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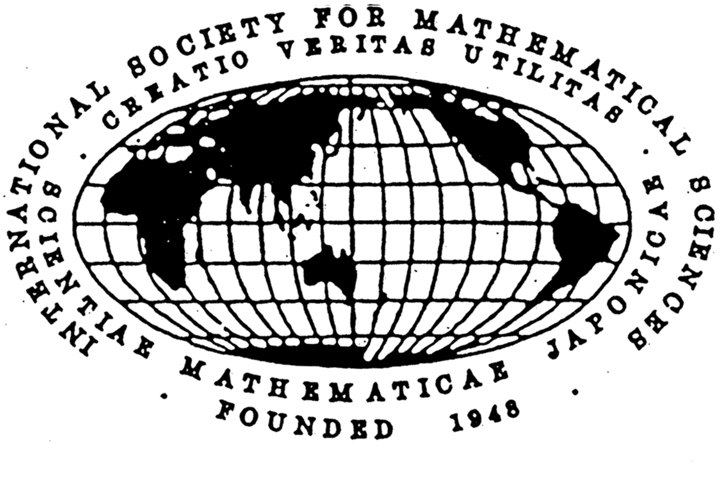

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ONE-DIMENSIONAL CACTOIDS AND UNIVERSALITY

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Abstract. We present some properties of one-dimensional cactoids and construct a universal element Z for the family of one-dimensional cactoids X such that a simple cyclic chain between any two cut points of X is a cactus. One-dimensional cactoids are partial case of planar totally regular curves and are investigated by Whyburn [13] under the term “boundary curves”.

1 Introduction.

In this paper under the term *continuum* is meant a nonempty, compact and connected metric space. A curve is a one-dimensional continuum.

A continuum Z is *universal* for a class $\mathcal{F}$ of continua provided that $Z \in \mathcal{F}$ and each member of $\mathcal{F}$ can be homeomorphically imbedded in Z. A space is *planar* if it is homeomorphic to a subset of the plane.

A *Peano continuum* is a locally connected continuum.

We will use the results of the papers of 1920s (see [2], [10], [11]) in which under the term *continuous curve* was meant a metric space X that is a continuous image of segment $[0,1]$. According to Hahn–Mazurkiewicz Theorem (see [13, (4.1). p. 92]) the above condition for X is equivalent to the property of X to be a Peano continuum.

The order of a space X at the point $p \in X$, written $\operatorname{ord}(p,X)$, is the least cardinal or ordinal number $\mathfrak{m}$ such that p has an arbitrary small open neighborhood in X with boundary of cardinality $\leq \mathfrak{m}$. In particular, $\operatorname{ord}(p,X) = \omega$, where ω denotes the least infinite ordinal number, if p has arbitrary small open neighborhoods in X with finite boundaries but $\operatorname{ord}(p,X) > n$ for every natural number n [6, §51, I, p. 274].

The points of $B(X) = \{x \in X : \operatorname{ord}(p,X) \geq 3\}$ are called *branch points* of X and the points of $E(X) = \{x \in X : \operatorname{ord}(p,X) = 1\}$ are called *end points* of X.

A point p of a connected space X is *a cut point* if $X \setminus \{p\}$ is not connected. The set of all cut points of a connected space X will be denoted by $c(X)$.

A *simple closed curve* is a space homeomorphic to the circle. An *arc* is a space A homeomorphic with a segment $[0,1]$. The arc A with end points p and q is written pq. An arc $pq \subseteq X$ is called *free* in X if the set $(pq) = pq \setminus \{p,q\}$ is an open subset of X.

A continuum X is said to be *cyclicly connected* provided that every two points of X lie together on some simple closed curve of X. By a *cyclic element* of Peano continuum X will be meant a cut point of X, an end point of X, or a nondegenerate cyclicly connected Peano subcontinuum M of X such that M is not a proper subset of any other cyclicly connected Peano subcontinuum of X. Any nondegenerate cyclic element of X is called *true cyclic element* of X.

A Peano continuum each true cyclic element of which is homeomorphic to a simple closed curve is called *a one-dimensional cactoid* [13]. The property of a Peano continuum M to be a one-dimensional cactoid is equivalent with any of following properties:

(i) No two simple closed curve of M have more than one point in common.

(ii) M contains no θ-curves.

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A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points [7]. A *cactus* is a graph in which any two simple closed curves have at most one point in common [9]. Clearly, a cactus is a cactoid that is a graph.

A simple cyclic chain of Peano continuum X between two of its cyclic elements E_1 and E_2 is a connected subset S that is a union of some family $\mathcal{F}$ of cyclic elements of X such that $E_1, E_2 \in \mathcal{F}$ and no proper connected subset of S containing E_1 and E_2 is the sum of cyclic elements (see [11]). Note that a simple cyclic chain between any two cyclic elements of Peano continuum is uniquely determined [11, Theorem 3].

The main result of the paper is a construction of a universal cactoid Z for the class of all one-dimensional cactoids X such that a simple cyclic chain between any two cut points of X is a cactus.

2 One-dimensional cactoids as a boundary curves. Let X is a Peano continuum of the plane $\mathbf{P}$. Any component of $\mathbf{P} \setminus X$ is called *complementary domain* of X. The boundary of any complementary domain of X is a subcontinuum of X and is called a *boundary curve.*

Wilder in [10, Theorem 17] proved the following result:

Theorem 2.1. *If a Peano continuum M is a boundary of complementary domain of a Peano continuum, then M is the union of disjoint families of sets S_1, S_2 and P, where:*

(1) *S_1 is a countable set of all simple closed curves contained in M no two of which have more than one point in common,*

(2) *S_2 is a countable set of arcs no two of which have in common an interior point of both, and*

(3) *$P = M \setminus (S_1 \cup S_2)$ is a totally disconnected set of limit points of $S_1 \cup S_2$.*

From Theorem 2.1 it follows that:

Corollary 2.1.1. *Each boundary curve is a one-dimensional cactoid.*

The fact that any one-dimensional cactoid is planar follows from the result of Ayres [2, Theorem in page 92]:

Theorem 2.2. *In order that a Peano continuum M be homeomorphic with a plane Peano continuum which is the boundary of one of its complementary domains it is necessary and sufficient that every true cyclic element of M be a simple closed curve.*

From Theorem 2.2 it also follows that:

Corollary 2.2.1. *A Peano continuum M is a one-dimensional cactoid if and only if M is homeomorphic with a plane Peano continuum which is the boundary of one of its complementary domains.*

A continuum K is said to be *regular* if K has a basis of open sets with finite boundaries. Any regular continuum is hereditarily locally connected [6, §51, IV, Theorem 2, p. 283]. Since a one-dimensional cactoid contains no θ-curves, it follows that (see [6, §52, IV, Theorem 3, p. 329]):

Corollary 2.2.2. *Any one-dimensional cactoid X is regular and any connected subset of X is arcwise connected.*

A metric space (X, d) is *uniformly locally arcwise connected* provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) \in (0, \delta)$, then x and y can be joined by an arc of diameter $< \varepsilon$. Any Peano continuum is uniformly locally arcwise connected [6, §50, II, Theorem 4, p. 257], hence:

Corollary 2.2.3. *Any one-dimensional cactoid is uniformly locally arcwise connected.*

3 Properties of one-dimensional cactoid. Let $\mathbb{N} = \{0, 1, \dots, n, \dots\}$. Given a subset G of a space X the closure and the boundary of G in X will be denoted by $cl_X(G)$ (or $cl(G)$) and $bd_X(G)$ (or $bd(G)$), respectively.

Proposition 3.1. *Each branch point of a one-dimensional cactoid X is a cut point.*

Proof. Let $r \in B(X)$. From the Menger n-Beinsatz (see [6, p. 277]), it follows that there exist arcs $A_1 = rx_1$, $A_2 = rx_2$, and $A_3 = rx_3$ of X having the unique point r in common.

Suppose, on the contrary, that r is not a cut point. Then the connected subset $X \setminus \{r\}$ of X is arcwise connected. Thus there exists an arc $A = x_1x_2 \subseteq X \setminus \{r\}$. Since $A \cap A_1$ is a compact subset of $A_1 \setminus r$, the component of $A_1 \setminus A$ containing r is a subarc $B_1 = rb_1$ of A_1 such that $B_1 \cap A = b_1$. Similarly, there exists a subarc $B_2 = rb_2$ of A_2 such that $B_2 \cap A = b_2$. Let $B = b_1b_2$ is a unique determined subarc of A joining the points b_1 and b_2. Then $B \cup B_1 \cup B_2$ is a closed curve containing the points r and b_1 .

Since $x_3, b_1 \in X\setminus\{r\}$ and $X\setminus\{r\}$ is arcwise connected, there is an arc $C = b_1x_3 \subseteq X\setminus\{r\}$. It is easy to see that the set $B \cup B_1 \cup B_2 \cup C \cup A_3$ contains a θ-curve. Hence, X is not a cactoid which is a contradiction. ☐

Proposition 3.2. *The set of branch points of one-dimensional cactoid is countable.*

Proof. Let X be a cactoid. Since all save possibly a countable number of cut points of X are of order 2 [13, (3.2), p. 49] in X, the cut points of X of order ≥ 3 are countable. Hence, $B(X)$ is countable from Theorem 3.1. ☐

Definition 3.1. A subcontinuum G of one-dimensional cactoid X is called *full* provided that each simple closed curve of X either is a subset of G, or does not intersect G, or intersects G in a single point.

Theorem 3.1. *If X is a one-dimensional cactoid, then for any full subcontinuum G of X and for any $x \in X \setminus G$ there exist a point $r_x \in G$ and an arc A_x from x to r_x such that:*

(1) *$A_x \cap G = \{r_x\}$ and r_x is a unique point that belongs any arc of X from x to any point of G.*

(2) *If G_x is a component of $X \setminus G$ containing x, then $G \cap cl(G_x) = \{r_x\}$.*

(3) *The map $r : X \to G$ by $r(x) = \begin{cases} x, & \text{if } x \in G \\ r_x, & \text{if } x \in X \setminus G \end{cases}$ is continuous.*

Proof. (1) Consider any $r_0 \in G$. Since X is arcwise connected there is an arc $A_0 = xr_0 \subseteq X$. Let S_x be a component of $A_0 \setminus G$ containing x. Clearly S_x is a half-open subarc $[xr_x)$ of A_0, where $r_x \in A_0 \cap G$. Hence, $A_x = cl(S_x)$ is an arc from x to r_x and $A_x \cap G = \{r_x\}$.

Let $A_1 = xr$ be an arc of X from x to $r \in G$ and $\widetilde{S}_x$ be a component of $A_1 \setminus G$ containing x. As above for a point r_0 we can find a point $g \in G$ and an arc $A_g = gx \subseteq A_1$ such that $A_g \cap G = \{g\}$. Suppose on the contrary that $r_x \notin A_1$. Then $r_x \neq g$. Let S_{r_x} be a component of $A_x \setminus A_g$ containing r_x. Then $A_2 = cl(S_{r_x})$ is an arc from r_x to $b \in A_g \cap A_x$. Since $b, g \in A_g$, there exists an arc $A_3 = bg \subseteq A_g$. Since $r_x, g \in G$ and G is arcwise connected, there is an arc $A_4 = gr_x \subseteq G$. From the above a simple closed curve $A_2 \cup A_3 \cup A_4$ of X intersects G in arc gr_x which is a contradiction, because G is full subcontinuum of X. Hence, $r_x \in A_1$.

Suppose that $r \in G$ and r belongs to any arc from x to any point of G. Then $r \in A_x$. Since $A_x \cap G = \{r_x\}$, $r = r_x$.

(2) Clearly, $r_x \in A_x \subseteq cl(G_x)$. Suppose that there exists $p \in G \cap cl(G_x)$ with $p \neq r_x$. Since $p \notin A_x$, there exists an open and connected subset V_p of X such that $p \in V_p \subseteq X \setminus A_x$. Since $p \in Cl(G_x)$, there exists $q \in V_p \cap G_x$. Since V_p is arcwise connected from Corollary 2.2.2, there exists an arc $qp \subseteq V_p$. Since $x, q \in G_x$ and G_x is arcwise connected, there exists arc $xq \in G_x$. Then $xq \cup qp$ contains an arc A from x to $p \in G$. Hence $r_x \in A$ from condition 1. On the other hand $r_x \notin qp \cup xq$. Hence $r_x \notin A$, which is a contradiction.

(3) Let $g \in G$ and W_g be an open and connected neighborhood of $r(g) = g$ in X. To prove that r is continuous at g it suffices to show that $r(W_g) \subseteq W_g$. Indeed, for $x \in W_g \cap G$ we have $r(x) = x \in W_g$. For $x \in W_g \setminus G$ there exists an arc $A \subseteq W_g$ from x to g. Since $r(x) = r_x \in A$ from 1, $r(x) \in W_g$.

Let $x \in X \setminus G$ and G_x be a component of $X \setminus G$ containing x. Since X is locally connected, G_x is open. To prove the continuity of r in x, it suffices to show that $r(G_x) = \{r(x)\}$. Indeed, if $p \in G_x \setminus \{x\}$, then G_x is a component of $X \setminus G$ containing p. From condition 2 of the Theorem it follows that $\{r_p\} = cl(G_x) \cap G = \{r_x\}$. Thus $r(p) = r_p = r_x = r(x)$. □

Remark 3.1. The map r defined in Theorem 3.1 is a retraction. We will call r the *first point map corresponding to full subcontinuum G of X.*

Lemma 3.1. *If a simple cyclic chain between any two cut points of one-dimensional cactoid X is a cactus, then any simple cyclic chain of X that is a subset of $X \setminus E(X)$ is a cactus.*

Proof. Let $C \subseteq X \setminus E(X)$ be a simple cyclic chain between cyclic elements E_1 and E_2 of X. Then each of E_1 and E_2 is either a cut point or a simple closed curve. Suppose that E_1 and E_2 are simple closed curves. Then $E_1 \cap E_2$ consists of at most one point. If $E_1 \cap E_2 = \{p\}$, then $C = E_1 \cup E_2$ is a cactus.

Suppose that $E_1 \cap E_2 = \emptyset$. Consider the first point maps $r_1 : X \to E_1$ and $r_2 : X \to E_2$. From Theorem 3.1 there are $p \in E_2$ and $q \in E_1$ such that $r_1(E_2) = r_1(p)$ and $r_2(E_1) = r_2(q)$. Obviously, $C^* = (C \setminus (E_1 \cup E_2)) \cup \{r_1(p), r_2(p)\}$ is a simple cyclic chain between cut points $r_1(p)$ and $r_2(q)$ of X. Hence C^* and, therefore, $C = C^* \cup E_1 \cup E_2$ are cactuses.

The proof is similar in the case that exactly one of E_1 and E_2 is a cut point. □

Lemma 3.2. *Let X be a one-dimensional cactoid, Y a full subcontinuum of X and $r : X \to Y$ a first point map.*

If $x \in X \setminus Y$, S is a cyclic element of X containing x, and C is a simple cyclic chain between $r(x)$ and S, then $Y \cup C$ is full.

Proof. Let L be a simple closed curve of X that intersects $Y \cup C$. If L intersects Y, then $L \cap Y = \{y\}$ because Y is full. If in addition L intersects C, then $y = r(y) = r(C) = r(x) \in C$. We conclude that $L \cap (Y \cup C) \subseteq L \cap C$.

Suppose, on the contrary, that $L \cap (Y \cup C)$ contains two points z and w. Then $z, w \in L \cap C$. Thus there exists an arc $A = zw \subseteq C$. Since X contains no θ-curves, $A \subseteq L$ and L is a unique simple closed curve containing A. Suppose that $q \in A$ with $\mathrm{ord}(q, X) = 2$. Since C is a union of cyclic elements, it follows that q is a cyclic element. Thus $q \in c(X)$. Hence, $X \setminus \{q\}$ contains at least two component. Since q does not separate L it follows that $L \setminus \{q\}$ is containing in some component W_1 of $X \setminus \{q\}$. Let w belongs to a component $W_2 \neq W_1$ of $X \setminus \{q\}$. Then there exists an arc $B = wq \subseteq W_2 \cup \{q\}$. Then $B \cap L = \{q\}$ and we conclude that $\mathrm{ord}(q, X) = 3$, which is a contradiction.

□

Lemma 3.3. *If X is a Peano continuum, then $X \setminus E(X)$ is dense in X.*

Proof. Let $U \neq \emptyset$ be an open subset of X. Since X is locally connected, there exists an open and connected set $V \neq \emptyset$ such that $V \subseteq U$. There exists an arc $ab \subseteq V$ [7, Theorem 8.26]. Then $ord(p, X) \geq ord(p, ab) = 2$ for $p \in (ab)$. Clearly, $p \in U \cap (X \setminus E(X))$. □

It is easy to prove the following Lemma.

Lemma 3.4. *If a cactus K is a simple cyclic chain between two of its cyclic elements, then $K = \bigcup_{j=1}^{n} C_j$, where $n \in \mathbb{N} \setminus \{0\}$ and each C_j is either a simple closed curve or a maximal free arc of K. Moreover, if $n \geq 2$, then*

(i) $C_j \cap C_{j+1} = \{b_j\}$ for $j = 1, ..., n-1$, where $b_j \in B(K)$, and

(ii) $C_j \cap C_i = \emptyset$ for $|i - j| > 2$.

Theorem 3.2. *Let X be a one-dimensional cactoid such that a simple cyclic chain between any two cut points of X is a cactus.*

Then there exists a sequence $\{Y_k\}_{k=1}^{\infty}$ of full cactuses of X such that

(i) $Y_1 = \{p_1\}$ or Y_1 is a simple closed curve;

(ii) $E(Y_k) \subseteq c(X)$ (including the case $E(Y_k) = \emptyset$);

(iii) $Y_k \subseteq Y_{k+1}$;

(iv) $cl(Y_{k+1} \setminus Y_k) \cap Y_k = \{p_k\}$ and $p_k \in c(X)$;

(v) $\lim Y_k = X$;

(vi) if $r_k : X \to Y_k$ is the first point map for $k = 1, 2, ...$, then the sequence of retractions $\{r_k\}_{k=1}^{\infty}$ converges uniformly to id_X.

Proof. Since X is separable, from Lemma 3.3 it follows that there exists a dense subset $\{x_i\}_{i=1}^{\infty}$ of X such that $\{x_i\}_{i=1}^{\infty} \subseteq X \setminus E(X)$.

Let Y_1 be a maximal cyclic element of X containing x_1. From definition of cyclic element it follows that either Y_1 is a simple closed curve or $Y_1 = \{x_1\}$ and $x_1 \in c(X)$.

Consider the first point map $r_1 : X \to Y_1$. Put $m_1 = \min\{i : x_i \notin Y_1\}$ and $r_1(x_{m_1}) = \{p_1\}$. Then either $p_1 = x_1$ or Y_1 is a simple closed curve and $p_1 \in Y_1 \cap B(X)$. In any case $p_1 \in c(X)$.

Let S_1 be the maximal cyclic element of X containing x_{m_1}. Either S_1 is a simple closed curve or $S_1 = \{x_{m_1}\}$ and $x_{m_1} \in c(X)$. Let C_1 be a cyclic chain between cyclic elements p_1 and S_1. From Lemma 3.1 C_1 is a cactus. Let $Y_2 = Y_1 \cup C_1$. By Lemma 3.2, Y_2 is a full subcontinuum of X. Since Y_1 is full, $x_{m_1} \in Y_2 \setminus Y_1$ and $Y_2 \setminus Y_1$ is a connected subset (see [11, Theorem 6]) of $X \setminus Y_1$, from Theorem 3.1(4) $Y_1 \cap cl(Y_2 \setminus Y_1) = \{p_1\}$. Obviously, $E(Y_2) \subseteq \{x_1, x_{m_1}\} \subseteq c(X)$.

Suppose that cactuses $Y_1, ..., Y_k$ with properties $(i) - (iv)$ have been defined.

Consider the first point map $r_k : X \to Y_k$. Let $m_k = \min\{i : x_i \notin Y_k\}$ and $r_k(x_{m_k}) = p_k \in Y_k$. If $p_k \in E(Y_k)$, then $p_k \in c(X)$ by inductive assumption. Otherwise, p_k is a branch point and, therefore, $p_k \in c(X)$ from Theorem 3.1. Let S_k be a maximal cyclic element of X containing x_{m_k} and C_k be a cyclic chain between cyclic elements p_k and S_k. Similarly as for Y_2 it can be shown that Y_{k+1} is full and satisfies the properties $(i) - (iv)$ of the Theorem.

To prove (v), set $A_k = \{x_1, ..., x_k\}$. Since $A_k \subseteq A_{k+1}$ and $cl(\{x_i\}_{i=1}^{\infty}) = X$, it follows that $\lim A_k = X$. Since $x_k \leq x_{m_k}$ and $A_{m_k} \subseteq Y_{k+1}$, it follows that $A_k \subseteq Y_{k+1} \subseteq X$. Thus $\lim Y_k = \lim A_k = X$.

In order to prove (vi) we consider the Hausdorff metric H_d generated on the set of closed subsets of X by metric d of X. Then

$$H_d(X, Y_k) = \inf\{\varepsilon^* > 0 : X \subseteq \bigcup_{y \in Y_k} B_d(y, \varepsilon^*)\},$$

where $B_d(y, \varepsilon^*) = \{x \in X : d(y, p) < \varepsilon^*\}$. Let $\varepsilon > 0$. Since X is uniformly locally arcwise connected from Corollary 2.2.3, there exists $\delta > 0$ such that if $x, y \in X$, and $0 < d(x, y) < \delta$, then there exists an arc $A = xy$ with diameter $< \varepsilon$. Since $\lim Y_k = X$ from (v), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $H_d(X, Y_k) < \delta$. Thus

$$X \subseteq \bigcup_{y \in Y_k} B_d(y, \delta) \text{ for any } k \geq k_0.$$

Let $x \in X$ and $k \geq k_0$. Then there exists $y_k \in Y_k$ such that $x \in B_d(y_k, \delta)$. Hence, x and y_k can be joined by arc A_x^k of diameter $< \varepsilon$. Since $y_k \in Y_k$ and $r_k(x)$ belongs to any arc from x to any point of Y_k, $r_k(x) \in A_x^k$. Since $x, r_k(x) \in A_x^k$, we conclude that

$$d(id_X(x), r_k(x)) = d(x, r_k(x)) \leq \operatorname{diam}(A_x^k) \leq \varepsilon.$$

□

Theorem 3.3. *[7, 2.29] Let Y be a compact metric space, and let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of Y such that, for each $i = 1, 2, ...$, there are continuous and onto functions $\psi_i : Y_{i+1} \to Y_i$ and $r_i : Y \to Y_i$ such that $\psi_i \circ r_{i+1} = r_i$. If $\{r_i\}_{i=1}^{\infty}$ converges uniformly to the identity map id_Y on Y, then Y is homeomorphic to inverse limit $\varprojlim\{Y_i, \psi_i\}_{i=1}^{\infty}$.*

The following Theorem follows directly from Theorems 3.2 and 3.3

Theorem 3.4. *If X is a one-dimensional planar cactoid such that any two cut points of X can be joined by a simple cyclic chain that is a cactus and $\{Y_k\}_{k=1}^{\infty}$ is the sequence of cactuses satisfying Theorem 3.2, then X is homeomorphic to $X_\infty = \varprojlim\{Y_k, \psi_k\}$, where $\psi_k = r_k\big|_{Y_{k+1}} : Y_{k+1} \to Y_k$, $k = 1, 2, ...$*

Theorem 3.5. *Let X be one-dimensional planar cactoid such that any two cut points can be joined by a simple cyclic chain that is a cactus.*

Then there exists an inverse sequence $\{X_i, g_i\}_{i=1}^{\infty}$ such that

- *(i) X_i is a full cactus and $g_i : X_{i+1} \to X_i$ is a monotone retraction;*
- *(ii) X_1 is a point or a simple closed curve;*
- *(iii) $X_i \subseteq X_{i+1}$ and there exists a unique point $t_i \in X_i$ such that $g_i^{-1}(t_i)$ is non degenerate and is either a simple closed curve or a free arc whose end points are in $c(X)$;*
- *(iv) X is homeomorphic to $\varprojlim\{X_i, g_i\}$.*

Proof. From Theorem 3.4, X is homeomorphic to $\varprojlim\{Y_k, \psi_k\}$, where $\{Y_k\}_{k=1}^{\infty}$ is the sequence of cactuses satisfying Theorem 3.2 and $\psi_k = r_k\big|_{Y_{k+1}}$.

Clearly, each $\psi_k : Y_{k+1} \to Y_k$ is a monotone retract.

From Theorem 3.2 there is a unique point $p_k \in Y_k$ for which $\psi_k^{-1}(p_k)$ is non degenerate. Also there exits $x_{m_k} \in X \setminus E(X)$ for which $\psi_k^{-1}(p_k) = cl(Y_{k+1} \setminus Y_k)$ is a cactus that is a simple cyclic chain from $p_k \in c(X)$ to the maximal cyclic element S_k of x_{m_k}. From Lemma

3.4 it follows that $\psi_k^{-1}(p_k) = \bigcup_{j=1}^{n_k} C_j^k$, where each C_j^k is either a simple closed curve or a maximal free arc of K. Moreover, if $n_k \geq 2$, then $C_j^k \cap C_{j+1}^k = \{b_j^k\}$ for $j = 1, ..., n_k - 1$ where $b_j \in B(X)$, and $C_j^k \cap C_i^k = \emptyset$ for $|i - j| > 2$.

For $k = 1$ we obtain $\psi_1^{-1}(p_1) = \bigcup_{j=1}^{n_1} C_j^1$. We define

$$X_1 = Y_1,\ X_2 = Y_1 \cup C_1^1,\ X_3 = X_2 \cup C_2^1, \dots, X_{1+n_1} = X_{n_1} \cup C_{n_1}^1 = Y_2.$$

From Theorem 3.2 the set X_1 is a point or a simple closed curve.

Put $t_1 = p_1$ and $t_j = b_{j-1}^1$ for $j = 2, ..., n_1$. Let $g_j : X_{j+1} \to X_j$, $j = 1, \dots, n_1$, be the first point map. Then $g_j^{-1}(t_j) = C_j^1$ for $j = 1, \dots, n_1$.

Let $i > n_1 + 1$ be a positive integer. There exist a unique $k(i) \in \{1, 2, \dots\}$ and a unique $m(i) \in \{1, \dots, n_{k(i)}\}$ such that $i = 1 + n_1 + \dots + n_{k(i)-1} + m(i)$. We define $X_i = Y_k \cup \left(\bigcup_{j=1}^{m(i)} C_j^{k(i)}\right)$. If $m(i) = 1$, then we define $t_i = p_{k(i)}$. Otherwise we define $t_i = b_{m(i)-1}^{k(i)}$. Let $g_{i-1} : X_i \to X_{i-1}$ be the first point map. Then $g_{i-1}^{-1}(t_i) = C_{m(i)}^{k(i)}$. Clearly, the condition $(i) - (iii)$ are satisfied.

To prove (iv) we observe that the inverse sequence $\{Y_k, \psi_k\}$ is confinal in the sequence $\{X_i, g_i\}$. Hence the inverse limits $\varprojlim\{X_i, g_i\}$ and $\varprojlim\{Y_k, \psi_k\}$ are homeomorphic [5, Corollary 2.5.11, page 102] . Since X is homeomorphic to $\varprojlim\{Y_k, \psi_k\}$, it follows that X is homeomorphic to $\varprojlim\{X_i, g_i\}$. □

4 Construction of universal space Z. Let $\mathbf{P}$ denote the plane with a system Oxy of orthogonal coordinates and a metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + y_1 - y_2)^2}$.

For any finite subset $\mathcal{V}$ of $\mathbf{P}$ we set

$$\mathrm{mesh}(\mathcal{V}) = \min\{d(x, y) : x, y \in \mathcal{V}, x \neq y\}.$$

For any finite family of subsets $\mathcal{G}$ of $\mathbf{P}$ we set

$$\mathrm{mesh}(\mathcal{G}) = \max\{\mathrm{diam}(G) : G \in \mathcal{G}\}.$$

Given a segment $E = \overline{pq}$ of $\mathbf{P}$ we denote by m_E the midpoint of E and define $\mathcal{E}(E) = \{\overline{pm}_E, \overline{qm}_E\}$.

Triangle of $\mathbf{P}$ with vertexes v_1, v_2, v_3 is the set $\overline{v_1v_2} \cup \overline{v_2v_3} \cup \overline{v_1v_3}$. For any triangle T of the plane we denote by $\mathcal{V}(T)$ the set of vertexes of T, by $\mathcal{E}(T)$ the set of sides of T, and by $\widehat{T}$ the 2-simplex of $\mathbf{P}$ with boundary T.

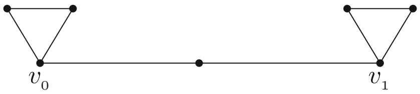
We will construct a sequence of cactuses $\{Z_i\}_{i=0}^{\infty}$ in $\mathbf{P}$ and monotone and surjective mappings $f_i : Z_{i+1} \to Z_i$ such that $Z_i \subseteq Z_{i+1}$ for each i. Our method is similar to construction of Ważewski's Universal Dendrite [7].

Consider the points $v_0 = (0, 0)$ and $v_1 = (1, 0)$ of $\mathbf{R}^2$. Set $Z_0 = \overline{v_0v_1}$, $\mathcal{E}_0 = \{\overline{v_0v_1}\}$, $\mathcal{V}_0 = \{v_0, v_1\}$, and $\varepsilon_0 = \frac{1}{2}$. Consider a family of disjoint triangles $\mathcal{T}_1 = \{T_v^1\}_{v \in V_0} \subseteq \mathbb{R}^2$ such that: v is a vertex of T_v^1, $T_v^1 \cap Z_0 = \{v\}$, and $T_v^1 \subseteq B(v, \frac{\varepsilon_0}{2})$. We define $Z_1 = Z_0 \cup \big(\bigcup_{v \in V_0} T_v^1\big)$ and $f_0 : Z_1 \to Z_0$ by

$$f_0(z) = \begin{cases} v, & \text{if } z \in T_v^1,\ v \in \mathcal{V}_0, \\ z, & \text{if } z \in Z_0. \end{cases}$$

Put

$$\begin{aligned}\mathcal{E}_1 &= \big(\bigcup_{E\in\mathcal{E}_0}\mathcal{E}(E)\big)\cup\big(\bigcup_{v\in\mathcal{V}_0}\mathcal{E}(T_v^1)\big)\\ \mathcal{V}_1 &= \{m_E\}_{E\in\mathcal{E}_0}\cup\big(\bigcup_{v\in\mathcal{V}_0}\mathcal{V}(T_v^1)\big)\\ \mu_1 &= \min\{d(v,E): v\in\mathcal{V}_1, E\in\mathcal{E}_1, v\notin E\}\end{aligned}$$



Z_1 and the set of vertexes $\mathcal{V}_1$

We fix a positive real number $\varepsilon_1 < \frac{1}{4}\min\{\varepsilon_0, \mathrm{mesh}(\mathcal{V}_1), \mu_1\}$.

Suppose that for $1 \le i \le n$ there are defined:

(a$_i$) the cactus Z_i with set of vertexes $\mathcal{V}_i$ and set o edges (segments) $\mathcal{E}_i$;
(b$_i$) a finite family of disjoint triangles $\mathcal{T}_i = \{T_v^i\}_{v\in V_{i-1}} \subseteq \mathbb{R}^2$;
(c$_i$) the numbers $\varepsilon_i > 0$ and $\mu_i = \min\{d(v,E): v\in\mathcal{V}_i, E\in\mathcal{E}_i, v\notin E\}$;
(d$_i$) a monotone surjective retraction $f_{i-1}: Z_i \to Z_{i-1}$;
such that
(1$_i$) $\mathcal{V}_{i-1} \subsetneqq \mathcal{V}_i$ and $Z_{i-1} \subsetneqq Z_i$;
(2$_i$) If $T_v^i \in \mathcal{T}_i$, then v is a vertex of T_v^i, $T_v^i \cap Z_{i-1} = \{v\}$, and $T_v^i \subseteq B(v, \frac{\varepsilon_{i-1}}{2})$;
(3$_i$) If $|f_{i-1}^{-1}(z)| > 1$, then $z \in \mathcal{V}_{i-1}$ and $f_{i-1}^{-1}(z) = T_z^i$;
(4$_i$) If $v \in \mathcal{V}_i \cap \mathcal{V}_j$ and $0 \le j < i$, then $\widehat{T}_v^i \cap \widehat{T}_v^j = \{v\}$.
(5$_i$) $\varepsilon_i < \frac{1}{4}\min\{\varepsilon_{i-1}, \mathrm{mesh}(\mathcal{V}_i), \mu_i\}$.

Since Z_n is a union of finite family of line segments and $\mathcal{V}_n$ is a finite subset of Z_n, there exists a finite family of disjoint triangles $\mathcal{T}_{n+1} = \{T_v^{n+1}\}_{v\in V_n} \subseteq \mathbb{R}^2$ such that: v is a vertex of T_v^{n+1}, $T_v^{n+1} \cap Z_n = \{v\}$, and $T_v^{n+1} \subseteq B(v, \frac{\varepsilon_n}{2})$.

We define $Z_{n+1} = Z_n \cup \big(\bigcup_{v\in V_n} T_v^{n+1}\big)$ and $f_n : Z_{n+1} \to Z_n$ by

$$f_n(z) = \begin{cases} v, & \text{if } z \in T_v^{n+1},\ v\in\mathcal{V}_n,\\ z, & \text{if } z\in Z_n.\end{cases}$$

Put

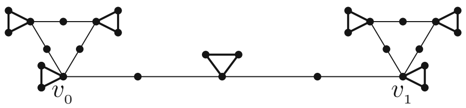
$$\begin{aligned}\mathcal{E}_{n+1} &= \big(\bigcup_{E\in\mathcal{E}_n}\mathcal{E}(E)\big)\cup\big(\bigcup_{v\in\mathcal{V}_n}\mathcal{E}(T_v^{n+1})\big),\\ \mathcal{V}_{n+1} &= \{m_E\}_{E\in\mathcal{E}_n}\cup\big(\bigcup_{v\in\mathcal{V}_n}\mathcal{V}(T_v^{n+1})\big),\\ \mu_{n+1} &= \min\{d(v,E): v\in\mathcal{V}_{n+1}, E\in\mathcal{E}_{n+1}, v\notin E\},\end{aligned}$$

and fix a positive real number $\varepsilon_{n+1} < \frac{1}{4}\min\{\varepsilon_n, \mathrm{mesh}(\mathcal{V}_{n+1}), \mu_{n+1}\}$.

It is easy to see that the above properties $(1_i) - (5_i)$ are satisfied for $i = n+1$. Denote $f_{ji} = f_j \circ f_{j+1} \circ \cdots \circ f_{i-1} : Z_i \to Z_j$ for $j < i-1$, $f_{jj+1} = f_j$, and $f_{jj} = id_{Z_j}$. Then for $0 < i$ we have the following property:

(6$_i$) If $0 \le i_0 \le j \le i$, then $f_{i_0 i} = f_{i_0 j} \circ f_{ji}$.

We will prove an additional property that holds for $i > 0$:



Z_2 and the set of vertexes $\mathcal{V}_2$

(7_i) If $u \in \mathcal{V}_{i_0}$, $0 \leq i_0 < i$, then $f^{-1}_{i_0 i}(u) \subseteq B(u, \varepsilon_{i_0})$.

Let $z \in f^{-1}_{i_0 i}(u)$. Then $z \in T^i_v \in \mathcal{T}_i$, where $v \in \mathcal{V}_{i-1}$, $0 \leq i_0 < i$ and $f_{i_0 i}(v) = u$. If $v = u$, then $T^i_u \subseteq B(u, \frac{\varepsilon_{i-1}}{2})$ from (2_i). Thus $z \in B(u, \frac{\varepsilon_{i-1}}{2}) \subseteq B(u, \varepsilon_{i_0})$.

Otherwise $z \in T^{i+1}_v \in \mathcal{T}_{i+1}$, where $v \in \mathcal{V}_i$, $0 \leq i_0 < i$ and $f_{i_0 i}(v) = u$. Let $i = i_0 + n$ and $f_{ji}(v) = u_j \in Z_j$ for $j = i_0 + 1, \ldots, i - 1$. Then $u = f_{i_0}(u_{i_0+1})$, $f_j(u_{j+1}) = u_j$ for any j, and $v = f_i(z)$.

From definition of f_n, the choice of ε_n, and (2_j) we obtain:

$$d(u,z) \leq d(u, u_{i_0+1}) + d(u_{i_0+1}, u_{i_0+2}) + \cdots + d(u_{i_0+n-1}, v) + d(v, z) <$$
$$< \frac{\varepsilon_{i_0}}{2} + \frac{\varepsilon_{i_0+1}}{2} + \cdots + \frac{\varepsilon_{i_0+n-1}}{2} + \frac{\varepsilon_i}{2} < \frac{\varepsilon_{i_0}}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n} + \ldots\right) < \varepsilon_{i_0}.$$

We set $Z = cl\,(\bigcup_{n=0}^{\infty} Z_n)$ and $Z_\infty = \underleftarrow{\lim}\{Z_n, f_n\}$.

Theorem 4.1. *$Z_\infty = \underleftarrow{\lim}\{Z_n, f_n\}$ is homeomorphic to $Z = cl\,(\bigcup_{n=0}^{\infty} Z_n)$.*

Proof. We define $h : Z_\infty \to Z$ by $h(\{z_i\}) = \lim z_i$. From [1, Theorem I] and its proof it follows that h is a homeomorphism if the following conditions are satisfied:

(a) For each $k_0 \in \mathbb{N}$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that if $k_0 < k$, $p, q \in Z_k$ and $d\,(f_{k_0 k}(p), f_{k_0 k}(q)) > \varepsilon$, then $d(p,q) > \delta$.

(b) For each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\operatorname{diam}\big(\bigcup_{k > k_0} f^{-1}_{k_0 k}(z)\big) < \varepsilon$ for any $z \in Z_{k_0}$.

To prove (a) note that $\lim\limits_{i\to\infty}(\operatorname{mesh}(\mathcal{E}_i)) = 0$. Thus there exists $m > k_0$ with $\operatorname{mesh}(\mathcal{E}_m) < \frac{\varepsilon}{4}$. We have

$$\varepsilon_m < \frac{1}{4}\operatorname{mesh}(\mathcal{V}_m) \leq \frac{1}{4}\operatorname{mesh}(\mathcal{E}_m) < \frac{\varepsilon}{4}$$

For each $k \geq k_0$ the map $f_{k_0 k} : Z_k \to Z_{k_0}$ is uniformly continuous. So, for each $k \in \{k_o, k_0 + 1, \ldots, m\}$ there exists $\delta_k > 0$ such that if $a, b \in Z_k$ and $d(a,b) \leq \delta_k$, then $d(f_{k_0 k}(a), f_{k_0 k}(b)) \leq 4\varepsilon_m$. Set

$$\delta = \min\{\varepsilon_m, \delta_{k_0}, \delta_{k_0+1}, \ldots, \delta_m\}.$$

Let $p, q \in Z_k$ and $d\,(f_{k_0 k}(p), f_{k_0 k}(q)) > \varepsilon$. Then $f_{k_0 k}(p) \neq f_{k_0 k}(q)$.

If $k \in \{k_0, k_0 + 1, \ldots, m\}$, then $d(f_{k_0 k}(p), f_{k_0 k}(q)) > 4\varepsilon_m$. So $d(p,q) > \delta_k > \delta$.

Suppose that $k > m$. Then $Z_{k_0} \subsetneqq Z_m \subsetneqq Z_k$. We have three cases to consider.

1^{st} *case* : $p, q \in Z_m$. Then $f_{mk}(p) = p$ and $f_{mk}(q) = q$. So, $f_{k_0 m}(p) = f_{k_0 k}(p)$ and $f_{k_0 m}(q) = f_{k_0 k}(q)$. Thus $d\,(f_{k_0 m}(p), f_{k_0 m}(q)) > \varepsilon > 4\varepsilon_m$ and, therefore, $d(p,q) > \delta_m \geq \delta$.

2^{nd} *case* : $p, q \in Z_k \setminus Z_m$. Then $f_{mk}(p), f_{mk}(q) \in \mathcal{V}_m$. Thus $d(f_{mk}(p), f_{mk}(q)) \geq \operatorname{mesh}(\mathcal{V}_m) > 4\varepsilon_m$. From (7_m): $d(p, f_{mk}(p)) < \varepsilon_{k_0}$ and $d(q, f_{mk}(q)) < \varepsilon_{k_0}$. Since $\varepsilon_{k_0} < \varepsilon_m$, it follows that

$$d(p,q) \geq d(f_{mk}(p), f_{mk}(q)) - d(q, f_{mk}(q)) - d(p, f_{mk}(p)) > 2\varepsilon_m > \delta.$$

3^d *case* : $p \in Z_m$ and $q \in Z_k \setminus Z_m$. Then $p = f_{mk}(p) \in E_p \in \mathcal{E}_m$ and $f_{mk}(q) = v_q \in \mathcal{V}_m$. Since $p, q \notin Z_{k_0}$, it follows that $E_p \subseteq f^{-1}_{k_0 m}(f_{k_0 k}(p))$ and $v_q \in f^{-1}_{k_0 m}(f_{k_0 k}(p))$. Since $f_{k_0 k}(p) \neq f_{k_0 k}(q)$, $f^{-1}_{k_0 m}(f_{k_0 k}(p)) \cap f^{-1}_{k_0 m}(f_{k_0 k}(q)) = \emptyset$. Hence, $v_q \notin E_p$. From the choice of μ_m it follows that

$$d(v_q, p) > d(v_q, E_p) > \mu_m > 4\varepsilon_m.$$

Since $d(v_q, q) < \varepsilon_m$ from (7_m), we conclude that

$$d(p, q) \geq d(p, v_q) - d(q, v_q) > 4\varepsilon_m - \varepsilon_m > \varepsilon_m > \delta.$$

To prove (b) take any $\varepsilon > 0$. Since $\lim\limits_{i \to \infty} \varepsilon_i = 0$, there exists $k_0 \in \mathbb{N}$ such that $2\varepsilon_{k_0} < \varepsilon$. If $z \in Z_{k_0} \setminus \left(\bigcup_{i \geq k_0} \mathcal{V}_i\right)$, then $\bigcup_{k > k_0} f^{-1}_{k_0 k}(z) = \{z\}$. So (a) holds.

Let $z \in Z_{k_0} \cap \left(\bigcup_{i \geq k_0} \mathcal{V}_i\right)$ and let $i_z \geq k_0$ be the least integer such that $z \in \mathcal{V}_{i_z}$. If $k_0 < k \leq i_z$, then $f^{-1}_{k_0 k}(z) = \{z\}$. Hence, $\bigcup_{k > k_0} f^{-1}_{k_0 k}(z) = \bigcup_{k > i_z} f^{-1}_{i_z k}(z)$.

From the properties (3_k) and (7_k) with $k > i_z$ it follows that $\bigcup_{k > i_z} f^{-1}_{i_z k}(z) \subseteq B(z, \varepsilon_{i_z})$. Thus again

$$\operatorname{diam}\Big(\bigcup_{k > k_0} f^{-1}_{k_0 k}(z)\Big) = \operatorname{diam}\Big(\bigcup_{k > i_z} f^{-1}_{i_z k}(z)\Big) < 2\varepsilon_{i_z} < 2\varepsilon_{k_0} < \varepsilon.$$

$\square$

Theorem 4.2. *Z is a one-dimensional cactoid such that any two cut points of Z can be joined by a simple cyclic chain that is a cactus.*

Proof. Since $Z_\infty = \varprojlim \{Z_n, f_n\}$, where each Z_n is locally connected and each f_n is a monotone surjection, it follows that Z_∞ is a locally connected continuum (see [7, 8.47]). Thus Z is a locally connected continuum from Theorem 4.1.

Let $a, b \in c(Z)$, $a \neq b$. If $a, b \in \bigcup_{i=0}^{\infty} Z_k$, then there exists a cactus Z_k such that $a, b \in Z_k$. Thus a and b can be joined by a simple cyclic chain that is a cactus. It suffices to show that $Z \setminus \bigcup_{i=0}^{\infty} Z_k$ contains no cut points of Z. Suppose, on the contrary, that there exists a cut point $z \in Z \setminus \bigcup_{i=0}^{\infty} Z_k$. Then $Z \setminus \{z\} = O_1 \cup O_2$, where O_1 and O_2 are disjoint, non empty, and open subsets of Z. Since $\bigcup_{i=0}^{\infty} Z_k$ is connected, we may suppose that $\bigcup_{i=0}^{\infty} Z_k \subseteq O_1$. Then $O_2 \cap (\bigcup_{i=0}^{\infty} Z_k) = \emptyset$. Hence, $cl\,(\bigcup_{i=0}^{\infty} Z_k) \neq Z$ which is a contradiction.

Let S be a true cyclic element of Z. Then $E(S) = \emptyset$. Hence, $ord_Z(x) \geq ord_S(x) > 1$ for each $x \in S$. Therefore, $S \cap E(Z) = \emptyset$. If $S \subseteq \bigcup_{i=0}^{\infty} Z_n$, then S is a simple closed curve from construction of Z_n. It suffices to prove that $Z \setminus \bigcup_{i=0}^{\infty} Z_n \subseteq E(Z)$.

Let $e \in Z \setminus \bigcup_{i=0}^{\infty} Z_i$ and $\varepsilon > 0$. It remains to find an open subset U_e of Z such that $e \in U_e \subseteq B(e, \varepsilon)$ and $bd_Z(U_e)$ consists of one point.

The map $h : Z_\infty \to Z$ defined by $h(\{z_i\}_{i=0}^{\infty}) = \lim z_i$ is a homeomorphism from the proof of Theorem 4.1. Let $h^{-1}(e) = \{e_i\}_{i=0}^{\infty}$. Then $f_i(e_{i+1}) = e_i \in Z_i$ for any i. Since $e = \lim e_i \notin \bigcup_{i=0}^{\infty} Z_i$ and each Z_i is compact, it follows that $\{e_i\}_{i=0}^{\infty} \nsubseteq Z_i$ for any i. Therefore, without loss of generality we may suppose that $e_i \neq e_{i+1}$ for any i. Since $f_i(e_{i+1}) = e_i \neq e_{i+1}$, it follows that $e_i \in \mathcal{V}_i$.

There exist $i_0, j_0 \in \mathbb{N}$ such that $e_i \in B(e, \frac{\varepsilon}{2})$ for any $i \geq i_0$ and $\varepsilon_j < \frac{\varepsilon}{2}$ for any $j \geq j_0$. Let $k_0 = \max\{i_0, j_0\}$. Then $e_k \in B(e, \frac{\varepsilon}{2})$ and $\varepsilon_k < \frac{\varepsilon}{2}$ for any $k \geq k_0$.

Let U_e be a component of $Z \setminus \{e_{k_0}\}$ containing e. Since Z is locally connected, U_e is open. Also $bd_Z(U_e) = \{e_{k_0}\}$. It is easy to see that $U_e = \{e\} \cup \left(\bigcup_{k=k_0}^{\infty} T^{k+1}_{e_k}\right)$. Let $z \in U_e$. Then $z \in T^{k+1}_{e_k}$ for some $k \geq k_0$. Therefore $d(z, e_k) < \frac{\varepsilon_{k_0}}{2} < \frac{\varepsilon}{2}$. Thus $d(e, z) \leq d(z, e_k) + d(e, e_k) < \varepsilon$. Hence, $z \in B(e, \varepsilon)$. $\square$

5 The proof of universality of Z

Theorem 5.1. *Z is a universal element in the family of all one-dimensional cactoids X such that any two cut points of X can be joined by a simple cyclic chain that is a cactus.*

Proof. The one-dimensional cactoid X, whose any two cut points can be joined by a simple cyclic chain that is a cactus, is homeomorphic to $X_\infty = \varprojlim\{X_k, g_k\}$, where the inverse sequence $\{X_k, g_k\}_{k=1}^{\infty}$ satisfies the conditions of Theorem 3.5. Also Z is homeomorphic to $Z_\infty = \varprojlim\{Z_k, f_k\}$ by Theorem 4.1. It suffices to find an embedding of X_∞ into Z_∞.

We set $Q(X) = \{t_k\}_{k=1}^{\infty}$ and $Q(Z) = \bigcup_{k=1}^{\infty} \mathcal{V}_k$, where the point t_k satisfies condition (iii) of Theorem 3.5 and $\mathcal{V}_k$ is a set of vertices of cactus Z_k. Note that $X_k \cap Q(X)$ is a countable subset of X_k and $Z_k \cap Q(Z)$ is countable and dense in Z_k for each k.

Observe that X_1 is either a point or a simple closed curve such that there exist a unique point $t_1 \in X_1$ with $|g_1^{-1}(t_1)| > 1$. We also observe that $Z_1 = \overline{v_0v_1} \cup T^1_{v_0} \cup T^1_{v_1}$, where $T^1_{v_i}$ are triangles. If $X_1 = \{t_1\}$, then $h_1 : X_1 \to Z_1$ with $h_1(t_1) = v_1$ is a homeomorphism. If X_1 is a closed curve, then there exist a homeomorphism $h_1 : X_1 \to T^1_{v_1}$ such that $h_1(t_1) = v_1$ and $h_1(X_1 \cap Q(X)) \subseteq T^1_{v_1} \cap Q(Z)$. We put $n_1 = 1$.

Suppose that $k \in \mathbb{N} \setminus \{0\}$ and for each $j \in 1, ..., k$ we have define an integer n_j and an embedding $h_j : X_j \to Z_{n_j}$ such that:

(1_j) $h_j(X_j \cap Q(X)) \subseteq Z_{n_j} \cap Q(Z)$;

(2_j) the following diagram is commutative for $j > 1$:

$$\begin{tikzcd}
X_{j-1} \arrow[d, "h_{j-1}"] & X_j \arrow[l, "g_{j-1}"'] \arrow[d, "h_j"] \\
Z_{n_{j-1}} & Z_{n_j} \arrow[l, "f_{n_{j-1}n_j}"]
\end{tikzcd}$$

(3_j) $n_j > n_{j-1}$ for $j > 1$.

We will define an integer n_{k+1} and an embedding $h_{k+1} : X_{k+1} \to Z_{n_{k+1}}$ that satisfy the properties $(1_{k+1}) - (3_{k+1})$.

Consider the monotone retraction $g_k : X_{k+1} \to X_k$ and the embedding $h_k : X_k \to Z_{n_k}$. By Theorem 3.5 there is a unique $t_k \in X_k$ such that $g_k^{-1}(t_k)$ is non degenerate. We denote $h_k(t_k) = z_k$. From (1_k) we have $z_k \in Z_{n_k} \cap (\bigcup_{i=1}^{\infty} \mathcal{V}_i)$. Since $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$ for all i, there exists $m > 1$ such that $z_k \in \mathcal{V}_{n_k+m}$. Put $n_{k+1} = n_k + m + 1$.

Since $Z_{n_k} \subseteq Z_{n_k+m}$, h_k is also embedding of X_k into Z_{n_k+m}. Observe that $Z_{n_k+m+1} = Z_{n_k+m} \cup \left(\bigcup_{v \in V_{n_k+m}} T_v^{n_k+m}\right)$. Thus z_k is a vertex of some triangle $T_{z_k}^{n_k+m+1} \subseteq Z_{n_k+m+1}$ such that $T_{z_k}^{n_k+m+1} \cap Z_{n_k+m} = \{z_k\}$.

If $g_k^{-1}(t_k) = A$ is a free arc of X_{k+1}, then $A \cap X_k = \{t_k\}$ and t_k is an end point of A. Let E be one of the sides of triangle $T_{z_k}^{n_k+m+1}$ with $z_k \in E$. There exists a homeomorphism $h_A : A \to E$ such that $h_A(t_k) = z_k$ and $h_A(A \cap Q(X)) \subseteq E \cap Q(Z)$, because $E \cap Q(Z)$ is dense in E.

We define a homeomorphism $h_{k+1} : X_{k+1} = X_k \cup A \to Z_{n_{k+1}}$ by

$$h_{k+1}(x) = \begin{cases} h_A(x), & x \in A \\ h_k(x), & x \in X_k \end{cases}$$

If $g_k^{-1}(t_k) = S$ is a closed curve of X_{k+1}, then $S \cap X_k = \{t_k\}$. There exists a homeomorphism $h_S : S \to T_{z_k}^{n_k+m+1}$ such that $h_S(t_k) = z_k$ and $h_S(S \cap Q(X)) \subseteq T_{z_k}^{n_k+m+1} \cap Q(Z)$, because $T_{z_k}^{n_k+m+1} \cap Q(Z)$ is dense in $T_{z_k}^{n_k+m+1}$.

We define a homeomorphism $h_{k+1} : X_{k+1} = X_k \cup S \to Z_{n_{k+1}}$ by

$$h_{k+1}(x) = \begin{cases} h_S(x), & x \in S \\ h_k(x), & x \in X_k \end{cases}$$

From (2_j) and (3_j), $j > 1$, the map $h_\infty : \varprojlim\{X_k, g_k\}_{k=1}^\infty \to \varprojlim\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty$ defined by $h_\infty\left((x_k)_{k=1}^\infty\right) = \left(f_{n_k}(x_k)\right)_{k=1}^\infty$ is continuous and one-to-one (see [7, 2.22]). Since X is a continuum, h_∞ is embedding. Since inverse sequence $\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty$ is confinal in the sequence $\{Z_k, f_k\}_{k=1}^\infty$, there exists a homeomorphism $H : \varprojlim\{Z_{n_k}, f_{n_k}\}_{k=1}^\infty \to \varprojlim\{Z_k, f_k\}_{k=1}^\infty = Z$. Hence, $H \circ h_\infty$ is an embedding of X into Z. □

6 Conclusions and problems. In this section we refer only to continua consisting of more than one point. A continuum X is called totally regular [8] if for any countable subset Q of X, each $x \in X$, and each $\varepsilon > 0$, there exists an open neighborhood U of x in X such that diam $(U) < \varepsilon$, $bd(U)$ is finite, and $bd(U) \cap Q = \emptyset$. Clearly, any graph is totally regular continuum. Totally regular continua were studied also [11] under the term "continua of finite degree". Since the property of being a totally regular continuum is cyclicly extensible and reducible [11, (4.2)], any cactoid is totally regular.

The order of totally regular continuum X is the ordinal number $ord(X) = \sup\{ord(p, X) : p \in X\}$. Note that [13, (3.2), p. 49] $ord(X) \geq 2$. If $ord(X) = 2$, then X is an arc or a simple closed curve [7, Theorem 9.5]. The cactoid Z constructed in section 4 is a totally regular planar continuum of order ω.

R. D. Buskirk proved that that there exists a universal totally regular continuum [4]. The natural problems arisen are the following:

1. Does there exists a universal one-dimensional cactoid.
2. Does there exists a universal one-dimensional cactoid in the family of one-dimensional cactoids of order $\leq n$, where $n > 2$.
3. Does there exists a universal planar totally regular continuum.
4. Does there exists a universal planar totally regular continuum in the family of totally regular continua of order $\leq n$, where $n > 2$.

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NONLINEAR NETWORK AUTOREGRESSIVE MODEL WITH APPLICATION TO NATURAL GAS NETWORK FORECASTING

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Abstract. We propose a nonlinear network autoregressive (NNAR) model to investigate the dynamics of complex network time series with high-dimensionality and nonlinear spatial-temporal dependence. We assume that the current network at a given time point non-linearly depends on the lagged values, neighborhood effect, and a set of node-specific covariates via a nonparametric smooth function. We conduct estimation using the profile least square method where the unknown link function is estimated using the local linear regression technique. We demonstrate the application of the NNAR with the daily natural gas flows in a real-life high-pressure gas pipeline network, where the response is the high dimensional vector of gas flows at 128 nodes. The NNAR model provides more accurate forecasts of the gas flow network compared to the linear network vector autoregression model proposed by Zhu et al. (2017) and some multivariate autoregression and naive benchmark models.

1. Introduction

In this data-rich era, the development in data acquisition and storage has made it available to collect large-scale network data in many fields varying from biomedical sciences (Wu et al. 2014) and physics (Benson et al. 2016) to finance (Chen et al. 2018; Zhu et al. 2019) and socialization (Wasserman & Faust 1994; Zhu et al. 2017). Network data contains rich information for statistical inference, while the complexity of data with high-dimensionality, spatial-temporal dependence structure, non-linearity, and dynamic evolution creates extra challenges for statistical modeling and computation. To describe the dynamics and make a prediction of this complex data effectively and efficiently, it requires more flexible time series modeling in addition to the conventional tools that are designed for linear and low-dimensional time series data.

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Let Y_{it} be the continuous response collected from subject (node) i at time point t with $0 < t \leq T$ (e.g., natural gas flows at certain location). Accordingly, denote $\mathbf{Y}_t = (Y_{1t}, ..., Y_{Nt}) \in \mathbb{R}^N$ as an ultra-high dimensional vector with a large number of total nodes N. We assume that Y_{it} exhibits serial dependence on previous values and has certain spatial correlation among the nodes, i.e., network structure, and the dependence is unnecessary to be linear. In the literature of time series, some common univariate models such as AutoRegressive (AR) and AutoRegressive Moving Average (ARMA) based models have been well studied to forecast the dynamics of serially dependent time series in both theory and applications. However, these models study each time series Y_{it} separately, and the rich correlation information across different time series, e.g. the lead-lag dependence, is lost. Multivariate models such as Vector Autoregression (VAR) (Lütkepohl 2005; Box et al. 2015) are proposed with the information of multiple time series fully considered. On the other hand, the linear relationship assumption may not be valid in practice. Many nonlinear models have been proposed, such as functional coefficient autoregressive models for univariate time series (Huang & Shen 2004; Cai et al. 2000) and nonlinear VAR models for multivariate time series (Härdle et al. 1998); See also Fan & Yao (2008) for a good reference. However, in a high-dimensional case with sufficiently large N, these linear and nonlinear multivariate regression models will suffer from the “curse of dimensionality”, which will cause a problem of overfitting and overparameterization, leading to poor out-of-sample forecast accuracy as well. Thus it is necessary to reduce dimension and many techniques have been therefore proposed, see e.g. factor modeling (Pan & Yao 2008; Park et al. 2009) and penalty estimation (Hsu et al. 2008). However, besides the serial and cross dependence, there could exist network information among the N nodes which needs to be taken into consideration.

The recent development of network modeling provides a wide variety of tools and methods to model the high-dimensional and complexly structured time series. In the literature, researchers mainly focus on conducting static analysis of network structure (Lee et al. 2010; Chen et al. 2013; Zhao et al. 2012; Zhou et al. 2017) or investigating the inherent dynamics of the network over time (Chen et al. 2018; Zhu & Pan 2018; Zhu et al. 2019). Among others, Zhu et al. (2017) proposed the linear network vector autoregression (NAR) model to study the dynamics of large scale network with the network information among individuals incorporated, while dependence between the response and explanatory variables is assumed to be linear. Though convenient, in reality, linear regression-based models may not capture the data-driven complex relationship with both high-dimensionality and non-linearity. The single-index model (Carroll et al. 1997) is a prevalent way to flexibly handle the data-driven non-linearity and circumvent the problem of high-dimensionality simultaneously. For example, Jia et al. (2019) proposed partial autoregression single-index (PASI) model to handle linear network dependence and nonlinear influence of static covariates; See also Wang & Yang (2009), Yu & Ruppert (2002), Li & Genton (2009), and Liang et al. (2010) for more reference of single-index model. However, Jia et al. (2019) did not consider the non-linearity of network dependence. In our work, we propose a flexible semiparametric model also by combining the single-index technique and dynamic autoregression network model, which inherits the advantages of both models and handles the non-linearity of network dependence as well.

We propose a nonlinear network autoregressive (NNAR) model to investigate the dynamics of network time series with nonlinear spatial-temporal dependence in high-dimensional framework, and simultaneously allow the measurement of the nonlinear impact of multivariate node-specific covariates, if applicable. In particular, we assume response Y_{it} depends on a single index defined on three items: its own lagged value, the weighted average of its neighbors, and exogenous covariates via a nonlinear link function, referring as the momentum effect, the network effect and the nodal effect by Zhu et al. (2017), respectively. The link function is assumed to be unknown and smooth. We conduct estimation using the profile least square method (Fan & Gijbels 1996) where the link function is estimated using the local linear regression technique. This paper makes contributions in two aspects: (1) We propose a flexible nonlinear network autoregressive model to investigate the dynamics of large-scale network with complex spatial-temporal dependence. The proposed model helps capture the nonlinear network dependence and node-specific exogenous covariates' impact. While Zhu et al. (2017) and Jia et al. (2019) only considered the linear network dependence. (2) We demonstrate the application of the NNAR model on forecasting gas flows at 128 distribution nodes of a high-pressure natural gas transmission network in Europe. It provides a more accurate out-of-sample forecast for the gas flows network compared with the linear network model and some multivariate time series and naive benchmark models.

The rest of the paper is organized as follows. Section 2 details the NNAR model and parameter estimation procedure using the profile least square method. Section 3 presents the gas flow network data on an energy transmission system. Section 4 implements the NNAR model to investigate the dynamics of gas flows network and conducts forecasting with comparison to several alternative models. Section 5 concludes.

2. Method

In this section, we present the nonlinear network autoregressive (NNAR) model and the estimation procedure using the profile least square method.

2.1. **The NNAR model.** Recall that the number of nodes in the network is N, and Y_{it} is the continuous response collected from node i at time point t with $0 < t \leq T$ and $1 \leq i \leq N$. Denote $\mathbf{Y}_t = (Y_{1t}, ..., Y_{Nt}) \in \mathbb{R}^N$ an ultra-high dimensional vector with a large number of N. In addition, for each node i, assume a $p-$dimensional node-specific random vector $\mathbf{Z}_{it} = (Z_{it}^{(1)}, \dots, Z_{it}^{(p)})^\top \in \mathbb{R}^p$ can be observed. To model $\mathbf{Y}_t$, we propose the following nonlinear network autoregressive model (NNAR):

$$Y_{it} = g\bigg(\beta_1 \sum_{j=1}^{N} w_{ij} Y_{j(t-1)} + \beta_2 Y_{i(t-1)} + \mathbf{Z}_{i(t-1)}^\top \gamma\bigg) + \epsilon_{it}, \tag{1}$$

where $g(\cdot)$ is an unknown link function which is assumed be smooth. $w_{ij} \in [0,1]$ is a given weight to measure the strength of the connection between node i and j for $i, j = 1, \cdots, N$. In specific, if we know the adjacency matrix of the network structure, which can be defined as $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, where $a_{ij} = 1$ if there exists a relationship between node i and j, and $a_{ij} = 0$ otherwise, then w_{ij} is commonly defined as $w_{ij} = a_{ij}/n_i$ where $n_i = \sum_{j \neq i} a_{ij}$ is the total number of nodes that node i is connected, i.e. out-degree (Wasserman & Faust 1994). Such a choice is quite common in many kinds of research like graphical and social network analysis (Bondy & Murty 1976; Zhu et al. 2017). However, in our real data implementation to natural gas flow network at next section, the adjacency matrix for the gas network is unknown, we thus define w_{ij} as the inverse of the shortest path between the gas node i and j, with further located nodes given smaller weight. The quantity $\sum_{j=1}^{N} \omega_{ij} Y_{j(t-1)}$

characterizes the average impact from the network to ith node at time $t-1$. Its associated parameter β_1 is referred as the network effect. The term $Y_{i(t-1)}$ is the autoregressive term which stands for the serial dependence and β_2 is the corresponding parameter. The term $\mathbf{Z}_{i(t-1)}^\top \gamma$ evaluates the influence of exogenous impact to the ith node at time $t-1$, where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)^\top \in \mathbb{R}^p$ is the associated coefficient (i.e. exogenous nodal effect). Moreover, ϵ_{it} is the error term, we assume that it is independent to response and exogenous covariates, i.e., $E(\epsilon_{it}, Y_{is}) = 0$ and $E(\epsilon_{it}, Z_{is}^{(j)}) = 0$ for any $s < t$ and $j = 1, \cdots, p$, and follows normal distribution with $\epsilon_{it} \sim_{\text{i.i.d}} \mathrm{N}(0, \sigma^2)$.

For notation simplicity, let $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top \in \mathbb{R}^2$, and $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\gamma}^\top)^\top \in \mathbb{R}^{p+2}$ standing for the vector of all the unknown parameters. We further employ the profile least-square estimation technique to estimate $\boldsymbol{\theta}$ and the unknown link function $g(\cdot)$. We rewrite the NNAR model in (1) as

$$\begin{aligned} Y_{it} &= g\Bigg(\beta_1 \sum_{j=1}^{N} w_{ij} Y_{j(t-1)} + \beta_2 Y_{i(t-1)} + \mathbf{Z}_{i(t-1)}^\top \boldsymbol{\gamma}\Bigg) + \epsilon_{it}, \\ &= g\Bigg(\mathbf{X}_{it}^\top \boldsymbol{\theta}\Bigg) + \epsilon_{it}, \end{aligned} \tag{2}$$

where $\mathbf{X}_{it} = (\sum_{j=1}^{N} w_{ij} Y_{j(t-1)}, Y_{i(t-1)}, \mathbf{Z}_{i(t-1)}^\top)^\top \in \mathbb{R}^{p+2}$ is the variable vector consisting of the network effect, lag effect and exogenous nodal effect to node i at time $t-1$. To ensure the identification, we set $||\boldsymbol{\theta}||_2 = 1$ with the first element positive, where $||\cdot||_2$ is the L_2 norm.

2.2. **Estimation.** In semiparametric models, it is popular to estimate the unknown parametric components using the profile likelihood approach, see Liang et al. (2010). We apply this technique for estimating parameter $\boldsymbol{\theta}$. First we start estimating the nonlinear link function $g(\cdot)$ for a given parameter value $\boldsymbol{\theta}$ using the local linear approximation method (Fan & Gijbels 1996). In particular, by defining $u_{it} = \mathbf{X}_{it}^\top \boldsymbol{\theta} \in \mathbb{R}$ we linearly approximate the function $g(\cdot)$ at a given point u_0. Since the unknown function $g(\cdot)$ is assumed to be smooth, we approximate it locally by a linear function (Taylor expansion),

$$g(u) \approx g(u_0) + g'(u_0)(u - u_0), \quad \text{for } u \in u_0 \pm h,$$

where h is a bandwidth referring to the size of the neighborhoods that the linear approximation holds. For notation simplicity, denote $a = g(u_0)$, and $b = g'(u_0)$. This leads to the following locally approximated NNAR model:

$$\begin{aligned} Y_{it} &= g(u_{it}) + \epsilon_{it}, \\ &\approx a + b(u_{it} - u_0) + \epsilon_{it},\ u_{it} \in u_0 \pm h. \end{aligned}$$

We consider to estimate the local parameters a and b by minimizing the below objective function using the weighted least squares method:

$$S(a,b) = \sum_{t=1}^{T}\sum_{i=1}^{N}\Big(Y_{it} - a - b(u_{it} - u_0)\Big)^2 K_h(u_{it} - u_0), \tag{3}$$

where $K_h(\cdot) = K(\cdot/h)/h$, and $K(\cdot)$ is a nonnegative unimodal kernel function. In our study, we use the Gaussian kernel function and select the optimal bandwidth via cross-validation (CV) method.

By computing the derivatives of $S(a,b)$ with respect to a and b, we find the minimizers of Eq.(3) for a given value of $\boldsymbol{\theta}$:

$$\hat{g}(u_0;\boldsymbol{\theta}) = \hat{a}|_{\boldsymbol{\theta}} = \frac{K_{20}(u_0,\boldsymbol{\theta})K_{01}(u_0,\boldsymbol{\theta}) - K_{10}(u_0,\boldsymbol{\theta})K_{11}(u_0,\boldsymbol{\theta})}{K_{00}(u_0,\boldsymbol{\theta})K_{20}(u_0,\boldsymbol{\theta}) - K_{10}^2(u_0,\boldsymbol{\theta})}, \tag{4}$$

where $K_{j\ell}(u,\boldsymbol{\theta}) = \sum_{t=1}^{T}\sum_{i=1}^{N}[K_h(\mathbf{X}_{it}^\top\boldsymbol{\theta} - u)](\mathbf{X}_{it}^\top\boldsymbol{\theta} - u)^j Y_{it}^\ell$ with exponents $j = 0, 1, 2$ and $\ell = 0, 1$. The proof is given in the appendix.

Next we estimate the parameter $\boldsymbol{\theta}$ using the estimates of the nonparametric component $\hat{g}(u_0;\boldsymbol{\theta})$ in Eq. (4). If assuming $u_0 = \mathbf{X}_{it}^\top\boldsymbol{\theta}$, we have $Y_{it} \approx g(\mathbf{X}_{it}^\top\boldsymbol{\theta};\boldsymbol{\theta}) + \epsilon_{it}$. We then obtain the estimator $\hat{\boldsymbol{\theta}}$ by minimizing the following profile least-square function following the assumption of Jennrich (1969):

$$Q(\boldsymbol{\theta}) = \sum_{t=1}^{T}\sum_{i=1}^{N}\Big\{Y_{it} - \hat{g}(\mathbf{X}_{it}^\top\boldsymbol{\theta};\boldsymbol{\theta})\Big\}^2. \tag{5}$$

Since there is no closed-form solution for the estimate $\boldsymbol{\theta}$ in above Eq.(5), we apply stochastic gradient descent algorithm (Kushner & Yin 2003) to iteratively update the estimations by minimizing the above objective function. The calculation stops when the parameters converge, and we set the final iterative estimator as $\hat{\boldsymbol{\theta}}$.

Finally, plugging in the optimal value $\hat{\boldsymbol{\theta}}$ to Eq.(4), and replacing the notation of u_0 with a general parameter symbol u, we have the following estimate of link function $g(\cdot)$

$$\hat{g}(u;\hat{\boldsymbol{\theta}}) = \hat{a}|_{\hat{\boldsymbol{\theta}}} = \frac{K_{20}(u,\hat{\boldsymbol{\theta}})K_{01}(u,\hat{\boldsymbol{\theta}}) - K_{10}(u,\hat{\boldsymbol{\theta}})K_{11}(u,\hat{\boldsymbol{\theta}})}{K_{00}(u,\hat{\boldsymbol{\theta}})K_{20}(u,\hat{\boldsymbol{\theta}}) - K_{10}^2(u,\hat{\boldsymbol{\theta}})}, \tag{6}$$

where $K_{j\ell}(u,\boldsymbol{\theta})$ has the same definition as before.

3. Data

Natural gas has become an important and clean energy source for power systems with its advantages varying from lower pollutant emission and smaller construction period, to higher efficiency of conversion and loading. About 24% of the worldwide energy demand is met by natural gas in 2018 (BP 2019). In particular, natural gas is a key energy resource for Europe and accounts for about 20% of the European energy demand (Petkovic et al. 2019). Natural gas is transported through transit countries and to the local distribution nodes through the high-pressure transmission pipeline network which is operated by so-called transmissions system operators or TSOs. The European gas market is moving to more short-term operations, for example, day-ahead contracts. This increases the necessity of modeling the underlying network dynamics of future gas flows for not only one node or a few nodes, but large-dimensional nodes in the transmission network. Accurate short-term forecasting of natural gas demand and supply is of importance for TSOs to monitor the situation and conduct operational decisions to ensure the safety of supply. There exists rich literature in natural gas forecasting, see e.g. Stoll & Wiebauer (2010), Soldo (2012), Banda & Herty (2008), Koch et al. (2015), Chen et al. (2018), and Chen et al. (2020). However, the dynamics of gas flow network structure has less been explored in the context of forecasting.

Our work is motivated by the challenging problem of short-term forecasting of gas supply and demand in the high-dimensional gas transmission network. We collect the high-resolution natural gas flow data at N=128 nodes in the gas pipeline network in one European country. The daily average gas in-flow or out-flow is observed for the consecutive T=637 days over 22 months. The gas flow network data is standardized with zero mean and unit variance to use because of the significant scale difference of flow values at various nodes. The response (i.e. Y_{it}) considered here is the daily average gas flow of node $i \in \{1, \cdots, 128\}$ at day $t \in [1, T]$. In addition, we consider the daily average air temperature at each node as an exogenous variable. Given that the natural gas is being widely used for heating purposes in European countries, the temperature is usually considered as a possible affected factor in forecasting gas flows (Chen et al. 2018). As gas flows in/out through all nodes in the network where gas nodes are connected with a pipeline, and it is unclear about the adjacency matrix $A = (a_{ij})$ of the network as well, we define the weight matrix, denoted as

W with the $(i, j)-$th element being the weight w_{ij}, as the inverse shortest path among each pair of nodes i and j. Here, the shortest path is defined as the Euclidean distance between two nodes.

Figure 1 displays the time series plot of gas flows at 25 arbitrarily selected nodes as a graphical demonstration. We can see similar dynamics of gas flow time series at these nodes, and synchronous behaviors associating with seasons. Figure 2 displays the lag-1 sample cross-correlation matrix of natural gas flows at 128 nodes. We can see a strong correlation among the 128 nodes. Figure 3 displays the correlation coefficients between variables from day 1 to day 637. In the figure, the columns represent the response variable of gas flow Y_t, network term WY_{t-1} referring to the network impact, lag-1 gas flow variable Y_{t-1} representing momentum impact, and the temperature variable Z_{t-1} representing node-specific exogenous covariate's impact from left to right respectively. As shown, the Pearson linear correlation coefficients between Y_t and three regressive terms are 0.48, 0.84, and -0.59 respectively, which indicate the existance of correlation among them. In addition, we can see some nonlinear dependence from the scatter plots. These motivate the use of the NNAR model which helps jointly analyze the nonlinearity, momentum impact, network impact, and exogenous variable effect simultaneously to utilize the rich information.

4. Forecasting Results

In this section, we demonstrate the forecasting performance of the NNAR model using the natural gas network data described in Section 3. We perform out-of-sample forecasts of daily natural gas and compare it with several alternative methods.

4.1. **Setup and evaluation.** We divide the network dataset into two phases with the first 500 days used as training period (T_1), which covers 80% of the total period, and 137 days from day 501 to the end at day 637 as the forecasting period (T_2). We train the model and estimate parameters in T_1, and select the optimal bandwidth as $h = 4$ via cross-validation.

As alternative methods, we consider the linear NAR model (Zhu et al. 2017), VAR and naive methods including Random Walk and Sample Mean to forecast 1-day ahead daily natural gas flows of 128 nodes. The NAR model considers linear network dependence among nodes. VAR method is popular in forecasting gas consumption, and we select the

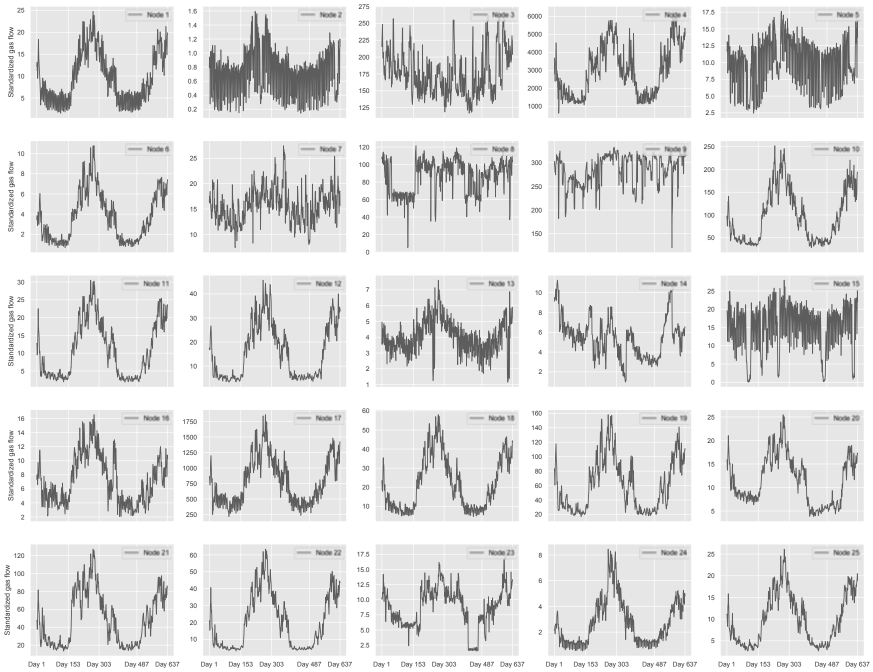

FIGURE 1. Time series plot of gas flows at 25 illustrative nodes from day 1 to day 637.

lag order of VAR via Bayesian Information Criteria (BIC). In addition, naive forecasting methods of Sample Mean (SM) and Random Walk (RW) are used as benchmark models. Here, at time t, the SM model forecasts h-step-ahead value by taking the average of all observed data up to time t, that is, $\hat{Y}_{i,t+h} = \frac{1}{t}\sum_{j=1}^{t} Y_{i,j}, \quad \text{for } i = 1, \ldots, N$.

We evaluate the relative forecast accuracy according to the average forecast error of individual nodes in the network. We use mean absolute percentage error (MAPE) as an error evaluation criteria. The smaller the MAPE, the better accuracy is obtained by the forecast model. First, for each node $i = 1 \ldots N$ we obtain daily predicted gas flow series $\hat{Y}_{it}$. The 1-day-ahead prediction performance is evaluated over the forecasting period of T_2 for each node. The MAPE for each node i is obtained as:

$$\text{MAPE}_i = \frac{1}{|T_2|}\sum_{t\in T_2}\left|\frac{\hat{Y}_{it} - Y_{it}}{Y_{it}}\right|,$$

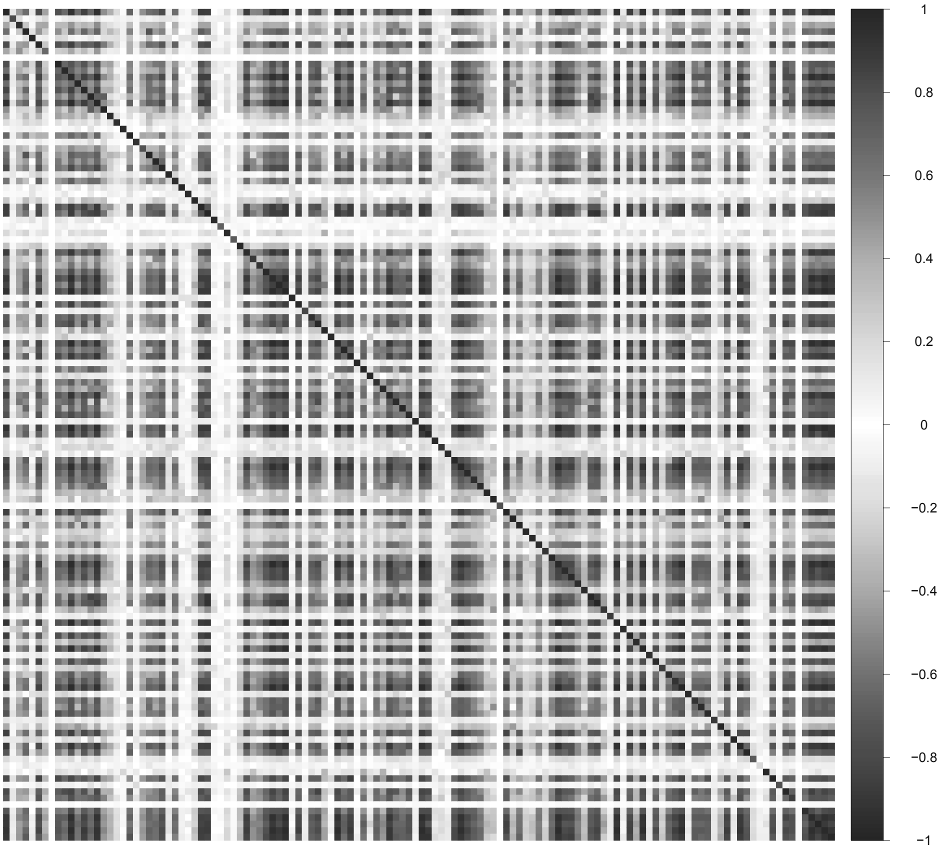

Figure 2. Sample lag 1 cross-correlation matrix of 128 nodes.

for $i = 1, \cdots, 128$, where $|T_2|$ is the length of forecasting period T_2. Average forecast performance evaluation is obtained via an average MAPE as

$$\text{aMAPE} = \frac{\sum_{i=1}^{N} \text{MAPE}_i}{N}.$$

4.2. **Results.** We demonstrate the 1-day-ahead out-of-sample forecasting results in the large scale gas network. Table 1 reports the aMAPE and its standard deviation (sd) as well as range over 128 nodes of de-standardized gas-flows for the NNAR model and the alternative models. The MAPE over different models is compared in the boxplot of Figure 4. As can be seen, the NNAR model performs much better than the VAR model, the RW, and the SM models with smaller aMAPE, smaller sd, and more narrow range. The NNAR model slightly outperforms the NAR model with aMAPE of 13.06%, sd of 11.589%, and

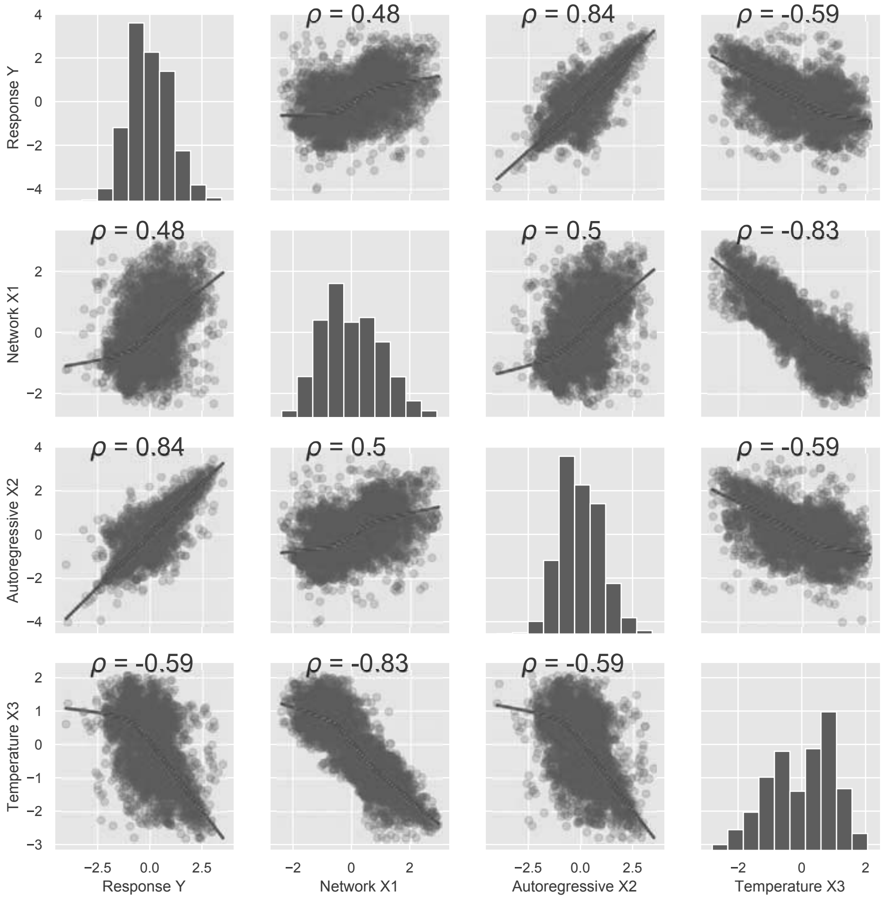

Figure 3. Correlation coefficients between variables from day 1 to day 637. The columns represent the response variable of gas flow Y_t, network term WY_{t-1} (the network impact), lag-1 gas flow Y_{t-1} (momentum impact), and exogenous variable Z_{t-1}(temperature) from left to right respectively. Here, ρ is the Pearson correlation coefficient.

range [2.206%; 84.452%] respectively. Our model is only slightly better than the linear NAR model, this could be because the non-linearity dependence in the gas network is not that significant, and the linear network model can fit the gas date well. It is worth mentioning that the NNAR model provides some advantage in flexibly capturing real network data with either linear or nonlinear dependence structure compared to the linear NAR model.

	aMAPE	(sd)	Range
NNAR	13.060%	(11.589%)	[2.206%; 84.452%]
NAR	13.570%	(12.398%)	[2.104%; 88.598%]
Random Walk	47.405%	(19.268%)	[3.817%; 98.546%]
VAR (BIC)	38.405%	(20.324%)	[3.878%; 149.517%]
Sample Mean	38.405%	(20.324%)	[3.878%; 149.517%]

Table 1. The 1-day-ahead out-of-sample forecast performance of the NNAR and alternative models. Average MAPE (aMAPE) and its standard deviation (sd) as well as range (Range) over 128 nodes are reported.

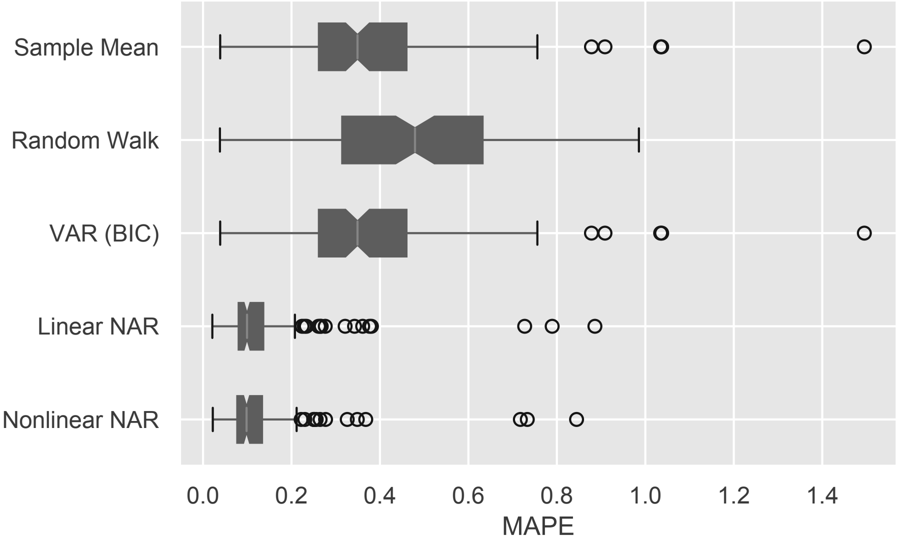

Figure 4. The boxplot of MAPE of the NNAR and four alternative models over 128 nodes.

Figure 5 displays the 1-day-ahead out-of-sample forecasts of the NNAR model for daily gas flow at the 25 nodes as an illustration from day 501 to the end at day 637. We can see that the NNAR model successfully captures the dynamic evolution of the gas flow time series at each individual node. The NNAR model also fits the observed gas flows well at the rest nodes. In general, we find that our proposed model delivers stable forecast performance no matter the node's type and dynamic. We have to mention that since we analyze the gas network partially i.e. with a selected number of gas nodes, and the nodes connection information in pipeline network is not clear as well, the conclusion from our modeling may have some limitations.

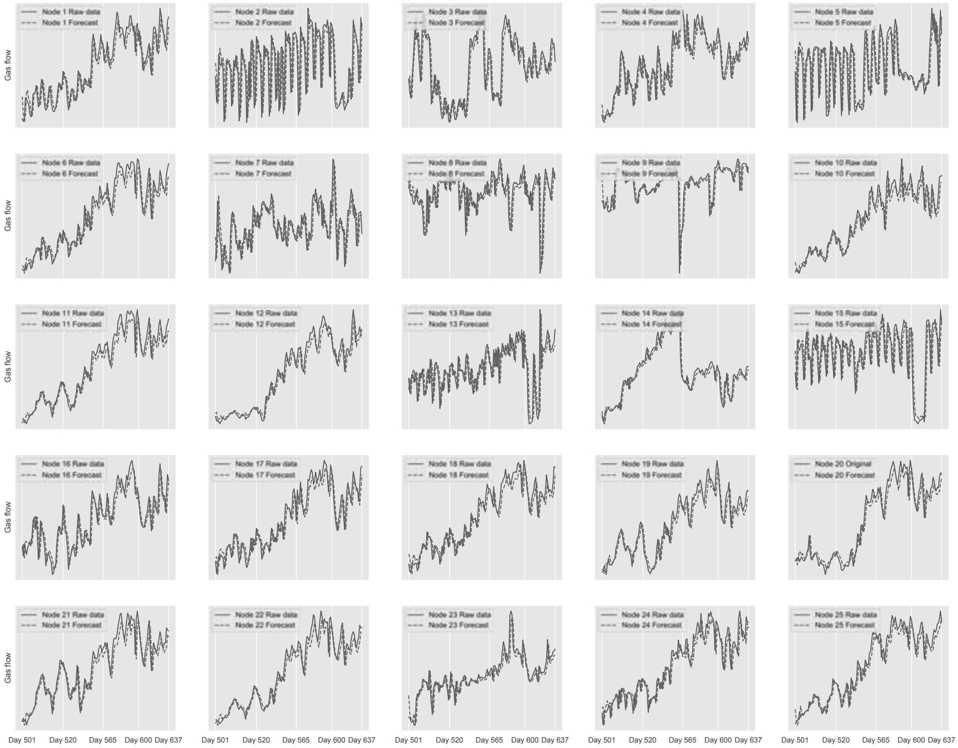

FIGURE 5. The daily gas flows and 1-step-ahead forecast of the NNAR model at 25 nodes from day 501 to day 637. The data are normalized to display.

5. CONCLUSION

We propose a nonlinear network autoregressive model to investigate the complex dynamics of high-dimensional network with nonlinear spatial-temporal dependence structure, where the nonlinear impact of node-specific exogenous covariates is incorporated simultaneously. The proposed model assumes that the current network at a given time point non-linearly depends on three items: the past values, network effect, and exogenous covariates via a nonlinear smooth function. We conduct estimation using the profile least square method where the unknown link function is estimated via the local linear regression technique. We demonstrate the application of the NNAR with the daily natural gas flows in a real-life high-pressure gas pipeline network, where the response is the high dimensional vector of gas flows at 128 nodes. The NNAR model provides more accurate forecasts of the gas flow network compared with several alternative models. It shows that the NNAR model has some advantages in flexibly capturing real-life network data with either linear or nonlinear dependence structure compared to linear models.

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Appendix: Estimation derivation

Here we describe the estimation of the nonlinear function g in more detail. First, we write the NNAR model for node i at time point t as follows,

$$Y_{it} = g(\mathbf{X}_{it}^\top \boldsymbol{\theta}) + \epsilon_{it},$$

and define $u_{it} = \mathbf{X}_{it}^\top \boldsymbol{\theta}$. Then we rewrite the model in terms of u_{it} as

$$Y_{it} = g(u_{it}) + \epsilon_{it}.$$

We assume unknown link function $g(\cdot)$ is second order differentiable. At a given point u_0, we consider to approximate the function $g(\cdot)$ by their first order Taylor's expansion with respect to u as

$$g(u) = g(u_0) + g'(u_0)(u - u_0), \text{ for } u \in u_0 \pm h,$$

where h is a bandwidth. Denote $a = g(u_0)$, and $b = g'(u_0)$, we can estimate the nonlinear function g by minimizing the following objective function using the weighted least square method,

$$S(a,b) = \sum_{t=1}^{T}\sum_{i=1}^{N}[Y_{it} - a - b(u_{it} - u_0)]^2 K_h(u_{it} - u_0). \tag{A.1}$$

To find the minimizer of local parameters a and b in (A.1), we take its derivatives with respect to a and b, respectively, then we have

$$\begin{aligned}\frac{\partial S}{\partial a} &= \sum_{t=1}^{T}\sum_{i=1}^{N}[Y_{it} - a - b(u_{it} - u_0)]K_h(u_{it} - u_0),\\ \frac{\partial S}{\partial b} &= \sum_{t=1}^{T}\sum_{i=1}^{N}[Y_{it} - a - b(u_{it} - u_0)](u_{it} - u_0)K_h(u_{it} - u_0).\end{aligned}$$

Setting the above two derivatives to zero, we obtain

$$\sum_{t=1}^{T}\sum_{i=1}^{N}[Y_{it} - a - b(u_{it} - u_0)]K_h(u_{it} - u_0) = 0, \tag{A.2}$$

and

$$\sum_{t=1}^{T}\sum_{i=1}^{N}[Y_{it} - a - b(u_{it} - u_0)](u_{it} - u_0)K_h(u_{it} - u_0) = 0. \tag{A.3}$$

From (A.2) we obtain the following estimate for b as

$$\hat{b} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{N} Y_{it}K_h(u_{it} - u_0) - a\sum_{t=1}^{T}\sum_{i=1}^{N} K_h(u_{it} - u_0)}{\sum_{t=1}^{T}\sum_{i=1}^{N}(u_{it} - u_0)K_h(u_{it} - u_0)}. \tag{A.4}$$

For notation simplicity, we denote $K_h(u_{it} - u_0) = K_h^{\star}$. Plug (A.4) into (A.3), we can derive the estimate of a as:

$$\hat{a} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{N}\{K_h^{\star}\big[\sum_{t=1}^{T}\sum_{i=1}^{N}(u_{it}-u_0)^2K_h^{\star} - (u_{it}-u_0)\sum_{t=1}^{T}\sum_{i=1}^{N}(u_{it}-u_0)K_h^{\star}\big]\}Y_{it}}{\sum_{t=1}^{T}\sum_{i=1}^{N}\{K_h^{\star}\big[\sum_{t=1}^{T}\sum_{i=1}^{N}(u_{it}-u_0)^2K_h^{\star} - (u_{it}-u_0)\sum_{t=1}^{T}\sum_{i=1}^{N}(u_{it}-u_0)K_h^{\star}\big]\}}.$$

Recall that $g(u_0) = a$, we can directly get

$$\hat{g}(u_0;\boldsymbol{\theta}) = \hat{a}|_{\boldsymbol{\theta}} = \frac{K_{20}(u_0,\boldsymbol{\theta})K_{01}(u_0,\boldsymbol{\theta}) - K_{10}(u_0,\boldsymbol{\theta})K_{11}(u_0,\boldsymbol{\theta})}{K_{00}(u_0,\boldsymbol{\theta})K_{20}(u_0,\boldsymbol{\theta}) - K_{10}^2(u_0,\boldsymbol{\theta})}, \tag{A.5}$$

where $K_{j\ell}(u_0,\boldsymbol{\theta}) = \sum_{t=1}^{T}\sum_{i=1}^{N} K_h^{\star}(\mathbf{X}_{it}^{\top}\boldsymbol{\theta} - u_0)^j Y_{it}^{\ell}$, with exponents $j = 0, 1, 2$ and $\ell = 0, 1$.

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RICCI CURVATURES AND SCALAR CURVATURES OF HOMOGENEOUS MINIMAL REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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Abstract. We compute Ricci curvatures and scalar curvatures of minimal homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$.

1. Introduction

Standard examples play an important role in geometry. We denote by $\widetilde{M}_n(c)$ a complex $n\,(\geqq 2)$-dimensional complete and simply connected nonflat complex space form of constant holomorphic sectional curvature $c(\neq 0)$, namely a complex projective space $\mathbb{C}P^n(c)(c>0)$ or a complex hyperbolic space $\mathbb{C}H^n(c)(c<0)$. In the theory of real hypersurfaces in $\widetilde{M}_n(c)$ it is interesting to investigate geometric properties of *homogeneous* examples. Here, a real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be homogeneous if M is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space. For example, we recall the following fact. In $\mathbb{C}H^n(c)$ there exist homogeneous real hypersurfaces with positive sectional curvature and also ones with negative sectional curvature. On the other hand, $\mathbb{C}P^n(c)$ admits homogeneous real hypersurfaces with positive sectional curvature, but does not admit those with nonpositive curvatures (*cf.* [9]).

Thus it is natural to study Ricci curvatures and scalar curvatures of homogeneous real hypersurfaces in $\widetilde{M}_n(c)$. In this paper, we pay particular attention to the case that M^{2n-1} is minimal in such ambient spaces. Our aim here is to compute Ricci curvatures and scalar curvatures of *minimal* homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$.

2. HOMOGENEOUS MINIMAL REAL HYPERSURFACES IN $\widetilde{M}_n(c)$

First of all we recall some fundamental notions on real hypersurfaces in a complete and simply connected nonflat complex space form. Let M be a real hypersurface of $\widetilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. Denote by g the standard Riemannian metric and by J the canonical Kähler structure of $\widetilde{M}_n(c)$. Then the hypersurface M can be equipped with an *almost contact metric structure* (ϕ, ξ, η, g) which consists of a tensor field

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ϕ of type $(1,1)$, a vector field ξ, a 1-form η and the induced Riemannian metric g. That is, we define ϕ, ξ and η on M by

$$\xi = -J\mathcal{N}, \quad \eta(X) = g(X,\xi) = g(JX,\mathcal{N}) \quad \text{and} \quad \phi X = JX - \eta(X)\mathcal{N} \tag{2.1}$$

for each tangent vector $X \in TM$. The structure satisfies

$$\begin{gathered}\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \\ \eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta(\phi X) = 0\end{gathered} \tag{2.2}$$

for all vectors $X, Y \in TM$. We call the vector field ξ the *characteristic vector field* on M.

The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M_n}(c)$ and ∇ of M are related by $\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)\mathcal{N}$ and $\widetilde{\nabla}_X \mathcal{N} = -AX$ for vector fields X and Y tangent to M, where A is the shape operator of M in $\widetilde{M_n}(c)$. Moreover, we have the following equations.

$$\nabla_X \xi = \phi AX, \tag{2.3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \tag{2.5}$$

The last one is known as the equation of Codazzi.

Eigenvalues and eigenvectors of the shape operator A of M are called *principal curvatures* and *principal curvature vectors* of M in $\widetilde{M_n}(c)$, respectively. We set $V_\lambda = \{X \in TM \mid AX = \lambda X\}$, which is called the *principal distribution* associated to the principal curvature λ. We call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M.

Next, we review the classification of homogeneous real hypersurfaces in $\widetilde{M_n}(c)$. Takagi ([12, 13]) classified homogeneous real hypersurfaces in $\mathbb{C}P^n(c)(c > 0)$ in an algebraic style. By virtue of the works of Cecil and Ryan ([4]) and Kimura ([6]), we can state geometrically that a homogeneous real hypersurface in $\mathbb{C}P^n(c)$ with $n \geqq 2$ is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant:

- (A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;
- (A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \pi/\sqrt{c}$;
- (B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
- (C) A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n ($\geqq 5$) is odd;
- (D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
- (E) A tube of radius r around a Hermitian symmetric space SO(10)/U(5), where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

Unifying types (A_1) and (A_2), we call them of type (A).

In Tables 1 and 2, we denote by δ the principal curvature associated with the characteristic vector ξ, that is, $A\xi = \delta\xi$. We also put $\tilde{r} := (\sqrt{|c|}\, r)/2$.

Hereinafter, we use these notations for simplicity. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (*cf.* [13]):

TABLE 1. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$

Type	Principal curvatures	Multiplicities
(A_1)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$	$2n-2$
(A_2)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$	$2n-2\ell-2$
	$\lambda_2 = -(\sqrt{c}/2)\tan\tilde{r}$	2ℓ
(B)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	$n-1$
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	$n-1$
(C)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	2
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	2
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	$n-3$
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	$n-3$
(D)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	4
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	4
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	4
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	4
(E)	$\delta = \sqrt{c}\cot 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{c}/2)\cot\{\tilde{r}-(\pi/4)\}$	6
	$\lambda_2 = (\sqrt{c}/2)\cot\{\tilde{r}+(\pi/4)\}$	6
	$\lambda_3 = (\sqrt{c}/2)\cot\tilde{r}$	8
	$\lambda_4 = -(\sqrt{c}/2)\tan\tilde{r}$	8

Note that by putting $\ell = 0$ in the case of homogeneous real hypersurfaces of type (A_2) we can obtain the case of type (A_1).

We describe the case of $\mathbb{C}H^n(c)$ ($n \geqq 2$). Let M be a homogeneous real hypersurface in such an ambient space. Then, thanks to [3], we know that M is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;
($A_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;
($A_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \infty$;
(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$;
(S) The homogeneous ruled real hypersurface HR determined by a horocycle in a totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$, or an equidistant hypersurface from HR at distance r, where $0 < r < \infty$;
(W_1) A tube of radius r around the minimal ruled submanifold W^{2n-k} with $k \in \{2, \dots, n-1\}$, where $0 < r < \infty$;

- (W$_2$) A tube of radius r around the minimal ruled submanifold W_{φ}^{2n-k} for some $\varphi \in (0, \pi/2)$ and $k \in \{2, \dots, n-1\}$, where k is even and where $0 < r < \infty$.

Unifying real hypersurfaces of types (A$_0$), (A$_{1,0}$), (A$_{1,1}$) and (A$_2$), we call them real hypersurfaces of type (A). In the above list, all examples of types (A) and (B) are Hopf hypersurfaces and others are non-Hopf. The principal curvatures of homogeneous hypersurfaces are given in Table 2 ([1, 2]).

TABLE 2. The principal curvatures of homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$

Type	Principal curvatures	Multiplicities
(A$_0$)	$\delta = \sqrt{\lvert c\rvert}$	1
	$\lambda_1 = \sqrt{\lvert c\rvert}/2$	$2n-2$
(A$_{1,0}$)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$2n-2$
(A$_{1,1}$)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-2$
(A$_2$)	$\delta = \sqrt{\lvert c\rvert}\coth 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$2n-2\ell-2$
	$\lambda_2 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	2ℓ
(B)	$\delta = \sqrt{\lvert c\rvert}\tanh 2\tilde{r}$	1
	$\lambda_1 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$n-1$
	$\lambda_2 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$n-1$
(S)	$\lambda_1 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} + (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_2 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} - (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_3 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-3$
(W$_1$)	$\lambda_1 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} - (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_2 = (3\sqrt{\lvert c\rvert}/4)\tanh\tilde{r} + (\sqrt{\lvert c\rvert}/2)\sqrt{1-(3/4)\tanh^2\tilde{r}}$	1
	$\lambda_3 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-k-2$
	$\lambda_4 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$k-1$
(W$_2$)	$\lambda_i = -(\sqrt{\lvert c\rvert}/6)\left\{\coth\tilde{r}\left(u^i_{\tilde{r},\varphi} + \frac{1}{u^i_{\tilde{r},\varphi}}\right) - \operatorname{csch}\tilde{r}\operatorname{sech}\tilde{r} - 4\tanh\tilde{r}\right\}$	1
	for $i = 1, 2, 3$. The number $u^i_{\tilde{r},\varphi}$ is the i-th cubic root of $\left(\beta_{\tilde{r},\varphi} + \sqrt{\beta^2_{\tilde{r},\varphi} - 4}\right)/2$, where $\beta_{\tilde{r},\varphi} = 27\sin^2\varphi\tanh^2\tilde{r}\operatorname{sech}^4\tilde{r} - 2$.	
	$\lambda_4 = (\sqrt{\lvert c\rvert}/2)\tanh\tilde{r}$	$2n-k-2$
	$\lambda_5 = (\sqrt{\lvert c\rvert}/2)\coth\tilde{r}$	$k-2$

Note that a real hypersurface of type (B) with radius $r = (1/\sqrt{\lvert c\rvert})\log(2+\sqrt{3})$ has two distinct principal curvatures $\lambda_1 = \delta = \sqrt{3\lvert c\rvert}/2$ and $\lambda_2 = \sqrt{\lvert c\rvert}/(2\sqrt{3})$. It has three distinct principal curvatures for other case. Moreover, the principal curvatures of the homogeneous ruled real hypersurface *HR* can be obtained as limits of those of hypersurfaces of type (S) given in Table 2 by taking $r \to 0$.

3. Ricci curvatures of minimal homogeneous real hypersurfaces

In this section, we investigate Ricci curvatures of minimal homogeneous real hypersurfaces M in nonflat complex space forms $\widetilde{M}_n(c)$ $(n \geqq 2)$. We first recall that the Ricci tensor S of an arbitrary real hypersurface M in $\widetilde{M}_n(c)$ is expressed as

$$SX = (c/4)\{(2n+1)X - 3\eta(X)\xi\} + (\text{trace } A)AX - A^2X \tag{3.1}$$

for all $X \in TM$. The Ricci curvature $Ric(X,X) = g(SX,X)$ in the direction of unit vector $X \in TM$ is given as

$$Ric(X,X) = (c/4)\{(2n+1) - 3\eta(X)^2\} + (\text{trace } A)g(AX,X) - \|AX\|^2. \tag{3.2}$$

On the other hand, by solving the equation $\text{Trace}\, A = 0$ one can find easily a minimal homogeneous real hypersurface in $\widetilde{M}_n(c)$. In fact, a homogeneous real hypersurface M in $\mathbb{C}P^n(c)$ $(n \geqq 2)$ is minimal if and only if it is congruent to either of type (A_1), (A_2), (B), (C), (D) or (E), and the radius r satisfies the following cases, respectively:

(A_1) $\cot\tilde{r} = 1/\sqrt{2n-1}$;
(A_2) $\cot\tilde{r} = \sqrt{(2\ell+1)/(2n-2\ell-1)}$;
(B) $\cot\tilde{r} = \sqrt{n} + \sqrt{n-1}$;
(C) $\cot\tilde{r} = (\sqrt{n} + \sqrt{2}\,)/\sqrt{n-2}$;
(D) $\cot\tilde{r} = \sqrt{5}$;
(E) $\cot\tilde{r} = (\sqrt{15} + \sqrt{6}\,)/3$.

In the case of $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a homogeneous real hypersurface M is minimal if and only if it is congruent to the homogeneous ruled real hypersurface HR determined by a horocycle in a totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$.

Since every homogeneous real hypersurface in $\mathbb{C}P^n(c)$ is Hopf, it is enough to check $Ric(\xi,\xi)$ and $Ric(X,X)$ for each unit principal curvature vector X orthogonal to ξ in order to compute Ricci curvatures. For the homogeneous ruled real hypersurface HR in $\mathbb{C}H^n(c)$ we note that the characteristic vector ξ is a eigenvector of the Ricci tensor S, although the real hypersurface HR is not Hopf. Then, a straightforward computation shows the following:

Theorem 1. (1) *The Ricci curvature Ric of a minimal homogeneous real hypersurface in complex projective space* $\mathbb{C}P^n(c)$ $(n \geqq 2)$ *satisfies the following sharp inequalities*:

(A_1) $(c/4)(2n-2)/(2n-1) \leqq Ric \leqq (c/4)(4n^2-2)/(2n-1)$;
(A_2) $(c/4)\{2n - (2n-2\ell-1)/(2\ell+1) - (2\ell+1)/(2n-2\ell-1)\} \leqq Ric \leqq (c/4)\{2n+1-(2n-2\ell-1)/(2\ell+1)\}$;
(B) $(c/4)(-2n+2) \leqq Ric \leqq (c/4)\{2n+1-(\sqrt{n}-1)/(\sqrt{n}+1)\}$;
(C) $(c/4)\{n+2-\sqrt{n(n-2)}\} \leqq Ric \leqq (c/4)\{n+2+\sqrt{n(n-2)}\}$;
(D) $(c/4)(31-3\sqrt{5})/2 \leqq Ric \leqq (c/4)(31+3\sqrt{5})/2$;
(E) $(c/4)(27-\sqrt{15}) \leqq Ric \leqq (c/4)(27+\sqrt{15})$.

(2) *The Ricci curvature Ric of the minimal homogeneous real hypersurface HR in complex hyperbolic space* $\mathbb{C}H^n(c)$ $(n \geqq 2)$ *satisfies the following*

sharp inequalities:

$$(c/4)(2n+2) \leqq Ric \leqq (c/4)(2n-1).$$

For a minimal real hypersurface in $\mathbb{C}P^n(c)$ the following theorem is known.

Theorem A ([8])**.** *Let M be a minimal real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 3)$. Suppose that the Ricci curvature Ric of M satisfies $c(n-1)/2 \leqq Ric \leqq cn/2$. Then M is locally congruent to the minimal homogeneous real hypersurface of type* (A_2) *with $2\ell = n-1$. In this case, M is a tube of radius $\pi/(2\sqrt{c})$ around a totally geodesic $\mathbb{C}P^\ell(c)$.*

Related to the above theorem, we pose the following.

Problem 1. Let M be a compact orientable minimal real hypersurface of $\mathbb{C}P^n(c)$ $(n \geqq 3)$. If every Ricci curvature of M is not less than $c(n-1)/2$, is M congruent to the tube of radius $\pi/(2\sqrt{c})$ around a totally geodesic $\mathbb{C}P^\ell(c)$ with $2\ell = n-1$?

4. Ricci curvatures of homogeneous real hypersurfaces of types (A) and (B)

In this section, we investigate Ricci curvatures $Ric(X,X)$ with $\|X\| = 1$ of homogeneous real hypersurfaces M of types (A) and (B) in a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. By a direct computation we have the following propositions.

Proposition 1. *Let M be a real hypersurface of type* (A_1) *in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Denote by Ric the Ricci curvature of M and put $\tilde{r} = (\sqrt{c}\, r)/2$. Then the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = (c/2)\{n + (n-1)\cot^2\tilde{r}\},$$
$$\min Ric = (c/2)(n-1)\cot^2\tilde{r}$$

Proposition 2. *Let M be a real hypersurface of type* (A_2) *in complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 3)$. Denote by Ric the Ricci curvature of M and by X_i a unit principal curvature vector with corresponding principal curvature λ_i $(i = 1, 2)$. Then, we have*

$$\begin{aligned} Ric(\xi,\xi) &= (c/2)\{(n-\ell-1)\cot^2\tilde{r} + \ell\tan^2\tilde{r}\},\\ Ric(X_1,X_1) &= (c/2)\{n-\ell+(n-\ell-1)\cot^2\tilde{r}\},\\ Ric(X_2,X_2) &= (c/2)(1+\ell\sec^2\tilde{r}), \end{aligned}$$

where $\tilde r = (\sqrt{c}\, r)/2$. *Moreover, the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = \begin{cases} Ric(X_1, X_1) & \textit{if}\ \ 0 < r \leqq (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell-1)/\ell}\,, \\ Ric(X_2, X_2) & \textit{if}\ \ (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell-1)/\ell}\ < r, \end{cases}$$

$$\min Ric = \begin{cases} Ric(X_2, X_2) & \textit{if}\ \ 0 < r \leqq (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell-1)/(\ell+1)}\,, \\ R(\xi, \xi) & \textit{if}\ \ (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell-1)/(\ell+1)}\ < r \\ & \qquad\qquad \leqq (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell)/\ell}\,, \\ Ric(X_1, X_1) & \textit{if}\ \ (2/\sqrt{c}) \tan^{-1}\sqrt{(n-\ell)/\ell}\ < r. \end{cases}$$

Proposition 3. *Let* M *be a real hypersurface* M *of type* (B) *in complex projective space* $\mathbb{C}P^n(c)$ $(n \geqq 2)$. *Denote by Ric the Ricci curvature of* M *and by* X_i *a unit principal curvature vector with corresponding principal curvature* λ_i $(i = 1, 2)$. *Then, we have*

$$\begin{aligned} Ric(\xi, \xi) =& - (c/2)(n-1), \\ Ric(X_1, X_1) =& (c/4)\{2n - 2 - \tan\tilde r - \cot\tilde r \\ & + 4(n-2)/(\tan\tilde r + \cot\tilde r - 2)\}, \\ Ric(X_2, X_2) =& (c/4)\{2n - 2 + \tan\tilde r + \cot\tilde r \\ & - 4(n-2)/(\tan\tilde r + \cot\tilde r + 2)\}, \end{aligned}$$

where $\tilde r = (\sqrt{c}\, r)/2$. *Moreover, we set* $T = \tan\tilde r + \cot\tilde r\,(> 2)$. *Then the maximum and the minimum values of Ric are given as follows*:

$$\max Ric = \begin{cases} Ric(X_1, X_1) & \textit{if}\ \ 2 < T < 2\sqrt{n-1}, \\ Ric(X_2, X_2) & \textit{if}\ \ 2\sqrt{n-1} \leqq T, \end{cases}$$

$$\min Ric = \begin{cases} Ric(\xi, \xi) & \textit{if}\ \ 2 < T < 2n - 1 + \sqrt{4n^2 - 8n + 1}, \\ Ric(X_1, X_1) & \textit{if}\ \ 2n - 1 + \sqrt{4n^2 - 8n + 1} \leqq T. \end{cases}$$

5. The derivative of the Ricci tensor on real hypersurfaces of type (A)

In this section, we calculate the length of the derivative of the Ricci tensor of real hypersurfaces of type (A) in a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$. It is known that there exist no real hypersurfaces with parallel Ricci tensor in nonflat complex space forms $\widetilde{M}_n(c)$ with $n \geqq 3$ (see [11]). We establish the following.

Proposition 4. *Let* M *be a real hypersurface* M *of type* (A) *in complex projective space* $\mathbb{C}P^n(c)$ $(n \geqq 2)$, *that is to say,* M *is a tube of radius* r *around a totally geodesic* $\mathbb{C}P^\ell(c)$ *with* $0 \leqq \ell \leqq n-2$ *and* $0 < r < \pi/\sqrt{c}$. *Denote by* S *the Ricci tensor of* M *in* $\mathbb{C}P^n(c)$ *and put* $\tilde r = (\sqrt{c}\, r)/2$. *Then we have*

$$\begin{aligned} \|\nabla S\|^2 =& (c^3/4)(n-\ell-1)\{(n-\ell)\cot\tilde r - \ell\tan\tilde r\}^2 \\ & + (c^3/4)\ell\{(n-\ell-1)\cot\tilde r - (\ell+1)\tan\tilde r\}^2. \end{aligned} \tag{5.1}$$

Set $x = \cot\tilde r$ *and denote the right-hand side of* (5.1) *by* $F(x)$. *Then the function* $F(x)$ *takes its minimum at* $x = \{\ell(n\ell + \ell + 1)/(n-\ell-1)(n^2 - n\ell - \ell)\}^{1/4}$.

Proof. Since trace A is constant, we have from (3.1) that

$$\begin{aligned}(\nabla_X S)Y =&\nabla_X(SY) - S\nabla_X Y\\ =&-(3c/4)\{g(\phi AX,Y)\xi + \eta(Y)\phi AX\}\\ &+(\text{trace } A)(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.\end{aligned} \tag{5.2}$$

We recall the fact that a connected real hypersurface M in a nonflat complex space form is locally congruent to a hypersurface of type (A) if and only if the shape operator A of M satisfies

$$(\nabla_X A)Y = -(c/4)\{g(\phi X,Y)\xi + \eta(Y)\phi X\} \tag{5.3}$$

for $X, Y \in TM$ ([10, 11]). The equation (5.2), together with (5.3), yields

$$\begin{aligned}(\nabla_X S)Y = -(c/4)\{&3g(\phi AX,Y)\xi + 3\eta(Y)\phi AX + (\text{trace } A)g(\phi X,Y)\xi\\ &+(\text{trace } A)\eta(Y)\phi X - g(\phi X,AY)\xi - \eta(AY)\phi X\\ &- g(\phi X,Y)A\xi - \eta(Y)A\phi X\}.\end{aligned} \tag{5.4}$$

We decompose the tangent bundle TM of M as the direct sum of principal distributions: $TM = V_\delta \oplus V_{\lambda_1} \oplus V_{\lambda_2}$, where $\delta = \sqrt{c}\cot 2\tilde{r}$, $\lambda_1 = (\sqrt{c}/2)\cot\tilde{r}$, $\lambda_2 = -(\sqrt{c}/2)\tan\tilde{r}$ and $\dim V_\delta = 1$, $\dim V_{\lambda_1} = 2n-2\ell-2$,$\dim V_{\lambda_2} = 2\ell$ $(0 \leqq \ell \leqq n-2)$ (see Table 1). Needless to say that $V_\delta = \mathbb{R}\xi$.

Now, we have the following lemma.

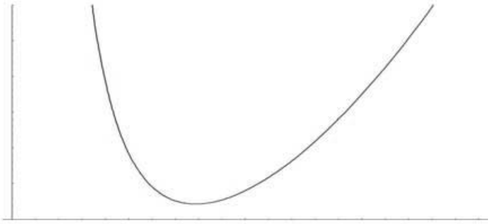
Lemma 1 ([5, 10]). *Let M be a Hopf hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. If a nonzero vector $X \in TM$ orthogonal to ξ satisfies $AX = \lambda X$, then $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$ holds, where δ is the principal curvature associated with ξ.*

By virtue of Lemma 1 we find that $\phi V_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2)$ for a hypersurface of type (A). Equation (5.4), combined with this fact, yields the following:

$$\begin{aligned}&(\nabla_X S)Y = -(c/4)(\text{trace } A + \lambda_1 - \lambda_2)g(\phi X,Y)\xi \quad \text{if } X,Y \in V_{\lambda_1},\\ &(\nabla_X S)Y = -(c/4)(\text{trace } A + \lambda_2 - \lambda_1)g(\phi X,Y)\xi \quad \text{if } X,Y \in V_{\lambda_2},\\ &(\nabla_X S)\xi = -(c/4)(\text{trace } A + \lambda_1 - \lambda_2)\phi X \quad \text{if } X \in V_{\lambda_1},\\ &(\nabla_X S)\xi = -(c/4)(\text{trace } A + \lambda_2 - \lambda_1)\phi X \quad \text{if } X \in V_{\lambda_2},\\ &(\nabla_X S)Y = 0 \quad \text{if } X \in V_{\lambda_i}, Y \in V_{\lambda_j} (i \neq j),\\ &(\nabla_\xi S)Z = 0 \quad \text{for any } Z \in TM.\end{aligned} \tag{5.5}$$

Let $\{\xi, X_1, \phi X_1, \dots, X_{n-\ell-1}, \phi X_{n-\ell-1}, Y_1, \phi Y_1, \dots, Y_\ell, \phi Y_\ell\}$ be an orthonormal basis of TM with $V_{\lambda_1} = Span\{X_1, \phi X_1, \dots, X_{n-\ell-1}, \phi X_{n-\ell-1}\}$ and $V_{\lambda_2} = Span\{Y_1, \phi Y_1, \dots, Y_\ell, \phi Y_\ell\}$. We apply equations in (5.5) to this basis. Then we can get (5.1) after a computation. The last statement can also be obtained by elementary calculation. □

Remark 1. The following is the graph of the function of $F(x)$ given in Proposition 4.



Remark 2. We put $x_0 = \cot \tilde{r}_0 = \{\ell(n\ell+\ell+1)/(n-\ell-1)(n^2-n\ell-\ell)\}^{1/4}$ ($0 \leqq \ell \leqq n-2$, $0 \leqq r_0 < \pi/\sqrt{c}$), which is the value where $F(x) = \|\nabla S\|^2$ takes its minimum. On the other hand, a homogeneous real hypersurface of type (A) is minimal if and only if $\cot \tilde{r} = \sqrt{(2\ell+1)/(2n-2\ell-1)}$. Denote this value by m_0. Then we have the following: $m_0 < x_0 \Leftrightarrow n-1 < 2\ell$; $m_0 = x_0 \Leftrightarrow n-1 = 2\ell$; $m_0 > x_0 \Leftrightarrow n-1 > 2\ell$.

6. Scalar curvatures of minimal homogeneous real hypersurfaces

We study all minimal homogeneous real hypersurfaces in nonflat complex space forms $\widetilde{M}_n(c)$ ($n \geqq 2$) by their scalar curvatures. Let M be an arbitrary real hypersurface in $\widetilde{M}_n(c)$. Then the scalar curvature $\rho = \text{trace } Ric$ is given by

$$\rho = c(n^2-1) + (\text{trace } A)^2 - \text{trace } (A^2). \tag{6.1}$$

Theorem 2. *Let M be a minimal homogeneous real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ ($n \geqq 2$). Then the scalar curvature ρ of M satisfies the following*:

(1) $\rho = (c/2)(2n^2-n-1)$ *when M is of type* (A) *in* $\mathbb{C}P^n(c)$;
(2) $\rho = (c/2)(2n^2-3n-1)$ *when M is of either type* (B), (C), (D) *or* (E) *in* $\mathbb{C}P^n(c)$;
(3) $\rho = (c/2)(2n^2-1)$ *when M is the minimal homogeneous real hypersurface HR in* $\mathbb{C}H^n(c)$.

Remark 3. Theorem 2 shows that we cannot distinguish minimal homogeneous real hypersurfaces of types (B), (C), (D) and (E) in $\mathbb{C}P^n(c)$ by their scalar curvatures.

The following theorem is known.

Theorem B ([7]). *Let M be a compact orientable minimal real hypersurface in $\mathbb{C}P^n(c)$ ($n \geqq 2$). Suppose that the scalar curvature ρ of M satisfies $\rho \geqq (c/2)(2n^2-n-1)$. Then $\rho = (c/2)(2n^2-n-1)$ and M is congruent to a homogeneous real hypersurfaces of type* (A).

The following problem is still open.

Problem 2. Let M be a minimal non-homogeneous real hypersurface in $\mathbb{C}P^n(c)$ ($n \geqq 2$). Does there exist M (even in local) with the scalar curvature ρ satisfying the following inequalities?

$$(c/2)(2n^2-3n-1) \leqq \rho \leqq (c/2)(2n^2-n-1).$$

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LIFE PERIOD ESTIMATION BY CELLULAR AUTOMATON IN CYCLE GRAPHS

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Abstract

Cellular automata have a configuration consisting of cells which may become a state “live (infected)” and “dead (non-infected)”, and a configuration evolves according to some rules with respect to time. Cellular automata also have been used for simulations of spreading some disease. We often have difficulty to estimate the evolution of configurations. In this manuscript, we focus on a cycle graph with $2^k(k > 1)$ cells and 1D cellular automaton rule 90. We first show that any initial configuration becomes a null configuration which consists of all “non-infected” cells with a time period of a finite number. Furthermore, some theorems give an estimation for the time period of an initial configuration until the null configuration by the position of the cells without any simulation or numerical computations.

1 Introduction

The system of cellular automata has been originally proposed by Stanislaw Ulam and John von Neumann for studying the growth of crystals [3] and building self-replicating robots [8]. Cellular automata have been used for representing some epidemic models [11, 7, 2, 9] with 2D models as well. Recently, we have faced some epidemic diseases such as influenza, MERS, SARS, and COVID-19, and it is important to estimate how diseases are spread with respect to time in real applications. Moreover, we often want to obtain an upper bound of the time period until disappearing infected patients or epidemics.

Before considering 2D models, we focus on the cellular automata in cycle graphs with 2^k cells ($k \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers). We have a cell whose stage can be either “live (1)” or “dead (0)”. We could consider the two stages which are “infected” and “cured (non-infected)” in some graphs whose edges represent the connection between people. A disease is often spread with respected to time by the interaction with people, i.e. an infected person interacts

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Table 1: Cellular automaton rule 90

state	111	110	101	100	011	010	001	000
future state	0	1	0	1	1	0	1	0

with a non-infected person. Therefore, we need to define a rule how a disease is spread in human society or some graphs.

In 1D cellular automata, there are 256 rules in total, and many rules have been studied in [5, 10, 13]. This manuscript focuses on rule 90 which has known as a generator of Sierpinski triangle [6, 12] (see Table 1). In the top of Table 1, we have three consecutive cells (state) at time t. The center of the three becomes 0 or 1 at future time $t+1$ (future state) according to the neighbor cells. We can interpret the state 101 at time t and the center cell 0 stays 0 at the future state as follows; when two infected people around a non-infected person, the non-infected person pays an attention of a disease carefully for not getting infected. This rule is named rule 90 because $0{*}2^7+1{*}2^6+0{*}2^5+1{*}2^4+1{*}2^3+0{*}2^2+1{*}2^1+0{*}2^0 = 90$.

We prepare an *initial configuration* which consists of cells in cycle graphs and observe how the initial configuration evolves with respect to time according to rule 90. This manuscript first shows that any initial configurations become a *null configuration* which consists of "dead" cells with a time period of t, $t \leq 2^{k-1}$. In real life applications, it is important to know the time period until all people get cured from some disease. However, we often have some difficulty of predicting the behavior of configuration in general cellular automata because the evolution of configurations often acts "randomly" or "repeatedly" [12]. That is because an initial configuration with a small number of "live" cells does not usually mean that it becomes a null space quickly. Therefore, we detect a set of initial configurations which become the null configuration with a time period of t, where $t = 2^{k-1}, t \leq 2^{k-2}$, etc. The objective of this manuscript is to estimate an upper bound of a time period of a given initial configuration until the null configuration without any simulations.

The remainder of this manuscript is organized as follows. First in Section 2, we introduce cycle graphs and show any initial configuration becomes a null configuration with a time period of some finite number. In Section 3, we detect a set of initial configurations which becomes a null configuration with a time period of t, $t = 2^{k-1}$, $t \leq 2^{k-2}$. In Section 4, we give simulations with some instances following our theorems. Finally, the manuscript is concluded in Section 5.

2 CYCLE GRAPHS

We let C_n be a cycle graph consisting n vertices (cells) and n edges. We let a vector of n consecutive cells in the cycle graph $B = [b_0, \ldots, b_{n-1}]$, where a cell at position i, $b_i \in \{0,1\}$. The cells b_i $(i = 0, \ldots, n-1)$ at the position i are located clockwise. If a cell $b_i = 0, 1$ holds, then a cell i is "dead", "live", respectively. We note that b_0 and b_{n-1} are connected with an edge. If all cells b_i $(i = 0, \ldots, n-1)$ are 0, then the configuration is called a null configuration B_{null}. We also describe a configuration $B^{(t)} = [b_0^{(t)}, \ldots, b_{n-1}^{(t)}]$ at time t.

We again introduce rule 90 of 1D cell automaton (see Table 1). A future state $b_i^{(t+1)}$ at time $t+1$ and position i can be defined by two cells $b_{i-1}^{(t)}, b_{i+1}^{(t)}$ at previous time t and position $i-1$, $i+1$ as follows, $b_i^{(t+1)} = b_{i-1}^{(t)} + b_{i+1}^{(t)} \pmod 2$. Therefore, the future configuration according to rule 90 can be obtained by the following matrix multiplication in modulo 2.

$$B^{(t+1)T} = Adj(C_n) B^{(t)T} \pmod 2, \tag{1}$$

where B^T represents the transpose of a vector B and $Adj(C_n)$ is an adjacency matrix of C_n.

For instance, if an initial configuration at time 0 in a cycle graph with eight cells is $B^{(0)} = [1,1,0,0,1,0,1,1]$, then the next configurations at time 1, 2, 3 are $B^{(1)} = [0,1,1,1,0,0,1,0]$, $B^{(2)} = [1,1,0,1,1,1,0,1]$, $B^{(3)} = [0,1,0,1,0,1,0,1]$, $B^{(4)} = [0,0,0,0,0,0,0,0]$ according to rule 90. This instance never becomes a null configuration because of "repeatedly".

Although this manuscript focuses on rule 90, rule 60, 102, and 150 can be expressed with some modifications for equation (1) (see details in Appendix A). We also obtain the following equation by induction;

$$B^{(t)T} = Adj(C_n)^{t-1} B^{(0)T} \pmod 2. \tag{2}$$

This manuscript focuses on circle graphs C_n, $n = 2^k$, a natural number $k > 1$. We show a theorem that any initial configurations $B^{(0)}$ become a null configuration B_{null} with a time period of t, $t \leq 2^{k-1}$. We prove the theorem combinatorically although the theorem has been proved polynomially [13]. For the theorem, we first prepare some lemmas. We first introduce the feature of an adjacency matrix. We note that the (i,j)th entry a_{ij} of $Adj(C_n)^m$ counts the number of walks (ways) of length m having start and end cells b_i and b_j, respectively [4].

Next, we consider Pascal's triangle and values with rows $2^k, 2^k - 1$ of the triangle. At first, we consider values with 2^kth row. We show that all values $\binom{2^k}{n}$ except the two ends $(n = 0, 2^k)$ are even by using mathematical induction.

Lemma 2.1. *If $n, k \in \mathbb{N}$ and $0 < n < 2^k$ holds, then $\binom{2^k}{n}$ is even.*

Proof. By induction, we assume that $(x+1)^{2^k} = x^{2^k} + 1 \pmod 2$. We obtain

$$(x+1)^{2^{k+1}} = ((x+1)^{2^k})^2 = (x^{2^k}+1)^2 = x^{2^{k+1}} + 2x^{2^k} + 1 = x^{2^{k+1}} + 1 \pmod 2.$$

□

We consider the values with $2^k - 1$th row of Pascal's triangle. We show that all values $\binom{2^k-1}{n}$ are odd by using lemma 2.1.

Lemma 2.2. *If $n, k \in \mathbb{N}$ and $1 \leq k$ holds, then $\binom{2^k-1}{n}$ is odd.*

Proof. All values $\binom{2^k-1}{n}$ in row 2^k-1 of Pascal's triangle are odd since all values in row 2^k is even except for the two ends $\binom{2^k}{0}$, $\binom{2^k}{2^k}$ by lemma 2.1. $\square$

Theorem 2.1. *For a natural number $k > 1$, any initial configurations $B^{(0)}$ of C_{2^k} become B_{null} with a time period t, $t \leq 2^{k-1}$.*

Proof. We first consider $Adj(C_{2^k})^{2^{k-1}}$, and (i,j)th entry of $Adj(C_{2^k})^{2^{k-1}}$ represents the number of walks (ways) to reach a cell b_i from a cell b_j with length 2^{k-1}. Without loss of generality, in cycle graphs we consider a cell b_0. For the convenience of proof, we consider the number of walks of length 2^{k-1} from b_0 to b_j, $0 \leq j \leq 2^{k-1}$ in a half circle.

The distance between b_j and b_0 is j, and we consider all possible walks to get to b_j from b_0 with length 2^{k-1}.

Let x and y, $x, y \in \mathbb{N}$, be the number of lengths to clockwise and counterclockwise, respectively. Then, we obtain the following equations;

$$x + y = 2^{k-1}, \ x - y = j. \tag{3}$$

We next obtain

$$x = \frac{j + 2^{k-1}}{2}, \ y = \frac{-j + 2^{k-1}}{2}. \tag{4}$$

When $j \in \mathbb{N}$ is odd, $x, y \notin \mathbb{N}$, which means that the number of walks from b_0 to b_j is 0. When $j \in \mathbb{N}$ is even, the number of walks from b_0 to b_j can be obtained as follow;

$$\binom{x+y}{x} = \binom{2^{k-1}}{x}. \tag{5}$$

In the case $j = 2^{k-1}$, $\binom{2^{k-1}}{2^{k-1}} = 1$, and there is another way from the other half circle, therefore there are two walks to $b_{2^{k-1}}$.

In the other case $0 \leq j \leq 2^{k-1} - 2$, we obtain $2^{k-2} \leq x \leq 2^{k-1} - 1$ and $\binom{2^{k-1}}{x} = 0$, (mod 2) by lemma 2.1. Finally, we obtain $Adj(C_{2^k})^{2^{k-1}} = O$ in modulo 2. We obtain $Adj(C_{2^k})^{2^{k-1}} B^{(0)T} = OB^{(0)T} = B_{null}^T \pmod 2$. $\square$

By a similar manner of Theorem 2.1, when $B^{(t+1)}$ becomes a null configuration at time $t+1$, the configuration $B^{(t)}$ has only two configurations. The first case is a configuration with all "live" cells. The second case is a configuration which locates "live" and "dead" cells alternately. We can confirm that with some instances in Section 4.

3 SURVIVAL PERIOD

We consider a set of initial configurations which takes a time period of exactly 2^{k-1} until a null configuration. We first obtain a set of initial configurations which a time period of t, $t \leq 2^{k-1} - 1$. The compliment of the obtained set is a set that we want to obtain.

After that, we consider a set of initial configurations which takes a time period of t, $t \leq 2^{k-2}$ until a null configuration.

Theorem 3.1. *If* $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)}$ *or* $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)}$ *is odd, then an initial configuration* $B^{(0)} = [b_0^{(0)}, \ldots, b_{2^k-1}^{(0)}]$ *becomes* B_{null} *with a time period of exactly* 2^{k-1}.

Proof. We again consider a half circle between a cell b_0 and a cell b_j, $0 \leq j \leq 2^{k-1}$. Let x and y, $x, y \in \mathbb{N}$ be the number of lengths to clockwise and counter-clockwise, respectively. The distance between b_j and b_0 is j, where $0 \leq j \leq 2^{k-1} - 1$. We obtain the following equations:

$$x + y = 2^{k-1} - 1, \; x - y = j. \tag{6}$$

We obtain

$$x = \frac{j + 2^{k-1} - 1}{2}, \; y = \frac{-j + 2^{k-1} - 1}{2}. \tag{7}$$

When j is even, $x, y \notin \mathbb{Z}$, which means that the number of walks from b_0 to b_j is 0. When j is odd, the number of walks from b_0 to b_j obtained as follows:

$$\binom{x+y}{x} = \binom{2^{k-1}-1}{x} \tag{8}$$

is odd by the lemma 2.2. Therefore, when we obtain the entry (i, j)th entry a_{ij} of $Adj(C_{2^k})^{2^{k-1}-1}$, $a_{ij} = 0$ if $i - j$ is even, otherwise $a_{ij} = 1$.

If $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)}$ and $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)}$ are even, its initial configuration $B^{(0)}$ becomes B_{null} with a time period of t, $t \leq 2^k - 1$. The compliment of the set of the initial configurations is a set which we want to obtain. □

Next, a set of initial configurations which become B_{null} with a time period of t, $t \leq 2^{k-2}$ is considered with the similar technique of Theorem 3.1.

Theorem 3.2. *If* $|i - j| = 2^{k-1}$ *and* $b_i^{(0)} + b_j^{(0)}$ *is even for any* i, j, *then an initial configuration* $B^{(0)}$ *becomes* B_{null} *with a time period of* t, $t \leq 2^{k-2}$.

Proof. This proof is obtained with a similar manner as the proof of Theorem 3.1. □

We believe that we can obtain the statement that an initial configuration becomes B_{null} with a time period of t, $t \leq 2^{k-3}$ with the similar manner and some modifications.

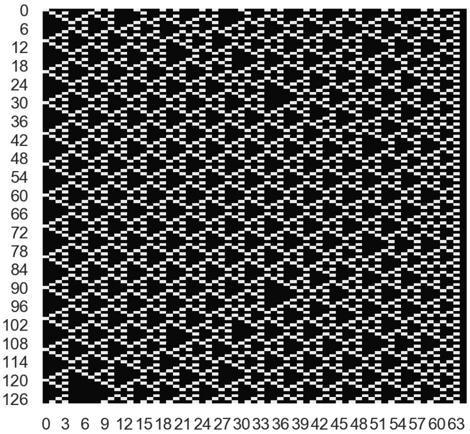
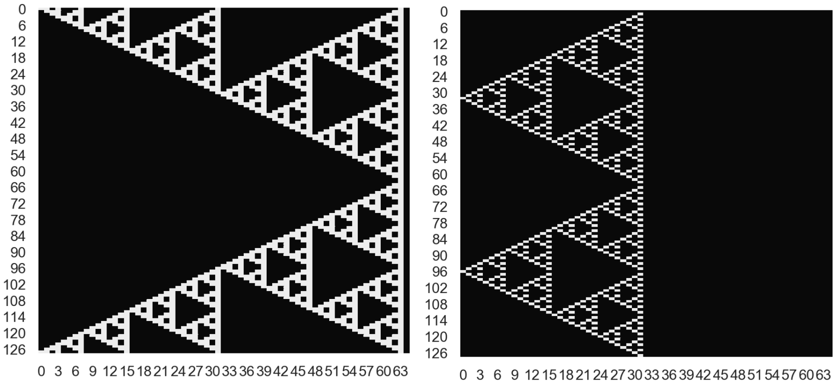


Figure 1: An initial configuration with thirteen "live" cells at {0, 10, 20, ..., 120}.



(a) (b)

Figure 2: (a) An initial configuration with two "live" cells at {0, 127}. (b) An initial configuration with two "live" cells at {32, 96}.

4 Experiments

To confirm our theorems, we prepare a cycle model $C_{2^7} = C_{128}$ and review our theorems with some initial configurations $B^{(0)}$. In Figure 1, 2, and 3, the ith column represents $B^{(i)}$ and the jth row represents b_j. Therefore, the value at ith column and jth row represents $b_j^{(i)}$. A white cell $b_j^{(i)}$ represents "live" or 1, and a black cell represents "dead" or 0.

At first, we prepare an initial configuration $B^{(0)}$ following Theorem 3.1,

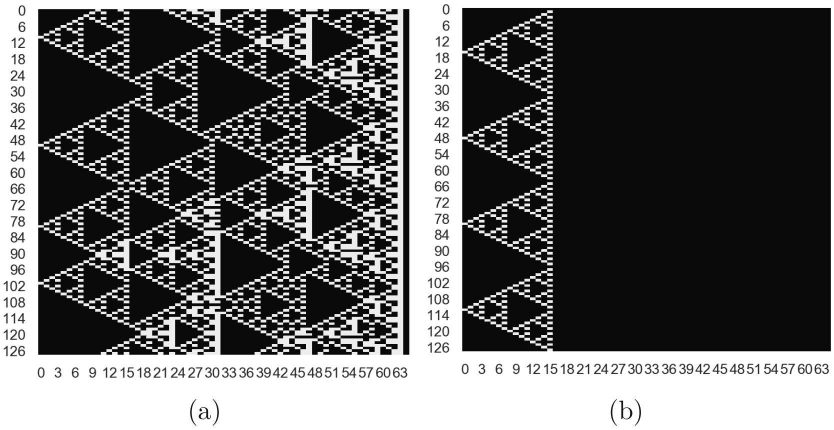


Figure 3: (a) An initial configuration with four "live" cells at {10, 50, 80, 101}. (b) An initial configuration with four "live" cells at {16, 48, 80, 112}.

which is $B^{(0)}$ with thirteen "live" cells; $b_j^{(0)} = 1, j \in \{0, 10, \ldots, 120\}$, otherwise $b_j^{(0)} = 0$. Since $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)} = 13$ (odd), we estimate the initial configuration $B^{(0)}$ becomes B_{null} with a time period of exactly $2^{7-1} = 64$ (see Figure 1). We also prepare an initial configuration $B^{(0)}$ following the Theorem 3.1. $B^{(0)}$ with two "live" cells $b_j^{(0)} = 1, j \in \{0, 127\}$, otherwise $b_j^{(0)} = 0$. Since $\sum_{i=0}^{2^{k-1}-1} b_{2i}^{(0)} = 1$ and $\sum_{i=0}^{2^{k-1}-1} b_{2i+1}^{(0)} = 1$, we also estimate the initial configuration becomes the null configuration with a time period of exactly 64 (see Figure 2(a)).

In Figure 2, we prepare two initial configurations with two "live" cells, which have different time periods until the null configuration. We prepare an initial configuration $B^{(0)}$ following the Theorem 3.2. $B^{(0)}$ with two "live" cells; $b_j^{(0)} = 1, j \in \{32, 96\}$, otherwise $b_j^{(0)} = 0$. Since $b_{32}^{(0)} + b_{96}^{(0)} = 2$ and the rest of $b_i^{(0)} + b_j^{(0)} = 0$, where $|i - j| = 2^{k-1} = 64$, the initial configuration becomes the null configuration with a time period of t, $t \leq 2^{7-2} = 32$ (see Figure 2(b)).

In Figure 3, we prepare two initial configurations with four "live" cells, which have different time periods until the null configuration. We prepare an initial configuration $B^{(0)}$; $b_j^{(0)} = 1, j \in \{10, 50, 80, 101\}$, otherwise $b_j^{(0)} = 0$ (see Figure 3(a)). We prepare another initial configuration $B^{(0)}$ with four "live" cells; $b_j^{(0)} = 1, j \in \{16, 48, 80, 112\}$, otherwise $b_j^{(0)} = 0$. Its initial configuration becomes B_{null} with a time period of $2^{7-3} = 16$ (see Figure 3(b)). Although we have the two similar initial configurations, we can see that the evolution of one configuration differs from that of the other due to the position of "live" cells.

5 CONCLUSION

In this manuscript, we study the cellular automata (rule 90) in the cycle graphs with $2^k, k > 1$ cells. First, we show that any initial configuration becomes the null configuration with a time period of exactly 2^{k-1} combinatorically. Next, we investigate the condition for a set of initial configurations with a time period of t, $t \leq 2^{k-2}$. With some modifications, we believe that we can obtain a set of initial configurations with a time period of 2^m, where $m < k$. According to simulations with some instances, we confirmed our theorems hold and the position of "live" and "dead" cells matters for the time period. For each initial configuration, we can estimate the time period until the null configuration without any simulations. We really hope this study gives some contribution for epidemic diseases in real applications.

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Appendix A Matrix Representations

Table 2: Cellular automaton rule 150

state	111	110	101	100	011	010	001	000
future state	1	0	0	1	0	1	1	0

With a given cycle graph C_n, we prepare a matrix A with an identity matrix I with size $n \times n$ such as $A = Adj(C_n) + I$. We can express rule 150 in Table 2 by using the matrix A. We obtain $B^{(t+1)}$ by matrix multiplication as follows;

$$B^{(t+1)T} = AB^{(t)T}. \tag{9}$$

By some modifications of the matrix A, we can express rule 60 and rule 102 by matrix multiplication of A.

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INVARIANT MEASURE OF ISOMETRIC ACTIONS ON METRIC SPACE

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Abstract. The late Prof. Y. Mibu proved that there exists an invariant measure under a condition that the action is transitive. Our aim in this note is to show that when space is a metric space, we do not need this condition.

1 Introduction. In [2] Prof. Matsumoto approched this problem from an angle of ergodic theory. Hence it is hard for us to understand its proof. Our proof is done by reducing the result of Mibu. Hence the proof can understand easily.

2 Notation and preparation. In this section we explain notation used in this note and do preparation for the proof of main theorem.

Definition 2.1. A metric space X is called proper if any closed ball $B_r(x) = \{y \in X \mid d(x,y) \leq r\}$ is compact for any $x \in X$ and $r > 0$. (The word comes from the properness of the distance function from x.)

Let X be a proper metric space and let $G = \{f \mid f\colon X \to X \text{ surjective, isometric}\}$ be the set of all the surjective isometries. We denote by $\mathcal{K}(X)$ (resp. $\mathcal{O}(X)$) the set of all compact (resp. open) subsets of X.

We introduce in G the compact-open topology [1]. Put

$$W(K,U) = \{f \in G \mid f(K) \subset U\} \tag{*}$$

for any K (resp. U) belonging $\mathcal{K}(X)$ (resp. $\mathcal{O}(X)$).

Definition 2.2. For any finite subset $\{W(K_i,U_i) \mid 1 \leq i \leq n\}$ of the form (*), all of their meet $\bigcap_{i=1}^{n} W(K_i,U_i)$ form an open base. One says this topology the compact-open topology in G.

For any compact sets $\{K_1, K_2\}$ and open sets $\{U_1, U_2\}$ such that $K_1 \subset K_2$ and $U_2 \subset U_1$, we have $W(K_2,U_2) \subset W(K_1,U_1)$.

Proposition 2.3. *(i) G is separable and metrizable with respect to the compact-open topology.*

(ii) For any $x \in X$ and a bounded closed ball $B_r(x_0)$ of X, $W(\{x\}, B_r(x_0))$ is a compact set with respect to the compact-open topology.

Proof. (i) Since X is a proper metric space, X is separable metric space. Let $D = \{d_k\}$ be a countable dense subset of X. Since X is a proper metric space, the closed ball $B_r(x_0)$ is a compact set in X.

Let V be a bounded open set in X. Let $B_r(x)$ be a closed ball. Since X is a proper metric space, for any $r > 0$ and $x \in X$, $B_r(x)$ is compact. For any $K \in \mathcal{K}(X)$ there exists

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an $r \in Q^+$ such that $K \subset B_r(x)$, where Q^+ is the set of all positive rational numbers. Since D is dense in X, there exists a d_k in D with $d_k \in V$. Hence there exists an open set $U_t(d_k)$ such that $d_k \in U_t(d_k) \subset \overline{U_t}(d_k) \subset V$ and $t \in Q^+$. We have $W(B_r(x), U_t(d_k)) \subset W(K, V)$. $\{W(B_r(d_l), U_t(d_k)) \,|\, r, t \in Q^+, d_k, d_l \in D\}$ is a countable open base and satisfies the regular condition. Hence G is metrizable.

(ii) $W(B_r(x_0), \overline{U_t}(d_k))$ is closed with respect to the compact-open topology, for $\overline{U_t}(d_k)$ is compact. Let $x \in B_r(x_0)$. By Ascoli's theorem $W(\{x\}, \overline{U_t}(d_k))$ is compact. Since $W(B_r(x), U_t(d_k)) \subset W(\{x\}, \overline{U_t}(d_k))$, $\overline{W}(B_r(x_0), U_t(d_k))$ is compact. Hence G is locally compact. □

3 Main theorem.

Let Y be a locally compact metric space with a distance function $d(x, y)$. Assume a group Γ acts on Y isometrically. For x in Y, put $\Gamma_x = \{\gamma x \,|\, \gamma \in \Gamma\}$, which is the orbit of y. We denote by X the closure $\overline{\Gamma_x}$ of the orbit.

Definition 3.1. When all the orbits are dense, one says that the action is minimal.

In this section we prove the main theorem.

Theorem 3.2 (main theorem)**.** *Let Y be a locally compact metric space. Assume a group Γ acts on Y isometrically. Then there is a Γ-invariant Radon measure on Y.*

The next proposition is a key to prove the main theorem.

Proposition 3.3. *Γ acts minimally on X. This means that any orbit contained in X is dense in X.*

Proof. We shall show that for any $y, z \in X$ and $\epsilon > 0$, there is $\gamma_y \in \Gamma$ such that $d({\gamma_y}^{-1} y, z) < \epsilon$. The point x used in the definition of X has the property that for any $\epsilon > 0$ and $y \in X$, there is γ_y such that $d(y, \gamma_y x) < \epsilon/2$, since X is the closure of the orbit of x. Hence we have

$$d(\gamma_z {\gamma_y}^{-1} y, z) \leq d(\gamma_z {\gamma_y}^{-1} y, \gamma_z x) + d(\gamma_z x, z) = d(y, \gamma_y x) + d(\gamma_z x, z) < \epsilon/2 + \epsilon/2 = \epsilon$$

and the proof is complete. □

This means that any orbit contained in X is dense in X. Since a closed subspace of a locally compact metric space is again locally compact, we have only to show the following.

Theorem 3.4. *Let X be a locally compact metric space. Assume a group Γ acts on X isometrically and minimally. Then there is a $\overline{\Gamma}$-invariant Radon measure on X.*

Proof. This is a special case of the main theorem of [2]. However if the space X is proper, we have a conceptually easier proof, which we shall discuss below.

Let G be the set of all the surjective isometries from X to X. By proposition 2.3, if X is a proper metric space, then standard argument shows that the group G of all the surjective isometries forms a locally compact metrizable group with respect to the compact-open topology (see proposition 2.3).

The closure $\overline{\Gamma}$ in G is also locally compact metrizable, and acts transitively on X, that is, for any $x, y \in X$ there exists an element g in $\overline{\Gamma}$ such that $gx = y$. Indeed, by proposition 3.3, for any positive integer n, there exists a $\gamma_n \in \overline{\Gamma} \subset G$ such that $d(\gamma_n x, y) < 1/n$. For any compact neighbourhood $\overline{U}(y)$ of y, $W(\{x\}, \overline{U}(y))$ is compact. Hence there exists a positive integer N such that $\gamma_n x \in \overline{U}(y)$ for any $n > N$. Accordingly, for any positive integer $n > N$, we have $\gamma_n \in W(\{x\}, \overline{U}(y))$. Hence there exists a convergent subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$. Its limit g is contained in the closure of $\Gamma \subset G$. Hence $gx = y$ holds, that is, the closure of Γ acts transitively on X. Therefore, Theorem 3.4 follows from a result of Mibu. □

The proof of main Theorem 3.2. Let Y be a locally compact metric space with a distance function $d(x,y)$. Put

$$G = C(Y,Y) = \{f \mid f\colon Y \to Y \text{ surjective, isometric}\}.$$

Let $\Gamma_x = \{\gamma x \mid \gamma \in \Gamma\}$. Γ acts isometrically on Y, that is, for any $x, y \in Y$ and $\gamma \in \Gamma$, $d(x,y) = d(\gamma x, \gamma y)$ holds. Γ acts minimally, that is, any orbit Γ_x is dense in Y. Assume the Y is proper. A group G is dense in Y. A group G is locally compact and metrizable with the compact-open topology and acts transitively on Y, that is, for any x, y in Y there exists g in $\overline{\Gamma}$ such that $gx = y$. By the result of Mibu we have Γ-invariant measure.

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TWO FORMS OF AIC BASED ON MODIFIED LASSO

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Abstract. The least absolute shrinkage and selection operator (LASSO) is a popular technique for variable selection and estimation in linear regression models. Introduction of information criteria for LASSO can decrease the computational cost efficiently. So far the forms of some classic information criteria for LASSO are derived. In fact, there exists some regression matrix such that the ordinary LASSO may not select the correct model efficiently even by information criteria. In such situation, [9] introduced modified LASSO approach. In this paper, we introduce two forms of Akaike information criterion (AIC) based on modified LASSO estimation to help find the optimal tuning parameters for prediction and variable selection purposes respectively. The properties of those two forms are shown and a simulation study comparing these two forms is conducted.

1 Introduction The least absolute shrinkage and selection operator (LASSO) is proposed by [7], and is a popular technique for variable selection and estimation in linear regression models. As we know, the performance of the LASSO relies heavily on the choice of tuning parameter λ to select the optimal model. For prediction purpose, the prediction error is estimated by using cross-validation (CV) or by information criteria ([2]). A drawback of using information criteria is that the degrees of freedom must be known. [8] showed that the number of nonzero coefficients is an unbiased estimate for the degree of freedom of the LASSO, and the unbiased estimator is shown to be asymptotically consistent. For variable selection purpose, choosing the optimal tuning parameter is more difficult since the prediction optimal value is inconsistent in the sense of correct selection. [4] shows that for certain high dimensional cases, generalized information criterion (GIC) on sub-models decided by LASSO is consistent in the sense of correct selection. In the paper, we consider a more general linear model, where the ordinary LASSO estimation may not work well. We consider the following linear model:

$$Y_i = \boldsymbol{x}_i'\boldsymbol{\beta}^* + \varepsilon_i,$$

where $1 \leq i \leq n$, $\boldsymbol{\beta}^* \in \mathbb{R}^p$, and $\{\varepsilon_i\}$ is independent and identically distributed process with $\varepsilon_i \sim N(0, \sigma^2)$. Let $\mathbf{x_i} = (x_{i1}, x_{i2}, \dots, x_{ip})'$ be a known nonrandom function of i. By $\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)'$, we discuss the estimation of $\boldsymbol{\beta}^*$ based on an observed stretch $\boldsymbol{Y} = (Y_1, \dots, Y_n)'$. Let $a_{jk}^n = \sum_{t=1}^n x_{tj}x_{tk}$, and we assume the following conditions on $\{\boldsymbol{x}_i\}$.

Assumption 1 *1.* $a_{jj}^n \to \infty$ $(n \to \infty)$, $(j = 1, \dots, p)$.

2. $\lim_{n\to\infty} \frac{x_{n+1,j}^2}{a_{jj}^n} = 0$, $(j = 1, \dots, p)$.

3. The limit

$$\lim_{n\to\infty}\frac{a_{jk}^n}{\sqrt{a_{jj}^n a_{kk}^n}}=\rho_{jk}$$

exists for $j,k=1,\ldots,p,\ h\in\mathbb{Z}$.

4. Letting $\Phi\equiv\{\rho_{jk}:j,k=1,\ldots,p\}$, Φ *is regular.*

The point of item 2 of Assumption 1 is to prevent that the last $x_{n+1,j}^2$ from being an appreciable part of the sum of squares for large n. Item 3 shows that the relations between regressors for all sufficiently large n are approximately fixed values. Item 4 is for avoidance of multicollinearity of the model. Obviously, the model includes the case that the norm of different column in regression matrix may have different order of sequence length n. For example, letting $x_{ij}=i^{j-1}$, $O(\sum_{i=1}^n x_{ij}^2)$ is greater than $O(n)$ when $j\geq 2$. In such condition, the ordinary LASSO estimation, where the estimators for the coefficients $\boldsymbol{\beta}^*$ are obtained by

$$\tilde{\boldsymbol{\beta}}(\lambda_n)=\arg\min_{\boldsymbol{\beta}}\sum_{i=1}^n(Y_i-\boldsymbol{x}_i'\boldsymbol{\beta})^2+\sum_{j=1}^p\lambda_n|\beta_j|,$$

might not work well in the sense of variable selection, where λ_n is a given tuning parameter. Correspondingly, it requires the modified LASSO estimation to match the different order of each column which was introduced by [9] as follows:

$$\hat{\boldsymbol{\beta}}(\lambda_n)=\arg\min_{\boldsymbol{\beta}}\sum_{i=1}^n(Y_i-\boldsymbol{x}_i'\boldsymbol{\beta})^2+\sum_{j=1}^p\lambda_n\sqrt{a_{jj}^n}|\beta_j|.$$

In the numerical results ([9]), it was shown that the estimation of modified LASSO has higher probability of correct selection of true model than that of the ordinary LASSO even by selecting an optimal λ_n with Akaike information criterion (AIC). In this paper, we construct two forms of AIC based on modified LASSO for prediction and variable selection purposes respectively in Section 2. In Section 3, the numerical analysis part, the selection and prediction performance of the modified LASSO when using the above two forms of AIC are analysised.

2 Main results

We first define some notations. Let $\hat{\boldsymbol{\mu}}_{\lambda_n}$ be the modified LASSO fit. $\hat{\mu}_i$ is the ith component of $\hat{\boldsymbol{\mu}}$. For convenience, we let $df(\lambda_n)$ stands for $df(\hat{\boldsymbol{\mu}}_{\lambda_n})$, the degrees of freedom of the modified LASSO. Suppose $\boldsymbol{W}$ is a matrix with p column. Let $\mathcal{S}$ be a subset of the indices set $\{1,2,\ldots,p\}$. Denote $\boldsymbol{W}_{\mathcal{S}}=[\cdots W_j\cdots]_{j\in\mathcal{S}}$, where W_j is the jth column of $\boldsymbol{W}$. Similarly, define $\boldsymbol{\beta}_{\mathcal{S}}=(\cdots\beta_j\cdots)_{j\in\mathcal{S}}$ for any vector $\boldsymbol{\beta}$ of length p. Let $\operatorname{sgn}(\cdot)$ be the sign function: $\operatorname{sgn}(x)=1$ if $x>0$; $\operatorname{sgn}(x)=0$ if $x=0$; $\operatorname{sgn}(x)=-1$, if $x<0$. Let $\mathcal{S}_0=\{j:\operatorname{sgn}(\boldsymbol{\beta}^*)_j\neq 0\}$ be the active set of $\boldsymbol{\beta}^*$, where $\operatorname{sgn}(\boldsymbol{\beta})$ is the sign vector of $\boldsymbol{\beta}$ given by $\operatorname{sgn}(\boldsymbol{\beta})_j=\operatorname{sgn}(\beta_j)$. We denote the active set of $\hat{\boldsymbol{\beta}}(\lambda_n)$ as $\mathcal{S}_0(\lambda_n)$ and the corresponding sign vector $\operatorname{sgn}(\hat{\boldsymbol{\beta}}(\lambda_n))$ as $\operatorname{sgn}(\lambda_n)$.

2.1 Prediction purpose

Prediction accuracy of a model can be assessed by calculating its prediction error, that is, the error when the model is used to predict a new sample of observations. Let $\hat{\boldsymbol{\mu}}$ be a model fit decided by $\boldsymbol{Y}$. The estimation of prediction error, covariance penalties (C_p) which was first introduced by [5], can be treated as a criterion to show how well $\hat{\mu}$ will predict a future dataset independently generated by the same linear regression model. Mallows shows that if $\hat{\boldsymbol{\mu}}=\boldsymbol{M}\boldsymbol{Y}$, where $\boldsymbol{M}$ is an $n\times n$ matrix not depending on $\boldsymbol{Y}$, $C_p(\hat{\boldsymbol{\mu}}) := \frac{\|\boldsymbol{Y}-\hat{\boldsymbol{\mu}}\|^2}{n} + \frac{2\,\mathrm{trace}(\boldsymbol{M})}{n}\sigma^2$ is an unbiased estimation for the expectation of prediction error. For a general estimate $\hat{\boldsymbol{\mu}} = m(\boldsymbol{Y})$, as in [10], C_p can be extended to

$$C_p(\hat{\boldsymbol{\mu}}) := \frac{\|\boldsymbol{Y}-\hat{\boldsymbol{\mu}}\|^2}{n} + \frac{2df(\hat{\boldsymbol{\mu}})}{n}\sigma^2,$$

where $df(\hat{\boldsymbol{\mu}}) := \frac{\sum_{i=1}^n \mathrm{cov}(\hat{\mu}_i, Y_i)}{\sigma^2}$. By the connection between Mallows's C_p ([5]) and AIC ([1]), we know

$$AIC(\hat{\boldsymbol{\mu}}) = \frac{C_p(\hat{\boldsymbol{\mu}})}{\sigma^2}.$$

In the following, we introduce the form of AIC for modified LASSO by following the line of [8].

From the properties of modified LASSO solution, for a given $\boldsymbol{Y}$, there is a finite sequence,

$$\lambda_{n0} > \lambda_{n1} > \lambda_{n2} > \cdots > \lambda_{nK} = 0,$$

such that for all $\lambda_n > \lambda_{n0}$, $\hat{\boldsymbol{\beta}}(\lambda_n) = \boldsymbol{0}$, and that for all $\lambda_n \in (\lambda_{n,m+1}, \lambda_{nm})$, the active set $\mathcal{S}_0(\lambda_n)$ and sign vector $\mathrm{sgn}(\lambda_n)$ are invariant with respect to λ_n. Thus we write them as $\mathcal{S}_m$ and sgn_m for simplicity. Noticing that for any $m = 0, \ldots, K-1$, when λ_n decreases from the right hand side of λ_{nm}, some predictors with zero coefficient at λ_{nm} are about to have nonzero coefficients, we call λ_{nm} as a transition point. Correspondingly, for any $\lambda_n \in [0, \infty) - \{\lambda_{nm}, m = 0, \ldots, K-1\}$, it is called as a non-transition point.

Theorem 1 *For any $\lambda_0 \geq 0$, the modified LASSO fit $\hat{\boldsymbol{\mu}}_{\lambda_n}(\boldsymbol{Y})$ is uniformly Lipschitz. Furthermore, under the condition that $\boldsymbol{X}$ is full rank, the degree of freedom of $\hat{\boldsymbol{\mu}}_{\lambda_n}(\boldsymbol{Y})$ equals the expectation of the cardinality of the active set $\mathcal{S}_0(\lambda_n)$, that is,*

$$df(\lambda_n) = \mathrm{E}|\mathcal{S}_0(\lambda_n)|.$$

Theorem 1 shows that $\widehat{df}(\lambda_n) \equiv |\mathcal{S}_0(\lambda_n)|$ is an unbiased estimate for $df(\lambda_n)$. In the following, we show that $\widehat{df}(\lambda_n)$ is also consistent.

Assumption 2 *There exists $\gamma > 0$ so that*

$$\min_i a_{ii}^n = O(n^\gamma), \quad for\ n \to \infty.$$

Lemma 1 *Assume that Assumptions 1 and 2 hold, and that $\lambda_n = O(n^\zeta)$ where $0 < \zeta < \gamma/2$, then,*

$$P(\mathcal{S}_0(\lambda_n) = \mathcal{S}_0) = 1, \quad for\ n \to \infty.$$

Theorem 2 *If $\frac{\lambda_n}{n^\zeta} \to \lambda^*$, then $\widehat{df}(\lambda_n) \to df(\lambda_n)$ in probability.*

Proof. From Lemma 1, $P(\mathcal{S}_0(\lambda_n) = \mathcal{S}_0) \to 1$. Immediately we see $\widehat{df}(\lambda_n) \to \mathcal{S}_0$ in probability. Then by the dominated convergence theorem, we have

$$df(\lambda_n) = E[\widehat{df}(\lambda_n)] \to |\mathcal{S}_0|.$$

Thus the theorem holds.

Based on the above discussion, the unbiased estimator for $\widehat{df}(\lambda_n)$ suffices to provide an unbiased estimate to the true prediction error of $\hat{\boldsymbol{\mu}}_{\lambda_n}$, as

$$C_p(\hat{\boldsymbol{\mu}}_{\lambda_n}) = \frac{\|\boldsymbol{Y}-\hat{\boldsymbol{\mu}}_{\lambda_n}\|^2}{n} + \frac{2}{n}|\mathcal{S}_0(\lambda_n)|\sigma^2.$$

Correspondingly, AIC for the modified LASSO is defined as follows:

$$AIC(\hat{\boldsymbol{\mu}}_{\lambda_n}) = \frac{\|\boldsymbol{Y} - \hat{\boldsymbol{\mu}}_{\lambda_n}\|^2}{n\sigma^2} + \frac{2}{n}|\mathcal{S}_0(\lambda_n)|.$$

Using AIC to find the optimal modified LASSO model, we introduce the following theorem to find the optimal λ_n where $AIC(\hat{\boldsymbol{\mu}}_{\lambda_n})$ get its minimum.

Theorem 3 *To find optimal* $\lambda_n(optimal)$, *we only need to solve*

$$m^* = \arg \min_{m\in\{0,1,\ldots,K\}} AIC(\hat{\boldsymbol{\mu}}_{\lambda_{nm}});$$

then $\lambda_n(optimal) = \lambda_{nm^*}$.

2.2 Variable selection porpuse In the least squares fit $\hat{\boldsymbol{\mu}}(\pi)$ for a given subset $\pi \subset \{1, \ldots, p\}$,

$$AIC(\hat{\boldsymbol{\mu}}(\pi)) = \frac{\|\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}(\pi)\|^2}{n\sigma^2} + \frac{2}{n}|\pi|,$$

where $\hat{\boldsymbol{\beta}}(\pi)$ is the lest squares estimate of $\boldsymbol{\beta}^*$ where $\hat{\boldsymbol{\beta}}(\pi)_{\{1,\ldots,p\}-\pi} \equiv \boldsymbol{0}$, that is

$$\hat{\boldsymbol{\beta}}(\pi) = \arg \min_{\boldsymbol{\beta}:\beta_j=0 for j\notin\pi} \sum_{i=1}^{n}(Y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2.$$

Then we define $\hat{\pi}$ as

$$\hat{\pi} = \arg \min_{\pi\subset\{1,\ldots,p\}} AIC(\hat{\boldsymbol{\mu}}(\pi)).$$

We say AIC is consistent if $P(\hat{\pi} = \mathcal{S}_0) \to 1$, as $n \to \infty$. Then by setting $\boldsymbol{x}_i^* = \boldsymbol{D}_n^{-1}\sqrt{n}\boldsymbol{x}_i$, where $\boldsymbol{D}_n = \text{diag}\{\sqrt{a_{11}^n}, \ldots, \sqrt{a_{pp}^n}\}$, for the original model

$$Y_i = \boldsymbol{x}_i'\boldsymbol{\beta}^* + \varepsilon_i,$$

it is transferred into

$$Y_i = (\boldsymbol{x}_i^*)'\boldsymbol{D}_n\boldsymbol{\beta}^*/\sqrt{n} + \varepsilon_i.$$

After the transformation, from Theorem 2 in [4], we can get that $AIC(\hat{\boldsymbol{\mu}}(\pi))$ is consistent. However, to find the $\hat{\pi}$, $O(2^n)$ times computational cost of a single least squares fit is needed. Define $\hat{m}$ as

$$\hat{m} = \arg\min_{m} AIC(\hat{\boldsymbol{\mu}}(\mathcal{S}_m)),$$

where m is the index of the transition point λ_{nm} of modified LASSO. Then the following theorem holds.

Theorem 4 $P(\mathcal{S}_{\hat{m}} = \mathcal{S}_0) = 1$, *as* $n \to \infty$.

Proof. From Lemma 1, among the transition points, the probability that there exists m such that $\mathcal{S}_0 = \mathcal{S}_m$ converges to 1. Noticing that $P(\hat{\pi} = \mathcal{S}_0) \to 1$, we get $P(\mathcal{S}_{\hat{m}} = \mathcal{S}_0) \to 1$.

From Theorem 4, the consistency of AIC on sub-models decided by modified LASSO approach is shown. Considering the computational cost can be reduced, it is reasonable to use $\mathcal{S}_{\hat{m}}$ to estimate $\mathcal{S}_0$.

3 Numerical results In this section, the simulation study analyses the selection and prediction performance of the modified LASSO when using the above two forms of AIC. We set $p = 8$ and $\boldsymbol{\beta} = \{1, 0, 1, 1, 0, 0, 0, 0\}$ i.e.

$$Y_i = x_{i1} + x_{i3} + x_{i4} + \varepsilon_i,$$

where the sequence $x_{i1} = 1$ for all $i \in \mathcal{N}$, $x_{i2} = i$, $x_{ij} = \cos \frac{\pi i j}{9}$ for $j = 3, \ldots, 8, i \in \mathcal{N}$ and $\{\varepsilon_i\}$ is generated by identically distributed Gaussian disturbances with length n going from 50 to 500 and variance $\sigma^2 = 0.1, 0.5, 1$ respectively. Here we use C_1 to stand for the $AIC(\hat{\boldsymbol{\mu}}_{\lambda_n})$, and use C_2 to stand for $AIC(\hat{\boldsymbol{\mu}}(\pi))$ for brevity. 100 replications are performed for each situation.

From table 1, we compare C_1 and C_2 by the bias and and mean squared error (MSE) of their estimators in the sense of parameter estimates. Here the bias and MSE are defined as follows:

$$Bias(\hat{\boldsymbol{\beta}}) = \frac{1}{s}\sum_{t=1}^{s}\sum_{j=1}^{8}(\hat{\beta}_{tj} - \beta_j^*);$$

$$MSE(\hat{\boldsymbol{\beta}}) = \frac{1}{s}\sum_{t=1}^{s}\sum_{j=1}^{8}(\hat{\beta}_{tj} - \beta_j^*)^2,$$

where s is the amount of replications. It is shown that the prediction performance of modified LASSO with C_1 is better than that of modified LASSO with C_2, noticing that both of absolute value of bias and MSE of C_1 are smaller than those of C_2. Besides, we can notice that, with sequence length n increases, the performance of C_2 gets worse. It agrees to the condition of consistency, that the optimal λ_n increases as n increases. From Table 2, the results from five aspects in the sense of variable selection are shown, which are the probability of correct selection, the probability of relevant variables included, the probability of irrelevant variables excluded, average number of included variables and average number of included irrelevant variables. It is shown that the results by C_2 are better than those by C_1 overall. From the probability of true model included, the probabilities by C_2 are greater than those by C_1. Besides, by comparing the values as n increases, it is shown that the the probability of correct selection of the true model increases, which keep consist with the consistency properties shown in Section 2. From the probability of relevant variables included, almost all the probabilities by C_1 and C_2 are 1, which means that by both C_1 and C_2, the probabilities that relevant variables are excluded are low.

Table 1: Parameter estimates

	C_1				C_2			
n	50	100	300	500	50	100	300	500
Bias								
N(0.1)	-0.1107	-0.0687	-0.0511	-0.0434	-1.4880	-1.4839	-1.5848	-1.6751
N(0.5)	-0.2390	-0.1712	-0.1010	-0.0903	-1.4646	-1.5184	-1.5618	-1.5790
N(1)	-0.3724	-0.2915	-0.1792	-0.1323	-1.5117	-1.5352	-1.5453	-1.5854
MSE								
N(0.1)	0.0221	0.0115	0.0042	0.0024	1.3915	1.3944	1.5361	1.6444
N(0.5)	0.1090	0.0522	0.0180	0.0120	1.2837	1.3877	1.4639	1.5000
N(1)	0.2365	0.1135	0.0412	0.0231	1.3124	1.3421	1.4166	1.4922

Table 2: Veriable selection

	C_1				C_2			
n	50	100	300	500	50	100	300	500
Probability of correct selection								
N(0.1)	0.53	0.55	0.55	0.57	0.74	0.73	0.79	0.84
N(0.5)	0.47	0.52	0.48	0.55	0.70	0.74	0.77	0.78
N(1)	0.35	0.41	0.46	0.49	0.68	0.71	0.75	0.78
Probability of relevant variables included								
N(0.1)	1	1	1	1	1	1	1	1
N(0.5)	1	1	1	1	1	1	1	1
N(1)	1	1	1	1	0.99	1	1	1
Probability of irrelevant excluded								
N(0.1)	0.53	0.55	0.55	0.57	0.74	0.73	0.79	0.84
N(0.5)	0.47	0.52	0.48	0.55	0.70	0.74	0.77	0.78
N(1)	0.35	0.41	0.46	0.49	0.68	0.71	0.75	0.78
Average number of included variables								
N(0.1)	3.81	3.80	3.66	3.56	3.33	3.34	3.23	3.16
N(0.5)	3.93	3.83	3.94	3.64	3.35	3.36	3.26	3.25
N(1)	4.11	3.96	3.79	3.75	3.41	3.33	3.25	3.24
Average number of included irrelevant variables								
N(0.1)	0.81	0.8	0.66	0.56	0.33	0.34	0.23	0.16
N(0.5)	0.93	0.83	0.94	0.64	0.35	0.36	0.26	0.25
N(1)	1.11	0.96	0.79	0.75	0.41	0.33	0.25	0.24

4 Conclusion In Sections 2 and 3, the prediction performance of C_1 is better than C_2. Whereas, the selection performance of the later is better than the former. Since these two forms of AIC are derived by the ideas of [8] and [4], in fact a more general form of the criteria can be derived. Besides, noticing that the consistency depends on the consistency in the sense that the probability of correct selection of the true model converges to 1 as the sequence length n goes to infinity, the consistency on high dimensions can be discussed furthermore.

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Appendix A. Proof of Theorem 1 To prove Theorem 1, we introduce the following lemma. For given $\boldsymbol{Y}$, there exists a set of transition point, $\{\lambda_{nm} : m = 0, \ldots, K\}$. Recall the definition of notations, $\mathcal{S}_m = \mathcal{S}_0(\lambda_{nm})$, and $\mathrm{sgn}_m = \mathrm{sgn}(\lambda_{nm})$.

Lemma 2 *Suppose* $\lambda_n \in (\lambda_{n,m+1}, \lambda_{nm})$. *Then we have*

$$\hat{\boldsymbol{\beta}}(\lambda_n)_{\mathcal{S}_m} = (\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{X}_{\mathcal{S}_m})^{-1}(\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{Y} - \frac{\lambda_n}{2}\boldsymbol{D}_n\mathrm{sgn}_m),$$

where $\boldsymbol{D}_n = \mathrm{diag}\{\sqrt{a^n_{11}(0)}, \ldots, \sqrt{a^n_{pp}(0)}\}$.

Lemma 3 *Consider the transition point* λ_{nm}, *when* λ_n *decreases from the right hand side of* λ_{nm} *to* λ^-_{nm}, i_{add} *is an index added into* $\mathcal{S}_m$, *and the order index of* i_{add} *is* i^*, *that is,* $i_{add} = (\mathcal{S}_m)_{i^*}$. *Denote the kth element of any vector* $\boldsymbol{a}$ *by* $(\boldsymbol{a})_k$. *We can express the transition point* λ_{nm} *as*

$$\lambda_{nm} = \frac{2((\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{X}_{\mathcal{S}_m})^{-1}\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{Y})_{i^*}}{((\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{X}_{\mathcal{S}_m})^{-1}\boldsymbol{D}^{-1}_n\mathrm{sgn}_m)_{i^*}}.$$

Lemma 4 *For any* $\lambda_n > 0$, *there exists a null set* $\mathcal{N}_{\lambda_n}$ *which is a finite collection of hyperplanes in* $\mathbb{R}^n$. *Let* $\mathrm{G}_{\lambda_n} = \mathbb{R}^n - \mathcal{N}_{\lambda_n}$. *Then* $\forall \boldsymbol{Y} \in \mathrm{G}_{\lambda_n}$, λ_n *is not any of the transition points for* $\boldsymbol{Y}$.

Lemma 5 $\forall\lambda_n > 0$, $\hat{\boldsymbol{\beta}}(\lambda_n)$ *is a continuous function with respect to* $\boldsymbol{Y}$.

Lemma 6 *Fix any* $\lambda_n > 0$ *and consider* $\boldsymbol{Y} \in \mathrm{G}_{\lambda_n}$ *as defined in Lemma 4. The active set* $\mathcal{S}_0(\lambda_n)$ *and the sign vector* $\mathrm{sgn}(\lambda_n)$ *are locally constant with respect to* $\boldsymbol{Y}$.

Lemma 7 *Let* $\mathrm{G}_0 = \mathbb{R}^n$. *For any* $\lambda_n \geq 0$, *on the set* G_{λ_n} *as defined in Lemma 4, the modified LASSO fit* $\hat{\boldsymbol{\mu}}_{\lambda_n}(\boldsymbol{Y})$ *is uniformly Lipschitz. Precisely,*

$$\|\hat{\boldsymbol{\mu}}_{\lambda_n}(\boldsymbol{Y} + \Delta\boldsymbol{Y}) - \hat{\boldsymbol{\mu}}_{\lambda_n}(\boldsymbol{Y})\| \leq \|\Delta\boldsymbol{Y}\|,$$

for sufficiently small $\Delta\boldsymbol{Y}$. *Moreover, we have the divergence formula*

$$\nabla \cdot \hat{\mu}_{\lambda_n}(\boldsymbol{Y}) = |\mathcal{S}_0(\lambda_0)|,$$

where $|\mathcal{S}_0(\lambda_n)|$ *stands for the cardinality of* $\mathcal{S}_0(\lambda_n)$.

The proofs of Lemma 2 to 7 are similar to those in [8], here we omit the proofs.
Proof of Theorem 1: By Lemma 4-7, $\hat{\mu}_{\lambda_n}(\boldsymbol{Y})$ is differentiable almost every where. Then by the Stein's unbiased risk estimation theory ([6]) and Lemma 7,

$$df(\lambda_n) = \mathrm{E}\nabla \cdot \hat{\mu}_{\lambda_n}(\boldsymbol{Y}) = \mathrm{E}|\mathcal{S}_0(\lambda_n)|.$$

Thus Theorem 1 holds.

Appendix B. Proof of Lemma 1

Noticing that $\{\varepsilon_i\}$ is independent and identically distributed process with $\varepsilon_i \sim N(0, \sigma^2)$, then $\sum_{i=1}^n b_{nij}\varepsilon_i$ satisfies the Bernstein inequality, where $b_{nij} = \frac{x_i^j}{\sqrt{a^n_{ii}(0)}}$ for $j = 1, \ldots, p$. Thus, Lemma 1 is a special case of Theorems 4 and 5 in [9], which implies it holds.

Appendix C. Proof of Theorem 3

From the forms of AIC, we know that

$$\lambda_n(optimal) = \arg\min_{\lambda_n} AIC(\hat{\boldsymbol{\mu}}_{\lambda_n}) = \arg\min_{\lambda_n} \frac{\|\boldsymbol{Y} - \hat{\boldsymbol{\mu}}_{\lambda_n}\|^2}{n\sigma^2} + \frac{2}{n}|\,\mathcal{S}_0(\lambda_n)|.$$

From Lemma 2, for $\lambda_n \in (\lambda_{n,m+1}, \lambda_{nm})$, we have

$$\|\boldsymbol{Y} - \hat{\boldsymbol{\mu}}_{\lambda_n}\|^2 = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{X}_{\mathcal{S}_m}(\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{X}_{\mathcal{S}_m})^{-1}\boldsymbol{X}'_{\mathcal{S}_m})\boldsymbol{Y} + \frac{\lambda_n^2}{4}\text{sgn}'_m\boldsymbol{D}_n(\boldsymbol{X}'_{\mathcal{S}_m}\boldsymbol{X}_{\mathcal{S}_m})^{-1}\boldsymbol{D}_n\text{sgn}_m,$$

where I is the $n \times n$ identity matrix. Thus we conclude that in the interval $(\lambda_{n,m+1}, \lambda_{nm})$, $\|\boldsymbol{Y} - \hat{\boldsymbol{\mu}}_{\lambda_n}\|^2$ is strictly increasing with respect to λ_n. On the other hand, note that $|\,\mathcal{S}_0(\lambda_{nm})| \geq |\,\mathcal{S}_0(\lambda_{n,m+1})|$. Therefore, the optimal choice of λ_n in $[\lambda_{n,m+1}, \lambda_{nm})$ is $\lambda_{n,m+1}$, which means $\lambda_n(optimal) \in \{\lambda_{nm} : m + 0, \ldots, K\}$. Thus Theorem 3 holds.

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NEW SPECTRAL MAPPING THEOREM OF THE TAYLOR SPECTRUM

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Abstract. We show new spectral mapping theorem of the Taylor spectrum for doubly commuting pairs of p-hyponormal operators and log-hyponormal operators. And we give Putnam inequality for log-hyponormal tuples.

1 Introduction Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\lambda \in \mathbb{C}$ belong to the residual spectrum $\sigma_r(T)$ of T if there exists $c > 0$ such that $\|(T-\lambda)x\| \geq c\|x\|$ for all $x \in \mathcal{H}$ and $(T-\lambda)\mathcal{H} \neq \mathcal{H}$. It is easy to see that if $\lambda \in \sigma_r(T)$, then $0 \in \sigma_p((T-\lambda)^*)$. It is well known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$. For an Hermitian operator $A \in B(\mathcal{H})$, we denote $A \geq 0$ if $(Ax, x) \geq 0$ for every $x \in \mathcal{H}$ and $A \geq B$ if $A - B \geq 0$. When $(Ax, x) > 0$ for every non-zero $x \in \mathcal{H}$, then we denote $T > 0$. For a given $p > 0$, $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. When $p = 1/2$, T is said to be semi-hyponormal. It means that T is semi-hyponormal if and only if $|T| \geq |T^*|$. T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$. It is well known that if T is invertible p-hyponormal for some $p > 0$, then T is log-hyponormal. If $\mathcal{M}$ is a reducing subspace for a p-hyponormal or log-hyponormal operator T, then so is $T|_{\mathcal{M}}$, respectively.

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we explain the Taylor spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ shortly. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, ..., e_n$. Let $E^n_k(\mathcal{H}) = \mathcal{H} \otimes E^n_k$. Define $d^n_k : E^n_k(\mathcal{H}) \longrightarrow E^n_{k-1}(\mathcal{H})$ by

$$d^n_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) := \sum_{i=1}^{k} (-1)^{i-1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

where $\check{e}_{j_i}$ means deletion. We denote d^n_k by d_k simply. We think Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$E(\mathbf{T}) \ : \ 0 \longrightarrow E^n_n(\mathcal{H}) \xrightarrow{d_n} E^n_{n-1}(\mathcal{H}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E^n_1(\mathcal{H}) \xrightarrow{d_1} E^n_0(\mathcal{H}) \longrightarrow 0.$$

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It is easy to see that $E_k^n(\mathcal{H}) \cong \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{\frac{n!}{(n-k)!\,k!}}$ $(k = 1, ..., n)$.

Definition 1.1. *A commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *is said to be singular if and only if the Koszul complex* $E(\mathbf{T})$ *of* $\mathbf{T}$ *is not exact.*

Definition 1.2. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, *the Taylor spectrum* $\sigma_T(\mathbf{T})$ *of* $\mathbf{T}$ *is the set of all* $z = (z_1, ..., z_n) \in \mathbb{C}^n$ *such that* $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$ *is singular.*

About the definition of the Taylor spectrum, see details J. L. Taylor [11] and [12].

For a commuting pair $\mathbf{T} = (T_1, T_2) \in B(\mathcal{H})^2$, it is well known that, for polynomials $f_1, ..., f_n$ of 2 variables, if $f(z_1, z_2) = (f_1(z_1, z_2), ..., f_n(z_1, z_2))$, then it holds

$$\sigma_T(f(T_1, T_2)) = f(\sigma_T(T_1, T_2)),$$

where $\sigma_T(T_1, T_2)$ is the Taylor spectrum of $\mathbf{T} = (T_1, T_2)$. See Theorem 4.7 in [12].

In this paper, we study other spectral mapping theorem, that is, let $T_j = U_j|T_j|$ $(j = 1, 2)$ be the polar decomposition of T_j and $f(t)$ be a continuous function on the non-negative real line. Let $S_j = U_j f(|T_j|)$ $(j = 1, 2)$ and $\mathbf{S} = (S_1, S_2)$. Then under some assumption does it hold

$$\sigma_T(\mathbf{S}) = \{(e^{i\theta_1} f(r_1), e^{i\theta_2} f(r_2)) \,:\, (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \sigma_T(\mathbf{T})\ \}\ ?$$

For a single operator, it holds for some classes of operators. For example, if $T = U|T|$ is a p-hyponormal operator or a log-hyponormal operator with $\log|T| > 0$ and $f(t) = t^{2p}$ or $f(t) = \log\, t$, then

$$\sigma(Uf(|T|)) = \{e^{i\theta} f(r) \,:\, re^{i\theta} \in \sigma(T)\,\}, \tag{1}$$

respectively by [7, 10].

Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ and let

$$\alpha(\mathbf{T} - \mathbf{z}) := \begin{pmatrix} T_1 - z_1 & T_2 - z_2 \\ -(T_2 - z_2)^* & (T_1 - z_1)^* \end{pmatrix} \quad \text{on} \quad \mathcal{H} \oplus \mathcal{H}.$$

Then Vasilescu proved the following result.

Proposition 1.3. (Theorem 1.1, Vasilescu [13]) *Let* $\mathbf{T} = (T_1, T_2) \in B(\mathcal{H})^2$ *be a commuting pair. Then*

$$\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T}) \textit{ if and only if } \alpha(\mathbf{T} - \mathbf{z}) \textit{ is not invertible.}$$

Therefore, we have

$$\mathbf{z} = (z_1, z_2) \in \sigma_T(\mathbf{T}) \text{ if and only if } \quad 0 \in \sigma(\alpha(\mathbf{T} - \mathbf{z})).$$

For an n-tuple $\mathbf{T} = (T_1, ..., T_n)$, the joint point spectrum $\sigma_{jp}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a non-zero vector $x \in \mathcal{H}$ which satisfies $T_j x = z_j x$ $(\forall j = 1, ..., n)$ and the joint approximate point spectrum $\sigma_{ja}(\mathbf{T})$ is the set of all numbers $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$ such that there exists a sequence $\{x_k\}$ of unit vectors of $\mathcal{H}$ which satisfies

$$(T_j - z_j)x_k \ \rightarrow \ 0 \text{ as } k \rightarrow \infty \ (\forall j = 1, ..., n).$$

Following proposition is due to Berberian [1] for a single operator case. It is easy to see a proof for n-tuples. See Berberian [1] and Chō [2].

Proposition 1.4. *Let* $B(\mathcal{H})$ *be the set of all bounded linear operators on* $\mathcal{H}$. *Then there exist an extension space* $\mathcal{K}$ *of* $\mathcal{H}$ *and a faithful* $*$*-representation of* $B(\mathcal{H})$ *into* $B(\mathcal{K})$: $T \ \rightarrow \ T^\circ$ *such that*

$$\sigma_{ja}(T_1, ..., T_n) = \sigma_{jp}(T^\circ, ..., T_n^\circ) = \sigma_p(T_1^\circ, ..., T_n^\circ).$$

We have Putnam inequalities of hyponormal tuples, semi-hyponormal tuples, and p-hyponormal tuples. See [2], [3], [4], [5], [8]. Finally we give Putnam inequality of log-hyponormal tuple.

2 New spectral mapping theorem

Following results are well known.

Proposition 2.1. *Let* $T = U|T|$ *be the polar decomposition of* T *and* f *be a continuous function on the non-negative real line which contains* $\sigma(|T|)$. *For a sequence* $\{x_n\}$ *of unit vectors, if* $(T - re^{i\theta})x_n \rightarrow 0$ *and* $(T - re^{i\theta})^* x_n \rightarrow 0$, *then* $(U - e^{i\theta})x_n \rightarrow 0, (|T| - r)x_n \rightarrow 0$ *and* $(f(|T|) - f(r))x_n \rightarrow 0$.

See Lemma 1.2.4 in [15].

Proposition 2.2. *Let* T *be semi-hyponormal. Then* $\sigma(T) = \{\overline{z} : z \in \sigma_a(T^*)\}$.

See Theorem 1.2.6 in [15].

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with unitary U and f be a continuous function on the non-negative real line which contains $\sigma(|T|)$. Let $\mathcal{K}$ be Berberian extension of $\mathcal{H}$ and $\circ : B(\mathcal{H}) \ni T \to T^\circ \in B(\mathcal{K})$ be a faithful $*$-representation. We set the following conditions (2) and (3):

$$\text{(2)} \qquad \text{For a sequence } \{x_n\} \text{ of unit vectors, if } (T-z)x_n \to 0, \text{ then } (T-z)^*x_n \to 0.$$

$$\text{(3)} \qquad \text{If a closed subspace } \mathcal{M} \text{ of } \mathcal{K} \text{ reduces } T^\circ \text{ and } re^{i\theta} \in \sigma(T^\circ|_{\mathcal{M}}),$$
$$\text{then } \mathcal{M} \text{ reduces } U^\circ, |T|^\circ \text{ and } e^{-i\theta}f(r) \in \sigma_p\left((U^\circ|_{\mathcal{M}} f(|T|)^\circ|_{\mathcal{M}})^*\right).$$

Remark. If T is p-hyponormal and $f(t) = t^{2p}$, then (2) holds by Theorem 4 of [5]. If T is log-hyponormal and $f(t) = \log t$, then (2) holds by Lemma 3 of [10]. About (3), since the mapping $\circ$ of Berberian method is a faithful $*$-representation, so is T° if T is p-hyponormal or log-hyponormal, respectively. Let $\mathcal{M}$ be a reducing subspace for T. It is clear that if T is p-hyponormal or log-hyponormal, then so is $T|_{\mathcal{M}}$, respectively.
(i) Let T be p-hyponormal and $T = U|T|$ be the polar decomposition of T and $f(t) = t^{2p}$. Then $S = U|T|^{2p}$ is semi-hyponormal and $\sigma(U|T|^{2p}) = \{r^{2p}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ by Theorem 3 of [7]. Hence (3) holds by Proposition 2.2.
(ii) Let $T = U|T|$ be log-hyponormal and $f(t) = \log t$. Then $S = U\log|T|$ is semi-hyponormal and $\sigma(U\log|T|) = \{e^{i\theta}\log r : re^{i\theta} \in \sigma(T)\}$ by Lemma 8 of [10]. Hence (3) holds by Proposition 2.2.
Therefore, if T is p-hyponormal or log-hyponormal and $f(t) = t^{2p}$ or $f(t) = \log t$, respectively, then T satisfies (2) and (3) for this f.

Theorem 2.3. *Let* $\mathbf{T} = (T_1, T_2)$ *be a doubly commuting pair of operators and* $T_j = U_j|T_j|$ $(j = 1, 2)$ *be the polar decomposition. Let* $f(t)$ *be a continuous function on a open interval in the non-negative real line which contains* $\sigma(|T_1|) \cup \sigma(|T_2|)$. *Let* $S_j = U_j f(|T_j|)$ $(j = 1, 2)$ *and* $\mathbf{S} = (S_1, S_2)$. *Let* T_1, T_2 *and* f *satisfy* (2) *and* (3). *If* $(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})$, *then* $(e^{i\theta_1}f(r_1), e^{i\theta_2}f(r_2)) \in \sigma_T(\mathbf{S})$.

Proof. Let $\mathbf{z} = (z_1, z_2) = (r_1e^{\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})$. Then $0 \in \sigma(\alpha(\mathbf{T} - \mathbf{z}))$ by Proposition 1.1.

Case 1. If $0 \in \sigma_a(\alpha(\mathbf{T} - \mathbf{z}))$, then there exists a sequence $\{x_n \oplus y_n\}$ of unit vectors of $\mathcal{H} \oplus \mathcal{H}$ such that

$$\alpha(\mathbf{T} - \mathbf{z})(x_n \oplus y_n) = \begin{pmatrix} (T_1 - z_1)x_n + (T_2 - z_2)y_n \\ -(T_2 - z_2)^*x_n + (T_1 - z_1)^*y_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since T_1, T_2 are doubly commuting, we have

$$(T_1 - z_1)^*(T_1 - z_1)x_n + (T_2 - z_2)(T_2 - z_2)^*x_n \to 0$$

and

$$(T_1 - z_1)(T_1 - z_1)^*y_n + (T_2 - z_2)^*(T_2 - z_2)y_n \to 0.$$

If $x_n \not\to 0$, then $(z_1, \overline{z_2}) \in \sigma_{ja}(T_1, T_2^*)$, and if $y_n \not\to 0$, then $(\overline{z_1}, z_2) \in \sigma_{ja}(T_1^*, T_2)$.

Case 2. If $0 \in \sigma_r(\alpha_2(\mathbf{T}-\mathbf{z})) \subset \sigma_p(\alpha(\mathbf{T}-\mathbf{z})^*)$, then there exists a non-zero vector $x \oplus y$ such that

$$\alpha(\mathbf{T}-\mathbf{z})^*(x \oplus y) = \begin{pmatrix} (T_1-z_1)^*x-(T_2-z_2)y \\ (T_2-z_2)^*x+(T_1-z_1)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we have

$$(T_1-z_1)(T_1-z_1)^*x+(T_2-z_2)^*(T_2-z_2)x=0$$

and

$$(T_1-z_1)^*(T_1-z_1)y+(T_2-z_2)(T_2-z_2)^*y=0.$$

If $x \neq 0$, then we have $(T_1-z_1)^*x=(T_2-z_2)x=0$ and if $y \neq 0$, then $(T_1-z_1)y=(T_2-z_2)^*y=0$.

Therefore, if necessarily by changing order, we may assume that there exists a sequence $\{x_n\}$ of unit vectors such that (the proof of case $(T_1-z_1)^*x_n \to 0$ and $(T_2-z_2)x_n \to 0$ is similar.)

$$(T_1-z_1)x_n \to 0 \text{ and } (T_2-z_2)^*x_n \to 0.$$

Hence

$$(U_1-e^{i\theta_1})x_n \to 0,\ (|T_1|-r_1)x_n \to 0, (S_1-e^{i\theta_1}f(r_1))x_n \to 0$$

by the assumption. Let $\mathcal{K}$ be the Berberian extension of $\mathcal{H}$. Then there exists $0 \neq x^\circ \in \mathcal{K}$ such that

$$(S_1^\circ-e^{i\theta_1}f(r_1))x^\circ=(T_2^\circ-z_2)^*x^\circ=0.$$

Let $\mathcal{M}=\ker(S_1^\circ-e^{i\theta_1}f(r_1))$. Since (S_1°,T_2°) are doubly commuting pair, $\mathcal{M}$ is a reducing subspace for T_2°. Since $x^\circ \in \mathcal{M}$, we have $z_2=r_2e^{i\theta_2} \in \sigma(T_2^\circ|_{\mathcal{M}})$. Let $S_2=U_2f(|T_2|)$. Then by the assumption (2), we have $T_2^\circ|_{\mathcal{M}}=U_2^\circ|_{\mathcal{M}}|T_2|^\circ|_{\mathcal{M}}$ and $e^{-i\theta_2}f(r_2) \in \sigma_p(S_2^{\circ *}|_{\mathcal{M}})$. Hence there exists non-zero $y^\circ \in \mathcal{M}$ such that $(S_2^\circ-e^{i\theta_2}f(r_2))^*y^\circ=0$. Since $y^\circ \in \mathcal{M}$, we have $(S_1^\circ-e^{i\theta_1}f(r_1))y^\circ=0$. Therefor there exists a sequence $\{y_n\}$ of unit vectors such that

$$(S_1-e^{i\theta_1}f(r_1))y_n \to 0 \text{ and } (S_2-e^{i\theta_2}f(r_2))^*y_n \to 0.$$

Then

$$\alpha(\mathbf{S}-(e^{i\theta_1}f(r_1),e^{i\theta_2}f(r_2)))\begin{pmatrix} y_n \\ 0 \end{pmatrix} = \begin{pmatrix} (S_1-e^{i\theta_1}f(r_1))y_n \\ -(S_2-e^{i\theta_2}f(r_2))^*y_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $0 \in \sigma\left(\alpha(\mathbf{S}-(e^{i\theta_1}f(r_1),e^{i\theta_2}f(r_2)))\right)$ and $\left(e^{i\theta_1}f(r_1),e^{i\theta_2}f(r_2)\right) \in \sigma_T(\mathbf{S})$. This completes the proof.

□

Corollary 2.4. *Let $\mathbf{T}=(T_1,T_2)$ be a doubly commuting pair of p-hyponormal operators $(0<p<1)$. Let U_j be unitary for the polar decomposition of $T_j=U_j|T_j|$ $(j=1,2)$ and $\mathbf{S}=(U_1|T_1|^{2p},U_2|T_2|^{2p})$. Then*

$$\sigma_T(\mathbf{S}) = \{(r_1^{2p}e^{i\theta_1},r_2^{2p}e^{i\theta_2}) : (r_1e^{i\theta_1},r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = t^{2p}$ on the non-negative real line. Since $\mathbf{T}$ is a doubly commuting pair of p-hyponormal operators and $f(t) = t^{2p}$, T_1, T_2 and f satisfy (2) and (3). Hence, by Theorem 2.3 we have

$$\sigma_T(\mathbf{S}) \supset \{(r_1^{2p}e^{i\theta_1}, r_2^{2p}e^{i\theta_2}) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, put $g(t) = t^{\frac{1}{2p}}$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting pair of semi-hyponormal operators, S_1, S_2 and g satisfy (2) and (3). Then we have the converse inclusion by Theorem 2.3 and similar argument. □

Corollary 2.5. *Let* $\mathbf{T} = (T_1, T_2)$ *be a doubly commuting pair of* log*-hyponormal operators with* $\log|T_j| > 0$. *Let* U_j *be unitary for the polar decomposition of* $T_j = U_j|T_j|$ $(j = 1, 2)$ *and* $\mathbf{S} = (U_1 \log|T_1|, U_2 \log|T_2|)$. *Then*

$$\sigma_T(\mathbf{S}) = \{e^{i\theta_1} \log r_1, e^{i\theta_2} \log r_2) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Proof. Let $f(t) = \log t$ on $(0, \infty)$. Since $\mathbf{T}$ is a doubly commuting pair of log-hyponormal operators and $f(t) = \log t$, T_1, T_2 and f satisfy (2) and (3). So by Theorem 2.3 we have

$$\sigma_T(\mathbf{S}) \supset \{e^{i\theta_1} \log r_1, e^{i\theta_2} \log r_2) : (r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \sigma_T(\mathbf{T})\}.$$

Conversely, let $g(t) = e^t$ on the non-negative real line. Since $\mathbf{S}$ is a doubly commuting pair of semi-hyponormal operators, S_1, S_2 and g satisfy (2) and (3). Hence, we have the converse inclusion by similar argument. □

3 Putnam inequality

In this section we study for Putnam inequality of log-hyponormal tuples. Let $\mathbf{U} = (U_1, ..., U_n)$ be an n-tuple of unitary operators. For $T \in B(\mathcal{H})$, an operator $\mathbf{Q}_j$ $(j = 1, ..., n)$ on $B(\mathcal{H})$ is defined by

$$\mathbf{Q}_j T := T - U_j T U_j^*.$$

Definition 3.1. *Let* $\mathbf{U} = (U_1, \cdots, U_n)$ *be a commuting* n*-tuple of unitary operators and* $A \geq 0$. *An* $(n+1)$*-tuple* $(\mathbf{U}, A)$ *is said to be a semi-hyponormal tuple if*

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m} A \geq 0 \ \ \textit{for all} \ \ 1 \leq j_1 < \cdots < j_m \leq n.$$

Definition 3.2. *Let* $\mathbf{U} = (U_1, \cdots, U_n)$ *be a commuting* n*-tuple of unitary operators and* $A > 0$ *with* $\log A \geq 0$. *An* $(n+1)$*-tuple* $(\mathbf{U}, A)$ *is said to be a* log*-hyponormal tuple if* $(\mathbf{U}, \log A)$ *is a semi-hyponormal tuple.*

Let $\mathbf{U} = (U_1, \cdots, U_n)$ be an n-tuple of unitary operators and $T \in B(\mathcal{H})$. If

$$\mathcal{S}_j^{\pm}(T) := \text{s}-\lim_{n \to \pm\infty} (U_j^{-n} T U_j^n)$$

exist, then the operators $\mathcal{S}_j^{\pm}(T)$ are called the polar symbols of T. If $U_j|A|$ is semi-hyponormal, then the polar symbols $\mathcal{S}_j^{\pm}(T)$ exist.

For $k \in [0,1]$ and $A \geq 0$, we denote

$$(k\mathcal{S}_j^+ + (1-k)\mathcal{S}_j^-)A := k\mathcal{S}_j^+(A) + (1-k)\mathcal{S}_j^-(A).$$

Let $\mathbf{k} = (k_1, ..., k_n) \in [0,1]^n$ and $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then the generalized polar symbols $A_{\mathbf{k}}$ of A are defined by

$$A_{\mathbf{k}} := \Pi_{j=1}^n \big(k_j\mathcal{S}_j^+ + (1-k_j)\mathcal{S}_j^-\big)A.$$

Since $A \geq 0$, then $A_{\mathbf{k}} \geq 0$. Hence it is clear that $(\mathbf{U}, A_{\mathbf{k}})$ is a commuting $(n+1)$-tuple of normal operators for every $\mathbf{k} \in [0,1]^n$.

Definition 3.3.
(1) *Let* $(\mathbf{U}, A)$ *be a semi-hyponormal tuple. The the Xia spectrum* $\sigma_X(\mathbf{U}, A)$ *is defined by*

$$\sigma_X(\mathbf{U}, A) := \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}).$$

(2) *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple. Since* $(\mathbf{U}, \log A)$ *is a semi-hyponormal tuple, Xia spectrum* $\sigma_X(\mathbf{U}, A)$ *of* $(\mathbf{U}, A)$ *is defined by*

$$\sigma_X(\mathbf{U}, A) := \{(z_1, ..., z_n, e^r) \,:\, (z_1, ..., z_n, r) \in \sigma_X(\mathbf{U}, \log A)\}.$$

Proposition 3.4. (Theorem 5, Xia [14]) *Let* $(\mathbf{U}, A)$ *be a semi-hyponormal tuple. Then*

$$||\mathbf{Q}_1 \cdots \mathbf{Q}_n A|| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U},A)} d\theta_1 \cdots d\theta_n \, dr.$$

Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple and $\mathbf{k} = (k_1, \cdots, k_n) \in [0,1]^n$. We define

$$A_{\log,\mathbf{k}} := \exp\{\Pi_{j=1}^n \big(k_j\mathcal{S}_j^+(\log A) + (1-k_j)\mathcal{S}_j^-(\log A)\big)\}.$$

Then $(\mathbf{U}, A_{\log,\mathbf{k}})$ is a commuting tuple and $(\mathbf{U}, \log A)$ is a semi-hyponormal tuple. Let

$$(\log A)_{\mathbf{k}} = \Pi_{j=1}^n \big(k_j\mathcal{S}_j^+(\log A) + (1-k_j)\mathcal{S}_j^-(\log A)\big).$$

Then $A_{\log,\mathbf{k}} = \exp\{(\log A)_{\mathbf{k}}\}$ and we have the following lemma.

Lemma 3.5. *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple and* $\mathbf{k} \in [0,1]^n$. *Then*

$$(z_1, \cdots, z_n, \log r) \in \sigma_{ja}(\mathbf{U}, (\log A)_{\mathbf{k}}) \text{ if and only if } (z_1, \cdots, z_n, r) \in \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}}).$$

Proof. It is easy from $A_{\log,\mathbf{k}} = \exp\{(\log A)_{\mathbf{k}}\}$.

□

Theorem 3.6. *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple. Then*

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}}).$$

Proof. Since $\sigma_X(\mathbf{U}, A) = \{(z_1, \cdots, z_n, e^r) : (z_1, \cdots, z_n, r) \in \sigma_X(\mathbf{U}, \log A)\}$ by the definition 3.3 (2), we have

$$\sigma_X(\mathbf{U}, \log A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, (\log A)_{\mathbf{k}}).$$

Hence we have

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\log,\mathbf{k}})$$

by Lemma 3.5.

□

Theorem 3.7. *Let* $(\mathbf{U}, A)$ *be a* log*-hyponormal tuple. Then*

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},A)} \frac{1}{r}\, d\theta_1 \cdots d\theta_n dr.$$

Proof. Since $(\mathbf{U}, \log A)$ is a semi-hyponormal tuple, it holds

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},\log A)} d\theta_1 \cdots d\theta_n dr$$

by proposition 3.4. Since

$$\sigma_X(\mathbf{U}, \log A) = \{(z_1, \cdots, z_n, \log s) : (z_1, \cdots, z_n, s) \in \sigma_X(\mathbf{U}, A)\}$$

by definition, we have

$$(z_1, \cdots, z_n, r) \in \sigma_X(\mathbf{U}, \log A) \Longleftrightarrow (z_1, \cdots, z_n, e^r) \in \sigma_X(\mathbf{U}, A).$$

Let $s = e^r$. Then $ds = e^r dr$ and $dr = \dfrac{1}{s} ds$. Hence

$$\frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},\log A)} d\theta_1 \cdots d\theta_n dr = \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma(\mathbf{U},A)} \frac{1}{s}\, d\theta_1 \cdots d\theta_n ds.$$

□

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of log-hyponormal operators and $T_j = U_j|T_j|$ be the polar decomposition of T_j with $\log|T_j| \geq 0$ $(j = 1, ..., n)$. Let $\mathbf{U} = (U_1, ..., U_n)$ and $A = \exp\big(\log|T_1| \cdots \log|T_n|\big)$. Then $\mathbf{U}$ is a commuting n-tuple of unitary operators and $A \geq 0$. By the definition of the operator $\mathbf{Q}_j$, it is easy to see that

$$\mathbf{Q}_j \log A = \big(\Pi_{k\neq j} \log|T_k|\big)(\log|T_j| - \log|T_j^*|)$$

for all $j = 1, ..., n$. Therefore $(\mathbf{U}, A)$ is a log-hyponormal tuple and

$$\mathbf{Q}_1 \cdots \mathbf{Q}_n \log A = \Pi_{j=1}^n \big(\log|T_j| - \log|T_j^*|\big) \geq 0.$$

Hence we have the following theorem.

Theorem 3.8. *Let* $\mathbf{T} = (T_1, ..., T_n)$ *be a doubly commuting n-tuple of* log*-hyponormal operators with* $\log|T_j| \geq 0$*. Let* $T_j = U_j|T_j|$ $(j = 1, ..., n)$ *be the polar decomposition and* $A = \exp\big(\log|T_1| \cdots \log|T_n|\big)$*. Then*

$$||\Pi_{j=1}^n(\log|T_j| - \log|T_j^*|)|| \leq \frac{1}{(2\pi)^n} \int ... \int_{\sigma(\mathbf{U},A)} \frac{1}{r}\, d\theta_1 \cdots d\theta_n dr.$$

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Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.
(73－age) × ￥3,000
Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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