Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 84, No. 1 Whole Number 297

April 2021

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

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TWO MACHINE FLEXIBLE SHOP SCHEDULING PROBLEM

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Received March 27,2019; revised September 13,2019

Abstract. We study two machine shop type scheduling problem in this paper. At shop type scheduling, one job is handled by plural machines. And it generally divided two types of problem. Flowshop type have been determined the order of operation at each machine. And it is not decided on Openshop type. In this paper, considering a problem with two machines and flexible jobs which have no strict order of machines, but have desirably order. The super shop problem which is mixed with flow shop and open shop is considered by Strusevich and suggests a solution method that based on 13 cases. In this paper, we extend this result and give more detailed condition on one case which include preemptive job. We also propose a solution to this problem by extending this result to flexible flowshop.

1 Introduction Shop type scheduling problem is one of the most major part of scheduling problem research. Especially the study for two machine flow shop scheduling problem by Johnson is one of the most famous results in scheduling problem. There are various types of constraints for shop scheduling problem. Two machine shop type scheduling problem is defined as follows.

- There are two machines M_1, M_2 and n jobs $1, \cdots, n$.
- Each job is processed by M_1 and M_2. And these processes are not allowed to overlap.
- There are constraints in the order of the processes for each job.
- The objective function is the completion time of all jobs.

Two machine flow shop problem is studied by Johnson [1]. In this problem the order of the processes for each job is fixed as $M_1 \to M_2$. He showed an optimizing procedure by sorting in processing time for each job as Johnson rule. The job shop type problem is defined as there are two types of processing order constraint as $M_1 \to M_2$ or $M_2 \to M_1$ for all jobs. This problem is studied by Jackson, J. R [2]. Also, there is no constraint in order of processes is called Open shop problem. This problem is studied by Gonzalez, T and Sahni, S [3]. Furthermore, it has been also considered a mixed problem that combines constraints of these three types. The problem including flow shop type and open shop type jobs has been studied by Masuda, T., Ishii, H. and Nishida, T [4] as a mixed shop problem. V. A. Strusevich studied two machine super shop including two types of flow shop jobs and open shop. He divided the problem to 13 cases based on the processing time and show the condition which optimal non-preemptive schedule is possibly different from the optimal preemptive schedule [5].

2000 *Mathematics Subject Classification.* Primary 65F10, 65F15; Secondary 65H10, 65F03.
Key words and phrases. conjugate gradient algorithm, Lanczos algorithm, variable metric algorithm.

2 Problem Definition In this paper, we consider the problem adding the flexible jobs to super shop problem. Flexible jobs are defined as : The order of processing by machine M_1, M_2 is unrestricted. However, the satisfaction degree is defined for the order. For example, in the painting jobs with two colors, any order of painting color is allowable, but there is the difference in the finish. Our problem settings are as follows;

- There exists a set of n jobs $N = J_{12} \cup J_{21} \cup O \cup F_{12} \cup F_{21}$.
- J_{12} : Flow shop type job set $M_1 \to M_2$.
- J_{21} : Flow shop type job set $M_2 \to M_1$.
- O : Open shop type job set which processing order is open i.e. either $M_1 \to M_2$ or $M_2 \to M_1$ is allowable.
- F_{12}: A flexible job preferably should be processed on M_1 but in some case first on M_2.
- F_{21} : A flexible job preferably should be processed on M_2 but in some case first on M_1.
- For each job, we define the two satisfaction degrees $\mu_1(j), \mu_2(j)$ of processing order on two machines.
- $\mu_1(j)$: the satisfaction degree in case that job j is processed M_1 first.
- $\mu_2(j)$: the satisfaction degree in case that job j is processed M_2 first.

The meanings of satisfaction degree are as follows;

- $j \in J_{12}$: $\mu_1(j) = 1, \mu_2(j) = 0$,
- $j \in J_{21}$: $\mu_1(j) = 0, \mu_2(j) = 1$,
- $j \in O$: $\mu_2(j) = 1, \mu_1(j) = 1$,
- $j \in F_{12}$: $\mu_1(j) = 1, 0 < \mu_2(j) < 1$,
- $j \in F_{21}$: $\mu_2(j) = 1, 0 < \mu_1(j) < 1$.

F_{12} and F_{21}: call flexible order job set.

For each job j, we define the processing times a_j, b_j on M_1, M_2 respectively. Each machine M_1 and M_2 processing at most one job at a time and each job is processed on at most one machine at a time. Under above setting, we seek a schedule minimizing the maximum completion time and maximizing the minimum satisfaction degree about processing order on machines, but usually there is no feasible schedule optimizing both criteria. We seek some non-dominated schedules after the definition of non-domination.

Non-dominated schedule For each schedule s, we define schedule vector $v^s = (v_1^s, v_2^s) = (C_{\max}^s, \mu^s)$ where $C_{\max}^s$ is the maximum completion time of schedule s and $\mu^s = \min\{\min\{\mu_1(j), j \in A(s)\}, \min\{\mu_2(j), j \in B(s)\}\}$, where $A(s)$: set of jobs processed on M_1 first in schedule s, $B(s)$: set of jobs processed on M_2 first in schedule s. For schedules s^1, s^2, we call s^1 dominate s^2 if $v_1^{s^1} \leq v_1^{s^2}, v_2^{s^1} \geq v_2^{s^2}$ and $v^{s^1} \neq v^{s^2}$ and we call a schedule s non-dominated schedule if no schedule dominates s. We seek some non-dominated schedules $a(J) = \Sigma_{j \in J} a_j, b(J) = \Sigma_{j \in J} b_j, \pi(J)$: arbitrary schedule of job set J.

3 Super shop problem The procedure for our problem is based on reducing to super shop problem corresponding to the satisfaction degree. Super shop problem is considered as a mixed model with two flow shop model and open shop model. The definition for super shop problem is as follows.

For subset of jobs $Q \subseteq N$, $a(Q) = \Sigma_{j_i \in Q} a_i$, $b(Q) = \Sigma_{j_i \in Q} b_i$, $a(\emptyset) = b(\emptyset) = 0$, define the subscripts a or b on upside of arbitrary job set Q as $Q^a = \{j_i \in Q \mid a_i < b_i\}$ or $Q^b = \{j_i \in Q \mid a_i \geq b_i\}$ respectively.

- N_{12} : Flow shop type job set with job processing order $M_1 \to M_2$,
- N_{21} : Flow shop type job set with job processing order $M_2 \to M_1$,
- N_O : Open shop job set,
- j_k : The job $j_k \in N_O^a$ and has maximum processing time on machine M_2, i.e. $b_k = \max\{b_j \mid \text{for } j \in N_O^a\}$. Here $N_O^a = \{j \in N_O \mid a_j < b_j\}$,
- j_r : The job $j_r \in N_O^b$ and has maximum processing time on machine M_1, i.e. $a_r = \max\{a_j \mid \text{for } j \in N_O^b\}$. Here $N_O^b = \{j \in N_O \mid a_j \geq b_j\}$,
- $N = N_{12} \cup N_{21} \cup N_O$. Job set of all jobs.

Let $T = \max\{a(N), b(N)\}$. Let the permutation $\pi(Q)$ be an arbitrary permutation of jobs from Q, permutation $\pi(\emptyset)$ be dummy permutation, and $\varphi(N_{12})$, $\varphi(N_{21})$ be an optimal processing order applying Johnson rule to flow shop job set N_{12} and N_{21} respectively.

In our problem, let $C_{\max}(s)$ be a maximum completion time of super shop schedule s. Lower bound of maximum completion is as follows

$$C_{\max} \geq \max\{a(N), b(N), C_{\max}(s_{12}^*), C_{\max}(s_{21}^*), \max\{a_i + b_i \mid J_i \in N_O\} + \tau\}$$

Here s_{12}^* and s_{21}^* are the optimal schedules for jobs of N_{12} and N_{21} respectively by Johnson's rule, $\tau = \min\{a(N_{12}^a) + b(N_{12}^b), a(N_{21}^a) + b(N_{21}^b)\}$. Strusevich develop the solution algorithm for this problem. In this algorithm the problem divided to 13 cases based on the sum of processing time for each part.

Case 1 : $a(N_{12}^a) \geq b(N_{21} \cup N_O)$

Optimal schedule is constructed by following procedure, where N_{12} is an optimal processing order applying Johnson rule to flow shop job set N_{12}, and s_{12}^* is the corresponding optimal schedule.

1. M_1: processing order $\varphi(N_{12}) \to \pi(N_{21} \cup N_O)$ from 0.
2. M_2: first processing order $\pi(N_{21} \cup N_O)$ from 0 and second processing order $\varphi(N_{12})$ from $\max\{b(N_{21} \cup N_O), C_{\max}(s_{12}^*) - b(N_{12})\}$.

Case 2 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O)$, and $a(N_O - \{j_k\}) \leq b(N_O - \{j_r\}), a(N - \{j_k\}) \geq b(N_{21} \cup N_O)$

Optimal schedule is constructed by following procedure. Let $\psi(N_O) = (j_r, \pi(N_O^b - \{j_r\}), \pi(N_O^a - \{j_k\}), j_k)$.

1. M_1: processing order $\pi(N_{12}) \to \pi(N_{21}) \to \psi(N_O)$ from 0
2. M_2: processing order $\pi(N_{21}) \to \psi(N_O) \to \pi(N_{12})$ from 0, where $\psi(N_O) = (j_r, \pi(N_O^b - \{j_r\}), \pi(N_O^a - \{j_k\}), j_k)$ and the corresponding maximum completion time of optimal schedule in this case is T.

Case 3 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N_O - \{j_k\}) \leq b(N_O - \{j_r\}), b(N - \{j_k\}) \geq b(N_{21} \cup N_O)$.

Optimal schedule is constructed by following procedure.

1. M_1: processing order $\psi(N_O - \{j_k\}) \to \pi(N_{12}) \to \pi(N_{21}) \to J$ from 0.
2. M_2: processing order $\pi(N_{21}) \to j_k \to \psi(N_O - \{j_k\}) \to \pi(N_{12})$ from 0, where $\psi(N_O - \{j_k\}) = (\pi(N_O^a - \{j_k\}), \pi(N_O^b - \{j_r\}))$ and the corresponding maximum completion time of optimal schedule in this case is T.

Case 4 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N_O - \{j_k\}) > b(N_O - \{j_r\}), a(N_{12} \cup N_{21}) \geq b(N \cup \{j_r\})$, corresponding maximum completion time of optimal schedule is T.

Case 5 : $b(N_{21} \cup N_O) > a(N_{12}) \geq b(N_{21}), a(N_O - \{j_r\}) \geq b(N_{21} \cup \{j_r\}) > a(N_{12} \cup N_{21})$, corresponding maximum completion time of optimal schedule is T.

Case 6 : first set m, $b(N_{21} \cup N_O) > a(N_O - \{j_k\})$, and $a(N_O - \{j_k\}) \leq b(N_O - \{j_r\}) \Rightarrow j_m = j_k$,

$a(N_{21} \cup N_{12}) < b(N_{21} \cup \{j_r\})$, and $a(N_O - \{j_k\}) > b(N_O - \{j_r\}) \Rightarrow j_m = j_r$, $b(N_{21}) \leq a(N_{21}) < b(N_{21} \cup N_O), a(N - \{m\}) < b(N_{12} - \{j_m\}), a_m \leq b(N_{12} \cup N_O - \{j_m\})$ corresponding maximum completion time of optimal schedule is $b(N)$.

Case 7 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N - \{j_m\}) < b(N_{21} \cup \{j_m\}), a_m > b(N_{12} \cup N_O - \{j_m\}), a(N_{12}^a) + b(N_{12}^b) \leq a(N_{21}^a) + b(n_{21}^b), a(N_{12}^b) < b_m$ corresponding maximum completion time of optimal schedule is $\max\{T, a_m + b_m + a(N_{12}^a) + b(N_{12}^b)\}$.

Case 8 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N - \{j_m\}) < b(N_{21} \cup \{j_m\}), a_m > b(N_{12} \cup N_O - \{j_m\})$, $a(N_{12}^a) + b(N_{12}^b) \leq a(N_{21}^a) + b(n_{21}^b), a(N_{12}^b) < b_m, a(N_{12} \cup N_O) > b(N_{21} \cup \{j_m\})$, corresponding maximum completion time of optimal schedule is T.

Case 9 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N - \{j_m\}) < b(N_{21} \cup \{j_m\}), a_m > b(N_{12} \cup N_O - \{j_m\})$, $a(N_{12}^a) + b(N_{12}^b) \leq a(N_{21}^a) + b(N_{21}^b), a(N_{12}^b) < b_m, a(N_{12} \cup N_O) > b(N_{21} \cup \{j_m\})$, corresponding maximum completion time of optimal schedule is T.

Case 10 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O), a(N - \{j_m\}) < b(N_{21} \cup \{j_m\}), a_m > b(N_{12} \cup N_O - \{j_m\})$, $a(N_{12}^a) + b(N_{12}^b) > a(N_{21}^a) + b(N_{21}^b), a_m \geq b(N_{21}^a)$, corresponding maximum completion time of optimal schedule is $\max\{T, a_m + b_m + a(N_{12}^a) + b(N_{12}^b)\}$.

Case 11 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O)$, $a(N - \{j_m\}) < b(N_{21} \cup \{j_m\})$, $b(N_{21}^a) > a_m > b(N_{12} \cup N_O - \{j_m\})$, $a(N_{12}^a) + b(N_{12}^b) > a(N_{21}^a) + b(N_{21}^b)$, $a(N_{12} \cup j_m) \geq b(N_{21} \cup N_O)$, the corresponding maximum completion time of optimal schedule in this case is $b(N)$.

Case 12 : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O)$, $a(N - \{j_m\}) < b(N_{21} \cup \{j_m\})$, $b(N_{21}^a) > a_m > b(N_{12} \cup N_O - \{j_m\})$, $a(N_{12}^a) + b(N_{12}^b) > a(N_{21}^a) + b(N_{21}^b)$, $a(N_{12} \cup \{j_m\}) \geq b(N_{21} \cup N_O)$, corresponding maximum completion time of optimal schedule in this case is T.

Strusevich proposed the 13 cases for Super shop problem. At only one case, the schedule includes the nonpreemptive jobs. We divide this case to two cases by precisely condition.

Case 13(i) : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O)$, $a(N - \{j_m\}) < b(N_{21} \cup \{j_m\})$, $a_m > b(N_{12} \cup N_O - \{j_m\})$, $b(N_{21}) + b_m > a(N_{21} \cup N_O - \{j_m\})$, the corresponding maximum completion time of optimal schedule in this case is $a(N)$.

Case 13(ii) : $b(N_{21}) \leq a(N_{12}) < b(N_{21} \cup N_O)$, $a(N - \{j_m\}) < b(N_{21} \cup \{j_m\})$, $a_m > b(N_{12} \cup N_O - \{j_m\})$, $b(N_{21}) + b_m \leq a(N_{21} \cup N_O - \{j_m\})$, the corresponding maximum completion time of optimal schedule in this case is $a(N)$. Only in this case, optimal non-preemptive schedule is possibly different from the optimal preemptive schedule. But in this case also optimal maximum completion time is one of $a_m + b_m, a(N), b(N)$. We should check only some cases among 14 cases.

4 Solution procedure of flexible shop scheduling problem In this section, we propose the solution procedure for our flexible shop model. This procedure is based on super shop problem. There are multiple optimal solutions for the value of satisfaction degree. Therefore, we seek the non-domination solution. The detail of the procedure is as follows.

Assignment of processing order and solve the super shop problems

1. Sort $\mu_2(j), j \in F_{12}, \mu_1(j), j \in F_{21}$ and result be $\mu(0) = 1 > \mu(1) > \mu(2) > \mu(3) > \cdots > \mu(u) > \mu(u+1) = 0$.
2. Consider the super shop problem $P(t)$ with parameter $\mu(t), t = 0, 1, 2, \cdots, u+1$ as the subproblem where u: the number of different values in $\mu_2(j), j \in F_{12}, \mu_1(j), j \in F_{21}$,
3. $P(t)$:the super shop problem with
   (a) $N_{12} = J_{12} \cup \{j \in F_{12} \mid \mu_2(j) < \mu(t)\}, N_{21} = N_{12} = J_{21} \cup \{j \in F_{12} \mid \mu(j) < \mu(t)\}$
   (b) $N_O = O \cup \{j \in F_{12} \mid \mu_2(j) \geq \mu(t)\} \cup \{j \in F_{21} \mid \mu_1(j) \geq \mu(t)\}$
   (c) Apply the super shop scheduling algorithm by checking 14 cases and obtain optimal scheduling $s(t)$. Note that $P(0)$: $N_{12} = J_{12} \cup F_{12}$, $N_{21} = J_{21} \cup F_{21}$, $N_O = O$, $P(u+1)$: $N_{12} = J_{12}$, $N_{21} = J_{21}$, $N_O = O \cup F_{12} \cup F_{21}$.
4. From $s(0), s(1), \cdots, s(u+1)$, choose non-dominated schedules. Note that N_O is non-decreasing about t and N_{12}, N_{21} is non-increasing.

5 Numerical Example In this section, we consider the some numerical example. The following jobs are considered.

	N_{12}	F_{12}		N_{21}	F_{21}		N_O	
i	1	2	3	4	5	6	7	8
a_i	3	4	1	1	2	2	17	2
b_i	2	5	4	2	3	1	12	1
μ_i^A	1	1	1	0	0.6	0.8	1	1
μ_i^B	0.3	0.7	0.5	1	1	1	1	1

Table 1: Numerical Example

We obtain 5 cases ($\mu = 1.0, 0.8, 0.7, 0.6, 0.5, 0.3$) in non-increasing order of satisfaction, and seek the optimal schedule in each case.

For $\mu = 1.0$: In this constraint, the processing order of flexible jobs is fixed.

This case corresponds the case 10 of super-shop from the following checking:

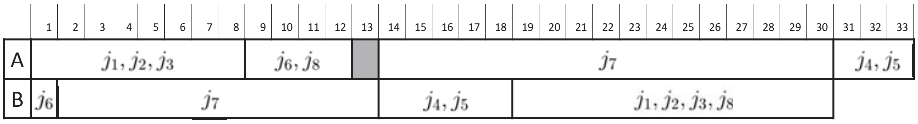
$N_O^a = \emptyset$, $N_O^b = \{j_7, j_8\}$, $j_r = j_7$, j_k is not defined. Therefore $a(N_o - \emptyset) = 19 > b(N_a - \{j_7\}) = 1$, $a(N_{12} \cup N_{21}) = 15 < b(N_{21} \cup \{j_7\}) = 18$ holds and so we set $j_m = j_7$. $a_7 = 17 > b(N_{12} \cup N_O - \{j_7\}) = 12$, $a(N_{12}^a) + b(N_{12}^b) + 4 + 1 + 2 = 7 > a(N_{21}^a) + b(N_{21}^b) = 1 + 2 + 1 = 4$, $a_7 = 17 > b(N_{21}^a) = 5$

	N_{12}			N_{21}			N_O	
i	1	2	3	4	5	6	7	8
a_i	3	4	1	1	2	2	17	2
b_i	2	5	4	2	3	1	12	1

Table 2: $\mu = 1.0$

Here $N_{12} = \{j_1, j_2, j_3\}$, $N_{21}^b \cup N_O - \{j_m\} = \{j_6, j_8\}$, $j_m = j_7$, $N_{21}^b = \{j_4, j_5\}$, and $N_{21}^b = \{j_6\}$, $N_{21}^b = \{j_4, j_5\}$, $N_{12} \cup N_O - \{j_m\} = \{j_1, j_2, j_3, j_8\}$.

If preemption of the processing is not allowed, the optimal completion time $C_{\max}(s^*) = 33$.

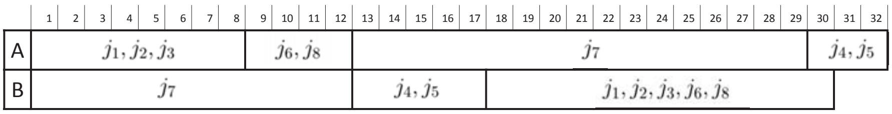


For $\mu = 0.8$: In this constraint, the processing order of flexible jobs is fixed.

	N_{12}			N_{21}		N_O		
i	1	2	3	4	5	6	7	8
a_i	3	4	1	1	2	2	17	2
b_i	2	5	4	2	3	1	12	1

Table 3: $\mu = 0.8$

We obtain the two optimal schedule which completion time $C_{\max}(s^*) = 32$ without preemption.



Since completion time 32 is a lower bound, we need not check $\mu = 0.7, \mu = 0.6, \mu = 0.5, \mu = 0.3$.

6 Discussion and Conclusion If case 13(ii) does not occur, we can obtain non-dominated solutions without preemption. Since $a(N), b(N), T = \max\{a(N), b(N)\}$ are constant independent from processing order of any jobs, we can utilize this fact to make our algorithm efficient. Anyway, we must consider the efficient method to solve each super shop scheduling problem using some sensitivity of the conditions about change on processing order of F_{12}, F_{21}. For that purpose, we should simplify the cases of the solution method due to Strusevich including investigation of further division in case(ii) though we divide case 13 into two subcases 13(i) and 13(ii).

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KATO'S INEQUALITIES UP TO THE BOUNDARY FOR A QUASILINEAR ELLIPTIC OPERATOR

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Received June 29, 2019

Abstract

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. By Δ_p with $1 < p < \infty$ we denote p-Laplacian. We prove that if $\Delta_p u$ is a finite measure in Ω, then under suitable assumptions on u, $\Delta_p u^+$ is also a finite measure in Ω up to the boundary $\partial\Omega$. *

1 Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. By Δ_p for $p \in (1, +\infty)$ we denote p-Laplacian. The classical Kato's inequality for a Laplacian in [12] asserts that given any function $u \in L^1_{\text{loc}}(\Omega)$ such that $\Delta u \in L^1_{\text{loc}}(\Omega)$, then $\Delta(u^+)$ is a Radon measure and the following holds:

$$\Delta(u^+) \geq \chi_{[u\geq 0]}\Delta u \qquad \text{in } D'(\Omega), \tag{1.1}$$

where $u^+ = \max\{u, 0\}$. In [5, 6], H.Brezis and A.Ponce intensively studied Kato's inequalities with Δu being a Radon measure and established the strong maximum principle, the improved Kato's inequality and the inverse maximum principle (See also [8, 10]). Then, in [13, 14] Kato's inequality was further studied for $\Delta_p u$ with $p \in (1, \infty)$ and most of the counter-parts were established under the assumption that u is admissible in $W^{1,p^*}_{\text{loc}}(\Omega)$, where $p^* := \max\{1, p-1\}$. For the admissibility in $W^{1,p^*}_{\text{loc}}(\Omega)$, see Definition 4.1 in Appendix and see also [15]. We remark that when $p = 2$, the notion of admissibility becomes trivial. On the other hand, H.Brezis and A. Ponce in [7] and A. Ancona in [1] studied Kato's inequality (1.1) up to the boundary for $p = 2$.

The purpose in the present paper is to study Kato's inequality for Δ_p up to the boundary of Ω. As a result, we will show that $\Delta_p u^+$ is also a finite measure under suitable assumptions on u. In these arguments it is crucial to introduce a class $\mathbb{X}_p$ in Definition 1.1, which was originally introduced in Brezis, Ponce [7] for Δ, and to use effectively a notion of admissibility in $\mathbb{X}_p$ for Δ_p.

Definition 1.1. We say $u \in \mathbb{X}_p$ if $u \in W^{1,p^*}(\Omega)$ and if there exists a constant $C > 0$ such that

$$\left|\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right| \leq C||\varphi||_{L^\infty(\Omega)}, \qquad \text{for any } \varphi \in C^1(\overline{\Omega}), \tag{1.2}$$

in which case we set

$$[u]_{\mathbb{X}_p} = \sup_{\substack{\psi\in C^1(\bar{\Omega}) \\ \|\psi\|_{L^\infty}\leq 1}} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi. \tag{1.3}$$

If $u \in \mathbb{X}_p$, then there exists a unique bounded linear functional $T \in [C(\bar{\Omega})]^* = \mathscr{M}_b(\bar{\Omega})$ such that

$$\langle T, \psi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \quad (\forall \psi \in C^1(\bar{\Omega})).$$

*Keywords: Kato's inequality, p-Laplace operator
2000 mathematics Subject Classification: Primary 35J70, Secondary 35J60
This research was partially supported by Grant-in-Aid for Scientific Research (No. 16K05189) and (No. 15H03621).

On the other hand, by the Riesz Representation Theorem any $T \in \mathscr{M}_b(\bar{\Omega})$ admits a unique decomposition

$$\langle T, \psi \rangle = \int_{\partial\Omega} \psi \, d\nu + \int_{\Omega} \psi \, d\mu \quad (\forall \psi \in C(\bar{\Omega})),$$

where $\mu \in \mathscr{M}_b(\Omega)$ and $\nu \in \mathscr{M}_b(\partial\Omega)$. By $\mathscr{M}_b(\Omega)$ and $\mathscr{M}_b(\partial\Omega)$ we denote the space of all bounded measures in Ω and $\partial\Omega$, equipped with the standard norms $\| \cdot \|_{\mathscr{M}_b(\Omega)}$ and $\| \cdot \|_{\mathscr{M}_b(\partial\Omega)}$ respectively. We remark that measures in $\mathscr{M}_b(\Omega)$ are identified with measures in Ω which do not charge $\partial\Omega$. More precisely we have

$$||\mu||_{\mathscr{M}_b(\Omega)} = \sup \left\{ \int_{\Omega} \varphi \, d\mu ; \varphi \in C_0(\bar{\Omega}) \text{ and } ||\varphi||_{L^\infty(\Omega)} \leq 1 \right\},$$

where by $C_0(\bar{\Omega})$ we denote the space of all continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. On the other hand $\mathscr{M}(\Omega)$ denotes the space of all Radon measures in Ω. In other words $\mu \in \mathscr{M}(\Omega)$ if and only if, for every $\omega \subset\subset \Omega$, there is $C_\omega > 0$ such that $| \int_\Omega \varphi \, d\mu | \leq C_\omega ||\varphi||_\infty$ for all $\varphi \in C_0(\overline{\omega})$. When $u \in \mathbb{X}_p$, we will denote

$$\mu = -\Delta_p u \, , \quad \nu = |\nabla u|^{p-2} \frac{\partial u}{\partial n},$$

where n denotes the outer normal. In this paper, for $u \in \mathbb{X}_p$ we always use the notations $\Delta_p u$ and $|\nabla u|^{p-2} \frac{\partial u}{\partial n}$ in the above sense. Hence if $u \in \mathbb{X}_p$, then we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi = \int_{\partial\Omega} \psi |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\Omega} \psi \Delta_p u \quad (\forall \psi \in C^1(\bar{\Omega})).$$

It follows from Theorem 3.1 that for every $u \in \mathbb{X}_p$

$$[u]_{\mathbb{X}_p} = \int_{\Omega} |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial n} \right|$$

and if u is admissible in $\mathbb{X}_p$, then $[u]_{\mathbb{X}_p} = 0$ if and only if $u = const.$ in Ω.

2 Preliminaries: Admissibilities in $\mathbb{X}_p$ and $W_0^{1,p^*}(\Omega)$

We will work with the standard Sobolev spaces; $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, where the space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = |||\nabla u|||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}, \tag{2.1}$$

and by $W_0^{1,p}(\Omega)$ we denote the completion of $C_c^\infty(\Omega)$ in the norm $|| \cdot ||_{W^{1,p}(\Omega)}$. Now we introduce two admissiblities for Δ_p to deal with Kato's inequalities up to the boundary. We note that these notions become trivial if $p = 2$ and a local version was already introduced in [14].

Definition 2.1. (**Admissibility in** $\mathbb{X}_p$) Let $1 < p < \infty$ and $p^* := \max\{1, p-1\}$. A function u is said to be admissible in $\mathbb{X}_p$ if $u \in \mathbb{X}_p$ and there exists a sequence $\{u_k\}_{k=1}^\infty \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W^{1,p^*}(\Omega)$ as $k \to \infty$.

2. $\Delta_p u_k \in L^1(\Omega)$ and $|\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \in L^1(\partial\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k ||\Delta_p u_k||_{\mathscr{M}_b(\Omega)} = \sup_k \int_{\Omega} |\Delta_p u_k| < \infty \tag{2.2}$$

$$\sup_k \left|\left| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right|\right|_{\mathscr{M}_b(\partial\Omega)} = \sup_k \int_{\partial\Omega} \left| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right| < \infty. \tag{2.3}$$

Definition 2.2. (**Admissibility in** $W_0^{1,p^*}(\Omega)$) Let $1 < p < \infty$ and $p^* := \max\{1, p-1\}$. A function u is said to be admissible in $W_0^{1,p^*}(\Omega)$ if $u \in W_0^{1,p^*}(\Omega)$, $\Delta_p u \in \mathscr{M}_b(\Omega)$ and there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W_0^{1,p^*}(\Omega)$ as $k \to \infty$.
2. $\Delta_p u_k \in L^1(\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k ||\Delta_p u_k||_{\mathscr{M}_b(\Omega)} = \sup_k \int_\Omega |\Delta_p u_k| < \infty. \tag{2.4}$$

Roughly speaking, if u is admissible in one of these definitions, then u can be approximated by a sequence of good functions not only in the sense of the distributions but also in the sense of measures. Moreover it is possible to approximate u by a sequence of C^1-functions provided that u is admissible. In fact in Proposition 4.1 in Appendix we collect such nice properties of admissible functions together with a local version of the admissibility in $W_{\text{loc}}^{1,p^*}(\Omega)$. In the subsequent we describe more remarks.

Remark 2.1. 1. For a general class of uniformly elliptic operators with a divergence form, one can define the admissibility and establish similar results in parallel to the present paper (c.f. [15]). Further it is possible to construct non-admissible functions in such cases. When $p = 2$, the existence of pathological solution, which is non-admissible, was initially shown by J Serrin in the famous paper [20] (See also [11]).

2. If $u \in W_{\text{loc}}^{1,p^*}(\Omega)$, then $\Delta_p u$, $\Delta_p(u^+)$ and $\Delta_p(u^-)$ are well-defined in $D'(\Omega)$. Let $\{u_k\}$ be the sequence in one of the definitions. It follows from the condition 1 that $\Delta_p u_k = \Delta_p(u_k^+) - \Delta_p(u_k^-)$ and $\Delta_p u_k \to \Delta_p u$ (i.e. $\Delta_p(u_k^{\pm}) \to \Delta_p(u^{\pm})$) in $D'(\Omega)$ as $k \to \infty$. Moreover, it follows from the condition 2 and the weak compactness of measures that we have $\Delta_p u_k \to \Delta_p u$ (i.e. $\Delta_p(u_k^{\pm}) \to \Delta_p(u^{\pm})$) in the sense of measures as $n \to \infty$. (In the case of Definition 2.1, $|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} \to |\nabla u|^{p-2}\frac{\partial u}{\partial n}$ in the sense of measures as well.) Therefore if u is admissible, then u^+ and u^- are so as well.

3. Let Ω be a unit ball $B_1(0)$ of R^N. Let $u = |x|^\alpha - 1$ for $\alpha = (p-N)/(p-1)$ and $p \in (1, N)$. Then u satisfies

$$\Delta_p u = \alpha|\alpha|^{p-2} c_N \delta \qquad \text{in } D'(\Omega),$$

where δ denotes a Dirac mass and c_N denotes the surface area of the N-dimensional unit ball B_1. Then u is admissible in $W_0^{1,p^*}(\Omega)$ if $p \in (2 - 1/N, N)$ with $N \geq 2$. We note that when $1 < p < 2 - \frac{1}{N}$, u is not admissible but regarded as a renormalized solution. For the detail see [2, 4, 17, 18, 19]

3 Main results

Given $M > 0$, we denote a truncation function T_M: $R \to R$ by

$$T_M(s) = \max\{-M, \min\{M, s\}\}. \tag{3.1}$$

Theorem 3.1. If $u \in \mathbb{X}_p$, then we have:

1.
$$[u]_{\mathbb{X}_p} = \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right|. \tag{3.2}$$

2. If u is admissible in $\mathbb{X}_p$, then for every $M > 0$ $T_M u \in W^{1,p}(\Omega)$ and we have

$$\int_\Omega |\nabla T_M(u)|^p \leq M[u]_{\mathbb{X}_p}. \tag{3.3}$$

3. If u is admissible in $\mathbb{X}_p$, then $[u]_{\mathbb{X}_p} = 0$ if and only if $u = const.\ in\ \Omega$.

Theorem 3.2. If u is admissible in $\mathbb{X}_p$, then $u^+ \in \mathbb{X}_p$ and we have

$$[u^+]_{\mathbb{X}_p} \leq [u]_{\mathbb{X}_p}. \tag{3.4}$$

Theorem 3.3. Assume that u is admissible in $W_0^{1,p^*}(\Omega)$. Then we have the followings:

1. u is admissible in $\mathbb{X}_p$ (hence, $u^+ \in \mathbb{X}_p$).
2. $$\int_\Omega |\Delta_p u^+| \leq \int_\Omega |\Delta_p u|. \tag{3.5}$$

Remark 3.1. If u does not vanish on $\partial\Omega$, then the assertion (3.5) fails. To see this it suffices to take a linear function u.

Theorem 3.4. Assume that u is admissible in $\mathbb{X}_p$. Moreover assume that $\Delta_p u \in L^1(\Omega)$, $|\nabla u|^{p-2}\frac{\partial u}{\partial n} \in L^1(\partial\Omega)$. Then

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+ \cdot \nabla\psi \leq \int_{\partial\Omega} H\psi - \int_\Omega G\psi \quad (\forall\psi \in C^1(\bar{\Omega}), \psi \geq 0\ in\ \Omega). \tag{3.6}$$

Here $G \in L^1(\Omega)$ and $H \in L^1(\partial\Omega)$ are given by

$$G = \begin{cases} \Delta_p u & on\ [u > 0] \\ 0 & on\ [u \leq 0] \end{cases}, \quad H = \begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial n} & on\ [u > 0] \\ 0 & on\ [u < 0] \\ \min\{|\nabla u|^{p-2}\frac{\partial u}{\partial n}, 0\} & on\ [u = 0]. \end{cases} \tag{3.7}$$

Thus, we have

$$\begin{cases} \Delta_p u^+ \geq G & in\ \Omega \\ |\nabla u|^{p-2}\frac{\partial u^+}{\partial n} \leq H & on\ \partial\Omega. \end{cases} \tag{3.8}$$

3.1 Proof of Theorem 3.1

Proof of Theorem 3.1 (1). This is a standard argument. Since $u \in \mathbb{X}_p$, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi = \int_{\partial\Omega} \psi\nu + \int_{\partial\Omega} \psi\mu \quad (\forall\psi \in C^1(\bar{\Omega})), \tag{3.9}$$

where $\mu = -\Delta_p u \in \mathscr{M}_b(\Omega)$ and $\nu = |\nabla u|^{p-2}\frac{\partial u}{\partial n} \in \mathscr{M}_b(\partial\Omega)$. From the definition we have

$$[u]_{\mathbb{X}_p} = \sup_{\substack{\psi \in C^1(\bar{\Omega}) \\ \|\psi\|_{L^\infty} \leq 1}} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \leq \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right|.$$

To see the opposite inequality, without the loss of generality we assume that $\mu \in C_c^\infty(\Omega)$ and $\nu \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \phi$. Define $\psi = \operatorname{sgn}(\mu) + \operatorname{sgn}(\nu)$, where $\operatorname{sgn}(t) = 1, t > 0; 0, t = 0; -1, t < 0$. Let ψ_ε be a mollification of ψ such that $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, $|\psi_\varepsilon| \leq 1$ and $\psi_\varepsilon \to \psi$ as $\varepsilon \downarrow 0$. Then for any $\eta > 0$ there exists a $\varepsilon > 0$ such that we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi_\varepsilon \geq \int_\Omega |\Delta_p u| + \int_{\partial\Omega} |\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right| - \eta.$$

Since η is an arbitrary positive number, the desired inequality holds. □

Proofs of (2) **and** (3)**.** The assertion (3) clearly follows from (2), we hence prove (2). Assume that u is admissible in $\mathbb{X}_p$. Then from Definition 2.1 there exists a sequence $\{u_k\} \subset W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying the properties 1 and 2. Noting that $\nabla(T_M u_k) = \chi_{|u_k| \le M} \nabla u_k$, we have

$$\begin{aligned}\int_\Omega |\nabla T_M(u_k)|^p\, dx &= \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla T_M(u_k) \\ &= \int_{\partial\Omega} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} T_M u_k - \int_\Omega \Delta_p u_k T_M u_k \\ &\le M [u_k]_{\mathbb{X}_p}.\end{aligned}$$

From the property 1 we see that $\Delta_p u_k \to \Delta_p u$ in $D'(\Omega)$ as $k \to \infty$. From the property 2 together with the weak compactness of Radon measures and the uniqueness of weak limit (see also Remark 2.1.2), $\lim_{k\to\infty} \Delta_p u_k = \Delta_p u$ in the sense of measures. Then by Fatou's lemma the assertion is proved. □

3.2 Proof of Theorem 3.2

First we prove Theorem 3.2 assuming that $u \in C^1(\overline{\Omega})$ and $\Delta_p u \in L^1(\Omega)$. Then we treat the general case.

Lemma 3.1. Assume that $u \in C^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$ (in the sense of distribution). Then

$$\int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla\phi \le \int_{\substack{\partial\Omega \\ [u \ge 0]}} \phi |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u\ge 0]}} \phi \Delta_p u \quad (\forall \phi \in C^1(\bar{\Omega}), \phi \ge 0 \ in\ \bar{\Omega}). \tag{3.10}$$

Proof. Let Φ is a C^2 convex function in $\mathbb{R}$, $\Phi' \ge 0$ in $\mathbb{R}$ and $\Phi' \in L^\infty(\mathbb{R})$.

First we assume that $p \ge 2$.

By a direct calculation we see that

$$\Delta_p \Phi(u) = \Phi'(u)^{p-1} \Delta_p u + (p-1)\Phi'(u)^{p-2} \Phi''(u) |\nabla u|^p \qquad \text{in } D'(\Omega). \tag{3.11}$$

Since $\Phi'' \ge 0$, we have

$$\Delta_p \Phi(u) \ge \Phi'(u)^{p-1} \Delta_p u \qquad \text{in } D'(\Omega), \tag{3.12}$$

in particular, $\Delta_p \Phi(u) \in L^1(\Omega)$. Hence, for any $\phi \in C^1(\bar{\Omega}), \phi \ge 0 \ in\ \bar{\Omega}$ we have

$$\begin{aligned}\int_\Omega |\nabla\Phi(u)|^{p-2} \nabla\Phi(u) \cdot \nabla\phi &= \int_{\partial\Omega} |\nabla\Phi(u)|^{p-2} \Phi'(u) \frac{\partial u}{\partial n} \phi - \int_\Omega \Delta_p \Phi(u) \phi \\ &\le \int_{\partial\Omega} \phi |\Phi'(u)|^{p-2} \Phi'(u) |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_\Omega \phi |\Phi'(u)|^{p-2} \Phi'(u) \Delta_p u.\end{aligned} \tag{3.13}$$

By the approximation argument, this is still valid for C^1 convex function Φ. Now we take a Φ in $\mathbb{R}$ such that $\Phi(t) = t$ if $t \ge 0$, $|\Phi(t)| < 1$ if $t < 0$, $0 \le \Phi' \le 1$ in $\mathbb{R}$ and $lim_{t\to-\infty} \Phi'(t) = 0$. Set $\Phi_n(t) = \Phi(nt)/n$ for $t \in \mathbb{R}$ and $n = 1, 2, \ldots$. Then we see that $\{\Phi_n\}$ is a sequence of C^1convex functions in $\mathbb{R}$ such that $\Phi_n(t) = t$ if $t \ge 0$, $|\Phi_n(t)| < \frac{1}{n}$ if $t < 0$, $0 \le \Phi_n' \le 1$ in $\mathbb{R}$. Then we see that $\Phi_n(t) \to t^+$ as $n \to \infty$. Replacing Φ by Φ_n in (3.13) and letting $n \to \infty$, we have the desired inequality by the dominated convergence theorem.

We proceed to the case where $1 < p < 2$. We set $\Phi^\eta(t) := \Phi(t) + \eta t$ for $t \in \mathbb{R}$ with $\eta > 0$. Then we see that for each $\eta > 0$

$$\sup_{t\in R} (\Phi^\eta)'(t)^{p-2} (\Phi^\eta)''(t) = \sup_{t \in R} (\Phi'(t) + \eta)^{p-2} \Phi''(t) \le \eta^{p-2} \sup_{t\in R} \Phi''(t) < \infty. \tag{3.14}$$

Hence we can apply he previous argument with Φ^η instead of Φ, so that in a similar way we reach to the inequality (3.13) replaced Φ by Φ^η. Letting $\eta \to 0$, we have (3.10) and this completes the proof. □

Lemma 3.2. Assume that $u \in C^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$ (in the sense of distribution). Then $u^+ \in \mathbb{X}_p$ and

$$[u^+]_{\mathbb{X}_p} \leq [u]_{\mathbb{X}_p} \ . \tag{3.15}$$

Proof. We note that $u^+ \in W^{1,p^*}(\Omega)$. For the proof of Lemma it suffices to show the following.

$$\Big| \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \psi \Big| \leq [u]_{\mathbb{X}_p} \|\psi\|_{L^\infty} \quad (\forall \psi \in C^1(\bar{\Omega})) \ . \tag{3.16}$$

For $\tilde{\psi} \in C^1(\bar{\Omega})$, we apply (3.10) with $\psi = \|\tilde{\psi}\|_{L^\infty} + \tilde{\psi}$. Then

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \tilde{\psi} \leq & \Big(\int_{\substack{\partial\Omega \\ [u \geq 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \geq 0]}} \Delta_p u \Big) \|\tilde{\psi}\|_{L^\infty} \\ & + \int_{\substack{\partial\Omega \\ [u \geq 0]}} \tilde{\psi} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \geq 0]}} \tilde{\psi} \Delta_p u \end{aligned} \tag{3.17}$$

Noting that

$$\int_{\substack{\partial\Omega \\ [u \geq 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \geq 0]}} \Delta_p u = - \int_{\substack{\partial\Omega \\ [u < 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} + \int_{\substack{\Omega \\ [u < 0]}} \Delta_p u$$

we have

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u^+ \cdot \nabla \tilde{\psi} \leq & - \Big(\int_{\substack{\partial\Omega \\ [u < 0]}} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u < 0]}} \Delta_p u \Big) \|\tilde{\psi}\|_{L^\infty} + \int_{\substack{\partial\Omega \\ [u \geq 0]}} \tilde{\psi} |\nabla u|^{p-2} \frac{\partial u}{\partial n} - \int_{\substack{\Omega \\ [u \geq 0]}} \tilde{\psi} \Delta_p u \\ \leq & \Big(\int_{\partial\Omega} |\nabla u|^{p-2} \Big| \frac{\partial u}{\partial n} \Big| + \int_\Omega |\Delta_p u| \Big) \|\tilde{\psi}\|_{L^\infty} = [u]_{\mathbb{X}_p} \|\tilde{\psi}\|_{L^\infty} \ . \end{aligned}$$

By replacing $\tilde{\psi}$ by $-\tilde{\psi}$, we have the desired inequality (3.15). □

Secondly we assume that u is admissible in $\mathbb{X}_p$. We recall a lemma on Neumann boundary problem for a monotone operator Δ_p.

Lemma 3.3. Let $\mu \in C_c^\infty(\Omega)$ and $\nu \in C_c^\infty(\mathbb{R}^N)$. Assume that $\int_\Omega \mu + \int_{\partial\Omega} \nu = 0$.
Then there exists a unique function $u \in C^{1,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$ such that

$$\begin{cases} -\Delta_p u = \mu & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \nu & \text{on } \partial\Omega, \\ \int_\Omega u = 0. \end{cases} \tag{3.18}$$

Proof. It follows from the standard theory that we have the unique solution u in $W^{1,p}(\Omega)$. For the detail, refer to [16]; theorems 2.1 and 2.2 for example. Since μ and ν smooth, we see that $u \in C^{1,\sigma}(\bar{\Omega})$ for some $\sigma \in (0,1)$ (See e.g. DiBenedetto [9]). Here we note that u is p-harmonic near the boundary as well. □

By Definition 2.1 of the admissibility in $\mathbb{X}_p$ we have for each $k \geq 1$ that

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \psi = \int_{\partial\Omega} \psi |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} - \int_{\partial\Omega} \psi \Delta_p u_k \quad (\forall \psi \in C^1(\bar{\Omega})) \ . \tag{3.19}$$

It follows from Remark 2.1(2) that in the sense of weak* topology as $n \to \infty$

$$\Delta_p u_k \stackrel{*}{\rightharpoonup} \Delta_p u \ in \ \mathscr{M}_b(\Omega), \quad \|\Delta_p u_k\|_{L^1(\Omega)} \to \|\Delta_p u\|_{\mathscr{M}_b(\Omega)} \ , \tag{3.20}$$

$$|\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \stackrel{*}{\rightharpoonup} |\nabla u|^{p-2} \frac{\partial u}{\partial n} \ in \ \mathscr{M}_b(\partial\Omega), \quad \||\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n}\|_{L^1(\partial\Omega)} \to \Big\| |\nabla u|^{p-2} \frac{\partial u}{\partial n} \Big\|_{\mathscr{M}_b(\partial\Omega)} \ . \tag{3.21}$$

By choosing $\psi = 1$ in (3.19), we have

$$\int_\Omega \Delta_p u_k = \int_{\partial\Omega} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n}. \tag{3.22}$$

Let us set $\mu_k = -\Delta_p u_k$ and $\nu_k = |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n}$. Let $\{\mu_k^j\} \subset C_c^\infty(\bar\Omega)$ and $\{\nu_k^j\} \subset C_c^\infty(\mathbb{R}^N)$ be two sequences such that as $j \to \infty$

$$\mu_k^j \overset{*}{\rightharpoonup} -\Delta_p u_k \textit{ weak}^* \textit{ in } L^1(\Omega), \qquad \|\mu_k^j\|_{L^1(\Omega)} \to \|\Delta_p u_k\|_{L^1(\Omega)}\ , \tag{3.23}$$

$$\nu_k^j \overset{*}{\rightharpoonup} |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \textit{ weak}^* \textit{ in } L^1(\partial\Omega), \qquad \|\nu_k^j\|_{L^1(\partial\Omega)} \to \left\| |\nabla u_k|^{p-2} \frac{\partial u_k}{\partial n} \right\|_{L^1(\partial\Omega)}. \tag{3.24}$$

From (3.22) we assume that

$$\int_{\partial\Omega} \nu_k^j = -\int_\Omega \mu_k^j \quad ({}^\forall j,k \geq 1).$$

It follows from Lemma 3.3 that for any $n \geq 1$ and $k \geq 1$, there exists $w_n^k \in C^{1,\sigma}(\bar\Omega)$ such that

$$\begin{cases} -\Delta_p w_k^j & = \mu_k^j \quad \textit{in } \Omega \\ |\nabla w_k^j|^{p-2} \frac{\partial w_k^j}{\partial n} & = \nu_k^j \quad \textit{on } \partial\Omega, \end{cases} \tag{3.25}$$

or equivalently

$$\int_\Omega |\nabla w_k^j|^{p-2} \nabla w_k^j \cdot \nabla\psi = \int_\Omega \psi\, d\mu_k^j + \int_{\partial\Omega} \psi\, d\nu_k^j, \quad \text{for any } \psi \in C^1(\bar\Omega), \tag{3.26}$$

and without the loss of generality we also assume that for any $j,k \geq 1$

$$\int_\Omega w_k^j = \int_\Omega u_k\ . \tag{3.27}$$

Under these preparations we have

Lemma 3.4. For each $n \geq 1$, there exists a function $w_k \in W^{1,q}(\Omega)$ for every $q \in [1, \frac{N(p-1)}{N-1})$ such that w_k^j converges to w_k in $w_k \in W^{1,q}(\Omega)$ as $k \to \infty$ and w_k satisfies (3.19).

Proof. Since for each $k \geq 1$, $\{\mu_k^j\}_{j=1}^\infty$ and $\{\nu_k^j\}_{j=1}^\infty$ are bounded in $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively, this assertion follows from the same argument in the proof of Theorem 1 in [3] with an obvious modification. In fact, one can show that $\{w_k^j\}_{j=1}^\infty$ is bounded in $W^{1,q}(\Omega)$, using similar test functions for ψ. Then by the weak compactness, Poincaré's inequality and the Rellich type theorem, one can see that there exists a function $w_k \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} &\nabla w_k^j \to \nabla w_k \quad \text{in } L^q \quad \text{(weak)} \\ &w_k^j \to w_k \quad \text{in } L^q \\ &w_k^j \to w_k \quad \text{a.e.}. \end{aligned}$$

Moreover one can see that $\nabla w_k^j \to \nabla w_k$ in $L^1(\Omega)$. Then by the dominated convergence theorem the conclusion follows in a quite similar way. For the detail see [3]. □

Lemma 3.5. We have $w_k = u_k$ a.e. for $k = 1, 2, \cdots$.

Proof. We claim that $w_k = u_k \in W^{1,q}(\Omega)$ for $q \in [1, \frac{N(p-1)}{N-1})$. Choose any $M > 0$. Recalling that $u_k \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we use $T_M(w_k^j - u_k) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function in (3.19) and (3.26). By a subtraction

$$\int_\Omega (|\nabla w_k^j|^{p-2} \nabla w_k^j - |\nabla u_k|^{p-2} \nabla u_k) \cdot \nabla (T_M(w_k^j - u_k)$$
$$= \int_\Omega T_M(w_k^j - u_k)\, d(\mu_k^j - \mu_k) + \int_{\partial\Omega} T_M(w_k^j - u_k)\, d(\nu_k^j - \nu_k).$$

The left hand side is estimated from below in the following way,

$$\int_\Omega (|\nabla w_k^j|^{p-2}\nabla w_k^j - |\nabla u_k|^{p-2}\nabla u_k)\cdot \nabla T_M(w_k^j - u_k) \ge C\int_\Omega |\nabla T_M(w_k^j - u_k)|^p \tag{3.28}$$

for some positive number C independent of each j, and the right hand side goes to 0 as $j \to \infty$. Since this holds for all $M > 0$, we conclude by the monotonicity of Δ_p that $\nabla w_k = \nabla u_k$ a.e. Taking into account that $w_k \in W^{1,q}(\Omega)$, $u_k \in W^{1,p}(\Omega)$ and (3.27), we conclude that $u_k = w_k$ a.e.. □

End of proof of Theorem 3.2. By applying Lemma 3.2 we have

$$\left|\int_\Omega |\nabla (w_k^j)^+|^{p-2}\nabla (w_k^j)^+ \cdot \nabla \psi\right| \le [w_k^j]_{\mathbb{X}_p}\|\psi\|_{L^\infty} \quad (\forall \psi \in C^1(\bar{\Omega}))\,. \tag{3.29}$$

From Lemma 3.4 and Lemma 3.5 we have, up to subsequence, that $w_k^j \to u_k$ a.e. and $(w_k^j)_+ \to (u_k)_+$ in $W^{1,q}(\Omega)$ as $j \to \infty$. Letting $j \to \infty$, we have

$$\left|\int_\Omega |\nabla u_k^+|^{p-2}\nabla u_k^+ \cdot \nabla \psi\right| \le [u_k]_{\mathbb{X}_p}\|\psi\|_{L^\infty} \quad (\forall \psi \in C^1(\bar{\Omega})).$$

Finally letting $k \to \infty$ we have the conclusion. □

3.3 Proof of Theorem 3.3

Proof of the assertion 1.
1st step. Assume that u is admissible in $W_0^{1,p^*}(\Omega)$, and hence both u^+ and u^- are admissible $W_0^{1,p^*}(\Omega)$. From the statement 4 of Proposition 4.1, we can assume that $\{u_k\}_{k=1}^\infty \subset W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ in Definition 2.2. We decompose $u_k \in W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ to obtain $u_k = u_k^+ - u_k^-$, where $u_k^+ = \max\{u_k, 0\}$, $u_k^- = \max\{-u_k, 0\}$. Then each $u_k^\pm \in W_0^{1,p}(\Omega)\cap C_0^{1,0}(\bar{\Omega})$. Since $u_k^+ \ge 0$ in Ω and $u_k^+ = 0$ on $\partial\Omega$, we see that $\frac{\partial u_k^+}{\partial n} \le 0$ on $\partial\Omega$. Similarly we have $\frac{\partial u_k^-}{\partial n} \le 0$ on $\partial\Omega$. Therefore

$$-\int_{\partial\Omega} |\nabla u_k^+|^{p-2}\left|\frac{\partial u_k^+}{\partial n}\right| = \int_{\partial\Omega} |\nabla u_k^+|^{p-2}\frac{\partial u_k^+}{\partial n} = \int_\Omega \Delta_p u_k^+,$$
$$-\int_{\partial\Omega} |\nabla u_k^-|^{p-2}\left|\frac{\partial u^-}{\partial n}\right| = \int_{\partial\Omega} |\nabla u_k^-|^{p-2}\frac{\partial u_k^-}{\partial n} = \int_\Omega \Delta_p u_k^-.$$

Hence

$$\int_{\partial\Omega} |\nabla u_k^+|^{p-2}\left|\frac{\partial u_k^+}{\partial n}\right| \le \left|\int_\Omega \Delta_p u_k^+\right|, \quad \int_{\partial\Omega} |\nabla u_k^-|^{p-2}\left|\frac{\partial u_k^-}{\partial n}\right| \le \left|\int_\Omega \Delta_p u_k^-\right|.$$

After all we have

$$\int_{\partial\Omega} |\nabla u_k|^{p-2}\left|\frac{\partial u_k}{\partial n}\right| \le \int_\Omega |\Delta_p u_k|, \tag{3.30}$$

in particular $|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} \in L^1(\partial\Omega)$. Hence we have

$$[u_k]_{\mathbb{X}_p} \le \int_{\partial\Omega} |\nabla u_k|^{p-2}\left|\frac{\partial u_k}{\partial n}\right| + \int_\Omega |\Delta_p u_k| \le 2\int_\Omega |\Delta_p u_k| < \infty. \tag{3.31}$$

2nd step. Again assume that $\{u_k\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)\cap C_0^1(\Omega)$ in Definition 2.2. By Definition 2.2 (1) we have

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla \psi \to \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \psi \quad \text{for any } \psi \in C_c^1(\Omega). \tag{3.32}$$

It follows from the weak compactness of bounded measures and the uniqueness of weak limit that $\Delta_p u_k \to \Delta_p u$ strongly in $\mathscr{M}(\Omega)$. By the previous step we have

$$|u_k|_{\mathbb{X}_p} \leq 2\int_\Omega |\Delta_p u_k| \qquad \text{for } k = 1, 2, \cdots. \tag{3.33}$$

Hence we see that $|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} \in L^1(\partial\Omega)$ converge to some measure ν in $M(\partial\Omega)$ up to subsequences. Therefore by the lower semicontinuity of the norm $||\cdot||_{M(\Omega)}$ with respect to the weak* convergence as $n \to \infty$, we have

$$[u]_{\mathbb{X}_p} \leq 2\int_\Omega |\Delta_p u|.$$

Therefore u is admissible in $\mathbb{X}_p$, and hence $u^+ \in \mathbb{X}_p$ by Theorem 3.2. □

Proof of the assertion 2. We claim that $\int_\Omega |\Delta_p u^+| \leq \int_\Omega |\Delta_p u|$.

Lemma 3.6. Assume that $u \in C_0^1(\bar{\Omega})$ and $\Delta_p u \in L^1(\Omega)$. Then $\Delta u^+ \in \mathscr{M}_b(\Omega)$ and

$$\|\Delta u^+\|_{\mathscr{M}_b(\Omega)} \leq \|\Delta u\|_{L^1(\Omega)}\ . \tag{3.34}$$

Proof. By applying Lemma 3.2 with $u+\varepsilon$, where $\varepsilon > 0$, we deduce that

$$|(u+\varepsilon)^+|_{\mathbb{X}_p} \leq |u+\varepsilon|_{\mathbb{X}_p} = |u|_{\mathbb{X}_p}. \tag{3.35}$$

Since $(u+\varepsilon)^+ = u+\varepsilon$ in a nelghborhood of $\partial\Omega$,

$$\frac{\partial}{\partial n}(u+\varepsilon)^+ = \frac{\partial u}{\partial n} \quad \text{ on } \partial\Omega. \tag{3.36}$$

Noting that

$$|(u+\varepsilon)^+|_{\mathbb{X}_p} = ||\Delta_p(u+\varepsilon)^+||_{\mathscr{M}(\Omega)} + |||\nabla(u+\varepsilon)^+|^{p-2}\frac{\partial}{\partial n}(u+\varepsilon)^+||_{L^1(\partial\Omega)}$$
$$|u|_{\mathbb{X}_p} = ||\Delta_p u||_{L^1(\Omega)} + |||\nabla u|^{p-2}\frac{\partial u}{\partial n}||_{L^1(\partial\Omega)},$$

we immediately have

$$||\Delta_p(u+\varepsilon)^+||_{\mathscr{M}(\Omega)} \leq ||\Delta_p u||_{L^1(\Omega)} \quad \text{ for any } \varepsilon > 0. \tag{3.37}$$

The results follows from the lower semicontinuity of the norm $||\cdot||_{\mathscr{M}(\Omega)}$ with respect to the weak* convergence as $\varepsilon \to 0$. □

3.4 Proof of Theorem 3.4

We prepare some fundamental lemmas.

Lemma 3.7. Let $u \in W^{1,p^*}(\Omega)$. Assume that for some $h \in L^1(\partial\Omega)$ and $g \in L^1(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \leq \int_{\partial\Omega} h\varphi + \int_\Omega g\varphi \quad \text{ for any } \varphi \in C^1(\bar{\Omega}), \varphi \geq 0. \tag{3.38}$$

Then $u \in \mathbb{X}_p$. Moreover $-\Delta_p u \leq g$ in $\mathscr{M}(\Omega)$ and $|\nabla u|^{p-2}\frac{\partial u}{\partial n} \leq h$ in $\mathscr{M}(\partial\Omega)$.

Proof. By (3.38) we have

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le \int_{\partial\Omega} h^{+}\varphi + \int_{\Omega} g^{+}\varphi \quad \text{for any } \varphi\in C^1(\bar{\Omega}), \varphi\ge 0. \tag{3.39}$$

Using nonnegative test functions $||\varphi||_{L^\infty}\pm\varphi$ as the argument in the proof of Lemma 3.2, it is easy to see that

$$\left|\int_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\right| \le 2(||h^{+}||_{L^1(\partial\Omega)} + ||g^{+}||_{L^1(\Omega)})||\varphi||_{L^\infty(\Omega)}. \tag{3.40}$$

Then we see $u\in\mathbb{X}_p$. The rest of the assertions are clear. □

Lemma 3.8. In the previous Lemma 3.7, we further assume that u is admissible in $\mathbb{X}_p$. Then we have

$$\int_{\Omega} |\nabla u^{+}|^{p-2}\nabla u^{+}\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} h\varphi + \int_{\substack{\Omega\\ [u\ge 0]}} g\varphi \quad \text{for any } \varphi\in C^1(\bar{\Omega}), \varphi\ge 0. \tag{3.41}$$

By the admissibility there exists a sequence $\{u_k\}\subset W^{1,p^*}(\Omega)$ having the properties in Definition 2.1. By virtue of Proposition 4.1 we can assume that $u_k\in C^1(\bar{\Omega})$. Then it follows from Lemma 3.1 that

$$\int_{\Omega} |\nabla u_k|^{p-2}\nabla u_k^{+}\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u_k\ge 0]}} \varphi|\nabla u_k|^{p-2}\frac{\partial u_k}{\partial n} - \int_{\substack{\Omega\\ [u_k\ge 0]}} \varphi\Delta_p u_k \quad (\forall\varphi\in C^1(\bar{\Omega}), \varphi\ge 0 \ in\ \bar{\Omega}) \tag{3.42}$$

Taking a limit as $k\to\infty$ we have

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u^{+}\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} \varphi|\nabla u|^{p-2}\frac{\partial u}{\partial n} - \int_{\substack{\Omega\\ [u\ge 0]}} \varphi\Delta_p u \quad (\forall\varphi\in C^1(\bar{\Omega}), \varphi\ge 0 \ in\ \bar{\Omega}) \tag{3.43}$$

Using Lemma 3.5 the conclusion holds. □

Lemma 3.9. Assume that $u\in C^1(\bar{\Omega})$ is admissible in $\mathbb{X}_p$ and $|\nabla u|^{p-2}\frac{\partial u}{\partial n}\in L^1(\partial\Omega)$. Then

$$|\nabla u^{+}|^{p-2}\frac{\partial u^{+}}{\partial n} \le \begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial n} & \text{on } [u>0] \\ 0 & \text{on } [u<0] \\ \min\{|\nabla u|^{p-2}\frac{\partial u}{\partial n}, 0\} & \text{on } [u=0]. \end{cases} \tag{3.44}$$

Proof. Put $\mu=(-\Delta_p u)^{+}$, $h=|\nabla u|^{p-2}\frac{\partial u}{\partial n}$. Then

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le \int_{\partial\Omega} h\varphi + \int_{\Omega}\varphi\, d\mu \quad (\forall\varphi\in C^1(\bar{\Omega}), \varphi\ge 0 \ in\ \bar{\Omega})$$

It follows from Lemma 3.8 that u^{+} satisfies

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u^{+}\cdot\nabla\varphi \le \int_{\substack{\partial\Omega\\ [u\ge 0]}} h\varphi + \int_{\Omega}\varphi\, d\mu \quad (\forall\varphi\in C^1(\bar{\Omega}),\ \varphi\ge 0 \ in\ \bar{\Omega}) \tag{3.45}$$

By Theorem 3.2 we have $u^{+}\in\mathbb{X}_p$, hence

$$|\nabla u|^{p-2}\frac{\partial u^{+}}{\partial n} \le \chi_{[u\ge 0]}h = \chi_{[u\ge 0]}|\nabla u|^{p-2}\frac{\partial u}{\partial n} \quad on\ \partial\Omega. \tag{3.46}$$

By using $u-\varepsilon$, where $\varepsilon>0$ instead of u we have in a similar way that

$$|\nabla u|^{p-2}\frac{\partial u^{+}}{\partial n} \le \chi_{[u>0]}h = \chi_{[u>0]}|\nabla u|^{p-2}\frac{\partial u}{\partial n} \quad on\ \partial\Omega. \tag{3.47}$$

In particular,

$$|\nabla u|^{p-2}\frac{\partial u^{+}}{\partial n} \le 0 \quad on\ [u=0]. \tag{3.48}$$

Hence the conclusion follows. □

Corollary 3.1. Assume that u is admissible in $\mathbb{X}_p$ and $u \in W_0^{1,p^*}(\Omega)$. If $u \geq 0$ *in* Ω, then

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} \leq 0 \quad on\ \partial\Omega.$$

Proof.
$u = u^+$ *in* Ω and $u = 0$ *on* $\partial\Omega$, hence applying the Lemma 3.9 we have

$$\frac{\partial u}{\partial n} = \frac{\partial u^+}{\partial n} \leq \min\{\frac{\partial u}{\partial n}, 0\} \leq 0 \quad on\ \partial\Omega.$$

□

Proof of Theorem 3.4. By Theorem 3.2 $u^+ \in \mathbb{X}_p$. By applying Kato's inequality (Corollary 1.1 in [13]) to $u - a \in \mathbb{X}_p$, we havre

$$\Delta_p(u-a)^+ \geq \chi_{[u \geq a]}\Delta_p u = G \quad in\ \Omega$$

for any $a \in \mathbf{R}$. Here we note thatt $(\Delta_p u)_d = \Delta_p u$, because $\Delta_p u \in L^1(\Omega)$. Letting $a \downarrow 0$ we have

$$\Delta_p u^+ \geq \chi_{[u>0]}\Delta_p u = G \quad in\ \Omega.$$

Combining this with Lemma 3.7, we have for any $\varphi \in C^1(\bar{\Omega}),\ \varphi \geq 0\ in\ \Omega$

$$\int_\Omega |\nabla u|^{p-2}\nabla u^+ \cdot \nabla\varphi = \int_{\partial\Omega} \varphi|\nabla u|^{p-2}\frac{\partial u^+}{\partial n} - \int_\Omega \varphi\Delta u^+ \leq \int_{\partial\Omega} H\varphi - \int_\Omega G\varphi.$$

□

4 Appendix (Proposition 4.1)

We begin with recalling a local version of Admissibility in [14].

Definition 4.1. (**Admissibility in** $W^{1,p^*}_{\mathrm{loc}}(\Omega)$) Let $1 < p < \infty$ and $p^* = \max\{1, p-1\}$. A function u is said to be admissible in in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, if $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$, $\Delta_p u \in \mathscr{M}(\Omega)$; the total measure is not necessarily finite, and if there exists a sequence $\{u_k\}_{k=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ such that:

1. $u_k \to u$ a.e. in Ω and $u_k \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $k \to \infty$.
2. $\Delta_p u_k \in L^1_{\mathrm{loc}}(\Omega)$ $(k = 1, 2, \cdots)$ and

$$\sup_k |\Delta_p u_k|(\omega) = \sup_k \int_\omega |\Delta_p u_k| < \infty \quad \text{ for all } \omega \subset\subset \Omega. \tag{4.1}$$

Here we describe the following fundamental results, parts of which are already known.

Proposition 4.1. Let Ω be a bounded smooth domain of $\mathbb{R}^N$.

1. Assume that u is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$. Then, for every $M > 0$, $T_M u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.
2. A function $u \in W_0^{1,p}(\Omega)$ is admissible in $W_0^{1,p^*}(\Omega)$, if $\Delta_p u \in \mathscr{M}_b(\Omega)$.
3. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, if $\Delta_p u \in \mathscr{M}(\Omega)$.
4. In Definition 2.1, the sequence $\{u_k\}$ can be taken in $C^1(\bar{\Omega})$.
5. In Definition 2.2, the sequence $\{u_k\}$ can be taken in $C^1_0(\bar{\Omega}) = \{\varphi \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

The proof of assertion 1 for $p=2$ is seen in [5] and [6]) and for $p>1$ in [14], and the proof of assertion 2 is seen in Appendix of [14]. The assertion 4 is already verified in the proof of Theorem 3.2. Therefore we establish the assertions 3 and 5 in the rest of this section.

Proof of assertion 3. To use a diagonal argument, we choose and fix a family of open set $\{\omega_k\}$ such that

$$\omega_1 \subset\subset \omega_2 \subset\subset \cdots \subset\subset \omega_k \subset\subset \omega_{k+1} \subset\subset \cdots \subset\subset \Omega \text{ and } \Omega = \cup_{k=0}^{\infty} \omega_k. \tag{4.2}$$

Let $\rho \in C_0^\infty(B_1)$ be a radial, nonnegative and decreasing mollifier. By extending $v \in L^1(\Omega)$ to the whole space so that $v \equiv 0$ outside Ω, we define a mollification of v with $\varepsilon > 0$ by

$$v^\varepsilon(x) := \rho_\varepsilon * v(x) = \int_\Omega \rho_\varepsilon(x-y)v(y)dy \qquad \text{for } x \in \Omega. \tag{4.3}$$

First we prove that $u \in W_0^{1,p}(\Omega)$ is admissible in $W_{\text{loc}}^{1,p^*}(\Omega)$, if $\Delta_p u$ is a Radon measure on Ω. Again by extending $u \in W_0^{1,p}(\Omega)$ and $\Delta_p u \in W^{-1,p'}$ to the whole space so that $u=0$ and $\Delta_p u = 0$ outside Ω respectively. Let $w_k \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ be the unique weak solution of the boundary value problem for the monotone operator Δ_p (see e.g. [16]): For $k = 1,2,\cdots$ and $\varepsilon_1 > \varepsilon_2 > \cdots \varepsilon_k > \cdots \to 0$, we set

$$\begin{cases} \Delta_p w_k = (\Delta_p u)^{\varepsilon_k} & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where $|\nabla u|^{p-2}\nabla u \in (L^{p'}(\Omega))^N$ with $p' = p/(p-1)$, $(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} \in (C^\infty(\mathbb{R}^N))^N$ and $(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k}$ is a mollification of $|\nabla u|^{p-2}\nabla u$ defined by (4.3). Let us set $\Delta_p u = \mu$. We note that $|\mu|(\omega) < \infty$ for any $\omega \subset\subset \Omega$. Then we have $\operatorname{div}(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} = (\operatorname{div}|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} = (\Delta_p u)^{\varepsilon_k} = \mu^{\varepsilon_k}$ in ω provided that ε_k is sufficiently small. Hence we clearly have

$$|\Delta_p w_k|(\omega) = |\mu^{\varepsilon_k}|(\omega) \to |\mu|(\omega) \text{ as } k \to \infty.$$

Since μ does not charge $\partial\Omega$, this proves the condition 2. Next we show

$$w_k \to u \text{ in } W_0^{1,p}(\Omega) \text{ as } k \to \infty. \tag{4.5}$$

Then we can choose a subsequence so that the condition 1 is satisfied. By using $w_k - u \in W_0^{1,p}(\Omega)$ as a test function, we have

$$\begin{aligned} -\langle \Delta_p w_k - \Delta_p u, w_k - u \rangle &= \int_\Omega |(\nabla w_k|^{p-2}\nabla w_k - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_k - u) \\ &\geq c_2 \int_\Omega |\nabla(w_k - u)|^p. \end{aligned} \tag{4.6}$$

In the left-hand side, using Young's inequality for $\delta > 0$ we have

$$\begin{aligned} -\langle \Delta_p w_k - \Delta_p u, w_k - u \rangle &= \int_\Omega ((|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_k - u) \\ &\leq C(\delta) \int_\Omega |(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} - |\nabla u|^{p-2}\nabla u|^{p'} + \delta \int_\Omega |\nabla(w_k - u)|^p, \end{aligned} \tag{4.7}$$

where $C(\delta) > 0$ is a constant depending only on δ.

We note that $||(|\nabla u|^{p-2}\nabla u)^{\varepsilon_k} - |\nabla u|^{p-2}\nabla u||_{L^{p'}(\Omega)} \to 0$ as $k \to \infty$. It follows from (4.6) and (4.7) that $\nabla w_k \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$, which implies (4.5). Then, taking a subsequence if necessary, $\{w_k\} \subset W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ satisfies the property $w_k \to u$ a.e. in Ω as $k \to \infty$.

Lastly we treat the case where $u \in W_{\text{loc}}^{1,p}(\Omega)$. For each k we choose $\eta_k \in C_c^\infty(\omega_{k+1})$ such that $0 \leq \eta_k \leq 1$ and $\eta_k = 1$ in some neighborhood of $\overline{\omega_k}$. Let us set $v_k = \eta_k u\,(k = 1,2,3,\cdots)$. Then we see that $v_k \in W_0^{1,p}(\omega_{k+1})$, $v_k \to u$ in $W_{\text{loc}}^{1,p}(\Omega)$ as $k \to \infty$ and $\Delta_p v_k \in W^{-1,p'}(\Omega) \cap M_b(\omega_k)$. Moreover we have

$|\Delta_p v_k|(\omega_j) = |\Delta_p u|(\omega_j)$ for any $k \geq j$. Hence u is admissible in $W^{1,p^*}_{\mathrm{loc}}(\omega_k)$ with $\Delta_p u \in \mathscr{M}_b(\omega_k)$ having an admissible sequence $\{v_k\}$. By the previous step with obvious modification, one can approximate each v_k inductively by $\xi_k \in W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega})$ such that $\xi_k \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $k \to \infty$ and $||\Delta_p \xi_k|(\omega_j) - |\Delta_p u|(\omega_j)| < \frac{1}{k}$ for $k \geq j$. Therefore the assertion is now proved. □

Proof of assertion 5. We assume that u is admissible in $W^{1,p^*}_0(\Omega)$. Then we have a sequence of functions $\{u_k\} \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ $(k = 1, 2, \dots)$ satisfying the properties 1 and 2 in Definition 2.2. By the previous step, we see that each u_k is approximated as $j \to \infty$ by a sequence of functions $\{w^j_k\} \subset W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega})$ defined by (4.4) with $w_k = w^j_k$, $u = u_k$ and $\varepsilon_k = \varepsilon_j$. Then we choose a suitable subsequence of $\{w^{j_k}_k\}$ as an approximation of u so that the assertion is verified. □

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Communicated by *Yoshitsugu Kabeya*

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FURUTA TYPE INEQUALITIES RELATED TO ANDO-HIAI INEQUALITY WITH NEGATIVE POWERS

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Received September 25, 2019; revised November 05, 2019

Abstract. Furuta inequality and Ando-Hiai inequality have been actively investigated since they were established about thirty years ago. Recently, Kian and Seo obtained the Ando-Hiai type inequality with negative powers as follows: For $A, B > 0$, $A \natural_{-\alpha} B \leq I$ for $\alpha \in [0,1]$ implies $A^r \natural_{-\alpha} B^r \leq I$ for $r \geq 1$. Related to this result, Fujii and Nakamoto obtained Furuta type inequality with negative powers. Moreover, they discussed these generalizations. In this paper, we improve their results based on properties of Furuta inequality and Ando-Hiai inequality.

1 Introduction

Throughout this paper, an operator means a bounded linear operator on a complex Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

First of all, we state Furuta inequality [10] established in 1987 (cf. [2, 11, 15, 19, 23]): If $A \geq B \geq 0$, then for each $r \geq 0$,

$$\text{(i)}\ (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \quad \text{and} \quad \text{(ii)}\ A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$. We remark that Furuta inequality is a generalization of Loewner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$,"* and also it is known that (i) is equivalent to (ii) under the assumption $A \geq B \geq 0$. As stated in [19] (cf. [11]), Furuta inequality can be arranged in terms of the weighted geometric mean $\sharp_\alpha$ defined by $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $A, B > 0$ and $\alpha \in [0,1]$:

$$\text{(F)} \quad A \geq B > 0 \quad \text{implies} \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \ \text{ for } p \geq 1 \text{ and } r \geq 0.$$

(F) is sometimes called the satellite theorem for Furuta inequality.

On the other hand, in 1994, Ando and Hiai [1] obtained the following inequality called Ando-Hiai inequality as follows: For $A, B > 0$,

$$\text{(AH)} \qquad A \sharp_\alpha B \leq I \text{ for } \alpha \in (0,1) \quad \text{implies} \quad A^r \sharp_\alpha B^r \leq I \text{ for } r \geq 1.$$

We remark that they obtained the log majorization theorem by using (AH).

As a generalization of Furuta and Ando-Hiai inequalities, Furuta established grand Furuta inequality in [13] (cf. [7, 14, 15, 16, 25]): If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

2010 *Mathematics Subject Classification.* Primary 47A63, 47A64.

Key words and phrases. Positive operators, operator mean, Furuta inequality, Ando-Hiai inequality and grand Furuta inequality.

*This work was supported by JSPS KAKENHI Grant Number JP16K05181.

holds for $r \geq t$ and $s \geq 1$. Similarly to (F), it is known in [13] that grand Furuta inequality can be arranged in terms of the weighted geometric mean, that is, we can get the satellite theorem for grand Furuta inequality:

$$\text{(SGF)} \quad A \geq B > 0 \quad \text{implies} \quad A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-t} \natural_s B^p) \leq A^{-r+t} \sharp_{\frac{1-t+r}{p-t+r}} B^p \leq B \leq A$$

for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$, where $A \natural_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in \mathbb{R}$. The notation $\natural_\alpha$ is the same as $\sharp_\alpha$ if $\alpha \in [0,1]$. We remark that (SGF) leads (F) by putting $t = 0$ and $s = 1$, and also (SGF) leads the equivalent inequality to (AH) by putting $t = 1$ and $s = r$. On Ando-Hiai inequality, its generalization was shown in [8], and also related topics were discussed in [5, 18].

Recently, Kian and Seo [22] obtained the Ando-Hiai type inequality with negative powers as follows:

Theorem 1.A ([22]). *For $A, B > 0$,*

$$\text{(KS)} \qquad A \natural_{-\alpha} B \leq I \quad \textit{for } \alpha \in [0,1] \quad \textit{implies} \quad A^r \natural_{-\alpha} B^r \leq I \quad \textit{for } 0 \leq r \leq 1.$$

Fujii and Nakamoto [9] discussed generalizations of Theorem 1.A, and also they obtained the Furuta type inequality with negative powers as follows:

Theorem 1.B ([9, Theorem 3.1]). *If $A \geq B > 0$, then $A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$ holds for $p \leq -1$ and $r \in [-1, 0]$.*

By replacing p, r by $-p, -r$ respectively, we can rewrite Theorem 1.B as follows:

$$\text{(FN)} \qquad A \geq B > 0 \quad \text{implies} \quad A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq A^{1-2r} \quad \text{for } p \geq 1 \text{ and } 0 \leq r \leq 1.$$

We remark that the equivalence between two inequalities

$$A^r \natural_{\frac{1-r}{-p-r}} B^{-p} \leq A \quad \text{and} \quad A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq A^{1-2r}$$

can be shown by using the relation

$$(*) \qquad A \natural_{-r} B = A(A^{-1} \natural_r B^{-1})A.$$

Fujii and Nakamoto [9] also discussed the grand Furuta type inequalities. We state them later.

In this paper, from the viewpoint of the satellite theorem for Furuta inequality, we improve some results in [9], and also we discuss relations among Theorem 1.A, Theorem 1.B and our results.

2 Furuta type inequalities and their grand Furuta type generalizations

Firstly, we show an improvement of (FN).

Theorem 2.1. *Let $A \geq B > 0$ and $r > 0$. Then for $p \geq 1$, the following inequalities hold.*

$$A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \begin{cases} \leq B^{1-2r} \leq A^{1-2r} & \text{if } 0 \leq r \leq \frac{1}{2}, \\ \leq A^{1-2r} \leq B^{1-2r} & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \tag{2.1}$$

$$A^{-r} \natural_{\frac{1-r}{p+r}} B^p \geq A^{1-2r} \ \text{ if } r > 1. \tag{2.2}$$

Proof. Firstly, we show (2.1). If $0 \leq r \leq 1$, then we have

$$A^{-r} \sharp_{\frac{1-r}{p+r}} B^p \leq B^{-r} \sharp_{\frac{1-r}{p+r}} B^p = B^{1-2r}$$

and

$$\begin{aligned} A^{-r} \sharp_{\frac{1-r}{p+r}} B^p &= A^{-r} \sharp_{\frac{1-r}{1+r}} (A^{-r} \sharp_{\frac{1+r}{p+r}} B^p) \\ &\leq A^{-r} \sharp_{\frac{1-r}{1+r}} (B^{-r} \sharp_{\frac{1+r}{p+r}} B^p) = A^{-r} \sharp_{\frac{1-r}{1+r}} B \leq A^{-r} \sharp_{\frac{1-r}{1+r}} A = A^{1-2r}. \end{aligned}$$

Therefore we obtain (2.1) since $B^{1-2r} \leq A^{1-2r}$ holds if $0 \leq r \leq \frac{1}{2}$ and $A^{1-2r} \leq B^{1-2r}$ holds if $\frac{1}{2} \leq r \leq 1$.

If $r > 1$, then we have (2.2) since

$$\begin{aligned} A^{-r} \natural_{\frac{1-r}{p+r}} B^p &= A^{-r}(A^r \sharp_{\frac{r-1}{p+r}} B^{-p})A^{-r} \\ &= A^{-r}(B^{-p} \sharp_{\frac{1+p}{r+p}} A^r)A^{-r} \geq A^{-r}AA^{-r} = A^{1-2r} \end{aligned}$$

holds by $(*)$ and (F). □

Next, we discuss grand Furuta type generalizations of Theorem 2.1. As a generalization of Theorem 1.B, Fujii and Nakamoto [9] showed the following result related to grand Furuta inequality.

Theorem 2.A ([9, Theorem 3.4])**.** *If $A \geq B > 0$ and $t \in [0, 1]$, then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \sharp_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ and $s \in \left[\max\left\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\right\}, 1\right]$.

Replacing p by $-p$ and using $(*)$, Theorem 2.A can be rewritten as follows: If $A \geq B > 0$ and $t \in [0, 1]$, then

$$A^{-r+t} \natural_{\frac{1-t+r}{r-(p+t)s}} (A^t \sharp_s B^{-p}) \leq A, \quad \text{that is,} \quad A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \leq A^{1-2(t-r)}$$

holds for $p \geq 1$, $r \in [0, t]$ and $s \in \left[\max\left\{\frac{t}{p+t}, \frac{1-t+2r}{p+t}\right\}, 1\right]$.

Here, we show an improvement of Theorem 2.A.

Theorem 2.2. *Let $A \geq B > 0$ and $0 \leq r \leq t \leq 1$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \begin{cases} \leq B^{1-2(t-r)} \leq A^{1-2(t-r)} & \text{if } 0 \leq t-r \leq \frac{1}{2}, \\ \leq A^{1-2(t-r)} \leq B^{1-2(t-r)} & \text{if } \frac{1}{2} \leq t-r \leq 1 \end{cases}$$

holds for $p \geq 1$ and $\frac{1-t+2r}{p+t} \leq s \leq 1$.

Proof. Noting that $0 \leq \frac{1-t+r}{(p+t)s-r} \leq 1$ and $0 \leq t-r \leq 1$ hold, we have

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq B^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = B^{1-2(t-r)}.$$

Next we show $A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{1-2(t-r)}$ by dividing into three cases. If $(p+t)s-t \geq 1$ holds, then

$$\begin{aligned} A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ &= A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} (A^{r-t} \natural_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} (B^{r-t} \natural_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ &= A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} B \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{1+t-r}} A = A^{1-2(t-r)}. \end{aligned}$$

If $0 \leq (p+t)s-t \leq 1$ holds, then

$$\begin{aligned} A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \natural_s B^p) = A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ &\leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}. \end{aligned}$$

If $(p+t)s-t \leq 0$ holds, then

$$\begin{aligned} A^{-t} \natural_s B^p &= A^{-t} \natural_{\frac{(p+t)s}{t}} (A^{-t} \natural_{\frac{t}{p+t}} B^p) \\ &\leq A^{-t} \natural_{\frac{(p+t)s}{t}} (B^{-t} \natural_{\frac{t}{p+t}} B^p) = A^{-t} \natural_{\frac{(p+t)s}{t}} I = A^{(p+t)s-t}, \end{aligned}$$

so that we have

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}.$$

Therefore we obtain the desired result since $B^{1-2(t-r)} \leq A^{1-2(t-r)}$ holds if $0 \leq t-r \leq \frac{1}{2}$ and $A^{1-2(t-r)} \leq B^{1-2(t-r)}$ holds if $\frac{1}{2} \leq t-r \leq 1$. □

We remark that Theorem 2.2 (grand Furuta type inequality) interpolates Theorem 2.1 (Furuta type inequality) and Theorem 1.A (Ando-Hiai type inequality) as follows: By putting $s = 1$ and $r = 0$ (and replacing t by r) in Theorem 2.2, we have (2.1) in Theorem 2.1.

On the other hand, by putting $t = 1$, Theorem 2.2 implies the following Theorem 2.3, which is an improvement of [9, Theorem 3.2].

Theorem 2.3. *Let* $A \geq B > 0$ *and* $0 \leq r \leq 1$. *Then*

$$A^{r-1} \sharp_{\frac{r}{(p+1)s-r}} (A^{-1} \sharp_s B^p) \begin{cases} \leq B^{2r-1} \leq A^{2r-1} & \text{if } \frac{1}{2} \leq r \leq 1, \\ \leq A^{2r-1} \leq B^{2r-1} & \text{if } 0 \leq r < \frac{1}{2} \end{cases}$$

holds for $p \geq 1$ *and* $\frac{2r}{p+1} \leq s \leq 1$.

Theorem 2.3 implies the following result by putting $s = r$.

Corollary 2.4. *Let* $A \geq B > 0$ *and* $0 \leq r \leq 1$. *Then*

$$A^{r-1} \sharp_{\frac{1}{p}} (A^{-1} \sharp_r B^p) \leq A^{2r-1}$$

holds for $p \geq 1$.

We understand that Theorem 1.A is equivalent to Corollary 2.4 by the replacements $S = A^{-1}$, $T = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha}$ and $p = \frac{1}{\alpha}$ as follows: For $\alpha \in [0, 1]$,

$$A \natural_{-\alpha} B \leq I \iff (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha} \leq A^{-1} \iff S \geq T.$$

Since $T = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\alpha}$ is equivalent to $B = (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^{-1}$, for $r \in [0, 1]$,

$$\begin{aligned} A^r \natural_{-\alpha} B^r \leq I &\iff S^{-r} \natural_{-\alpha} (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^{-r} \leq I \\ &\iff S^{-r} \left\{ S^r \sharp_\alpha (S^{\frac{1}{2}} T^{\frac{1}{\alpha}} S^{\frac{1}{2}})^r \right\} S^{-r} \leq I \quad \text{by } (*) \\ &\iff S^{\frac{1}{2}-r} \left\{ S^{r-1} \sharp_\alpha (S^{-1} \sharp_r T^{\frac{1}{\alpha}}) \right\} S^{\frac{1}{2}-r} \leq I \\ &\iff S^{r-1} \sharp_{\frac{1}{p}} (S^{-1} \sharp_r T^p) \leq S^{2r-1}. \end{aligned}$$

3 Inequalities for chaotic order

In this section, we show a generalization of Theorems 2.1 and 2.2.

Theorem 3.1. *Let* $\log A \geq \log B$ *for* $A, B > 0$ *and* $0 \leq r \leq t$.

(i) *For* $p > 0$ *and* $\frac{1-t+2r}{p+t} \leq s \leq 1$,

$$A^{r-t} \sharp_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \begin{cases} \leq B^{1-2(t-r)} & \text{if } 0 \leq t-r \leq \frac{1}{2}, \\ \leq A^{1-2(t-r)} & \text{if } \frac{1}{2} \leq t-r \leq 1. \end{cases}$$

(ii) *For* $p > 0$ *and* $\frac{t-1}{p+t} \leq s \leq 1$,

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \geq A^{1-2(t-r)} \text{ if } t-r > 1.$$

We remark that inequalities in Theorem 3.1 hold for chaotic order $\log A \geq \log B$, which is weaker assumption than usual order $A \geq B$, and Theorem 3.1 holds for some looser conditions of parameters than Theorem 2.2. Moreover, Theorem 3.1 gives a generalization of Theorem 2.1 by putting $s = 1$ and $r = 0$ (and replacing t by r).

In order to prove Theorem 3.1, we use the following theorem in [16] (see also [3, 6, 12, 21, 26]).

Theorem 3.A ([16])**.** *Let $A, B > 0$. Then the following assertions are mutually equivalent.*

(i) $\log A \geq \log B$.

(ii) *For any fixed $q \geq 0$, $F_q(p,r) = A^{-r} \natural_{\frac{q+r}{p+r}} B^p$ is decreasing for $p \geq q$ and $r \geq 0$.*

(iii) *For any fixed $q \leq 0$, $F_q(p,r) = A^{-r} \natural_{\frac{q+r}{p+r}} B^p$ is decreasing for $p \geq 0$ and $r \geq -q$.*

Since $\log A \geq \log B$ is equivalent to $\log B^{-1} \geq \log A^{-1}$, Theorem 3.A ensures that $\log A \geq \log B$ implies the following two statements.

(i) For any fixed $q \geq 0$, $\widehat{F}_q(p,r) = B^{-r} \natural_{\frac{q+r}{p+r}} A^p$ is increasing for $p \geq q$ and $r \geq 0$,

(ii) For any fixed $q \leq 0$, $\widehat{F}_q(p,r) = B^{-r} \natural_{\frac{q+r}{p+r}} A^p$ is increasing for $p \geq 0$ and $r \geq -q$.

We remark that $\log A \geq \log B$ implies that

$$\begin{aligned} &F_q(p,r) \leq F_q(p,0) = B^q \quad \text{for } p \geq q \text{ and } r \geq 0 \text{ if } q \geq 0, \\ &F_q(p,r) \leq F_q(p,-q) = A^q \quad \text{for } p \geq 0 \text{ and } r \geq -q \text{ if } q \leq 0 \end{aligned} \tag{3.1}$$

by Theorem 3.A, and also the similar inequalities hold for $\widehat{F}_q(p,r)$.

Proof of Theorem 3.1. By (3.1), $\log A \geq \log B$ implies

$$A^{-t} \natural_s B^p = A^{-t} \natural_{\frac{(p+t)s-t+t}{p+t}} B^p \begin{cases} \leq B^{(p+t)s-t} & \text{if } (p+t)s-t \geq 0, \\ \leq A^{(p+t)s-t} & \text{if } (p+t)s-t \leq 0 \end{cases} \tag{3.2}$$

for $p > 0$, $t \geq 0$ and $0 \leq s \leq 1$.

Firstly, we show (i). We may assume $t - r < 1$. We note that $1-t+r > 0$, $(p+t)s-r > 0$ and $0 < \frac{1-t+r}{(p+t)s-r} \leq 1$ hold. If $(p+t)s-t \geq 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \begin{cases} \leq B^{1-2(t-r)} & \text{if } 1-2(t-r) \geq 0, \\ \leq A^{1-2(t-r)} & \text{if } 1-2(t-r) \leq 0 \end{cases}$$

holds for $t - r \geq 0$, where the first inequality holds by (3.2) and the second ones hold by (3.1) since $\frac{1-t+r}{(p+t)s-r} = \frac{1-2(t-r)+(t-r)}{(p+t)s-t+(t-r)}$. If $(p+t)s-t \leq 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \natural_s B^p) \leq A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}$$

holds by (3.2). In this case, $\frac{1-t+2r}{p+t} \leq s \leq \frac{t}{p+t}$ holds, so that $1-2(t-r) \leq 0$ holds. Therefore the proof of (i) is complete.

Next we show (ii). We note that $1-t+r<0$, $(p+t)s-r>0$ and $-1 \le \frac{1-t+r}{(p+t)s-r} < 0$ hold. If $(p+t)s-t \ge 0$, then

$$\begin{aligned} A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) &= A^{r-t} \left\{ A^{t-r} \sharp_{\frac{-1+t-r}{(p+t)s-r}} (A^{-t} \sharp_s B^p)^{-1} \right\} A^{r-t} \\ &\ge A^{r-t} \left\{ A^{t-r} \sharp_{\frac{-1+t-r}{(p+t)s-r}} B^{-((p+t)s-t)} \right\} A^{r-t} \\ &= A^{r-t} \left\{ B^{-((p+t)s-t)} \sharp_{\frac{1+(p+t)s-t}{t-r+(p+t)s-t}} A^{t-r} \right\} A^{r-t} \\ &\ge A^{r-t} A A^{r-t} = A^{1-2(t-r)} \end{aligned}$$

holds for $t-r>1$ by (3.2) and Theorem 3.A. If $(p+t)s-t \le 0$, then

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \ge A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}$$

holds by (3.2). Therefore the proof of (ii) is complete. □

4 Inequalities for $s < \frac{1-t+2r}{p+t}$

In [9], Fujii and Nakamoto also considered the case $s = \frac{t}{p+t} < \frac{1-t+2r}{p+t}$. As a generalization of [9, Theorems 3.6 and 3.8], we obtain the following results.

Theorem 4.1. *Let $A \ge B > 0$ and $0 \le r \le t$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \le A^{-t} \sharp_{\frac{1-t+2r}{p+t}} B^p \le A^{1-2(t-r)}$$

holds for $p \ge 1$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \le s \le \frac{1-t+2r}{p+t}$ with $(p+t)s-r \ne 0$.

Theorem 4.2. *Let $\log A \ge \log B$ for $A, B > 0$ and $0 \le r \le t$ with $t-r \ge \frac{1}{2}$. Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \le A^{-t} \natural_{\frac{1-t+2r}{p+t}} B^p \le A^{1-2(t-r)}$$

holds for $p > 0$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \le s \le \frac{1-t+2r}{p+t}$ with $(p+t)s-r \ne 0$.

In order to prove Theorems 4.1 and 4.2, we use the following inequalities in [20] known as the Furuta type inequalities with negative powers (cf. [4, 17, 24, 27]).

Theorem 4.A ([20]). *If $A \ge B \ge 0$ with $A > 0$, then the following inequalities hold.*

(i) $A^t \natural_{\frac{1-t}{p-t}} B^p \le B \le A$ *holds for* $0 \le t < p \le 1$ *with* $p \ge \frac{1}{2}$.

(ii) $A^t \natural_{\frac{2p-t}{p-t}} B^p \le B \le A$ *holds for* $0 \le t < p \le \frac{1}{2}$.

By replacing A, B, p, t by $A^q, B^q, \frac{p}{q}, \frac{t}{q}$ respectively, we have the following proposition.

Proposition 4.3. *Let $A > 0$ and $B \geq 0$. If $A^q \geq B^q$ for $q > 0$, then the following inequalities hold.*

(i) $A^t \natural_{\frac{q-t}{p-t}} B^p \leq B^q \leq A^q$ *holds for* $0 \leq t < p \leq q$ *with* $p \geq \frac{q}{2}$.

(ii) $A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^q \leq A^q$ *holds for* $0 \leq t < p \leq \frac{q}{2}$.

Proof of Theorem 4.1. By Furuta inequality, $A \geq B > 0$ implies

$$A^{1+t} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1+t}{p+t}} \tag{4.1}$$

for $p \geq 1$ and $t \geq 0$. Put $A_1 = A^{1+t}$ and $B_1 = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1+t}{p+t}}$. Then $A_1^q \geq B_1^q$ holds for $0 \leq q \leq 1$ by (4.1) and Loewner-Heinz theorem. Then by putting $p_1 = \frac{(p+t)s}{1+t}$, $t_1 = \frac{r}{1+t}$ and $q = \frac{1-t+2r}{1+t}$, (i) in Proposition 4.3 ensures that

$$A_1^{t_1} \natural_{\frac{q-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1^q \leq A_1^q$$

holds for $0 \leq t_1 < p_1 \leq q \leq 1$ with $p_1 \geq \frac{q}{2}$, that is,

$$A^r \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s \leq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1-t+2r}{p+t}} \leq A^{1-t+2r}$$

holds for $0 \leq r \leq t$, $p \geq 1$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \leq s \leq \frac{1-t+2r}{p+t}$ with $(p+t)s - r \neq 0$. Therefore we have the desired result. □

Proof of Theorem 4.2. By Theorem 3.A, $\log A \geq \log B$ implies

$$A^t \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}} \tag{4.2}$$

for $p > 0$ and $t \geq 0$. Put $A_1 = A^t$ and $B_1 = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}}$. Then $A_1^q \geq B_1^q$ holds for $0 \leq q \leq 1$ by (4.2) and Loewner-Heinz theorem. Then by putting $p_1 = \frac{(p+t)s}{t}$, $t_1 = \frac{r}{t}$ and $q = \frac{1-t+2r}{t}$, (i) in Proposition 4.3 ensures that

$$A_1^{t_1} \natural_{\frac{q-t_1}{p_1-t_1}} B_1^{p_1} \leq B_1^q \leq A_1^q$$

holds for $0 \leq t_1 < p_1 \leq q \leq 1$ with $p_1 \geq \frac{q}{2}$, that is,

$$A^r \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s \leq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1-t+2r}{p+t}} \leq A^{1-t+2r}$$

holds for $0 \leq r \leq t$, $p > 0$, $t-r \geq \frac{1}{2}$ and $\max\left\{\frac{1-t+2r}{2(p+t)}, \frac{r}{p+t}\right\} \leq s \leq \frac{1-t+2r}{p+t}$ with $(p+t)s-r \neq 0$. Therefore we have the desired result. □

Theorems 3.1 and 4.1 ensure the following, which is a slight extension of [9, Theorems 3.6 and 3.8].

Theorem 4.4. *Let* $A \geq B > 0$ *and* $0 \leq r < t$ *with* $-1 \leq 1-2(t-r) \leq t$. *Then*

$$A^{r-t} \natural_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \sharp_s B^p) \leq A^{1-2(t-r)} \tag{4.3}$$

holds for $p \geq 1$ *and* $s = \frac{t}{p+t}$.

Proof. By putting $s = \frac{t}{p+t}$ in Theorem 4.1, (4.3) holds for $0 \leq 1-2(t-r) \leq t$. By putting $s = \frac{t}{p+t}$ in (i) of Theorem 3.1, (4.3) holds for $-1 \leq 1-2(t-r) \leq 0$. □

Acknowledgements. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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ON SUBALMOST CONTRA-B-CONTINUOUS FUNCTIONS

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Received July 15, 2018

Abstract. The purpose of this paper is to introduce a new class functions called, subalmost contra-b-continuous functions. Also, we obtain its characterizations and its basic properties.

1 Introduction Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of b-open sets introduced by Andrijević in 1996. Andrijević studied several fundamental and interesting properties of b-open sets. The purpose of this paper is to introduce a new class functions called, subalmost contra-b-continuous functions. Also, we obtain its characterizations and its basic properties.

2 Preliminaries Throughout the paper (X,τ) and (Y,σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space (X,τ), $Cl(A)$, $Int(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A of X is said to be regular open [14] (resp. semi-open [8], α-open [10], b-open [2](= γ-open [6])) if $A = Int(Cl(A))$ (resp. $A \subset Cl(Int(A)), A \subset Int(Cl(Int(A))), A \subset (Int(Cl(A)) \cup Cl(Int(A)))$. The family of all α-open (resp. semi-open, regular open, b-open) subsets of X is denoted by $\alpha(X)$ (resp. $SO(X)$, $RO(X)$, $BO(X)$). The family of all semi-open (resp. regular open, b-closed) subsets of X containing the point x is denoted by $SO(X,x)$ (resp. $RO(X,x)$, $BC(X,x)$). The complement of a semi-open (resp. regular open, b-open) set is called a semiclosed [4] (resp. regular closed, b-closed) set. The intersection of all semi-closed (resp. b-closed) sets containing A is called the semi-closure [3] (resp. b-closure [2]) of A and is denoted by $sCl(A)$ (resp. $bCl(A)$). A subset A is b-closed if and only if A=$bCl(A)$. The θ-semi-closure [7] (resp. the semi-θ-closure [5]) of A, denoted by θ-$sCl(A)$ (resp. $sCl_\theta(A)$), is defined to be the set of all $x \in X$ such that $A \cap Cl(U) \neq \varnothing$ (resp. $A \cap sCl(U) \neq \emptyset$) for every $U \in SO(X,x)$. A subset A is called θ-semi-closed [7] (resp. semi-θ-closed [5]) if and only if $A = \theta$-$sCl(A)$ (resp. $A = sCl_\theta(A)$). The complement of a θ-semi-closed set (resp. semi-θ-closed set) is called a θ-semi-open [7] (resp. semi-θ-open [5]) set. It is well known that θ-$sCl(A) \neq sCl_\theta(A)$ for some subset A of a topological space (X,τ). A function $f:(X,\tau)\to(Y,\sigma)$ is said to be b-continuous [6] (resp. contra-b-continuous [9]) if $f^{-1}(V)$ is b-open (resp. b-closed) set in (X,τ) for each open set V of (Y,σ).

Definition 2.1 [1] A function $f:(X,\tau)\to(Y,\sigma)$ is said to be *almost contra-b-continuous* if $f^{-1}(V) \in BC(X)$ for each $V \in RO(Y)$.

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*2010 Math. Subject Classification – : 54C10, 54C08, 54C05.
Key words and phrases – :b-open set, b-closed set, subalmost contra-b-continuous functions

Lemma 2.2 [11, Lemma 5.3] *If $B \subset A \subset X$ and A is α-open in (X, τ), then $bCl_A(B) = bCl(B) \cap A$.*

Lemma 2.3 [5, Lemma 2.1] *If V is an open set, then $sCl(V) = Int(Cl(V))$.*

Lemma 2.4 [5, Proposition 2.1(a)] *If V is a semi-open set, then $sCl_\theta(V) = sCl(V)$.*

3 Subalmost contra-b-continuous functions

Definition 3.1 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *subalmost contra-b-continuous* if there exists an open base $\mathcal{B}$ for the topology on Y for which $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Sometimes, f is called *subalmost contra-b-continuous with respect to an open base $\mathcal{B}$.*

Theorem 3.2 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(1) *f is subalmost contra-b-continuous with respect to an open base $\mathcal{B}$.*
(2) *$bCl(f^{-1}(V)) \subset f^{-1}(sCl_\theta(V))$ for every $V \in \mathcal{B}$.*
(3) *$bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))$ for every $V \in \mathcal{B}$.*

Proof. (1) $\Leftrightarrow$ (2): The proof follows from Lemma 2.4 and a well known property that $\tau \subset SO(X)$.
(1) $\Leftrightarrow$ (3): The proof follows from Lemma 2.3. □

Theorem 3.3 *If $f : (X, \tau) \to (Y, \sigma)$ is subweakly b-continuous [12] and satisfies the additional property that images of b-closed sets are open, then f is subalmost contra-b-continuous.*

Proof. By the definition of subweakly b-continuity [12, Definition 3.1], there exists an open base $\mathcal{B}$ for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since images of b-closed sets are open, $f(bCl(f^{-1}(V))) \subset Int(Cl(V))$ or $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$. Therefore, by Theorem 3.2, f is subalmost contra-b-continuous. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)); x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 3.4 A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *regular b-closed* if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Theorem 3.5 *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is regular b-closed if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \emptyset$.* □

Theorem 3.6 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous and (Y, σ) is a Hausdorff space, then the graph of f, $G(f)$ is regular b-closed.*

Proof. Let $(x, y) \in X \times Y \setminus G(f)$. Then $y \neq f(x)$. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff, there exist disjoint open sets V and W such that $f(x) \in V$, $y \in W$, and $V \in \mathcal{B}$. Then, since $Int(Cl(V)) \cap Int(Cl(W)) = \emptyset$, it follows that $(x, y) \in bCl(f^{-1}(V)) \times Int(Cl(W)) \subset (X \times Y) \setminus G(f)$, which proves that $G(f)$ is regular b-closed. □

Corollary 3.7 *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and (Y, σ) is a Hausdorff space, then the graph $G(f)$ is regular b-closed.* □

Theorem 3.8 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and let $\mathcal{B}$ be an open base for σ. Let $\mathcal{C} := \{U \times V : U \in \tau, V \in \mathcal{B}\}$. If the graph function of $f, g : X \to X \times Y$ is subalmost contra-b-continuous with respect to $\mathcal{C}$, then f is subalmost contra-b-continuous with respect to $\mathcal{B}$.*

Proof. If $V \in \mathcal{B}$, then $bCl(f^{-1}(V)) = bCl(g^{-1}(X \times V)) \subset g^{-1}(sCl(X \times V)) = g^{-1}(X \times sCl(V)) = f^{-1}(sCl(V))$. Hence f is subalmost contra-b-continuous with respect to $\mathcal{B}$. □

Recall that a space (X, τ) is said to be *zero-dimensional* provided that (X, τ) has a clopen base (cf. [15]).

Theorem 3.9 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous and X is zero-dimensional, then the graph function of $f, g : X \to X \times Y$ is subalmost contra-b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$ for every $V \in \mathcal{B}$. Then $\mathcal{B}_1 = \{U \times V : U \subset X \text{ is clopen and } V \in \mathcal{B}\}$ is a base for the topology on $X \times Y$. For $U \times V \in \mathcal{B}_1$, we have $bCl(g^{-1}(U \times V)) = bCl(U \cap f^{-1}(V)) \subset U \cap bCl(f^{-1}(V)) \subset Int(Cl(U)) \cap f^{-1}(Int(Cl(V))) = g^{-1}(Int(Cl(U)) \times Int(Cl(V))) = g^{-1}(Int(Cl(U \times V)))$. Therefore the graph function g is subalmost contra-b-continuous. □

Definition 3.10 A topological space (X, τ) is said to be weakly Hausdorff [13] if each element of X is an intersection of regular closed sets.

Definition 3.11 A topological space (X, τ) is said to be b-T_1 [11] if for each pair of distinct points x and y of X, there exist b-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

Theorem 3.12 *If $f : (X, \tau) \to (Y, \sigma)$ is a subalmost contra-b-continuous injection and (Y, σ) is weakly Hausdorff, then (X, τ) is b-T_1.*

Proof. Let x_1 and x_2 be distinct points in X. Then $f(x_1) \neq f(x_2)$, and since Y is weakly Hausdorff, there exists a regular closed subset F of Y such that $f(x_1) \in F$ and $f(x_2) \notin F$. Then $f(x_2) \in X \setminus F$, which is regular open. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Then let $V \in \mathcal{B}$ such that $f(x_2) \in V \subset Y \setminus F$. Then $x_2 \notin X \setminus bCl(f^{-1}(V))$, which is b-open. Also $f(x_1) \in F$, which is regular closed and therefore also semi-open. Since $F \cap V = \emptyset$, it follows that $f(x_1) \notin sCl(V)$, and hence $x_1 \notin f^{-1}(sCl(V))$. Then $x_1 \in X \setminus f^{-1}(sCl(V)) \subset X \setminus bCl(f^{-1}(V))$. Hence $X \setminus bCl(f^{-1}(V))$ is a b-open set containing x_1 but not x_2, which proves that X is b-T_1. □

Theorem 3.13 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous with respect to the open base $\mathcal{B}$ for the topology on Y and A is an α-open subset of X, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is subalmost contra-b-continuous with respect to $\mathcal{B}$, where τ_A is the relative topology for A and f_A is the restriction of f to A.*

Proof. Let $V \in \mathcal{B}$. Then $bCl_A(f_A^{-1}(V)) \subset A \cap bCl(f_A^{-1}(V)) = A \cap bCl(f^{-1}(V) \cap A) \subset A \cap bCl(f^{-1}(V)) \cap bCl(A) = A \cap bCl(f^{-1}(V)) \subset A \cap f^{-1}(sCl(V)) = f_A^{-1}(sCl(V))$. Hence, $f_A : A \to Y$ is subalmost contra-b-continuous with respect to $\mathcal{B}$. □

If we take $\mathcal{B}$ to be the topology on Y in the above theorem, we obtain the following result.

Corollary 3.14 *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and A is an α-open subset of X, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is subalmost contra-b-continuous .* □

Theorem 3.15 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous and A is an open subset of (Y, σ) with $f(X) \subset A$, then $f : (X, \tau) \to (A, \sigma_A)$ is subalmost contra-b-continuous.*

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Then $\mathcal{B}_A := \{V \cap A : V \in \mathcal{B}\}$ is an open base for the relative topology σ_A on A. For $V \cap A \in \mathcal{B}_A$, where $V \in \mathcal{B}$, we have $bCl(f^{-1}(V \cap A)) = bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))) = f^{-1}(Int(Cl(V)) \cap A) \subset f^{-1}(Int_A(Cl_A(V \cap A)))$, which proves that $f : (X, \tau) \to (A, \sigma_A)$ is subalmost contra-b-continuous with respect to the base $\mathcal{B}_A$. □

Definition 3.16 The θ-closure [16] of A, denoted by $Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $Cl(U) \cap A \neq \emptyset$ for every open set U containing x. A subset A is called θ-closed [16] if and only if $A = Cl_\theta(A)$. The complement of a θ-closed set is called a θ-open set [16].

Theorem 3.17 *If $f : (X, \tau) \to (Y, \sigma)$ is subalmost contra-b-continuous, then for every θ-open* (resp. *θ-closed*) *subset W of Y, $f^{-1}(W)$ is a union of b-closed sets* (resp. *an intersection of b-open sets*).

Proof. Let $\mathcal{B}$ be an open base for the topology on Y such that $bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every $V \in \mathcal{B}$. Let W be a θ-open set of Y and let $x \in f^{-1}(W)$. Let $V \in \mathcal{B}$ such that $f(x) \in V \subset Cl(V) \subset W$. Then $x \in bCl(f^{-1}(V)) \subset f^{-1}(sCl(V)) \subset f^{-1}(Cl(V)) \subset f^{-1}(W)$. Since $bCl(f^{-1}(V))$ is b-closed, it follows that $f^{-1}(W)$ is a union of b-closed sets. An argument using complements will prove the remaining part of the theorem. □

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Communicated by *Haruo Maki*

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ESTIMATION OF TRIGONOMETRIC MOMENTS FOR CIRCULAR DISTRIBUTION OF MA(p)TYPE BY USING BINARY SERIES

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Received September 11,2019 ; revised November 30,2019

Abstract. Directional statistics have received a great deal of interest in recent years, and a variety of distributions on the circle have been proposed. In this paper, we propose circular distributions of a moving average model of order p type which includes the cardioid distribution, and discuss estimation of trigonometric moments based on binary series. We give an explicit form of the root n consistent estimator based on clipped series, which enables us to construct an efficient estimator by the Newton–Raphson iterative method. We also show a robustness of the proposed estimator when the probability density function is contaminated with a noise term.

1 Introduction Directional statistics is an important field which deals with directional data. The history of directional statistics dates back to 1950s. Fisher (1953) had large influence and appealed the necessity of directional statistics. After that, many authors tackled the problem (see Mardia (1975); Watson (1983); Fisher et al. (1993)). In recent years, directional statistics has attracted attention because of Mardia and Jupp (2000).

Many distributions on the circle have been developed (e.g. uniform, cardioid, wrapped Cauchy, von Mises distribution). These distributions are closely related to the spectral density functions in time series with complex valued coefficients. For example, the spectral density of the autoregressive model of order 1, that of the moving average model of order 1, and that of the autoregressive model of order 2 correspond to wrapped Cauchy distribution, cardioid distribution, and the more flexible distribution proposed by Kato and Jones (2013), respectively.

Binary series are processes that each of realizations takes value 0 or 1. The execution time of methods based on clipped processes are significantly short, and estimation accuracy of methods based on 0-1 valued processes are high when the original processes are contaminated with outliers (see Bagnall and Janacek (2005), Kedem (1994, p.172), Goto and Taniguchi (2019), Goto and Taniguchi). Methods based on binary series have been applied to various fields including biology and linguistics. For example, analysis of vocal sounds of humpback whales of Kedem and Li (1989); speech discrimination of Panagiotakis and Tziritas (2005)); and emotion recognition using brain signals of Petrantonakis and Hadjileontiadis (2010).

Binary series have been studied by many researchers (see Rice (1944), Lomnicki and Zaremba (1955), McNeil (1967), Kedem (1980), Kedem (1994)). Rice (1944) gave a pioneer study in this field, and showed a relationship between correlations of Gaussian processes and correlations of a binary series generated by the Gaussian processes. Kedem (1980) showed the asymptotic normality of the estimator of autocorrelation based on clipped processes. In recent years, related to binary data, the categorical time series (see Fokianos and Kedem (2003)) and the quantile based spectra (e.g. Li (2014) and Dette et al. (2015)) have been developed. However, binary series in directional statistics have not yet been investigated.

2010 *Mathematics Subject Classification.* 62H11, 62H12 .

Key words and phrases. asymptotic normality; binary series; circular distribution; Cramér–Rao lower bound; efficient estimator; robustness; trigonometric moment.

In this paper, we propose a family of circular distributions of a moving average model of order p type, and discuss estimation of trigonometric moments based on binary series. We derive an explicit form of the root n consistent estimator. Although the estimator based on clipped series does not attain Cramér–Rao lower bound, it enables us to construct efficient estimator by the Newton–Raphson iterative method. We also show a robustness of the estimator when the true probability density function is contaminated with noise. The finite sample performance of proposed estimator is also investigated.

The paper is organized as follows: In Section 2, we introduce circular distributions of the moving average model of order p type and the estimator of trigonometric moments based on binary series for the proposed distribution. We show the asymptotic normality and compare the asymptotic variance with Cramér–Rao lower bound. In Section 3, we elucidate a robustness of the estimator when the probability density function is contaminated with noise. The finite sample performance of proposed estimator is investigated, and asymptotic normality of the proposed estimator is illustrated by computer simulation in Section 4. Finally, we conclude this paper with proofs of the theorems and the proposition in Sections 2 and 3.

2 Settings and Main Result In this section, we define a family of circular distributions of MA(p) type and propose a root n consistent estimator based on binary series. After that, we show the asymptotic normality and compare the asymptotic variance of the proposed estimator with Cramér–Rao lower bound.

Throughout this paper, we consider a family of circular distributions of MA(p) type whose probability density function is defined by

$$p(\theta) = \frac{1}{2\pi(1+\phi_1^2+\cdots+\phi_p^2)}|\phi(e^{i\theta})|^2 \tag{1}$$

where $\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \cdots + \phi_p z^p$ and $\phi_j \in \mathbb{R}$ for any j.

Let $\{\Theta_k : k \in \mathbb{N}\}$ be independent random variables with a common circular distribution defined by (1). From the residue theorem and symmetry of (1), the j-th sine and cosine moments can be obtained as

$$\mathrm{E}\{\sin(j\Theta_k)\} = 0 \quad \text{for } j \in \mathbb{Z},$$
$$\mathrm{E}\{\cos(j\Theta_k)\} = \begin{cases} \frac{\phi_j+\phi_{j+1}\phi_1+\cdots+\phi_p\phi_{p-j}}{1+\phi_1^2+\cdots+\phi_p^2} & \text{for } |j| \le p, \\ 0 & \text{for } |j| \ge p+1, \end{cases}$$

respectively. Then, the mean resultant length and the mean direction of $\{\Theta_k : k \in \mathbb{N}\}$ can be obtained as

$$|\mathrm{E}\{e^{i\Theta_k}\}| = \left|\frac{\phi_1+\phi_2\phi_1+\cdots+\phi_p\phi_{p-1}}{1+\phi_1^2+\cdots+\phi_p^2}\right|,$$
$$\arg \mathrm{E}\{e^{i\Theta_k}\} = \begin{cases} 0 & \phi_1+\phi_2\phi_1+\cdots+\phi_p\phi_{p-1} > 0, \\ \pi & \phi_1+\phi_2\phi_1+\cdots+\phi_p\phi_{p-1} < 0, \\ \text{undefined} & \phi_1+\phi_2\phi_1+\cdots+\phi_p\phi_{p-1} = 0, \end{cases}$$

respectively. From Mardia and Jupp (2000, p.31), (1) can be written as

$$p(\theta) = \frac{1}{2\pi}\left(1+\sum_{j=1}^{p}\rho_j\cos(j\theta)\right), \tag{2}$$

where $\rho_j = 2(\phi_j + \phi_{j+1}\phi_1 + \cdots + \phi_p\phi_{p-j})/(1 + \phi_1^2 + \cdots + \phi_p^2)$. If we take $p = 1$, (2) is the well-known cardioid distribution (see Mardia and Jupp (2000, p.45)). Clearly, if $\phi_j = 0$ for any $j \in \{1, \ldots, p\}$, (2) is a uniform distribution. The proposed model (1) is generally non-identifiable. Actually, for $p = 2$ and $(\phi_1, \phi_2, \psi_1, \psi_2) := (0, -\frac{1}{2}, \pm\sqrt{\frac{1}{2}}, -1)$, we have $p(\theta; \phi_1, \phi_2) = p(\theta; \psi_1, \psi_2)$.

In this paper, we discuss the estimation problem of $\rho_1, \ldots, \rho_p$ of the proposed probability density function by using clipped series. Hereafter, we confine ourselves to the case that $(\phi_1, \ldots, \phi_p)$ satisfies $\phi_1 + \phi_2\phi_1 + \cdots + \phi_p\phi_{p-1} \geq 0$. Define $(\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_p < \pi$. For each α_j, $j = 1, \ldots, p$, binary series $\{X_k^j\}$ are defined, for any $j = 1, \ldots, p$,

$$X_k{}^j = \begin{cases} 1 & -\alpha_j \leq \Theta_k \leq \alpha_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Applying the technique for the derivation of an orthant probability for normal distribution (see Kedem (1994, p.48)), we have the following equation

$$\begin{pmatrix} P(-\alpha_1 \leq \Theta_1 \leq \alpha_1) \\ P(-\alpha_2 \leq \Theta_1 \leq \alpha_2) \\ \vdots \\ P(-\alpha_p \leq \Theta_1 \leq \alpha_p) \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{\pi} \\ \frac{\alpha_2}{\pi} \\ \vdots \\ \frac{\alpha_p}{\pi} \end{pmatrix} + \frac{1}{2\pi} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pp} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_p \end{pmatrix},$$

where

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pp} \end{pmatrix} = \begin{pmatrix} \int_{-\alpha_1}^{\alpha_1} \cos\theta d\theta & \int_{-\alpha_1}^{\alpha_1} \cos 2\theta d\theta & \cdots & \int_{-\alpha_1}^{\alpha_1} \cos p\theta d\theta \\ \int_{-\alpha_2}^{\alpha_2} \cos\theta d\theta & \int_{-\alpha_2}^{\alpha_2} \cos 2\theta d\theta & \cdots & \int_{-\alpha_2}^{\alpha_2} \cos p\theta d\theta \\ \vdots & \vdots & \ddots & \vdots \\ \int_{-\alpha_p}^{\alpha_p} \cos\theta d\theta & \int_{-\alpha_p}^{\alpha_p} \cos 2\theta d\theta & \cdots & \int_{-\alpha_p}^{\alpha_p} \cos p\theta d\theta \end{pmatrix}.$$

Here, we suppose the observed stretch $\{\Theta_1, \ldots, \Theta_n\}$ is available. We choose $(\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ adequately so that $(b_{ij})_{i,j=1}^p$ is a nonsingular matrix, and substitute $\left(1/n \sum_{k=1}^n X_k{}^1, \ldots, 1/n \sum_{k=1}^n X_k{}^p\right)^{\mathrm{T}}$ for $\left(P(-\alpha_1 \leq \Theta_1 \leq \alpha_1), \ldots, P(-\alpha_p \leq \Theta_1 \leq \alpha_p)\right)^{\mathrm{T}}$. Then, the binary estimator $(\hat{\rho}_1, \ldots, \hat{\rho}_p)^{\mathrm{T}}$ can be defined as

$$\begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \vdots \\ \hat{\rho}_p \end{pmatrix} = 2\pi \begin{pmatrix} b^{11} & b^{12} & \cdots & b^{1p} \\ b^{21} & b^{22} & \cdots & b^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b^{p1} & b^{p2} & \cdots & b^{pp} \end{pmatrix} \begin{pmatrix} \frac{1}{n}\sum_{k=1}^n X_k{}^1 - \frac{\alpha_1}{\pi} \\ \frac{1}{n}\sum_{k=1}^n X_k{}^2 - \frac{\alpha_2}{\pi} \\ \vdots \\ \frac{1}{n}\sum_{k=1}^n X_k{}^p - \frac{\alpha_p}{\pi} \end{pmatrix},$$

where $(b^{ij})_{i,j=1}^p$ is the inverse matrix of $(b_{ij})_{i,j=1}^p$.

Before we derive the asymptotic distribution of the proposed estimator, we give some examples that $(b_{ij})_{i,j=1}^p$ is a nonsingular matrix for specific models.

Example 2.1. MA(2) case: if we take $\alpha_1 = \frac{\pi}{4}$ and $\alpha_2 = \frac{\pi}{2}$, then

$$(b_{ij})_{i,j=1}^2 = \begin{pmatrix} \sqrt{2} & 1 \\ 2 & 0 \end{pmatrix}, \quad (b^{ij})_{i,j=1}^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Example 2.2. MA(3) case: if we take $\alpha_1 = \frac{\pi}{4}$, $\alpha_2 = \frac{\pi}{2}$, and $\alpha_3 = \frac{3\pi}{4}$, then

$$(b_{ij})_{i,j=1}^3 = \begin{pmatrix} \sqrt{2} & 1 & \frac{\sqrt{2}}{3} \\ 2 & 0 & -\frac{2}{3} \\ \sqrt{2} & -1 & \frac{\sqrt{2}}{3} \end{pmatrix}, \quad (b^{ij})_{i,j=1}^3 = \begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4\sqrt{2}} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{3}{4\sqrt{2}} & -\frac{3}{4} & \frac{1}{4\sqrt{2}} \end{pmatrix}.$$

The following theorem shows that the asymptotic normality of the proposed estimator.

Theorem 2.1. *It holds that*

$$\sqrt{n} \begin{pmatrix} \hat{\rho}_1 - \rho_1 \\ \hat{\rho}_2 - \rho_2 \\ \vdots \\ \hat{\rho}_p - \rho_p \end{pmatrix} \Rightarrow N(0, \boldsymbol{V}),$$

where $\boldsymbol{V} = (v_{ij})_{i,j=1}^p$ *and*

$$\begin{aligned} v_{ij} = \quad & 4\pi^2 \textstyle\sum_{s,k=1}^p b^{is} b^{jk} \{P(-\alpha_s \leq \Theta_1 \leq \alpha_s, -\alpha_k \leq \Theta_1 \leq \alpha_k) \\ & -P(-\alpha_s \leq \Theta_1 \leq \alpha_s) P(-\alpha_k \leq \Theta_1 \leq \alpha_k)\}. \end{aligned}$$

Next, we investigate whether our proposed method attains the Cramér–Rao lower bound or not. For simplicity, we confine ourselves to the case of circular distributions of MA(1) type.

Proposition 2.1. *The Cramér–Rao lower bound is given by*

$$\mathcal{I}^{-1}(\rho_1) = 1 - \rho_1^2 + \sqrt{1 - \rho_1^2}.$$

Proposition 2.1 enables us to compare the asymptotic variance of the proposed estimator with the Cramér–Rao lower bound. Thus, we have the following statement.

Remark 2.1. *The Binary estimator is not efficient.*

Actually, If we consider the case $\rho_1 = 1$, then it is easy to see that

$$(\text{Covariance of } \hat{\rho}_1) - \mathcal{I}^{-1}(\rho_1) > 0.$$

Remark 2.1 is not a preferable property of the estimator. However, from Hosoya and Taniguchi (1982, Theorem 5.1), we can construct an efficient estimator from $\hat{\rho}_1, \ldots, \hat{\rho}_p$ by the Newton–Raphson iterative method. In the next section, we show a robust property of the estimator when the true probability density function is contaminated.

3 Robustness of proposed estimator against contamination In the previous section, we showed that proposed estimator is root n consistent, and it enable us to construct the efficient estimator by the Newton–Raphson iterative method. In this section, we show our estimator is robust when the true probability density function is contaminated with noise. Let $q(\cdot)$ be a contaminated probability density function defined, for $\theta \in [-\pi, \pi]$ and some $\beta \in (0, \pi/2)$, as

$$q(\theta) = \begin{cases} p(\theta) & \text{if } -\pi + \beta \leq \theta \leq \pi - \beta, \\ cg(\theta) & \text{otherwise,} \end{cases}$$

where $p(\theta)$ is defined by (1), $g(\theta)$ is a non-negative function with $\int_{\pi-\beta}^{\pi+\beta} g(\theta)\mathrm{d}\theta > 0$, c is some constant such that $q(\theta)$ is probability density function. In the above setting, $cg(\theta)$ corresponds to a noise. Assume that the process $\{\Theta_k : k \in \mathbb{N}\}$ is misspecified, that is, the true model of $\{\Theta_k : k \in \mathbb{N}\}$ comes from $q(\theta)$, but we fit the process to $p(\theta)$.

Theorem 3.1. *If α_p and β satisfy $\alpha_p < \pi - \beta$, then the our estimator does not be influenced by the contamination.*

Thus, the proposed method is robust against the contamination of probability density.

4 Simulation Study In this section, we study finite sample performance of the proposed method, and confirm the asymptotic normality of the proposed estimator based on binary process. In this simulation, the circular distributions of MA(1) and MA(2) types are discussed. First, we illustrate finite sample performance. The procedure is the following; first, we generate random variables $\{U_i : i = 1, \ldots, n\}$ $(n = 100, 300, 500, 1000)$, which follows i.i.d. standard uniform distribution. Next, we compute $\{\Theta_i = 1 \ldots, n\} := \{F^{-1}(U_i) : i = 1, \ldots, n\}$, where F^{-1} is the generalized inverse of a distribution function of (1), which follows the circular distribution of MA(p) type for $p = 1, 2$. Then, we calculate the proposed estimators of ρ_1 and ρ_2 for the each set of parameters $\phi_1 = 0.4, 0.7, -0.5$ and angulars $\alpha_1 = \pi/4, \pi/2, 3\pi/4$ for MA(1) type distribution, and $(\phi_1, \phi_2) = (0.7, 0.4), (1.0, 0.7), (0.9, -0.3)$ and $(\alpha_1, \alpha_2) = (\pi/4, \pi/2), (\pi/2, 3\pi/4)$ for MA(2) type distribution. We iterate 1000 times and calculate mean absolute error, defined as $\mathrm{MAE_j} := \sum_{k=1}^{1000} |\hat{\rho}_j^{(k)} - \rho_j|/n$ for $j = 1, 2$, where $\hat{\rho}_j^{(k)}$ is the estimator of ρ_j of k-th iteration. Next, we calculate, for $n = 1000$, $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1);\ k = 1, \ldots, 10000\}$ and $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1), \sqrt{n}(\hat{\rho}_2^{(k)} - \rho_2);\ k = 1, \ldots, 10000\}$ for circular distributions of MA(1) type with $\phi_1 = 0.7$ and MA(2) type with $(\phi_1, \phi_2) = (0.7, 0.4)$, respectively to confirm the asymptotic normality of the proposed estimator. Then, we give the Q-Q plots in Figures 1, 2, and 3. We also provide the Kolmogorov–Smirnov test of normality to check the asymptotic normality of the proposed estimator. The null hypothesis is that $\{\sqrt{n}(\hat{\rho}_1 - \rho_1)\}$ follows the normal distribution for large n. For $n = 100,\ 1000,\ 10000$, $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1); k = 1, \ldots, 100\}$ and $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1), \sqrt{n}(\hat{\rho}_2^{(k)} - \rho_2);\ k = 1, \ldots, 100\}$ are calculated for circular distributions of MA(1) type with $\phi_1 = 0.7$ and MA(2) type with $(\phi_1, \phi_2) = (0.7, 0.4)$. Then, we compute p-value by using R-function ks.test() when $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1); k = 1, \ldots, 100\}$ regarded as a set of i.i.d. observations with respect to k. Note that, from the definition of binary estimator, we possibly have the exact same value $\hat{\rho}_j^{(k)} = \hat{\rho}_j^{(k')}$ for some k and $k'(\neq k)$. Therefore, we added a small perturbation to $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1); k = 1, \ldots, 100\}$ by R function jitter() in order to compute p-value (see Robert et al. (2010, p.17-18)).

The results are shown in Tables 1 and 2 and Figures 1, 2, and 3. Tables 1 and 2 show the proposed estimator works well, and the mean absolute errors get smaller as the sample size increases. In Table 1, for $\phi_1 = 0.4$ and 0.7 in MA(1) type model, MAE_1 is smallest when $\alpha_1 = 3\pi/4$ among $\alpha_1 = \pi/4,\ \pi/2,\ 3\pi/4$. On the other hand, for $\phi_1 = -0.5$ in MA(1) type model, MAE_1 is smallest when $\alpha_1 = \pi/4$ among three angulars. It is because MA(1) model with $\phi_1 = -0.5$ has a mean direction π. The mean directions of the proposed model are 0 in the other cases. In Table 2, MAE_1 are smaller than MAE_2. For better estimation of ϕ_2, the set of angulars $(\pi/2, 3\pi/4)$ is better than $(\pi/4, \pi/2)$. Regarding to estimation of ϕ_1, both sets of angulars $(\pi/2, 3\pi/4)$ and $(\pi/4, \pi/2)$ provide almost the same MAE_1. Figures 1 2, and 3 show that almost of all points are on the reference line, that is, we could confirm that our estimator has asymptotic normality. Moreover, for MA(1) model, the p-values of the KS test are obtained as 0.582, 0.987, 0.981 for $n = 100,\ 1000\ 10000$, respectively. For

MA(2) model, the p-values of the KS test for $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1); k = 1, \ldots, 100\}$ are obtained as 0.528, 0.507, 0.718 and that for $\{\sqrt{n}(\hat{\rho}_2^{(k)} - \rho_2); k = 1, \ldots, 100\}$ are obtained as 0.990, 0.799, 0.989 for $n = 100,\ 1000\ 10000$, respectively. As a result, it shows that we cannot reject the null hypothesis in all cases we investigated.

Table 1: MAE for circular distributions of MA(1) type

ϕ_1	α_1	n	MAE_1	ϕ_1	α_1	n	MAE_1
0.4	$\pi/4$	100	0.175	0.7	$\pi/4$	100	0.171
		300	0.103			300	0.108
		500	0.076			500	0.077
		1000	0.054			1000	0.055
	$\pi/2$	100	0.115		$\pi/2$	100	0.098
		300	0.648			300	0.060
		500	0.050			500	0.044
		1000	0.036			1000	0.032
	$3\pi/4$	100	0.106		$3\pi/4$	100	0.071
		300	0.059			300	0.038
		500	0.046			500	0.030
		1000	0.034			1000	0.022
-0.5	$\pi/4$	100	0.087				
		300	0.052				
		500	0.041				
		1000	0.029				
	$\pi/2$	100	0.113				
		300	0.065				
		500	0.049				
		1000	0.034				
	$3\pi/4$	100	0.180				
		300	0.103				
		500	0.077				
		1000	0.055				

Table 2: MAE for circular distributions of MA(2) type

(ϕ_1, ϕ_2)	(α_1, α_2)	n	MAE_1	MAE_2
(0.7,0.4)	$(\pi/4, \pi/2)$	100	0.081	0.215
		300	0.045	0.121
		500	0.036	0.099
		1000	0.026	0.069
	$(\pi/2, 3\pi/4)$	100	0.083	0.094
		300	0.046	0.055
		500	0.037	0.041
		1000	0.026	0.029
(1.0,0.7)	$(\pi/4, \pi/2)$	100	0.060	0.222
		300	0.035	0.129
		500	0.028	0.096
		1000	0.020	0.070
	$(\pi/2, 3\pi/4)$	100	0.064	0.070
		300	0.036	0.039
		500	0.027	0.032
		1000	0.019	0.021
(0.9,-0.3)	$(\pi/4, \pi/2)$	100	0.115	0.210
		300	0.066	0.125
		500	0.051	0.095
		1000	0.035	0.069
	$(\pi/2, 3\pi/4)$	100	0.111	0.155
		300	0.067	0.094
		500	0.052	0.075
		1000	0.036	0.052

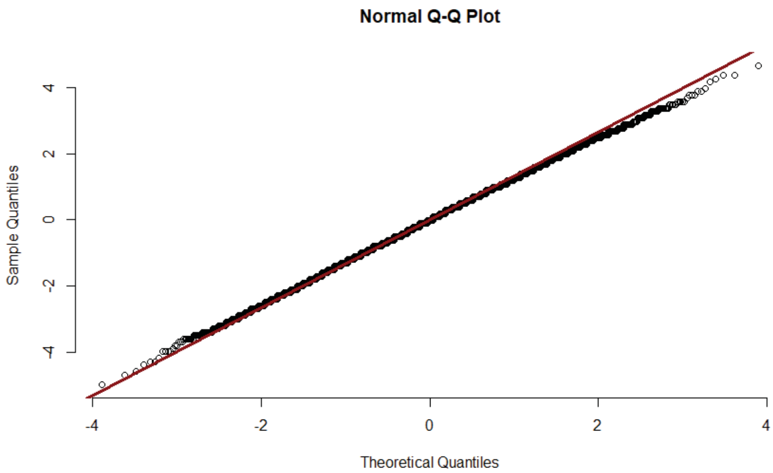

Figure 1: Q-Qplots of $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1);\ k = 1, \ldots, 10000\}$ for a circular distribution of MA(1) type with $\phi_1 = 0.7$ for $n = 1000$.

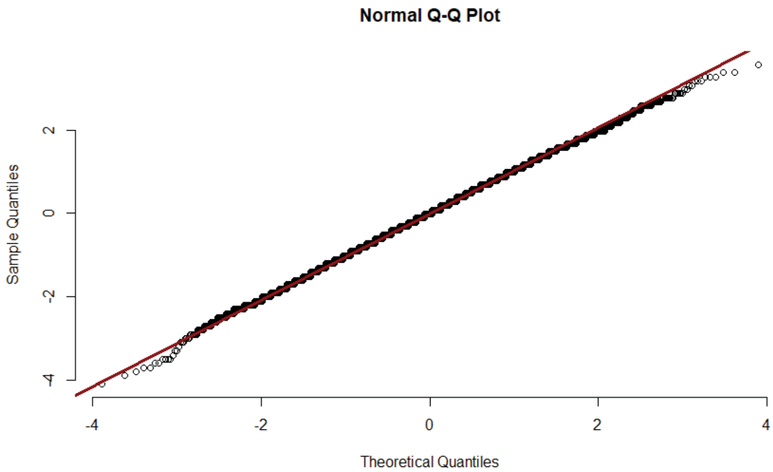

Figure 2: Q-Qplots of $\{\sqrt{n}(\hat{\rho}_1^{(k)} - \rho_1);\ k = 1, \ldots, 10000\}$ for a circular distribution of MA(2) type $(\phi_1, \phi_2) = (0.7, 0.4)$ for $n = 1000$.

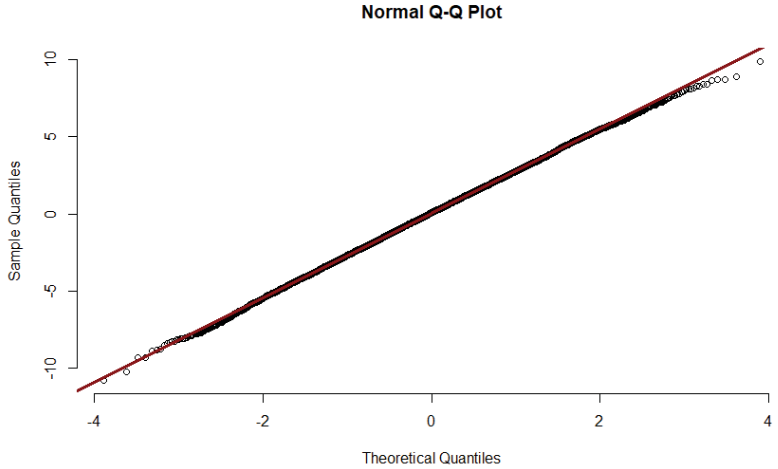

Figure 3: Q-Qplots of $\{\sqrt{n}(\hat{\rho}_2^{(k)} - \rho_2);\ k = 1, \ldots, 10000\}$ for a circular distribution of MA(2) type $(\phi_1, \phi_2) = (0.7, 0.4)$ for $n = 1000$.

5 Proof In this section, we provide the proofs of Theorems 2.1 and 3.1 and Proposition 2.1.

Proof of Theorem 2.1. First, we show the binary estimator is centered. For each $j \in \{1, \ldots, p\}$,

$$\mathrm{E}\{\sqrt{n}(\hat{\rho}_j - \rho_j)\} = \sqrt{n} 2\pi (b^{j1}, \ldots, b^{jp}) \begin{pmatrix} \frac{1}{n}\sum_{k=1}^n \mathrm{E}{X_k}^1 - P(-\alpha_1 \leq \Theta_1 \leq \alpha_1) \\ \vdots \\ \frac{1}{n}\sum_{k=1}^n \mathrm{E}{X_k}^p - P(-\alpha_p \leq \Theta_1 \leq \alpha_p) \end{pmatrix} = 0.$$

Next, we evaluate the variance of estimator. For $i, j \in \{1, \ldots, p\}$,

$$\begin{aligned} &\mathrm{cum}\{\sqrt{n}(\hat{\rho}_i - \rho_i), \sqrt{n}(\hat{\rho}_j - \rho_j)\} \\ =&\frac{4\pi^2}{n} \sum_{s,k=1}^{p} b^{is} b^{jk} \sum_{v=1}^{n} \mathrm{cum}\{{X_v}^s, {X_v}^k\} \\ =&4\pi^2 \sum_{s,k=1}^{p} b^{is} b^{jk} \mathrm{cum}\{X_1^s, X_1^k\}. \end{aligned}$$

Finally, we elucidate the L-th order cumulant $(L \geq 3)$ of the binary estimator is of order $O(n^{-L/2+1})$. For $i_1, \ldots, i_L \in \{1, \ldots, p\}$,

$$\begin{aligned} &\mathrm{cum}\{\sqrt{n}(\hat{\rho}_{i_1} - \rho_{i_1}), \ldots, \sqrt{n}(\hat{\rho}_{i_L} - \rho_{i_L})\} \\ =&n^{L/2}(2\pi)^L \sum_{s_1,\ldots,s_L=1}^{p} b^{i_1 s_1} \cdots b^{i_L s_l} \mathrm{cum}\{\frac{1}{n}\sum_{k=1}^{n} {X_k}^{s_1}, \ldots, \frac{1}{n}\sum_{k=1}^{n} {X_k}^{s_L}\} \\ =&n^{-L/2+1}(2\pi)^L \sum_{s_1,\ldots,s_L=1}^{p} b^{i_1 s_1} \cdots b^{i_L s_l} \mathrm{cum}\{X_1^{s_1}, \ldots, X_1^{s_L}\} \\ =&O(n^{-L/2+1}), \end{aligned}$$

thus, we have the desired result. □

Proof of Proposition 2.1. It is sufficient to show the Fisher information $\mathcal{I}$, defined by

$$\mathcal{I}(\rho_1) = \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \rho_1} \log p(\theta) \right)^2 p(\theta) \mathrm{d}\theta,$$

becomes the following

$$\mathcal{I}(\rho_1) = \begin{cases} \frac{1}{2} & (\rho_1 = 0), \\ \frac{1}{\rho_1^2} \left(\frac{1}{\sqrt{1-\rho_1^2}} - 1 \right) & (0 < |\rho_1| < 1), \\ \infty & (\rho_1 = \pm 1). \end{cases}$$

First, for $\rho_1 = 0$, by a straightforward calculation. Second, the residue theorem yields the assertion when ρ_1 satisfies $0 < |\rho_1| < 1$. Third, for $\rho_1 = \pm 1$, it is easy to see the integral diverges. □

Proof of Theorem 3.1. For any $a_j (< \pi - \beta)$, $j = 1, \ldots, p$, we have

$$\int_{-a_j}^{a_j} q(\theta) \mathrm{d}\theta = \int_{-a_j}^{a_j} p(\theta) \mathrm{d}\theta,$$

from which the statement follows. □

6 Acknowledgement The author Y.G. is grateful to the referee for instructive comments and kindness. Y.G. also thanks Professor Masanobu Taniguchi and Doctor Yan Liu for their encouragements and comments. Y.G.'s research was supported by Grant-in-Aid for JSPS Research Fellow Grant Number JP201920060.

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Communicated by *Masanobu Taniguchi*

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THE n-TH OPERATOR VALUED DIVERGENCES $\Delta_{i,x}^{[n]}(A|B)$

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Received February 6,2020 ; revised April 20, 2020

Abstract. Let A and B be strictly positive linear operators on a Hilbert space $\mathcal{H}$. As a generalization of the relative operator entropy $S(A|B) \equiv A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ and the Tsallis relative operator entropy $T_x(A|B) \equiv A^{\frac{1}{2}}\dfrac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^x - I}{x}A^{\frac{1}{2}}$, we have introduced the n-th relative operator entropy $S^{[n]}(A|B)$ and the n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In this paper, we define the n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B) \equiv T_x^{[n]}(A|B) - S^{[n]}(A|B)$ $(x \in R)$ corresponding to the operator valued divergence $\Delta_{1,\alpha}(A|B) \equiv T_\alpha(A|B) - S(A|B)$ $(\alpha \in [0,1])$ which is a generalization of Petz-Bregman divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Similatly, by using $\mathscr{D}_x^{[n]}(A|B)$, we introduce the n-th operator valued divergences $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ corresponding to $\Delta_{2,\alpha}(A|B) \equiv S_\alpha(A|B) - T_\alpha(A|B)$, $\Delta_{3,\alpha}(A|B) \equiv -T_{1-\alpha}(B|A) - S_\alpha(A|B)$ and $\Delta_{4,\alpha}(A|B) \equiv S_1(A|B) + T_{1-\alpha}(B|A)$, respectively, and show their properties and relations among them.

1 Introduction. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is positive (denoted by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in \mathcal{H}$, and T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive. Throughout this paper, A and B denote strictly positive operators.

Based on the concept of the α-divergence introduced by Amari [1], Fujii [2] defined the operator valued α-divergence:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)} \quad (\alpha \in (0,1)),$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean and $A \sharp_\alpha B \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$ is the weighted geometric operator mean [18]. We use the representation $A \natural_\alpha B$ instead of $A \sharp_\alpha B$ below if $\alpha \in \mathbb{R}$ ([17]).

Aside from this, Petz [19] introduced the Bregman divergence for an operator valued smooth function $\psi : C \to B(\mathcal{H})$ as

$$\psi(x) - \psi(y) - \lim_{t \to +0} \frac{\psi(y + t(x-y)) - \psi(y)}{t},$$

where C is a convex set in a Banach space. As an analogy of this kind of divergence, we had given an operator valued divergence

$$\psi(1) - \psi(0) - \frac{d}{dt}\,\psi(t)\Big|_{t=0} = B - A - S(A|B)$$

for $\psi(t) \equiv A \natural_t B$. We call it the Petz-Bregman divergence and denote it by

$$D_{FK}(A|B) = B - A - S(A|B),$$

2010 *Mathematics Subject Classification.* 47A63, 47A64 and 94A17.

Key words and phrases. operator valued divergence, operator valued α-divergence, Petz-Bregman divergence, the n-th operator valued divergence, the n-th relative operator entropy.

where

$$S(A|B) \equiv A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$$

is the relative operator entropy introduced by Fujii and Kamei [3, 9, 11]. Fujii, et. al. [5, 6] showed that the operator valued α-divergence coincides with the Petz-Bregman divergence at the end points for interval $(0, 1)$. That is,

$$D_0(A|B) \equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B) = D_{FK}(A|B).$$

In addition, since $D_1(A|B) \equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = D_{FK}(B|A)$ holds, $D_\alpha(A|B)$ combines $D_{FK}(A|B)$ with $D_{FK}(B|A)$. This is a symmetric property for $D_\alpha(A|B)$ in the sense of [4].

In [10], we had given the following relations among relative operator entropies:

$$(1) \qquad S(A|B) \le T_\alpha(A|B) \le S_\alpha(A|B) \le -T_{1-\alpha}(B|A) \le S_1(A|B) \ for\ \alpha \in (0, 1),$$

where $S_x(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^x \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$ $(x \in \mathbb{R})$ is the generalized relative operator entropy defined by Furuta [8] and $T_\alpha(A|B) \equiv \dfrac{A \sharp_\alpha B - A}{\alpha}$ $(\alpha \in (0, 1])$ is the Tsallis relative operator entropy defined by Yanagi, Kuriyama and Furuichi [20]. The Tsallis relative operator entropy $T_x(A|B)$ can be defined for all $x \in \mathbb{R}$ and the inequalities (1) hold also at $\alpha = 0$ and 1.

In [12], we obtained the following representations of the operator valued α-divergence and the Petz-Bregman divergence:

$$\begin{aligned} D_\alpha(A|B) &= -T_{1-\alpha}(B|A) - T_\alpha(A|B) \quad (\alpha \in (0, 1)), \\ D_{FK}(A|B) &= -T_1(B|A) - T_0(A|B) = T_1(A|B) - S(A|B). \end{aligned}$$

Since these are differences between the terms in (1), we also regarded other differences as operator divergences [14]: For $\alpha \in (0, 1)$,

$$\begin{aligned} \Delta_{1,\alpha}(A|B) &\equiv T_\alpha(A|B) - S(A|B), & \Delta_{2,\alpha}(A|B) &\equiv S_\alpha(A|B) - T_\alpha(A|B), \\ \Delta_{3,\alpha}(A|B) &\equiv -T_{1-\alpha}(B|A) - S_\alpha(A|B), & \Delta_{4,\alpha}(A|B) &\equiv S_1(A|B) + T_{1-\alpha}(B|A) \end{aligned}$$

and so on.

Since the relative operator entropy $S(A|B)$ is given as the derivative of the path $A \natural_t B$ at $t = 0$, Fujii et. al. [7] gave the viewpoint that $S(A|B)$ is the velocity on the path $A \natural_t B$ at $t = 0$. Similarly, we regarded $S_\alpha(A|B)$ as the velocity on $A \natural_t B$ at $t = \alpha$ and based on this viewpoint, we tried to introduce a notion of the acceleration on the path $A \natural_t B$ at $t = \alpha$ which was given as the second derivative of the path at $t = \alpha$ in [15]. As an extension of such perspective, we regarded the Tsallis relative operator entropy $T_x(A|B)$ as the average rate of change of the path $A \natural_t B$ over the interval $[0, x]$ and $S(A|B) = \lim_{x \to 0} T_x(A|B)$ as the rate of change of the path at $t = 0$ in [16].

The n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ is constructed inductively as follows:

$$T_x^{[1]}(A|B) \equiv T_x(A|B)$$

and for $n \ge 2$,

$$T_x^{[n]}(A|B) \equiv \frac{T_x^{[n-1]}(A|B) - S^{[n-1]}(A|B)}{x} \quad (x \in \mathbb{R} \backslash \{0\}),$$

where $S^{[n]}(A|B)$ is defined by

$$S^{[n]}(A|B) \equiv \frac{1}{n!}A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^n A^{\frac{1}{2}} = \frac{1}{n!}A(A^{-1}S(A|B))^n$$

and we call it the n-th relative operator entropy. Since $T_x^{[n]}(A|B)$ is represented specifically as

$$T_x^{[n]}(A|B) = \frac{1}{x^n}\left(A \natural_x B - A - \sum_{k=1}^{n-1} x^k S^{[k]}(A|B)\right) \quad (x \in \mathbb{R}\backslash\{0\}),$$

the corresponding functions to $T_x^{[n]}(A|B)$ and $S^{[n]}(A|B)$ are

$$\frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n-1}\frac{x^k}{k!}(\log\lambda)^k\right) \quad \text{and} \quad \frac{1}{n!}(\log\lambda)^n \quad (\lambda > 0),$$

respectively. Since $\lim_{x\to 0}\frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n-1}\frac{x^k}{k!}(\log\lambda)^k\right) = \frac{1}{n!}(\log\lambda)^n$, we obtain $\lim_{x\to 0} T_x^{[n]}(A|B) = S^{[n]}(A|B)$ for all $n \in \mathbb{N}$. Therefore, we defined $T_0^{[n]}(A|B)$ by

$$T_0^{[n]}(A|B) \equiv S^{[n]}(A|B).$$

For $n \geq 2$, the n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ is regarded as the average rate of change of $T_x^{[n-1]}(A|B)$ over the interval $[0, x]$.

In addition, we defined $S_y^{[n]}(A|B)$ by

$$S_y^{[n]}(A|B) \equiv \frac{1}{n!}\frac{d^n}{dx^n}\ A \natural_x B\ \Big|_{x=y} = (A \natural_y B)A^{-1}S^{[n]}(A|B) \quad (y \in \mathbb{R})$$

and call it the n-th generalized relative operator entropy. We remark that $S_0^{[n]}(A|B)$ coincides with $S^{[n]}(A|B)$ and $(A \natural_x B)A^{-1}S_y^{[n]}(A|B) = S_{x+y}^{[n]}(A|B)$ holds for $x, y \in \mathbb{R}$.

In [16], we defined the n-th Petz-Bregman divergence $D_{FK}^{[n]}(A|B)$ and the n-th operator valued divergence $\mathscr{D}_\alpha^{[n]}(A|B)$ by

$$D_{FK}^{[n]}(A|B) \equiv T_1^{[n]}(A|B) - S^{[n]}(A|B) = B - A - \sum_{k=1}^{n} S^{[k]}(A|B),$$

$$\mathscr{D}_\alpha^{[n]}(A|B) \equiv T_\alpha^{[n]}(A|B) - S^{[n]}(A|B) = \frac{1}{\alpha^n}\left(A \natural_\alpha B - A - \sum_{k=1}^{n}\alpha^k S^{[k]}(A|B)\right) \quad (\alpha \in [0,1]),$$

and showed their properties. We remark $\mathscr{D}_1^{[1]}(A|B) = D_{FK}^{[1]}(A|B) = D_{FK}(A|B)$ and $\mathscr{D}_\alpha^{[1]}(A|B) = \Delta_{1,\alpha}(A|B)$. So we think $\mathscr{D}_\alpha^{[n]}(A|B)$ is a generalization of the Petz-Bregman divergence. In addition, it is natural that $\mathscr{D}_\alpha^{[n]}(A|B)$ is regarded as the n-the operator valued divergence corresponding to $\Delta_{1,\alpha}(A|B)$. In this paper, we propose the definitions of the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ and to show properties of them. For this purpose, we need to extend $\mathscr{D}_\alpha^{[n]}(A|B)$ $(\alpha \in (0,1))$ to $\mathscr{D}_x^{[n]}(A|B)$ $(x \in \mathbb{R})$. We call $\mathscr{D}_x^{[n]}(A|B)$ the n-th generalized Petz-Bregman divergence and show some properties of it in section 2. In section 3, we define the n-th operator valued divergences $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ with $x \in \mathbb{R}$ which correspond to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ by using the n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B)$ and show some properties for them.

2 The n-th Generalized Petz-Bregman Divergence. Our idea of defining the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ is to use $\mathscr{D}_\alpha^{[n]}(A|B)$ defined in [16]. In order to achieve such purpose, we need to broaden the range of α for $\mathscr{D}_\alpha^{[n]}(A|B)$ from $[0,1]$ to $\mathbb{R}$. For strictly positive operators A and B, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we define $\mathscr{D}_x^{[n]}(A|B)$ as follows:

$$\mathscr{D}_x^{[n]}(A|B) \equiv T_x^{[n]}(A|B) - S^{[n]}(A|B).$$

We call it the n-th generalized Petz-Bregman divergence.

By Proposition 4.5 in [16], the following proposition holds for the n-th generalized Petz-Bregman divergence.

Proposition 2.1. *Let n be a fixed natural number and x be a fixed real number in $\mathbb{R}\backslash\{0\}$. Then the following holds for any strictly positive operators A and B:*

$$\mathscr{D}_x^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B.$$

Remark 1. Since $\mathscr{D}_1^{[n]}(A|B) = D_{FK}^{[n]}(A|B)$,

$$D_{FK}^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B$$

holds for any fixed natural number n.

The following are fundamental properties for $\mathscr{D}_x^{[n]}(A|B)$.

Theorem 2.2. *Let A and B be strictly positive operators and $x \in \mathbb{R}$. Then the following hold for $n \in \mathbb{N}$:*

(a) $\mathscr{D}_0^{[n]}(A|B) = O$.

(b) *If $x > 0$ then*

$$\mathscr{D}_x^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(c) *If $x < 0$ then*

$$\mathscr{D}_x^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. Since it is obvious that $\mathscr{D}_0^{[n]}(A|B) = O$ holds, we suppose $x \in \mathbb{R}\backslash\{0\}$. Let $\lambda > 0$. Since $\mathscr{D}_x^{[n]}(A|B) = \dfrac{1}{x^n}\left(A \natural_x B - A - \sum_{k=1}^{n} x^k S^{[k]}(A|B)\right)$, the corresponding function $f^{[n]}(\lambda, x)$ for $\mathscr{D}_x^{[n]}(A|B)$ is given as follows:

$$f^{[n]}(\lambda, x) = \frac{1}{x^n}\left(\lambda^x - 1 - \sum_{k=1}^{n} \frac{x^k}{k!}(\log\lambda)^k\right).$$

On the other hand, λ^x can be represented by using some $\theta \in (0,1)$ as

$$\lambda^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!}(\log\lambda)^k + \frac{x^{n+1}}{(n+1)!}\lambda^{\theta x}(\log\lambda)^{n+1}.$$

Hence, by using this θ, we get

$$f^{[n]}(\lambda, x) = \frac{x}{(n+1)!}\lambda^{\theta x}(\log \lambda)^{n+1}.$$

Let $x > 0$. Then $f^{[n]}(\lambda, x) \geq 0$ if n is odd or $\lambda \geq 1$, and $f^{[n]}(\lambda, x) \leq 0$ if n is even and $0 < \lambda \leq 1$. Therefore, (b) holds. Let $x < 0$. We obtain (c) since $f^{[n]}(\lambda, x) \leq 0$ if n is odd or $\lambda \geq 1$, and $f^{[n]}(\lambda, x) \geq 0$ if n is even and $0 < \lambda \leq 1$. □

In [16], we have obtained the following properties for the n-th relative operator entropies.

Lemma 2.3. (Theorem 2.4 and Theorem 3.4 in [16]) *Let A and B be strictly positive operators, r, $s \in \mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$. Then*

(a) $T_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_r B)A^{-1}T_{(s-r)x}^{[n]}(A|B)$,

(b) $S_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_r B)A^{-1}S_{(s-r)x}^{[n]}(A|B) = (s-r)^n S_{(1-x)r+xs}^{[n]}(A|B)$

hold for all $n \in \mathbb{N}$. In particular, $S^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_r B)A^{-1}S^{[n]}(A|B)$.

By using Lemma 2.3, the n-th generalized Petz-Bregman divergence has also similar properties.

Proposition 2.4. (cf. Theorem 4.8 in [16]) *Let A and B be strictly positive operators and r, s, $x \in \mathbb{R}$. Then*

$$\mathscr{D}_x^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_r B)A^{-1}\mathscr{D}_{(s-r)x}^{[n]}(A|B)$$

holds for $n \in \mathbb{N}$.

Corollary 2.5. *Let A and B be strictly positive operators and r, x, $y \in \mathbb{R}$. Then the following holds for $n \in \mathbb{N}$:*

(a) $\mathscr{D}_x^{[n]}(A \natural_r B|A) = (-r)^n (A \natural_r B)A^{-1}\mathscr{D}_{-rx}^{[n]}(A|B)$,

(b) $\mathscr{D}_x^{[n]}(B|A) = (-1)^n BA^{-1}\mathscr{D}_{-x}^{[n]}(A|B)$,

(c) $(A \natural_y B)A^{-1}\mathscr{D}_x^{[n]}(A|B) = (-1)^n (B \natural_{1-y} A)B^{-1}\mathscr{D}_{-x}^{[n]}(B|A)$.

Since $A \natural_y B = B \natural_{1-y} A$ holds for $y \in \mathbb{R}$, we obtain (c) by (b) in Corollary 2.5.

Remark 2. By putting $x = 1$ in (a) in Lemma 2.3, we have

$$D_{FK}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^{n+1}(A \natural_r B)A^{-1}T_{s-r}^{[n+1]}(A|B).$$

The following relation between $\mathscr{D}_x^{[n]}(A|B)$ and $D_{FK}^{[n]}(A|B)$ holds, which is an extension of Corollary 4.9 in [16].

Proposition 2.6. *Let A and B be strictly positive operators and $x \in \mathbb{R}\backslash\{0\}$. Then the following holds for all $n \in \mathbb{N}$:*

$$\mathscr{D}_x^{[n]}(A|B) = \frac{1}{x^n}D_{FK}^{[n]}(A|A \natural_x B).$$

3 The n-th Operator Valued Divergences corresponding to $\Delta_{i,\alpha}(A|B)$. The n-th generalized Petz-Bregman divergence $\mathscr{D}_x^{[n]}(A|B)$ defined in section 2 coincides with $\Delta_{1,\alpha}(A|B)$ when $n = 1$, $0 < x < 1$ and $x = \alpha$. So it is natural to regard $\mathscr{D}_x^{[n]}(A|B)$ as the n-the operator valued divergence corresponding to $\Delta_{1,\alpha}(A|B)$ and we can write it as $\Delta_{1,x}^{[n]}(A|B)$. In this section, we define the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ by using $\mathscr{D}_x^{[n]}(A|B)$ and show some properties of them.

For r, $s \in \mathbb{R}$, $(A \natural_r B)A^{-1}(A \natural_s B) = A \natural_{r+s} B$ holds (cf. [13]). Then the Tsallis relative operator entropy $T_x(A|B)$ can be rewritten as

$$T_x(A|B) = \frac{A \natural_x B - A}{x} = (A \natural_x B)A^{-1}\frac{A \natural_{-x} B - A}{-x}$$
$$= (A \natural_x B)A^{-1}T_{-x}(A|B) \quad (x \in \mathbb{R}\backslash\{0\}).$$

In addition, since $S_x(A|B) = (A \natural_x B)A^{-1}S(A|B)$ holds for $x \in \mathbb{R}$, we can rewrite $\Delta_{2,\alpha}(A|B)$ as follows:

$$\Delta_{2,\alpha}(A|B) = S_\alpha(A|B) - T_\alpha(A|B) = -(A \natural_\alpha B)A^{-1}(T_{-\alpha}(A|B) - S(A|B))$$
$$= -(A \natural_\alpha B)A^{-1}\mathscr{D}_{-\alpha}^{[1]}(A|B) \quad (\alpha \in (0,1)).$$

Similarly, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ can be rewritten as follows ($\alpha \in (0,1)$):

$$\Delta_{3,\alpha}(A|B) = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = (A \natural_\alpha B)A^{-1}(T_{1-\alpha}(A|B) - S(A|B))$$
$$= (A \natural_\alpha B)A^{-1}\mathscr{D}_{1-\alpha}^{[1]}(A|B),$$
$$\Delta_{4,\alpha}(A|B) = S_1(A|B) + T_{1-\alpha}(B|A) = -(A \natural_1 B)A^{-1}(T_{\alpha-1}(A|B) - S(A|B))$$
$$= -(A \natural_1 B)A^{-1}\mathscr{D}_{\alpha-1}^{[1]}(A|B).$$

Based on such representations, we define the n-th operator valued divergences corresponding to $\Delta_{2,\alpha}(A|B)$, $\Delta_{3,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$.

Definition 1. Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We define the n-th operator valued divergence $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ as follows:

$$\Delta_{2,x}^{[n]}(A|B) \equiv -(A \natural_x B)A^{-1}\mathscr{D}_{-x}^{[n]}(A|B), \qquad \Delta_{3,x}^{[n]}(A|B) \equiv (A \natural_x B)A^{-1}\mathscr{D}_{1-x}^{[n]}(A|B),$$
$$\Delta_{4,x}^{[n]}(A|B) \equiv -(A \natural_1 B)A^{-1}\mathscr{D}_{x-1}^{[n]}(A|B).$$

We remark that $\Delta_{2,x}^{[n]}(A|B)$, $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ are defined for all $x \in \mathbb{R}$ as $\Delta_{1,x}^{[n]}(A|B) = \mathscr{D}_x^{[n]}(A|B)$ was. They are also written as follows by (c) in Corollary 2.5.

Proposition 3.1. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{1,x}^{[n]}(A|B) = (-1)^n(B \natural_1 A)B^{-1}\mathscr{D}_{-x}^{[n]}(B|A)$,

(b) $\Delta_{2,x}^{[n]}(A|B) = (-1)^{n+1}(B \natural_{1-x} A)B^{-1}\mathscr{D}_x^{[n]}(B|A)$,

(c) $\Delta_{3,x}^{[n]}(A|B) = (-1)^n(B \natural_{1-x} A)B^{-1}\mathscr{D}_{x-1}^{[n]}(B|A)$,

(d) $\Delta_{4,x}^{[n]}(A|B) = (-1)^{n+1}(B \natural_0 A)B^{-1}\mathscr{D}_{1-x}^{[n]}(B|A)$.

Properties shown in Theorem 3.2 and Theorem 3.3 are fundamental where the n-th operator valued divergences have in common.

Theorem 3.2. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold for $i = 1, 2$:*

(a) $\Delta_{i,0}^{[n]}(A|B) = O$.

(b) *If $x \neq 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B.$$

(c) *If $x > 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(d) *If $x < 0$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. By Proposition 2.1 and Theorem 2.2, $\Delta_{1,x}^{[n]}(A|B)$ satisfies (a), (b), (c) and (b).

Let $\lambda > 0$. As with the proof of Theorem 2.2, the corresponding function $f_2^{[n]}(\lambda, x)$ for $\Delta_{2,x}^{[n]}(A|B)$ is represented by using $\theta_2 \in (0, 1)$ as follows:

$$f_2^{[n]}(\lambda, x) = \frac{x}{(n+1)!}\lambda^{(1-\theta_2)x}(\log \lambda)^{n+1}.$$

We obtain (a) since $f_2^{[n]}(\lambda, 0) = 0$ holds. Let $x \neq 0$. Then we have (b) since $f_2^{[n]}(\lambda, x) = 0$ if and only if $\lambda = 1$ holds. Assume that $x > 0$. Since $f_2^{[n]}(\lambda, x) \geq 0$ holds if n is odd or $\lambda \geq 1$, and $f_i^{[n]}(\lambda, x) \leq 0$ holds if n is even and $0 < \lambda \leq 1$. Hence, we have (c). We can get (d) in the same way as (c). □

Theorem 3.3. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold for $i = 3, 4$:*

(a) $\Delta_{i,1}^{[n]}(A|B) = O$.

(b) *If $x \neq 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) = O \quad \textit{if and only if} \quad A = B.$$

(c) *If $x < 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \geq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \leq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

(d) *If $x > 1$ then*

$$\Delta_{i,x}^{[n]}(A|B) \begin{cases} \leq O, & \textit{if } n \textit{ is odd or } A \leq B, \\ \geq O, & \textit{if } n \textit{ is even and } A \geq B. \end{cases}$$

Proof. The corresponding functions $f_3^{[n]}(\lambda,x)$ and $f_4^{[n]}(\lambda,x)$ for $\Delta_{3,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ are represented by using θ_2, $\theta_3 \in (0,1)$ as follows, respectively ($\lambda > 0$):

$$f_3^{[n]}(\lambda,x) = \frac{1-x}{(n+1)!}\lambda^{(1-\theta_3)x+\theta_3}(\log\lambda)^{n+1},$$

$$f_4^{[n]}(\lambda,x) = \frac{1-x}{(n+1)!}\lambda^{\theta_4 x+(1-\theta_4)}(\log\lambda)^{n+1}.$$

We obtain the assertions in the same way as Theorem 3.2. □

Corollary 3.4. (Proposition 2.1 and Proposition 4.2 in [16]) *Let A and B be strictly positive operators and $n \in \mathbb{N}$. Then the following holds:*

(a) $D_{FK}^{[n]}(A|B) = O \iff A = B$,

(b) $D_{FK}^{[n]}(A|B) = O \begin{cases} \geq O, & \text{if } n \text{ is odd or } A \leq B, \\ \leq O, & \text{if } n \text{ is even and } A \geq B. \end{cases}$

By Proposition 2.4, $\Delta_{1,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)x}^{[n]}(A|B)$ holds for $n \in \mathbb{N}$ and r, s, $x \in \mathbb{R}$. We can also obtain similar results for remaining.

Theorem 3.5. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and r, s, $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{2,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B)$,

(b) $\Delta_{3,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B)$,

(c) $\Delta_{4,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n(A \natural_s B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B)$.

Proof. For r, s, $x \in \mathbb{R}$, $(A \natural_r B) \natural_x (A \natural_s B) = A \natural_{(1-x)r+xs} B$ holds (cf. (1) in Lemma 2.2 in [13]). By using Proposition 2.4, these are shown as follows:

(a)
$$\begin{aligned} &\Delta_{2,x}^{[n]}(A \natural_r B|A \natural_s B) \\ &= -((A \natural_r B) \natural_x (A \natural_s B))(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B) \\ &= -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{-(s-r)x}^{[n]}(A|B). \end{aligned}$$

(b)
$$\begin{aligned} &\Delta_{3,x}^{[n]}(A \natural_r B|A \natural_s B) \\ &= ((A \natural_r B) \natural_x (A \natural_s B))(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B) \\ &= (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\mathscr{D}_{(s-r)(1-x)}^{[n]}(A|B). \end{aligned}$$

(c)
$$\begin{aligned} &\Delta_{4,x}^{[n]}(A \natural_r B|A \natural_s B) \\ &= -(A \natural_s B)(A \natural_r B)^{-1}(s-r)^n(A \natural_r B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B) \\ &= -(s-r)^n(A \natural_s B)A^{-1}\mathscr{D}_{(s-r)(x-1)}^{[n]}(A|B). \end{aligned}$$
□

In the following sense, $\Delta_{1,\alpha}(A|B)$ and $\Delta_{4,\alpha}(A|B)$ are symmetric as well as $\Delta_{2,\alpha}(A|B)$ and $\Delta_{3,\alpha}(A|B)$ are:

$$\Delta_{1,1-\alpha}(B|A) = T_{1-\alpha}(B|A) - S(B|A) = T_{1-\alpha}(B|A) + S_1(A|B) = \Delta_{4,\alpha}(A|B),$$
$$\Delta_{2,1-\alpha}(B|A) = S_{1-\alpha}(B|A) - T_{1-\alpha}(B|A) = -T_{1-\alpha}(B|A) - S_\alpha(A|B) = \Delta_{3,\alpha}(A|B).$$

These properties are some kind of duality. By Proposition 3.1, similar properties hold between $\Delta_{1,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ and between $\Delta_{2,x}^{[n]}(A|B)$ and $\Delta_{3,x}^{[n]}(A|B)$.

Theorem 3.6. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{1,1-x}^{[n]}(B|A) = (-1)^{n+1}\Delta_{4,x}^{[n]}(A|B)$,

(b) $\Delta_{2,1-x}^{[n]}(B|A) = (-1)^{n+1}\Delta_{3,x}^{[n]}(A|B)$.

In [14], we have shown the following relations between $\Delta_{i,\alpha}(A|B)$ and the Petz-Bregman divergence:

$$\Delta_{1,\alpha}(A|B) = \frac{1}{\alpha}D_{FK}(A|A \natural_\alpha B), \qquad \Delta_{2,\alpha}(A|B) \equiv \frac{1}{\alpha}D_{FK}(A \natural_\alpha B|A),$$
$$\Delta_{3,\alpha}(A|B) \equiv \frac{1}{1-\alpha}D_{FK}(A \natural_\alpha B|B), \qquad \Delta_{4,\alpha}(A|B) \equiv \frac{1}{1-\alpha}D_{FK}(B|A \natural_\alpha B).$$

By Proposition 2.6, the corresponding relation between the n-th operator valued divergence $\Delta_{1,x}^{[n]}(A|B)$ and the n-th Petz-Bregman divergence holds:

$$\Delta_{1,x}^{[n]}(A|B) = \frac{1}{x^n}D_{FK}^{[n]}(A|A \natural_x B).$$

We show the corresponding relations between remaining $\Delta_{i,x}^{[n]}(A|B)$ $(2 \leq i \leq 4)$ and the n-th Petz-Bregman divergence. The next theorem comes from Corollary 2.5, Proposition 2.6 and Theorem 3.6.

Theorem 3.7. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the followings hold:*

(a) $\Delta_{2,x}^{[n]}(A|B) = (-1)^{n+1}\dfrac{1}{x^n}D_{FK}^{[n]}(A \natural_x B|A)$,

(b) $\Delta_{3,x}^{[n]}(A|B) = \dfrac{1}{(1-x)^n}D_{FK}^{[n]}(A \natural_x B|B)$,

(c) $\Delta_{4,x}^{[n]}(A|B) = (-1)^{n+1}\dfrac{1}{(1-x)^n}D_{FK}^{[n]}(B|A \natural_x B)$.

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A NEW CLASS IN BCK-ALGEBRAS

Dedicated to the late professor Shôtarô Tanaka (1928-2019)

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Received April 3,2020

ABSTRACT. In BCK-algebras, it is well known that a commutative BCK-algebra is as a lower semilattice with respect to the operation $\wedge$. In this paper, we show that a new class in BCK-algebras which is a proper large class than the class of the commutative BCK-algebras exists, and this class is as a lower semilattice with respect to the new operation $\times$.

1 Introduction

A BCK-algebra is a generalization of the following two concepts. It is generalized from, one hand the concept of the algebra of sets only with the set-difference (see W. Sierpiński [5]), the other hand the concept of the propositional calculi which contain the only implication functor among the logical functors (Meredith's System B-C-K, see A. N. Prior [4]).

BCK-algebras were introduced by K. Iséki in the article [2].

These algebras are partially ordered sets. Further, S. Tanaka showed that the commutative BCK-algebras of a special class in BCK-algebras are lower semilattices with respect to the operation $\wedge$ in the article [6].

In this paper, we will define a new concept which is called the condition $(\mathrm{I})_{x,y}$ for x, y in a BCK-algebra X. Using this concept, we shall clarify a difference between the existence of a commutative element $x \wedge y = y \wedge x$ and the existence of the greatest lower bound of x and y in a BCK-algebra X. And more, we will define the BCK-algebras with Condition (I) X which is a special class satisfying the single condition $(\mathrm{I})_{x,y}$ for any x, y in X. We shall show that the BCK-algebras with Condition (I) is a proper large class than the class of the commutative BCK-algebras, and this class is as a class of lower semilattices in BCK-algebras with respect to the new operation $\times$.

2 Preliminaries and Problems

We will start out to recall the definition and some basic properties of BCK-algebras.

Definition 2.1 An algebra $X =< X; *, 0 >$ of type$< 2, 0 >$ satisfying the following five conditions is called *a BCK-algebra* :

2000 Mathematics Subject Classification. 03G25, 06A12, 06F35.

Keywords and phrases. BCK-algebra, commutative BCK-algebra, lower semilattice, the condition $(\mathrm{I})_{x,y}$, the single condition $(\mathrm{I})_{x,y}$, BCK-algebra with Canonical Condition (I), BCK-algebra with Condition (I).

For any x, y, z in X,

(I) $\{(x * y) * (x * z)\} * (z * y) = 0$,
(II) $\{x * (x * y)\} * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$, $y * x = 0$ imply $x = y$.

In this algebra, we denote $x \leqq y$ when $x * y = 0$.

We will state basic properties of BCK-algebras.

Proposition 2.2 *For any x, y, z in a BCK-algebra X, we have the following properties hold.*

(1) *$X =< X; \leqq >$ is a partially ordered set with respect to $\leqq$.*
(2) *$x \leqq y$ implies $x * z \leqq y * z$, $z * y \leqq z * x$.*
(3) *$(x * y) * z = (x * z) * y$.*
(4) *$(x * y) * (z * y) \leqq x * z$.*
(5) *$x * y \leqq z$ implies $x * z \leqq y$.*
(6) *$x * y \leqq x$.*
(7) *$x * 0 = x$.*

For details of the proofs, see K. Iséki, S. Tanaka [3].

We define the operation $x \wedge y$ by $y * (y * x)$. A BCK-algebra X is said to be *commutative* when $x \wedge y = y \wedge x$ holds for any x, y in X.

For a commutative BCK-algebra X, the following theorem holds.

Theorem 2.3(Tanaka's Theorem)(S. Tanaka [6], K. Iséki, S. Tanaka [3]) *Any commutative BCK-algebra $X =< X; *, 0 >$ is a lower semilattice with respect to the operation $\wedge$ in X.*

This theorem asserts that, if a BCK-algebra X is commutative, then the greatest lower bound exists for x and y in X, and is identical to $x \wedge y$.

Inspired by this theorem, we consider the following problem.

Problem Under what condition on x and y in a non-commutative BCK-algebra X, does there exist the greatest lower bound of x and y, and when is obtained by $x \wedge y$?

3 Basic properties of the Condition $(\mathrm{I})_{x,y}$ in a BCK-algebra

We will give an additional condition in BCK-algebras.

For x, y in a BCK-algebra X, we put the condition $(\mathrm{I})_{x,y}$ in the following.

Condition $(\mathrm{I})_{x,y}$ For x, y in a BCK-algebra X, z exist in X, we say that z *satisfies the condition* $(\mathrm{I})_{x,y}$ when z satisfies the following conditions (i)∼(iii).

(i) $z \leqq x$, $z \leqq y$,
(ii) $x * z \leqq x * y$,

(iii) $\quad y * z \leqq y * x.$

Under the condition $(\mathrm{I})_{x,y}$, the following basic properties hold for z in a BCK-algebra X.

Proposition 3.1 *If z is the greatest lower bound of x and y in X, then z satisfies the condition* $(\mathrm{I})_{x,y}$.

Proof Suppose that z is the greatest lower bound of x and y in X. Clearly, z satisfies the inequalities (i). We will show the inequality (ii).

Now, we will show that $y \wedge x$ is a common lower bound of x and y. By (6) in Proposition 2.2, (II) in Definition 2.1, we obtain

$$y \wedge x = x * (x * y) \leqq x, \; y \wedge x = x * (x * y) \leqq y \tag{3.1}$$

Then, $y \wedge x$ is a common lower bound of x and y.

Let $0 \neq u \in X$ be a common lower bound of x and y. Here, we will show that

$$y \wedge x \leqq u \tag{3.2}$$

for any u in X.

First, if $y \wedge x > u$ implies,

$$u * (y \wedge x) = 0 \tag{3.3}$$

On the other hand, by (2), (6) in Proposition 2.2,

$$u * (y \wedge x) \leqq u * (x * x) = u \neq 0 \tag{3.4}$$

Hence, (3.3) contradicts (3.4).

Second, if $y \wedge x \ngeqq u$, $y \wedge x \nleqq u$ imply that $y \wedge x$ and u are anti-chain. Then, two maximal lower bound of x and y exist. However, z is the greatest lower bound of x and y. This is contradiction. Hence, we obtain (3.2).

Here, z is the special element in the set of the common lower bounds of x and y, by (3.2), we have

$$(x * z) * (x * y) = (y \wedge x) * z = 0. \tag{3.5}$$

Therefore, we will prove the inequality (ii). By the same way, we have the inequality (iii).

□

Proposition 3.2 *If $x \wedge y = y \wedge x$, then $x \wedge y$ satisfies the condition* $(\mathrm{I})_{x,y}$.

Proof Noting that $x \wedge y$ is a common lower bound of x and y. This implies the inequalities (i).

We will show the inequality (ii). Put

$$z = x \wedge y = y \wedge x.$$

By (3) in Proposition 2.2,

$$\begin{aligned}(x * z) * (x * y) &= \{x * (y \wedge x)\} * (x * y) \\ &= \{x * (x * y)\} * (y \wedge x) \\ &= (y \wedge x) * (y \wedge x) \\ &= 0.\end{aligned}$$

This implies the inequality (ii). By the same way, we have the inequality (iii).

□

We will give a necessary and sufficient condition on (ii) in the condition $(\mathrm{I})_{x,y}$.

Proposition 3.3 *Let $z \leqq x,\ z \leqq y$. Then the following the conditions* (1)∼(3) *are equivalent.*

(1) $x * z \leqq x * y$,
(2) $x * z = x * y$,
(3) $y \wedge x \leqq z$.

Proof Clearly, the condition (2) implies (1). In a BCK-algebra, from (2) in Proposition 2.2, $z \leqq y$ implies $x * y \leqq x * z$. Hence $x * z = x * y$. Then the conditions (1) and (2) are equivalent.

The inequality (1) means

$$(x * z) * (x * y) = 0. \tag{3.6}$$

On the other hand, by (3) in Proposition 2.2, the condition (3) implies

$$(y \wedge x) * z = \{x * (x * y)\} * z = (x * z) * (x * y) = 0. \tag{3.7}$$

The equalities (3.6), (3.7) imply that the condition (1) and the condition (3) are equivalent.

□

It is more convenient to substitute the condition (3) for the condition (1). This substitution is often used in the proofs.

Let $I(x, y)$ denote the set of elements satisfying the condition $(\mathrm{I})_{x,y}$.

Proposition 3.4 *$I(x, y) \neq \phi$, then there exists the greatest lower bound of x and y in $I(x, y)$.*

Proof Assume that z exists in $I(x, y) \neq \phi$.

For any u in X such that $u \leqq x,\ u \leqq y$, we will show $u \leqq z$.
By the assumption, z satisfies the condition $(\mathrm{I})_{x,y}$. From Proposition 3.3,

$$y \wedge x \leqq z. \tag{3.8}$$

Further, by (2) in Proposition 2.2 and (3.8),

$$u * z \leqq u * (y \wedge x). \tag{3.9}$$

On the other hand, $y \wedge x$ is a common lower bound of x and y. By (3.9), replace u with $y \wedge x$,

$$u * z \leq (y \wedge x) * (y \wedge x) = 0.$$

Then, we get $u \leqq z$. Therefore, z is the greatest lower bound for x and y.

□

Next, we shall prove an important proposition which is applied to theorems in Section 4.

Proposition 3.5 *If there exists the only one element z satisfying the condition $(\mathrm{I})_{x,y}$, then z is the greatest lower bound of x and y.*

Proof Let z be the only one element satisfying the condition $(\mathrm{I})_{x,y}$. Suppose that the greatest lower bound u of x and y exists, which is not equal to z, by Proposition 3.1 and Proposition 3.3, then

$$y \wedge x \leqq u, \ x \wedge y \leqq u. \tag{3.10}$$

0n the other hand, by the fact that z is the only element with the condition $(\mathrm{I})_{x,y}$, this implies that u doesn't satisfying the condition $(\mathrm{I})_{x,y}$. Then, we have that u doesn't satisfy the inequality (ii),

$$y \wedge x > u \qquad \text{or} \qquad y \wedge x \ngeqq u, \ y \wedge x \nleqq u, \tag{3.11}$$

and more u doesn't satisfy the condition (iii), then

$$x \wedge y > u \qquad \text{or} \qquad x \wedge y \ngeqq u, \ x \wedge y \ngeqq u. \tag{3.12}$$

The inequality (3.10) contradict with (3.11), (3.12). Hence u doesn't exist. Therefore, by Proposition 3.4, z is the greatest lower bound x and y.

□

For any ordered two elements in a BCK-algebra have several basic properties.

Proposition 3.6 *If $z \leqq x$, then $z = x \wedge z$.*

Proof By the assumption $z \leqq x$ and (7) in Proposition 2.2, we have

$$z = z * 0 = z * (z * x) = x \wedge z.$$

□

Proposition 3.7 *Let $x \wedge y = y \wedge x$. If there exists an element u satisfying the condition $(\mathrm{I})_{x,y}$ and $u \neq x \wedge y$, then $x \wedge y < u$.*

Proof By the assumption, u satisfies the condition $(\mathrm{I})_{x,y}$. By Proposition 3.3, this implies

$$x \wedge y \leqq u. \tag{3.13}$$

By (3.13) and $u \neq x \wedge y$, we have

$$x \wedge y = y \wedge x < u.$$

□

Let $|X|$ denote the cardinal of X for a set X.

Corollary 3.8 *If $x \wedge y = y \wedge x$ and $|I(x,y)| \geqq 2$, then $x \wedge y$ is the least lower element in $I(x,y)$.*

Lemma 3.9 *If x and y satisfy the order $x \leqq y$, and z satisfies the condition* $(\mathrm{I})_{x,y}$, *then $x \wedge y \leqq z = x = y \wedge x$.*

Proof Since z satisfies the condition $(\mathrm{I})_{x,y}$, we have

$$x * z \leqq x * y. \tag{3.14}$$

Further, by $x \leqq y$, we have

$$x * y = 0. \tag{3.15}$$

The inequality (3.14) satisfies (3.15), This implies $x * z \leqq 0$.

Then, we obtain, $x * z = 0$. This implies

$$x \leqq z. \tag{3.16}$$

Conversely, since z is a common lower bound of x and y,

$$x \geqq z. \tag{3.17}$$

Then, by (3.16), (3.17), we have

$$z = x. \tag{3.18}$$

By $x \leqq y$,

$$x = x * (x * y) = y \wedge x. \tag{3.19}$$

On the other hand, by (II) in Definition 2.1, we have

$$x \wedge y = y * (y * x) \leqq x \tag{3.20}$$

From (3.18), (3.19), (3.20),

$$x \wedge y \leqq z = x = y \wedge x.$$

□

Here, the next Proposition 3.10 is derived naturally from Lemma 3.9.

Proposition 3.10 *If x and y satisfy the order $x \leqq y$, then $I(x,y) = \{x\}$.*

When we add the condition $z \leqq x \wedge y$ to the hypothesis of Lemma 3.9, the next Proposition 3.11 holds.

Proposition 3.11 *If x and y satisfy the order $x \leqq y$, z satisfies the condition* $(\mathrm{I})_{x,y}$ *and $z \leqq x \wedge y$, then $z = x \wedge y = y \wedge x$.*

Proposition 3.12 *Let $z \leqq x$, $z \leqq y$. Then $z \wedge x = x \wedge z$ is equivalent to $z \leqq y \wedge x$.*

Proof By $z \leqq x$ and Proposition 3.6, we have $z = x \wedge z$.
Let $x \wedge z = z \wedge x$. Then

$$z \wedge x = x \wedge z = z. \tag{3.21}$$

By $z \leqq y$ and (2) in Proposition 2.2, we have $x * y \leqq x * z$.

Again, using (2) in Proposition 2.2, we have $x * (x * z) \leqq x * (x * y)$.
Then, we obtain

$$z \wedge x \leqq y \wedge x \tag{3.22}$$

Hence, (3.21), (3.22) lead to $z \leqq y \wedge x$.

Conversely, by $z \leqq x$ and Proposition 3.6, we have

$$z = x \wedge z. \tag{3.23}$$

By (3) in Proposition 2.2,

$$\begin{aligned}(z \wedge x) * z &= \{x * (x * z)\} * z \\ &= (x * z) * (x * z) \\ &= 0\end{aligned}$$

Then, we have

$$z \wedge x \leqq z. \tag{3.24}$$

On the other hand, for any y with $z \leqq y$, by the assumption $z \leqq y \wedge x$, we have

$$z * (y \wedge x) = 0.$$

If $y = z$, then $z * (z \wedge x) = 0$, this implies

$$z \leqq z \wedge x. \tag{3.25}$$

Hence, by (3.24), (3.25), we obtain

$$z = z \wedge x. \tag{3.26}$$

Therefore, by (3.23), (3.26), $x \wedge z = z \wedge x$.

□

Using Proposition 3.3 and Proposition 3.12, we have the next theorem. This theorem gives a characterization of the commutativity of any x, y in a BCK-algebra X.

Theorem 3.13 *For x, y, z in a BCK-algebra X, x and y are commutative if and only if there exists an element z satisfying the condition $(I)_{x,y}$ and $z \wedge x = x \wedge z$, $z \wedge y = y \wedge z$.*

We wish to characterize $x \wedge y = y \wedge x$ by using only the condition $(\mathrm{I})_{x,y}$, but the above result is obtained at present.

4 BCK-algebras with Condition (I) and lower semilattices

As are described in (1) of Proposition 2.2, BCK-algebras are partially ordered sets with respect to the relation $\leqq$. Then, we define the next.

Definition 4.1 In a BCK-algebra X, we called that the relation $\leqq$ is *BCK-order.* A BCK-algebra X is called *a lower BCK-semilattice* if when X is a lower semilattice with respect to BCK-order $\leqq$.

We will classify this class further using the condition $(\mathrm{I})_{x,y}$. In the following, let X be a BCK-algebra and any x, y, z in X.

Definition 4.2 If z is the only one element satisfying the condition $(\mathrm{I})_{x,y}$, then we say that z *satisfies the single condition* $(\mathrm{I})_{x,y}$.

Definition 4.3 If z satisfies the following two requirement i) and ii) ;

i) z is the only one element satisfying the condition $(\mathrm{I})_{x,y}$,
ii) $z = x \wedge y = y \wedge x$.

then we say that z *satisfies the canonical condition* $(\mathrm{I})_{x,y}$.

By Definition 4.2 and Definition 4.3, we will define the two special classes in BCK-algebras.

Let X be a BCK-algebra.

Definition 4.4 For any x, y in X, if z exists in X and satisfies the canonical condition $(\mathrm{I})_{x,y}$, then X is called that *a BCK-algebra with Canonical Condition (I).*

Definition 4.5 For any x, y in X, if z exists in X and satisfies the single condition $(\mathrm{I})_{x,y}$, then X is called that *a BCK-algebra with Condition (I).*
At this time, we define $x \times y$ by z. Surely, $x \times y = y \times x$.

First, we will show the fundamental properties of BCK-algebras with Canonical Condition (I).

Theorem 4.6 *A commutative BCK-algebra is a BCK-algebra with Canonical Condition (I), and the converse also holds.*

Proof Let X be a commutative BCK-algebra. For any x, y in X, put

$$z = x \wedge y = y \wedge x. \tag{4.1}$$

Clearly, the element z satisfies requirement ii).
And more, by Proposition 3.2, the element z satisfies the condition $(\mathrm{I})_{x,y}$. Here, if any u exists in X and satisfies the condition $(\mathrm{I})_{x,y}$, by (3) in Proposition 3.3, we have

$$y \wedge x \leqq u, x \wedge y \leqq u. \tag{4.2}$$

In addition, since X is a commutative BCK-algebra, we have $x \wedge u = u \wedge x, y \wedge u = u \wedge y$. By Proposition 3.12, this implies

$$u \leqq y \wedge x,\ u \leqq x \wedge y. \tag{4.3}$$

Hence, by (4.2), (4.3), implies

$$u = y \wedge x = x \wedge y. \tag{4.4}$$

By (4.1), (4.4), we have $u = z$. Therefore, this shows the requirement i).

The other, the converse is clear. We complete the proof of Theorem 4.6.

□

Theorem 4.7 *A lower semilattice with respect to the operation $\wedge$ is a BCK-algebra with Canonical Condition (I), and the converse also holds.*

Proof Let $X =< X; \wedge, 0 >$ be a lower semilattice for the operation $\wedge$. For any x, y in X, $x \wedge y$ is the greatest lower bound of x and y. Then, put $z = x \wedge y = y \wedge x$. The requirement ii) is clear.

And more, by Proposition 3.1, the greatest lower bound $x \wedge y$ satisfies the condition $(\mathrm{I})_{x,y}$. If any u satisfies the condition $(\mathrm{I})_{x,y}$, by (3) in Proposition 3.3, this implies

$$x \wedge y \leqq u. \tag{4.5}$$

Since X is a lower semilattice for the operation $\wedge$, and $u \leqq x$, we have

$$u = u \wedge x = x \wedge u. \tag{4.6}$$

From (4.6) and Proposition 3.12,

$$u \leqq y \wedge x = x \wedge y. \tag{4.7}$$

Thus, by (4.5), (4.7), $u = x \wedge y$. This shows the requirement i).

Conversely, let X be a BCK-algebra with Canonical Condition (I). For any x, y in X, the element z exists in X and satisfies the requirement i), ii) of Definition 4.2. By Proposition 3.5, we have that the commutative element z is the greatest lower bound of x and y. Therefore, we complete the proof of Theorem 4.7.

□

Clearly, Theorem 4.6 and Theorem 4.7 implies Tanaka's theorem (Theorem 2.3) and the inverse of Theorem 2.3. Consequently, we get the canonical condition $(\mathrm{I})_{x,y}$ in Definition 4.3 is an equivalent to the commutativity in BCK-algebras.

Secondly, we will show the following fundamental property of BCK-algebras with Condition (I).

Theorem 4.8 *A lower semilattice with respect to the operation $\times$ is a BCK-algebra with Condition (I), and the converse also holds.*

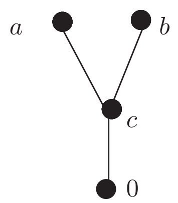
Proof Let $X =< X; \times, 0 >$ be a lower semilattice for the operation $\times$. For any x, y in X, the greatest lower bound of x and y exists in X. This element is defined by $x \times y$ in Definition 4.5. By the assumption, $x \times y$ is the only one element satisfying the condition $(\mathrm{I})_{x,y}$.

Conversely, let X be a BCK-algebra with Condition (I). For any x, y in X, there exists the only one element z in $I(x, y)$. By Proposition 3.5, this implies that z is the greatest lower bound of x and y. Then, we denote $z = x \times y$. The converse is showed. Therefore, we complete the proof of Theorem 4.8.

□

5 Examples of BCK-algebras

Example 5.1 Let $X = \{0, a, b, c\}$ be a set with four elements. And more, the set X satisfies the following Hasse daiagram: $Figure.1$, as BCK-order.



$Figure.1$

At this time, there are four types in the role of algebraic structure of non-isomorphic BCK-algebras on the partially ordered set X: $Figure.1$, as follow four Cayley Tables 1.A, 1.B, 1.C and 1.D exist.

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	c	0	c
b	b	c	c	0

Table 1.A

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	c	0	c
b	b	b	b	0

Table 1.B

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	a	0	c
b	b	c	c	0

Table 1.C

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	a	0	a
b	b	b	b	0

Table 1.D

The above four BCK-algebras are written as follow $X_A =< X; *, 0 >, X_B =< X; *, 0 >, X_C =< X; *, 0 >$ and $X_D =< X; *, 0 >$ in the order of Cayley Table 1.A, 1.B, 1.C and 1.D.

First, for the BCK-algebra X_A given in Table 1.A, the next Table1.A-$\wedge$ is available for the operation $\wedge$.

$\wedge$	0	c	a	b
0	0	0	0	0
c	0	c	c	c
a	0	c	a	c
b	0	c	c	b

Table 1.A-$\wedge$

Thus, the BCK-algebra X_A is commutative. Further, the operation $\wedge$ matches the lattice-meet $\cap$ obtain from the Hasse diagram: $Figure.1$.

Next, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_A. The results is as follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{c\}$

Table 1.A-(I)

Therefore, the BCK-algebra X_A is a BCK-algebra with Condition (I). Clearly, the operation$\times$ matches the lattice-meet $\cap$, and this algebra also coincides with the operation $\wedge$. Then, the BCK-algebra X_A is a BCK-algebra with Canonical Condition (I).

Second, for the BCK-algebra X_B given in the Table 1.B, the following relation holds for the operation $\wedge$.

$$c = c * (c * b) = b \wedge c \neq c \wedge b = b * (b * c) = 0,$$

$$c = a * (a * b) = b \wedge a \neq a \wedge b = b * (b * a) = 0.$$

Therefore, the BCK-algebra X_B is not commutative. However, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_B. The result is as follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{c\}$

Table 1.B-(I)

From this table, any two elements in the BCK-algebra X_B satisfy the single condition $(\mathrm{I})_{x,y}$. Then, the BCK-algebra X_B is a BCK-algebra with Condition (I). Therefore, the following the Table 1.B-$\times$ is obtained with respect to the operation $\times$.

$\times$	0	c	a	b
0	0	0	0	0
c	0	c	c	c
a	0	c	a	c
b	0	c	c	b

Table 1.B-$\times$

This table matches the lattice-meet $\cap$ that obtained from the Hasse diagram: $Figure.1$.

Similarly, for the BCK-algebra X_C is given in the Table 1.C, the following relations hold for the operation $\wedge$.

$$c = c * (c * a) = a \wedge c \neq c \wedge a = a * (a * c) = 0,$$

$$c = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a.$$

Therefore, the BCK-algebra X_C is not commutative. However, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_C. The result is follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,y\}$	$\{c\}$

Table 1.C-(I)

From this table, X_C is a BCK-algebra with Condition (I). This table is consistent with the lattice-meet $\cap$ obtained from the Hasse diagram: $Figure.1$.

Thirdly, for the operation $\wedge$ in the BCK-algebra X_D, we have the following relation.

$$0 = a * (a * c) = c \wedge a \neq a \wedge c = c * (c * a) = c,$$

$$0 = b * (b * c) = c \wedge b \neq b \wedge c = c * (c * b) = c$$

Then, the BCK-algebra X_D is not commutative. Further, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_D, the following is obtained.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{0,c\}$

Table 1.D-(I)

The following can be understood. In the BCK-algebra X_D, there are two elements satisfying the condition $(\mathrm{I})_{x,y}$ for a and b.

$$a = a * 0 \leqq a * b = a,\ \ b = b * 0 \leqq b * a = b,$$

$a = a * c \leqq a * b = b, \ b = b * c \leqq b * a = b.$

Therefore, two elements a and b don't satisfy the single condition $(\mathrm{I})_{x,y}$, then the BCK-algebra X_D is not a BCK-algebra with Condition (I).

Example 5.2 Let $Y = \{0, x, y, 1\}$ be a set with four elements. And more, the set Y satisfies the following totally order $0 \leqq a \leqq b \leqq 1$, as BCK-order of Y.

At this time, there are six types in the role of algebraic structure of non-isomorfic BCK-algebras on the totally ordered set Y, as follow Cayley Table 2.A, 2.B, 2.C, 2.D, 2.E and 2.F exist.

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	b	a	0

Table 2.A

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Table 2.B

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

Table 2.C

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	b	0

Table 2.D

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	1	1	0

Table 2.E

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	1	1

Table 2.F

The above six BCK-algebras are written as follow $Y_A =< Y; *, 0 >, Y_B =< Y; *, 0 >, Y_C =< Y; *, 0 >, Y_D =< Y; *, 0 >, Y_E =< Y; *, 0 >$ *and* $Y_F =< Y; *, 0 >$ in order of Cayley Table 2.A, 2.B, 2.C, 2.D, and 2.F.

First, for the BCK-algebra Y_A given in Table 2.A, the following Table 2. A-$\wedge$ is available for the operation $\wedge$.

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 2.A-$\wedge$

Therefore, the BCK-algebra Y_A is commutative. Further, the operation $\wedge$ matches the lattice-meet $\cap$ obtain from BCK-order of Y.

Next, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in Y_A. The result is the following.

$\{,\}$	$I(x,y)$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{0,1\}$	$\{0\}$
$\{a,b\}$	$\{a\}$
$\{a,1\}$	$\{a\}$
$\{b,1\}$	$\{b\}$

Table 2.A-(I)

Therefore, the BCK-algebra Y_A is a BCK-algebra with Condition (I). Clearly, the operation $\times$ matches the lattice-meet $\cap$, and this algebra also coincides with the operation $\wedge$. Then, the BCK-algebra Y_A is a BCK-algebra with Canonical Condition (I).

Second, for the BCK-algebras Y_B, Y_C, Y_D, Y_E, and Y_F given in the Table 2.B, Table 2.C, Table 2.D, Table 2.E and Table 2.F, the following relation holds for the operation $\wedge$.

In the BCK-algebra Y_B,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a\ ;$$

In the BCK-algebra Y_C,

$$a = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b\ ;$$

In the BCK-algebra Y_D,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a,$$

$$0 = 1 * (1 * a) = x \wedge 1 \neq 1 \wedge a = a * (a * 1) = b\ ;$$

In the BCK-algebra Y_E,

$$0 = 1 * (1 * a) = a \wedge 1 \neq 1 \wedge a = a * (a * 1) = a,$$

$$0 = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b\ ;$$

In the BCK-algebra Y_F,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a,$$

$$0 = 1 * (1 * a) = a \wedge 1 \neq 1 \wedge a = a * (*1) = a.$$

$$0 = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b\ ;$$

Therefore, the BCK-algebras Y_B, Y_C, Y_D, Y_E and Y_F are not commutative.

Next, we consider that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in Y_B. This is clear from the next Corollary 5.3. Because of that the set Y is a totally ordered set, the Proposition 3.10 implies the Corollary 5.3.

Corollary 5.3 *For the totally ordered sets, the algebraic structure of the BCK-algebra with Condition (I) is always given.*

Then, the BCK-algebras Y_B, Y_C, Y_D, Y_E, and Y_F are BCK-algebras with Condition (I).

6 Additional Remarks

P. M. Idziak showed that a lower BCK-semilattice is as a variety in the article [1]. Then, a BCK-algebra with Condition (I) should be as a variety. Therefore, we should be able to define this class with the condition for only identities including the operation $\times$. However, this condition expression has not been provided yet. If we can this condition expression, we will investigate into BCK-algebras with Condition (I) more deeply.

In addition, we don't know whether the BCK-algebras with Condition (I) is the maximum class in lower BCK-semilattices.

Acknowledgement The author would like to express his hearty thanks to Professor Kimiaki Narukawa in Naruto University of Education for his constant encouragement and kind discussion.

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Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
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(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73 − age) × ￥3,000

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