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# STABILITY OF INHOMOGENEOUS STATIONARY SOLUTIONS TO RACETRACK MODEL IN SPATIAL ECONOMY

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ABSTRACT. We continue our study on the racetrack model. In the previous paper, we have shown that the global solution has an $\omega$-limit which is a stationary solution. In this paper, we introduce a simplified racetrack model and study stability and instability of stationary solutions by using the linearization principle.

**1 Introduction.** We continue our study on the racetrack model which has been presented in [9] by M. Fujita, P. Krugman, A. Venables in order to describe the dynamics of a tutorial economic system on a circumference driven by economic incentives. The model is written by

$$
(1.1)\quad \begin{cases}
w(t,x) = \left[\int_S \{\mu\lambda(t,y)w(y,t) + (1-\mu)\phi(y)\} G(t,y)^{\sigma-1} e^{-(\sigma-1)\tau|x-y|} dy\right]^{\frac{1}{\sigma}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [0,\infty)\times S, \\
G(t,x) = \left[\int_S \lambda(t,y) w(t,y)^{1-\sigma} e^{-(\sigma-1)\tau|x-y|} dy\right]^{\frac{1}{1-\sigma}} \quad (t,x) \in [0,\infty)\times S, \\
\omega(t,x) = w(t,x) G(t,x)^{-\mu} \quad (t,x) \in [0,\infty)\times S, \\
\dfrac{\partial\lambda}{\partial t}(t,x) = \gamma\left[\omega(t,x) - \int_S \omega(t,y)\lambda(t,y) dy\right]\lambda(t,x) \quad (t,x) \in [0,\infty)\times S, \\
\lambda(0,x) = \lambda_0(x) \quad x \in S.
\end{cases}
$$

Here, $S$ is a circumference on which economic regions exist continuously and $x$ is a spatial variable varying on $S$. The unknown function $\lambda(t,x)$ is a function such that $\mu\lambda(t,x)$ denotes population density of manufacturing workers at time $t \in [0,\infty)$ at a position $x \in S$. The other unknown function $w(t,x)$ denotes nominal wage at $(t,x) \in [0,\infty)\times S$. The function $G(t,x)$ and $\omega(t,x)$ denote respectively, price index and real wage at $(t,x) \in [0,\infty)\times S$. The function $\phi$ is a given function such that $(1-\mu)\phi(x)$ denotes the density of agricultural workers on $S$. It is assumed that $0 \le \phi \in L^1(S)$ and $\int_S \phi(x)dx = 1$. The function $|x-y|$ denotes a symmetric distance between $x,\ y \in S$ along $S$. The exponent $0 < \mu \le 1$ denotes a ratio of the manufacturing workers on $S$ to the total number of (manufacturing and agricultural) workers. Meanwhile $\sigma > 1$ stands for an index of preference for manufacturing goods, and $\tau > 0$ stands for a parameter concerning the transportation cost.

In the previous paper [11], we have studied (1.1) mathematically and numerically. In fact, we have shown, after discussing the global existence, that the global solution has an $\omega$-limit which is a stationary solution of (1.1) and that any stationary solution to (1.1) is either the homogeneous solution on $S$ or an inhomogeneous solution whose manufacturing density is a sum of Dirac delta functions.

We are then interested in investigating stability of stationary solutions to (1.1). As mentioned in [9] (and indeed reviewed in [11]), the homogeneous stationary solution is always unstable. So, in this paper, our interest is addressed to considering inhomogeneous stationary solutions. Meanwhile, our numerical computations suggest that there are no continuous shifts of manufacturing regions along $S$ but there is growth or decline in manufacturing population at each manufacturing region. In order to simplify the situations, we are here led to introduce a modified racetrack model of a finite number of manufacturing regions. We assume that the whole manufacturing population agglomerates only in a finite number of fixed positions $x_1, \cdots, x_M \in S$ which will be called manufacturing regions of the model. Then, (1.1) reduces to

$$
(1.2)\qquad \begin{cases}
w_i(t)^\sigma = \mu \displaystyle\sum_{j=1}^{M} \lambda_j(t) w_j G_j(t)^{\sigma-1} e^{-(\sigma-1)\tau|x_i - x_j|} \\
\qquad\qquad + (1-\mu) \displaystyle\int_S \phi(y) G(t,y)^{\sigma-1} e^{-(\sigma-1)\tau|x_i-y|} dy, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,\infty),\ i = 1,2,\cdots,M, \\
G(t,x)^{1-\sigma} = \displaystyle\sum_{j=1}^{M} \lambda_j(t) w_j(t)^{1-\sigma} e^{-(\sigma-1)\tau|x_i-y|}, \quad t \in [0,\infty),\ x \in S, \\
\omega_i = w_i(t) G_i(t)^{-\mu}, \qquad\qquad t \in [0,\infty),\ i = 1,2,\cdots,M, \\
\dfrac{d}{dt}\lambda_i(t) = \left\{ \omega_i(t) - \displaystyle\sum_{j=1}^{M} \lambda_j(t)\omega_j(t) \right\} \lambda_i(t), \qquad t \in [0,\infty),\ i = 1,2,\cdots,M, \\
\lambda_i(0) = \lambda_{i,0}, \qquad\qquad i = 1,2,\cdots,M.
\end{cases}
$$

Here, $w_i(t)$, $\lambda_i(t)$, and $\omega_i(t)$ are, respectively, nominal wage, manufacturing population size, and real wage, at time $t \in [0,\infty)$ and at manufacturing region $x_i \in S$.

We begin with deriving (1.2) from (1.1) by putting $\lambda(t,x) = \sum_{k=1}^{M} \lambda_k(t)\delta_k(x)$, $\delta_i(x)$ being the Dirac delta function with center $x_i$. Then, after formulating (1.2) mathematically in a suitable way, we construct a global solution as well its numerical computations. The second part of the paper is devoted to studying stability of stationary solutions to (1.2). As will been seen, (1.2) is finally reduced to an $M$ dimensional ordinary differential equation but including complex implicit functions. Anyway, we can use the usual linearization principle for studying the stability. The linearization matrices at stationary solutions are calculated. In the case of $M = 2$ or 3, the signs of their eigenvalues are known, provided that the exponent $\alpha := \tau(\sigma - 1)$ is sufficiently small or sufficiently large. More refined results are obtained by numerical computations.

Generally speaking, increase of $\alpha$ provides existence of more stable stationary solutions under the no black hole condition

$$\frac{\sigma - 1}{\sigma} > \mu.$$

This condition suppresses too strong centripetal forces in spatial economy, and thus prevents the manufacturing workers from agglomerating to only single economic region (see [9, p59]). On the other hand, as $\alpha$ decreases, stationary solutions lose their stability and only stationary solutions with single manufacturing region remain stable (i.e., $M = 1$).

When $M$ is fixed, say $M = 3$, the configuration of $x_1,\ x_2,\ x_3$ on $S$ also influences the stability of its stationary solution. As will be seen, the symmetric configuration solution has the highest degree of stability, that is, can remain stable as $\alpha$ decreases than any other non-symmetric stationary solution.

Let us finally refer to some papers related to our paper. Actually, the racetrack model (1.1) is one of the many models which may be divided into discrete space models and continuous space models. The discrete space models have already been studied in many papers.

Especially, the bifurcation property of stationary solutions is well studied. We want to quote Castro-Correia da Silva-Mossay [5], Ikeda-Akamatsu-Kono [10], Akamatsu-Takayama [3], Akamatsu-Takayama-Ikeda [1], Akamatsu-Mori-Takayama [2]. Tabuchi-Thisse [13] consider the racetrack model in which the agricultural sector is distributed continuously, and the manufacturing sector is distributed discretely. This setting is similar to our model (1.2), however due to their assumption on a utility function of consumers, their model is quite different from (1.2). Most of the papers on stationary solutions to the racetrack model treat only the symmetric stationary solutions except a few paper Fabinger [7]. Using a discrete space model, Barbero-Zofío [4] discussed the relation between stability and a configuration (they call it space topology) of economic regions.

**2 Modeling.** In this section, we will sketch the derivation of (1.2) from (1.1). In what follows, $\alpha$ stands for $\alpha = \tau(\sigma-1)$. The manufacturing regions $x_1, \cdots, x_M \in S$ are positions at which all the manufacturing workers accumulate, and $\lambda_1(t), \cdots, \lambda_M(t)$ denote the manufacturing population size at time $t \in [0,\infty)$ at each manufacturing region. Then, the manufacturing population density $\lambda(t,x)$ on $S$ is written in the form

$$\lambda(t,x) = \sum_{k=1}^{M} \lambda_k(t)\delta_k(x), \quad t \in [0,\infty), \quad x \in S, \tag{2.1}$$

where $\delta_k(x)$ is the Dirac delta function with center $x_k$. From $\int_S \lambda(t,x)dx = 1$, it holds that $\sum_{k=1}^{M} \lambda_k(t) = 1$ for any time $t$. By (2.1), the first equation of (1.1) becomes

$$\begin{aligned} w(t,x)^\sigma &= \mu \sum_{j=1}^{M} \lambda_j(t) w_j G(t,x_j)^{\sigma-1} e^{-\alpha|x-x_j|} \\ &+ (1-\mu)\int_S \phi(y) G(t,y)^{\sigma-1} e^{-\alpha|x-y|} dy, \quad t \in [0,\infty), \quad x \in S. \end{aligned} \tag{2.2}$$

So, the first equation of (1.2) is verified.

Let us write $w(t,x_i) = w_i(t)$ for $i = 1, \cdots, M$. By (2.1), the second equation of (1.1) becomes

$$G(t,x)^{1-\sigma} = \sum_{j=1}^{M} \lambda_j(t) w_j(t)^{1-\sigma} e^{-\alpha|x_i - x|}, \quad t \in [0,\infty), \quad x \in S, \tag{2.3}$$

hence the second equation of (1.2).

Let us write $\omega(t,x_i) = \omega_i(t)$ and $G(t,x_i) = G_i(t)$ for $i = 1, \cdots, M$. Then, the real wage at each manufacturing region is given by

$$\omega_i(t) = w_i(t) G_i(t)^{-\mu}, \quad t \in [0,\infty), \quad i = 1, \cdots, M. \tag{2.4}$$

Finally, the fourth equation of (1.1) reduces to

$$\frac{d}{dt}\lambda_i(t) = \left[\omega_i(t) - \sum_{k=1}^{M} \omega_k(t)\lambda_k(t)\right] \lambda_i(t), \quad i = 1, \cdots, M. \tag{2.5}$$

This is the fourth equation of (1.2).

**3 Mathematical Formulation** In this section, let us make mathematical formulation for (1.2).

**3.1 Norms on $\mathbb{R}^M$.** As seen, the unknown functions $w(t)$ and $\lambda(t)$ of (1.2) take both their values in $\mathbb{R}^M$. It is however convenient to use different norms of $\mathbb{R}^M$ for $w(t)$ and $\lambda(t)$.

We denote the space $\mathbb{R}^M$ equipped with the maximum norm $\|\cdot\|_\infty$ as $E^\infty$, i.e.,

$$E^\infty = \left(\mathbb{R}^M, \|w\|_\infty = \max\{|w_1|, \cdots, |w_M|\}\right).$$

We further denote a positive subset of $E^\infty$ as

$$E^\infty_+ = \{w \in E^\infty | w_i > 0,\ i = 1, \cdots, M\}.$$

It is reasonable to expect that $w(t) \in E^\infty_+$ for any $t > 0$.

On the other hand, we denote the space $\mathbb{R}^M$ equipped with the summation norm $\|\cdot\|_1$ as $E^1$, i.e.,

$$E^1 = \left(\mathbb{R}^M, \|\lambda\|_1 = |\lambda_1| + \cdots + |\lambda_M|\right).$$

We further consider a subset of $E^1$ such that

$$\mathcal{M} = \left\{\lambda \in E^1 | \lambda_i \geq 0,\ i = 1, \cdots, M, \quad \|\lambda\|_1 = 1\right\}.$$

It is reasonable to expect that $\lambda(t) \in \mathcal{M}$ for any $t > 0$.

**3.2 Formation.** We begin with formulating the first equation of (1.2) as a fixed point problem in $E^\infty$. To do so, let us introduce the operator $G$ which maps $E^\infty_+ \times \mathcal{M}$ into the space of continuous functions $\mathcal{C}(S)$ defined by

$$(3.1) \qquad [G(f,\lambda)](x) = \left[\sum_j \lambda_j f_j^{\frac{1}{\sigma}-1} e^{-\alpha|x-x_j|}\right]^{\frac{1}{1-\sigma}}, \quad x \in S.$$

Put $[G(f,\lambda)](x_i) = G(f,\lambda)_i$ for $i = 1, \cdots, M$. We also introduce the operator $\Phi : E^\infty_+ \times \mathcal{M} \to E^\infty_+$ by

$$(3.2) \quad \Phi(f,\lambda)_i = \sum_{j=1}^M \frac{\mu\lambda_j f_j^{\frac{1}{\sigma}} e^{-\alpha|x_i-x_j|}}{G(f,\lambda)_j^{1-\sigma}} + (1-\mu)\int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{[G(f,\lambda)](y)^{1-\sigma}}dy, \qquad i = 1, \cdots, M.$$

Then, by putting $f = w^\sigma$, it is observed that at each $t$ the first and second equations of (1.2) are confined into

$$(3.3) \qquad f = \Phi(f,\lambda), \qquad f \in E^\infty_+, \quad \lambda \in \mathcal{M}.$$

Next, we formulate the fourth equation of (1.2) as an ordinary differential equation in $E^1$. To do so, let us introduce the operator $\omega : E^\infty_+ \times \mathcal{M} \to E^\infty_+$ given by

$$(3.4) \qquad \omega(f,\lambda)_i = f_i^{\frac{1}{\sigma}} [G(f,\lambda)_i]^{-\mu}, \qquad i = 1, \cdots, M,$$

and the operator $\Psi : E^\infty_+ \times \mathcal{M} \to E^1$ given by

$$(3.5) \qquad \Psi(f,\lambda)_i = \left[\omega(f,\lambda)_i - \sum_{k=1}^M \omega(f,\lambda)_k \lambda_k\right]\lambda_i, \quad i = 1, \cdots, M.$$

Then, the fourth equation of (1.2) becomes

$$\frac{d\lambda}{dt}(t) = \Psi(w(t), \lambda(t)).$$

In this way, putting $f(t) = w(t)^\sigma$, the problem (1.2) has been formulated as stationary and evolution equations:

$$\begin{cases} f(t) = \Phi(f(t), \lambda(t)), & 0 \le t < \infty, \\ \dfrac{d\lambda}{dt}(t) = \Psi(f(t), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0 \end{cases} \tag{3.6}$$

in the product space

$$E^\infty \times E^1 = \left\{ (f, \lambda) \middle| f \in E^\infty,\ \lambda \in E^1 \right\}.$$

The initial value $\lambda_0$ is taken in $\mathcal{M}$.

**4 Global solution.** In this section, we construct a global solution for (3.6). This section consists of two subsections. In Subsection 4.1, the fixed point problem (3.3) is handled for each fixed $\lambda \in \mathcal{M}$. Based on the results, a local solution is constructed in Subsection 4.2 and is extended to global one in Subsection 4.3.

**4.1 Fixed Point Problem (3.3).** For real numbers $0 < r_1 < r_2$, we set a bounded closed subset $E^\infty_{r_1,r_2}$ of $E^\infty$ by

$$E^\infty_{r_1,r_2} := \{ u \in E^\infty | r_1 \le u_i \le r_2,\ i = 1, \cdots, M \}.$$

In addition, denote the maximal value of the distance between the manufacturing regions as

$$\overline{d} = \max_{i,j} |x_i - x_j|.$$

**Theorem 4.1.** *Assume that $\sigma > 1$ and $\tau > 0$ are sufficiently small so that*

$$e^{\alpha\pi} < 1/\mu. \tag{4.1}$$

*And, put numbers $a$ and $b$ as*

$$\begin{aligned} a &= \left[ \frac{(1-\mu)e^{-\alpha\pi}}{1 - \mu e^{-\alpha\overline{d}}} \right]^\sigma, \\ b &= \left[ \frac{(1-\mu)e^{\alpha\pi}}{1 - \mu e^{\alpha\overline{d}}} \right]^\sigma, \end{aligned} \tag{4.2}$$

*respectively. Then, for any $\lambda \in \mathcal{M}$, (3.3) has at least one solution $f$ in $E^\infty_{a,b}$.*

*Proof.* The proof is based on the Brouwer fixed point theorem.

The bounded closed subset $E^\infty_{a,b}$ is convex. In fact, for any $u, v \in E^\infty_{a,b}$ and $\theta \in (0, 1)$, we have

$$\begin{aligned} [\theta u + (1-\theta)v]_i &= \theta u_i + (1-\theta)v_i \\ &\le \theta b + (1-\theta)b = b, \quad i = 1, \cdots, M. \end{aligned}$$

Similarly,

$$[\theta u + (1-\theta)v]_i \ge \theta a + (1-\theta)a = a, \quad i = 1, \cdots, M.$$

The operator $\Phi(\cdot,\lambda)$ defined by (3.2) maps $E^\infty_{a,b}$ into itself. In fact, for $i=1,\cdots,M$,

$$\begin{aligned}\Phi(f,\lambda)_i &\le \mu b\sum_j \frac{\lambda_j e^{-\alpha|x_i-x_j|}}{\sum_k \lambda_k e^{-\alpha|x_j-x_k|}}\\ &\quad + (1-\mu)b^{1-\frac{1}{\sigma}}\int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{\sum_k \lambda_k e^{-\alpha|y-x_k|}}dy\\ &\le \mu b e^{\alpha\bar d} + (1-\mu)b^{1-\frac{1}{\sigma}}e^{\alpha\pi}\\ &= b\end{aligned}$$

due to the definition of $b$. Similarly, for $i=1,\cdots,M$,

$$\begin{aligned}\Phi(f,\lambda)_i &\ge \mu a\sum_j \frac{\lambda_j e^{-\alpha|x_i-x_j|}}{\sum_k \lambda_k e^{-\alpha|x_j-x_k|}}\\ &\quad + (1-\mu)a^{1-\frac{1}{\sigma}}\int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{\sum_k \lambda_k e^{-\alpha|y-x_k|}}dy\\ &\ge \mu a e^{-\alpha\pi} + (1-\mu)a^{1-\frac{1}{\sigma}}e^{-\alpha\pi}\\ &= a.\end{aligned}$$

The operator $\Phi(\cdot,\lambda)$ is continuous in $E^\infty_{a,b}$. More strongly, it is actually Lipschitz continuous. Indeed, for any $f,g\in E^\infty_{a,b}$,

$$\begin{aligned}&|\Phi(f,\lambda)_i-\Phi(g,\lambda)_i|\\ &\le \mu\left|\sum_{j=1}^M \frac{\lambda_j f_j^{\frac{1}{\sigma}}e^{-\alpha|x_i-x_j|}}{\sum_{k=1}^M \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}} - \sum_{j=1}^M \frac{\lambda_j g_j^{\frac{1}{\sigma}}e^{-\alpha|x_i-x_j|}}{\sum_{k=1}^M \lambda_k g_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}}\right|\\ &\quad + (1-\mu)\left|\int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{\sum_{k=1}^M \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|y-x_k|}}dy - \int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{\sum_{k=1}^M \lambda_k g_k^{\frac{1}{\sigma}-1}e^{-\alpha|y-x_k|}}dy\right|\\ &\le \mu\sum_j \frac{\lambda_j f_j^{\frac{1}{\sigma}}\sum_k \lambda_k\left|g_k^{\frac{1}{\sigma}-1}-f_k^{\frac{1}{\sigma}-1}\right|e^{-\alpha|x_j-x_k|}}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}\right)}e^{-\alpha|x_i-x_j|}\\ &\quad + \mu\sum_s \frac{\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}\cdot\lambda_s\left|f_s^{\frac{1}{\sigma}}-g_j^{\frac{1}{\sigma}}\right|}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_s-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1}e^{-\alpha|x_j-x_k|}\right)}e^{-\alpha|x_i-x_j|}\\ &\quad + (1-\mu)\int_S \frac{\phi(y)\sum_k \lambda_k\left|g_k^{\frac{1}{\sigma}-1}-f_k^{\frac{1}{\sigma}-1}\right|e^{-\alpha|y-x_k|}}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1}e^{-\alpha|y-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1}e^{-\alpha|y-x_k|}\right)}dy\\ &\le \left\{\frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(\frac{1}{\sigma}-1)}e^{\alpha\bar d} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{\frac{1}{\sigma}-2}e^{\alpha\bar d}\right.\\ &\qquad\qquad \left. + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{\frac{1}{\sigma}-2}}{b^{2(\frac{1}{\sigma}-1)}}e^{\alpha\pi}\right\}\|f-g\|_\infty.\end{aligned}$$

Therefore, $\Phi(\cdot,\lambda)$ is Lipschitz continuous with the Lipschitz constant

$$L = \frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(\frac{1}{\sigma}-1)} e^{\alpha\bar{d}} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{\frac{1}{\sigma}-2} e^{\alpha\bar{d}} + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{\frac{1}{\sigma}-2}}{b^{2(\frac{1}{\sigma}-1)}}e^{\alpha\pi}. \tag{4.3}$$

As shown, $\Phi(\cdot,\lambda)$ is a Lipschitz continuous operator from the bounded closed convex subset $E^\infty_{a,b}$ into itself. Then, by the Brouwer fixed point theorem, (3.3) has at least one solution $f \in E^\infty_{a,b}$. □

Uniqueness of the solution is obtained by the following theorem.

**Theorem 4.2.** *In addition to* (4.1)*, assume that*

$$\frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(1/\sigma-1)} e^{\alpha\bar{d}} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{1/\sigma-2} e^{\alpha\bar{d}} + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{1/\sigma-2}}{b^{2(1/\sigma-1)}}e^{\alpha\pi} < 1. \tag{4.4}$$

*Then, for any* $\lambda \in \mathcal{M}$*, the solution* $f \in E^\infty_{a,b}$ *to* (3.3) *is unique.*

*Proof.* Since (4.4) means that $L < 1$, (4.4) implies that $\Phi(\cdot,\lambda)$ is a contraction on $E^\infty_{a,b}$. □

Because of the following theorem, the solution $f$ constructed in Theorem 4.2 is unique in the whole space $E^\infty_+$.

**Theorem 4.3.** *Under* (4.1)*, any solution to* (3.3) *in* $E^\infty_+$ *actually lies in* $E^\infty_{a,b}$.

*Proof.* Let $f \in E^\infty_+$ be a solution to (3.3). Then, an upper estimate such as

$$\begin{aligned} f_i &= \Phi(f,\lambda)_i \\ &\le \mu \max_i |f_i| e^{\alpha\bar{d}} + (1-\mu)\left(\max_i |f_i|\right)^{1-\frac{1}{\sigma}} e^{\alpha\pi} \end{aligned}$$

holds. By solving this inequality for $\max_i |f_i|$, we see that $\max_i |f_i| \le b$.

On the other hand, a lower estimate such as

$$\begin{aligned} f_i &= \Phi(f,\lambda)_i \\ &\ge \mu \min_i |f_i| e^{-\alpha\bar{d}} + (1-\mu)\left(\min_i |f_i|\right)^{1-\frac{1}{\sigma}} e^{-\alpha\pi} \end{aligned}$$

holds, too. By solving this inequality for $\min_i |f_i|$, we see that $\min_i |f_i| \ge a$. □

The following proposition gives upper and lower bounds for $G(f,\lambda)$ and $\omega(f,\lambda)$ when $(f,\lambda)$ varies in $E^\infty_{a,b} \times \mathcal{M}$.

**Proposition 4.1.** *For* $i = 1,\cdots,M$*, we have the estimates*

$$a^{\frac{1}{\sigma}} \le G(f,\lambda)_i \le b^{\frac{1}{\sigma}} e^{\tau\bar{d}}, \quad (f,\lambda) \in E^\infty_{a,b} \times \mathcal{M}, \tag{4.5}$$

$$a^{\frac{1}{\sigma}} b^{-\frac{\mu}{\sigma}} e^{-\mu\tau\bar{d}} \le \omega(f,\lambda)_i \le b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}}, \quad (f,\lambda) \in E^\infty_{a,b} \times \mathcal{M}. \tag{4.6}$$

*Proof.* These estimates are verified by direct calculations in view of the definitions of $G(f,\lambda)$ and $\omega(f,\lambda)$ and the range condition $a \le f_i \le b$, $i = 1, \cdots, M$. For example, the upper bound for $G(f,\lambda)$ is verified by

$$\begin{aligned} G(f,\lambda)_i &= \left[\sum_{j=1}^{M} \lambda_j f_j^{\frac{1}{\sigma}-1} e^{-\alpha|x_i-x_j|}\right]^{\frac{1}{1-\sigma}} \\ &\le \left[b^{\frac{1}{\sigma}-1} e^{-\alpha\overline{d}}\right]^{\frac{1}{1-\sigma}} \\ &= b^{\frac{1}{\sigma}} e^{\tau\overline{d}}. \end{aligned}$$

□

In the case when the fixed point problem (3.3) admits a unique solution $f \in E^\infty_+$ for $\lambda \in \mathcal{M}$, we denote it by $f = \Phi_{\rm f}(\lambda)$. Then, (3.6) ultimately reduces to the Cauchy problem

$$\begin{cases} \dfrac{d\lambda}{dt}(t) = \Psi(\Phi_{\rm f}(\lambda(t)), \lambda(t)), \qquad 0 \le t < \infty, \\ \lambda(0) = \lambda_0 \end{cases} \tag{4.7}$$

in $E^1$ with an initial value $\lambda_0 \in \mathcal{M}$.

**4.2 Local Solution.** We construct a local solution to (4.7) using the Banach fixed point theorem. The following proposition plays an important role.

**Proposition 4.2.** *Under* (4.1) *and* (4.4), *the estimates*

$$\|\Phi_{\rm f}(\lambda) - \Phi_{\rm f}(\kappa)\|_\infty \le \beta_1 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S), \tag{4.8}$$

$$\|G(\Phi_{\rm f}(\lambda), \lambda) - G(\Phi_{\rm f}(\kappa), \kappa)\|_\infty \le \beta_2 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S), \tag{4.9}$$

$$\|\omega(\Phi_{\rm f}(\lambda), \lambda) - \omega(\Phi_{\rm f}(\kappa), \kappa)\|_\infty \le \beta_3 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S) \tag{4.10}$$

*hold true with some constants* $\beta_1,\ \beta_2,\ \beta_3 > 0$.

*Proof.* It suffices to prove (4.8), because (4.9) and (4.10) are easily verified from (4.8).

For $\lambda,\ \kappa \in \mathcal{M}$, we write $f = \Phi_{\rm f}(\lambda),\ g = \Phi_{\rm f}(\kappa)$, and we use the following notations

$$\begin{aligned} A_j &= \sum_k \kappa_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}, \quad j = 1, \cdots, M, \\ B_j &= \sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}, \quad j = 1, \cdots, M, \\ A(y) &= \sum_k \kappa_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}, \quad y \in S, \\ B(y) &= \sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}, \quad y \in S. \end{aligned}$$

Then,

$$\begin{aligned} |f_i - g_i| \le{}& \mu \sum_j \frac{\left|\lambda_j f_j^{\frac{1}{\sigma}} A_j - \kappa_j g_j^{\frac{1}{\sigma}} B_j\right|}{A_j B_j} e^{-\alpha|x_i-x_j|} \\ &+ (1-\mu) \int_S \frac{|A(y) - B(y)|}{A(y)B(y)} \phi(y) e^{-\alpha|x_i-y|} dy. \end{aligned} \tag{4.11}$$

Furthermore, it follows that

$$
\begin{aligned}
&\left|\lambda_j f_j^{\frac{1}{\sigma}} A_s - \kappa_j g_j^{\frac{1}{\sigma}} B_s\right| \\
&\le \lambda_j f_j^{\frac{1}{\sigma}} \sum_k \left|\kappa_k g_k^{\frac{1}{\sigma}-1} - \lambda_k f_k^{\frac{1}{\sigma}-1}\right| e^{-\alpha|x_j-x_k|} + B_s \left|\lambda_j f_j^{\frac{1}{\sigma}} - \kappa_j g_j^{\frac{1}{\sigma}}\right| \\
&\le \lambda_j f_j^{\frac{1}{\sigma}} \sum_k \left\{\kappa_k |g_k^{\frac{1}{\sigma}-1} - f_k^{\frac{1}{\sigma}-1}| + f_k^{\frac{1}{\sigma}-1}|\kappa_k - \lambda_k|\right\} e^{-\alpha|x_j-x_k|} \\
&\qquad + B_j \left\{\lambda_j |f_j^{\frac{1}{\sigma}} - g_j^{\frac{1}{\sigma}}| + g_j^{\frac{1}{\sigma}}|\lambda_j - \kappa_j|\right\} \\
&\le \left(\frac{\sigma-1}{\sigma}\right) a^{\frac{1}{\sigma}-2} b^{\frac{1}{\sigma}} \lambda_j \|f-g\|_\infty \sum_k \kappa_k e^{-\alpha|x_j-x_k|} \\
&\qquad + a^{\frac{1}{\sigma}-1} b^{\frac{1}{\sigma}} \lambda_s \|\kappa - \lambda\|_1 \\
&\qquad + \frac{1}{\sigma} a^{2\left(\frac{1}{\sigma}-1\right)} \lambda_j \|f-g\|_\infty \sum_k \lambda_k e^{-\alpha|x_j-x_k|} \\
&\qquad + a^{\frac{1}{\sigma}-1} b^{\frac{1}{\sigma}} |\lambda_j - \kappa_j| \sum_k \lambda_k e^{-\alpha|x_j-x_k|}.
\end{aligned}
\tag{4.12}
$$

It is also verified by the similar calculations that

$$
\begin{aligned}
&|A(y) - B(y)| \le \\
&\left(\frac{\sigma-1}{\sigma}\right) a^{\frac{1}{\sigma}-2} \|f-g\|_\infty \sum_k \kappa_k e^{-\alpha|y-x_k|} + a^{\frac{1}{\sigma}-1}\|\lambda - \kappa\|_1.
\end{aligned}
\tag{4.13}
$$

In addition, the estimates

$$
\begin{aligned}
A_j B_j &\ge b^{2\left(\frac{1}{\sigma}-1\right)} \left(\sum_k \kappa_k e^{-\alpha|x_j-x_k|}\right)\left(\sum_k \lambda_k e^{-\alpha|x_j-x_k|}\right), \\
A(y)B(y) &\ge b^{2\left(\frac{1}{\sigma}-1\right)} \left(\sum_k \kappa_k e^{-\alpha|y-x_k|}\right)\left(\sum_k \lambda_k e^{-\alpha|y-x_k|}\right)
\end{aligned}
\tag{4.14}
$$

hold obviously. Using (4.12), (4.13), and (4.14), and noticing (4.4), we conclude from (4.11) that

$$
\|f-g\|_\infty \le \beta_1 \|\lambda - \kappa\|_1 ,
$$

i.e., (4.8), where

$$
\begin{aligned}
\beta_1 = &\left\{1 - \frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(1/\sigma-1)} e^{\alpha\overline{d}} - \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{1/\sigma-2} e^{\alpha\overline{d}}\right. \\
&\qquad \left. - \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{1/\sigma-2}}{b^{2(1/\sigma-1)}} e^{\alpha\pi}\right\}^{-1} \\
&\times \left\{\mu \frac{a^{\frac{1}{\sigma}-1}}{b^{\frac{1}{\sigma}-2}}\left(e^{2\alpha\overline{d}} + e^{\alpha\overline{d}}\right) + (1-\mu)\frac{a^{\frac{1}{\sigma}-1}}{b^{2\left(\frac{1}{\sigma}-1\right)}} e^{2\alpha\pi}\right\}.
\end{aligned}
$$

□

To construct a local solution to (4.7), we have to introduce an auxiliary problem for (4.7). For a given $\tilde{\lambda} \in \mathcal{M}$, let $\tilde{\Psi}$ be an operator from $E^\infty \times \mathcal{M}$ to $E^1$ defined by

$$\tilde{\Psi}(w,\lambda)_i = \left[\omega(w,\tilde{\lambda})_i - \sum_{k=1}^{M} \omega(w,\tilde{\lambda})_k \lambda_k\right] \lambda_i, \quad i = 1, \cdots, M. \tag{4.15}$$

For a given $\tilde{\lambda} \in \mathcal{C}([0,\infty);\mathcal{M})$, consider an auxiliary problem

$$\begin{cases} \dfrac{d\lambda}{dt}(t) = \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(t)), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0. \end{cases} \tag{4.16}$$

**Proposition 4.3.** *Under* (4.1) *and* (4.4)*, let* $\tilde{\lambda}$ *be given as above. Then,* (4.16) *possesses a unique local solution* $\lambda \in \mathcal{C}^1([0,c];\mathcal{M})$*, provided that* $(1 \ge)$ $c > 0$ *is sufficiently small, but* $c$ *being independent of the given function* $\tilde{\lambda}$ *and the initial value* $\lambda_0$*.*

*Proof.* Set a closed subset of $E^1$ given by

$$E^1_1 := \left\{ \lambda \in E^1 \,\middle|\, \sum_{j=1}^{M} \lambda_j = 1 \right\},$$

and define an operator $\tilde{T} : \mathcal{C}([0,c];E^1_1) \to \mathcal{C}([0,c];E^1)$ by

$$\left[\tilde{T}(\lambda)\right](t) = \lambda_0 + \int_0^t \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s))ds.$$

Using $\tilde{T}$, we rewrite (4.16) into an equivalent problem

$$\lambda(t) = [\tilde{T}(\lambda)](t), \quad 0 \le t < \infty.$$

It is verified that $\tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda)$ is Lipschitz continuous with respect to $\lambda \in E^1_1$. Indeed, by (4.6) and (4.15), we see that

$$\left\| \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda) - \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \kappa) \right\|_1 \le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \|\lambda - \kappa\|_1, \quad \lambda, \kappa \in E^1_1. \tag{4.17}$$

Meanwhile, $\tilde{T}$ maps $\mathcal{C}([0,c];E^1_1)$ into itself. To verify this, it is sufficient to see that $\sum_j \tilde{T}(\lambda)_j = 1$, because $\tilde{T}$ obviously maps $\mathcal{C}([0,c];E^1_1)$ into $\mathcal{C}([0,c];E^1)$. Then,

$$\begin{aligned} \sum_j \left[\tilde{T}(\lambda)\right]_j (t) - \sum_j \lambda_{0,j} &= \sum_j \int_0^t \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s))_j ds \\ &= \int_0^t \sum_j \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s)) ds = 0 \end{aligned}$$

due to (4.15).

From (4.17),

$$\begin{aligned} \left\| \tilde{T}(\lambda) - \tilde{T}(\kappa) \right\|_{\mathcal{C}([0,c];E^1)} &\le \max_{t\in[0,c]} e^{-t} \int_0^t \left\| \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda) - \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \kappa) \right\|_1 ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \max_{t\in[0,c]} e^{-t} \int_0^t \|\lambda(s) - \kappa(s)\|_1 ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \max_{t\in[0,c]} e^{-t} \int_0^t \|\lambda(s) - \kappa(s)\|_1 e^{-s} e^{s} ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} (1 - e^{-c}) \|\lambda - \kappa\|_{\mathcal{C}([0,c];E^1)}. \end{aligned}$$

Therefore, if $c$ is sufficiently small, then $\tilde{T}$ becomes a contraction mapping. Thus, (4.16) has a unique fixed point $\lambda \in \mathcal{C}^1([0,c];E_1^1)$ for sufficiently small $c>0$.

As a matter of fact, this $\lambda \in \mathcal{C}([0,c];E_1^1)$ is in $\mathcal{C}([0,c];\mathcal{M})$. Indeed, it is sufficient to verify $\lambda_i(t) \geq 0,\ \forall i=1,\cdots,M$ for $t\in[0,c]$. Since the solution to (4.16) can be written as

$$\lambda_i(t) = \lambda_{0,i}\exp\left[\int_0^t \left\{\omega(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)),\tilde{\lambda}(s))_i - \sum_k \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)),\tilde{\lambda}(s))_k\lambda_k(s)\right\} ds\right],$$

$\lambda_{0,i}\geq 0,\ i=1,\cdots,M$, imply that $\lambda_i(t)\geq 0$ for all $i=1,\cdots,M$.

As seen above, the time $c>0$ was determined independently of $\tilde{\lambda}$ and $\lambda_0$. □

Now, we are ready to construct a local solution to (4.7).

**Theorem 4.4.** *Under* (4.1) *and* (4.4)*, for each* $\lambda_0\in\mathcal{M}$*, there exists a unique local solution* $\lambda\in\mathcal{C}^1([0,c];\mathcal{M})$ *to* (4.7)*, provided that* $(1\geq)\ c>0$ *is sufficiently small, but* $c$ *being independent of the initial value* $\lambda_0$*.*

*Proof.* By virtue of Proposition 4.3, for each $\lambda_0$, we can define an operator $F_{\lambda_0}$ which corresponds $\tilde{\lambda}\in\mathcal{C}^1([0,c];\mathcal{M})$ to the local solution $\lambda\in\mathcal{C}^1([0,c];\mathcal{M})$ of the auxiliary problem (4.16). By the definition of $F_{\lambda_0}$, it immediately follows that

$$\left[F_{\lambda_0}(\tilde{\lambda})\right](t) = \lambda_0 + \int_0^t \tilde{\Psi}\left(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)),[F_{\lambda_0}(\tilde{\lambda})](s)\right) ds.$$

If there exists a fixed point of $F_{\lambda_0}$, then it is obviously a local solution to (4.7). So, we will prove that $F_{\lambda_0}$ is a contraction mapping from $\mathcal{C}^1([0,c];\mathcal{M})$ into itself.

For $\tilde{\lambda},\ \tilde{\kappa}\in\mathcal{C}([0,c];\mathcal{M})$,

$$\begin{aligned}
&\left\|\left[F_{\lambda_0}(\tilde{\lambda})\right](t)-\left[F_{\lambda_0}(\tilde{\kappa})\right](t)\right\|_1\\
&\leq \int_0^t \left\|\tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}),F_{\lambda_0}(\tilde{\lambda}))-\tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\kappa}),F_{\lambda_0}(\tilde{\kappa}))\right\|_1 ds\\
&\leq \int_0^t \left\|\omega(\Phi_{\mathrm{f}}(\tilde{\lambda}),\tilde{\lambda})F_{\lambda_0}(\tilde{\lambda})-\omega(\Phi_{\mathrm{f}}(\tilde{\kappa}),\tilde{\kappa})F_{\lambda_0}(\tilde{\kappa})\right\|_1 ds\\
&\quad+\int_0^t \left\|\sum_k \omega(\Phi_{\mathrm{f}}(\tilde{\kappa}),\tilde{\kappa})_k F_{\lambda_0}(\tilde{\kappa})_k\cdot F_{\lambda_0}(\tilde{\kappa})\right.\\
&\qquad\left.-\sum_k \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}),\tilde{\lambda})_k F_{\lambda_0}(\tilde{\lambda})_k\cdot F_{\lambda_0}(\tilde{\lambda})\right\|_1 ds.
\end{aligned}\tag{4.18}$$

Note that $\sum_i F_{\lambda_0}(\tilde{\kappa})_i=1$, then it follows from (4.6) and (4.10) that

$$\begin{aligned}
&\left\|\omega(\Phi_{\mathrm{f}}(\tilde{\lambda}),\tilde{\lambda})F_{\lambda_0}(\tilde{\lambda})-\omega(\Phi_{\mathrm{f}}(\tilde{\kappa}),\tilde{\kappa})F_{\lambda_0}(\tilde{\kappa})\right\|_1\\
&\leq b^{\frac{1}{\sigma}}a^{-\frac{\mu}{\sigma}}\left\|F_{\lambda_0}(\tilde{\lambda})(t)-F_{\lambda_0}(\tilde{\kappa})(t)\right\|_1+\beta_3\left\|\tilde{\lambda}-\tilde{\kappa}\right\|_1
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sum_k \omega(\Phi_{\mathrm{f}}(\tilde{\kappa}),\tilde{\kappa})_k F_{\lambda_0}(\tilde{\kappa})_k\cdot F_{\lambda_0}(\tilde{\kappa})\right.\\
&\qquad\left.-\sum_k \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}),\tilde{\lambda})_k F_{\lambda_0}(\tilde{\lambda})_k\cdot F_{\lambda_0}(\tilde{\lambda})\right\|_1\\
&\leq 2b^{\frac{1}{\sigma}}a^{-\frac{\mu}{\sigma}}\left\|F_{\lambda_0}(\tilde{\lambda})-F_{\lambda_0}(\tilde{\kappa})\right\|_1+\beta_3\left\|\tilde{\lambda}-\tilde{\kappa}\right\|_1.
\end{aligned}$$

By applying these estimates to (4.18), it follows that

$$\left\|[F_{\lambda_0}(\tilde\lambda)](t) - [F_{\lambda_0}(\tilde\kappa)](t)\right\|_1 \le 3b^{\frac{1}{\sigma}}a^{-\frac{\mu}{\sigma}}\int_0^t \left\|F_{\lambda_0}(\tilde\lambda)(s) - F_{\lambda_0}(\tilde\kappa)(s)\right\|_1 ds + 2\beta_3\int_0^t \left\|\tilde\lambda(s) - \tilde\kappa(s)\right\|_1 ds, \quad 0 \le t \le c.$$

As a result, we obtain that

$$\left\|F_{\lambda_0}(\tilde\lambda) - F_{\lambda_0}(\tilde\kappa)\right\|_{\mathcal{C}([0,c];E^1)} \le k\left\|\tilde\lambda - \tilde\kappa\right\|_{\mathcal{C}([0,c];E^1)},$$

with

$$k = \frac{2\beta_3(1-e^{-c})}{1 - 3b^{\frac{1}{\sigma}}a^{-\frac{\mu}{\sigma}}(1-e^{-c})}.$$

Therefore, if $c > 0$ is sufficiently small, then $k < 1$, and $F_{\lambda_0}$ is a contraction mapping in $\mathcal{C}([0,c];\mathcal{M})$. □

**4.3 Global Solution.** We can now easily extend the local solution of (4.7) to global one.

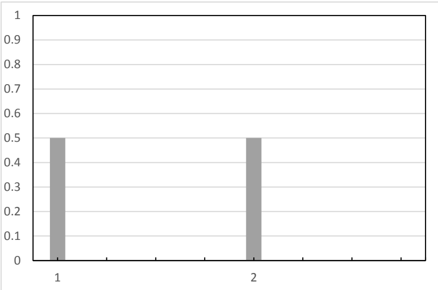
**Theorem 4.5.** *Under* (4.1) *and* (4.4), *for each* $\lambda_0 \in \mathcal{M}$, *there exists a unique global solution* $\lambda \in \mathcal{C}^1([0,\infty);\mathcal{M})$ *to* (4.7).

*Proof.* Note that the interval $[0,c]$ on which we construct a local solution is independent of the initial value $\lambda_0$. Then, the uniqueness of the local solution shows that the unique local solution $\lambda \in \mathcal{C}^1([0,2c];\mathcal{M})$ is obtained by repeating the same argument but with the initial value $\lambda(c)$. By repeating this procedure, we finally obtain a unique global solution to (4.7). □

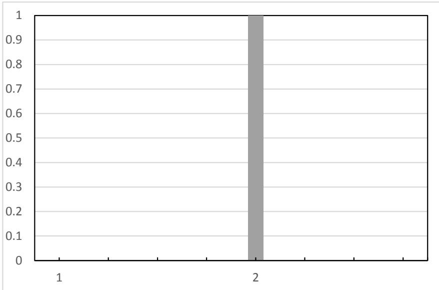
**5 Numerical Results.** In this section, some examples of numerical computations are illustrated. In Subsection 5.1, the case of $M = 2$, and in Subsection 5.2, the case of $M = 3$ is handled, respectively. Throughout this section, the parameters $\mu$ and $\sigma$ are fixed as $\mu = 0.5$ and $\sigma = 3$. And $\tau > 0$ is changed as a control parameter. The density of agricultural workers is assumed to be constant, i.e., $\phi(x) \equiv \frac{1}{2\pi}$. The initial value for the manufacturing population size $\lambda = (\lambda_1, \cdots, \lambda_M)$ is given by adding small perturbations to the uniform population size $\lambda_i = \overline{\lambda} \equiv 1/M$, $i = 1, \cdots, M$. The circumference $S$ is identified with the interval $[-\pi,\pi]$. In the following, we refer the manufacturing region as the region and the manufacturing population as the population for simplicity.

**5.1 Case of $M = 2$.** We consider two kinds of configurations of two regions such that $|x_1 - x_2| = \pi$ and $|x_1 - x_2| = \pi/4$. Figures 1 and 2 illustrate the stationary solutions $\overline{\lambda}$ to which the solutions $\lambda(t)$ converge as $t \to \infty$ for the cases $\pi$ and $\pi/4$, respectively. Here, the horizontal axis and the vertical axis denote the interval $[-\pi,\pi]$ and the population size, respectively.

Figure 1(a) shows when $\tau = 1.3$ that the population is separated in the two regions uniformly. However, Figure 1(b) shows when $\tau = 1.2$ that the population is accumulated into a single region. On the other hand, Figure 2(a) shows when $\tau = 1.5$ that the population is separated in the two regions equally, and Figure 2(b) shows when $\tau = 1.45$ that the population is accumulated into a single region. In any case, there exists a threshold $\hat\tau$ such that, if $\tau > \hat\tau$ the population is equally divided between the regions, and if $\tau < \hat\tau$ the population is concentrated in a single region. Moreover, it is observed that the threshold $\widehat\tau$ differs by configurations. In fact, Figures 1 and 2 show that $1.2 < \widehat\tau < 1.3$ when $|x_1 - x_2| = \pi$ and $1.45 < \widehat\tau < 1.5$ when $|x_1 - x_2| = \pi/4$.

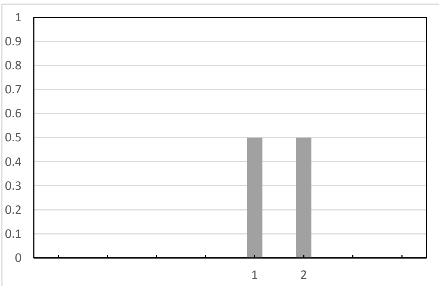



(a) $\tau = 1.3$

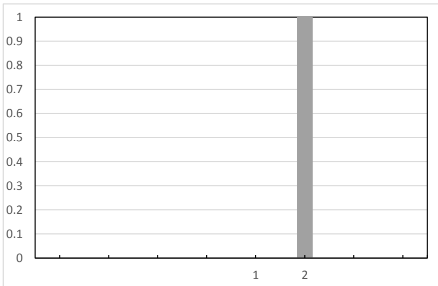



(b) $\tau = 1.2$

Fig. 1: $\overline{\lambda}$ when $|x_1 - x_2| = \pi$




(a) $\tau = 1.5$




(b) $\tau = 1.45$

Fig. 2: $\overline{\lambda}$ when $|x_1 - x_2| = \pi/4$

**5.2 Case of $M = 3$.** Let us put nine points on $S$ equidistantly. Then, we select three manufacturing regions from them. By doing so, we consider the seven cases of configurations of the regions as below.

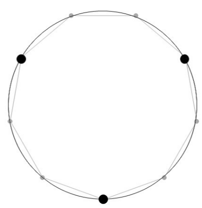
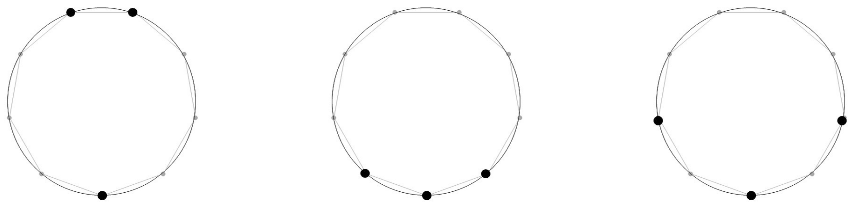
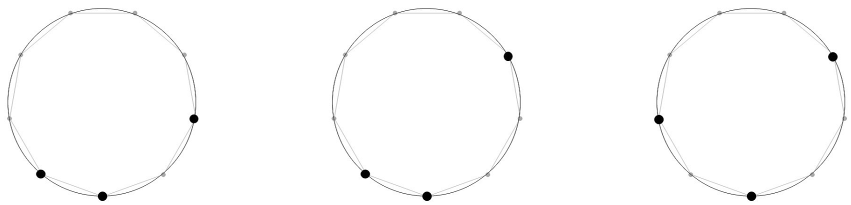


Fig. 3: Equilateral triangle



(a) Isosceles 1 (b) Isosceles 2 (c) Isosceles 3

Fig. 4: Isosceles triangles

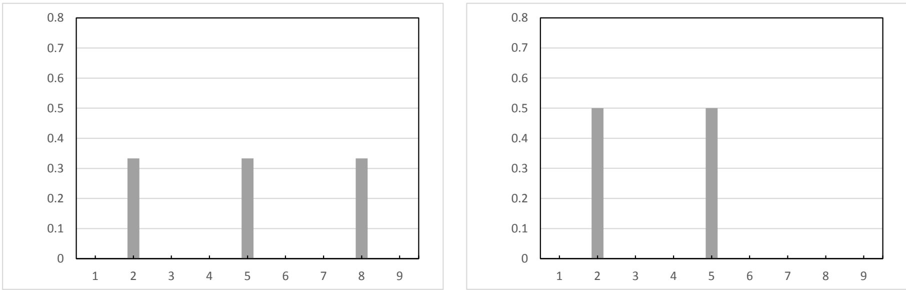


(a) Scalene 1 (b) Scalene 2 (c) Scalene 3

Fig. 5: Scalene triangles

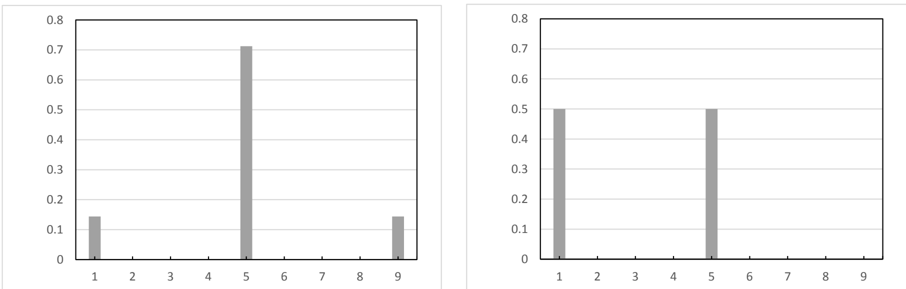
Figures 6 - 12 illustrate the stationary solutions $\overline{\lambda}$ to which the solutions $\overline{\lambda}(t)$ converges as $t \to \infty$ for the seven cases. Here, the horizontal axis and the vertical axis denote the interval $[-\pi, \pi]$ and the population size, respectively.

Figure 6(a) shows under the equilateral configuration, when $\tau = 1.5$ that the population is separated in three regions equally. However, Figure 6(b) shows when $\tau = 1.45$ that the population is accumulated into two regions only. Under the isosceles configuration 1, although the population is dispersed to the three regions when $\tau = 3$, the population is separated into two regions when $\tau = 2.9$ as shown by Figures 7(a) and 7(b). The numerical results illustrated in Figures 8 - 12 are similar, i.e., there exists a threshold $\hat{\tau}$ such that, $\tau > \hat{\tau}$ the population is equally divided among the three regions, and if $\tau < \hat{\tau}$ the population is concentrated in two regions. Moreover, it is observed that the threshold $\widehat{\tau}$ differs by the types of configurations. In fact, there is more than seven times difference in the value of $\widehat{\tau}$ between the equilateral configuration and the isosceles configuration 2.



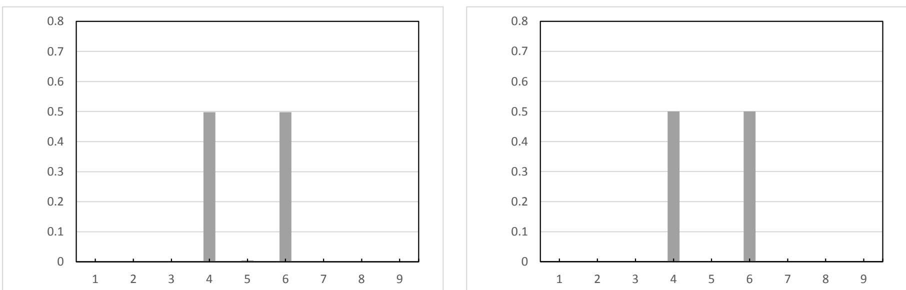

(a) $\tau = 1.5$ (b) $\tau = 1.45$

Fig. 6: $\overline{\lambda}$ for equilateral triangle



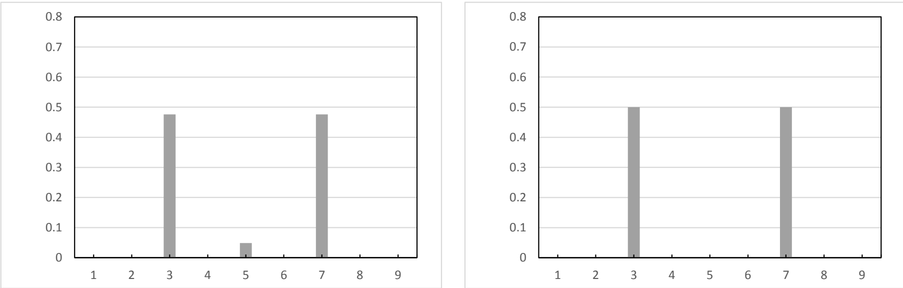

(a) $\tau = 3$ (b) $\tau = 2.9$

Fig. 7: $\overline{\lambda}$ for isosceles 1



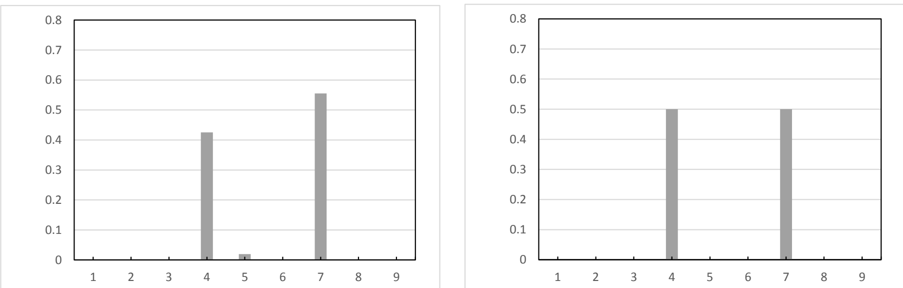

(a) $\tau = 11.5$ (b) $\tau = 10.9$

Fig. 8: $\overline{\lambda}$ for isosceles 2



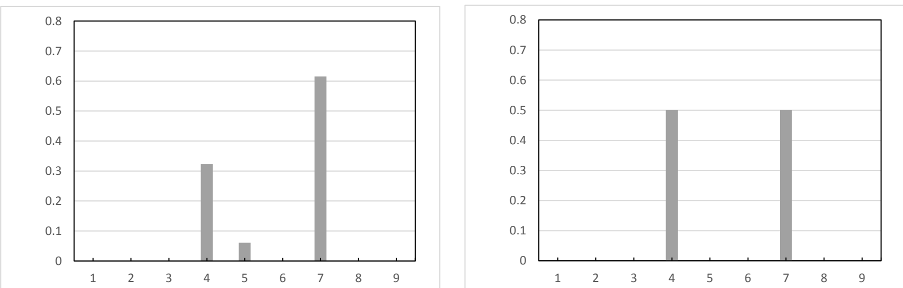

(a) $\tau = 4$ (b) $\tau = 3.9$

Fig. 9: $\overline{\lambda}$ for isosceles 3



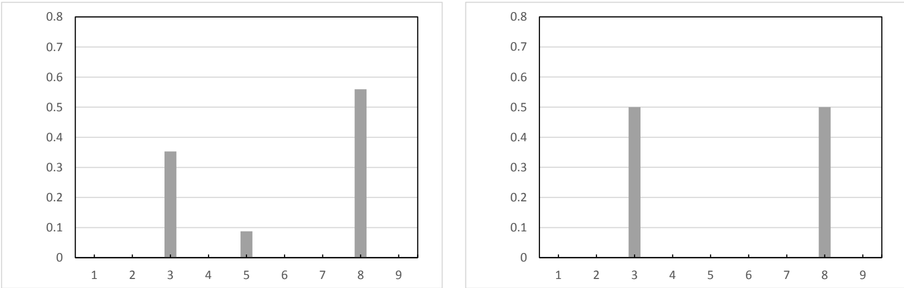

(a) $\tau = 6.8$ (b) $\tau = 6.7$

Fig. 10: $\overline{\lambda}$ for scalene 1




(a) $\tau = 4.8$ (b) $\tau = 4.7$

Fig. 11: $\overline{\lambda}$ for scalene 2




(a) $\tau = 2.7$ (b) $\tau = 2.6$

Fig. 12: $\overline{\lambda}$ for scalene 3

**6 Stability of Stationary Solutions.** In this section, we want to investigate stability of stationary solutions for (1.2). As in the previous section, the agricultural population density is assumed to be constant, i.e., $\phi(x) \equiv \overline{\phi}$. After discussing existence of stationary solutions, we investigate their stability in the cases of $M = 2$ and 3.

**6.1 Existence of Stationary Solutions.** By $(\overline{f}, \overline{\lambda})$ we denote a stationary solution to (3.6), where $\overline{f} = (\overline{f}_1, \cdots, \overline{f}_M) \in \mathbb{R}^M$ and $\overline{\lambda} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_M) \in \mathbb{R}^M$. We also denote $\overline{w} = (\overline{w}_1, \cdots, \overline{w}_M)$, where $\overline{f}_i = \overline{w}_i^\sigma$ for $i = 1, \cdots, M$. Then, the price index and the real wage of stationary state are given by $\overline{G} = \overline{G}(x)$, $x \in S$ and $\overline{\omega} = (\overline{\omega}_1, \cdots, \overline{\omega}_M)$, respectively.

From (3.3), the stationary solution must satisfy

$$
\begin{aligned}
\overline{w}_i^\sigma = &\sum_{j=1}^{M} \frac{\overline{\lambda}_j \overline{w}_j e^{-\alpha|x_i - x_j|}}{\sum_{k=1}^{M} \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j - x_k|}} \\
&+ (1-\mu)\overline{\phi} \int_S \frac{e^{-\alpha|x_i - y|}}{\sum_{k=1}^{M} \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|y - x_k|}} dy, \quad i = 1, 2, \cdots, M.
\end{aligned} \tag{6.1}
$$

Moreover, by the fact that $\Psi(\overline{f}, \overline{\lambda}) = 0$ in (3.6), the following equations

$$
\begin{aligned}
&\overline{w}_i \left\{ \sum_{k=1}^{M} \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_i - x_k|} \right\}^{\frac{\mu}{\sigma-1}} \\
&\quad = \sum_{j=1}^{M} \overline{\lambda}_j \left\{ \sum_{k=1}^{M} \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j - x_k|} \right\}^{\frac{\mu}{\sigma-1}}, \quad i = 1, 2, \cdots, M
\end{aligned} \tag{6.2}
$$

must be satisfied. Thereby, the number of unknowns $\overline{\lambda}_1, \cdots, \overline{\lambda}_M$ and $\overline{w}_1, \cdots, \overline{w}_M$ is equal to the number of equations. This fact suggests that there may exist a stationary solution to (3.6) under any location of $M$ manufacturing regions on $S$. But it is very difficult to demonstrate this assertion. We consider only the symmetric solutions.

**Definition 6.1.** *A stationary solution satisfying the conditions:*

1. *all the distances between adjacent manufacturing regions are equal,*
2. *the population size and the nominal wages are uniform for the regions,*

*is called a symmetric stationary solution.*

When $M = 2$ or 3, the symmetric stationary solution is obtained analytically.

**Theorem 6.1.** *When $M = 2$, for any configuration of $x_1$ and $x_2$,* (3.6) *has a symmetric stationary solution such that*

$$w_i \equiv \bar{w} = 1, \quad i = 1, 2, \qquad \lambda_i \equiv \bar{\lambda} = 1/2, \quad i = 1, 2. \tag{6.3}$$

*Proof.* It is easy to verify that (6.3) is a stationary solution of (3.6) in view of

$$\int_S \frac{e^{-\alpha|x_i-y|}}{e^{-\alpha|x_1-y|} + e^{-\alpha|x_2-y|}} dy = \pi, \qquad i = 1, 2.$$

□

**Theorem 6.2.** *When $M = 3$, let $|x_2 - x_1|$, $|x_3 - x_2|$ and $|x_1 - x_3|$ be equal to $2\pi/3$. Then,* (1.2) *has a symmetric stationary solution such that*

$$w_i = \overline{w} = 1, \quad i = 1, 2, 3, \qquad \lambda_i = \overline{\lambda} = 1/3, \quad i = 1, 2, 3. \tag{6.4}$$

*Proof.* It is easy to verify that (6.4) is a stationary solution of (3.6) due to the fact that

$$\int_S \frac{e^{-\alpha|x_i-y|}}{\sum_{k=1}^{3} e^{-\alpha|x_k-y|}} dy = \frac{2\pi}{3}, \qquad i = 1, 2, 3.$$

□

**6.2 Linearization Matrix.** Let a stationary solution $(\overline{\lambda}, \overline{w})$ be given. We want to linearize (3.6) around it. Let $\Delta w$, $\Delta\lambda \in \mathbb{R}^M$ be small perturbations added to $\overline{w}$ and $\overline{\lambda}$, respectively, but satisfying the restriction $\sum_{i=1}^{M} \Delta\lambda_i = 0$. The linearized equations are given by

$$\begin{cases} (I - A)\,\Delta w = B\Delta\lambda, \\ \dfrac{d}{dt}\Delta\lambda = L\big[I - \overline{\Lambda}\big]\big[(E + FC)\,\Delta w + \big(FD - \overline{R}\big)\,\Delta\lambda\big], \end{cases} \tag{6.5}$$

where $I$ stands for the identity matrix. Here, the $M \times M$ matrices $A$, $B$, $C$, $D$, $E$, and $F$ are given by

$$\begin{aligned} A_{ij} = {} & \frac{\mu}{\sigma}\overline{w}_i^{1-\sigma} \frac{\overline{\lambda}_j e^{-\alpha|x_i-x_j|}}{\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j-x_k|}} \\ & + \frac{\mu(\sigma-1)}{\sigma}\overline{w}_i^{1-\sigma}\overline{\lambda}_j\overline{w}_j^{-\sigma} \sum_s \frac{\overline{\lambda}_s\overline{w}_s e^{-\alpha|x_s-x_j|}e^{-\alpha|x_i-x_s|}}{\big[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|x_s-x_k|}\big]^2} \\ & + \frac{(1-\mu)(\sigma-1)}{2\pi\sigma}\overline{w}_i^{1-\sigma}\overline{\lambda}_j\overline{w}_j^{-\sigma} \int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\big[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|y-x_k|}\big]^2} dy, \end{aligned} \tag{6.6}$$

$$\tag{6.7}\begin{aligned} B_{ij} = &\frac{\mu}{\sigma}\overline{w}_i^{1-\sigma}\frac{\overline{w}_j e^{-\alpha|x_i-x_j|}}{\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j-x_k|}} \\ &- \frac{\mu}{\sigma}\overline{w}_i^{1-\sigma}\sum_s \frac{\overline{\lambda}_s \overline{w}_s \overline{w}_j^{1-\sigma} e^{-\alpha|x_s-x_j|} e^{-\alpha|x_i-x_s|}}{\left[\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_s-x_k|}\right]^2} \\ &- \frac{1-\mu}{2\pi\sigma}\overline{w}_i^{1-\sigma}\overline{w}_j^{1-\sigma}\int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\left[\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|y-x_k|}\right]^2}dy, \end{aligned}$$

$$\tag{6.8}\begin{aligned} C_{ij} &= \overline{G}_i^{\sigma}\overline{\lambda}_j \overline{w}_j^{-\sigma} e^{-\alpha|x_i-x_j|}, \\ D_{ij} &= -\frac{\overline{G}_i^{\sigma}}{\sigma-1}\overline{w}_j^{1-\sigma} e^{-\alpha|x_i-x_j|}, \end{aligned}$$

$$\tag{6.9}\begin{aligned} E &= \mathrm{diag}(\overline{G}_1^{-\mu}, \cdots, \overline{G}_M^{-\mu}), \\ F &= \mathrm{diag}(-\mu\overline{w}_1\overline{G}_1^{-\mu-1}, \cdots, -\mu\overline{w}_M\overline{G}_M^{-\mu-1}) \end{aligned}$$

respectively. The matrix $\overline{\Lambda}$ denotes

$$\overline{\Lambda} = \begin{pmatrix} \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \\ \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \\ \vdots & \vdots & \vdots \\ \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \end{pmatrix},$$

and the matrix $L$ is $L := \mathrm{diag}(\overline{\lambda}_1, \cdots, \overline{\lambda}_M)$. Finally,

$$\overline{R} = \overline{\omega}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

As a matter of fact, by using the matrix

$$\Omega := \left[E(I-A)^{-1}B + F\left\{C(I-A)^{-1}B + D\right\}\right],$$

the linearized equations (6.5) is reduced to

$$\tag{6.10} \frac{d}{dt}\Delta\lambda = J\Delta\lambda,$$

where $J = L\left[(I-\overline{\Lambda})\Omega - \overline{R}\right]$.

Since $\sum_{i=1}^M \lambda_i = 1$, it is natural to impose the condition that $\sum_{i=1}^M \Delta\lambda_i = 0$; therefore, $\Delta\lambda_M = -(\Delta\lambda_1 + \cdots + \Delta\lambda_{M-1})$. The $M$-dimensional ordinary equation (6.10) is actually reduced to an ordinary differential equation for $\Delta\lambda' = (\Delta\lambda_1, \cdots, \Delta\lambda_{M-1})^{\mathrm{T}}$. Introduce an $(M-1)\times M$ matrix $P_1$ and an $M\times(M-1)$ matrix $P_2$ as

$$P_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix},$$

respectively. Then, (6.10) is reduced to

$$\frac{d}{dt}\Delta\lambda' = P_1 J P_2 \Delta\lambda'.$$

Hence, stability of the stationary solution is determined by the eigenvalues of the matrix $J' := P_1 J P_2$. In general, it is very complicate to calculate exactly the eigenvalues of $J'$, and we have to rely on numerical computations. However, in the case of symmetric stationary solutions with $M = 2$ or 3, we can compute $J'$ analytically.

Define the $M \times M$ matrices $X$ and $Y$ by

$$X_{ij} = e^{-\alpha|x_i - x_j|}, \tag{6.11}$$

$$Y_{ij} = \int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\sum_{k=1}^M e^{-\alpha|y-x_k|}} dy, \tag{6.12}$$

respectively. Then, from (6.6) and (6.7), $A$ and $B$ are described as

$$A = \frac{\mu(\sigma-1)}{\sigma}\overline{\lambda}^2\overline{w}^{2-2\sigma}\overline{G}^{2\sigma-2}X^2 + \frac{\mu}{\sigma}\overline{\lambda}\overline{w}^{1-\sigma}\overline{G}^{\sigma-1}X + \frac{(1-\mu)(\sigma-1)}{2\pi\sigma}\overline{\lambda}^{-1}\overline{w}^{-1}Y, \tag{6.13}$$

$$B = -\frac{\mu}{\sigma}\overline{\lambda}\overline{w}^{3-2\sigma}\overline{G}^{2\sigma-2}X^2 + \frac{\mu}{\sigma}\overline{w}^{2-\sigma}\overline{G}^{\sigma-1}X - \frac{1-\mu}{2\pi\sigma}\overline{\lambda}^{-2}Y. \tag{6.14}$$

Similarly, from (6.8) and (6.9), $C,\ D,\ E$ and $F$ are described as

$$C = \overline{G}^{\sigma}\overline{\lambda}\overline{w}^{-\sigma}X,$$
$$D = -\frac{\overline{G}^{\sigma}\overline{w}^{1-\sigma}}{\sigma-1}X,$$
$$E = \overline{G}^{-\mu}I,$$
$$F = \overline{w}\overline{G}^{-\mu-1}I,$$

respectively. Thereby, $\Omega$ is given by

$$\Omega = \overline{G}^{-\mu}(I-A)^{-1}B - \mu\overline{\lambda}\overline{w}^{1-\sigma}\overline{G}^{\sigma-\mu-1}X(I-A)^{-1}B + \frac{\mu}{\sigma-1}\overline{w}^{2-\sigma}\overline{G}^{\sigma-\mu-1}X. \tag{6.15}$$

Note that all the diagonal components of $\Omega$ are equal each other, and all the non-diagonal components are also equal, i.e., $\Omega$ takes the form:

$$\text{when } M = 2, \qquad \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_1 \end{pmatrix};$$

$$\text{when } M = 3, \qquad \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_2 \\ \Omega_2 & \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_2 & \Omega_1 \end{pmatrix}.$$

We call such a matrix as "*a strong diagonal matrix*". It is easy to see that sum, product, or linear combination of strong diagonal matrices is also a strong diagonal matrix. Moreover, the inverse of a strong diagonal matrix is also a strongly diagonal. By these facts, $\Omega$ is seen to be strongly diagonal, because $X$ and $Y$ are strongly diagonal (See (6.11), (6.12)). As a result, the matrix $J'$ is simply given by

$$\text{when } M = 2, \qquad J' = \frac{1}{2}\left(\Omega_1 - \Omega_2\right), \tag{6.16}$$
$$\text{when } M = 3, \qquad J' = \frac{1}{3}\left(\Omega_1 - \Omega_2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

in the symmetric stationary solution.

**6.3 Case of $M = 2$.** When $M = 2$, we have the following theorem.

**Theorem 6.3.** *Let the no black hole condition $(\sigma - 1)/\sigma > \mu$ be satisfied. If $\tau > 0$ or $\sigma > 1$ is sufficiently small, then the stationary solution given by* (6.3) *is unstable. On the other hand, if $\tau > 0$ or $\sigma > 1$ is sufficiently large, then the stationary solution given by* (6.3) *is stable.*

*Proof.* In this proof, the circumference $S$ is identified with the interval $[-\pi, \pi]$ and two regions $x_1$, $x_2$ are set as $x_1 = 0$, $x_2 = d \in (0, \pi]$.

First, for sufficiently small $\tau > 0$ or $\sigma > 1$, i.e., for sufficiently small $\alpha$, we consider the Taylor expansion for $J'$. Since $J'$ is composed of the matrices $X, Y, A, B$, we calculate the Taylor expansion for them. As $M = 2$, $X$ is given by

$$X = \begin{pmatrix} 1 & e^{-\alpha d} \\ e^{-\alpha d} & 1 \end{pmatrix},$$

thereby

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.17}$$

The matrix $Y$ is given by

$$Y_{11} = Y_{22} = d + \frac{1 - e^{\alpha d}}{\alpha(1 + e^{\alpha d})} + (\pi - d)\frac{1 + e^{2\alpha d}}{(1 + e^{\alpha d})^2},$$
$$Y_{12} = Y_{21} = \frac{e^{\alpha d} - 1}{\alpha(1 + e^{\alpha d})} + (\pi - d)\frac{2e^{\alpha d}}{(1 + e^{\alpha d})^2}$$

thereby

$$Y = \frac{\pi}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + O(\alpha^2). \tag{6.18}$$

In addition, $\overline{G}^{1-\sigma}$ is expanded as

$$\overline{G}^{1-\sigma} = 1 - \frac{d}{2}\alpha + O(\alpha^2). \tag{6.19}$$

Hence, $A$ is expanded as

$$A = A_1 + A_2\alpha + O(\alpha^2),$$

where

$$A_1 = \frac{\sigma+\mu-1}{2\sigma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$A_2 = -\frac{\mu d}{4\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $\|A\| < 1$, we have

$$(I-A)^{-1} = I + A + A^2 + A^3 + \cdots .$$

Note that $A_1A_2 = A_2A_1$ is the null matrix. It then follows that

$$A^n = A_1^n + O(\alpha^2), \quad \text{for } n = 2, 3, \cdots .$$

So,

$$\begin{aligned}(I-A)^{-1} &= I + A + A^2 + A^3 + \cdots \\ &= I + (A_1 + A_1^2 + A_1^3 + \cdots) + A_2\alpha + O(\alpha^2).\end{aligned}$$

Moreover,

$$\begin{aligned}A_1 + A_1^2 + A_1^3 + \cdots &= \sum_{n=1}^{\infty}\left(\frac{\sigma-1+\mu}{2\sigma}\right)^n 2^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{\mu+\sigma-1}{2(1-\mu)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.\end{aligned}$$

Hence, we obtain that

$$(I-A)^{-1} = I + \frac{\mu+\sigma-1}{2(1-\mu)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\mu d}{4\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.20}$$

Meanwhile, $B$ is expanded as

$$B = -\frac{1-\mu}{\sigma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\mu d}{2\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.21}$$

By (6.17), (6.18), (6.19), (6.20), and (6.21), it is observed from (6.15) that

$$\begin{aligned}\Omega = &\frac{1+\mu\sigma-\sigma}{\sigma-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &+ \left[-\frac{\mu d(2\sigma-1)}{2\sigma(\sigma-1)}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{\mu d(1+\mu\sigma-\sigma)}{2(\sigma-1)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right]\alpha + O(\alpha^2).\end{aligned}$$

Thus, $J'$ is given by

$$J' = \frac{\mu d(2\sigma-1)}{\sigma(\sigma-1)}\alpha + O(\alpha^2).$$

The first order term obviously takes positive value for $\alpha > 0$. Therefore, the symmetric stationary solution (6.3) is unstable for sufficiently small $\alpha > 0$.

Next, let us verify that when $\tau$ or $\sigma$ is sufficiently large, i.e., when $\alpha$ is sufficiently large, $J'$ is negative. From (6.3), (6.11) and (6.12), it follows that

$$\lim_{\alpha\to\infty} \overline{G}^{1-\sigma} = 1/2,$$
$$\lim_{\alpha\to\infty} X = I,$$
$$\lim_{\alpha\to\infty} Y = \pi I.$$

It follows from these results and (6.13), (6.14) that

$$\lim_{\alpha\to\infty} A = \frac{\sigma - 1 + \mu}{\sigma} I,$$
$$\lim_{\alpha\to\infty} (I - A)^{-1} = \frac{\sigma}{1-\mu} I,$$
$$\lim_{\alpha\to\infty} B = -\frac{2(1-\mu)}{\sigma} I.$$

Therefore, we obtain from (6.15) that

$$\lim_{\alpha\to\infty} \Omega = 2^{\frac{\mu}{1-\sigma}} \frac{-\sigma + 1 + \mu\sigma}{\sigma - 1} I.$$

Then,

$$\lim_{\alpha\to\infty} J' = 2^{\frac{\mu}{1-\sigma}} \frac{-\sigma + 1 + \mu\sigma}{\sigma - 1}.$$

Obviously, this value is negative under the assumption of no black hole $(\sigma - 1)/\sigma > \mu$. □

Figure 13 illustrates the value of $J'$ as a function of $\alpha$ obtained numerically. Here, the horizontal axis and the vertical axis are taken as $\alpha > 0$ and the value of $J'$, respectively. The red line indicates the case when $d = 1$; similarly, the green line $d = 2$, the blue line $d = \pi$. This shows that there exists a threshold $\alpha = \alpha^*$ where the sign of $J'$ changes. Then, smaller $\alpha^*$ means higher degree of stability. Since the longer $d$ results in smaller $\alpha^*$ according to this figure, it follows that the longer distance between two regions is, the higher degree of stability is.

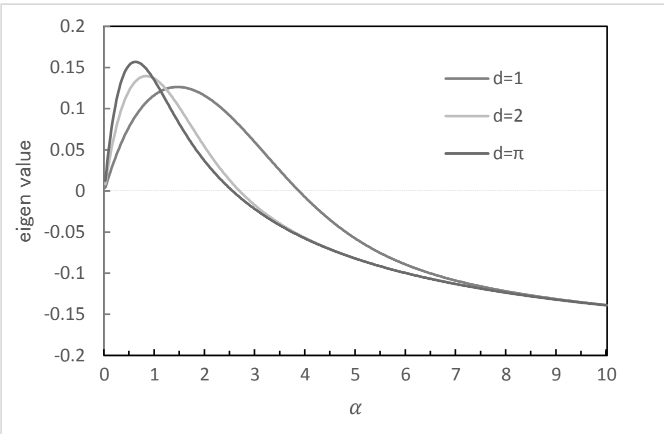



Fig. 13: Value of $J'$

**6.4 Case of $M = 3$.** In this subsection we consider the case of $M = 3$.

**Theorem 6.4.** *Assume the no black hole condition $(\sigma - 1)/\sigma > \mu$. If $\tau > 0$ or $\sigma > 1$ is sufficiently small, then the stationary solution given by (6.4) is unstable. On the other hand for sufficiently large $\tau$ or $\sigma$, the stationary solution given by (6.4) is stable.*

*Proof.* In this proof, $S$ is identified with the interval $[-\pi, \pi]$ and three manufacturing regions $x_1$, $x_2$ and $x_3$ are set as $x_1 = -\frac{2\pi}{3}$, $x_2 = 0$, $x_3 = \frac{2\pi}{3}$.

First, for sufficiently small $\tau > 0$ or $\sigma > 1$, i.e., for sufficiently small $\alpha$, we consider the Taylor expansion for the matrix $J$ as in the proof of Theorem 6.3. Since the matrix $J$ is composed of the matrices $X, Y, A, B$, the Taylor expansions for them should be calculated. The matrix $X$ is given by

$$X = \begin{pmatrix} 1 & e^{-\alpha\frac{2\pi}{3}} & e^{-\alpha\frac{2\pi}{3}} \\ e^{-\alpha\frac{2\pi}{3}} & 1 & e^{-\alpha\frac{2\pi}{3}} \\ e^{-\alpha\frac{2\pi}{3}} & e^{-\alpha\frac{2\pi}{3}} & 1 \end{pmatrix},$$

and its Taylor expansion is

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi}{3} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \alpha + O(\alpha^2). \tag{6.22}$$

In general, the function given by

$$F(\alpha) = \int_{-\pi}^{\pi} f(\alpha, x)dx$$

can be expanded as

$$\begin{aligned} F(\alpha) &= F(0) + F'(0)\alpha + O(\alpha^2) \\ &= \int_{-\pi}^{\pi} f(0, x)dx + \int_{-\pi}^{\pi} \frac{\partial f}{\partial \alpha}(0, x)dx \cdot \alpha + O(\alpha^2). \end{aligned}$$

We then set

$$f(\alpha, x) = \frac{e^{-2\alpha|x-x_1|}}{\left[e^{-\alpha|x-x_1|} + e^{-\alpha|x-x_2|} + e^{-\alpha|x-x_3|}\right]^2}.$$

It is easy to see that

$$f(0, x) = \frac{1}{9},$$

and

$$\frac{\partial f}{\partial \alpha}(0, x) = \frac{-4|x - x_1| + 2|x - x_2| + 2|x - x_3|}{27}.$$

Hence, $Y_{11}$ is expanded as

$$\begin{aligned} Y_{11} &= \frac{1}{9} \int_{-\pi}^{\pi} dy + \frac{1}{27} \int_{-\pi}^{\pi} \left[-4|y - x_1| + 2|y - x_2| + 2|y - x_3|\right] dx \cdot \alpha + O(\alpha^2) \\ &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

Other diagonal elements are also expanded as

$$\begin{aligned} Y_{22} &= \frac{2\pi}{9} + O(\alpha^2), \\ Y_{33} &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

As a non-diagonal element, let us consider $Y_{12}$. If we set

$$f(\alpha, x) = \frac{e^{-\alpha|x-x_1|}e^{-\alpha|x-x_2|}}{\left[e^{-\alpha|x-x_1|} + e^{-\alpha|x-x_2|} + e^{-\alpha|x-x_3|}\right]^2},$$

then it is easy to see that

$$f(0, x) = \frac{1}{9},$$

and

$$\frac{\partial f}{\partial \alpha}(0, x) = \frac{-|x-x_1| - |x-x_2| + 2|x-x_3|}{27}.$$

Hence, $Y_{12}$ is expanded as

$$\begin{aligned} Y_{12} &= \frac{1}{9}\int_{-\pi}^{\pi} dx + \frac{1}{27}\int_{-\pi}^{\pi} \left[-|x-x_1| - |x-x_2| + 2|x-x_3|\right] dx \cdot \alpha + O(\alpha^2) \\ &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

Other non-diagonal elements are also expanded as

$$\begin{aligned} Y_{13} &= \frac{2\pi}{9} + O(\alpha^2), \\ Y_{23} &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

After all, $Y$ is expanded as

$$Y = \frac{2\pi}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + O(\alpha^2). \tag{6.23}$$

In addition, $\overline{G}^{1-\sigma}$ is expanded as

$$\overline{G}^{1-\sigma} = 1 - \frac{4\pi}{9}\alpha + O(\alpha^2). \tag{6.24}$$

Hence, $A$ is expanded as

$$A = \frac{\sigma+\mu-1}{3\sigma}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi\mu}{27\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.25}$$

Since $\|A\| < 1$, we have

$$(I-A)^{-1} = I + A + A^2 + A^3 + \cdots.$$

Repeating the same argument as for the case of $M = 2$, we obtain that

$$\begin{aligned} (I-A)^{-1} = I &+ \frac{\sigma+\mu-1}{3(1-\mu)}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &- \frac{2\pi\mu}{27\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \end{aligned} \tag{6.26}$$

Moreover, the matrix $B$ is expanded as

$$B = -\frac{1-\mu}{\sigma}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi\mu}{9\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.27}$$

By (6.22), (6.23), (6.24), (6.26), (6.27), (6.15) provides that

$$\Omega = \frac{1+\mu\sigma-\sigma}{\sigma-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \left[\frac{2\pi\mu(2\sigma-1)}{9\sigma(\sigma-1)}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} - \frac{4\pi\mu(1+\mu\sigma-\sigma)}{9(\sigma-1)^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right]\alpha + O(\alpha^2).$$

By this and (6.16), it is easy to see that the eigenvalue of $J'$ is expanded as

$$\frac{1}{3}(\Omega_1 - \Omega_2) = \frac{2\pi(2\sigma-1)\mu}{3\sigma(\sigma-1)}\alpha + O(\alpha^2).$$

The first order term obviously takes positive value for $\alpha > 0$. Therefore, the symmetric stationary solution (6.3) is proved to be unstable for sufficiently small $\alpha > 0$.

Next, let us verify that when $\tau \to \infty$ or $\sigma \to \infty$, i.e., when $\alpha \to \infty$, the eigenvalue of $J'$ is negative. From (6.4), (6.11) and (6.12), it follows that

$$\lim_{\alpha\to\infty} \overline{G}^{1-\sigma} = 1/3,$$
$$\lim_{\alpha\to\infty} X = I,$$
$$\lim_{\alpha\to\infty} Y = \frac{2\pi}{3}I.$$

It follows from these and (6.13), (6.14) that

$$\lim_{\alpha\to\infty} A = \frac{\sigma-1+\mu}{\sigma}I,$$
$$\lim_{\alpha\to\infty} (I-A)^{-1} = \frac{\sigma}{1-\mu}I,$$
$$\lim_{\alpha\to\infty} B = -\frac{3(1-\mu)}{\sigma}I.$$

By these results, (6.15) provides that

$$\lim_{\alpha\to\infty} \Omega = 3^{\frac{\mu}{1-\sigma}}\frac{(-\sigma+1+\mu\sigma)}{\sigma-1}I.$$

Then, as $\alpha \to \infty$, the eigenvalue of $J'$ converges to the limit

$$3^{\frac{\mu}{1-\sigma}}\frac{(-\sigma+1+\mu\sigma)}{\sigma-1}$$

which is obviously negative under the of no black hole condition. □

In fact, Figure 14 illustrates a graph of the eigenvalue of $J'$ as a function of $\alpha$ obtained numerically. Here, the horizontal axis and the vertical axis are taken as $\alpha > 0$ and the eigenvalue of $J'$, respectively. It is observed that the sign of the eigenvalue changes at some threshold $\alpha = \alpha^*$.

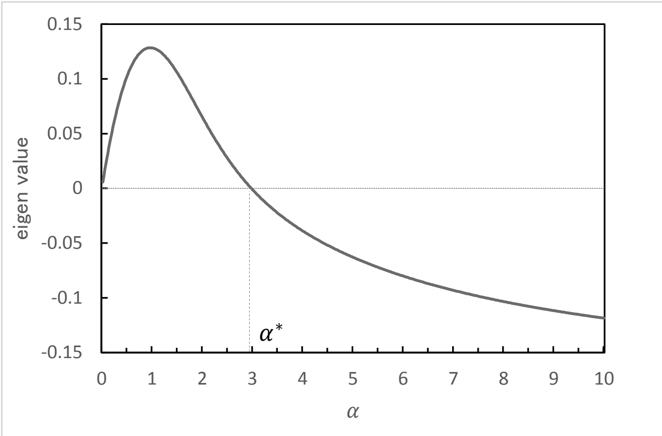



Fig. 14: Eigenvalue of $J'$ when $M = 3$

Even for non-symmetric stationary solutions, we can compute the eigenvalue of $J'$ and investigate its sign. These results show good agreement to the numerical computations performed in Subsection 5.2. But we will omit the details.

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# MOORE-PENROSE INVERSE AND OPERATOR MEAN

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*Dedicated to the memory of the late professor Takayuki Furuta*

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Abstract. Recently the geometric operator mean is extended to the multi-variable one; the Karcher mean. Including these multivarible means, we discuss a construction method by the Moore-Penrose inverse. The key concept is the orthogonality of operator means.

## 1 Introduction.

Let m be *an operator mean* in the sense of Kubo-Ando [12] which is defined by a positive operator monotone function $f_{\mathrm{m}}$ on the half interval $(0, \infty)$ with $f_{\mathrm{m}}(1) = 1$;

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}} f_{\mathrm{m}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$$

for positive invertible operators $A$ and $B$ on a Hilbert space. Thus every operator mean can be constructed by a numerical function $f_{\mathrm{m}}(x) = 1 \,\mathrm{m}\, x$ which is called the representing function of m. Among common properties for operator means, we pay attention to the *orthogonality*:

$$(A_1 \oplus A_2) \,\mathrm{m}\, (B_1 \oplus B_2) = (A_1 \,\mathrm{m}\, B_1) \oplus (A_2 \,\mathrm{m}\, B_2)$$

and the *transformer inequality*:

$$T^*(A \,\mathrm{m}\, B)T \leq (T^*AT) \,\mathrm{m}\, (T^*BT).$$

Recall that the Karcher mean $X = \mathsf{G}(\omega_j; A_j)$ for invertible $A_j \geq 0$ with a weight $\{\omega_j\}$ is defined as a unique solution of the *Karcher equation* [11, 13, 14]:

$$\sum_j \omega_j S(X|A_j) = \sum_j \omega_j X^{\frac{1}{2}} \log \left( X^{-\frac{1}{2}} A_j X^{-\frac{1}{2}} \right) X^{\frac{1}{2}} = 0.$$

We extend it to non-invertible case in [11], which is an extension of the weighted geometric mean. Moreover in [11], we extended such multi-variable *operator mean* $\mathrm{M}(A_j) = \mathrm{M}(\omega_j; A_1, ...., A_n)$ including the Karcher mean: Define an ($n$-variable) *general operator mean* $\mathrm{M}(\omega_j; A_j)$ as an $n$-ary operation on positive invertible operators on $\mathcal{H}$ satisfying the following properties where each weight $\omega_j$ is assumed to be positive here:

(M1) **transformer equality**: $T^* \mathrm{M}(\omega_j; A_j)T = \mathrm{M}(\omega_j; T^*A_jT)$ for all invertible $T$.

(M1') **homogeneity**: $\mathrm{M}(\omega_j; tA_j) = t\,\mathrm{M}(\omega_j; A_j)$ for $t > 0$.

(M2) **normalization**: $\mathrm{M}(\omega_j; A) = A$.

(M3) **monotonicity**: $A_j \leq B_j$ implies $\mathrm{M}(\omega_j; A_j) \leq \mathrm{M}(\omega_j; B_j)$.

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(M4) **sub-additivity**: $\mathrm{M}(\omega_j; A_j + B_j) \geq \mathrm{M}(\omega_j; A_j) + \mathrm{M}(\omega_j; B_j)$.

(M5) **adjoint sub-additivity**: $\mathrm{M}(\omega_j; A_j : B_j) \leq \mathrm{M}(\omega_j; A_j) : \mathrm{M}(\omega_j; B_j)$.

(M6) **orthogonality**: $\mathrm{M}(\omega_j; \bigoplus_m A_j^{(m)}) = \bigoplus_m \mathrm{M}(\omega_j; A_j^{(m)})$.

Here : stands for the parallel sum defined by

$$A : B = \left(A^{-1} + B^{-1}\right)^{-1}.$$

In addition, we can define

$$\mathrm{M}(\omega_j; A_j) = \operatorname*{s\text{-}lim}_{\varepsilon \to 0} \mathrm{M}(\omega_j; (A_j + \varepsilon))$$

for (non-invertible) positive operators $A_j$ where the above properties preserve, which includes our extended Karcher mean. Also for $t \in [0, 1]$, note that

(M7) **joint concavity:** $\mathrm{M}(\omega_j; (1-t)A_j + tB_j) \leq (1-t)\,\mathrm{M}(\omega_j; A_j) + t\,\mathrm{M}(\omega_j; B_j)$

follows from the sub-additivity and homogeneity. Here we pay attention to the orthogonality for operator means as in the below.

On the other hand, for the parallel sum (the half of the harmonic mean), rephrasing them into the harmonic mean, we have

$$A \,\mathrm{h}\, B = A \left(\frac{A+B}{2}\right)^{\dagger} B$$

if $A + B$ has the generalized inverse [1]. Incidentally the Moore-Penrose generalized inverse $^\dagger$ for operators was discussed in [9, 15]: It is known that if $\operatorname{ran} X$ is closed, then $\operatorname{ran} X^*$, $\operatorname{ran} XX^*$ and $\operatorname{ran} X^*X$ are also closed, and $(X^*X)^\dagger = \left(X^*X|_{\operatorname{ran} X^*}\right)^{-1} \oplus 0_{(\operatorname{ran} X^*)^\perp}$ and $X^\dagger = (X^*X)^\dagger X^* = X^*(XX^*)^\dagger$.

In this note, we observe operator means from the viewpoint of the generalized inverse, which includes our extended version of the Karcher mean. We discuss the constructing formulae for operator means using the Moore-Penrose inverses if they exist:

$$A^{\frac{1}{2}}(I \,\mathrm{m}\, A^{\dagger\frac{1}{2}} B A^{\dagger\frac{1}{2}}) A^{\frac{1}{2}} \qquad \text{or} \qquad B^{\frac{1}{2}}(B^{\dagger\frac{1}{2}} A B^{\dagger\frac{1}{2}} \,\mathrm{m}\, I) B^{\frac{1}{2}}.$$

Our equality condition [6] for the transformer iequality shows that it represents the operator mean $A \,\mathrm{m}\, B$ if $\ker A \subset \ker B$ or $\ker A \supset \ker B$ respectively. We also show that they are not less than the original one if the kernel of the mean $A \,\mathrm{m}\, B$ includes those for $A$ and $B$.

**2 Transformer equality.** In [6], we gave an equality condition for transformer inequality for certain means:

**Theorem F.** *If* $\ker T^* \subset \ker A \cap \ker B$, *then* $T^*(A \,\mathrm{m}\, B)T = (T^*AT) \,\mathrm{m}\, (T^*BT)$ *for an operator mean* $\mathrm{m}$.

This assures the Izumino construction of operator means: Let $R = (A + B)^{\frac{1}{2}}$, then, there exist the derivatives $D$ and $E$ with $A^{\frac{1}{2}} = RD$ and $B^{\frac{1}{2}} = RE$ by the range inclusion theorem in [3, 4]. So we have $D^*D + E^*E = I_{\overline{\operatorname{ran} R}}$ and an operator mean is reduced into the commutative case [6]:

$$A \,\mathrm{m}\, B = R(D^*D \,\mathrm{m}\, E^*E)R,$$

which is a space-free version of the Pusz-Woronowics means [16, 17].

But the original proof of the above was based on the integral representation of operator means, so that we cannot extend the equality in Theorem F to multi-variable means. Under the closedness of the ranges for operators, we show the equality for our extended (multi-variable) operator means including the Karcher operator mean:

**Theorem 1.** *Let* $\mathrm{M}(A_j) = \mathrm{M}(\omega_j; A_1, .., A_n)$ *be an operator mean (satisfying the orthogonality). If an operator* $T$ *on* $H$ *satisfies* $\ker T^* \subset \bigcap_j \ker A_j$ *and* $\operatorname{ran} T$ *is closed, then the transformer equality holds:*

$$T^* \mathrm{M}(A_j)T = \mathrm{M}(T^* A_j T).$$

*Proof.* Note that $\operatorname{ran} T^*$ is also closed. Recall that $P = TT^\dagger$ and $Q = T^\dagger T$ are projections onto $\operatorname{ran} T$ and $\operatorname{ran} T^*$ respectively, see e.g. [9, 15]. By the assumption $\operatorname{ran} T^\perp = \ker T^* \subset \ker A_j$, we have $PA_jP = A_j$ for all $j$. Also $QT^*A_jTQ = T^*A_jT$ implies $QM(T^*A_jT)Q = M(T^*A_jT)$ for all $j$ by the orthogonality. Then we have

$$\begin{aligned} T^* \mathrm{M}(A_j)T \le \mathrm{M}(T^*A_jT) &= Q\,\mathrm{M}(TA_jT)Q = T^*T^{\dagger *}\,\mathrm{M}(T^*A_jT)T^\dagger T \\ &\le T^*\,\mathrm{M}\big(T^{\dagger *}T^*A_jTT^\dagger\big)T = T^*\,\mathrm{M}(PA_jP)T = T^*\,\mathrm{M}(A_j)T, \end{aligned}$$

which shows the required equality. □

*Remark.* The assumption $\ker T^* \subset \bigcap_j \ker A_j$ in the above is equivalent to $\operatorname{ran} T \supset \bigvee_j \operatorname{ran} A_j$ under the closedness of operators.

**Corollary 2.** *Let* $\mathrm{m}$ *be an* (2*-variable*) *operator mean. If* $\ker A \subset \ker B$ *and* $\operatorname{ran} A$ *is closed, then*

$$A\,\mathrm{m}\,B = A^{\frac{1}{2}}(I\,\mathrm{m}\,A^{\dagger\frac{1}{2}}BA^{\dagger\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}f_{\mathrm{m}}(A^{\dagger\frac{1}{2}}BA^{\dagger\frac{1}{2}})A^{\frac{1}{2}}.$$

*Remark.* Contrastively we have

$$A\,\mathrm{m}\,B = B^{\frac{1}{2}}(B^{\dagger\frac{1}{2}}AB^{\dagger\frac{1}{2}}\,\mathrm{m}\,I)B^{\frac{1}{2}}$$

if $\ker B \subset \ker A$ and $\operatorname{ran} B$ is closed.

## 3 Means satisfying the kernel condition.

Initiated by [5], we observe the kernel conditions for operator means, see also [7, 8]:

$$\ker A\,\mathrm{m}\,B \supset \ker A \vee \ker B \tag{1}$$

if and only if $1\,\mathrm{m}\,0 = 0\,\mathrm{m}\,1 = 0$. The geometric or harmonic mean satisfies this, while the arithmetic mean does not. In [11], we showed $\ker A\#B = \ker A \vee \ker B$. Moreover, based on this property, we introduced the Karcher mean $X = \mathsf{G}(\omega_j; A_j)$ for non-invertible positive operators $A_j$ under this kernel condition: $\ker X = \vee_j \ker A_j$.

For invertible operators, we have two expressions:

$$A\,\mathrm{m}\,B = A^{\frac{1}{2}}(I\,\mathrm{m}\,A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\,\mathrm{m}\,I)B^{\frac{1}{2}}. \tag{2}$$

Then we discuss the means where the inverses in (2) are exchanged into the Moore-Penrose inverse:

**Theorem 3.** *Let* $\mathrm{m}$ *be an operator mean satisfying the above kernel condition* (1)*. If* $\operatorname{ran} A$ (*resp.* $\operatorname{ran} B$) *is closed, then*

$$A\,\mathrm{m}\,B \le A^{\frac{1}{2}}(I\,\mathrm{m}\,A^{\dagger\frac{1}{2}}BA^{\dagger\frac{1}{2}})A^{\frac{1}{2}} \qquad \left(\textit{resp.}\ \le B^{\frac{1}{2}}(B^{\dagger\frac{1}{2}}AB^{\dagger\frac{1}{2}}\,\mathrm{m}\,I)B^{\frac{1}{2}}\right).$$

*Proof.* Let $P$ be the projections onto $(\ker A)^\perp$, that is, $P = A^\dagger A = A^{\dagger\frac{1}{2}}A^{\frac{1}{2}}$. The kernel condition shows $\operatorname{ran} A \operatorname{m} B \subset \operatorname{ran} P$ and hence Theorem 1 implies

$$\begin{aligned} A \operatorname{m} B &= P(A \operatorname{m} B)P \\ &\leq (PAP) \operatorname{m} (PBP) = A \operatorname{m} (A^{\frac{1}{2}}A^{\dagger\frac{1}{2}}BA^{\dagger\frac{1}{2}}A^{\frac{1}{2}}) \\ &= A^{\frac{1}{2}}\left(P \operatorname{m} (A^{\dagger\frac{1}{2}}AA^{\dagger\frac{1}{2}})\right)A^{\frac{1}{2}} \leq A^{\frac{1}{2}}\left(I \operatorname{m} (A^{\dagger\frac{1}{2}}BA^{\dagger\frac{1}{2}})\right)A^{\frac{1}{2}}. \end{aligned}$$

Similarly we have the other case. □

*Remark.* The kernel condition (1) is necessary in the above theorem. In fact, the arithmetic mean $A\nabla B = (A+B)/2$ does not satisfy (1). Let $P(= B^{\frac{1}{2}})$ be a projection that does not commute with $A$. Then $PAP \not\geq A$, so that

$$P(P^\dagger AP^\dagger \nabla I)P = PAP\nabla P = \frac{PAP + P}{2} \not\geq \frac{A+P}{2} = A\nabla B.$$

The difference in the inequality in the above theorem is somewhat larger than we expected as in the following examples:

**Example.** For $0 < a < 1$, we define a positive-definite matrix $A$ and a projection $P$ : Put

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}^2 = \begin{pmatrix} 1+a^2 & 2a \\ 2a & 1+a^2 \end{pmatrix}.$$

Then we have $A^{-\frac{1}{2}} = A^{\dagger\frac{1}{2}} = \dfrac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$ and

$$P^{\frac{1}{2}}\sqrt{P^{\dagger\frac{1}{2}}AP^{\dagger\frac{1}{2}}}P^{\frac{1}{2}} = P\sqrt{PAP}P = \sqrt{1+a^2}P\,(\geq P).$$

On the other hand,

$$A^{\dagger\frac{1}{2}}PA^{\dagger\frac{1}{2}} = \frac{1}{(1-a^2)^2}\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix} = \frac{1+a^2}{(1-a^2)^2}Q,$$

where $Q = \dfrac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix}$ is a rank 1 projection. Hence we have

$$A\#P = P\#A = A^{\frac{1}{2}}\sqrt{A^{\dagger\frac{1}{2}}PA^{\dagger\frac{1}{2}}}A^{\frac{1}{2}} = \frac{\sqrt{1+a^2}}{1-a^2}A^{\frac{1}{2}}QA^{\frac{1}{2}} = \frac{1-a^2}{\sqrt{1+a^2}}P\ (\leq P).$$

These differences are under the kernel inclusion as in Corollary 2.

To see a general case, we put $B = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}^2$ for $0 < b < 1$. For $X = P \oplus B$, $Y = A \oplus P$, the orthogonality shows

$$X\#Y = (P\#A) \oplus (B\#P) = \frac{1-a^2}{\sqrt{1+a^2}}P \oplus \frac{1-b^2}{\sqrt{1+b^2}}P.$$

Thus we have

$$X\#Y \leq P \oplus \frac{1-b^2}{\sqrt{1+b^2}}P \equiv M_1 \quad \text{and} \quad X\#Y \leq \frac{1-a^2}{\sqrt{1+a^2}}P \oplus P \equiv M_2,$$

while

$$X^{\frac{1}{2}}\sqrt{X^{\dagger\frac{1}{2}}YX^{\dagger\frac{1}{2}}}X^{\frac{1}{2}} = \sqrt{1+a^2}P \oplus \frac{1-b^2}{\sqrt{1+b^2}}P \geq M_1 \quad \text{and}$$
$$Y^{\frac{1}{2}}\sqrt{Y^{\dagger\frac{1}{2}}XY^{\dagger\frac{1}{2}}}Y^{\frac{1}{2}} = \frac{1-a^2}{\sqrt{1+a^2}}P \oplus \sqrt{1+b^2}P \geq M_2.$$

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# POSITIVE DEFINITE SEQUENCES WITH CONSTANT MODULUS

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Abstract. Let $a_0, a_1, ..., a_N$ be complex numbers. We consider the Toeplitz matrix $T_N$, where the $(i,j)$-th component is $a_{i-j}$ if $i \geq j$ and $\overline{a_{j-i}}$ if $i < j$. If $T_N$ is positive and $|a_0| = |a_1| \neq 0$, then $a_2, a_3, ..., a_N$ can be represented in terms of $a_0$ and $a_1$ and there exists a unique positive definite sequence $f$ such that $f(i) = a_i$ for any $i = 0, 1, 2, ..., N$. In particular, it holds $|f(n)| = |a_0|$ for any $n$. We also provides some applications related to this fact.

## 1 Introduction

Let $\mathbb{N} = \{0, 1, 2, ...\}$ and $f$ is a complex-valued function on $\mathbb{N}$. An $n \times n$ matrix $A = (a_{ij})$ with complex entries is said to be positive and it is denoted by $A \geq 0$ if

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j a_{ij} \geq 0 \text{ for all } \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}.$$

It is well-known that $A \geq 0$ if and only if there exists a $k \times n$ matrix $B$ in which $A = B^*B$ for some $k \in \mathbb{N} \setminus \{0\}$. We call that $f$ is a positive definite sequence if, for any positive integer $N$, the following $(N+1)\times(N+1)$ Toeplitz matrix

$$T_N = \begin{pmatrix} f(0) & \overline{f(1)} & \cdots & \overline{f(N)} \\ f(1) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(1)} \\ f(N) & \cdots & f(1) & f(0) \end{pmatrix}$$

is positive, where the $(i,j)$-th component of $T_N$ is $f(i-j)$ if $i \geq j$ and $\overline{f(j-i)}$ if $i < j$. We remark that the positivity of $T_N$ implies $|f(i)| \leq f(0)$ for any $i = 1, 2, ..., N$.

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For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, the function $f$ given by $f(n) = e^{n\theta\sqrt{-1}}$ is a positive definite sequence. In fact, for any positive integer $N$, $T_N$ is positive since

$$T_N = \begin{pmatrix} 1 & e^{-\theta\sqrt{-1}} & \cdots & e^{-N\theta\sqrt{-1}} \\ e^{\theta\sqrt{-1}} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-\theta\sqrt{-1}} \\ e^{N\theta\sqrt{-1}} & \cdots & e^{\theta\sqrt{-1}} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ e^{\theta\sqrt{-1}} \\ \vdots \\ e^{N\theta\sqrt{-1}} \end{pmatrix} \begin{pmatrix} 1 & e^{-\theta\sqrt{-1}} & \cdots & e^{-N\theta\sqrt{-1}} \end{pmatrix} \geq 0.$$

This function is a typical example of positive definite sequence.

Our result is as follows:

**Theorem 1.** *Let $N \geq 1$. If $|a_0| = |a_1| \neq 0$ and*

$$T = \begin{pmatrix} a_0 & \overline{a_1} & \cdots & \overline{a_N} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{a_1} \\ a_N & \cdots & a_1 & a_0 \end{pmatrix} \geq 0,$$

*then there exists a unique positive definite sequence $f$ such that*

$$f(i) = a_i \qquad \textit{for any } i = 0, 1, ..., N.$$

*Moreover, it holds*

$$f(n) = f(0)\left(\frac{f(1)}{f(0)}\right)^n \ (\textit{in particular, } |f(n)| = f(0)) \textit{ for any } n \in \mathbb{N} \setminus \{0\}.$$

# 2 Proof of Theorem and Application

Let $T = \begin{pmatrix} 1 & \overline{\alpha} & \overline{\gamma} \\ \alpha & 1 & \overline{\beta} \\ \gamma & \beta & 1 \end{pmatrix}$ where $\alpha, \beta, \gamma$ are complex numbers and $|\alpha| = 1$. The following fact is known and is used in this paper.

(†) $\qquad T \geq 0$ if and only if $|\beta| \leq 1$ and $\gamma = \alpha\beta$.

The statement (†) for operators had been considered in [6], and we extend as follows:

**Lemma 2.** *Let $u, v, w$ be bounded linear operators on a Hilbert space $\mathcal{H}$ and $u$ isometric (that is, $u^*u = 1$). Then*

$$T = \begin{pmatrix} 1 & u^* & w^* \\ u & 1 & v^* \\ w & v & 1 \end{pmatrix} \geq 0 \textit{ if and only if } \|v\| \leq 1 \textit{ and } w = vu.$$

*Proof.* Assume $\|v\| \leq 1$ and $w = vu$. Since $(1 - uu^*)^2 = 1 - uu^*$, we have

$$T = \begin{pmatrix} 1 \\ u \\ vu \end{pmatrix} \begin{pmatrix} 1 & u^* & u^*v^* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 - uu^* \\ v(1 - uu^*) \end{pmatrix} \begin{pmatrix} 0 & 1 - uu^* & (1 - uu^*)v^* \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - vv^* \end{pmatrix} \geq 0.$$

Conversely, Assume $T \geq 0$. Since $\begin{pmatrix} 1 & v^* \\ v & 1 \end{pmatrix}$ is positive, we have $\|v\| \leq 1$. For any vectors $x, y \in \mathcal{H}$, we have

$$0 \leq \left\langle T \begin{pmatrix} x \\ -ux \\ y \end{pmatrix}, \begin{pmatrix} x \\ -ux \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w^*y \\ v^*y \\ wx - vux + y \end{pmatrix}, \begin{pmatrix} x \\ -ux \\ y \end{pmatrix} \right\rangle$$
$$= \langle w^*y, x\rangle + \langle v^*y, -ux\rangle + \langle wx - vux + y, y\rangle$$
$$= \langle y, wx - vux\rangle + \langle wx - vux + y, y\rangle.$$

Set $y = -(wx - vux)$, then $-\|wx - vux\|^2 \geq 0$. This implies $w = vu$. □

**Proof of Theorem 1.** By the assumption, we have $a_0 > 0$. We want to show that $a_i = a_0(a_i/a_0)^i$ for any $i = 1, 2, ..., N$.

For each $i = 2, 3, ..., N$, the matrix

$$\begin{pmatrix} 1 & \overline{(a_1/a_0)} & \overline{(a_i/a_0)} \\ a_1/a_0 & 1 & \overline{(a_{i-1}/a_0)} \\ a_i/a_0 & a_{i-1}/a_0 & 1 \end{pmatrix} = \frac{1}{a_0} E_{3,i} T E_{3,i}^* \geq 0,$$

where $E_{3,i}$ is a $3 \times (N + 1)$ matrix and its $(a, b)$-th component is

$$e_{a,b} = \begin{cases} 1 & ; \text{ if } (a, b) = (1, 1), (2, 2), (3, i + 1) \\ 0 & ; \text{ otherwise} \end{cases}.$$

Since $|a_1/a_0| = 1$, we have $a_i/a_0 = (a_1/a_0)(a_{i-1}a_0)$ for any $i = 2, 3, ..., N$ by (†). This implies $a_i = a_0(a_1/a_0)^i$ for all $i = 1, 2, ..., N$. By setting $f(n) =$

$a_0(a_1/a_0)^n$ for any $n \in \mathbb{N}$ and continuing the above argument to the larger number than $N$, then $f$ is a positive definite sequence with $f(i) = a_i$ for any $i = 0, 1, ..., N$ and

$$f(n) = f(0)\left(\frac{f(1)}{f(0)}\right)^n \quad \text{for any } n \in \mathbb{N} \setminus \{0\}.$$

We assume that there exists another positive definite sequence $g$ with $g(i) = a_i$ for any $i = 0, 1, ..., N$. For $M > N$, we then have

$$\begin{pmatrix} g(0) & \overline{g(1)} & \cdots & \overline{g(M)} \\ g(1) & g(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{g(1)} \\ g(M) & \cdots & g(1) & g(0) \end{pmatrix} \geq 0.$$

Since $f(0) = g(0), f(1) = g(1)$ and $|g(0)| = |g(1)|$, we have

$$g(M) = g(0)\left(\frac{g(1)}{g(0)}\right)^M = f(0)\left(\frac{f(1)}{f(0)}\right)^M = f(M)$$

by the above argument. So, $f = g$. □

**Corollary 3.** *Let $f$ be a positive definite sequence. If there exists a positive integer $K$ in which $f(0) = |f(K)|$, then*

$$f(nK) = f(0)\left(\frac{f(K)}{f(0)}\right)^n \quad \textit{for any } n = 1, 2, \ldots .$$

*Proof.* We may assume that $f(0) > 0$. Define the $(n+1) \times (nK+1)$ matrix $F_{n,K}$ whose $(a, b)$-th component is

$$f_{a,b} = \begin{cases} 1 & ; \text{ if } (a,b) = (i+1, iK+1) \text{ for } i = 0, 1, ..., n \\ 0 & ; \text{ otherwise} \end{cases}.$$

Then we have

$$F_{n,K} \begin{pmatrix} f(0) & \overline{f(1)} & \cdots & \overline{f(nK)} \\ f(1) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(1)} \\ f(nK) & \cdots & f(1) & f(0) \end{pmatrix} F_{n,K}^* \geq 0$$

and this matrix is equal to

$$\begin{pmatrix} f(0) & \overline{f(K)} & \cdots & \overline{f(nK)} \\ f(K) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(K)} \\ f(nK) & \cdots & f(K) & f(0) \end{pmatrix}.$$

Hence, by Theorem 1 we have

$$f(nK) = f(0)\left(\frac{f(K)}{f(0)}\right)^n.$$

□

In the setting of Corollary 3 we have $f(0) = |f(nK)|$ for all $n \in \mathbb{N}$. In general, the sequence $\{|f(n)|\}$ is not necessarily constant. For instance, consider the function $f(n) = e^{\frac{2}{3}\pi n\sqrt{-1}}$. It is clear that $f$ and $\overline{f}$ are positive definite sequences. Then, so is

$$g(n) = \frac{f(n) + \overline{f(n)}}{2} = \cos\left(\frac{2n\pi}{3}\right),$$

here we have

$$g(n) = \begin{cases} 1 & ;\ \text{if } n = 0, 3, 6, 9, ... \\ -\frac{1}{2} & ;\ \text{if } n = 1, 2, 4, 5, 7, 8, ... \end{cases}.$$

Let $G$ be a group and $e$ the unit of $G$. We say a complex-valued function $\varphi$ on $G$ is positive definite if for any positive integer $N$ and for any $g_1, g_2, ..., g_N \in G$, the following $N \times N$ matrix

$$\begin{pmatrix} \varphi(g_1^{-1}g_1) & \varphi(g_2^{-1}g_1) & \cdots & \varphi(g_N^{-1}g_1) \\ \varphi(g_1^{-1}g_2) & \varphi(g_2^{-1}g_2) & \cdots & \varphi(g_N^{-1}g_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(g_1^{-1}g_N) & \varphi(g_2^{-1}g_N) & \cdots & \varphi(g_N^{-1}g_N) \end{pmatrix} \geq 0.$$

By definition, $\varphi(e) \geq 0$, $\varphi(g^{-1}) = \overline{\varphi(g)}$, and $|\varphi(g)| \leq \varphi(e)$ and for any $g \in G$.

**Corollary 4.** *Let $\varphi$ be a positive definite function on $G$ with $\varphi(e) \neq 0$ and $K$ a subgroup of $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then*

$$H = \left\{ g \in G \,\middle|\, \frac{\varphi(g)}{\varphi(e)} \in K \right\}$$

*is a subgroup of $G$ and the function $\frac{1}{\varphi(e)}\varphi$ is multiplicative on $H$.*

*Proof.* It is obvious that $e \in H$ and if $g \in H$, then $g^{-1} \in H$. Given $g, h \in H$, then by the assumption we have that the following matrix

$$\begin{pmatrix} \varphi(e) & \varphi(g^{-1}) & \varphi((gh)^{-1}) \\ \varphi(g) & \varphi(g^{-1}g) & \varphi((gh)^{-1}g) \\ \varphi(gh) & \varphi(g^{-1}(gh)) & \varphi((gh)^{-1}(gh)) \end{pmatrix} = \begin{pmatrix} \varphi(e) & \overline{\varphi(g)} & \overline{\varphi(gh)} \\ \varphi(g) & \varphi(e) & \overline{\varphi(h)} \\ \varphi(gh) & \varphi(h) & \varphi(e) \end{pmatrix} \geq 0.$$

By (†), we have

$$\frac{\varphi(gh)}{\varphi(e)} = \frac{\varphi(g)}{\varphi(e)} \frac{\varphi(h)}{\varphi(e)}.$$

It follows that $\varphi(gh)/\varphi(e) \in K$. That is, $gh \in H$. □

Let $\varphi$ be a positive definite sequence, that is, a positive definite function on $\mathbb{Z}$. By Bochner's theorem (or Herglotz's theorem [5]), there exists a positive finite measure $\mu$ on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ which is identified by $[0, 1)(\cong \mathbb{R}/\mathbb{Z})$ such that

$$\varphi(n) = \int_0^1 e^{2\pi\sqrt{-1}nx} d\mu(x) \quad \text{for all } n \in \mathbb{Z}.$$

It is known that

$$\mu(\{0\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n).$$

To see this, it suffices to show that

$$\mu(\{0\}) = 0 \Rightarrow \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) = 0$$

by considering $\mu - \mu(\{0\})\delta_0$ instead of $\mu$, where $\delta_0$ is a Dirac measure at 0. Since $|\sin x| \leq |x|$ and $\frac{2x}{\pi} \leq \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned} \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) \right| &= \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \int_0^1 e^{2\pi\sqrt{-1}nx} d\mu(x) \right| \\ &= \left| \int_0^1 \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} d\mu(x) \right| \\ &\leq \int_{-\delta}^{\delta} \left| \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} \right| d\mu(x) \\ &\quad + \int_{\delta}^{1-\delta} \left| \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} \right| d\mu(x) \\ &\leq \frac{\pi}{2}\mu((-\delta, \delta)) + \frac{1}{(2N+1)\sin \pi\delta}\mu(\mathbb{T}). \end{aligned}$$

for any $\delta \in \left(0, \frac{1}{2}\right)$. Hence, $\limsup_{N\to\infty} \left|\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n)\right| \leq \frac{\pi}{2}\mu((-\delta,\delta))$. Since $\delta$ is arbitrary and $\mu(\{0\}) = 0$, then we have $\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n) = 0$.

**Proposition 5.** *Let $\varphi$ be a positive definite function on $\mathbb{Z}$. If $\lim_{n\to\infty}\varphi(n) = \varphi(0)$, then $\varphi(n) = \varphi(0)$ for all $n \in \mathbb{Z}$.*

*Proof.* Let $\varphi(n) = \int_0^1 e^{2\pi\sqrt{-1}nx}d\mu(x)$ $(n \in \mathbb{Z})$. Then, $\varphi(0) = \mu(\mathbb{T})$. Also, we have $\mu(\{0\}) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n) = \varphi(0)$. This means that $\mu$ is a non-negative scalar multiple of Dirac measure at 0 and so we have $\varphi(n) = \varphi(0)$ for all $n \in \mathbb{Z}$. □

**Corollary 6.** *Let $\varphi$ be a positive definite function on a group $G$ and $G$ is generated by $\{g_i \mid i \in I\}$. If*

$$\lim_{n\to\infty}\varphi(g_i^n) = \varphi(e) \quad \textit{for all } i \in I,$$

*then $\varphi(g) = \varphi(e)$ for all $g \in G$.*

*Proof.* We may assume that $\varphi(e) \neq 0$. By assumption and since $\lim_{n\to\infty}\varphi(g_i^n) = \varphi(e)$, we have $\varphi(g_i^n) = \varphi(e)$ for all $n$ by Proposition 5. In particular $\varphi(g_i) = \varphi(e)$ for all $i \in I$. Set

$$H = \left\{ g \in G \;\middle|\; \frac{\varphi(g)}{\varphi(e)} \in \{1\} \right\}.$$

Using Corollary 4, we conclude that $H$ is a subgroup of $G$. Since $G$ is generated by $\{g_i \mid i \in I\}$ and $g_i \in H$ for any $i \in I$, we have $\varphi(g) = \varphi(e)$ for all $g \in G$. □

**Remark.** Let $\varphi$ be a positive definite function on the additive group $\mathbb{R}$. We assume that the sequence $\{\varphi(nx)\}_{n=1}^{\infty}$ converges to $\varphi(0)$ for any $x \in \mathbb{R}$. If $\varphi$ is continuous, then there exists a finite positive measure $\mu$ on $\mathbb{R}$ such that $\varphi(x) = \int_{\mathbb{R}} e^{\sqrt{-1}tx}d\mu(t)$ $(x \in \mathbb{R})$ by Bochner's theorem. We can prove $\varphi(x) = \varphi(0)$ by using the fact

$$\mu(\{0\}) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x)d\mu(x)$$

(see [3]:Appendices A.4). Without the assumption of the continuity of $\varphi$, we can also have $\varphi(x) = \varphi(0)$ by Corollary 6.

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# EXPONENTIAL ATTRACTORS FOR SELF-REGULATING HOMEOSTASIS MODEL ON A SPHERE

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Abstract. This paper is devoted to studying a complete two-dimensional Daisyworld model on a sphere. The Daisyworld model which has been originally introduced by Andrew Watson and James Lovelock (1983) describes the process of planetary self-regulating homeostasis by a biota and its environment. After formulating our two-dimensional model, we construct global solutions, dynamical systems and exponential attractors. We also show some numerical results suggesting pattern formation of stripe segregation.

**1 Introduction** We are concerned with the initial-boundary value problem for a reaction-diffusion system

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + [(1-u-v)\Phi(u,v,w) - f]\,u & \text{in} \quad S\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = d\Delta v + [(1-u-v)\Psi(u,v,w) - f]\,v & \text{in} \quad S\times(0,\infty), \\ \dfrac{\partial w}{\partial t} = D\Delta w + [1-g(u,v)]R(\omega) - \sigma w^4 & \text{in} \quad S\times(0,\infty), \\ u(\omega,0)=u_0(\omega),\;\; v(\omega,0)=v_0(\omega),\;\; w(\omega,0)=w_0(\omega) & \text{in} \quad S, \end{cases}$$

on a sphere $S \subset \mathbb{R}^3$. This is a tutorial mathematical model originally introduced by Watson-Lovelock [20] in order to investigate how the mechanism of global homeostasis works in Daisyworld which was ideally set as a biological and climatological system. Daisyworld is an imaginary planet that has only two types of daisies with contrasting brightness. They are expressly referred to as white and black daisy. On the planet, there are enough water and nutrients to sustain daisies, and thus the temperature is an only factor affecting the growth of daisies (for the details, see the review of Wood-Ackland-Dyke-Williams-Lenton [22]).

Unknown functions $u = u(\omega,t)$ and $v = v(\omega,t)$ denote a coverage rate of white and black daisy, respectively, at position $\omega \in S$ and time $t$. Therefore, $u \geq 0$, $v \geq 0$ and $u+v \leq 1$ at any $(\omega,t)$, and $1-u-v$ denotes a rate of uncovered ground. The third unknown function $w = w(\omega,t)$ denotes a surface temperature. We assume that $u$ and $v$ satisfy a diffusion equation on $S$ with diffusion rate $d > 0$. It is the same for $w$ with diffusion rate $D > 0$. So, $\Delta$ denotes a Laplace operator on the sphere $S$. It is natural to assume that $0 < d < D$. The function $g(u,v)$ stands for an averaged albedo of the surface that is given at each point as a function of $u$, $v$ in the form

$$\begin{aligned} g(u,v) &= a_w u + a_b v + a_g(1-u-v) \\ &= (a_w - a_g)u + (a_b - a_g)v + a_g, \end{aligned}$$

where $a_w$, $a_b$ and $a_g$ denote the proper albedo of white daisy, black daisy and bare ground, respectively. In general, we have $0 < a_b < a_g < a_w < 1$; as a consequence, it is always the case that

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$a_b \le g(u,v) \le a_w$. Furthermore, $\Phi(u,v,w)$ and $\Psi(u,v,w)$ denote a growth rate of white and black daisy, respectively. According to [20], they are set as

$$\Phi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_w])^2\}_+,$$
$$\Psi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_b])^2\}_+.$$

Here, $\overline{w}$ is a fixed optimal temperature for growing for both white daisy and black daisy. The term $q[g(u,v) - a_w]$ (resp. $q[g(u,v) - a_b]$) means some suitable adjustment on a local temperature to the global one (i.e., $w$) at each position where white daisy (resp. black daisy) grows, $q > 0$ being some coefficient. Since $g(u,v) \le a_w$ (resp. $g(u,v) \ge a_b$), we see that $w$ is always adjusted negatively (resp. positively) where white daisy (resp. black daisy) grows. The notation $\{w\}_+ = \max\{w, 0\}$ denotes a positive cutoff of the function $w$ for $-\infty < w < \infty$; consequently, $\{1 - \delta(\overline{w} - w)^2\}_+$ is a positive cutoff of the square function $1 - \delta(\overline{w} - w)^2$ for $-\infty < w < \infty$, $\delta > 0$ being some coefficient. Both white daisy and black daisy die at a rate $f > 0$. Finally, the term $[1 - g(u,v)]R(x)$ denotes an increasing rate of the global temperature which is determined by the averaged albedo $g(u,v)$ mentioned above and the incoming energy $R(\omega)$ from the sun which is a function of $\omega \in S$ hitting its maximum on the equator and vanishing at the two poles. And, the term $-\sigma w^4$ denotes a decaying rate of the temperature due to the Stefan-Boltzmann law, $\sigma > 0$ being the Stefan-Boltzmann constant of the surface.

A planetary biota modifies its environment and its environment regulates a biota by natural selection. Self-regulating homeostatic system is an idea that the feedback between a biota and its environment keeps the planetary surface environment stable and habitable for a biota. Daisyworld has been introduced by Lovelock [13] as a simple parable to verify a hypothesis that the Earth maintain self-regulating homeostasis (see Lovelock-Margulis [14] and Lenton [12]). In the original Daisyworld model due to Watson-Lovelock [20], the whole planet is regarded as a single point. The model is governed by rather simple rules: black daisies absorb more incoming energy, while white daisies reflect more. They showed that the competition of daisies controls the global albedo of the planet and regulate the global temperature to be more suitable for daisies. Their results suggested a possibility that the Daisyworld model is very valuable for understanding the mechanisms of self-regulating homeostasis of the Earth. The Daisyworld model was analyzed by several authors (e.g., [7, 16]), on the other hand, that inspired many modifications and extensions. Adams-Carr [2] and Adams-Carr-Lenton-White [3] extended the original model to one-dimensional one including variation of incoming solar energy and heat diffusion on the sphere. The one-dimensional model retains the temperature regulation and shows a stripe pattern that shows two types of daisies segregate. A two-dimensional extension of the Daisyworld model based on cellular automata was introduced by von Bloh-Block-Schellnhuber [19]. In [19], the equation for heat transfer on Daisyworld is governed by a simple energy balance equation and the spatial distribution of daisies are determined by discretized equations. Additional extensional models based on the two-dimensional one were presented (e.g., [1, 23]).

In this paper, we want to consider a complete two-dimensional version of the Daisyworld model adding diffusion terms of daisies on a sphere. After formulating our model as a reaction-diffusion equations on the sphere, we will analytically construct local solutions, global solutions, dynamical systems and exponential attractors. In the last section, we will show some numerical results suggesting two-dimensional pattern formation of the Turing type.

We denote by $S$ a sphere given by

$$S \equiv \{\omega = (x,y,z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = \ell^2\}$$

with radius $\ell > 0$. And, $\Delta$ denotes a Laplace operator on $S$, namely, $\Delta$ is a Laplace-Beltrami operator on $S$ whose definition will be reviewed in the next section. Following [3], we assume that the solar energy incident on the surface is parallel to the latitude lines. The incoming energy $R(\omega)$ thus arrives symmetrically with respect to the equator, and it is given by

$$R(\omega) = R_0\sqrt{1-(z/\ell)^2}, \qquad \omega = (x, y, z) \in S, \tag{1.2}$$

with some coefficient $R_0 > 0$.

**2 Diffusion equations on $S$.** The theory of diffusion equations on Riemannian manifolds is already well known (see, e.g., [5]). It is however constructed in a very general context only using Riemannian metrics and without using any information in which Euclidean spaces the manifolds are embedded. On the other hand, in order to treat nonlinear diffusion equations like (1.1), the functional analytical approach has a great advantage over other ones.

By this reason, we want to review in this section the theory of diffusion equations on $S$ using the fact that $S$ is a special Riemannian manifold embedded in $\mathbb{R}^3$.

**2.1 Local coordinates of $S$.** We will use two polar coordinates in $\mathbb{R}^3$. First one is the usual one. Let $H_1 = \{(x, y, z) \in \mathbb{R}^3;\ y = 0,\ x \geq 0\}$. For $P = (x, y, z) \in \mathbb{R}^3 - H_1$, put

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta, \end{cases} \tag{2.1}$$

where $r = \sqrt{x^2+y^2+z^2}$, $\theta$ is the zenithal angle of $OP$ and $z$-axis and $\phi$ is the azimuthal angle of $OQ$ and $x$-axis, $Q = (x, y, 0)$ being the projected point of $P$ on the $xy$-plane. Therefore $(r, \theta, \phi)$ varies in $0 < r < \infty$, $0 < \theta < \pi$ and $0 < \phi < 2\pi$.

Second one is defined in $\mathbb{R}^3 - H_2$, where $H_2 = \{(x, y, z) \in \mathbb{R}^3;\ z = 0, x \leq 0\}$. For $P = (x, y, z) \in \mathbb{R}^3 - H_2$, put

$$\begin{cases} x = r\sin\vartheta\cos\varphi, \\ z = r\sin\vartheta\sin\varphi, \\ y = r\cos\vartheta, \end{cases} \tag{2.2}$$

where $r = \sqrt{x^2+y^2+z^2}$, $\vartheta$ is the zenithal angle of $OP$ and $y$-axis and $\varphi$ is the azimuthal angle of $OQ$ and $x$-axis, $Q = (x, 0, z)$ being the projected point of $P$ on the $xz$-plane. Now $(r, \vartheta, \varphi)$ varies in $0 < r < \infty$, $0 < \vartheta < \pi$ and $-\pi < \varphi < \pi$.

These polar coordinates immediately provide a local coordinate system for $S$. Let $S_1 = S - \Gamma_1$, where $\Gamma_1 = S \cap H_1$. Then, fixing $r = \ell$, we get from (2.1) a homeomorphism $\Theta_1 : S_1 \to D_1$ with $D_1 = \{(\theta, \phi);\ 0 < \theta < \pi, 0 < \phi < 2\pi\}$. Similarly, setting $S_2 = S - \Gamma_2$, where $\Gamma_2 = S \cap H_2$, we get from (2.2) a homeomorphism $\Theta_2 : S_2 \to D_2$ with $D_2 = \{(\vartheta, \varphi);\ 0 < \vartheta < \pi, -\pi < \varphi < \pi\}$. By $\{(S_i, \Theta_i)\}_{i=1,2}$, $S$ becomes a differentiable manifold.

Let $\{\psi_i(\omega)\}_{i=1,2}$ be a partition of unity subordinate to $\{S_i, \Theta_i\}$, that is, $\psi_i(\omega)$ are smooth functions on $S$ such that $0 \leq \psi_i(\omega) \leq 1$, $\psi_1(\omega) + \psi_2(\omega) \equiv 1$ on $S$ and $\operatorname{supp}\psi_i \subset S_i$. We need also suitable rectangular domains $G_i \subset D_i$. For $i = 1, 2$, let $G_i$ be a rectangular domain such that

$$\Theta_i(\operatorname{supp}\psi_i) \subset G_i \subset \overline{G}_i \subset D_i.$$

We equip $S$ with the surface measure $d\omega$. For $1 \le p \le \infty$, $L_p(S)$ is the space of all measurable functions such that $|f(\omega)|^p$ is integrable on $S$. By the usual $L_p$-norm, $L_p(S)$ is a Banach space. When $p = 2$, $L_2(S)$ is a Hilbert space with the usual inner product. Of course, $f \in L_p(S)$ if and only if $\psi_i f \in L_p(S_i)$ for $i = 1, 2$; furthermore, $\psi_i f \in L_p(S_i)$ if and only if $(\psi_i f) \circ \Theta_i^{-1} \in L_p(G_i)$ with norm equivalence of $\|\psi_i f\|_{L_p(S_i)}$ and $\|(\psi_i f) \circ \Theta_i^{-1}\|_{L_p(G_i)}$.

**2.2 Laplace-Beltrami operator on $S$.** Let us denote by $\nabla_S$ the gradient operator acting to the differentiable functions on $S$.

In $S_1$, $\nabla_S u$ is described by the polar coordinate (2.1) in the form

$$\nabla_S u = \frac{1}{\ell}\left(\cos\theta\cos\phi\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial\phi},\ \cos\theta\sin\phi\frac{\partial u}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial u}{\partial\phi},\ -\sin\theta\frac{\partial u}{\partial\theta}\right). \tag{2.3}$$

If $\omega \in S_1$, then the normal vector for $S$ at $\omega$ is given by $n_\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Thereby, it is directly verified that $\nabla_S u(\omega) \cdot n_\omega = 0$, i.e., $\nabla_S u(\omega)$ is a tangential vector of $S$ at $\omega$.

It is the same for the description of $\nabla_S$ on $S_2$.

We can then give a definition of the first order Sobolev space $H^1(S)$ on $S$ using $\nabla_S$. In fact, $H^1(S)$ is the space of all functions $u \in L_2(S)$ for which $|\nabla_S u|$ also belong to $L_2(S)$. It is easy to see that $u \in H^1(S)$ if and only if $\psi_i u \in H^1(S)$ for $i = 1, 2$. Furthermore, in $S_1$ it follows from (2.3) that

$$|\nabla_S u|^2 = \frac{1}{\ell^2}\left[\left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\phi}\right)^2\right].$$

Hence, $\psi_1 u \in H^1(S)$ if and only if $(\psi_1 u) \circ \Theta_1^{-1} \in H^1(G_1)$. It is the same for $\psi_2 u \in H^1(S)$. We equip $H^1(S)$ with the inner product

$$(u, v)_{H^1} = \int_S (\nabla_S u \cdot \nabla_S \overline{v} + u\overline{v}) d\omega, \qquad u,\, v \in H^1(S).$$

Then, $H^1(S)$ becomes a Hilbert space. The norm $\|\psi_i u\|_{H^1}$ is equivalent to $\|(\psi_i u) \circ \Theta_i^{-1}\|_{H^1(G_i)}$ for $i = 1, 2$.

We are now led to define the Laplace-Beltrami operator $\Delta_S$ by

$$\Delta_S = \nabla_S \cdot \nabla_S. \tag{2.4}$$

In view of (2.3), in $S_1$ we observe that

$$\Delta_S u = \frac{1}{\ell^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]. \tag{2.5}$$

In $S_2$, too, the similar description of $\Delta_S$ by means of (2.2) is verified.

**2.3 Realization of $\Delta_S$ in $L_2(S)$.** In order to formulate (1.1) in the space $L_2(S)$, we have to define $\Delta_S$ as a linear operator acting in $L_2(S)$. For this purpose, we consider the sesquilinear form $a(u, v) = (u, v)_{H^1}$, $u, v \in H^1(S)$. Trivially, $a(u, v)$ is continuous and coercive on $H^1(S)$. For each $u \in H^1(S)$, the mapping $v \mapsto a(u, v)$ is an anti-linear continuous functional on $H^1(S)$. Then, elements $u \in H^1(S)$ for which the mappings are continuous in the topology of $L_2(S)$ are picked up. By the Riesz theorem, for such a $u$, there exists a unique element $f \in L_2(S)$ such that $a(u, v) = (f, v)_{L_2}$ for all $v \in H^1(S)$. We then set $\Lambda u = f$, that is,

$$\begin{cases} \mathcal{D}(\Lambda) = \{u \in H^1(S);\ (u, v)_{H^1} = (f, v)_{L_2} \text{ for all } v \in H^1(S)\}, \\ \Lambda u = f. \end{cases}$$

It is immediate to see that $u \mapsto f = \Lambda u$ is a linear operator from $\mathcal{D}(\Lambda)$ into $L_2(S)$. Moreover, the theory of variation (see Dautray-Lions [6]) provides that $\mathcal{D}(\Lambda)$ is dense in $L_2(S)$ and $\Lambda$ is a positive definite self-adjoint operator of $L_2(S)$. Furthermore, $\mathcal{D}(\Lambda)$ is shown to coincide with the second order Sobolev space $H^2(S)$ which consists of functions $u \in L_2(S)$ such that $(\psi_i u) \circ \Theta_i^{-1} \in H^2(G_i)$ for $i = 1, 2$.

We here set $A = \Lambda - 1$ with $\mathcal{D}(A) = \mathcal{D}(\Lambda) = H^2(S)$. By definition, it holds for $u \in \mathcal{D}(A)$ that

$$(Au, v)_{L_2} = \int_S \nabla_S u \cdot \nabla_S \overline{v} d\omega \qquad \text{for all } v \in H^1(S).$$

Therefore, $A$ is a nonnegative self-adjoint operator of $L_2(S)$. And, $Au = 0$ implies $|\nabla_S u|^2 \equiv 0$ and hence $u \equiv$ const. In the meantime, by integration by parts we verify that $Au = -\Delta_S u$ for $u \in \mathcal{D}(A)$. Hence, $A$ is a realization of $-\Delta_S$ in the space $L_2(S)$. Since $\mathcal{D}(A)$ is compactly embedded in $L_2(\Omega)$, $A$ can be decomposed of the form

$$Au = \sum_{k=0}^{\infty} \lambda_k (u, e_k)_{L_2} e_k, \tag{2.6}$$

where $\lambda_k$ are eigenvalues of $A$ and $e_k(\omega)$ are eigenfunctions of $A$ corresponding to $\lambda_k$, respectively. Clearly,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

Meanwhile, $e_k(\omega)$ compose an orthonormal basis of $L_2(S)$. As noticed above, $e_0(\omega)$ must be a constant function on $S$, hence $e_0(\omega) \equiv \frac{1}{2\sqrt{\pi}\ell}$.

**2.4 Semigroup generated by $-A$.** Since $A$ is a nonnegative self-adjoint operator, $-A$ generates an analytic and contraction semigroup $e^{-tA}$, $0 \leq t < \infty$, on $L_2(S)$. As the minimal eigenvalue $\lambda_0$ is zero, we have just

$$\|e^{-tA}\|_{\mathcal{L}(L_2(S))} = 1 \qquad \text{for any } 0 \leq t < \infty. \tag{2.7}$$

For any initial function $u_0 \in L_2(S)$, $e^{-tA}$ gives a unique solution to the Cauchy problem of diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_S u & \text{in} \quad S \times (0, \infty), \\ u(\omega, 0) = u_0(\omega) & \text{in} \quad S, \end{cases} \tag{2.8}$$

on $S$. Indeed, $u(t) = e^{-tA} u_0$ is a unique solution to (2.8) in the function space:

$$u \in \mathcal{C}((0, \infty); H^2(S)) \cap \mathcal{C}([0, \infty); L_2(S)) \cap \mathcal{C}^1((0, \infty); L_2(S)).$$

From (2.6), $e^{-tA}u_0$ can be expressed by

$$e^{-tA}u_0 = \sum_{k=0}^{\infty} e^{-\lambda_k t}(u_0, e_k)_{L_2} e_k, \qquad u_0 \in L_2(S). \tag{2.9}$$

The formula (2.9) immediately provides various properties of the solution $u(t)$ to (2.8) as follows:

(1) If $u_0 \geq 0$, then $e^{-tA}u_0 \geq 0$ for any $0 < t < \infty$.

(2) It holds that $\int_S e^{-tA}u_0 d\omega = \int_S u_0 d\omega$ for any $0 < t < \infty$.

(3) Let $P_0 u_0 = (u_0, e_0)_{L_2} e_0$ be the projection from $L_2(S)$ onto the eigenspace of $\lambda_0 = 0$. Then,

$$\|e^{-tA} - P_0\|_{\mathcal{L}(L_2(S))} \leq e^{-\lambda_1 t} \qquad \text{for any } 0 \leq t < \infty. \tag{2.10}$$

(4) As an operator from $L_2(S)$ into $H^1(S)$, $e^{-tA}$ satisfies

$$\|\nabla_S e^{-tA}\|_{\mathcal{L}(L_2(S))} \leq \left(\lambda_1 + \frac{1}{et}\right)^{\frac{1}{2}} e^{-\lambda_1 t} \qquad \text{for any } 0 < t < \infty. \tag{2.11}$$

(5) As an operator from $L_2(S)$ into $H^2(S)$, $e^{-tA}$ satisfies

$$\|\Delta_S e^{-tA}\|_{\mathcal{L}(L_2(S))} \leq \left(\lambda_1 + \frac{1}{et}\right) e^{-\lambda_1 t} \qquad \text{for any } 0 < t < \infty. \tag{2.12}$$

The estimate (2.10) follows from

$$\|e^{-tA}u_0 - P_0 u_0\|_{L_2} = e^{-\lambda_1 t}\left(\sum_{k=1}^{\infty} e^{-2(\lambda_k - \lambda_1)t}|(u_0, e_k)_{L_2}|^2\right)^{\frac{1}{2}} \leq e^{-\lambda_1 t}\|u_0\|_{L_2}.$$

Similarly, the estimate (2.12) follows from

$$\begin{aligned}\|\Delta_S e^{-tA}u_0\|_{L_2} = \|Ae^{-tA}u_0\|_{L_2} &= \left\|\sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k t}(u_0, e_k)_{L_2} e_k\right\|_{L_2} \\ &= e^{-\lambda_1 t}\left\|\sum_{k=1}^{\infty}[t^{-1}(\lambda_k - \lambda_1)t + \lambda_1]e^{-(\lambda_k - \lambda_1)t}(u_0, e_k)_{L_2} e_k\right\|_{L_2} \\ &\leq e^{-\lambda_1 t}[t^{-1}e^{-1} + \lambda_1]\|u_0\|_{L_2}.\end{aligned}$$

Finally, the estimate (2.11) is observed by

$$\begin{aligned}\|\nabla_S e^{-tA}u_0\|_{L_2}^2 &= (Ae^{-tA}u_0, [e^{-tA} - P_0]u_0)_{L_2} \leq \|Ae^{-tA}u_0\|_{L_2}\|[e^{-tA} - P_0]u_0\|_{L_2} \\ &\leq \|Ae^{-tA}\|_{\mathcal{L}(L_2(S))}\|e^{-tA} - P_0\|_{\mathcal{L}(L_2(S))}\|u_0\|_{L_2}^2.\end{aligned}$$

**3 Construction of Solutions.** In this section, we shall construct global solutions to the Cauchy problem (1.1) and a dynamical system generated by them. We begin with formulating (1.1) in an abstract form (cf. [11, 17, 24]).

**3.1 Abstract formulation.** We consider (1.1) in the product $L_2$-space

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u \in L_2(S),\, v \in L_2(S),\, w \in L_2(S) \right\},$$

$X$ being a complex Hilbert space by the usual inner product. Let $\mathcal{A}$ be a linear operator acting in $X$ which is given by

$$\mathcal{A} = \begin{pmatrix} dA & 0 & 0 \\ 0 & dA & 0 \\ 0 & 0 & DA \end{pmatrix},$$

where $A$ is the realization of $-\Delta_S$ in $L_2(S)$ introduced in Section 2. Then, $\mathcal{A}$ is a nonnegative self-adjoint operator of $X$ and generates an analytic semigroup $e^{-t\mathcal{A}}$ which is expressed by diag $\{e^{-tdA}, e^{-tdA}, e^{-tDA}\}$ on $X$. It then follows from (2.7) that

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \le 1 \qquad \text{for } 0 \le t < \infty. \tag{3.1}$$

Moreover, it is seen from (2.11) that

$$\|\nabla_S e^{-tdA}\|_{\mathcal{L}(L_2(S))} \le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} \qquad \text{for } 0 < t < \infty,$$

$$\|\nabla_S e^{-tDA}\|_{\mathcal{L}(L_2(S))} \le \left(\lambda_1 + \frac{1}{eDt}\right)^{\frac{1}{2}} e^{-D\lambda_1 t} \qquad \text{for } 0 < t < \infty.$$

As a consequence,

$$\|\nabla_S e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} \qquad \text{for } 0 < t < \infty. \tag{3.2}$$

From the view point of modeling, we may expect that the solutions exist in the ranges of $u \ge 0$, $v \ge 0$, $u + v \le 1$ and $0 \le w \le (R_0/\sigma)^{\frac{1}{4}}$. On account of these range conditions, we introduce a nonlinear operator $\mathcal{F}$ of $X$ by

$$\mathcal{F}(U) = \begin{pmatrix} [\chi_1(1 - \mathrm{Re}\, u - \mathrm{Re}\, v)\Phi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w)) - f]\chi_1(\mathrm{Re}\, u) \\ [\chi_1(1 - \mathrm{Re}\, u - \mathrm{Re}\, v)\Psi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w)) - f]\chi_1(\mathrm{Re}\, v) \\ [1 - g(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v))]R(\omega) - \sigma\chi_2(\mathrm{Re}\, w)^4 \end{pmatrix}.$$

Here, $\chi_1(\xi)$ and $\chi_2(\xi)$ are cutoff functions defined by

$$\chi_1(\xi) = \begin{cases} 0, & -\infty < \xi \le 0, \\ \xi, & 0 < \xi \le 1, \\ 1, & 1 < \xi < \infty, \end{cases} \qquad \chi_2(\xi) = \begin{cases} 0, & -\infty < \xi \le 0, \\ \xi & 0 < \xi \le (R_0/\sigma)^{\frac{1}{4}}, \\ (R_0/\sigma)^{\frac{1}{4}}, & (R_0/\sigma)^{\frac{1}{4}} < \xi < \infty, \end{cases}$$

respectively.

The problem (1.1) is then formulated as the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U = \mathcal{F}(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{3.3}$$

in $X$, where $U(t) = {}^t(u(t),\, v(t),\, w(t))$ is an unknown function and $U_0$ is an initial value. As for the space of initial values, we set

$$K = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X;\ u_0 \geq 0,\ v_0 \geq 0,\ u_0 + v_0 \leq 1, 0 \leq w_0 \leq \left(\frac{R_0}{\sigma}\right)^{\frac{1}{4}} \right\}. \tag{3.4}$$

It is clear that $\chi_1(\xi)$ and $\chi_2(\xi)$ are uniformly bounded and globally Lipschitz continuous functions for $-\infty < \xi < \infty$. Consequently, $\Phi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w))$ and $\Psi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w))$ are uniformly bounded and globally Lipschitz continuous functions for $(u, v, w) \in \mathbb{C}^3$. Therefore, it is easily verified that $\mathcal{F}$ is a bounded operator and satisfies the Lipschitz condition, i.e.,

$$\|\mathcal{F}(U)\|_X \leq C_1, \qquad U \in X, \tag{3.5}$$

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_X \leq C_2 \|U - V\|_X, \qquad U, V \in X, \tag{3.6}$$

with suitable constants $C_i > 0\, (i = 1, 2)$.

It is then possible to apply the general theory of abstract parabolic evolution equations, see [24, Theorem 4.4], to (3.3) to obtain that, for any $U_0 \in K$, there exists a unique local solution to (3.3) in the function space:

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(\mathcal{A})) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X).$$

Here, the time $T_{U_0} > 0$ is determined by the norm $\|U_0\|_X$ alone.

**3.2 Global solutions.** We can verify that the local solution $U(t)$ constructed above takes its values in $K$.

**Proposition 3.1.** *The condition $U_0 \in K$ implies that $U(t) \in K$ for any $0 < t \leq T_{U_0}$.*

*Proof.* It is easy to verify that, if $U(t)$ is a local solution of (3.3), then its complex conjugate $\overline{U(t)}$ is also a local solution with the same initial condition. Therefore, $U(t) = \overline{U(t)}$ and $U(t)$ is real valued.

Firstly, let us see that $u(t) \geq 0$. For this purpose, we use a $\mathcal{C}^2$-cutoff function $H(u)$ given by

$$H(u) = \begin{cases} \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}, & -\infty < u \leq -1, \\ -\frac{1}{6}u^3, & -1 \leq u < 0, \\ 0, & 0 \leq u < \infty. \end{cases}$$

Put $h(t) = \int_S H(u(\omega, t)) d\omega$. Then, for $0 < t \leq T_{U_0}$,

$$\begin{aligned} \frac{dh_1}{dt}(t) = \int_S H'(u) \frac{\partial u}{\partial t}\, d\omega = d \int_S H'(u) \varDelta_S u\, d\omega \\ + \int_S H'(u)[\chi_1(1 - u - v)\Phi(\chi_1(u), \chi_1(v), \chi_2(w)) - f]\chi_1(u)\, d\omega. \end{aligned}$$

Since

$$\int_S H'(u) \varDelta_S u d\omega = -\int_S \nabla_S H'(u) \cdot \nabla_S u d\omega = -\int_S H''(u) |\nabla_S u|^2 d\omega \leq 0$$

and since $H'(u)\chi_1(u) \equiv 0$ for $-\infty < u < \infty$, it follows that $\frac{dh}{dt}(t) \le 0$, i.e., $h(t) \le h(0) = 0$ for any $0 < t \le T_{U_0}$.

The same arguments for $v(t)$ conclude that $v(t) \ge 0$ for any $0 < t \le T_{U_0}$.

Secondly, in order to see that $u(t) + v(t) \le 1$, we notice that $z(t) = 1 - u(t) - v(t)$ is regarded as a solution to

$$\frac{\partial z}{\partial t} = d\Delta_S z - [\Phi(\chi_1(u), \chi_1(v), \chi_2(w)) + \Psi(\chi_1(u), \chi_1(v), \chi_2(w))]\chi_1(z) + f[\chi_1(u) + \chi_1(v)].$$

We can then repeat the same arguments as for $u(t)$ and $v(t)$ to conclude that $z(t) \ge 0$ for any $0 < t \le T_{U_0}$.

Thirdly, let us observe that $0 \le w \le (R_0/\sigma)^{\frac{1}{4}}$ for any $0 < t \le T_{U_0}$. The verification that $w(t) \ge 0$ is the same as for $u(t)$ and $v(t)$. Putting $w_1(t) = (R_0/\sigma)^{\frac{1}{4}} - w(t)$, we notice due to (1.2) that

$$\frac{\partial w_1}{\partial t} = D\Delta_S w_1 - \sigma[R_0/\sigma - \chi_2(w)^4] + R_0 \left\{1 + [g(u,v) - 1]\sqrt{1 - (z/\ell)^2}\right\}.$$

Then, put $h_1(t) = \int_S H(w_1(\omega, t))d\omega$. Since $H'((R_0/\sigma)^{\frac{1}{4}} - w)[R_0/\sigma - \chi_2(w)^4] \equiv 0$ for $-\infty < w < \infty$, it follows that $\frac{dh_1}{dt}(t) \le 0$, i.e., $h_1(t) \le h_1(0) = 0$. Hence, $(R_0/\sigma)^{\frac{1}{4}} - w(t) \ge 0$ for any $0 < t \le T_{U_0}$. □

This proposition shows that the norm $\|U(t)\|_X$ remains uniformly bounded on the interval $[0, T_{U_0}]$. This then means that one can always extend any local solution with a uniform time interval. Therefore, we obtain the following existence result.

**Theorem 3.1.** *For any $U_0 \in K$,* (3.3) *possesses a unique global solution $U(t)$ in the function space:*

$$U \in \mathcal{C}((0,\infty); \mathcal{D}(\mathcal{A})) \cap \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X).$$

As verified by Proposition 3.1 $U(t)$ takes its values in $K$ for all $0 < t < \infty$. Thereby, $\chi_1(u(t)) = u(t)$, $\chi_1(v(t)) = v(t)$, $\chi_1(1 - u(t) - v(t)) = 1 - u(t) - v(t)$ and $\chi_2(w(t)) = w(t)$ for all $0 < t < \infty$. This in turn shows that the global solution $U(t)$ of (3.3) can be considered as a global solution to the original problem (1.1), too.

Let us finally verify global norm estimate and continuous dependence of solutions on initial values.

**Theorem 3.2.** *Let $U_0 \in K$ and let $U(t)$ be the global solution of* (3.3). *Then,*

$$\|\nabla_S U(t)\|_X \le C_3 \left[\left(1 + \frac{1}{t}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + 1\right] \qquad \text{for} \quad 0 < t < \infty \tag{3.7}$$

*with some constant $C_3$.*

*Proof.* By Duhamel's formula, $U(t)$ can be written as

$$U(t) = e^{-t\mathcal{A}}U_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{F}(U(s))ds.$$

Thereby,

$$\nabla_S U(t) = \nabla_S e^{-t\mathcal{A}}U_0 + \int_0^t \nabla_S e^{-(t-s)\mathcal{A}}\mathcal{F}(U(s))ds.$$

Due to (3.2) and (3.5), we have

$$\begin{aligned}\|\nabla_S U(t)\|_X &\le \left(\lambda_1+\frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + C_1\int_0^t \left(\lambda_1+\frac{1}{ed(t-s)}\right)^{\frac{1}{2}} e^{-d\lambda_1(t-s)}ds\\ &\le \left(\lambda_1+\frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + C_1\int_0^\infty \left(\lambda_1+\frac{1}{eds}\right)^{\frac{1}{2}} e^{-d\lambda_1 s}ds.\end{aligned}$$

Hence, (3.7) is verified. □

**Theorem 3.3.** *Let $U_0$, $V_0 \in K$ and let $U(t)$ and $V(t)$ be the global solutions of* (3.3) *with initial values $U_0$ and $V_0$, respectively. Then,*

$$\|U(t)-V(t)\|_X \le e^{C_2 t}\|U_0-V_0\|_X \qquad \text{for} \quad 0\le t<\infty, \tag{3.8}$$

$$\|\nabla_S[U(t)-V(t)]\|_X \le C_4\left[\left(1+\frac{1}{t}\right)^{\frac{1}{2}} + te^{C_2 t}\right]\|U_0-V_0\|_X \qquad \text{for} \quad 0<t<\infty \tag{3.9}$$

*with some constant $C_4$.*

*Proof.* By Duhamel's formula again, we have

$$U(t)-V(t) = e^{-t\mathcal{A}}[U_0-V_0] + \int_0^t e^{-(t-s)\mathcal{A}}[\mathcal{F}(U(s))-\mathcal{F}(V(s))]ds.$$

In view of (3.1) and (3.6),

$$\|U(t)-V(t)\|_X \le \|U_0-V_0\|_X + C_2\int_0^t \|U(s)-V(s)\|_X ds.$$

Hence, (3.8) is obtained.

Similarly, from

$$\nabla_S[U(t)-V(t)] = \nabla_S e^{-t\mathcal{A}}[U_0-V_0] + \int_0^t \nabla_S e^{-(t-s)\mathcal{A}}[\mathcal{F}(U(s))-\mathcal{F}(V(s))]ds,$$

it is estimated by (3.2), (3.6) and (3.8) that

$$\begin{aligned}&\|\nabla_S[U(t)-V(t)]\|_X\\ &\quad\le \left[\left(\lambda_1+\frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} + C_2\int_0^t \left(\lambda_1+\frac{1}{ed(t-s)}\right)^{\frac{1}{2}} e^{-d\lambda_1(t-s)+C_2 s}\right]\|U_0-V_0\|_X\\ &\quad\le \left[\left(\lambda_1+\frac{1}{edt}\right)^{\frac{1}{2}} + C_2 e^{C_2 t}\int_0^t \left(\lambda_1+\frac{1}{ed(t-s)}\right)^{\frac{1}{2}} ds\right]\|U_0-V_0\|_X.\end{aligned}$$

Hence, (3.9) has been verified. □

**4 Dynamical Systems** This section is devoted to constructing a dynamical system determined by (3.3) and showing existence of attractor sets. For this purpose, however, it suffices to simply follow the general procedure that is known for the Cauchy problems of semilinear abstract parabolic evolution equations, see [24, Section 6.5].

For $U_0 \in K$, let $U(t; U_0)$ denote the global solution of (3.3), and set

$$S(t)U_0 = U(t; U_0), \qquad 0 \le t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on $K$, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \le s, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into $X$. Indeed, fix $(t, U_0) \in [0,\infty) \times K$. Due to (3.8),

$$\begin{aligned}\|S(t')U_0' - S(t)U_0\|_X &\le \|S(t')U_0' - S(t')U_0\|_X + \|S(t')U_0 - S(t)U_0\|_X \\ &\le e^{C_2 t'}\|U_0' - U_0\|_X + \|S(t')U_0 - S(t)U_0\|_X.\end{aligned}$$

Then, $(t', U_0') \to (t, U_0)$ implies $S(t')U_0' \to S(t)U_0$ in $X$. Hence, $S(t)$ defines a dynamical system in the space $X$ which is denoted by $(S(t), K, X)$ (cf. [4, 18]). The phase space $K$ is a bounded, closed subset of $X$.

As well known, the dissipative estimate (3.7) provides existence of the global attractor. Set a subset $B$ of $K$ by

$$B = K \cap \left\{U \in [H^1(S)]^3; \|\nabla U\|_X \le C_3 + 1\right\}.$$

Then, (3.7) means that there is a time $T$ such that $S(t)K \subset B$ for every $t \ge T$, i.e., $B$ is an absorbing set. In addition, $B$ is a compact set of $X$. Therefore, $B$ is a compact absorbing set of $(S(t), K, X)$. It is clear that $S(t)B \subset B$ for every $t \ge T$. So, we set again a subset of $K$ by

$$\mathcal{K} = \bigcup_{0 \le t \le T} S(t)B.$$

Then, $S(t)\mathcal{K} \subset \mathcal{K}$ for every $t > 0$, i.e., $\mathcal{K}$ is an invariant set. Therefore, $\mathcal{K}$ is not only compact and absorbing but also invariant. This means that the asymptotic behavior of trajectories of $(S(t), K, X)$ can be reduced to a sub dynamical system $(S(t), \mathcal{K}, X)$ in which the phase space $\mathcal{K}$ is a compact set of $X$.

By the usual arguments, it is seen that $\mathcal{B} = \bigcap_{0 \le t < \infty} S(t)\mathcal{K}$ becomes a global attractor of $(S(t), \mathcal{K}, X)$.

Furthermore, thanks to the estimate (3.9), we can construct the exponential attractors. Remember that a subset $\mathcal{M} \subset \mathcal{K}$ satisfying the following conditions is called an exponential attractor of $(S(t), \mathcal{K}, X)$:

1. $\mathcal{M}$ is a compact subset of $X$ with finite fractal dimension.
2. $\mathcal{M}$ includes the global attractor $\mathcal{B}$.
3. $\mathcal{M}$ is an invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$.
4. There exists an exponent $k > 0$ such that

$$h(S(t)\mathcal{K}, \mathcal{M}) \le C_5 e^{-kt}, \qquad 0 < t < \infty,$$

with a constant $C_5 > 0$.

Here, $h(K_1, K_2) = \sup_{F\in K_1} \inf_{G\in K_2} \|F - G\|_X$ is a semi-distance of two subsets $K_1$ and $K_2$ of $\mathcal{K}$.

As explained in [24, Section 6.4], the compact smoothing property

$$\|S(t^*)U_0 - S(t^*)V_0\|_{H^1(S)} \le C_6\|U_0 - V_0\|_X, \qquad U_0,\, V_0 \in \mathcal{K},$$

of $S(t^*)$ with any fixed time $t^* > 0$ provides existence of exponential attractors. But, in the present case, this property is nothing more than the estimates (3.8) and (3.9).

In this way, we have obtained the following theorem.

**Theorem 4.1.** *The dynamical system $(S(t), K, X)$ possesses exponential attractors.*

*Proof.* As explained above, we already know that there exists an exponential attractor $\mathcal{M}$ for $(S(t), \mathcal{K}, X)$. Then, as $S(T)K \subset B \subset \mathcal{K}$, it is readily verified that $\mathcal{M}$ is an exponential attractor for $(S(t), K, X)$, too. □

**5 Some Numerical Results** We shall conclude this paper with illustrating some numerical examples. Let us consider (1.1) in the sphere $S$ with $\ell = 1$. Numerical methods for partial differential equations on the spheres have been widely developed in the field of geodynamo simulations. For example, Yin-Yang grid by Kageyama-Sato [9], Cubed Sphere grid by Ronchi-Iacono-Paolucci [15], Half-Step-Shifted grid (e.g., [8]) and a method of applying l'Hospital's rule on the pole grids (e.g., [10]), see also the review of Williamson [21]. These numerical methods have in general a trade-off between computational cost and their accuracy.

We use the explicit Half-Step-Shifted grid scheme. As surveyed below, this scheme is based on the traditional finite difference methods with the spherical polar coordinate system. For spatial discretization, the $i$-th colatitude grid point $\theta_i$ and the $j$-th longitude grid point $\phi_j$ are defined by

$$\begin{aligned}\theta_i &= \left(i - \frac{1}{2}\right)\Delta\theta_i, \qquad && (i = 1, 2, \cdots, N),\\ \phi_j &= j\Delta\phi, && (j = 0, 1, \cdots, M),\end{aligned}$$

respectively, where $N$ and $M$ denote the numbers of grid points. And the $n$-th time step is defined by $t_n = n\Delta t$. We assume that $\Delta\theta_i$ is a non-uniform grid spacing which is smaller near the poles, while $\Delta\phi$ is a uniform one ($\Delta\phi = 2\pi/M$). This scheme is a simple idea which the horizontal grid lines are shifted by a distance of $\Delta\theta_i/2$ in order to remove coordinate singularity problems at the poles.

Let us denote the approximate values by $U^n_{i,j} \approx u(\theta_i, \phi_j, t_n)$, $V^n_{i,j} \approx v(\theta_i, \phi_j, t_n)$ and $W^n_{i,j} \approx w(\theta_i, \phi_j, t_n)$, respectively. Then, (1.1) is discretized as follows:

$$\begin{aligned}\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} &= \left[\left(1 - U^n_{i,j} - V^n_{i,j}\right)\Phi(U^n_{i,j}, V^n_{i,j}, W^n_{i,j}) - f\right]U^n_{i,j}\\ &\quad + d\left[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{U^n_{i+1,j} - U^n_{i,j}}{\Delta\theta_i} - \sin\theta_{i-\frac{1}{2}}\frac{U^n_{i,j} - U^n_{i-1,j}}{\Delta\theta_i}\right)\right.\\ &\qquad\qquad \left. + \frac{1}{\sin^2\theta_i}\left(\frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{(\Delta\phi)^2}\right)\right],\end{aligned}$$

$$\begin{aligned}\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta t} &= \left[\left(1-U_{i,j}^{n}-V_{i,j}^{n}\right)\Psi(U_{i,j}^{n},V_{i,j}^{n},W_{i,j}^{n})-f\right]V_{i,j}^{n} \\ &\quad + d\left[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{V_{i+1,j}^{n}-V_{i,j}^{n}}{\Delta\theta_i}-\sin\theta_{i-\frac{1}{2}}\frac{V_{i,j}^{n}-V_{i-1,j}^{n}}{\Delta\theta_i}\right)\right. \\ &\quad \left. + \frac{1}{\sin^2\theta_i}\left(\frac{V_{i,j+1}^{n}-2V_{i,j}^{n}+V_{i,j-1}^{n}}{(\Delta\phi)^2}\right)\right],\end{aligned}$$

$$\begin{aligned}\frac{W_{i,j}^{n+1}-W_{i,j}^{n}}{\Delta t} &= \left[1-g(U_{i,j}^{n},V_{i,j}^{n})\right]R(\theta_i)-\sigma\left(W_{i,j}^{n}\right)^4 \\ &\quad + D\left[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{W_{i+1,j}^{n}-W_{i,j}^{n}}{\Delta\theta_i}-\sin\theta_{i-\frac{1}{2}}\frac{W_{i,j}^{n}-W_{i-1,j}^{n}}{\Delta\theta_i}\right)\right. \\ &\quad \left. + \frac{1}{\sin^2\theta_i}\left(\frac{W_{i,j+1}^{n}-2W_{i,j}^{n}+W_{i,j-1}^{n}}{(\Delta\phi)^2}\right)\right].\end{aligned}$$

Meanwhile, we set the parameters in (1.1) as: $d = 10^{-6}, D = 1.0, a_w = 0.75, a_g = 0.50, a_b = 0.25, \delta = 0.003265, f = 0.3, \overline{w} = 295.5, q = 40$ and $\sigma = 5.67 \times 10^{-8}$. The incoming energy $R(\theta)$ is taken as

$$R(\theta) = \frac{4 \cdot 917}{\pi} L \sin\theta$$

where $L = 0.85$ is the same as in Watson-Lovelock [20]. Initial functions $u_0(\theta,\phi)$, $v_0(\theta,\phi)$, $w_0(\theta,\phi)$ are constructed by slightly perturbing constant functions $\bar{u}_0(\theta,\phi) \equiv 0.321, \bar{v}_0(\theta,\phi) \equiv 0.291$ and $\bar{w}_0(\theta,\phi) \equiv 290.96$ for $(\theta,\phi) \in (0,\pi) \times [0,2\pi)$. In numerical computations, we apply the periodic boundary conditions at $j = 0$ and $j = M$ and the latitudinal boundary conditions:

$$\begin{aligned} &U_{0,j} = U_{1,\frac{M}{2}+j}, && U_{N+1,j} = U_{N,\frac{M}{2}+j}, && (j = 0,1,...,M/2), \\ &U_{0,j} = U_{1,-\frac{M}{2}+j}, && U_{N+1,j} = U_{N,-\frac{M}{2}+j}, && (j = M/2+1,...,J), \end{aligned}$$

at $i = 1$ and $i = N$. It is the same for $V_{i,j}$ and $W_{i,j}$.

The numerical solution to (1.1) stabilizes asymptotically. About $t = 600$, its evolution shows down evidently. This may mean that the solution is attracted by the global attractor. Fig.1 illustrates the graphs of $u(\theta,\phi,t)$, $v(\theta,\phi,t)$, $w(\theta,\phi,t)$ at $t = 600$.

Their graphs show a clear segregation strip pattern. The interface is given by zigzag curves which are almost parallel with the equator.

Similar results are obtained by another numerical method.

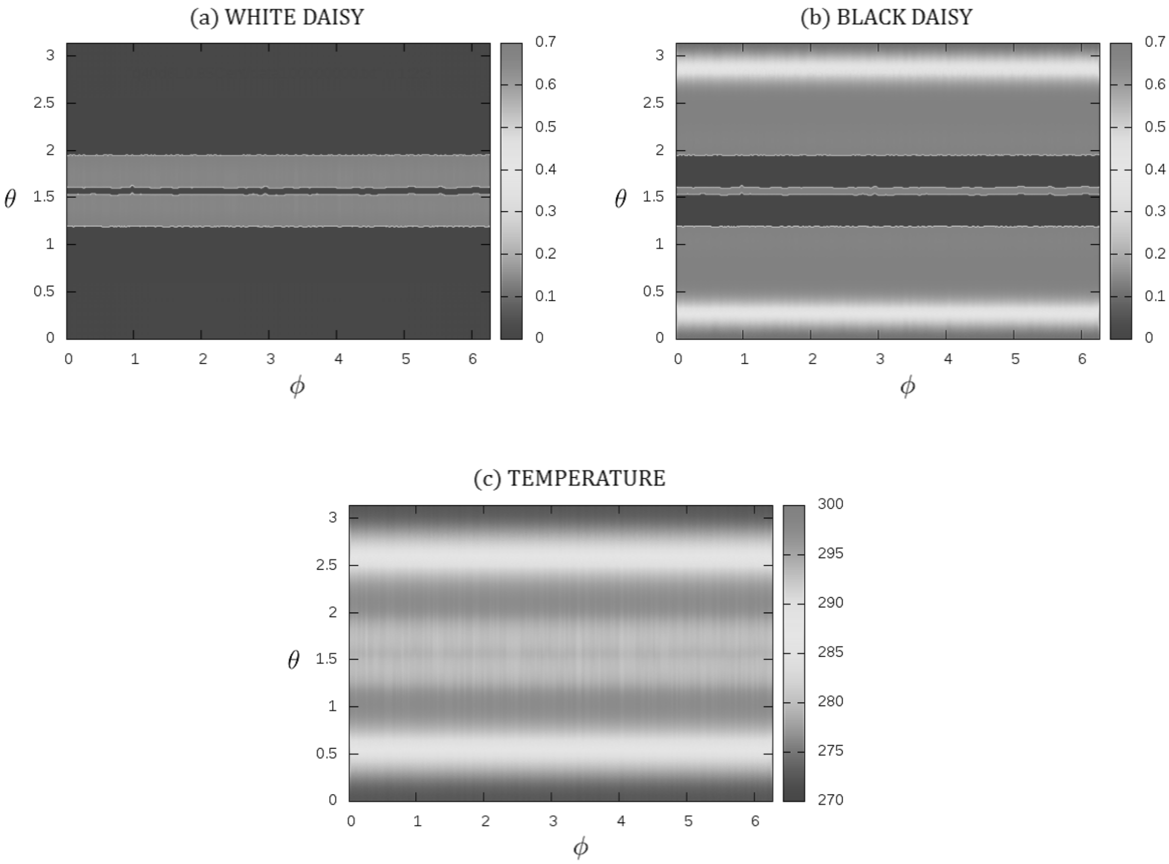


Fig. 1: (a) Graph of $u(\theta, \phi)$, (b) Graph of $v(\theta, \phi)$ and (c) Graph of $w(\theta, \phi)$ at time $t = 600$.

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# BIFURCATIONS WITH MULTI-DIMENSIONAL KERNEL IN A CHEMOTAXIS-GROWTH SYSTEM

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Abstract. We study the bifurcation problem for a chemotaxis-growth system with logistic growth in a two-dimensional rectangular domain. We apply the local bifurcation theorem by Ambrosetti and Prodi that does not require one-dimensional degeneration of the linearized operator around trivial solutions. We then obtain bifurcation solutions with two- and three-dimensional degeneration indicating spatially regular nesting patterns.

## 1 Introduction.

Budrene and Berg [2, 3] found that the chemotactic bacteria *E. coli* form remarkable macroscopic regular patterns in their colony, resulting from the interplay between diffusion, chemotaxis and growth. Mimura and Tsujikawa [12] studied the following chemotaxis-growth system to elucidate the mechanisms for pattern formation processes:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - \chi\nabla\cdot(u\nabla\rho) + f(u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \rho}{\partial t} = \Delta\rho - b\rho + cu & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$, and $\partial/\partial n$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The function $u(x,t)$ is the population density of the chemotactic bacteria at position $x \in \Omega$ and time $t \in [0,\infty)$, and $\rho(x,t)$ is the concentration of chemical substance that is produced by the individuals. The function $f(u)$ denotes the growth of $u$, and several different forms have been proposed for $f(u)$ [7, 16]. We assume in this paper that $f(u)$ is a logistic saturating growth function,

$$f(u) = au(1-\mu u),$$

where $a$ and $\mu$ are positive constants. The other coefficients $b$, $c$, $d$ and $\chi$ are also positive constants. The advection term $-\chi\nabla\cdot(u\nabla\rho)$ corresponds to chemotaxis of bacteria, and the coefficient $\chi$ indicates the intensity of chemotaxis.

In this article, we consider a bifurcation problem for the stationary state of (E). In a two-dimensional rectangular domain, Kuto et al. [11] proved that one-mode bifurcations occurred for the uniform state $(u,\rho) = (1/\mu, c/(\mu b))$, that is, stripe and rectangle patterns occurred along destabilized $x$ and $y$-directional double Fourier modes. Kuto et al. [11] also showed solutions for a hybrid mode bifurcation that formed hexagonal patterns. In the

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analytic proof, Kuto et al. [11] applied the classical local bifurcation theorem by Crandall and Rabinowitz [4], which requires one-dimensional degeneration of a linearized operator around the uniform state. In other words, the kernel of linearized operator is prohibited from containing any hybrid modes in the Crandall and Rabinowitz theorem [4]. Hexagonal pattern formation, however, does use two destabilized hybrid Fourier modes (see Section 4). Kuto et al. [11] therefore introduced a restricted functional space possessing $2\pi/3$-rotational symmetry, which is a closed one-dimensional subspace in the universal space, and applied the Crandall and Rabinowitz theorem in this subspace (see also [13]). An equivalent approach for hybrid mode bifurcation is the bifurcation theorem considered in [5], which is a similar bifurcation theorem that considers the symmetry of patterns.

In this article, we apply another type of bifurcation theorem introduced by Ambrosetti and Prodi [1], which admits multi-dimensional kernels of linearized operators. Indeed, we obtain another hexagonal pattern solution to (E) which does not have $2\pi/3$-rotational symmetry (see Figure 2). In addition, we study the bifurcation problem for (E) with three-dimensional kernel and obtain novel spatially regular nesting patterns. These patterns are not demonstrated in [11]. The advantage of no one-dimensional restrictions is, for instance, there is no need to know all of the symmetries in the hybrid mode in advance. Another advantage is the simpler the sufficient conditions for bifurcation are, the easier we can implement the algorithm for obtaining bifurcating solutions in a computer program.

We conclude this introduction by referring as other results as follows: In the one-dimensional chemotaxis-growth system (E), Kurata et al. [10] demonstrated that the instability of solutions occurred for the uniform state, and time-periodic solutions successively bifurcated from the nontrivial stationary solutions by utilizing the bifurcation software AUTO. Painter and Hillen [15] also showed that one-dimensional periodic solutions successively bifurcated and resulted in chaotic dynamics. If the logistic term is absent for (E) ($f(u) \equiv 0$), the chemotaxis-growth system (E) reduces to the celebrated Keller-Segel chemotaxis system [9], which admits the blow-up of solutions by overcrowding due to chemotaxis [6, 8, 18]. Meanwhile, for the two-dimensional chemotaxis-growth system (E), Osaki et al. [14] showed the existence of global-in-time solutions and a compact global attractor for the dynamical system generated by these solutions. For the case of more than three dimensions, Winkler [17] obtained the global-in-time existence of solutions for (E) under the quadratic suppression of $f(u)$ for sufficiently large $\mu$. Zheng [19] also recently extended the valid region of $\mu$ for the global-in-time existence of solutions to the $n$-dimensional chemotaxis-growth system (E).

This paper is organized as follows. First, we provide a brief review of the multi-dimensional bifurcation theorem by Ambrosetti and Prodi [1] as a preliminary. In Section 3 we frame the chemotaxis-growth system (E) as a nonlinear bifurcation problem. In Sections 4 and 5 we study the bifurcation problems with two- and three-dimensional degeneration of the linearized operators, respectively.

## 2 A brief review of the multi-dimensional bifurcation theorem.

Let $F$ be a nonlinear operator such that $F \in \mathcal{C}^\infty((\lambda_1, \lambda_2) \times X; Y)$. Here, $X$ and $Y$ are Banach spaces, and $(\lambda_1, \lambda_2)$ is an interval in $\mathbb{R}$. We consider a bifurcation problem for a functional equation in the Banach space $Y$:

$$F(\lambda, u) = 0 \ \in Y. \tag{2.1}$$

Assume that the nonlinear equation (2.1) has a trivial solution $u = 0$ for arbitrary $\lambda$, i.e., $F(\lambda, 0) = 0$, ${}^\forall \lambda \in (\lambda_1, \lambda_2)$. We denote a bifurcation point by $\lambda = \lambda^*$. Then the linearized operator of $F(\lambda, u)$ around $(\lambda, u) = (\lambda^*, 0)$, $L = F_u(\lambda^*, 0) \in \mathcal{L}(X; Y)$, should degenerate, that is, $L$ is not invertible, and then $V := \mathcal{K}(L) \neq \{0\}$. Let us denote $R := \mathcal{R}(L)$. Assume also that $V$ has a topological complement $W$ in $X$, and $R$ is closed and also has a topological complement $Z$ in $Y$:

$$X = V \oplus W, \quad Y = R \oplus Z.$$

The Taylor expansion of $F(\lambda, u)$ around $(\lambda, u) = (\lambda^*, 0)$ is expressed as

$$F(\lambda^* + \mu, u) = Lu + \mu M u + \frac{1}{2}\mathcal{B}[u, u] + \psi(\mu, u), \tag{2.2}$$

where $M := F_{u\lambda}(\lambda^*, 0)$, $\mathcal{B} := F_{uu}(\lambda^*, 0)$, and $\psi(\mu, u)$ is a smooth function such that $\psi(\mu, 0) \equiv 0$, $\psi_u(0, 0) = 0$, $\psi_{uu}(0, 0) = 0$, and $\psi_{\lambda u}(0, 0) = 0$.

By denoting the solution as $u = \mu(v+w)$, Ambrosetti and Prodi [1] derived a bifurcation equation with conjugate projections

$$P : Y \to Z, \quad Q : Y \to R.$$

By substituting $u = \mu(v + w)$ into the equation, we have

$$PM(v + w) + \frac{1}{2}P\mathcal{B}[v + w, v + w] + \mu P\tilde{\psi}(\mu, v, w) = 0, \tag{2.3}$$

$$\tilde{\Phi}(\mu, v, w) := Lw + \mu QM(v + w) + \frac{1}{2}\mu Q\mathcal{B}[v + w, v + w] + \mu^2 Q\tilde{\psi}(\mu, v, w) = 0. \tag{2.4}$$

Here, $\psi(\mu, \mu(v + w)) = \mu^3\tilde{\psi}(\mu, v, w)$ for a smooth function $\tilde{\psi}(\mu, v, w)$. Since $\tilde{\Phi}(0, v, 0) = 0$ for any $v \in V$ and $\tilde{\Phi}_w(0, v, 0) = L \neq 0$, the nonlinear equation $\tilde{\Phi}(\mu, v, w) = 0$ (which generally has an infinite number of dimensions) can be uniquely solved in $w$ around the neighborhood $\Lambda \times \mathcal{V} \times \mathcal{W}$ of $(\mu, v, w) = (0, v^*, 0)$, where $v^* \in V$ is arbitrarily fixed in $V$. Then, the component $w$ can be expressed uniquely as $w = \mu\,\gamma(\mu, v) \in \mathcal{W}$, $(\mu, v) \in \Lambda \times \mathcal{V}$, with a smooth function $\gamma$ depending on $v^*$. Substituting this into the equation (2.3) (which is finite dimensional in a favorable case, e.g. $L$ is a Fredholm operator), we obtain the bifurcation equation for $\Lambda \times \mathcal{V}$:

$$\begin{aligned} N(\mu, v) := PM(v + \mu\gamma(\mu, v)) + \frac{1}{2}P\mathcal{B}[v + \mu\gamma(\mu, v), v + \mu\gamma(\mu, v)] \\ + \mu P\tilde{\psi}(\mu, v, \mu\gamma(\mu, v)) = 0 \in Z, \end{aligned} \tag{2.5}$$

where $N(\mu, v)$ is smooth. We here note again that when the dimension of the subspace $Z \subset Y$ is finite, the bifurcation equation (2.5) consists of a finite number of equations.

The multi-dimensional bifurcation theorem introduced by Ambrosetti and Prodi [1] is as follows:

**Theorem 2.1.** [1, Theorem 5.1, p.102] Assume that two Banach spaces $X$ and $Y$ satisfy the conditions that $V = \mathcal{K}(L)$ has a topological complement in $X$, and $R = \mathcal{R}(L)$ is closed and has a topological complement in $Y$. Assume also that: for the nonlinear problem (2.5), there exists $v^* \in V$, $v^* \neq 0$, such that

(a) $N(0, v^*) = PMv^* + \frac{1}{2}P\mathcal{B}[v^*, v^*] = 0$;

(b) the linear operator $N_v(0, v^*) = S : V \to Z$, $Sv = PMv + P\mathcal{B}[v^*, v]$, is invertible.

Then, there exists a local branch of nontrivial solutions $(\lambda, u(\lambda))$ to (2.1) which bifurcates from $(\lambda^*, 0)$ such that

$$\lambda = \lambda^* + \mu, \quad u = \mu\,[v^* + \mu\tilde{v}(\mu)],$$

where $\tilde{v}(\mu)$ is a smooth function of $\mu$. □

For the complete proof of Theorem 2.1 we refer to [1, p.102], but the above results are clear from the implicit function theorem. Indeed, for the bifurcation equation $N(\mu, v) = 0$, there exists a unique solution $(\mu, v(\mu))$ for small $\mu$ and $v(0) = v^*$. Then, by substituting $v = v(\mu)$ into $u = \mu(v + w)$ and $w = \mu\,\gamma(\mu, v)$ we obtain a nontrivial solution $u = \mu\,[\,v(\mu) + \mu\gamma(\,\mu, v(\mu)\,)\,]$ near $\mu = 0$. From the Taylor expansion of $v(\mu)$ around $\mu = 0$, we obtain the local solution $(\lambda, u(\lambda))$ to (2.1). □

## 3 Bifurcation equation of chemotaxis-growth model.

We return to the bifurcation problem for the following stationary system of (E):

$$\text{(SE)} \qquad \begin{cases} d\Delta u - \chi\nabla\cdot(u\nabla\rho) + au(1-\mu u) = 0 & \text{in } \Omega, \\ \Delta\rho - b\rho + cu = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega, \\ u \geq 0,\ \rho \geq 0 & \text{in } \Omega. \end{cases}$$

The spatial domain is specified as

$$\Omega = \left(0, \frac{\pi}{l}\right) \times \left(0, \frac{\pi}{\sqrt{3}l}\right). \tag{3.1}$$

Here $l > 0$ is a control parameter for bifurcation. The setting of Banach (Hilbert) spaces $X$ and $Y$ is

$$X = H^2_N(\Omega) \times H^2_N(\Omega), \quad Y = L^2(\Omega) \times L^2(\Omega)$$

with norms:

$$\|U\|_X := \sqrt{\|u\|^2_{H^2} + \|\rho\|^2_{H^2}}, \quad \|U\|_Y := \sqrt{\|u\|^2_{L^2} + \|\rho\|^2_{L^2}}, \quad U = {}^T[u\ \rho],$$

where $H^2_N(\Omega) = \left\{w \in H^2(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega\right\}$. Then, the inner product of $Y$ is:

$$\langle U_1, U_2\rangle_Y := \langle u_1, u_2\rangle_{L^2} + \langle \rho_1, \rho_2\rangle_{L^2}, \quad U_1 = {}^T[u_1\ \rho_1],\ U_2 = {}^T[u_2\ \rho_2] \in Y.$$

We will show the existence of nontrivial solutions bifurcating from the positive trivial equilibrium to (SE):

$$U^* = \begin{bmatrix} u^* \\ \rho^* \end{bmatrix} := \begin{bmatrix} 1/\mu \\ c/(\mu b) \end{bmatrix}.$$

We set $\chi$ as a bifurcation parameter, and $F : (0,\infty) \times X \to Y$ by

$$F(\chi, U) := \begin{bmatrix} d\Delta u - \chi\nabla\cdot(u\nabla\rho) + au(1-\mu u) \\ \Delta\rho - b\rho + cu \end{bmatrix}. \tag{3.2}$$

Indeed, the nonlinear terms are $L^2$-valued functions in view of $\|\nabla\cdot(u\nabla\rho)\|_{L^2} \leq C\|u\nabla\rho\|_{H^1} \leq C\|u\|_{H^2}\|\rho\|_{H^2}$ and $\|u^2\|_{L^2} = \|u\|^2_{L^4} \leq C\|u\|_{H^1}\|u\|_{L^2}$. Then, the nonlinear bifurcation problem for (SE) can be expressed as

$$F(\chi, U) = 0 \ \in Y, \quad (\chi, U) \in (0,\infty) \times X.$$

The linearized operator

$$L := F_U\,(\chi^*, U^*) \in \mathcal{L}(X; Y)$$

around $U = U^*$ is calculated as

$$L \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} d\Delta h - \frac{\chi^*}{\mu}\Delta k - ah \\ \Delta k + ch - bk \end{bmatrix} = \begin{bmatrix} d\Delta - a & -\frac{\chi^*}{\mu}\Delta \\ c & \Delta - b \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}, \quad \begin{bmatrix} h \\ k \end{bmatrix} \in X; \tag{3.3}$$

and we also obtain the second order derivatives of $F(\chi, U)$ as follows:

$$M \begin{bmatrix} h \\ k \end{bmatrix} = F_{U,\chi}(\chi^*, U^*) \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mu}\Delta k \\ 0 \end{bmatrix},$$

$$\mathcal{B}\left( \begin{bmatrix} h \\ k \end{bmatrix}, \begin{bmatrix} \tilde{h} \\ \tilde{k} \end{bmatrix} \right) = F_{U,U}(\chi^*, U^*) \left( \begin{bmatrix} h \\ k \end{bmatrix}, \begin{bmatrix} \tilde{h} \\ \tilde{k} \end{bmatrix} \right) = \begin{bmatrix} -\chi^*[\nabla \cdot (h\nabla\tilde{k}) + \nabla \cdot (\tilde{h}\nabla k)] - 2a\mu h\tilde{h} \\ 0 \end{bmatrix}$$

for ${}^T[h\ k]$, ${}^T[\tilde{h}\ \tilde{k}] \in X$. From the above we can set the bifurcation equation $N(\lambda, U) = 0$ for (SE) in the neighborhood of $(0, U^*) \in (-\varepsilon, \varepsilon) \times V$ with $\chi = \chi^* + \lambda$ and small $\varepsilon$.

We introduce double cosine functions for the usual orthogonal basis of $L^2(\Omega)$ under homogeneous Neumann boundary conditions:

$$\{\phi_m(x)\,\psi_n(y) \mid m, n \geq 0\}, \ \phi_m(x) = \cos(lmx), \ \psi_n(y) = \cos(\sqrt{3}lny).$$

Then, the orthogonal basis of $Y$ is induced as:

$$\left\{ {}^T\left[ h_{mn}\,\phi_m(x)\psi_n(y) \quad k_{mn}\,\phi_m(x)\psi_n(y) \right] \mid m, n \geq 0 \right\}.$$

**Proposition 3.1.** The linearized operator $L = F_U(\chi^*, U^*)$ degenerates at $\chi^* = \chi(m, n)$, where $\chi(m, n)$ is defined as

$$\chi(m, n) := \frac{\mu}{c}\left[ dl^2(m^2 + 3n^2) + \frac{ab}{l^2(m^2 + 3n^2)} + a + bd \right]. \tag{3.4}$$

**Proof.** Consider the linearized equation $L\,{}^T[k\ h] = 0$ with homogeneous Neumann boundary condition $\frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0$ on $\partial\Omega$. Substituting the two cosine Fourier series for $h(x, y)$ and $k(x, y)$:

$$\begin{bmatrix} h \\ k \end{bmatrix} = \sum_{m,n=0}^{\infty} \begin{bmatrix} h_{mn} \\ k_{mn} \end{bmatrix} \phi_m(x)\psi_n(y) \tag{3.5}$$

to the linearized equation, we have an equivalent equation for each Fourier coefficient ${}^T[h_{mn}\ k_{mn}]$ such that

$$\begin{bmatrix} -dl^2(m^2 + 3n^2) - a & \frac{\chi}{\mu} l^2(m^2 + 3n^2) \\ c & -l^2(m^2 + 3n^2) - b \end{bmatrix} \begin{bmatrix} h_{mn} \\ k_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad m, n \in \mathbb{N}. \tag{3.6}$$

This indicates that there exists a nontrivial solution ${}^T[h_{mn}\ k_{mn}]$ to (3.6), if and only if the following characteristic equation holds:

$$\begin{vmatrix} -dl^2(m^2 + 3n^2) - a & \frac{\chi}{\mu} l^2(m^2 + 3n^2) \\ c & -l^2(m^2 + 3n^2) - b \end{vmatrix} \tag{3.7}$$
$$= [\, dl^2(m^2 + 3n^2) + a\,][\, l^2(m^2 + 3n^2) + b\,] - \chi\frac{c}{\mu} l^2(m^2 + 3n^2) = 0.$$

Solving this for $\chi$, we have (3.4). □

Let $V$ be the kernel of the linearized operator $L$: $V = \mathcal{K}(L)$. Then, for the composition of $V$ we have:

**Proposition 3.2.** The kernel $V$ is the linear span of the cosine Fourier basis

$$\Phi_{mn} := \begin{bmatrix} 1 \\ \eta_{mn} \end{bmatrix} \phi_m(x)\psi_n(y)$$

of which modes $(m,n)$ satisfy the characteristic equation (3.7), where

$$\eta_{mn} = \frac{c}{l^2(m^2+3n^2)+b}.$$

**Proof.** The proof is given in Kuto et al. [11, Theorem 5.1]. □

## 4 Two-dimensional kernel bifurcation of chemotaxis-growth model.

Kuto et al. [11] studied the bifurcation with two-dimensional kernel for (SE) by restricting the functional space to $2\pi/3$-rotational symmetry. The multiplicity occurred in the lowest Fourier modes $(m,n)=(2,0)$ and $(1,1)$, in the sense that $m^2+3n^2 = 2^2+3\cdot 0^2 = 1^2+3\cdot 1^2 = 4$, and there are not multiple solutions of $(m,n)$ for $m^2+3n^2 \le 3$.

In this section, we study the two-dimensional kernel bifurcation under the multiplicity of Fourier modes $(m,n)=(2,0)$ and $(1,1)$ without a one-dimensional kernel restriction. The kernel $V$ is actually the linear span of the two Fourier bases:

$$V = \operatorname{span}\{\Phi_{20},\ \Phi_{11}\},$$

and hence $\dim V = 2$. Since $R$ and $W$ are isomorphic on $L|_W$, $Z$ is the same linear span:

$$Z = \operatorname{span}\{\Phi_{20},\ \Phi_{11}\}.$$

The projection $P : Y \to Z$ is naturally introduced as:

$$P\,\Phi = \frac{\langle \Phi, \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2}\,\Phi_{20} + \frac{\langle \Phi, \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2}\,\Phi_{11} \in Z, \quad \Phi \in Y, \tag{4.1}$$

Where $\|\Phi_{20}\|_Y^2 = \frac{(1+\pi^2)\pi^2}{2\sqrt{3}l^2}$ and $\|\Phi_{11}\|_Y^2 = \frac{(1+\pi^2)\pi^2}{4\sqrt{3}l^2}$.

We extract $v^* \in V$ satisfying the sufficient conditions (a) and (b) in Theorem 2.1. By denoting

$$v^* = \alpha\,\Phi_{20} + \beta\,\Phi_{11} := \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \in V; \quad \alpha,\ \beta \in \mathbb{R}, \tag{4.2}$$

we first determine $\alpha$ and $\beta$, as $v^*$ satisfies the condition (a). The values $Mv^*$ and $\mathcal{B}[v^*, v^*]$ are calculated as:

$$Mv^* = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2^* \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4l^2\eta_{20}}{\mu}\left[\alpha\phi_2(x) + \beta\phi_1(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{4.3}$$

$$\mathcal{B}[v^*, v^*] = \begin{bmatrix} -2\left[\chi^*\left(\nabla\cdot(v_1^*\nabla v_2^*)\right) + a\mu(v_1^*)^2\right] \\ 0 \end{bmatrix} = \begin{bmatrix} -\eta_{20}\chi^*\Delta\left(v_1^*\right)^2 - 2a\mu(v_1^*)^2 \\ 0 \end{bmatrix}. \tag{4.4}$$

By straightforward calculation we have

$$\frac{\langle Mv^*, \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\alpha, \quad \frac{\langle Mv^*, \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\beta.$$

We then obtain

$$PMv^* = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left(\alpha\Phi_{20}+\beta\Phi_{11}\right). \tag{4.5}$$

Similarly, since

$$\frac{\langle \mathcal{B}[v^*,v^*],\Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{2\chi^* l^2\eta_{20}-a\mu}{2(1+\eta_{20}^2)}\beta^2, \quad \frac{\langle \mathcal{B}[v^*,v^*],\Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{2(2\chi^* l^2\eta_{20}-a\mu)}{1+\eta_{20}^2}\alpha\beta,$$

we have

$$P\mathcal{B}[v^*,v^*] = \frac{2\chi^* l^2\eta_{20}-a\mu}{2(1+\eta_{20}^2)}\left(\beta^2\Phi_{20}+4\alpha\beta\Phi_{11}\right). \tag{4.6}$$

From the above, the condition (a) of Theorem 2.1 results in

$$\begin{aligned} PMv^* + \frac{1}{2}P\mathcal{B}[v^*,v^*] &= \frac{1}{4(1+\eta_{20}^2)\mu}\left[16l^2\eta_{20}\alpha + \mu(2\chi^* l^2\eta_{20}-a\mu)\beta^2\right]\Phi_{20} \\ &\quad + \frac{\beta}{(1+\eta_{20}^2)\mu}\left[4l^2\eta_{20} + \mu(2\chi^* l^2\eta_{20}-a\mu)\alpha\right]\Phi_{11} = 0. \end{aligned} \tag{4.7}$$

As $\Phi_{20}$ and $\Phi_{11}$ are linearly independent in $Y$, we obtain the coefficients $(\alpha,\beta)\neq(0,0)$ under the condition $2\chi^* l^2\eta_{20}-a\mu\neq 0$ where $(\alpha,\beta)=(A,-2A)$, $(A,2A)$ with $A=-\frac{4l^2\eta_{20}}{\mu(2\chi^* l^2\eta_{20}-a\mu)}$. This shows that the $v^*$ satisfying condition (a) are the following two candidates:

$$v^* = A\left(\Phi_{20}-2\Phi_{11}\right), \quad A\left(\Phi_{20}+2\Phi_{11}\right), \quad \text{where } A = -\frac{4l^2\eta_{20}}{\mu(2\chi^* l^2\eta_{20}-a\mu)}. \tag{4.8}$$

We display the profiles of these functions in Figure 1.

We here note that the latter result was first demonstrated in [11], and the former is newly derived in this paper, indeed, the former result does not have $2\pi/3$-rotational symmetry (see Figure 2).

Next, we consider the condition (b), that is, the invertibility of the operator $S: V\to Z$, $Sv = PMv + P\mathcal{B}[v^*,v]$, $v\in V$, with $v^*\in V$ fixed as in (4.8). Let us denote $v\in V$ as

$$v = \eta\,\Phi_{20} + \zeta\,\Phi_{11} := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \quad \eta,\ \zeta\in\mathbb{R}. \tag{4.9}$$

Then, we have

$$Mv = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4l^2\eta_{20}}{\mu}\left[\eta\phi_2(x)+\zeta\phi_1(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{4.10}$$

$$\begin{aligned} \mathcal{B}[v^*,v] &= \begin{bmatrix} -\chi^*\left[\nabla\cdot(v_1^*\nabla v_2)+\nabla\cdot(v_1\nabla v_2^*)\right]-2a\mu v_1^* v_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\eta_{20}\chi^*\Delta\left(v_1^* v_1\right) - 2a\mu v_1^*\, v_1 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.11}$$

Since

$$\frac{\langle Mv,\Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta, \quad \frac{\langle Mv,\Phi_{20}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta,$$

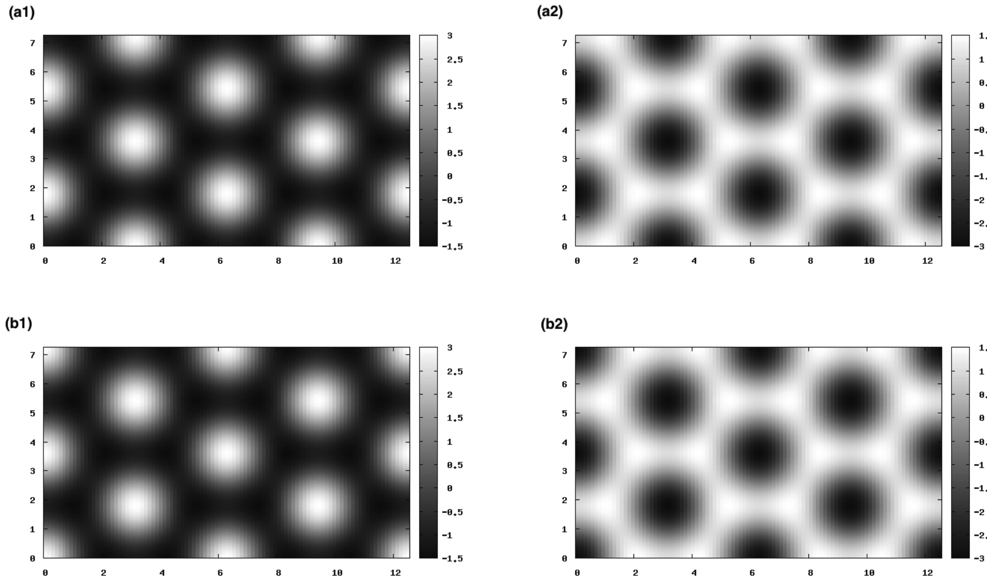


Figure 1: Plots of the functions belonging to the two-dimensional kernel $V = \mathrm{span}\{\Phi_{20}, \Phi_{11}\}$. The spatial domain is $\Omega = (0, 4\pi) \times (0, 4\sqrt{3}\pi)$. (a1) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = 1 > 0$. (a2) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = -1 < 0$. (b1) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = 1 > 0$. (b2) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = -1 < 0$.

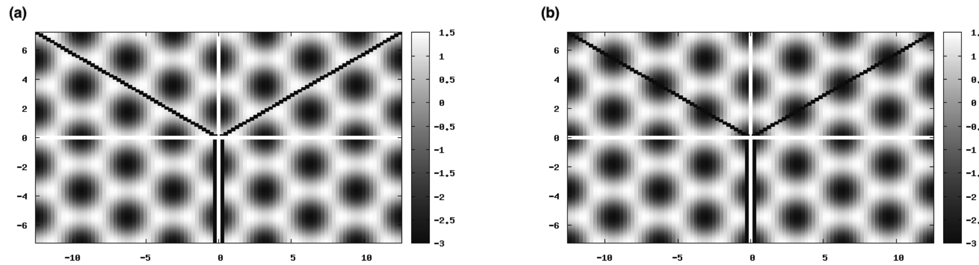


Figure 2: The spanning $v^*$ in the spatial domain $(-4\pi, 4\pi) \times (-4\sqrt{3}\pi, 4\sqrt{3}\pi)$. The white horizontal and vertical lines represent the $x$ and $y$ axes, respectively. The black lines are auxiliary axes in the directions of $\pi/6$, $5\pi/6$, $3\pi/2$. (a) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = -1 < 0$, which does not have $2\pi/3$-rotational symmetry. And, (b) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = -1 < 0$, which have $2\pi/3$-rotational symmetry.

we then obtain

$$PMv = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left(\eta\Phi_{20} + \zeta\Phi_{11}\right).$$

Similarly, since

$$\frac{\langle \mathcal{B}[v^*, v], \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta,$$
$$\frac{\langle \mathcal{B}[v^*, v], \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}(\eta-\zeta),$$

we have

$$P\mathcal{B}[v^*, v] = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left[\zeta\Phi_{20} + 2(\eta-\zeta)\Phi_{11}\right].$$

From this, it follows that

$$Sv = \left(\frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta + \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta\right)\Phi_{20} + \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta\,\Phi_{11} := [\Phi_{20}\ \Phi_{11}]\,\widetilde{S}\begin{bmatrix}\eta\\ \zeta\end{bmatrix}.$$

Here, $\widetilde{S}$ is the representation matrix of $S$:

$$\widetilde{S} = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}.$$

Since

$$\det\widetilde{S} = -\frac{8l^4\eta_{20}^2}{\mu^2(1+\eta_{20}^2)^2} \neq 0,$$

the operator $S$ is isomorphic.

We finally arrive at the main result of this section.

**Theorem 4.1.** Let $v^* \in V$ be the functions defined in (4.8), and $\chi^* = \chi(m,n)$. Then, under the conditions

$$2\chi^* l^2\eta_{20} - a\mu \neq 0,$$

there exists a local branch of nontrivial solutions $(\chi(\lambda), U(\lambda)) \in (0,\infty)\times X$ to (SE), with small parameter $\lambda \in (-\varepsilon, \varepsilon)$, which bifurcate from $(\chi^*, U^*)$ such that

$$\chi(\lambda) = \chi^* + \lambda, \quad U(\lambda) = U^* + \lambda[\,v^* + \lambda\tilde{v}(\lambda)\,],$$

where $\tilde{v}(\lambda)$ is a smooth function of $\lambda$.

## 5 Three-dimensional kernel bifurcation of chemotaxis-growth model.

In this section, we study the lowest dimension-three bifurcation along the Fourier modes $(m,n) = (1,3)$, $(4,2)$, $(5,1)$. Indeed, these are the triple solutions for $m^2+3n^2 = 28$, and there are no triple solutions for the case $m^2+3n^2 \leq 27$. Three-dimensional bifurcation is not analyzed in [11].

The kernel $V$ is the linear span of the three Fourier bases:

$$V = \operatorname{span}\{\Phi_{13},\ \Phi_{42},\ \Phi_{51}\},$$

and hence $\dim V = 3$. Since $R$ and $W$ are isomorphic on $L|_W$, $Z$ is the same linear span:

$$Z = \operatorname{span}\{\Phi_{13},\ \Phi_{42},\ \Phi_{51}\}.$$

The projection $P : Y \to Z$ is naturally introduced as:

$$P\,\Phi = \frac{\langle \Phi, \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2}\,\Phi_{13} + \frac{\langle \Phi, \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2}\,\Phi_{42} + \frac{\langle \Phi, \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2}\,\Phi_{51} \in Z, \quad \Phi \in Y, \tag{5.1}$$

where $\|\Phi_{13}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$, $\|\Phi_{42}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$ and $\|\Phi_{51}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$.

We set $v^* \in V$ so as to satisfy the sufficient conditions (a) and (b) in Theorem 2.1. By denoting

$$v^* = \alpha\,\Phi_{13} + \beta\,\Phi_{42} + \gamma\,\Phi_{51} \coloneqq \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \in V; \quad \alpha,\ \beta,\ \gamma \in \mathbb{R}, \tag{5.2}$$

we first determine $\alpha$, $\beta$ and $\gamma$, as $v^*$ satisfies the condition (a). The values $Mv^*$ and $\mathcal{B}[v^*, v^*]$ are calculated as:

$$Mv^* = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2^* \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{28l^2\eta_{13}}{\mu}\left[\alpha\phi_1(x)\psi_3(y) + \beta\phi_4(x)\psi_2(y) + \gamma\phi_5(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{5.3}$$

$$\mathcal{B}[v^*, v^*] = \begin{bmatrix} -2\left[\chi^*\left(\nabla\cdot(v_1^*\nabla v_2^*)\right) + a\mu(v_1^*)^2\right] \\ 0 \end{bmatrix} = \begin{bmatrix} -\eta_{13}\chi^*\Delta\,(v_1^*)^2 - 2a\mu(v_1^*)^2 \\ 0 \end{bmatrix}. \tag{5.4}$$

By straightforward calculation we have

$$\frac{\langle Mv^*, \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\alpha, \quad \frac{\langle Mv^*, \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\beta, \quad \frac{\langle Mv^*, \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\gamma.$$

We then obtain

$$PMv^* = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\alpha\,\Phi_{13} + \beta\,\Phi_{42} + \gamma\,\Phi_{51}\right). \tag{5.5}$$

Similarly, since

$$\begin{aligned} \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\beta\gamma, \\ \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\gamma\alpha, \\ \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\alpha\beta, \end{aligned}$$

we have

$$P\mathcal{B}[v^*, v^*] = \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\left(\beta\gamma\,\Phi_{13} + \gamma\alpha\,\Phi_{42} + \alpha\beta\,\Phi_{51}\right). \tag{5.6}$$

From the above, the condition (a) of Theorem 2.1 gives

$$\begin{aligned} PMv^* + \frac{1}{2}P\mathcal{B}[v^*, v^*] = \frac{1}{2\mu(1+\eta_{13}^2)}\Big(&\left[56l^2\eta_{13}\alpha + \mu(14\chi^* l^2\eta_{13} - a\mu)\beta\gamma\right]\Phi_{13} \\ &+ \left[56l^2\eta_{13}\beta + \mu(14\chi^* l^2\eta_{13} - a\mu)\gamma\alpha\right]\Phi_{42} \\ &+ \left[56l^2\eta_{13}\gamma + \mu(14\chi^* l^2\eta_{13} - a\mu)\alpha\beta\right]\Phi_{51}\Big) = 0. \end{aligned} \tag{5.7}$$

As $\Phi_{13}$, $\Phi_{42}$ and $\Phi_{51}$ are linearly independent in $Y$, we obtain the coefficients $(\alpha,\beta,\gamma)\neq(0,0,0)$ under the condition $14\chi^* l^2\eta_{13}-a\mu\neq 0$ that $(\alpha,\beta,\gamma)=(\widetilde{A},\widetilde{A},\widetilde{A})$, $(\widetilde{A},-\widetilde{A},-\widetilde{A})$, $(-\widetilde{A},\widetilde{A},-\widetilde{A})$, $(-\widetilde{A},-\widetilde{A},\widetilde{A})$ with $\widetilde{A}=-\frac{56l^2\eta_{13}}{\mu(14\chi^* l^2\eta_{13}-a\mu)}$. This shows that the $v^*$ satisfying the condition (a) are the following four candidates:

$$
\begin{aligned}
(5.8)\quad v^* &= \widetilde{A}\,(\Phi_{13}+\Phi_{42}+\Phi_{51}),\quad \widetilde{A}\,(\Phi_{13}-\Phi_{42}-\Phi_{51}),\\
&\widetilde{A}\,(-\Phi_{13}+\Phi_{42}-\Phi_{51}),\quad \widetilde{A}\,(-\Phi_{13}-\Phi_{42}+\Phi_{51}),\quad \text{where } \widetilde{A}=-\frac{56l^2\eta_{13}}{\mu(14\chi^* l^2\eta_{13}-a\mu)}.
\end{aligned}
$$

We display the profiles of these functions in Figure 3. We here note that only the first $v^*=\widetilde{A}\,(\Phi_{13}+\Phi_{42}+\Phi_{51})$ has $2\pi/3$-rotational symmetry (see Figure 4).

Next, we consider the condition (b), that is, the invertibility of the operator $S:V\to Z$, $Sv=PMv+P\mathcal{B}[v^*,v]$, $v\in V$, with $v^*\in V$ fixed as in (5.8). Let us denote $v\in V$ as

$$
(5.9)\qquad v=\eta\,\Phi_{13}+\zeta\,\Phi_{42}+\xi\,\Phi_{51}:=\begin{bmatrix}v_1\\ v_2\end{bmatrix};\quad \eta,\ \zeta,\ \xi\in\mathbb{R}.
$$

Then, we have

$$
(5.10)\qquad Mv=\begin{bmatrix}-\frac{1}{\mu}\Delta v_2\\ 0\end{bmatrix}=\begin{bmatrix}\frac{28l^2\eta_{13}}{\mu}\,[\eta\,\phi_1(x)\psi_3(y)+\zeta\,\phi_4(x)\psi_2(y)+\xi\,\phi_5(x)\psi_1(y)]\\ 0\end{bmatrix},
$$

$$
\begin{aligned}
(5.11)\quad \mathcal{B}[v^*,v] &= \begin{bmatrix}-\chi^*\,[\nabla\cdot(v_1^*\nabla v_2)+\nabla\cdot(v_1\nabla v_2^*)]-2a\mu v_1^* v_1\\ 0\end{bmatrix}\\
&= \begin{bmatrix}-\eta_{13}\chi^*\Delta\,(v_1^* v_1)-2a\mu v_1^* v_1\\ 0\end{bmatrix}.
\end{aligned}
$$

Since

$$
\frac{\langle Mv,\Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\eta,\quad \frac{\langle Mv,\Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\zeta,\quad \frac{\langle Mv,\Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\xi,
$$

we then obtain

$$
PMv=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\,(\eta\Phi_{13}+\zeta\Phi_{42}+\xi\Phi_{51})\,.
$$

Similarly, since

$$
\begin{aligned}
\frac{\langle\mathcal{B}[v^*,v],\Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\,(\zeta+\xi)\,,\\
\frac{\langle\mathcal{B}[v^*,v],\Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\,(\xi+\eta)\,,\\
\frac{\langle\mathcal{B}[v^*,v],\Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\,(\eta+\zeta)\,,
\end{aligned}
$$

we have

$$
P\mathcal{B}[v^*,v]=-\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}[(\zeta+\xi)\Phi_{13}+(\xi+\eta)\Phi_{42}+(\eta+\zeta)\Phi_{51}].
$$

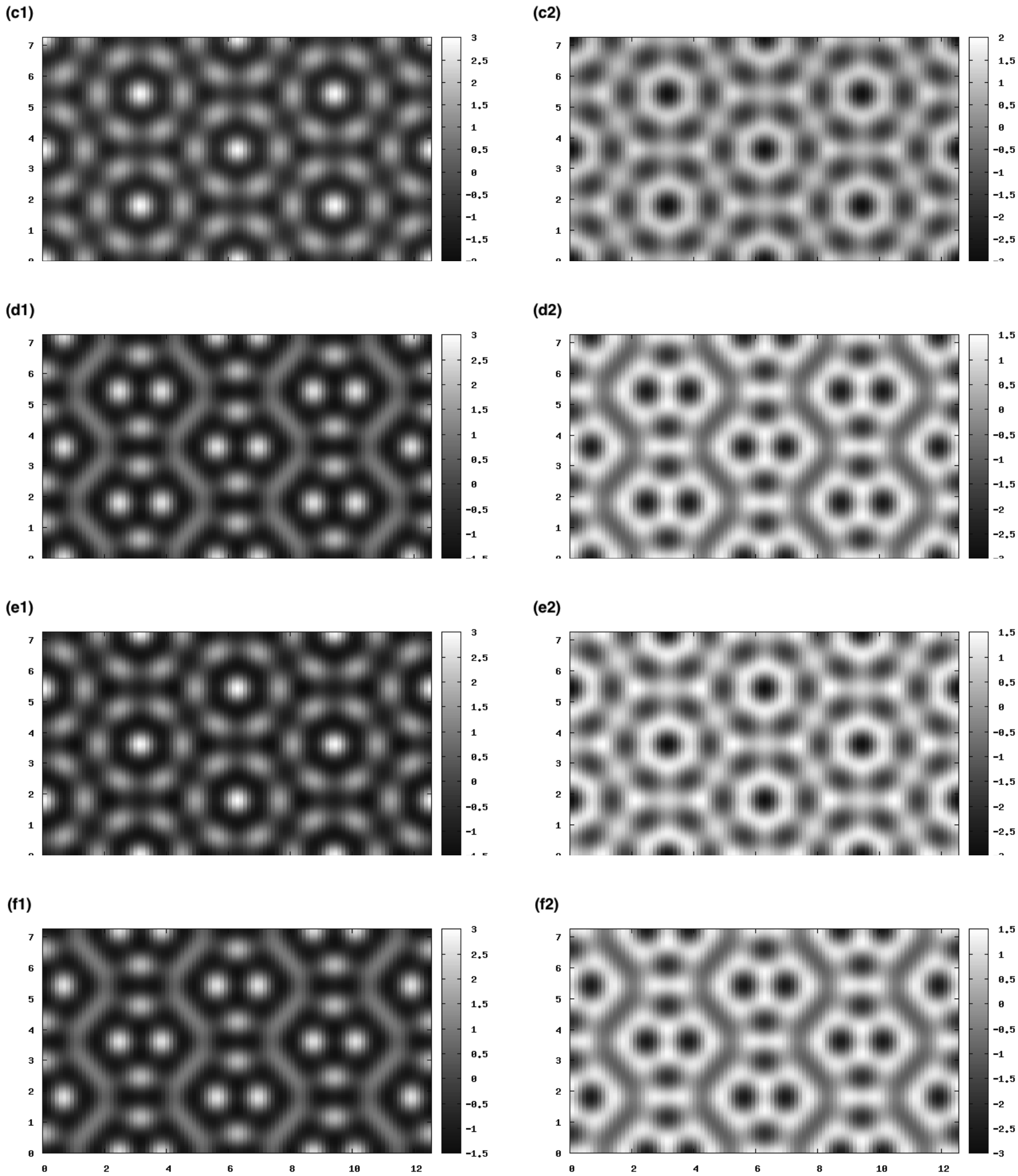


Figure 3: Plots of the functions for the three-dimensional kernel $V = \mathrm{span}\{\Phi_{13}, \Phi_{42}, \Phi_{51}\}$. The spatial domain is $\Omega = (0, 4\pi)\times(0, 4\sqrt{3}\pi)$. (c1) $v^* = \widetilde{A}(\Phi_{13}+\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (c2) $v^* = \widetilde{A}(\Phi_{13}+\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = -1 < 0$. (d1) $v^* = \widetilde{A}(\Phi_{13}-\Phi_{42}-\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (d2) $v^* = \widetilde{A}(\Phi_{13} - \Phi_{42} - \Phi_{51})$ with $\widetilde{A} = -1 < 0$. (e1) $v^* = \widetilde{A}(-\Phi_{13} + \Phi_{42} - \Phi_{51})$ with $\widetilde{A} = 1 > 0$. (e2) $v^* = \widetilde{A}(-\Phi_{13}+\Phi_{42}-\Phi_{51})$ with $\widetilde{A} = -1 < 0$. (f1) $v^* = \widetilde{A}(-\Phi_{13}-\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (f2) $v^* = \widetilde{A}(-\Phi_{13} - \Phi_{42} + \Phi_{51})$ with $\widetilde{A} = -1 < 0$.

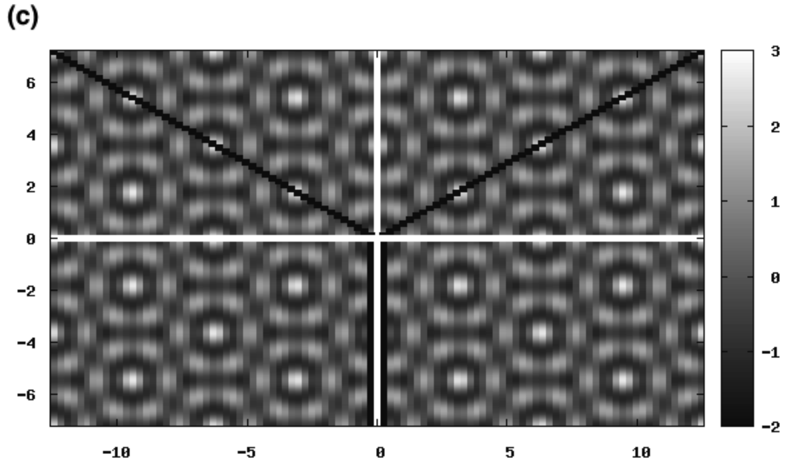


Figure 4: The spanning $v^*$ for the spatial domain $(-4\pi, 4\pi) \times (-4\sqrt{3}\pi, 4\sqrt{3}\pi)$. The white horizontal and vertical lines represent the $x$ and $y$ axes, respectively. The black lines are the auxiliary axes in the directions of $\pi/6$, $5\pi/6$, $3\pi/2$. (c) $v^* = \widetilde{A}(\Phi_{13} + \Phi_{42} + \Phi_{51})$ with $\widetilde{A} = 1 > 0$, which has $2\pi/3$-rotational symmetry.

From this, it follows that

$$Sv = \widehat{S}\left([\eta - \zeta - \xi]\,\Phi_{13} + [-\eta + \zeta - \xi]\,\Phi_{42} + [-\eta - \zeta + \xi]\,\Phi_{51}\right)$$
$$:= [\Phi_{13}\ \Phi_{42}\ \Phi_{51}]\,\widetilde{S}\begin{bmatrix}\eta\\ \zeta\\ \xi\end{bmatrix}, \quad \widehat{S} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}.$$

Here, $\widetilde{S}$ is the representation matrix of S:

$$\widetilde{S} = \begin{bmatrix}\widehat{S} & -\widehat{S} & -\widehat{S}\\ -\widehat{S} & \widehat{S} & -\widehat{S}\\ -\widehat{S} & -\widehat{S} & \widehat{S}\end{bmatrix}.$$

Because of

$$\det \widetilde{S} = -4\widehat{S}^{\,3} = -4\left(\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\right)^3 \neq 0,$$

the operator $S$ is isomorphic.

We finally arrive at the main result of this section.

**Theorem 5.1.** Let $v^* \in V$ be the functions defined in (5.8). Then, under the conditions

$$14\chi^* l^2\eta_{13} - a\mu \neq 0,$$

there exists a local branch of nontrivial solutions $(\chi(\lambda), U(\lambda)) \in (0,\infty) \times X$ to (SE), with small parameter $\lambda \in (-\varepsilon, \varepsilon)$, which bifurcates from $(\chi^*, U^*)$ such that

$$\chi(\lambda) = \chi^* + \lambda, \quad U(\lambda) = U^* + \lambda[\, v^* + \lambda\tilde{v}(\lambda)\,],$$

where $\tilde{v}(\lambda)$ is a smooth function of $\lambda$.

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# BIFURCATIONS WITH MULTI-DIMENSIONAL KERNEL IN A CHEMOTAXIS-GROWTH SYSTEM

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# ALMOST CONTRA-$b$-CONTINUOUS FUNCTIONS

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Abstract. In [1], the authors introduced and studied the notion of almost contra-$b$-continuity in topological spaces. In this paper, we investigate some more properties of this type of continuity.

**1 Introduction** Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of $b$-open [2] sets introduced by Andrijevic in 1996. This class is a subset of the class of semi-preopen sets [3], that is a subset of a topological space which is contained in the closure of the interior of its closure. Also, a class of $b$-open sets is a superset of the class of semi-open sets [17], that is a set which is contained in the closure of its interior, and the class of preopen sets [19], that is a set which is contained in the interior of its closure. Andrijevic studied several fundamental and interesting properties of $b$-open sets. In [1], the authors introduced and studied the notion of almost contra-$b$-continuity in topological spaces. In this paper, we investigate some more properties of this type of continuity.

**2 Preliminaries** Throughout the paper $(X,\tau)$ and $(Y,\sigma)$ (or simply $X$ and $Y$) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X,\tau)$, $Cl(A), Int(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively. A subset $A$ of $X$ is said to be regular open [26] (resp. semi-open [17], preopen [19], $\alpha$-open [21], $b$-open [2](= $\gamma$-open [13])) if $A = Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Int(Cl(Int(A)))$, $A \subset Int(Cl(A)) \cup Cl(Int(A))$). The family of all $\alpha$-open (resp. regular open, $b$-open) subsets of $X$ is denoted by $\alpha O(X)$ (resp. $RO(X)$, $BO(X)$). The complement of semi-open (resp. regular open, preopen, $b$-open) is called semi-closed [7] (resp. regular closed, preclosed [19], $b$-closed [2]). The family of all regular closed sets (resp. $b$-closed sets) of $(X,\tau)$ is denoted by $RC(X)$ (resp. $BC(X)$). The intersection of all regular open sets containing $A$ is called the $r$-kernal [9] of $A$ and is denoted by $rKer(A)$. The intersection of all semi-closed (resp. preclosed, $b$-closed) sets containing $A$ is called the semi-closure [6] (resp. pre-closure [19], $b$-closure [2]) of $A$ and is denoted by $sCl(A)$ (resp. $pCl(A)$, $bCl(A)$). A subset $A$ is $b$-closed if and only if $A= bCl(A)$. For each $x \in X$, the family of all $b$-open (resp. $b$-closed, semi-open, regular open, regular closed) sets containing $x$ is denoted by $BO(X,x)$ (resp. $BC(X,x)$, $SO(X,x)$, $RO(X,x)$, $RC(X,x)$). The $\theta$-semi-closure [16] of $A$, denoted by $\theta$-$sCl(A)$, is defined to be the set of all $x \in X$ such that $A \cap Cl(U) \neq \varnothing$ for every $U \in SO(X,x)$. A subset $A$ is called $\theta$-semi-closed [16] if and only if $A = \theta$-$sCl(A)$.

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The complement of $\theta$-semi-closed set is called $\theta$-semi-open [16]. For a subset $A$ of $X$, $sCl(A)=A\cup Int(Cl(A))$ [3], $pCl(A) = A\cup Cl(Int(A))$ [3] and $bCl(A) = sCl(A)\cap pCl(A)$ [2]. If $A$ is open in a space $X$, then $sCl(A) = Int(Cl(A))$ [3]. It follows that, if $A$ is open in a space $X$, then $bCl(A) = Int(Cl(A))$. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $b$-continuous [13] (resp. contra-$b$-continuous [20]) if $f^{-1}(V)$ is $b$-open (resp. $b$-closed) set in $X$ for each open set $V$ of $Y$. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be weakly $b$-continuous [24] (or almost weakly $b$-continuous [1]) if for every $x \in X$ and every open set $V$ of $Y$ containing $f(x)$, there exists $U \in BO(X,x)$ such that $f(U) \subset Cl(V)$.

## 3 Almost contra-$b$-continuous functions

**Definition 3.1** [1] A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *almost contra-b-continuous* if $f^{-1}(V) \in BC(X)$ for each $V \in RO(Y)$ (cf. Remark 3.4 below).

It is clear that every contra-$b$-continuous function is almost contra-$b$-continuous but the converse is not true in general.

**Example 3.2** Let $X = \{a,b,c\}$ and $\tau = \{\varnothing,\{a\},X\}$. Then the identity function $f : (X,\tau) \to (X,\tau)$ is almost contra-$b$-continuous but not contra-$b$-continuous.

**Theorem 3.3** *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*
(i) *$f$ is almost contra-b-continuous;*
(ii) *$f^{-1}(F) \in BO(X)$ for every $F \in RC(Y)$;*
(iii) *for each $x \in X$ and each $F \in RC(Y, f(x))$, there exists $U \in BO(X,x)$ such that $f(U) \subset F$;*
(iv) *$f^{-1}(Int(Cl(G))) \in BC(X)$ for every open subset $G$ of $(Y,\sigma)$;*
(v) *$f^{-1}(Cl(Int(F))) \in BO(X)$ for every closed subset $F$ of $(Y,\sigma)$;*
(vi) *$f(bCl(A)) \subset rKer(f(A))$ for every subset $A$ of $(X,\tau)$;*
(vii) *$bCl(f^{-1}(B)) \subset f^{-1}(rKer(B))$ for every subset $B$ of $(Y,\sigma)$.*

*Proof* (i)⇔(ii): Let $F \in RC(Y)$. Then $Y\backslash F \in RO(Y)$. By (i), $f^{-1}(Y\backslash F) = X\backslash f^{-1}(F) \in BC(X)$. We have $f^{-1}(F) \in BO(X)$. The proof of the reverse in similar.
(ii)⇒(iii): Let $F \in RC(Y, f(x))$. By (ii), $f^{-1}(F) \in BO(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$, then $f(U) \subset F$.
(iii)⇒(ii): Let $F \in RC(Y)$ and $x \in f^{-1}(F)$. From (iii), there exists a $b$-open set $U_x$ in $X$ containing $x$ such that $U_x \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup\{U_x | x \in f^{-1}(F)\}$. Since any union of $b$-open sets is $b$-open, $f^{-1}(F)$ is $b$-open in $X$.
(i) ⇔(iv): Let $G$ be an open subset of $Y$. Since $Int(Cl(G))$ is regular open, then by (i), it follows that, $f^{-1}(Int(Cl(G))) \in BC(X)$. The converse can be shown similarly.
(iii)⇒(vi): Let $A \subset X$ and let $x \in bCl(A)$ and $F \in RC(Y, f(x))$. By (iii), there exists $U \in BO(X,x)$ such that $f(U) \subset F$. Since $x \in bCl(A)$, we have $U \cap A \neq \emptyset$. Hence, $f(U) \cap f(A) \neq \emptyset$ and therefore $F \cap f(A) \neq \emptyset$. It follows from Proposition 24(i) of [9] that $f(x) \in rKer(f(A))$ and hence $f(bCl(A)) \subset rKer(f(A))$.
(vi)⇒(vii): Let $B \subset Y$. By (vii), $f(bCl(f^{-1}(B))) \subset rKer(f(f^{-1}(B))) \subset rKer(B)$. Hence $bCl(f^{-1}(B)) \subset f^{-1}(rKer(B))$.
(vii)⇒(i): Let $V \in RO(Y)$. Then by (vii), $bCl(f^{-1}(V)) \subset f^{-1}(rKer(V))$. Since $V \in RO(Y), rKer(V) = V$ and hence $bCl(f^{-1}(V)) \subset f^{-1}(V)$, which shows that $f^{-1}(V)$ is $b$-closed. Consequently, $f$ is almost contra-$b$-continuous. □

**Remark 3.4** (i) A function $f : (X, \tau) \to (Y, \sigma)$ is called: *almost contra-b-continuous at a point* $x \in X$, if for each regular closed subset $V$ of $(Y, \sigma)$ containing $f(x)$, there exists a $b$-open subset $U$ of $(X, \tau)$ containing $x$ such that $f(U) \subset V$.

(ii) By Theorem 3.3 and definitions, it is shown that a function $f : (X, \tau) \to (Y, \sigma)$ is almost contra-$b$-continuous if and only if $f$ is almost contra-$b$-continuous at each point of $X$.

**Theorem 3.5** (i) *If* $f : (X, \tau) \to (Y, \sigma)$ *is weakly-b-continuous and* $(Y, \sigma)$ *is regular, then* $f$ *is b-continuous.*

(ii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is almost contra-b-continuous and* $(Y, \sigma)$ *is regular, then* $f$ *is b-continuous.*

(iii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is contra-b-continuous and* $(Y, \sigma)$ *is regular, then* $f$ *is b-continuous.*

*Proof.* Clear. □

Sometimes, the concept of a $b$-open set (resp. $b$-closed set) of a topological space $(X, \tau)$ is called a $\gamma$-open set (resp. $\gamma$-closed set); and so the family $BO(X)$ (resp. $BC(X)$) is denoted by $\gamma O(X)$ (resp. $\gamma C(X)$).

**Lemma 3.6** [13] *Let* $A$ *and* $B$ *be subsets of a topological space* $(X, \tau)$.

(i) *If* $A \in \gamma O(X)$ *and* $B \in \alpha O(X)$, *then* $A \cap B \in \gamma O(B)$.

(ii) *Let* $A \subset B \subset X$, $A \in \gamma O(B)$ *and* $B \in \alpha O(X)$, *then* $A \in \gamma O(X)$.

**Theorem 3.7** *If* $f : (X, \tau) \to (Y, \sigma)$ *is almost contra-b-continuous and* $U \in \alpha O(X)$, *then* $f|U : (U, \tau|U) \to (Y, \sigma)$ *is almost contra-b-continuous.*

*Proof.* Let $V$ be a regular closed subset of $Y$. We have $(f|U)^{-1}(V) = f^{-1}(V) \cap U$. Since $f^{-1}(V)$ is $b$-open and $U$ is $\alpha$-open, it follows from the Lemma 3.6 (i) that $(f|U)^{-1}(V)$ is $b$-open in the relative topology of $U$. Thus, $f|U$ is almost contra-$b$-continuous. □

**Theorem 3.8** *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function and* $x \in X$. *If there exists* $U \in BO(X, x)$ *and* $f|U : (U, \tau|U) \to (Y, \sigma)$ *is almost contra-b-continuous at* $x$, *then* $f$ *is almost contra-b-continuous at* $x$.

*Proof.* Suppose that $F \in RC(Y, f(x))$. Since $f|U$ is almost contra-$b$-continuous at $x$, there exists $V \in BO(U, x)$ such that $f(V) = (f|U)(V) \subset F$. Since $U \in \alpha O(X, x)$, it follows from Lemma 3.6 (ii) that $V \in BO(X, x)$. This shows that $f$ is almost contra-$b$-continuous at $x$. □

**Theorem 3.9** *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function and* $\Sigma = \{U_i : i \in I\}$ *be a cover of* $X$ *by* $\alpha$*-open sets of* $(X, \tau)$. *If for each* $i \in I$, $f|U_i : (U_i, \tau|U_i) \to (Y, \sigma)$ *is almost contra-b-continuous, then* $f : (X, \tau) \to (Y, \sigma)$ *is almost contra-b-continuous.*

*Proof.* Let $V \in RC(Y)$. Since $f|U_i$ is almost contra-$b$-continuous for each $i \in I$, $(f|U_i)^{-1}(V) \in BO(U_i)$, since $U_i \in \alpha O(X)$, by Lemma 3.6 (2), $(f|U_i)^{-1}(V) \in BO(X)$ for each $i \in U$. Then $f^{-1}(V) = \bigcup\{(f|U_i)^{-1}(V) \in BO(X) | i \in I\}$. This gives $f$ is almost contra-$b$-continuous. □

**Theorem 3.10** *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function and let* $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ *be the graph function of* $f$, *defined by* $g(x) = (x, f(x))$ *for every* $x \in X$. *If* $g$ *is almost contra-b-continuous, then* $f$ *is almost contra-b-continuous.*

*Proof.* Let $V \in RC(Y)$, then $X \times V = X \times Cl(Int(V)) = Cl(Int(X)) \times Cl(Int(V)) = Cl(Int(X \times V))$. Then $X \times V \in RC(X \times Y)$. Since $g$ is almost contra-$b$-continuous, then $f^{-1}(V) = g^{-1}(X \times V) \in BO(X)$. Thus, $f$ is almost contra-$b$-continuous. □

**Definition 3.11** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
(i) *pre-b-open* if $f(U) \in BO(Y)$ for each $U \in BO(X)$,
(ii) *b-irresolute* [13] if for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$,
(iii) *θ-irresolute* [13] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(Cl(U)) \subset Cl(V)$.

**Theorem 3.12** *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective pre-b-open and $g : (Y, \sigma) \to (Z, \gamma)$ is a function such that $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous, then $g$ is almost contra-b-continuous.*

*Proof.* Let $V$ be any regular closed set in $Z$. Since $g \circ f$ is almost contra-$b$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $b$-open. Since $f$ is surjective pre-$b$-open, $f(f^{-1}(g^{-1}(V)))$ $= g^{-1}(V)$ is $b$-open. Therefore, $g$ is almost contra-$b$-continuous. □

**Theorem 3.13** (i) *If $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute and $g : (Y, \sigma) \to (Z, \gamma)$ is almost contra-b-continuous, then $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous.*
(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is θ-irresolute, then $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous.*

*Proof.* (i) Let $x \in X$ and $W \in SO(Z)$. Then there exists a set $U \in BO(X, x)$ such that $(g \circ f)(U) \subset Cl(W)$. Therefore, $g \circ f$ is almost contra-$b$-continuous.
(ii) Similar to (i). □

**Definition 3.14** A filter base $\Lambda$ is said to be *b-convergent* (resp. *rc-convergent* [12]) *to a point* $x \in X$ if for any $U \in BO(X, x)$ (resp. $U \in RC(X, x)$), there exists $B \in \Lambda$ such that $B \subset U$.

**Theorem 3.15** *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous function, then for each point $x \in X$ and each filter base $\Lambda$ in $X$ b-converging to $x$, the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.*

*Proof.* Let $x \in X$ and $\Lambda$ be any filter base in $X$ $b$-converging to $x$. Since $f$ is almost contra-$b$-continuous, then for any $V \in RC(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$. Since $\Lambda$ is $b$-converging to $x$, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is $rc$-convergent to $f(x)$. □

## 4 Separation axioms and covering properties

**Definition 4.1** A topological space $(X, \tau)$ is said to be
(i) $P_{\sum}$ [30] if for any open set $V$ of $(X, \tau)$ and each $x \in V$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$,
(ii) *weakly* $P_{\sum}$ [22] if for any $V \in RO(X, x)$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$.

**Theorem 4.2** *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous function and $(Y, \sigma)$ is $P_{\sum}$, then $f$ is b-continuous.*

*Proof.* Let $V$ be any open set in $Y$. Since $Y$ is $P_{\sum}$, there exists a subfamily $\mathcal{A}$ of $RC(Y)$ such that $V = \cup\{F : F \in \mathcal{A}\}$. Since $f$ is almost contra-$b$-continuous, $f^{-1}(F)$ is $b$-open in $X$ for each $F \in \mathcal{A}$ and $f^{-1}(V)$ is $b$-open in $X$. Therefore, $f$ is $b$-continuous. □

**Theorem 4.3** *If $f : (X,\tau) \to (Y,\sigma)$ is an almost contra-b-continuous function and $(Y,\sigma)$ is weakly $P_{\sum}$, then $f$ is almost b-continuous.*

*Proof.* Similar to the proof of Theorem 4.2. □

**Definition 4.4** A topological space $(X,\tau)$ is said to be

(i) *weakly Hausdorff* [28] if each element of $X$ is an intersection of regular closed sets,

(ii) *b-$T_0$* [5] if for each pair of distinct points in $X$, there exists a $b$-open set of $(X,\tau)$ containing one point but not the other,

(iii) *b-$T_1$* [5] if for each pair of distinct points $x$ and $y$ of $X$, there exist $b$-open sets $U$ and $V$ of $(X,\tau)$ containing $x$ and $y$, respectively such that $y \notin U$ and $x \notin V$,

(iv) *b-$T_2$* [10] if for each pair of distinct points $x$ and $y$ of $X$, there exist $b$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

**Theorem 4.5** *If $f : (X,\tau) \to (Y,\sigma)$ is an almost contra-b-continuous injection and $(Y,\sigma)$ is weakly Hausdorff, then $(X,\tau)$ is b-$T_1$.*

*Proof.* Suppose that $Y$ is weakly Hausdorff. For any two distinct points $x$ and $y$ in $X$, there exist $V, W \in RC(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since $f$ is almost contra-$b$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $b$-open subsets of $X$ such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $X$ is $b$-$T_1$. □

**Definition 4.6** A topological space $(X,\tau)$ is said to be

(i) *hyperconnected* [27] if every open set is dense,

(ii) *ultra b-connected* if every two non-void $b$-closed subsets of $(X,\tau)$ intersect,

(iii) *b-connected* [13] provided that $X$ is not the union of two disjoint nonempty $b$-open sets.

**Theorem 4.7** *If $(X,\tau)$ is ultra b-connected and $f : (X,\tau) \to (Y,\sigma)$ is an almost contra-b-continuous surjection, then $(Y,\sigma)$ is hyperconnected.*

*Proof.* Assume that $Y$ is not hyperconnected. Then there exists an open set $V$ such that $V$ is not dense in $Y$. Then there exist disjoint nonempty regular open subsets $B_1$ and $B_2$ in $Y$, namely $Int(Cl(V))$ and $Y \setminus Cl(V)$. Since $f$ is almost contra-$b$-continuous surjection, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint nonempty $b$-closed subsets of $X$. By assumption, the ultra-$b$-connectedness of $X$ implies that $A_1$ and $A_2$ must intersect. By contradiction, $Y$ is hyperconnected. □

**Theorem 4.8** (i) [24] *If $f : (X,\tau) \to (Y,\sigma)$ is weakly b-continuous surjection and $(X,\tau)$ is b-connected, then $(Y,\sigma)$ is connected.*

(ii) *If $f : (X,\tau) \to (Y,\sigma)$ is a almost contra-b-continuous surjection and $(X,\tau)$ is b-connected, then $(Y,\sigma)$ is connected.*

(iii) [20] *If $f : (X,\tau) \to (Y,\sigma)$ is a contra-b-continuous surjection and $(X,\tau)$ is b-connected, then $(Y,\sigma)$ is connected.*

**Theorem 4.9** (i) [24] *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly b-continuous injection and $(Y, \sigma)$ is Urysohn, then $(X, \tau)$ is b-$T_2$.*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous injection and $(Y, \sigma)$ is Urysohn, then $(X, \tau)$ is b-$T_2$.*

(iii) [20] *If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous injection and $(Y, \sigma)$ is Urysohn, then $(X, \tau)$ is b-$T_2$.*

**Definition 4.10** [15] A topological space $(X, \tau)$ is said to be *$\theta$-irreducible* if every pair of nonempty regular closed sets of $(X, \tau)$ has a nonempty intersection.

**Theorem 4.11** *If $(X, \tau)$ is b-connected and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous surjection, then $(Y, \sigma)$ is $\theta$-irreducible.*

*Proof.* Similar to that proof of Theorem 4.7. □

**Definition 4.12** [13] A topological space $(X, \tau)$ is said to be *b-normal* provided that every pair of nonempty disjoint closed sets can be separated by disjoint $b$-open sets.

**Theorem 4.13** (i) *If $(Y, \sigma)$ is normal and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous closed injection, then $(X, \tau)$ is b-normal.*

(ii) [20] *If $(Y, \sigma)$ is normal and $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous closed injection, then $(X, \tau)$ is b-normal.*

*Proof.* (i) Let $F_1$ and $F_2$ be disjoint nonempty closed sets of $X$. Since $f$ is injective and closed, $f(F_1)$ and $f(F_1)$ are disjoint closed sets of $Y$. Since $Y$ is normal, there exist open sets $V_1$ and $V_2$ of $Y$ such that $f(F_1) \subset V_1$, $f(F_2) \subset V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. Then, since $Cl(V_1), Cl(V_2) \in RC(Y)$ and $f$ is almost contra-$b$-continuous, $f^{-1}(Cl(V_1)), f^{-1}(Cl(V_2)) \in BO(X)$. Since $F_1 \subset f^{-1}(V_1)$, $F_2 \subset f^{-1}(V_2)$ and $f^{-1}(Cl(V_1))$ and $f^{-1}(Cl(V_2))$ are disjoint, $X$ is $b$-normal. □

**Definition 4.14** A cover $\sum = \{U_i : i \in I\}$ of subsets of $X$ is called a *b-cover* if $U_i$ is $b$-open in $(X, \tau)$ for each $i \in I$.

**Definition 4.15** A topological space $(X, \tau)$ is said to be

(i) *b-compact* [23] (resp. *S-closed* [29]) if every $b$-open (resp. regular closed) cover of $X$ has a finite subcover,

(ii) *countably b-compact* [11] (resp. *countably S-closed* [8]) if every countable cover of $X$ by $b$-open (resp. regular closed) sets has a finite subcover,

(iii) *b-Lindelöf* [11] (resp. *S-Lindelöf* [18]) if every $b$-open (resp. regular closed) cover of $X$ has a countable subcover.

**Definition 4.16** A topological space $(X, \tau)$ is said to be

(i) *b-closed compact* [11] (resp. *nearly compact* [25]) if every $b$-closed (resp. regular open) cover of $X$ has a finite subcover,

(ii) *countably b-closed compact* [11] (resp. *nearly countably compact* [14]) if every countable cover of $X$ by $b$-closed (resp. regular open) sets has a finite subcover,

(iii) *b-closed Lindelöf* [11] (resp. *nearly Lindelöf* [14]) if every $b$-closed (resp. regular open) cover of $X$ has a countable subcover.

**Theorem 4.17** *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost contra-b-continuous surjection. Then the following statements hold.*

(i) *If $(X, \tau)$ is b-closed compact, then $Y$ is nearly compact.*

(ii) *If $(X, \tau)$ is b-closed Lindelöf, then $Y$ is nearly Lindelof.*

(iii) *If $(X, \tau)$ is countably b-closed compact, then $Y$ is nearly countably compact.*

*Proof.* We prove only (i), the proofs of (ii) and (iii) being entirely analogous.
Let $\{V_i : i \in I\}$ be any regular open cover of $Y$. Since $f$ is almost contra-$b$-continuous, $\{f^{-1}(V_i)$: $i \in I\}$ is a $b$-closed cover of $X$. Since $X$ is $b$-closed compact, there exists a a finite subset $I_0$ of $I$ such that $X = \cup\{f^{-1}(V_i) : i \in I_0\}$. Therefore, we have $Y = \cup\{V_i : i \in I_0\}$ and $Y$ is $S$-closed. □

**Definition 4.18** [26] A topological space $(X, \tau)$ is said to be *mildly compact* (resp. *mildly contably compact, mildly Lindelöf*) if every clopen (resp. countable clopen, clopen) cover of $X$ has a finite (resp. finite, countable) subcover.

**Theorem 4.19** *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous and almost continuous surjection and $(X, \tau)$ is mildly compact* (resp. *mildly countably compact, mildly Lindelöf), then $Y$ is nearly compact* (resp. *nearly countably compact, nearly Lindelöf*) *and $S$-closed* (resp. *countably $S$-closed, $S$-Lindelöf*).

*Proof.* Let $V \in RC(Y)$. Then since $f$ is almost contra-$b$-continuous and almost continuous, $f^{-1}(V)$ is $b$-open and closed in $X$ and hence $f^{-1}(V)$ is clopen (resp. open). Let $\{V_i : i \in I\}$ be any regular closed (resp. regular open) cover of $Y$. Then $\{f^{-1}(V_i) : i \in I\}$ is a clopen cover of $X$ and since $X$ is mildly compact, there exists a finite subset $I_0$ of $I$ such that $X = \cup\{f^{-1}(V_i) : i \in I_0\}$. Since $f$ is surjective, we obtain $Y = \cup\{V_i : i \in I_0\}$. This shows that $Y$ is $S$-closed (resp. nearly compact).
The other proofs can be obtained similarly. □

**Definition 4.20** A topological space $(X, \tau)$ is said to be *s-Urysohn* [4] if for each pair of distinct points $x$ and $y$ in $X$, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$.

**Theorem 4.21** *If $(Y, \sigma)$ is s-Urysohn and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous injection, then $(X, \tau)$ is b-$T_2$.*

*Proof.* It is similar to Proof of Theorem 4.5. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)); x \in X\} \subset X \times Y$ is called the *graph of* $f$ and is denoted by $G(f)$.

**Definition 4.22** A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *regular b-closed* if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

**Theorem 4.23** *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is regular b-closed if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \emptyset$.*

*Proof.* This is an immediate consequence of Definition 4.22. □

**Theorem 4.24** *Let $f : (X, \tau) \to (Y, \sigma)$ have a regular b-closed graph $G(f)$. If $f$ is injective, then $(X, \tau)$ is b-$T_1$.*

*Proof.* Let $x$ and $y$ be any two distinct points of $X$. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. By definition of regular $b$-closed graph, there exist $U \in BC(X)$ and $V \in RO(Y)$ such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$; hence $U \cap f^{-1}(V) = \emptyset$. Therefore, we have $y \notin U$. Thus, $y \in X\backslash U$ and $x \notin X\backslash U$. We obtain that $X\backslash U \in BO(X)$. This implies that $X$ is $b$-$T_1$. □

**Theorem 4.25** *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and $(Y, \sigma)$ is $T_2$, then the graph $G(f)$ is regular b-closed.*

*Proof.* Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $f(x) \neq y$. Snce $Y$ is $T_2$, there exist open sets $V$ and $W$ containing $f(x)$ and $y$, respectively, such that $V \cap W = \emptyset$; hence $Int(Cl(V)) \cap Int(Cl(W)) = \emptyset$. Since $f$ is almost contra-$b$-continuous, $f^{-1}(Int(Cl(V)))$ is $b$-closed containing $x$. Take $U = f^{-1}(Int(Cl(V)))$. Then $f(U) \subset Int(Cl(V))$. Therefore, $f(U) \cap Int(Cl(W)) = \emptyset$ and hence the graph $G(f)$ is regular $b$-closed. □

**Theorem 4.26** *Let $f : (X, \tau) \to (Y, \sigma)$ have a regular b-closed graph $G(f)$. If $f$ is surjective, then $(Y, \sigma)$ is weakly Hausdorff.*

*Proof.* Let $y \in Y$. Since $f$ is surjective, $f(x) = y$ for some $x \in X$ and $(x, a) \in (X \times Y)\backslash G(f)$ for any point $a \in Y$ such that $a \neq y$. For the points $y$ and $a$, by definition of regular $b$-closed graph $G(f)$, there exists a $b$-closed set $U_a$ of $X$ and $F(a) \in RO(Y)$ such that $(x, a) \in U_a \times F(a)$ and $f(U_a) \cap F(a) = \emptyset$; hence $y \notin F(a)$. Then, $\{y\} \subset A$, where $A = \cap\{Y\backslash F(z) : z \neq y\}$. In order to prove $\{y\} \supset A$, let $w \in A$ and suppose that $w \notin \{y\}$. The, for any point $z$ with $z \neq y$, we have that $w \in Y\backslash F(z)$. Since $w \neq y$, we can take $z = w$ and so $w \in F(w)$. This is a contradicition. Hence we show that $\{y\} = A$; and so $\{y\}$ is an intersection of regular closed sets $Y\backslash F(z)$, where $z \neq y$, that is $(Y, \sigma)$ is weakly Hausdorff. □

## 5 Additional Properties

**Theorem 5.1** *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *$f$ is almost contra-b-continuous;*

(ii) *$f^{-1}(V) \in BO(X)$ for each $\theta$-semi-open set $V$ of $(Y, \sigma)$;*

(iii) *$f^{-1}(F) \in BC(X)$ for each $\theta$-semi-closed set $F$ of $(Y, \sigma)$;*

(iv) *for each $x \in X$ and each $U \in SO(Y, f(x))$, there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(U)$;*

(v) *$f^{-1}(U) \subset bInt(f^{-1}(Cl(U)))$ for every $U \in SO(Y)$;*

(vi) *$f(bCl(A)) \subset \theta\text{-}sCl(f(A))$ for every subset $A$ of $(X, \tau)$;*

(vii) *$bCl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}sCl(B))$ for every subset $B$ of $(Y.\sigma)$;*

(viii) *$bCl(f^{-1}(V)) \subset f^{-1}(\theta\text{-}sCl(V))$ for every open subset $V$ of $(Y, \sigma)$;*

(ix) *$bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every open subset $V$ of $(Y, \sigma)$;*

(x) *$bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))$ for every open subset $V$ of $(Y, \sigma)$.*

*Proof.* (i)⇒(ii): This follows from the fact that every $\theta$-semi-open set is the union of regular closed sets.
(ii)⇔(iii): This is obvious.
(ii)⇒(iv): Let $x \in X$ and $U \in SO(Y, f(x))$. Since $Cl(U)$ is $\theta$-semi-open in $Y$, there exists $V \in BO(X, x)$ such that $x \in V \subset f^{-1}(Cl(U))$ and hence $f(V) \subset Cl(U)$.
(iv)⇒(v): Let $U \in SO(Y)$ and $x \in f^{-1}(U)$. Then $f(x) \in U$. By (iv), there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(U)$. It follows that $x \in V \subset f^{-1}(Cl(U))$. Hence, $x \in bInt(f^{-1}(Cl(U)))$.
(v)⇒(i): Let $F \in RC(Y)$. Since $F \in SO(Y)$, then by (v), $f^{-1}(F) \subset bInt(f^{-1}(Cl(F)))$ and consequently, $f^{-1}(F) \in BO(X)$. Hence, by Theorem 3.3, (i) holds.
(iv)⇒(vi): Let $A$ be any subset of $X$. Suppose that $x \in bCl(A)$ and $G \in SO(Y, f(x))$. By (v), there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(G)$. Since $x \in bCl(A)$, $V \cap A \neq \emptyset$ and hence $\emptyset \neq f(V) \cap f(A) \subset Cl(G) \cap f(A)$. Therefore, we obtain $f(x) \in \theta\text{-}sCl(f(A))$ and hence $f(bCl(A)) \subset \theta\text{-}sCl(f(A))$.

(vi)⇒(vii): Let $B$ be any subset of $Y$. Then $f(bCl(f^{-1}(B))) \subset (\theta\text{-}sCl(f(f^{-1}(B))) \subset \theta\text{-}sCl(B)$ and hence $bCl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}sCl(B))$.
(vii)⇒(viii): Obvious.
(viii)⇒(ix): Follows from the fact that $\theta\text{-}sCl(V) = sCl(V)$ for every open subset $V$ of $Y$.
(ix)⇒(x): Obvious.
(x)⇒(i): Let $V \in RO(Y)$. By (x), $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))) = f^{-1}(V)$ and hence $f^{-1}(V) \in BC(X)$, which proves that $f$ is almost contra-$b$-continuous. □

**Theorem 5.2** *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(i) *$f$ is almost contra-$b$-continuous;*
(ii) *$f^{-1}(Cl(V)) \in BO(X)$ for every $V \in SPO(Y)$;*
(iii) *$f^{-1}(Cl(V)) \in BO(X)$ for every $V \in SO(Y)$;*
(iv) *$f^{-1}(Int(Cl(V))) \in BC(X)$ for every $V \in PO(Y)$.*

*Proof.* (i)⇒(ii): Let $V$ be any semi-preopen set of $Y$. It follows from Theorem 2.4 of [3] that $Cl(V)$ is regular closed. Then by Theorem 3.3 $f^{-1}(Cl(V)) \in BO(X)$.
(ii)⇒(iii): This is obvious since $SO(Y) \subset SPO(Y)$.
(iii)⇒(iv): Let $V \in PO(Y)$. Then $Y \setminus Int(Cl(V))$ is regular closed and hence it is semi-open. Then $X \setminus f^{-1}(Int(Cl(V)))=f^{-1}(Y \setminus Int(Cl(V))) = f^{-1}(Cl(Y \setminus Int(Cl(V)))) \in BO(X)$. Hence $f^{-1}(Int(Cl(V))) \in BC(X)$.
(iv)⇒(i): Let $V$ be any regular open set of $Y$. Then $V \in PO(Y)$ and hence $f^{-1}(V) = f^{-1}(Int(Cl(V)))$ is $b$-closed in $X$. □

**Definition 5.3** [11] The *$b$-frontier* of a subset $A$ of a topological space $(X, \tau)$, $bFr(A)$, is defined by $bFr(A) = bCl(A) \cap bCl(X \setminus A) = bCl(A) \cap (X \setminus bInt(A))$.

**Theorem 5.4** *For a function $f : (X, \tau) \to (Y, \sigma)$, we introduce the following notations relating to $f$:*
$A_f := \{x \in X : f$ *is not almost contra-$b$-continuous at* $x\}$,
$B_f(x) := \cup\{bFr(f^{-1}(F_x)) : F_x \in RC(Y, f(x))\}$, *where* $x \in A_f$, *and*
$B_f := \bigcup\{B_f(x) | x \in A_f\}$.
*Then, we have the following properties:*
*If $z \in A_f$, then $z \in B_f(z)$; and so $A_f \subset B_f$ holds in $(X, \tau)$.*

*Proof.* Let $z \in A_f$. Namely, we suppose that $f$ is not almost contra-$b$-continuous at $z \in X$. By Theorem 3.3, there exists a subset $F_z \in RC(Y, f(z))$ such that $f(U) \cap (Y \setminus F_z) \neq \emptyset$ for every $U \in BO(X, z)$. By the property (5) in Proposition 5 of [10], $z \in bCl(f^{-1}(Y \setminus F_z))$ holds: and so $z \in bCl(X \setminus f^{-1}(F_z))$. On the other hand, we obtain $z \in f^{-1}(F_z) \subset bCl(f^{-1}(F_z))$; and hence $z \in bFr(f^{-1}(F_z))$, $F_z \in RC(Y, f(z))$ and $z \in A_f$. Namely, if $z \in A_f$, then $z \in B_f(z)$ holds; and so we have $A_f \subset \bigcup\{B_f(z) | z \in A_f\} = B_f$ holds in $(X, \tau)$.

□

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**FUZZY INTERIOR IDEALS IN HYPERSEMIGROUPS**

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**Abstract**

We introduce the concept of interior ideal and the concept of fuzzy interior ideal in hypersemigroups and we prove, among others, that in regular also in intra-regular hypersemigroups the interior ideals and the fuzzy interior ideals coincide. We also prove that an hypergroupoid $H$ is simple if and only if every fuzzy ideal of $H$ is a constant function; and that an hypersemigroup $H$ is simple if and only if every fuzzy interior ideal of $H$ is a constant function, equivalently if, for every element $a$ of $H$, we have $H = H * \{a\} * H$.

# 1 Introduction

This paper is based on our paper [5] and partly on [6]. We first introduce the concept of an interior ideal and the concept of a fuzzy interior ideal of an hypersemigroup and we prove that if $H$ is an hypersemigroup and $A$ an interior ideal of $H$, then the characteristic mapping $f_A$ is a fuzzy interior ideal of $H$. "Conversely", if $A$ is a nonempty subset of $H$ and $f_A$ a fuzzy interior ideal of $H$, then the set $A$ is an interior ideal of $H$. Then we prove that any fuzzy ideal of an hypersemigroup $H$ is a fuzzy interior ideal of $H$ and in regular, also in intra-regular hypersemigroups the concepts of interior ideals and fuzzy interior ideals coincide. We also prove that in a regular and in an intra-regular hypersemigroup $H$ the interior ideals are subsemigroups of $H$. Following Kuroki, we call an hypergroupoid $H$ fuzzy simple if every fuzzy ideal of $H$ is a constant function. We prove that an hypergroupoid is simple if and only if it is fuzzy simple, and an hypersemigroup $H$ is simple if and only $H = H*\{a\}*H$ for every $a \in H$, equivalently, if every fuzzy interior ideal of $H$ is a constant function. As a consequence, for an hypersemigroup $H$, the following are equivalent: (1) $H$ is simple. (2) $H = H * \{a\} * H$ for every $a \in H$. (3) $H$ is fuzzy simple. (4) every fuzzy interior ideal of $H$ is a constant function.

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# 2 Prerequisites

For the sake of completeness, we will give some definitions already given in [2]. An *hypergroupoid* is a nonempty set $H$ with an hyperoperation

$\circ : H \times H \to \mathcal{P}^*(H) \mid (a,b) \to a \circ b$ on $H$ and an operation

$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A,B) \to A * B$ on $\mathcal{P}^*(H)$ (induced by the operation of $H$) such that $A * B = \bigcup\limits_{(a,b)\in A\times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ being the set of nonempty subsets of $H$). As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be denoted by $(H,\circ)$ (instead of $(H,\circ,*)$). If $(H,\circ)$ is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = \bigcup\limits_{a\in\{x\},\, b\in\{y\}} (a \circ b) = x \circ y$. The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

**Proposition 2.1.** *Let $(H,\circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

(1) *If $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$ and*

(2) *If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.*

**Proposition 2.2.** *If $(H,\circ)$ is an hypergroupoid then, for every $A, B, C, D \in \mathcal{P}^*(H)$, we have*

(1) *$A \subseteq B \Rightarrow A * C \subseteq B * C$ and $C * A \subseteq C * B$, equivalently,*
*$A \subseteq B$ and $C \subseteq D \Rightarrow A * C \subseteq B * D$.*

(2) *$H * A \subseteq H$ and $A * H \subseteq H$.*

**Definition 2.3.** Let $(H,\circ)$ be an hypergroupoid. A nonempty subset $A$ of $H$ is called a *left* (resp. *right*) *ideal* of $H$ if $H * A \subseteq A$ (resp. $A * H \subseteq A$). If $A$ is both a left and a right ideal of $H$, then it is called an *ideal* of $H$. A nonempty subset $A$ of $H$ is called a *subgroupoid* of $H$ if $A * A \subseteq A$.
Clearly, every left (resp. right) ideal of $H$ is a subgroupoid of $H$.

**Definition 2.4.** An hypergroupoid $(H,\circ)$ is called *hypersemigroup* if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

for every $x, y, z \in H$. Since $\{x\} * \{y\} = x \circ y$ for every $x, y \in H$, this is equivalent to saying that $\{x\} * \Big(\{y\} * \{z\}\Big) = \Big(\{x\} * \{y\}\Big) * \{z\}$ for every $x.y, z \in H$.

**Proposition 2.5.** ([1,2]; for its proof we refer to [4]) If $(H,\circ)$ be an hypersemigroup, then $(\mathcal{P}^*(H), *)$ is a semigroup.

As a result, for any $A, B, C \in \mathcal{P}^*(H)$, we write $A * (B * C) = (A * B) * C := A * B * C$; and in an expression of the form $A_1 * A_2 * \ldots.. * A_n$, where the $A_i$ $(i = 1, 2, ..., n)$ are elements of $\mathcal{P}^*(H)$ we can put parentheses in any place beginning with some $A_i$ and ending in some $A_j$ $(1 \le i, j \le n)$.

Following Zadeh, any mapping $f : H \to [0,1]$ of an hypergroupoid $H$ into the closed interval $[0,1]$ of real numbers is called a *fuzzy subset* of $H$ (or a *fuzzy* *set* in $H$) and, for any nonempty subset $A$ of $H$, the characteristic function $f_A$ of $A$, is the fuzzy subset of $H$ defined by

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The concepts of fuzzy left ideals and fuzzy right ideals of semigroups due to Kuroki [6], are the following: A fuzzy subset $f$ of a semigroup $S$ is called a fuzzy left (resp. fuzzy right) ideal of $S$ if, for every $x, y \in S$, we have $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$). It is called a fuzzy ideal of $S$ if it is both a fuzzy left and a fuzzy right ideal of $S$. These concepts can be transferred, in a natural way, to an hypergroupoid as follows:

**Definition 2.6.** [3] Let $(H, \circ)$ be an hypergroupoid. A fuzzy subset $f$ of $H$ is called a *fuzzy left ideal* of $H$ if

$$f(x \circ y) \geq f(y) \text{ for all } x, y \in H,$$

in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$.
A fuzzy subset $f$ of $H$ is called a *fuzzy right ideal* of $H$ if

$$f(x \circ y) \geq f(x) \text{ for all } x, y \in H,$$

meaning that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.
A fuzzy subset $f$ of $H$ is called a *fuzzy ideal* of $H$ it is both a fuzzy left ideal and a fuzzy right ideal of $H$. As one can easily see, a fuzzy subset $f$ of $H$ is a fuzzy ideal of $H$ if and only $f(x \circ y) \geq \max\{f(x), f(y)\}$ for all $x, y \in H$, in the sense that $x, y \in H$ and $u \in x \circ y$ implies $f(u) \geq \max\{f(x), f(y)\}$.

# 3 Main results

**Definition 3.1.** Let $H$ be an hypersemigroup. A nonempty subset $A$ of $H$ is called an *interior ideal* of $H$ if

$$H * A * H \subseteq A.$$

By a *subidempotent interior ideal* of $H$ we mean an interior ideal of $H$ which is at the same time a subsemigroup of $H$.

The concept of fuzzy interior ideal of semigroups is also due to Kuroki [6], and it is the following: A fuzzy subset $f$ of a semigroup $S$ is called a fuzzy interior ideal of $S$ if, for any $x, a, y \in S$, we have $f(xay) \geq f(a)$. This concept can be naturally transferred to an hypersemigroup as follows:

**Definition 3.2.** Let $H$ be an hypersemigroup. A fuzzy subset $f$ of $H$ is called a *fuzzy interior ideal* of $H$ if

$$f\Big((x \circ a) * \{y\}\Big) \geq f(a) \text{ for every } x, a, y \in H,$$

in the sense that if $x, a, y \in H$ and $u \in (x \circ a) * \{y\}$, then $f(u) \geq f(a)$.
For an hypersemigroup, we clearly have

$$(x \circ a) * \{y\} = \{x\} * (a \circ y) = \{x\} * \{a\} * \{y\}.$$

**Proposition 3.3.** *Let $H$ be an hypersemigroup. If $A$ is an interior ideal of $H$, then the characteristic function $f_A$ is a fuzzy interior ideal of $H$. "Conversely", if $A$ is a nonempty subset of $H$ such that $f_A$ is a fuzzy interior ideal of $H$, then $A$ is an interior ideal of $H$.*

**Proof.** $\Longrightarrow$. Let $x, a, y \in H$. Then $f_A\Big((x \circ a) * \{y\}\Big) \geq f_A(a)$. In fact: Let $u \in (x \circ a) * \{y\}$. If $a \in A$, then $f_A(a) = 1$. Since $A$ is an interior ideal of $H$, we have $H * A * H \subseteq A$. So we have $u \in \{x\} * \{a\} * \{y\} \subseteq H * A * H \subseteq A$. Then $u \in A$, and $f_A(u) = 1$. Thus we get $f_A(u) \geq f_A(a)$. Let now $a \notin A$. Then $f_A(a) = 0$. Since $f_A$ is a fuzzy subset of $H$ and $u \in H$, we have $f_A(u) \geq 0$. Thus we have $f_A(u) \geq f_A(a)$.
$\Longleftarrow$. Let $A$ be a nonempty subset of $H$ and $f_A$ a fuzzy interior ideal of $H$. Then $H * A * H \subseteq A$. Indeed: Let $u \in H * A * H$. Then $u \in v \circ y$ for some $v \in H * A$, $y \in H$ and $v \in x \circ a$ for some $x \in H$, $a \in A$. Since $v \circ y \subseteq (x \circ a) * \{y\}$, we have $u \in (x \circ a) * \{y\}$, where $x, y \in H$ and $a \in A$. Since $f_A$ a fuzzy interior ideal of $H$, we have $f_A(u) \geq f_A(a) = 1$. Since $f_A$ is a fuzzy subset of $H$ and $u \in H$, we have $f_A(u) \leq 1$. So we have $f_A(u) = 1$, and $u \in A$. □

**Proposition 3.4.** *Let $H$ be an hypersemigroup. If $f$ is a fuzzy ideal of $H$, then $f$ is a fuzzy interior ideal of $H$.*

**Proof.** Let $x, a, y \in H$. Then $f\Big((x \circ a) * \{y\}\Big) \geq f(a)$. In fact:
Let $u \in (x \circ a) * \{y\}$. By Proposition 2.1, there exists $v \in x \circ a$ such that $u \in v \circ y$. Since $v \in x \circ a$ and $f$ is a fuzzy left ideal of $H$, we have $f(v) \geq f(a)$. Since $u \in v \circ y$ and $f$ is a fuzzy right ideal of $H$, we have $f(u) \geq f(v)$. Then we have $f(u) \geq f(a)$, and the proof is complete. □

**Definition 3.5.** (cf. also [3]) An hypersemigroup $H$ is called *regular* if for every $a \in H$ there exists $x \in H$ such that $a \in \{a\} * (x \circ a)$.

**Lemma 3.6.** [3; Lemma 1.2] *Let $H$ be an hypersemigroup. The following are equivalent:*
(1) *$H$ is regular.*
(2) *$a \in \{a\} * \{x\} * \{a\}$ for every $a \in H$.*
(3) *$A \subseteq A * H * A$ for every nonempty subset $A$ of $H$.*

**Proposition 3.7.** *Let $H$ be a regular hypersemigroup and $A$ an interior ideal of $H$. Then $A$ is a subsemigroup of $H$.*

**Proof.** Since $A$ is an interior ideal of $H$, we have $H * A * H \subseteq A$. Since $H$ is regular, we have $A \subseteq A * H * A$. Then we have

$$A * A \subseteq (A * H * A) * A = (A * H) * A * A \subseteq H * A * H \subseteq A,$$

so $A$ is a subsemigroup of $H$. □

**Proposition 3.8.** *Let $H$ be a regular hypersemigroup and $f$ a fuzzy interior ideal of $H$. Then $f$ is a fuzzy ideal of $H$.*

**Proof.** Let $a, b \in H$. Then $f(a \circ b) \geq f(a)$ and $f(a \circ b) \geq f(b)$. In fact:
Let $u \in a \circ b$. Then $f(u) \geq f(a)$. Indeed: Since $a \in H$ and $H$ is regular, there exists $x \in H$ such that $a \in \{a\} * \{x\} * \{a\}$. Then

$$a \circ b \subseteq \{a\} * \{x\} * \{a\} * \{b\} = (a \circ x) * (a \circ b),$$

from which $u \in v \circ w$ for some $v \in a \circ x$, $w \in a \circ b$. We have $u \in v \circ w \subseteq \{v\} * (a \circ b)$ and $f\Big(\{v\} * (a \circ b)\Big) \geq f(a)$, thus we have $f(u) \geq f(a)$, and $f$ is a fuzzy right ideal of $H$. We also have $f(u) \geq f(b)$. Indeed: Since $b \in H$ and $H$ is regular, there exists $y \in H$ such that $b \in \{b\} * \{y\} * \{b\}$. Then we have

$$u \in a \circ b \subseteq \{a\} * \{b\} * \{y\} * \{b\} = (a \circ b) * (y \circ b).$$

Then $u \in s \circ t$ for some $s \in a \circ b$, $t \in y \circ b$. Then we have

$$u \in s \circ t \subseteq (a \circ b) * \{t\} = \{a\} * (b \circ t).$$

Since $f\Big(\{a\} * (b \circ t)\Big) \geq f(b)$, we obtain $f(u) \geq f(b)$, and $f$ is a fuzzy left ideal of $H$. Therefore $f$ is a fuzzy ideal of $H$. $\square$
From Propositions 3.4 and 3.8 we have the following

**Theorem 3.9.** *In regular hypersemigroups the concepts of fuzzy ideals and fuzzy interior ideals coincide.*

**Definition 3.10.** (cf. also [3]) An hypersemigroup $H$ is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ such that $a \in (x \circ a) * (a \circ y)$.

**Lemma 3.11.** *Let $H$ be an hypersemigroup. The following are equivalent:*
(1) *$H$ is intra-regular.*
(2) *$a \in H * \{a\} * \{a\} * H$ for every $a \in H$.*
(3) *$A \subseteq H * A * A * H$ for every nonempty subset of $H$.*

**Proof.** The implication (1) $\Rightarrow$ (2) and the equivalence (2) $\Leftrightarrow$ (3) are obvious. Let us prove the implication (2) $\Rightarrow$ (1). Let $a \in H$. By (2), we have $a \in \Big(H * \{a\}\Big) * \Big(\{a\} * H\Big)$. By Proposition 2.1, $a \in u \circ v$ for some $u \in H * \{a\}$, $v \in \{a\} * H$, $u \in x \circ a$ and $v \in a \circ y$ for some $x, y \in H$. Then we have $a \in u \circ v \subseteq (x \circ a) * (a \circ y)$, then $a \in (x \circ a) * (a \circ y)$, where $x, y \in H$ and so $H$ is intra-regular. $\square$

**Proposition 3.12.** *Let $H$ be an intra-regular hypersemigroup and $A$ an interior ideal of $H$. Then $A$ is a subsemigroup of $H$.*

**Proof.** Since $A$ is an interior ideal of $H$, we have $H * A * H \subseteq A$. Since $H$ is intra-regular, we have $A \subseteq H * A * A * H$. Then we have

$$\begin{aligned} A * A &\subseteq (H * A * A * H) * A = (H * A) * A * (H * A) \\ &\subseteq H * A * H \subseteq A, \end{aligned}$$

so $A$ is a subsemigroup of $H$. □

By Propositions 3.7 and 3.12, we have the following

**Corollary 3.13.** *In regular and in intra-regular hypersemigroups the interior ideals and the subidempotent interior ideals coincide.*

**Proposition 3.14.** *Let $H$ be an intra-regular hypersemigroup and $f$ is a fuzzy interior ideal of $H$. Then $f$ is a fuzzy ideal of $H$.*

**Proof.** Let $a, b \in H$ and $u \in a \circ b$. Since $a \in H$ and $H$ is intra-regular, there exist $x, y \in H$ such that $a \in \{x\} * \{a\} * \{a\} * \{y\}$. Then

$$a \circ b \subseteq \{x\} * \{a\} * \{a\} * \{y\} * \{b\} = (x \circ a) * \Big((a \circ y) * \{b\}\Big).$$

Then $u \in v \circ w$ for some $v \in x \circ a$, $w \in (a \circ y) * \{b\}$. We have

$$u \in v \circ w \subseteq (x \circ a) * \{w\}$$

and, since $f$ is a fuzzy interior ideal of $H$, $f\Big((x \circ a) * \{w\}\Big) \geq f(a)$. Thus we get $f(u) \geq f(a)$, and $f$ is a fuzzy right ideal of $H$. Since $b \in H$ and $H$ is intra-regular, there exist $z, t \in H$ such that $b \in \{z\} * \{b\} * \{b\} * \{t\}$, then we have

$$a \circ b \subseteq \{a\} * \{z\} * \{b\} * \{b\} * \{t\} = \Big((a \circ z) * \{b\}\Big) * (b \circ t).$$

Then $u \in c \circ d$ for some $c \in (a \circ z) * \{b\}$, $d \in b \circ t$. Since $u \in c \circ d \subseteq \{c\} * (b \circ t)$ and $f\Big(\{c\} * (b \circ t)\Big) \geq f(b)$, we have $f(u) \geq f(b)$, and $f$ is a fuzzy left ideal of $H$. Hence $f$ is a fuzzy ideal of $H$. □

By Propositions 3.4 and 3.14, we have the following theorem

**Theorem 3.15.** *In intra-regular hypersemigroups the concepts of fuzzy ideals and fuzzy interior ideals coincide.*

An ideal $A$ of an hypergroupoid $H$ is called *proper* if $A \neq H$.

**Definition 3.16.** An hypergroupoid $H$ is called *simple* if does not contain proper ideals, that is, for every ideal $A$ of $H$, we have $A = H$.

The concept of fuzzy simple semigroups due to Kuroki [6] can be naturally transferred to hypergroupoids as follows:

**Definition 3.17.** An hypergroupoid $H$ is called *fuzzy simple* if every fuzzy ideal of $H$ is a constant function, that is, for every fuzzy ideal $f$ of $H$ and every $a, b \in H$, we have $f(a) = f(b)$.

**Notation 3.18.** Let $H$ be an hypergroupoid and $a \in H$. We denote by $I_a$ the subset of $H$ defined as follows:

$$I_a = \{b \in H \mid f(b) \geq f(a)\}.$$

**Lemma 3.19.** *Let $H$ be an hypergroupoid and $f$ a fuzzy right (resp. fuzzy left) ideal of $H$. Then the set $I_a$ is a right (resp. left) ideal of $H$ for every $a \in H$.*

**Proof.** Let $a \in H$ and $f$ a fuzzy right ideal of $H$. The set $I_a$ is a right ideal of $H$. Indeed: Since $a \in I_a$, the set $I_a$ is a nonempty subset of $H$. Moreover, $I_a * H \subseteq I_a$. Indeed: Let $x \in I_a * H$. Then $x \in u \circ v$ for some $u \in I_a$, $v \in H$. Since $x \in u \circ v$ and $f$ is a fuzzy right ideal of $H$, we have $f(x) \geq f(u)$. Since $u \in I_a$, we have $f(u) \geq f(a)$, thus we have $f(x) \geq f(a)$. Since $u \in I_a$, we have $u \in H$. Since $u, v \in H$, we have $u \circ v \subseteq H * H \subseteq H$, so $x \in H$. Since $x \in H$ and $f(x) \geq f(a)$, we have $x \in I_a$. Thus $I_a$ is a right ideal of $H$. Similarly, if $f$ is a fuzzy left ideal of $H$, then the set $I_a$ is a left ideal of $H$ for every $a \in H$. □

**Corollary 3.20.** *If $H$ is an hypergroupoid and $f$ a fuzzy ideal of $H$, then the set $I_a$ is an ideal of $H$ for every $a \in H$.*

**Lemma 3.21.** *Let $H$ be an hypergroupoid. If $A$ a left (resp. right) ideal or an ideal of $H$, then the characteristic function $f_A$ is a fuzzy left (resp. fuzzy right) ideal or a fuzzy ideal of $H$. "Conversely", if $A$ is a nonempty subset of $H$ and $f_A$ a fuzzy left (resp. fuzzy right) ideal or a fuzzy ideal of $H$, then $A$ is a left (resp. right) ideal or an ideal of $H$.*

**Proof.** Let $A$ be a left ideal of $H$, $x, y \in H$ and $u \in x \circ y$. Then $f_A(u) \geq f_A(y)$. Indeed: If $y \in A$, then $x \circ y \subseteq H * A \subseteq A$, then $u \in A$ and $f_A(u) = 1 \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0 \leq f_A(u)$, so $f_A$ is a fuzzy left ideal of $H$. Let now $f_A$ be a fuzzy left ideal of $H$. Then $H * A \subseteq A$. Indeed: Let $u \in H * A$. Then $u \in x \circ y$ for some $x \in H$, $y \in A$. Since $u \in x \circ y$, we have $f_A(u) \geq f_A(y) = 1$. Then $f_A(u) = 1$, and $u \in A$. The "dual" (for right-fuzzy right ideals) can be proved in a similar way, this completes the proof. □

**Theorem 3.22.** *An hypergroupoid $H$ is simple if and only if it is fuzzy simple.*

**Proof.** $\Longrightarrow$. Let $f$ be a fuzzy ideal of $H$ and $a, b \in H$. Since $f$ is a fuzzy ideal of $H$ and $a \in H$, by Corollary 3.20, the set $I_a$ is an ideal of $H$. Since $H$ is simple, we have $I_a = H$. Then $b \in I_a$, so $f(b) \geq f(a)$. By symmetry, we get $f(a) \geq f(b)$. Thus we have $f(a) = f(b)$, and $H$ is fuzzy simple.
$\Longleftarrow$. Let $H$ be fuzzy simple and $I$ an ideal of $H$. Then $I = H$. Indeed: Let $x \in H$. Since $I$ is an ideal of $H$, by Lemma 3.21, the characteristic function $f_I$ is a fuzzy ideal of $H$. Since $H$ is fuzzy simple, $f_I$ is a constant function, that is, $f_I(y) = f_I(z)$ for every $y, z \in H$. Take an element $a \in I$ ($I \neq \emptyset$). Then we have $f_I(x) = f_I(a) = 1$, so $x \in I$. Thus $H$ is simple. □

**Theorem 3.23.** *If $H$ is an hypersemigroup, then the following are equivalent:*
(1) *$H$ is simple.*
(2) *$H = H * \{a\} * H$ for every $a \in H$.*
(3) *Every fuzzy interior ideal of $H$ is a constant function.*

**Proof.** (1) $\Longrightarrow$ (2). Let $a \in H$. The set $H * \{a\} * H$ is an ideal of $H$. Indeed, it is a nonempty subset of $H$, and we have
$H * (H * \{a\} * H) = (H * H) * \{a\} * H \subseteq H * \{a\} * H$ and
$(H * \{a\} * H) * H = H * \{a\} * (H * H) \subseteq H * \{a\} * H$.
Since $H$ is simple, we have $H * \{a\} * H = H$.
(2) $\Longrightarrow$ (3). Let $f$ be a fuzzy interior ideal of $H$ and $a, b \in H$. Then $f(a) = f(b)$.

Indeed: Since $b \in H$, by hypothesis, we have $b \in (x \circ a) * \{y\}$ for some $x, y \in H$. Since $f$ is a fuzzy interior ideal of $H$, we have $f(b) \geq f(a)$. By symmetry, we get $f(a) \geq f(b)$, so $f(a) = f(b)$.
(3) $\Longrightarrow$ (1). Let $f$ is a fuzzy ideal of $H$. By Proposition 3.4, $f$ is a fuzzy interior ideal of $H$. By hypothesis, $f$ is a constant function. Thus $H$ is fuzzy simple. Then, by Theorem 3.22, $H$ is simple. □

Summarizing, in case of an hypersemigroup the following are equivalent: (1) $H$ is simple; (2) $H = H * \{a\} * H$ for every $a \in H$; (3) $H = H * A * H$ for every $A \in \mathcal{P}^*(H)$; (4) $H$ is fuzzy simple; (5) every fuzzy interior ideal of $H$ is a constant function. Clearly $H = H * \{a\} * H$ for every $a \in H$ is equivalent to $H = H * A * H$ for every nonempty subset $A$ of $H$.

With my best thanks to Prof. Klaus Denecke for his interest in my work and his prompt reply.

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