for the C*-integral.

Theorem 3.1. *If $f \in (\mathbf{C}^*)([a,b])$, then there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C^*) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ *whenever* $x_k \in N$.

Proof. Since $f \in (\mathbf{C}^*)([a,b])$, there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} f(x_k)|I_k| - (C^*) \int_{[a,b]} f(x)dx \right| < \frac{\varepsilon}{4}$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_1\}$ satisfying

$$\sum_{k=1}^{k_1} d(I_k, x_k) < \frac{2}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$. Let $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ be a δ-fine partial McShane partition satisfying

$$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$, and let $\{I_k \mid k = k_0 + 1, \dots, k_1\}$ be the sequence of intervals satisfying

$$\bigcup_{k=1}^{k_1} I_k = [a, b]$$

and $I_{k_2}^i \cap I_{k_3}^i = \emptyset$ if $k_2 \neq k_3$. Since f is C*-integrable on each I_k $(k = k_0 + 1, \dots, k_1)$, there exists a gauge δ_k such that

$$\left| \sum_{\ell=1}^{\ell(k)} \left(f(x_{k,\ell}) |I_{k,\ell}| - (C^*) \int_{I_{k,\ell}} f(x) dx \right) \right| < \frac{\varepsilon}{4(k_1 - k_0)}$$

for any δ_k-fine McShane partition $\{(I_{k,\ell}, x_{k,\ell}) \mid \ell = 1, \dots, \ell(k)\}$ satisfying

$$\sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon(k_1 - k_0)}$$

and $x_{k,\ell} \in I_{k,\ell}$ whenever $x_{k,\ell} \in N$. Without loss of generality, it may be assumed that $\delta_k \leq \delta$ for any $k = k_0 + 1, \dots, k_1$. Note that

$$\sum_{k=1}^{k_0} d(I_k, x_k) + \sum_{k=k_0+1}^{k_1} \sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon} + \sum_{k=k_0+1}^{k_1} \frac{1}{\varepsilon(k_1 - k_0)} = \frac{2}{\varepsilon}.$$

Therefore we obtain

$$\begin{aligned}
&\left| \sum_{k=1}^{k_0} \left(f(x_k) |I_k| - (C^*) \int_{I_k} f(x) dx \right) \right| \\
&\quad \leq \left| \sum_{k=1}^{k_1} f(x_k) |I_k| - (C^*) \int_{[a,b]} f(x) dx \right| \\
&\qquad + \sum_{k=k_0+1}^{k_1} \left| \sum_{\ell=1}^{\ell(k)} \left(f(x_{k,\ell}) |I_{k,\ell}| - (C^*) \int_{I_{k,\ell}} f(x) dx \right) \right| \\
&\quad < \frac{\varepsilon}{4} + \sum_{k=k_0+1}^{k_1} \frac{\varepsilon}{4(k_1 - k_0)} = \frac{\varepsilon}{2}.
\end{aligned}$$

Moreover we obtain

$$\begin{aligned}
&\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx \right| \\
&= \left| \sum_{f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx > 0} \left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx \right) \right| \\
&\quad + \left| \sum_{f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx < 0} \left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx \right) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

The following is the Saks-Henstock type lemma for the $\tilde{L}$-integral. The proof is similar to Theorem 3.1.

Theorem 3.2. *If $f \in (\tilde{\mathbf{L}})([a,b])$, then there exists a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (\tilde{L})\int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

The Saks-Henstock type lemma for the improper Lebesgue integral also holds, see [5].

Theorem 3.3. *If $f \in (\mathbf{L}^*)([a,b])$, then there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (L^*)\int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

We show that the above relations of inclusion are proper.

Theorem 3.4. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\mathbf{C})([0,1])$.*

Proof. Let f_1 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_1(x) = \begin{cases} (1-2x)\left(\sin\frac{1}{x(1-x)} - \frac{1}{x(1-x)}\cos\frac{1}{x(1-x)}\right), & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}, \end{cases}$$

and let F_1 be a function defined by

$$F_1(x) = \begin{cases} x(1-x)\sin\frac{1}{x(1-x)}, & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}. \end{cases}$$

Since f_1 is continuous on $(0,1)$ and

$$\lim_{\alpha\downarrow 0,\beta\uparrow 1}(L)\int_{[\alpha,\beta]} f_1(x)dx = \lim_{\alpha\downarrow 0,\beta\uparrow 1}(F_1(\beta)-F_1(\alpha)) = 0,$$

we obtain $f_1 \in \mathbf{(L^*)}([0,1])$ and hence $f_1 \in \mathbf{(C^*)}([0,1])$. However $f_1 \notin \mathbf{(C)}([0,1])$. Indeed, assume that $f_1 \in \mathbf{(C)}([0,1])$. Then by [2, Lemma 6] for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_1(x_k)(b_k-a_k) - (F_1(b_k)-F_1(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k,b_k],x_k) \mid k=1,\dots,k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d([a_k,b_k],x_k) < \frac{1}{\varepsilon}.$$

For any natural number n let

$$\begin{aligned} a_n &= \frac{1-\sqrt{1-\frac{4}{\frac{3}{2}\pi+2n\pi}}}{2}, \\ b_n &= \frac{1-\sqrt{1-\frac{4}{\frac{\pi}{2}+2n\pi}}}{2}. \end{aligned}$$

Note that $\{[a_n,b_n]\}$ is mutually disjoint and

$$\begin{aligned} F_1(a_n) &= -a_n(1-a_n) = -\frac{1}{\frac{3}{2}\pi+2n\pi}, \\ F_1(b_n) &= b_n(1-b_n) = \frac{1}{\frac{\pi}{2}+2n\pi}. \end{aligned}$$

Since the sequence $\{b_n(1-b_n)+a_n(1-a_n) \mid n\in\mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$\begin{aligned} &0 < b_n(1-b_n)+a_n(1-a_n), \\ &\sum_{n=1}^{\infty}(b_n(1-b_n)+a_n(1-a_n)) = \infty, \end{aligned}$$

we can take a strictly increasing finite sequence $\{n(k) \mid k=1,\dots,k_0\}$ satisfying $b_{n(1)} < \delta(0)$ and

$$\varepsilon < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)})+a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\{([a_{n(k)},b_{n(k)}],0) \mid k=1,\dots,k_0\}$ is a δ-fine partial McShane partition and satisfies

$$\sum_{k=1}^{k_0} d([a_{n(k)},b_{n(k)}],0) = \sum_{k=1}^{k_0} a_{n(k)} < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)})+a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

However

$$\begin{aligned}
&\sum_{k=1}^{k_0} |f_1(0)(b_{n(k)} - a_{n(k)}) - (F_1(b_{n(k)}) - F_1(a_{n(k)}))| \\
&= \sum_{k=1}^{k_0} |F_1(b_{n(k)}) - F_1(a_{n(k)})| \\
&= \sum_{k=1}^{k_0} (b_{n(k)}(1 - b_{n(k)}) + a_{n(k)}(1 - a_{n(k)})) \\
&> \varepsilon
\end{aligned}$$

and hence it is a contradiction. □

Theorem 3.5. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\mathbf{C}^*)([0,1])$.*

Proof. Let f_2 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_2(x) = \begin{cases} n(n+1)f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_2 be a function defined by

$$F_2(x) = \begin{cases} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 3.4. Since $F_2'(x) = f_2(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, we obtain $f_2 \in (\mathbf{N}^*)([0,1])$ and hence $f_2 \in (\tilde{C})([0,1])$. However $f_2 \notin (\mathbf{C}^*)([0,1])$. Indeed, assume that $f_2 \in (\mathbf{C}^*)([0,1])$. Then by Theorem 3.1 there exists a finite subset $N \subset [0,1]$ such that for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_2(x_k)(b_k - a_k) - (F_2(b_k) - F_2(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d([a_k, b_k], x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in [a_k, b_k]$ whenever $x_k \in N$.

Since N is finite, there exists a natural number p such that $\left[\frac{1}{p+1}, \frac{1}{p}\right] \cap N = \emptyset$. For any natural number n let

$$\begin{aligned}
a_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}}{2p(p+1)}, \\
b_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}}{2p(p+1)}.
\end{aligned}$$

Note that $\{[a_n, b_n]\}$ is mutually disjoint and

$$\begin{aligned} F_2(a_n) &= -(p(p+1)a_n - p)(p+1-p(p+1)a_n) \\ &= -p(p+1)((p+1)a_n - 1)(1-pa_n) \\ &= -\frac{1}{\frac{3}{2}\pi + 2n\pi}, \\ F_2(b_n) &= (p(p+1)b_n - p)(p+1-p(p+1)b_n) \\ &= p(p+1)((p+1)b_n - 1)(1-pb_n) \\ &= \frac{1}{\frac{\pi}{2} + 2n\pi}. \end{aligned}$$

Since the sequence $\{p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)) \mid n \in \mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$0 < p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)),$$
$$\sum_{n=1}^{\infty} p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)) = \infty,$$

we can take a strictly increasing finite sequence $\{n(k) \mid k = 1, \ldots, k_0\}$ satisfying $b_{n(1)} < \frac{1}{p+1} + \delta\left(\frac{1}{p+1}\right)$ and

$$\varepsilon < \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\left\{\left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \,\middle|\, k = 1, \ldots, k_0\right\}$ is a δ-fine partial McShane partition and

$$\begin{aligned} &\sum_{k=1}^{k_0} d\left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \\ &= \sum_{k=1}^{k_0}\left(a_{n(k)} - \frac{1}{p+1}\right) \\ &< \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) \\ &< \frac{1}{\varepsilon}. \end{aligned}$$

However

$$\begin{aligned} &\sum_{k=1}^{k_0}\left|f_2\left(\frac{1}{p+1}\right)(b_{n(k)} - a_{n(k)}) - (F_2(b_{n(k)}) - F_2(a_{n(k)}))\right| \\ &= \sum_{k=1}^{k_0}|F_2(b_{n(k)}) - F_2(a_{n(k)})| \\ &= \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) \\ &> \varepsilon \end{aligned}$$

and hence it is a contradiction. □

Theorem 3.6. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin (\mathbf{L}^*)([0,1])$.*

Proof. Let f_3 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_3(x) = \begin{cases} f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_3 be a function defined by

$$F_3(x) = \begin{cases} \frac{1}{n(n+1)} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 3.4. Then $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin (\mathbf{L}^*)([0,1])$. Indeed, since f_3 is improper Lebesgue integrable on each $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ and

$$(L^*) \int_{\left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x)dx = 0,$$

by Theorem 2.1 there exists a finite subset $N_n \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]$ such that for any positive number ε there exists a gauge δ_n such that

$$\left|\sum_{k=1}^{k_n} f_3(x_{n,k})|I_{n,k}|\right| < \frac{\varepsilon}{2^{n+1}}$$

for any δ_n-fine McShane partition $\{(I_{n,k}, x_{n,k}) \mid k = 1, \ldots, k_n\}$ of $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ satisfying $x_{n,k} \in I_{n,k}$ whenever $x_{n,k} \in N_n$. It is obvious that $N_n = \left\{\frac{1}{n+1}, \frac{1}{n}\right\}$. Let

$$M_n = \max\left\{|F_3(x)| \;\middle|\; x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]\right\}.$$

It holds that $M_n = \frac{2}{n(n+1)} M_1$. Without loss of generality, it may be assumed that $(x - \delta_n(x), x + \delta_n(x)) \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$. Let $N = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, $\delta(x) = \delta_n(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $\delta\left(\frac{1}{n}\right) = \min\left\{\delta_n\left(\frac{1}{n}\right), \delta_{n-1}\left(\frac{1}{n}\right)\right\}$ for any $n \in \mathbb{N}$ with $n \geq 2$ and $\delta(0) < \frac{1}{p}$ with $M_p < \frac{\varepsilon}{2}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Let

$q = \min\left\{ n \;\middle|\; I_1 \cap \left[\frac{1}{n+1}, \frac{1}{n}\right) \neq \emptyset \right\}$. Then

$$\begin{aligned}
&\left|\sum_{k=1}^{k_0} f_3(x_k)|I_k|\right| \\
&\quad = \left| f_3(0)|I_1| + \sum_{n=1}^{q} \sum_{I_k \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x_k)|I_k| + \sum_{n=2}^{q} \sum_{\frac{1}{n} \in I_k} f_3\left(\frac{1}{n}\right)|I_k| \right| \\
&\quad \leq |f_3(0)|I_1|| \\
&\qquad + \left| \sum_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} f_3(x_k)|I_k| + \sum_{\frac{1}{q} \in I_k} f_3\left(\frac{1}{q}\right) \left| I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]\right| \right| \\
&\qquad + \sum_{n=2}^{q-1} \left| \sum_{\frac{1}{n+1} \in I_k} f_3\left(\frac{1}{n+1}\right) \left| I_k \cap \left[\frac{1}{n+1}, \frac{1}{n}\right]\right| \right. \\
&\qquad\quad + \sum_{I_k \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x_k)|I_k| \\
&\qquad\quad \left. + \sum_{\frac{1}{n} \in I_k} f_3\left(\frac{1}{n}\right) \left| I_k \cap \left[\frac{1}{n+1}, \frac{1}{n}\right]\right| \right| \\
&\qquad + \left| \sum_{\frac{1}{2} \in I_k} f_3\left(\frac{1}{2}\right) \left| I_k \cap \left[\frac{1}{2}, 1\right]\right| + \sum_{I_k \subset \left[\frac{1}{2}, 1\right]} f_3(x_k)|I_k| \right| \\
&\quad \leq 0 + \left| \sum_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} f_3(x_k)|I_k| + \sum_{\frac{1}{q} \in I_k} f_3\left(\frac{1}{q}\right) \left| I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]\right| \right| + \sum_{n=2}^{q-1} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^2}.
\end{aligned}$$

By Theorem 3.3 we obtain

$$\begin{aligned}
&\left| \sum_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} f_3(x_k)|I_k| + \sum_{\frac{1}{q} \in I_k} f_3\left(\frac{1}{q}\right) \left| I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]\right| \right| \\
&\quad \leq \sum_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} \left| f_3(x_k)|I_k| - (L^*)\int_{I_k} f_3(x)dx \right| \\
&\qquad + \sum_{\frac{1}{q} \in I_k} \left| f_3(x_k) \left| I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]\right| - (L^*)\int_{I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]} f_3(x)dx \right| \\
&\qquad + \left| (L^*)\int_{\left(\bigcup_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} I_k\right) \cup \left(\bigcup_{\frac{1}{q} \in I_k} I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right]\right)} f_3(x)dx \right| \\
&\quad < \frac{\varepsilon}{2^{q+1}} + M_q.
\end{aligned}$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} f_3(x_k)|I_k|\right| &< \frac{\varepsilon}{2^{q+1}} + M_q + \sum_{n=2}^{q-1}\frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^2} \\ &< M_p + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} \\ &< \varepsilon\end{aligned}$$

and hence $f_3 \in (\tilde{L})([0,1])$. However, since it can be shown similarly to Theorem 3.5 that $f_3 \notin (\mathbf{C}^*)([0,1])$, we obtain $f_3 \notin (\mathbf{L}^*)([0,1])$. □

Theorem 3.7. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\mathbf{L}^*)([0,1])$.*

Proof. Let C be the Cantor set in $[0,1]$, let $\{(\alpha_p, \beta_p) \mid p \in \mathbb{N}\}$ be the sequence of all connected components of $[0,1] \setminus C$, let f_4 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_4(x) = \begin{cases} \frac{2(\alpha_p+\beta_p-2x)}{(\beta_p-\alpha_p)^2}\left(\frac{(x-\alpha_p)(\beta_p-x)}{(\beta_p-\alpha_p)^2}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)} - \cos\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}\right), & \text{if } x \in (\alpha_p,\beta_p), p \in \mathbb{N}, \\ 0, & \text{if } x \in C, \end{cases}$$

and let F_4 be a function defined by

$$F_4(x) = \begin{cases} \frac{(x-\alpha_p)^2(\beta_p-x)^2}{(\beta_p-\alpha_p)^4}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}, & \text{if } x \in (\alpha_p,\beta_p), p \in \mathbb{N}, \\ 0, & \text{if } x \in C. \end{cases}$$

Since $F_4'(x) = f_4(x)$ for any $x \in [0,1]$, we obtain $f_4 \in (\mathbf{N})([0,1])$ and hence $f_4 \in (\mathbf{C}^*)([0,1])$. However $f_4 \notin (\tilde{L})([0,1])$ and hence $f_4 \notin (\mathbf{L}^*)([0,1])$. We show $f_4 \notin (\tilde{L})([0,1])$. Assume that $f_4 \in (\tilde{L})([0,1])$. Then by Theorem 3.2 there exists a countable subset $N \subset [0,1]$ such that for any positive number ε there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_4(x_k)(b_k - a_k) - (F_4(b_k) - F_4(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in [a_k, b_k]$ whenever $x_k \in N$. Since N is countable and C is perfect, there exist $z \in C$ and $\{(\alpha_{p(q)}, \beta_{p(q)}) \mid q \in \mathbb{N}\} \subset \{(\alpha_p, \beta_p) \mid p \in \mathbb{N}\}$ such that $z \notin N$ and $\left(\alpha_{p(q)}, \frac{\alpha_{p(q)}+\beta_{p(q)}}{2}\right) \subset [z, z+\delta(z))$ for any q. For any natural numbers q and n let

$$\begin{aligned} a_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)} - \alpha_{p(q)})\left(1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}\right)}{2}, \\ b_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)} - \alpha_{p(q)})\left(1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}\right)}{2}. \end{aligned}$$

Note that $\{[a_{q,n}, b_{q,n}]\}$ is mutually disjoint and

$$\begin{aligned} F_4(a_{q,n}) &= -\frac{(a_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-a_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= -\frac{1}{\left(\frac{3}{2}\pi+2n\pi\right)^2}, \\ F_4(b_{q,n}) &= \frac{(b_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-b_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= \frac{1}{\left(\frac{\pi}{2}+2n\pi\right)^2}. \end{aligned}$$

Since $\{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ is a δ-fine partial McShane partition and

$$\sum_{q=1}^{\infty}\sum_{n=1}^{\infty} |f_4(z)(b_{q,n}-a_{q,n}) - (F_4(b_{q,n}) - F_4(a_{q,n}))| = \sum_{q=1}^{\infty}\sum_{n=1}^{\infty} |F_4(b_{q,n}) - F_4(a_{q,n})| = \infty,$$

there exists $\{([a_k, b_k], z) \mid k = 1, \ldots, k_0\} \subset \{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{k_0} |f_4(z)(b_k - a_k) - (F_4(b_k) - F_4(a_k))| > \varepsilon.$$

It is a contradiction. □

Theorem 3.8. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 3.7 that $f_4 \in \mathbf{(N)}([0,1])$ and hence $f_4 \in (\tilde{C})([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 3.9. *There exists a function f such that $f \in \mathbf{(C^*)}([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 3.7 that $f_4 \in \mathbf{(N)}([0,1])$ and hence $f_4 \in \mathbf{(C^*)}([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 3.10. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin \mathbf{(C^*)}([0,1])$.*

Proof. We show in the proof of Theorem 3.6 that $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin \mathbf{(C^*)}([0,1])$. □

4 Properties of the C*-integral

In this section we give a criterion for the C*-integrability.

Definition 4.1. Let F be an interval function on $[a,b]$ and let N be a finite subset of $[a,b]$. Then F is said to be C*-absolutely continuous on $E \subset [a,b]$ with respect to N if for any positive number ε there exist a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_{\mathrm{C}^*}(E, N)$ the class of all C*-absolutely continuous interval functions on E with respect to N. Moreover F is said to be C*-generalized absolutely continuous on $[a, b]$ if there exist a finite subset N and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$ for any m. We denote by $\mathbf{ACG}_{\mathrm{C}^*}([a, b])$ the class of all C*-generalized absolutely continuous interval functions on $[a, b]$.

Lemma 4.1. *If $F \in \mathbf{ACG}_{C^*}([a, b])$ and $E \subset [a, b]$ with $|E| = 0$, then there exists a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) *$x_k \in E$ for any k;*

(2) *$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;*

(3) *$x_k \in I_k$ whenever $x_k \in N$.*

Proof. Since $F \in \mathbf{ACG}_{\mathrm{C}^*}([a, b])$, there exist a finite subset $N \subset [a, b]$ and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$ for any m. Therefore for any positive number ε and for any natural number m there exist a gauge δ_m and a positive number η_m such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E_m$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta_m$.

Since $|E \cap E_m| = 0$, there exists an open set $O_m \supset E \cap E_m$ such that $|O_m| < \eta_m$. Define $\delta_m^*(x) = \min\{\delta_m(x), d(O_m^c, x)\}$, where O_m^c is the complement of O_m. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m^*-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying (1), (2), (3) and (4). Define $\delta(x) = \delta_m^*(x)$ for any $x \in E \cap E_m$ $(m \in \mathbb{N})$. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| = \sum_{n=1}^{\infty} \sum_{x_k \in E_m} |F(I_k)| \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^{m+1}} = \frac{\varepsilon}{2} < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$.

□

Lemma 4.2. *If F is differentiable at $x \in [a,b]$, then for any positive number ε there exists a positive number δ such that*

$$|F(t) - F(s) - F'(x)(t-s)| < \varepsilon(2d([s,t],x) + t - s)$$

for any interval $[s,t] \subset (x-\delta, x+\delta) \cap [a,b]$.

Proof. Since F is differentiable at $x \in [a,b]$, there exists a positive number δ such that

$$|F(\xi) - F(x) - F'(x)(\xi - x)|| < \varepsilon|\xi - x|$$

for any $\xi \in (x-\delta, x+\delta) \cap [a,b]$. Therefore for any interval $[s,t] \subset (x-\delta, x+\delta) \cap [a,b]$ we obtain

$$\begin{aligned}
&|F(t) - F(s) - F'(x)(t-s)| \\
&\quad \leq |F(t) - F(x) - F'(x)(t-x)| + |F(x) - F(s) - F'(x)(x-s)| \\
&\quad < \varepsilon|t-x| + \varepsilon|s-x| \\
&\quad = \varepsilon(2d([s,t],x) + t - s).
\end{aligned}$$

□

Theorem 4.1. *For any $F \in \mathbf{ACG}_{C^*}([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in \mathbf{(C^*)}([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (C^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in \mathbf{(C^*)}([a,b])$ satisfies $F \in \mathbf{ACG}_{C^*}([a,b])$.

Proof. Note that, if $F \in \mathbf{ACG}_{\mathrm{C}^*}([a,b])$, then $F \in \mathbf{ACG}_\delta([a,b])$, see [7, Definition 9.14]. By [7, Theorem 9.17] there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$. Let

$$E = \left\{ x \;\middle|\; \frac{d}{dx}F([a,x]) \text{ does not exist at } x \in [a,b] \right\}.$$

Then $|E| = 0$, and by Lemma 4.1 there exists a finite subset $N \subset [a,b]$ such that for any positive number ε with $\varepsilon < \frac{4}{b-a}$ there exists a gauge δ_1 such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{4}$$

for any δ_1-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$.

If $x \notin E$, then by Lemma 4.2 there exists a positive number $\delta_2(x)$ such that

$$\left|F(t) - F(s) - \frac{d}{dx}F([a,x])(t-s)\right| < \frac{\varepsilon^2}{8}(2d([s,t],x) + t - s)$$

for any interval $[s,t] \subset (x - \delta_2(x), x + \delta_2(x)) \cap [a,b]$. Let

$$\delta(x) = \begin{cases} \delta_1(x), & \text{if } x \in E, \\ \delta_2(x), & \text{if } x \notin E, \end{cases}$$

and let

$$f(x) = \begin{cases} 0, & \text{if } x \in E, \\ \frac{d}{dx}F([a,x]), & \text{if } x \notin E. \end{cases}$$

Then we obtain

$$\begin{aligned} \left|\sum_{k=1}^{k_0} f(x_k)|I_k| - F(I)\right| &\le \left|\sum_{x_k \in E} F(I_k)\right| + \left|\sum_{x_k \notin E} f(x_k)|I_k| - F(I_k)\right| \\ &\le \sum_{x_k \in E} |F(I_k)| + \sum_{x_k \notin E} |f(x_k)|I_k| - F(I_k)| \\ &< \frac{\varepsilon}{4} + \sum_{x_k \notin E} \frac{\varepsilon^2}{8}(2d(I_k, x_k) + |I_k|) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} \cdot 2 \cdot \frac{1}{\varepsilon} + \frac{\varepsilon^2}{8}(b-a) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for any interval $I \subset [a,b]$ and for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ of I satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

Conversely let $f \in \mathbf{(C^*)}([a,b])$ and let

$$F(I) = (C^*) \int_I f(x)dx$$

for any interval $I \subset [a,b]$. For any natural number m let $E_m = \{x \mid x \in [a,b], |f(x)| \le m\}$. Then $\bigcup_{m=1}^{\infty} E_m = [a,b]$. We show that $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$, where N is an excepting finite subset of $[a,b]$ in the definition of the C*-integral of f. Let ε be a positive number. By Theorem 3.1 there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

Let $\eta = \frac{\varepsilon}{2m}$. If $x_k \in E_m$ for any k and $\sum_{k=1}^{k_0} |I_k| < \eta$, then we obtain

$$\begin{aligned} \sum_{k=1}^{k_0} |F(I_k)| &\leq \sum_{k=1}^{k_0} |f(x_k)||I_k| + \sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| \\ &< m\sum_{k=1}^{k_0} |I_k| + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

□

5 Criteria for the C*-integrability We consider the following four criteria for the pair of a function f from $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$.

$(\mathrm{A})_{\mathrm{C}^*}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a, b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a finite subset $N \subset [a, b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \dots, k_0, k_0 + 1, \dots, k_1,\ 0 \leq k_0 \leq k_1\}$ of non-overlapping intervals in $[a, b]$ which consists of a finite family $\{I_k \mid k = 1, \dots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0 + 1, \dots, k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0 + 1, \dots, k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in I_k$ whenever $x_k \in N$.

$(\mathrm{B})_{\mathrm{C}^*}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a finite subset $N \subset [a, b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \dots, k_0, k_0+1, \dots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a, b]$ which consists of a finite family $\{I_k \mid k = 1, \dots, k_0\}$ with $I_k \cap M_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \dots, k_1\}$ satisfying

(4.1) $x_k \in M_n$ for any $k = k_0+1, \dots, k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(4.3) $x_k \in I_k$ whenever $x_k \in N$.

$(\mathrm{C})_{\mathrm{C}^*}$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a, b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a finite subset $N \subset [a, b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ in $[a, b]$ satisfying

(3.1) $x_k \in F_n$ for any k;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in I_k$ whenever $x_k \in N$;

(3.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

$(\mathrm{D})_{\mathrm{C}^*}$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a finite subset $N \subset [a, b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ in $[a, b]$ satisfying

(4.1) $x_k \in M_n$ for any k;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;
(4.3) $x_k \in I_k$ whenever $x_k \in N$;
(4.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(\mathrm{A})_{\mathrm{C}^*}$ implies $(\mathrm{B})_{\mathrm{C}^*}$ and $(\mathrm{C})_{\mathrm{C}^*}$ implies $(\mathrm{D})_{\mathrm{C}^*}$. Now we give the following theorems for the C*-integral.

Theorem 5.1. *Let $f \in (\mathbf{C}^*)([a,b])$ and let F be an additive interval function on $[a,b]$ defined by*

$$F(I) = (C^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies $(\mathrm{A})_{\mathrm{C}^*}$.

Proof. Since $f \in (\mathbf{C}^*)([a,b])$, we obtain $f \in (\mathbf{D}^*)([a,b])$. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Since by Theorem 2.2 the pair of f and F satisfies (A), for $\left\{\frac{\varepsilon_n}{2}\right\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1,\dots,k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$. By Theorem 3.1 there exists a finite subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}(f(x_k)|I_k| - F(I_k))\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1,\dots,k_1\}$ in $[a,b]$ satisfying (3.2) and (3.3). Since $f\chi_{F_n} \in (\mathbf{L})([a,b])$, where χ_{F_n} means the characteristic function of F_n, by the Saks-Henstock lemma for the McShane integral, for instance see [7, Lemma 10.6], for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}(x_k)|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1,\dots,k_1\}$ in $[a,b]$. Since $f = f\chi_{F_n}$ on F_n, for any n there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&= \left|\sum_{k=k_0+1}^{k_1}(F(I_k) - f(x_k)|I_k|)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}(x_k)|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&< \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.1), (3.2) and (3.3). Therefore

$$\begin{aligned}&\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\ &\quad\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\ &\quad< \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying (3.1), (3.2) and (3.3), that is, (3) holds. □

Theorem 5.2. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies $(\mathrm{A})_{\mathrm{C}^*}$, then the pair of f and F satisfies $(\mathrm{C})_{\mathrm{C}^*}$. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies $(\mathrm{B})_{\mathrm{C}^*}$, then the pair of f and F satisfies $(\mathrm{D})_{\mathrm{C}^*}$.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_{\mathrm{C}^*}$ hold. We show (3) of $(\mathrm{C})_{\mathrm{C}^*}$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n,\varepsilon)$ such that, if $|E| < \rho(n,\varepsilon)$, then

$$\left|(L)\int_{E\cap F_n} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n,\varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n,\varepsilon) \ge n$, and put $\eta = \rho(m(n,\varepsilon),\varepsilon)$. By (3) of $(\mathrm{A})_{\mathrm{C}^*}$ there exists a subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for $m(n,\varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a,b]$ satisfying (3.1), (3.2), (3.3) and (3.4) of $(\mathrm{C})_{\mathrm{C}^*}$. Then we obtain

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0} |I_k| < \eta = \rho(m(n,\varepsilon),\varepsilon)$, we obtain

$$\left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} F(I_k)\right| &\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| + \left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Next we show (4) of $(\mathrm{C})_{\mathrm{C}^*}$. Let I be a subinterval of $[a,b]$. In the case of $I \cap F_n = \emptyset$ (4) of $(\mathrm{C})_{\tilde{C}}$ is clear. Consider the case of $I \cap F_n \neq \emptyset$. Let $\{J_p \mid p = 1, 2, \ldots\}$ be the sequence of all connected components of $I^i \setminus F_n$. Since $I \cap F_m \neq \emptyset$ holds for any $m \geq n$, by (3) of $(\mathrm{A})_{\mathrm{C}^*}$ we obtain

$$\left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| < \varepsilon_m.$$

Since $\overline{J_p} \cap F_m \neq \emptyset$ holds for any p, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left|\sum_{p=1}^{\infty}\left(F(\overline{J_p}) - (L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \leq \varepsilon_m$$

for any $m \geq n$. On the other hand, we obtain

$$(L)\int_{I\cap F_m} f(x)dx = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx$$

for any $m \geq n$. Therefore we obtain

$$\begin{aligned}
&\left|F(I) - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p})\right)\right| \\
&\quad \leq \left|F(I) - (L)\int_{I\cap F_m} f(x)dx\right| \\
&\qquad + \left|(L)\int_{I\cap F_m} f(x)dx - \left((L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right)\right| \\
&\qquad + \left|-\sum_{p=1}^{\infty} F(\overline{J_p}) + \sum_{p=1}^{\infty}(L)\int_{\overline{J_p}\cap F_m} f(x)dx\right| \\
&\quad < \varepsilon_m + 0 + \varepsilon_m = 2\varepsilon_m
\end{aligned}$$

for any $m \geq n$ and hence

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}).$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_{\mathrm{C}^*}$, then the pair of f and F satisfies $(\mathrm{D})_{\mathrm{C}^*}$. □

Theorem 5.3. *If the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies $(\mathrm{D})_{\mathrm{C}^*}$, then $f \in (\mathbf{C}^*)([a, b])$ and*

$$F(I) = (C^*)\int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Proof. By (1) and (4) there exist a finite subset $N \subset [a, b]$ and a increasing sequence $\{M_n\}$ of non-empty measurable sets such that $\bigcup_{n=1}^{\infty} M_n = [a, b]$ and for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a, b]$ satisfying (4.1), (4.2), (4.3) and (4.4). Therefore we obtain

$$\begin{aligned} \sum_{k=1}^{k_0} |F(I_k)| &= \left| \sum_{F(x_k)>0} F(I_k) \right| + \left| \sum_{F(x_k)<0} F(I_k) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and hence $F \in \mathbf{ACG}_{\mathrm{C}^*}([a, b])$. By Theorem 4.1 there exists $\frac{d}{dx}F([a, x])$ for almost every $x \in [a, b]$, and there exists $g \in \mathbf{(C^*)}([a, b])$ such that

$$F(I) = (C^*) \int_I g(x)dx$$

for any interval $I \subset [a, b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [16, Theorem (5.1)] $g_n \in \mathbf{(D^*)}(I)$ for any interval $I \subset [a, b]$ and by (3)

$$\begin{aligned} (D^*) \int_I g_n(x)dx &= (D^*) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (D^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (C^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}), \end{aligned}$$

where $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*) \int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a, x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a, x]) = f(x)$ for almost every $x \in [a, b]$. □

By Theorems 5.1, 5.2 and 5.3 we obtain the following criteria for the C*-integrability.

Theorem 5.4. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is C*-integrable if and only if there exists an additive interval function F on $[a, b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_{\mathrm{C}^*}$, $(\mathrm{B})_{\mathrm{C}^*}$, $(\mathrm{C})_{\mathrm{C}^*}$ *and* $(\mathrm{D})_{\mathrm{C}^*}$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_{\mathrm{C}^*}$, $(\mathrm{B})_{\mathrm{C}^*}$, $(\mathrm{C})_{\mathrm{C}^*}$ *and* $(\mathrm{D})_{\mathrm{C}^*}$, *then*

$$F(I) = (C^*) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

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LEFT REGULAR AND INTRA-REGULAR ORDERED HYPERSEMIGROUPS IN TERMS OF SEMIPRIME AND FUZZY SEMIPRIME SUBSETS

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To the memory of Professor Kiyoshi Iséki

Abstract

We prove that an ordered hypersemigroup H is left (resp. right) regular if and only if every left (resp. right) ideal of H is semiprime and it is intra-regular if and only if every ideal of H is semiprime. Then we prove that an ordered hypersemigroup H is left (resp. right) regular if and only if every fuzzy left (resp. right) ideal of H is fuzzy semiprime and it is intra-regular if and only if every fuzzy ideal of H is fuzzy semiprime.

1 Introduction and prerequisites

A semigroup $(S, \cdot)$ is left (resp. right) regular if and only if every left (resp. right) ideal of S is semiprime, it is intra-regular if and only if every ideal of S is semiprime (cf. [1; Theorems 4.2, 4.4]). For an ordered semigroup $(S, \cdot, \leq)$ and a subset A of S, we denote by $(A]$ the subset of S defined by $(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$. An ordered semigroup $(S, \cdot, \leq)$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$. This is equivalent to saying that $a \in (Sa^2]$ for every $a \in S$ or $A \subseteq (SA^2]$ for every $A \subseteq S$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2x$, equivalently if $a \in (a^2S]$ for every $a \in S$ or $A \subseteq (A^2S]$ for every $A \subseteq S$. An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. This is equivalent to saying that $a \in (Sa^2S]$ for every $a \in S$ or $A \subseteq (SA^2S]$ for every $A \subseteq S$. We have seen in [10] that an ordered semigroup

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S is left (resp. right) regular if and only if the left (resp. right) ideals of S are semiprime and it is intra-regular if and only if the ideals of S are semiprime. We have also seen that an ordered semigroup S is left (resp. right) regular if and only if the fuzzy left (resp. fuzzy right) ideals of S are semiprime and it is intra-regular if and only if the fuzzy ideals of S are semiprime. In the present paper we examine these results for an hypersemigroup. For the sake of completeness, let us first give some definitions-remarks already given in [7, 8].

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a,b) \to a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A,B) \to A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b)\in A\times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ is the set of nonempty subsets of H). As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be denoted by $(H, \circ)$ (instead of $(H, \circ, *)$). If $(H, \circ)$ is an hypergroupoid and $A, B, C, D \in \mathcal{P}^*(H)$, then

$A \subseteq B$, implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Equivalently,

$A \subseteq B$ and $C \subseteq D$ implies $A * C \subseteq B * D$ and $C * A \subseteq D * B$.

We also have $H * H \subseteq H$.

If H is an hypergroupoid then, for every $x, y \in H$, we have

$$\{x\} * \{y\} = x \circ y.$$

Indeed, $\{x\} * \{y\} = \bigcup\limits_{u\in\{x\},v\in\{y\}} (u \circ v) = x \circ y$.

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 1.1. *Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

1. *$x \in A * B \iff x \in a \circ b$ for some $a \in A$, $b \in B$.*
2. *If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.*

Lemma 1.2. [7] *Let $(H, \circ)$ be an hypergroupoid and $A_i, B \in \mathcal{P}^*(H)$, $i \in I$. Then we have the following:*

(1) $(\bigcup\limits_{i\in I} A_i) * B = \bigcup\limits_{i\in I} (A_i * B)$.

(2) $B * (\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I} (B * A_i)$.

An hypergroupoid H is called *hypersemigroup* if, for every $x, y, z \in H$, we have

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

which is equivalent to saying that $\{x\} * \Big(\{y\} * \{z\}\Big) = \Big(\{x\} * \{y\}\Big) * \{z\}$ for every $x, y, z \in H$. If we like, we can identify the $\{x\}$ by x and the $\{z\}$ by z and write $x * (y \circ z)$ instead of $\{x\} * (y \circ z)$ and $(x \circ y) * z$ instead of $(x \circ y) * \{z\}$. So the associativity relation of an hypergroupoid can be also given, for short, as $x * (y \circ z) = (x \circ y) * z$.

Lemma 1.3 [7] *If $(H, \circ)$ is an hypersemigroup and $A, B, C \in \mathcal{P}^*(H)$, then we have*

$$\begin{aligned}(A * B) * C &= \bigcup_{(a,b,c)\in A\times B\times C} \Big((a \circ b) * \{c\}\Big)\\ &= \bigcup_{(a,b,c)\in A\times B\times C} \Big(\{a\} * (b \circ c)\Big) = A * (B * C)\\ &= \bigcup_{(a,b,c)\in A\times B\times C} \Big(\{a\} * \{b\} * \{c\}\Big).\end{aligned}$$

Thus we can write $(A * B) * C = A * (B * C) = A * B * C$. As a consequence, for any product $A_1 * A_2 * ..... * A_n$ of elements of $\mathcal{P}^*(H)$ we can put the parentheses in any place beginning with some A_i and ending in some A_j $(1 \le i, j \le n)$. In addition, using induction, we have the following which gives the form of the elements of the set $A_1 * A_2 * ..... * A_n$.

Lemma 1.4. For any finite family $A_1, A_2, ..., A_n$ of elements of $\mathcal{P}^*(H)$, we have

$$A_1 * A_2 * ..... * A_n = \bigcup_{(a_1,a_2...a_n)\in A_1\times A_2\times ...\times A_n} \Big(\{a_1\} * \{a_2\} * ... * \{a_n\}\Big).$$

For an hypergroupoid H, we denote by $(A]$ the subset of H defined by

$$(A] := \{t \in H \mid t \le a \text{ for some } a \in A\}.$$

Exactly as in ordered semigroups, we have $(H] = H$ and $((A]] = (A]$ for any nonempty subset A of H.
The results of the present paper hold not only for the elements but for the subsets of H as well which shows the pointless character of the results.

2 A characterization of left regular (resp. intra-regular) ordered hypersemigroups in terms of semiprime left ideals (resp. ideals)

Notation 2.1. Let $(H, \circ)$ be an hypergroupoid and "$\le$" an order relation on H. Denote by "$\preceq$" the relation on $\mathcal{P}^*(H)$ defined by

$$\preceq := \{(A, B) \mid \forall\, a \in A\ \exists\, b \in B \text{ such that } (a, b) \in \le\}.$$

So, for $A, B \in \mathcal{P}^*(H)$, we write $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \le b$. This is a reflexive and transitive relation on $\mathcal{P}^*(H)$, that is, a preorder on $\mathcal{P}^*(H)$.

A semigroup $(S, \cdot)$ is called an ordered semigroup if there exists an order relation "$\le$" on S such that $(a, b) \in\le$ implies $(ac, bc) \in\le$ and $(ca, cb) \in\le$ for every $c \in S$. Using the notation $a \le b$ instead of $(a, b) \in\le$, we write $a \le b$ implies $ac \le bc$ and $ca \le cb$ for every $c \in S$. The definition of the ordered semigroup can be naturally transferred to hypersemigroups as follows:

Definition 2.2. (cf. also [13]) Let $(H, \circ)$ be an hypergroupoid and "$\le$" an order relation on H. Then H is called an *ordered hypergroupoid*, denoted by $(H, \circ, \le)$, if given an element $(x, y) \in\le$, we have $(x \circ z, y \circ z) \in\preceq$ and $(z \circ x, z \circ y) \in\preceq$ for every $z \in H$. In other words,

$$x \le y \text{ implies } x \circ z \preceq y \circ z \text{ and } z \circ x \preceq z \circ y \text{ for all } z \in H.$$

The concept of right regular ordered semigroups introduced by Kehayopulu in [4] is as follows: An ordered semigroup $(S, \cdot, \le)$ is called right regular if for every $a \in S$ there exists $x \in S$ such that $a \le a^2x$. Later, in an analogous manner she defined and studied the left regular ordered semigroups: An ordered semigroup S is called left regular if for every $a \in S$ there exists $x \in S$ such that $a \le xa^2$. The concept of left regular ordered semigroups is naturally transferred to an ordered hypersemigroup H as follows: for every $a \in H$, there exists $x \in H$ such that $\{a\} \preceq \{x\}*(a\circ a)$. (Clearly $\{x\}*(a\circ a) = (x\circ a)*\{a\} = \{x\}*\{a\}*\{a\}$). This leads to the following definition.

Definition 2.3. An ordered hypersemigroup H is called *left regular* if for every $a \in H$ there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \le t$.
It is called *right regular* if for every $a \in H$ there exist $x, t \in H$ such that $t \in (a \circ a) * \{x\}$ and $a \le t$.

Proposition 2.4. *Let H be an ordered hypersemigroup. The following are equivalent:*

1. *H is left regular.*
2. *$a \in \big(H * (a \circ a)\big]$ for every $a \in H$.*
3. *$A \subseteq (H * A * A]$ for every $A \in \mathcal{P}^*(H)$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in H$. Since H is left regular, there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \le t$. We have

$$a \le t \in \{x\} * (a \circ a) \subseteq H * (a \circ a),$$

so $a \in \big(H * (a \circ a)\big]$.
(2) $\Longrightarrow$ (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. Since $a \in H$, by (2), we have

$$a \in \big(H * (a \circ a)\big] = \big(H * \{a\} * \{a\}\big] \subseteq (H * A * A],$$

thus we get $a \in (H * A * A]$, and (3) holds.
(3) $\Longrightarrow$ (1). Let $a \in H$. Since $\{a\} \in \mathcal{P}^*(H)$, by (3), we have

$$a \in \{a\} \subseteq \Big(H * \{a\} * \{a\}\Big].$$

Then $a \le t$ for some $t \in \Big(H * \{a\}\Big) * \{a\}$. By Proposition 1.1, there exists $y \in H * \{a\}$ such that $t \in y \circ a$. Since $y \in H * \{a\}$, again by Proposition 1.1, there exists $x \in H$ such that $y \in x \circ a$. We have

$$t \in y \circ a \subseteq (x \circ a) * \{a\} = \{x\} * (a \circ a).$$

Since $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \le t$, H is left regular. $\square$
In a similar way we prove the following:

Proposition 2.5. *Let H be an ordered hypersemigroup. The following are equivalent:*

1. *H is right regular.*
2. *$a \in \Big((a \circ a) * H\Big]$ for every $a \in H$.*
3. *$A \subseteq (A * A * H]$ for every $A \in \mathcal{P}^*(H)$.*

A subset A of a groupoid or an ordered groupoid S is called semiprime if $x^2 \in A$ $(x \in S)$ implies $x \in A$ [1, 2, 5]. This concept is naturally transferred in case of hypergroupoids in the definition below:

Definition 2.6. Let H be an hypergroupoid. A nonempty subset A of H is called *semiprime* if for every $t \in H$ such that $t \circ t \subseteq A$, we have $t \in A$.

Proposition 2.7. *Let $(H, \circ)$ be an hypergroupoid and $A \in \mathcal{P}^*(H)$. The following are equivalent:*

1. *A is semiprime.*
2. *For every $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$, we have $T \subseteq A$.*

Proof. (1) $\Longrightarrow$ (2). Let $T \in \mathcal{P}^*(H)$, $T * T \subseteq A$ and $t \in T$. Since $t \in H$ and $t \circ t \subseteq T * T \subseteq A$, by (1), we have $t \in A$.
(2) $\Longrightarrow$ (1). Let $t \in H$ such that $t \circ t \subseteq A$. Since $\{t\} \in \mathcal{P}^*(H)$ and $\{t\} * \{t\} = t \circ t \subseteq A$, by (2), we have $\{t\} \subseteq A$, so $t \in A$, and T is semiprime. $\square$

Lemma 2.8. *Let H be an ordered hypergroupoid and $A, B \in \mathcal{P}^*(H)$. Then we have*

$$(A] * (B] \subseteq (A * B].$$

Proof. Let $t \in (A] * (B]$. Then $t \in x \circ y$ for some $x \in (A]$, $y \in (B]$. Since $x \in (A]$, we have $x \le a$ for some $a \in A$. Since $y \in (B]$, we get $y \le b$ for some $b \in B$. Since $x \le a$ and $y \le b$, we have $t \in x \circ y \preceq a \circ b$. Then, there exists $z \in a \circ b$ such that $t \le z$. We get $t \le z \in a \circ b$, so $t \in (a \circ b]$. On the other hand, since $a \in A$ and $b \in B$, we have $a \circ b \subseteq A * B$. Then $(a \circ b] \subseteq (A * B]$, and $t \in (A * B]$. $\square$

The concepts of left and right ideals of ordered groupoids introduced by Kehayopulu in [3] are naturally transferred in case of ordered hypergroupoids as follows: A nonempty subset A of an ordered hypergroupoid H is called a *left* (resp. *right*) *ideal* of H if

1. $H * A \subseteq A$ (resp. $A * H \subseteq A$) and
2. if $a \in A$ and $H \ni b \leq a$, then $b \in A$, that is if $(A] = A$.

It is called an *ideal* of H if it is both a left and a right ideal of H.

Theorem 2.9. *An ordered hypersemigroup H is left regular if and only if every left ideal of H is semiprime.*

Proof. $\Longrightarrow$. Let A be a left ideal of H and $a \in H$ such that $a \circ a \subseteq A$. Since H is left regular and $a \in H$, there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \leq t$. We have $t \in \{x\} * (a \circ a) \subseteq H * A \subseteq A$. Then $a \leq t \in A$, and $a \in A$.
$\Longleftarrow$. Let $a \in H$. We have

$$(a \circ a) * (a \circ a) \subseteq H * (a \circ a) \subseteq \Big(H * (a \circ a)\Big].$$

The set $\Big(H * (a \circ a)\Big]$ is a left ideal of H. Indeed, it is a nonempty subset of H and we have

$$\begin{aligned} H * \Big(H * (a \circ a)\Big] &= (H] * \Big(H * (a \circ a)\Big] \subseteq \bigg(H * \Big(H * (a \circ a)\Big)\bigg] \\ &= \Big((H * H) * (a \circ a)\Big] \subseteq \Big(H * (a \circ a)\Big], \end{aligned}$$

and

$$\bigg(\Big(H * (a \circ a)\Big]\bigg] = \Big(H * (a \circ a)\Big]$$

(as $((A]] = (A]$ for any subset A of S). Since $\Big(H * (a \circ a)\Big]$ is semiprime, we have $(a \circ a) \subseteq \Big(H * (a \circ a)\Big]$, and $a \in \Big(H * (a \circ a)\Big]$. Then, by Proposition 2.4, H is left regular.

Now we will give a second proof of the Theorem using only sets: $\Longrightarrow$. Let A be a left ideal of H and $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$. Since H is left regular, by Proposition 2.4, we have $T \subseteq (H * T * T] \subseteq (H * A] \subseteq (A] = A$. $\Longleftarrow$. Let $A \in \mathcal{P}^*(H)$. We have

$$(A * A) * (A * A) \subseteq (H * H) * A * A \subseteq H * A * A \subseteq (H * A * A].$$

Since $(H*A*A]$ is a left ideal of H, it is semiprime, and we have $A*A \subseteq (H*A*A]$ and $A \subseteq (H * A * A]$. Thus H is left regular. $\square$

In a similar way we prove the following:

Theorem 2.10. *An ordered hypersemigroup H is right regular if and only if every right ideal of H is semiprime.*

Our aim now is to characterize the intra-regular ordered hypersemigroups in terms of semiprime ideals. The concept of an intra-regular ordered semigroup introduced by Kehayopulu in [6] is as follows: An ordered semigroup $(S, \cdot, \leq)$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. This concept is naturally transferred to an ordered hypersemigroup as follows: For every $a \in H$, $\{a\} \preceq \{x\} * (a \circ a) * \{y\}$. This leads to the following definition

Definition 2.11. An ordered hypersemigroup $(H, \circ, \leq)$ is called *intra-regular* if for every $a \in H$ there exist $x, y, t \in H$ such that $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$.
Clearly, $\{x\} * (a \circ a) * \{y\} = (x \circ a) * (a \circ y) = \{x\} * \{a\} * \{a\} * \{y\}$.

Proposition 2.12. *Let $(H, *, \leq)$ be an ordered hypersemigroup. The following are equivalent:*

1. *H is intra-regular.*
2. *$a \in \Big(H * (a \circ a) * H\Big]$ for every $a \in H$.*
3. *$A \subseteq (H * A * A * H]$ for every $A \in \mathcal{P}^*(H)$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in H$. Since H is intra-regular, there exist $x, y, t \in H$ such that $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$. We have

$$a \leq t \in \{x\} * (a \circ a) * \{y\} \subseteq H * (a \circ a) * H,$$

so $a \in \Big(H * (a \circ a) * H\Big]$.
(2) $\Longrightarrow$ (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. By (2), we have

$$a \in \Big(H * (a \circ a) * H\Big] = \Big(H * \{a\} * \{a\} * H\Big] \subseteq (H * A * A * H],$$

so $a \in (H * A * A * H]$ and (3) is satisfied.
(3) $\Longrightarrow$ (1). Let $a \in H$. Since $\{a\} \in \mathcal{P}^*(H)$, by (3), we have

$$a \in \{a\} \subseteq \Big(H * \{a\} * \{a\} * H\Big].$$

Then $a \leq t$ for some $t \in H * \{a\} * \{a\} * H = \Big(H * (a \circ a)\Big) * H$. Then there exist $u \in H * (a \circ a)$ and $y \in H$ such that $t \in u \circ y$. Since $u \in H * (a \circ a)$, there exist $x \in H$ and $w \in (a \circ a)$ such that $u \in x \circ w$. We have

$$t \in u \circ y \subseteq (x \circ w) * \{y\} = \{x\} * \{w\} * \{y\} \subseteq \{x\} * (a \circ a) * \{y\}.$$

For the elements $x, y, t \in H$, we have $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$, so H is intra-regular. □

Theorem 2.13. *An ordered hypersemigroup H is intra-regular if and only if every ideal of H is semiprime.*

$\Longrightarrow$. Let A be an ideal of H and $a \in H$ such that $a \circ a \subseteq A$. Since $a \in H$ and H is intra-regular, there exist $x, y, t \in H$ such that $t \in H * (a \circ a) * \{y\}$ and $a \leq t$. Then $t \in H * (a \circ a) * \{y\} \subseteq H * A * H \subseteq A$. Since $a \in H$ and $a \leq t \in A$, we have $a \in A$. Thus H is semiprime.
$\Longleftarrow$. Let $a \in H$. We have

$$(a \circ a) * (a \circ a) \subseteq H * \{a\} * \{a\} * H \subseteq \Big(H * \{a\} * \{a\} * H\Big].$$

The set $\Big(H * \{a\} * \{a\} * H\Big]$ is an ideal of H, so it is semiprime. Hence we have $a \circ a \subseteq \Big(H * \{a\} * \{a\} * H\Big]$, and $a \in \Big(H * \{a\} * \{a\} * H\Big] = \Big(H * (a \circ a) * H\Big]$. By Proposition 2.12, H is intra-regular.

A second proof of the theorem using only sets is as follows: $\Longrightarrow$. Let A be an ideal of H and $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$. Since H is intra-regular, by Proposition 2.12, we have

$$T \subseteq (H * T * T * H] \subseteq (H * A * H] \subseteq (A] = A,$$

so A is semiprime. $\Longleftarrow$. Let A be a nonempty subset of H. Since $(A * A) * (A * A) \subseteq (H*A*A*H]$ and $(H*A*A*H]$ is semiprime, we have $A*A \subseteq (H*A*A*H]$, and $A \subseteq (H * A * A * H]$. $\square$

3 A characterization of left regular and intra-regular ordered hypersemigroups in terms of fuzzy semiprime subsets

Following Zadeh, any mapping $f : H \to [0,1]$ of a ordered hypergroupoid H into the closed interval $[0,1]$ of real numbers is called a *fuzzy subset* of H (or a *fuzzy set* in H) and f_A (: the characteristic function of A) is the mapping

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The concepts of fuzzy right and fuzzy left ideals of an ordered groupoid due to Kehayopulu-Tsingelis [9] are naturally transferred to an ordered hypersemigroup as follows:

Definition 3.1. Let H be an ordered hypergroupoid. A fuzzy subset f of H is called a *fuzzy left ideal* of H if

1. $f(x \circ y) \geq f(y)$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$ and
2. $x \leq y$ implies $f(x) \geq f(y)$.

A fuzzy subset f of H is called a *fuzzy right ideal* of H if

1. $f(x \circ y) \geq f(x)$ for all $x, y \in H$, meaning that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$ and
2. $x \leq y$ implies $f(x) \geq f(y)$.

A fuzzy subset of H is called a *fuzzy ideal* of H it is both a fuzzy left and a fuzzy right ideal of H. As one can easily see, a fuzzy subset f of H is a fuzzy ideal of H if and only if

1. if $f(x \circ y) \geq \max\{f(x), f(y)\}$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq \max\{f(x), f(y)\}$ and
2. if $x \leq y$, then $f(x) \geq f(y)$.

The concept of fuzzy semiprime subsets of groupoids introduced by Kuroki in [12] is as follows: A fuzzy subset f of a groupoid S is called semiprime if $f(a) \geq f(a^2)$ for every $a \in S$, and remains the same in case of ordered groupoids as well [11]. This concept is naturally transferred in case of an hypergroupoid as follows:

Definition 3.2. Let $(H, \circ)$ be an hypergroupoid. A fuzzy subset f of H is called *fuzzy semiprime* if

$$f(a) \geq f(a \circ a) \text{ for every } a \in H,$$

in the sense that if $u \in a \circ a$, then $f(a) \geq f(u)$.

Remark 3.3. Let $(H, \circ)$ be an hypergroupoid and f a semiprime fuzzy left ideal (or fuzzy right ideal) of H. Then, for every $a \in H$, we have $f(a) = f(a \circ a)$, meaning that if $u \in a \circ a$, then $f(a) = f(u)$. Indeed: Let $u \in a \circ a$. Since f is a fuzzy left (or right) ideal of H, we have $f(a \circ a) \geq f(a)$, then $f(u) \geq f(a)$. Since f is semiprime, we have $f(a) \geq f(a \circ a)$, then $f(a) \geq f(u)$. Thus we have $f(a) = f(u)$.

Lemma 3.4. *Let $(H, \circ, \leq)$ be an ordered hypergroupoid. If A is a left (resp. right) ideal of H, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy left (resp. fuzzy right) ideal of H, then A is a left (resp. right) ideal of H.*

Proof. For the hypergroupoid $(H, \circ)$ the lemma is satisfied (cf. [8; Proposition 7]). It remains to prove that the following are equivalent:

(1) $y \in A$ and $H \ni x \leq y \Longrightarrow x \in A$ and

(2) $x \leq y \Longrightarrow f_A(x) \geq f_A(y)$.

(1) $\Longrightarrow$ (2). Let $x \leq y$. If $y \in A$ then, by (1), we have $x \in A$. Then $f_A(x) = 1 \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0 \leq f_A(x)$.

(2) $\Longrightarrow$ (1). Let $y \in A$ and $H \ni x \leq y$. Since $x \leq y$, by (2), we have $f_A(x) \geq f_A(y) = 1$. Then $f_A(x) = 1$, and $x \in A$. $\square$

Lemma 3.5. *Let H be an ordered hypergroupoid. A nonempty subset A of H is an ideal of H if and only if the characteristic function f_A is a fuzzy ideal of H.*

Lemma 3.6. *Let $(H, \circ, \leq)$ be an ordered hypergroupoid. If I is a subset of H such that f_I is fuzzy semiprime, then I is semiprime. "Conversely", let I be a subset of H such that, for every $a \in H$, either $a \circ a \subseteq I$ or $(a \circ a) \cap I = \emptyset$. If I is semiprime, then f_I is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let $a \in H$ such that $a \circ a \subseteq I$. Then $a \in I$. In fact: Since $a \circ a \subseteq I$, we have $f_I(a \circ a) = 1$. This is because if $u \in a \circ a$, then $u \in I$, so $f_I(u) = 1$. Since f_I is fuzzy semiprime, we have $f_I(a) \geq f_I(a \circ a) = 1$. Since f_I is a fuzzy subset of H, we have $f_I(a) \leq 1$. Thus we have $f_I(a) = 1$, and $a \in I$, so I is semiprime.

$\Longleftarrow$. Let I be semiprime. Then $f_I(a) \geq f_I(a \circ a)$. Indeed: If $a \circ a \subseteq I$ then, since I is semiprime, we have $a \in I$, then $f_I(a) = 1 \geq f_I(a \circ a)$. If $a \circ a \nsubseteq I$ then, by hypothesis, we have $(a \circ a) \cap I = \emptyset$, then $f_I(a \circ a) = 0$. This is because if $u \in a \circ a$, then $a \notin I$, so $f_I(a) = 0$. Hence we obtain $f_I(a \circ a) = 0 \leq f_I(a)$. □

Lemma 3.7. *Let H be an ordered hypergroupoid. A nonempty subset A of H is a semiprime subset of H if and only if the fuzzy subset f_A of H is fuzzy semiprime.*

Theorem 3.8. *An ordered hypersemigroup $(H, \leq)$ is left regular if and only if the fuzzy left ideals of H are fuzzy semiprime.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of H and $a \in H$. Then $f(a) \geq f(a \circ a)$. In fact: Let $u \in a \circ a$. Then $f(a) \geq f(u)$. Indeed: Since $u \in H$ and H is left regular, there exist $x, t \in H$ such that $t \in (x \circ u) * \{u\}$ and $a \leq t$. Since $t \in (x \circ u) * \{u\}$, we have $t \in w \circ u$ for some $w \in x \circ u$. Since f is a fuzzy left ideal of H, we have $f(w \circ u) \geq f(u)$. Since $t \in w \circ u$, we have $f(t) \geq f(u)$. Since $a \leq t$, we have $f(a) \geq f(t)$. Thus we have $f(a) \geq f(u)$.
$\Longleftarrow$. By Theorem 2.9, it is enough to prove that every left ideal of H is semiprime. Let now A be a left ideal of H. By Lemma 3.4, f_A is a fuzzy left ideal of H. By hypothesis, f_A is semiprime. Then, by Lemma 3.6, A is semiprime. □
The right analogue of the above theorem also holds, and we have

Theorem 3.9. *An ordered hypersemigroup H is right regular if and only if every fuzzy right ideal of H is fuzzy semiprime.*

Theorem 3.10. *An ordered hypersemigroup $(H, \circ, \leq)$ is intra-regular if and only if every fuzzy ideal of H is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let f be a fuzzy ideal of H and $a \in H$. Then $f(a) \geq f(a \circ a)$. In fact: Let $u \in a \circ a$. Then $f(a) \geq f(u)$. Indeed: Since $u \in H$ and H is intra-regular, there exist $x, y, t \in H$ such that $t \in \{x\} * (u \circ u) * \{y\}$ and $a \leq t$. Since $t \in (x \circ u) * (u \circ y)$, there exist $v \in x \circ u$ and $w \in u \circ y$ such that $t \in v \circ w$. Since f is a fuzzy left ideal of H, we have $f(v \circ w) \geq f(w)$. Since $t \in v \circ w$, we have $f(t) \geq f(w)$. Since f is a fuzzy right ideal of H, we have $f(u \circ y) \geq f(u)$. Since $w \in u \circ y$, we have $f(w) \geq f(u)$. Since $a \leq t$, we have $f(a) \geq f(t)$. Thus we get $f(a) \geq f(u)$.

$\Longleftarrow$. By Theorem 2.13, it is enough to prove that every ideal of H is semiprime. Let now A be an ideal of H. By Lemma 3.5, the characteristic function f_A is a fuzzy ideal of H. By hypothesis, f_A is fuzzy semiprime. Then, by Lemma 3.6, A is semiprime. □

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FUZZY SETS IN $\leq$–HYPERGROUPOIDS

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To the memory of Professor Kiyoshi Iséki

Abstract

This paper serves as an example to show the way we pass from ordered groupoids (ordered semigroups) to ordered hypergroupoids (ordered hypersemigroups), from groupoids (semigroups) to hypergroupoids (hypersemigroups). The results on semigroups (or on ordered semigroups) can be transferred to hypersemigroups (or to ordered hypersemigroups) in the way indicated in the present paper.

1 Introduction and prerequisites

An ordered groupoid (: *po*-groupoid) is a nonempty set S endowed with an order "$\leq$" and a multiplication "$\cdot$" such that $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for every $c \in S$. Given a set S, a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping of S into the closed interval $[0, 1]$ of real numbers (Zadeh). Fuzzy sets in ordered groupoids have been first considered in 2002 in Semigroup Forum [7], where the following concepts have been introduced and studied: A fuzzy subset f of an ordered groupoid $(S, \cdot, \leq)$ is called a *fuzzy left* (resp. *fuzzy right*) *ideal* of S if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) if $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for every $x, y \in S$. It is called a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S. A fuzzy subset f of a groupoid (or an ordered groupoid) S is called a *fuzzy subgroupoid* of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$. A fuzzy subset f of an ordered groupoid S is called a *fuzzy filter* of S if (1) $x \leq y$ implies $f(x) \leq f(y)$ and (2) if $f(xy) = \min\{f(x), f(y)\}$ for all $x, y \in S$. A fuzzy subset f of a groupoid S is called *fuzzy prime* if $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$. For a groupoid S and a fuzzy subset f of S, the complement of f is the fuzzy subset $f' : S \rightarrow [0, 1]$ of S defined by $f'(x) = 1 - f(x)$ for all $x \in S$. We have seen in

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[7] that a nonempty subset A of an ordered groupoid S is a left (resp. right) ideal of S if and only if its characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of S. A nonempty subset F of an ordered groupoid S is a filter of S if and only if the fuzzy subset f_F is a fuzzy filter of S. A fuzzy subset f of an ordered groupoid S is a fuzzy filter of S if and only if the complement f' of f is a fuzzy prime ideal of S. Later, fuzzy ordered semigroups have been widely studied by many authors.

In the present paper we examine the results of ordered groupoids given in [7] in case of some hypergroupoids. We deal with an hypergroupoid $(H, \circ)$ endowed with a relation "$\le$" (not order relation, and so not compatible with the hyperoperation "$\circ$" in general). Though we could call σ that relation and $\sigma-$hypergroupoid the hypergroupoid endowed with the relation σ, we will show by "$\le$" the relation and use the term $\le$–hypergroupoid, to emphasize the fact that our results hold for ordered hypergroupoids as well. As a consequence, the results in [7] also hold in groupoids endowed with a relation "$\le$" which is not an order in general and so no compatible with the multiplication in general. Our aim is to show the way we pass from ordered groupoids to ordered hypergroupoids.

For a groupoid $(S, \cdot)$ we have one operation corresponding to each $(a, b) \in S \times S$ the unique element ab of S. For an hypergroupoid H we have two "operations". One of them is the "operation" between the elements of H which is called "hyperoperation" as it maps the set $H \times H$ into the set of nonempty subsets of H and the other is the operation between the nonempty subsets of H. We use the terms left (right) ideal, bi-ideal, quasi-ideal instead of left (right) hyperideal, bi-hyperideal, quasi-hyperideal and so on, and this is because in this structure there are no two kind of left ideals, for example, to distinguish them as left ideal and left hyperideal. The left ideal in this structure is that one which corresponds to the left ideal of groupoids.

2 Main results

An *hypergroupoid* is a nonempty set H with an hyperoperation

$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$ on H

and an operation

$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$ on $\mathcal{P}^*(H)$

(induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$, $\mathcal{P}^*(H)$ being the set of (all) nonempty subsets of H. As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be also denoted by $(H, \circ)$ (instead of $(H, \circ, *)$). If H is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = x \circ y$.

By the definition of the hypergroupoid we have the following proposition which, though clear, plays an essential role in the theory of hypersemigroups.

Proposition 1. [4, 5] *Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$.*

Then we have the following:

(1) $x \in A * B \iff x \in a \circ b$ *for some* $a \in A$, $b \in B$.

(2) *If* $a \in A$ *and* $b \in B$, *then* $a \circ b \subseteq A * B$.

It is well known that a nonempty subset A of a groupoid $(S, \cdot)$ is called a left (right) ideal of S if $SA \subseteq A$ (resp. $AS \subseteq A$). It is called a subgroupoid of S if $A^2 \subseteq A$ (cf., for example [1]). These concepts are naturally transferred in case of hypergroupoids as follows: A nonempty subset A of an hypergroupoid $(H, \circ)$ is called a *left* (resp. *right*) *ideal* of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). A subset of H which is both a left ideal and a right ideal of H is called an *ideal* of H. A nonempty subset A of H is called a *subgroupoid* of H if $A * A \subseteq A$. Clearly, every left ideal, right ideal or ideal of H is a subgroupoid of H.

Lemma 2. [5] *Let* $(H, \circ)$ *be an hypergroupoid. If* A *is a left (resp. right) ideal of* H *then, for every* $h \in H$ *and every* $a \in A$, *we have* $h \circ a \subseteq A$ *(resp.* $a \circ h \subseteq A$*). "Conversely", if* A *is a nonempty subset of* H *such that* $h \circ a \subseteq A$ *(resp.* $a \circ h \subseteq A$*) for every* $h \in H$ *and every* $a \in A$, *then the set* A *is a left (resp. right) ideal of* H.

Lemma 3. *Let* $(H, \circ)$ *be an hypergroupoid. If* A *is a subgroupoid of* H *then, for every* $a, b \in A$, *we have* $a \circ b \subseteq A$. *"Conversely", if* A *is a nonempty subset of* H *such that* $a \circ b \subseteq A$ *for every* $a, b \in A$, *then* A *is a subgroupoid of* H.

Definition 4. By a $\leq$–*hypergroupoid* we mean an hypergroupoid H endowed with a relation denoted by "$\leq$".
We write $b \geq a$ if $a \leq b$ (i.e. if (a, b) belongs to the relation $\leq$).

The concepts of fuzzy left (right) ideals of ordered groupoids introduced by Kehayopulu and Tsingelis in [7] are naturally transferred to $\leq$–hypergroupoids in the following definition:

Definition 5. (cf. also [5]) Let H be a $\leq$–hypergroupoid. A fuzzy subset f of H is called a *fuzzy left ideal* of H if

1. $x \leq y \Rightarrow f(x) \geq f(y)$ and
2. if $f(x \circ y) \geq f(y)$ for all $x, y \in H$, meaning that $x, y \in H$ and $u \in x \circ y$ implies $f(u) \geq f(y)$.

A fuzzy subset f of H is called a *fuzzy right ideal* of H if

1. $x \leq y \Rightarrow f(x) \geq f(y)$ and
2. if $f(x \circ y) \geq f(x)$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.

A fuzzy subset f of H is called a *fuzzy ideal* of H it is both a fuzzy left and a fuzzy right ideal of H. As one can easily see, a fuzzy subset f of H is a fuzzy ideal of H if and only if

1. $x \leq y$ implies $f(x) \geq f(y)$ and
2. if $f(x \circ y) \geq \max\{f(x), f(y)\}$ for all $x, y \in H$, in the sense that $x, y \in H$ and $u \in x \circ y$ implies $f(u) \geq \max\{f(x), f(y)\}$.

Following Zadeh, any mapping $f : H \to [0,1]$ of a $\leq$–hypergroupoid H into the closed interval $[0,1]$ of real numbers is called a *fuzzy subset* of H (or a *fuzzy set* in H) and f_A (: the characteristic function of A) is the mapping

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The concepts of left and right ideals of ordered groupoids introduced by Kehayopulu in [3] are as follows: If $(S,\cdot,\leq)$ is an ordered groupoid, a nonempty subset A of S is called a left (resp. right) ideal of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) (that is, if A is a left (resp. right) ideal of the groupoid $(S,\cdot)$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and right ideal of $(S,\cdot,\leq)$, then it is called an ideal of S. These concepts are naturally transferred in case of an $\leq$–hypergroupoid as follows:

Definition 6. [5] Let H be a $\leq$–hypergroupoid. A nonempty subset A of H is called a *left* (resp. *right*) *ideal* of H if

(1) $H * A \subseteq A$ (resp. $A * H \subseteq A$) and

(2) if $a \in A$ and $H \ni b \leq a$, then $b \in A$.

Proposition 7. (cf. also [5]) *Let H be a $\leq$–hypergroupoid. If L is a left ideal of H, then f_L is a fuzzy left ideal of H. "Conversely", if L is a nonempty subset of H such that f_L is a fuzzy left ideal of H, then L is a left ideal of H.*

Proof. $\Longrightarrow$. Let L be a left ideal of H. By definition, f_L is a fuzzy subset of H. Let $x \leq y$. If $y \notin L$, then $f_L(y) = 0$, so $f_L(x) \geq f_L(y)$. If $y \in L$, then $H \ni x \leq y \in L$ and, since L is a left ideal of H, we have $x \in L$. Then $f_L(x) = f_L(y) = 1$, so $f_L(x) \geq f_L(y)$. Let now $x, y \in H$ and $u \in x \circ y$. Then $f_L(u) \geq f_L(y)$. Indeed: If $y \in L$ then, by by Proposition 1, we have $x \circ y \subseteq H * L \subseteq L$, so $u \in L$, then $f_L(y) = f_L(u) = 1$, so $f_L(u) \geq f_L(y)$. If $y \notin L$, then $f_L(y) = 0 \leq f_L(u)$.
$\Longleftarrow$. Let $x \in H$ and $y \in L$. Then $x \circ y \subseteq L$. Indeed: Let $x \circ y \nsubseteq L$. Then there exists $u \in x \circ y$ such that $u \notin L$. Since $u \in x \circ y$, by hypothesis, we have $f_L(u) \geq f_L(y)$. Since $u \notin L$, we have $f_L(u) = 0$. Since $y \in L$, we have $f_L(y) = 1$, then $0 \geq 1$ which is impossible. Let now $x \in L$ and $H \ni y \leq x$. Then $y \in L$. Indeed: Since f_L is a fuzzy left ideal of H and $y \leq x$, we have $f_L(y) \geq f_L(x)$. Since $x \in L$, we have $f_L(x) = 1$. Then we have $f_L(y) \geq 1$. On the other hand, $f_L(y) \leq 1$, so we have $f_L(y) = 1$, then $y \in L$, and the proof is complete. □

In a similar way we prove the following:

Proposition 8. *Let H be a $\leq$–hypergroupoid. If R is a right ideal of H, then f_R is a fuzzy right ideal of H. "Conversely", if R is a nonempty subset of H such that f_R is a fuzzy right ideal of H, then R is a right ideal of H.*

Proposition 9. *If H is a $\leq$–hypergroupoid, a nonempty subset I of H is an ideal of H if and only if f_I is a fuzzy ideal of H.*

The concept of a filter of an ordered groupoid introduced by Kehayopulu in 1987 [2] is as follows: If $(S,\cdot,\leq)$ is an ordered groupoid, a nonempty subset F of S is called a filter of S if (1) if $a, b \in F$, then $ab \in F$. (2) if $a, b \in F$ such that

$ab \in F$, then $a \in F$ and $b \in F$. (3) if $a \in F$ and $S \ni b \geq a$, then $b \in F$ (that is, if F is a subgroupoid of the groupoid $(S, \cdot)$ satisfying the properties (2) and (3)). This concept is naturally transferred to $\leq$–hypergroupoids in Definition 10 below. It might be noted that the properties (1) and (2) of Definition 10 correspond to the properties (1) and (2) of filters of ordered groupoids but, in contrast to the case of ordered groupoids, they are not enough to prove basic results on ordered hypergroupoids. To overcome this difficulty, the property (3) in Definition 10 has been added. Our aim now is to characterize the filters of $\leq$–hypergroupoids in terms of fuzzy filters.

Definition 10. Let H be a $\leq$–hypergroupoid. A nonempty subset F of H is called a *filter* of H if

(1) if $x, y \in F$, then $x \circ y \subseteq F$.
(2) if $x, y \in H$ and $x \circ y \subseteq F$, then $x \in F$ and $y \in F$.
(3) if $x, y \in H$, then $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.
(4) if $x \in F$ and $H \ni y \geq x$, then $y \in F$.

So a filter of H is a subgroupoid of H satisfying the conditions (2)–(4).

Remark 11. Let H be a $\leq$–hypergroupoid, F a filter of H and $x, y \in H$. The following are equivalent:

(1) $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.
(2) if $x \notin F$ or $y \notin F$, then $(x \circ y) \cap F = \emptyset$.

Indeed: (1) $\Longrightarrow$ (2). Let $x \notin F$ or $y \notin F$. If $x \circ y \subseteq F$ then, since F is a filter of H, we have $x \in F$ and $y \in F$ which is impossible. Thus we have $x \circ y \nsubseteq F$. Then, by (1), we get $(x \circ y) \cap F = \emptyset$ and (2) is satisfied.
(2) $\Longrightarrow$ (1). Let $x \circ y \nsubseteq F$. If $x, y \in F$ then, since F is a filter of H, we have $x \circ y \subseteq F$ which is impossible. Thus we have $x \notin F$ or $y \notin F$. Then, by (2), we have $(x \circ y) \cap F = \emptyset$, and (1) holds true.

The concept of a fuzzy filter of an ordered groupoid introduced by Kehayopulu and Tsingelis in [7] is naturally transferred to a $\leq$–hypergroupoid in the following definition:

Definition 12. Let H be a $\leq$–hypergroupoid. A fuzzy subset f of H is called a *fuzzy filter* of H if

1. if $x \leq y$ implies $f(x) \leq f(y)$ and
2. if $f(x \circ y) = \min\{f(x), f(y)\}$ for every $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) = \min\{f(x), f(y)\}$.

Proposition 13. *Let H be a $\leq$–hypergroupoid. If F is a filter of H, then the fuzzy subset f_F is a fuzzy filter of H. "Conversely", if F is a nonempty subset of H such that f_F is a fuzzy filter of H, then F is a filter of H.*

Proof. $\Longrightarrow$. Let $x \leq y$. If $x \notin F$, then $f_F(x) = 0$, so $f_F(x) \leq f_F(y)$. If $x \in F$, then $f_F(x) = 1$. Since $y \in H$ and $y \geq x \in F$, we have $y \in F$. Then $f_F(y) = 1$, and $f_F(x) \leq f_F(y)$. Let now $x, y \in H$ and $u \in x \circ y$. Then $f_F(u) = \min\{f_F(x), f_F(y)\}$. Indeed: (a) If $x \circ y \subseteq F$, then $x \in F$ and $y \in F$. Also $u \in F$. Then $f_F(x) = f_F(y) = f_F(u) = 1$, so $f_F(u) = \min\{f_F(x), f_F(y)\}$.

(b) Let $x \circ y \nsubseteq F$. Then $x \notin F$ or $y \notin F$ (since $x, y \in F$ implies $x \circ y \subseteq F$ which is impossible), then $f_F(x) = 0$ or $f_F(y) = 0$, and $\min\{f_F(x), f_F(y)\} = 0$. On the other hand, since $x \circ y \nsubseteq F$, we have $(x \circ y) \cap F = \emptyset$. Since $u \in x \circ y$, we have $u \notin F$. Then $f_F(u) = 0$, so $f_F(u) = \min\{f_F(x), f_F(y)\}$.

$\Longleftarrow$. Let $x, y \in F$. Then $x \circ y \subseteq F$. Indeed: Let $u \in x \circ y$. By hypothesis, we have $f_F(u) = \min\{f_F(x), f_F(y)\}$. Since $x, y \in F$, we have $f_F(x) = f_F(y) = 1$. Then $f_F(u) = 1$, and $u \in F$, so F is a subgroupoid of H. Let $x, y \in H$ such that $x \circ y \subseteq F$. Then $x \in F$ and $y \in F$. Indeed: Let $x \notin F$ or $y \notin F$. Then $f_F(x) = 0$ or $f_F(y) = 0$, hence $\min\{f_F(x), f_F(y)\} = 0$. Since $x \circ y \in \mathcal{P}^*(H)$, the set $x \circ y$ is nonempty. Take an element $u \in x \circ y$. Since f_F is a fuzzy filter of H, we have $f_F(u) = \min\{f_F(x), f_F(y)\}$, so $f_F(u) = 0$. On the other hand, since $u \in x \circ y \subseteq F$, we have $f_F(u) = 1$. We get a contradiction. Let $x, y \in H$ such that $x \circ y \nsubseteq F$. Then $(x \circ y) \cap F = \emptyset$. Indeed: Let $u \in (x \circ y) \cap F$. Since $u \in x \circ y$, by hypothesis, we have $f_F(u) = \min\{f_F(x), f_F(y)\}$. If $x \notin F$, then $f_F(x) = 0$, thus $f_F(u) = 0$. On the other site, since $u \in F$, we have $f_F(u) = 1$. We get a contadiction, so we have $x \in F$. In a similar way we prove that $y \in F$ and, since F is a subgroupoid of H, we have $x \circ y \subseteq F$, which is impossible. Finally, let $x \in F$ and $H \ni y \geq x$. Since f_F is a fuzzy filter of H, we have $1 \geq f_F(y) \geq f_F(x) = 1$, then $f_F(y) = 1$, so $y \in F$. Thus F is a filter of H. □

In what follows, for a fuzzy subset f of an $\leq$–hypergroupoid H we introduce the concept of the complement f' of f (which again is analogous to that one defined for ordered groupoids in [6]) and we prove that a fuzzy subset f of a $\leq$–hypergroupoid H is a fuzzy filter of H if and only if the complement f' of H is a fuzzy prime ideal of H.

Definition 14. Let H be an hypergroupoid or $\leq$–hypergoupoid and f a fuzzy subset of H. The fuzzy subset

$$f' : S \to [0,1] \text{ defined by } f'(x) = 1 - f(x)$$

is called the *complement* of f in H.

We remark the following:

(a) If $x \in H$, then $(f')'(x) = 1 - f'(x) = f(x)$. Thus we have $f'' := (f')' = f$.

(b) $f(x) \leq f(y) \Longleftrightarrow f'(x) \geq f'(y)$ $(x, y \in H)$.

(c) $f(x) = f(y) \Longleftrightarrow f'(x) = f'(y)$ $(x, y \in H)$.

Lemma 15. (cf. also [6]) *Let H be an hypergroupoid, f a fuzzy subset of H and $x, y \in H$. Then we have*

$$1 - \min\{f(x), f(y)\} = \max\{f'(x), f'(y)\} \qquad (*)$$

Proof. Let $f(x) \leq f(y)$. Then $\min\{f(x), f(y)\} = f(x)$, thus $1 - \min\{f(x), f(y)\} = 1 - f(x) = f'(x)$. On the other hand, $f(x) \leq f(y)$ implies $f'(x) \geq f'(y)$, so we have $\max\{f'(x), f'(y)\} = f'(x)$ and $(*)$ is satisfied. If $f(y) \leq f(x)$, by symmetry, the relation $(*)$ also holds. □

Lemma 16. *Let H be an hypergroupoid, f a fuzzy subset of H and $x, y \in H$. The following are equivalent:*

(1) $f(x \circ y) = \min\{f(x), f(y)\}$.
(2) $f'(x \circ y) = \max\{f'(x), f'(y)\}$.

Proof. (1) $\Longrightarrow$ (2). Let $u \in x \circ y$. By (1), we have $f(u) = \min\{f(x), f(y)\}$. Then, by Lemma 15, we have

$$f'(u) = 1 - f(u) = 1 - \min\{f(x), f(y)\} = \max\{f'(x), f'(y)\},$$

and (2) holds true.
(2) $\Longrightarrow$ (1). Let $u \in x \circ y$. By (2) and Lemma 15, we have

$$f'(u) = \max\{f'(x), f'(y)\} = 1 - \min\{f(x), f(y)\}.$$

Then $f(u) = 1 - f'(u) = \min\{f(x), f(y)\}$, and (1) is satisfied. □

The concept of fuzzy prime subsets of groupoids or of ordered groupoids [7] is naturally transferred to hypergroupoids and to ordered hypergroupoids in the following definition.

Definition 17. [5] A fuzzy subset f of a groupoid (or a ≤–hypergroupoid) H is called a *fuzzy prime subset* of H if $f(x \circ y) \leq \max\{f(x), f(y)\}$ for all $x, y \in H$. That is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) \leq \max\{f(x), f(y)\}$.

If H is a ≤–hypergroupoid, by a *fuzzy prime ideal* of H we clearly mean a fuzzy prime subset of H which is at the same time a fuzzy ideal of S. So a fuzzy subset of H is a fuzzy prime ideal of H if and only if the following assertions are satisfied:

1. $x \leq y \Rightarrow f(x) \geq f(y)$ and
2. if $f(x \circ y) = \max\{f(x), f(y)\}$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) = \max\{f(x), f(y)\}$.

Proposition 18. *Let H be a ≤–hypergroupoid and f a fuzzy subset of H. Then f is a fuzzy filter of H if and only if the complement f' of f is a fuzzy prime ideal of H.*

Proof. $\Longrightarrow$. Let $x \leq y$. Since f is a fuzzy filter of H, we have $f(x) \leq f(y)$, then $f'(x) \geq f'(y)$. Let now $x, y \in H$. Since f is a fuzzy filter of H, we have $f(x \circ y) = \min\{f(x), f(y)\}$. Then, by Lemma 16, $f'(x \circ y) = \max\{f'(x), f'(y)\}$, thus f' is a fuzzy prime ideal of H.
$\Longleftarrow$. Let $x \leq y$. Since f' is a fuzzy ideal of H, we have $f'(x) \geq f'(y)$. Then $f(x) = f''(x) \leq f''(y) = f(y)$. Let now $x, y \in H$. Since f' is a fuzzy prime ideal of H, we have $f'(x \circ y) = \max\{f'(x), f'(y)\}$ then, by Lemma 16, $f(x \circ y) = \min\{f(x), f(y)\}$, thus f is a fuzzy filter of H. □

Let us finish with an example.

Example. We consider the ≤–hypergroupoid $H = \{a, b, c, d, e\}$ defined by the hyperoperation given in the table and the relation "≤" below.

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$	$\{a, b\}$
c	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$	$\{a, b, c\}$
d	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$	$\{a, b, c\}$
e	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, d\}$	$\{a, b, e\}$

$$\leq := \{(a,a),(a,b),(b,b),(b,d)\}.$$

One can easily check that the set $A = \{a,b,d\}$ is a left ideal of $(H,\circ,\leq)$ and that the characteristic function f_A of A, that is, the mapping

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is a fuzzy left ideal of H (the latest being also a consequence of Proposition 7).

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