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GROUPOID FACTORIZATIONS IN THE SEMIGROUP OF BINARY SYSTEMS

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Abstract. Let $(X, \bullet)$ be a groupoid (binary algebra) and $Bin(X)$ denote the collection of all groupoids defined on X. We introduce two methods of factorization for this binary system under the binary groupoid product "$\diamond$" in the semigroup $(Bin(X), \diamond)$. We conclude that a strong non-idempotent groupoid can be represented as a product of its *similar*- and *signature*- derived factors. Moreover, we show that a groupoid with the orientation property is a product of its *orient*- and *skew*- factors. These unique factorizations can be useful for various applications in other areas of study. Application to algebras such as $B/BCH/BCI/BCK/BH/BI/d$-algebra are widely given throughout this paper.

1. Introduction

Algebraic structures play a vital role in mathematical applications such as information science, network engineering, computer science, cell biology, etc. This encourages sufficient motivation to study abstract algebraic concepts and review previously obtained results. One such concept of interest to many mathematicians over the past two decades or so is that of a simple yet very interesting notion of a single set with one binary operation, historically known as magma and more recently referred to as groupoid. Bruck [8] published the book,"A Survey of Binary Systems" in which the theory of groupoids, loops, quasigroups, and several algebraic structures were discussed. Borůvka in [7] explained the foundations for the theory of groupoids, set decompositions and their application to binary systems.

Given a binary operation "$\bullet$" on a non-empty set X, the groupoid $(X, \bullet)$ is a generalization of the very well-known structure of a group. H. S. Kim and J. Neggers in [33] investigated the structure $(Bin(X), \diamond)$ where $Bin(X)$ is the collection of all binary systems (groupoids or algebras) defined on a non-empty set X along with an associative binary product $(X, *) \diamond (X, \circ) = (X, \bullet)$ such that $x \bullet y = (x * y) \circ (y * x)$ for all $x, y \in X$. They recognized that the left-zero-semigroup serves as the identity of this semigroup. The present author in [11] introduced the notion of the center $ZBin(X)$ in the semigroup $(Bin(X), \diamond)$, and proved that $(X, \bullet) \in ZBin(X)$, if and only if $(X, \bullet)$ is locally-zero. Han and Kim in [13] introduced the notion of hypergroupoids $HBin(X)$, and showed that $(HBin(X), \diamond)$ is a supersemigroup of the semigroup $(Bin(X), \diamond)$ via the identification $x \longleftrightarrow \{x\}$. They proved that $(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra.

In this paper, we investigate the following problems:

Main Problem: Consider the semigroup $(Bin(X), \diamond)$. Let the left-zero-semigroup be denoted as $id_{Bin(X)}$. Given a groupoid (binary system) $(X, \bullet) \in Bin(X)$, is it possible to find two groupoid factors $(X, *)$ and $(X, \circ)$ such that

$$(X, \bullet) = (X, *) \diamond (X, \circ)?$$

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Key words and phrases. groupoid factorization, groupoid decomposition, composite groupoid, prime groupoid, Bin(X), idempotent groupoid, signature-factor, similar-factor, orient-factor, skew-factor, u-normal, j-normal, Ψ-type-factor, τ-type-factor.

If so,

Problem 1 (Uniqueness). *Are the corresponding groupoid factors:*

(1) *Distinct, i.e.,* $(X,*) \neq (X,\circ)$*?*
(2) *Unique, i.e., if* $(X,\bullet) = (X,*) \diamond (X,\circ)$*, is it possible for* $(X,\bullet) = (X,\lhd) \diamond (X,\rhd)$ *such that* $(X,*) \neq (X,\lhd)$ *and* $(X,\circ) \neq (X,\rhd)$*?*
(3) *Different from* $(X,\bullet)$*, i.e.,* $(X,*) \neq (X,\bullet)$ *and* $(X,\circ) \neq (X,\bullet)$*?*
(4) *Different from the left-zero-semigroup, i.e.,* $(X,*) \neq id_{Bin(X)}$ *and* $(X,\circ) \neq id_{Bin(X)}$*?*

Problem 2 (Derivation). *How do we find the groupoid-factors? Are they:*

(1) *Derived (related to, based off of, dependent on) from: the parent groupoid* $(X,\bullet)$*?*
(2) *Derived from the identity* $id_{Bin(X)}$*?*

Problem 3 (Factorization). *If we use a certain method to find the two groupoid-factors, what is the nature of this factorization?*

(1) *Is it unique?*
(2) *When is it commutative?*

We begin answering these questions by introducing two methods for factoring a random groupoid in $Bin(X)$ using the product "$\diamond$". We will show that both methods result in unique factorizations (Problem 3.1) of a given groupoid and hence we answer Problem 1.2 with a definite yes! Section two provides some definitions and preliminary ideas which are necessary in this context. We also present a summarized table of "logic" algebras for a clear view. Section three describes AU- and UA-factorizations, which comprises the first method (method-1) of factoring. In fact, method-1 factors a groupoid $(X,\bullet)$ by obtaining two *derived* factors from it (Problem 2.1) and from the left-zero-semigroup (Problem 2.2), the *signature*- and *similar*-factors, respectively. We prove that a strong groupoid has a commutative method-1 factorization (Problem 3.2). The possibility of this first method is shown to be feasible and produces non-trivial decompositions (Problem 1.4), however, it is restricted to non-idempotent groupoids only. Hence, section four introduces an OJ- and a JO-factorization, which constitutes our second method (method-2). We will demonstrate that the latter method is sufficient for idempotent as well as non-idempotent groupoids. In addition, an interesting outcome of method-2 is that one of the factors is not *derived* from the parent groupoid (Problems 2.1 and 2.2) while the other factor is; we name them *orient*- and *skew*-factors, respectively. We show that a given groupoid $(X,\bullet)$ with $x \bullet y \in \{x, y\}$, for all x, y in X, has a commutative method-2 factorization (Problem 3.2). Section five briefly applies our two methods to some of the algebras listed in section two; and discusses a promising relationship to graph theory.

Finally, in our last section we generalize and summarize our findings that certain groupoids/algebras decompose into distinct groupoids via (1) an operation on the parent groupoid and the left-zero-semigroup simultaneously, which is a generalization of our first method; or (2) an operation which acts on the parent-groupoid and the left-zero-semigroup separately, hence resulting in a generalization of our second method.

Notions of "method"-*composite,* "method"-*normal*, "factor"-*prime* and "partially"-*left/right-prime* are used to classify and analyze various groupoids as well as other familiar algebras. For simplicity, the left-zero-semigroup will be denoted as $id_{Bin(X)}$.

2. Preliminaries

A *groupoid* [8] $(X, \bullet)$ consists of a non-empty set X together with a binary operation $\bullet : X \times X \to X$ where $x \bullet y \in X$ for all $x, y \in X$.

A groupoid $(X, \bullet)$ is *strong* [33] if and only if for all $x, y \in X$,

$$x \bullet y = y \bullet x \text{ implies } x = y. \tag{2.1}$$

A groupoid $(X, \bullet)$ is *idempotent* if $x \bullet x = x$ for all $x \in X$.

Example 2.1 [12] Let $X = [0, \infty)$ and let $x \bullet y = \max\{0, x - y\}$ for any $x, y \in X$. Then $(X, \bullet)$ is a strong groupoid. To visualize this, let's consider the associated Cayley product table for "$\bullet$". For simplicity, its partial table is displayed below which shows that $x \bullet y = 0$ for all $x \leq y$ and $x \bullet y \neq 0$ for all $x > y$:

$\bullet$	0	1	2	3	4	$\cdots$
0	0	0	0	0	0	$\cdots$
1	1	0	0	0	0	$\cdots$
2	2	1	0	0	0	$\cdots$
3	3	2	1	0	0	$\cdots$
4	4	3	2	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Hence, the strong or anti-commutative property holds for all $x, y \in X$.

Example 2.2 [12] Let $X = \mathbb{R}$ be the set of all real numbers and let $x, y, e \in \mathbb{R}$. If we define a binary operation "$\bullet$" on X by $x \bullet y = (x - y)(x - e) + e$,
then the groupoid $(X, \bullet, e)$ is not strong, since $x = e + \alpha$, $y = e - \alpha$, $\alpha \neq \pm e$ implies $x \bullet y = y \bullet x$, but $x \neq y$.

A groupoid $(X, \bullet)$ is a *left-zero-semigroup* if $x \bullet y = x$ for all $x, y \in X$. Similarly, $(X, \bullet)$ is a *right-zero-semigroup* if $x \bullet y = y$ for all $x, y \in X$. For the theory of semigroups, we refer to [10, 30].

A groupoid $(X, \bullet)$ is *locally-zero* [11] if
(i) $x \bullet x = x$ for all $x \in X$; and
(ii) for any $x \neq y$ in X, $(\{x, y\}, \bullet)$ is either a left-zero-semigroup or a right-zero-semigroup.

Example 2.3 Given a set $X = \{0, 1, 2\}$, let the binary operation "$\bullet$" be defined by the following Cayley product table:

$\bullet$	0	1	2
0	0	0	2
1	1	1	1
2	0	2	2

Then the binary system $(X, \bullet)$ is locally-zero and has the following subtables:

$\bullet$	0	1
0	0	0
1	1	1

$\bullet$	1	2
1	1	1
2	2	2

$\bullet$	0	2
0	0	2
2	0	2

where $(\{0, 1\}, \bullet)$ is a left-zero-semigroup; $(\{1, 2\}, \bullet)$ is also a left-zero-semigroup; and $(\{0, 2\}, \bullet)$ is a right-zero-semigroup.

The notion of the semigroup $(Bin(X), \diamond)$ was introduced by J. Neggers and H.S. Kim in [33]. Given a non-empty set X, let $Bin(X)$ denote the collection of all groupoids $(X, \bullet)$, where $\bullet : X \times X \to X$ is a map. Given elements $(X, *)$ and $(X, \circ)$ of $Bin\,(X)$, define a binary product "$\diamond$" on these groupoids as follows:

$$(X, *) \diamond (X, \circ) = (X, \bullet) \tag{2.2}$$

where

$$x \bullet y = (x * y) \circ (y * x) \tag{2.3}$$

for all $x,\ y \in X$. This turns $(Bin(X), \diamond)$ into a semigroup with identity, the left-zero-semigroup, and an analog of negative one in the right-zero-semigroup.

The present author [11] showed that a groupoid $(X, \bullet)$ commutes, relative to the product "$\diamond$", if and only if any 2-element subset of $(X, \bullet)$ is a subgroupoid that is either a left-zero-semigroup or a right-zero-semigroup. Thus, $(X, \bullet)$ is an element of the *center* $ZBin(X)$ of the semigroup $(Bin(X), \diamond)$, defined as follows:

$$ZBin\,(X) = \{(X, \bullet) \in Bin\,(X) \mid (X, \bullet) \diamond (X, *) = (X, *) \diamond (X, \bullet),\ \forall (X, *) \in Bin(X)\}.$$

In turn, several properties were obtained.

Theorem 2.4 [33] *The collection $(Bin(X), \diamond)$ of all binary systems (groupoids or algebras) defined on X is a semigroup, i.e., the operation "$\diamond$" as defined in general is associative. Furthermore, the left-zero-semigroup is an identity for this operation.*

Proposition 2.5 [33] *Let $(X, \bullet)$ be the right-zero-semigroup on X. Then $(X, \bullet) \in Str(X)$, the collection of all strong groupoids on X.*

Proposition 2.6 [11] *The left-zero semigroup and right-zero semigroup on X are both in $ZBin(X)$.*

Corollary 2.7. [11] *The collection of all locally-zero groupoids on X forms a subsemigroup of $(Bin(X), \diamond)$.*

Proposition 2.8 [11] *Let $(X, \bullet)$ be a locally-zero groupoid. Then $(X, \bullet) \diamond (X, \bullet) = id_{Bin(X)}$, the left-zero-semigroup on X.*

Let $(X, \bullet)$ be an element of the semigroup $(Bin(X), \diamond)$, we say that $(X, \bullet)$ is a *unit* if and only if there exists an element $(X, *) \in Bin\,(X)$ such that

$$(X, \bullet) \diamond (X, *) = id_{Bin(X)} = (X, *) \diamond (X, \bullet). \tag{2.4}$$

Subsequently, by Proposition 2.8, a locally-zero-groupoid is a unit in $Bin\,(X)$.

The logic-based BCK/BCI-algebras were introduced by Iséki and Imai in [15] as propositional calculus, but later in [16] developed into the present notion of BCK/BCI which have since then been investigated thoroughly by numerous researchers. J. Neggers and H. S. Kim generalized a BCK-algebra [26] by introducing the notion of a d-algebra in [32]. They also introduced B-algebras in [2]. C. B. Kim and H. S. Kim generalized a B-algebra by defining a BG-algebra in [21].

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a B-*algebra* [2] if for all $x, y, z \in X$, it satisfies the following axioms:

B1: $x \bullet x = 0$,
B2: $x \bullet 0 = x$, and
B: $(x \bullet y) \bullet z = x \bullet [z \bullet (0 \bullet y)]$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a BG-*algebra* [21] if for all $x, y, z \in X$, it satisfies **B1**, **B2**, and

BG: $x = (x \bullet y) \bullet (0 \bullet y)$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *BCI-algebra* [36] if for all $x, y, z \in X$, it satisfies **B2**, **I:** $((x \bullet y) \bullet (x \bullet z)) \bullet (z \bullet y) = 0$, and **BH:** $x \bullet y = 0$ and $y \bullet x = 0$ implies $x = y$.

Example 2.9 [36] Let $X = \{0, 1, a, b\}$. Define a binary operation "$\bullet$" on X by the following product table:

$\bullet$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Then $(X, \bullet, 0)$ is a BCI-algebra.

A BCI-algebra $(X, \bullet, 0)$ is a *BCK-algebra* [26] if it satisfies the next additional axiom:

K: $0 \bullet x = 0$ for all $x \in X$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *d-algebra* provided that for all x, $y \in X$, it satisfies (B1), (K) and (BH).

A d-algebra is *strong* if for all x, $y \in X$:

d-3′: $x \bullet y = y \bullet x$ implies $x = y$.

Otherwise we consider the d-algebra to be *exceptional*. For more information on d-algebras we refer to [5, 6, 32, 31].

Example 2.10 [32] Let $(X, \bullet) = (\mathbb{Z}_5, \bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Then $(\mathbb{Z}_5, \bullet, 0)$ is a d-algebra which is not a BCK-algebra. For details on BCK-algebras, see [14, 26, 36].

Y. B. Jun, E. H. Roh and H. S. Kim in [18] introduced the notion of a BH-algebra which is a generalization of $BCK/BCI/BCH$-algebras. There are many other generalizations of similar algebras. We summarize several properties which are used as axioms to define each algebraic structure . Let $(X, \bullet, 0)$ be an algebra of type $(2, 0)$, for any x, y, $z \in X$:

B1: $x \bullet x = 0$,
B2: $x \bullet 0 = x$,
B: $(x \bullet y) \bullet z = x \bullet (z \bullet (0 \bullet y))$,
BG: $x = (x \bullet y) \bullet (0 \bullet y)$,
BM: $(z \bullet x) \bullet (z \bullet y) = y \bullet x$,
BH: $x \bullet y = 0$ and $y \bullet x = 0 \Rightarrow x = y$,
BF: $0 \bullet (x \bullet y) = y \bullet x$,
BN: $(x \bullet y) \bullet z = (0 \bullet z) \bullet (y \bullet x)$,
BO: $x \bullet (y \bullet z) = (x \bullet y) \bullet (0 \bullet z)$,
BP1: $x \bullet (x \bullet y) = y$,
BP2: $(x \bullet z) \bullet (y \bullet z) = x \bullet y$,
Q: $(x \bullet y) \bullet z = (x \bullet z) \bullet y$,
CO: $(x \bullet y) \bullet z = x \bullet (y \bullet z)$,
BZ: $((x \bullet z) \bullet (y \bullet z)) \bullet (x \bullet y) = 0$,

K: $0 \bullet x = 0$,
I: $((x \bullet y) \bullet (x \bullet z)) \bullet (z \bullet y) = 0$,
BI: $x \bullet (y \bullet x) = x$.

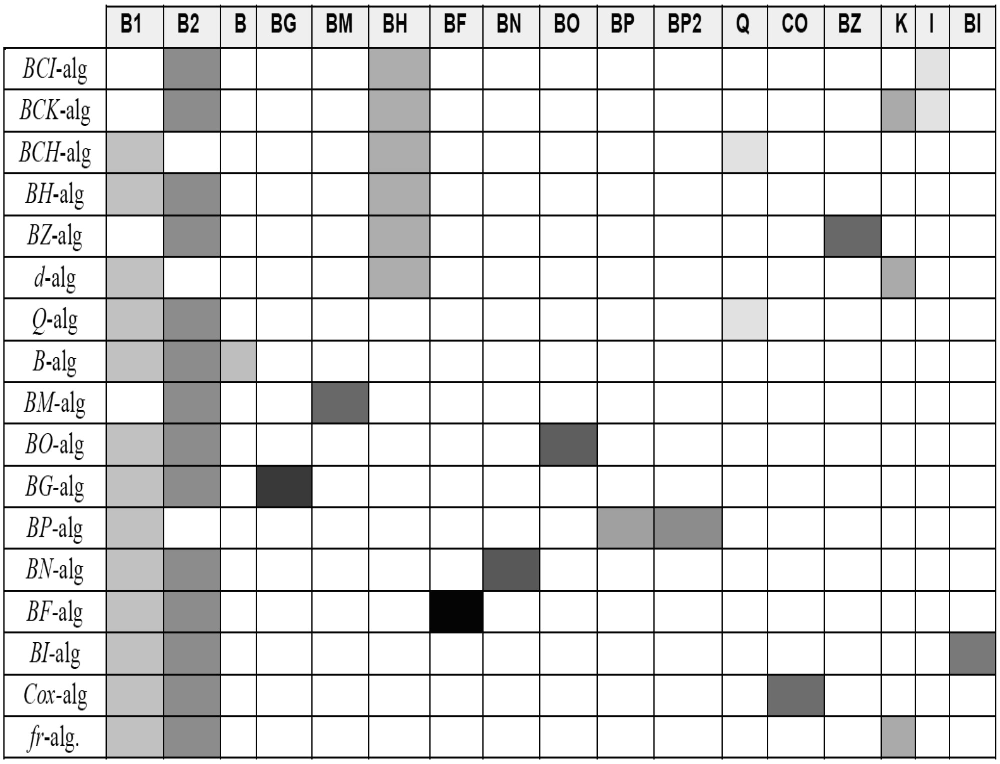


Figure 1. Comparison of Algebras

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is classified according to a combination of axioms **B1** through **BI** as noted in "**Figure 1**" above. For instance, $(X, \bullet, 0)$ is a BN-algebra [19] if it satisfies **B1**, **B2** and **BN**. For detailed information on each, please see [2-6, 14-26, 31, 32, 34, 36].

3. Similar-Signature Factorization

In this section, we present a unique factorization of a given groupoid by "deriving" two factors from it and from the left-zero-semigroup simultaneously.

Let $(X, \bullet)$ be a groupoid of finite order, i.e., $|X| = n$. Then $d^\bullet$ is the *diagonal function* of $(X, \bullet)$ such that $d^\bullet : \mathbb{N} \longrightarrow X$ where $d^\bullet(i) = x_i \bullet x_i$, $i = 1, 2, ..., n$ for all $x_i \in X$.
Example 3.1 Let $(X, \bullet, 0)$ and $(X, *)$ be a d-algebra and an idempotent algebra, respectively. Then $x \bullet x = 0$ and $x * x = x$; or $d^\bullet = 0$ and $d^* = x$ for all $x \in X$.

Two binary systems $(X, *)$ and $(X, \bullet)$ are said to be *similar* if they have the same diagonal function, that is, $d^* = d^\bullet$.

Two binary systems $(X, *)$ and $(X, \bullet)$ are said to be *signature* if
(i) $x * y = x \bullet y$ when $x \neq y$; and
(ii) $x * x \neq x \bullet x$ for all $x \in X$.

Let $(X,\bullet)$ be a groupoid. *Derive* groupoids $(X,*)$ and $(X,\circ)$ from $(X,\bullet)$ and $id_{Bin(X)}$, simultaneously, such that for all $x,y\in X$,

$$x*y=\begin{cases}x & \text{if } x=y,\\ x\bullet y & \text{otherwise.}\end{cases}\quad\text{and}\quad x\circ y=\begin{cases}x\bullet x & \text{if } x=y,\\ x & \text{otherwise.}\end{cases}\tag{3.1}$$

The groupoids $(X,*)$ and $(X,\circ)$ are said to be the *signature*- and the *similar-factors* of $(X,\bullet)$, respectively, denoted by $U(X,\bullet)$ and $A(X,\bullet)$. The product "$\diamond$" is associative but not commutative. Hence, for $(X,\bullet)\in Bin(X)$, we may have a *UA-factorization* such that

$$(X,\bullet)=U(X,\bullet)\diamond A(X,\bullet)\tag{3.2}$$

or an *AU-factorization* such that

$$(X,\bullet)=A(X,\bullet)\diamond U(X,\bullet).\tag{3.3}$$

By the equations in 3.1, it follows that for any given groupoid $(X,\bullet)$,

(1) $U(X,\bullet)$ is similar to $id_{Bin(X)}$ while $A(X,\bullet)$ is similar to $(X,\bullet)$; and
(2) $U(X,\bullet)$ is signature with $(X,\bullet)$ while $A(X,\bullet)$ is signature with $id_{Bin(X)}$.

Proposition 3.2 *The similar-factor of a groupoid is strong.*
Proof. Given $(X,\bullet)\in Bin(X)$, let $(X,\circ)=A(X,\bullet)$.

(i) If $x=y$, then $x\circ y=x\circ x=x\bullet x=y\bullet y=y\circ y=y\circ x$.
(ii) If $x\neq y$ and $x\circ y=y\circ x$ for any $x,y\in X$. Then $x\circ y=x$ and $y\circ x=y$. Thus, $x=y$, a contradiction.

Therefore, $(X,\circ)$ is strong.

■

Example 3.3 Let $(X,\bullet,0)$ be the BCI-algebra defined in Example 2.9. In accordance with equation 3.1, derive its *signature*- and *similar*- factors $U(X,\bullet,0)$ and $A(X,\bullet,0)$, respectively. Let groupoids $(X,*,0):=U(X,\bullet,0)$ and $(X,\circ,0):=A(X,\bullet,0)$ be given. We obtain:

$*$	0	1	a	b
0	0	0	a	a
1	1	1	a	a
a	a	a	a	0
b	b	a	1	b

and

$\circ$	0	1	a	b
0	0	0	0	0
1	1	0	1	1
a	a	a	0	a
b	b	b	b	0

It remains to verify that $(X,\bullet,0)=(X,*,0)\diamond(X,\circ,0)$ and/or $(X,\bullet,0)=(X,\circ,0)\diamond(X,*,0)$. This will be discussed in more detail in the next section. However, there is a very interesting fact in this example: the two factors are distinct from each other, their parent groupoid, and the left-zero-semigroup. In summary:

(1) $(X,*,0)\neq(X,\circ,0)$; (Problem 1.1)
(2) $(X,*,0)\neq(X,\bullet,0)\neq(X,\circ,0)$; (Problem 1.3)
(3) $(X,*,0)\neq id_{Bin(X)}\neq(X,\circ,0)$. (Problem 1.4)

This is important since it is not always the case that all three distinctions hold as the following example demonstrates.

Example 3.4 Let $(X,\bullet)=(\mathbb{Z}_3,\bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X, \bullet, 0)$ is a BI-algebra. Derive its *signature*- and *similar*-factors $U(X, \bullet, 0)$ and $A(X, \bullet, 0)$, respectively, in accordance to the equations in 3.1. Let $(X, *, 0) := U(X, \bullet, 0)$ and $(X, \circ, 0) := A(X, \bullet, 0)$, hence:

$*$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

and

$\circ$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Here we observe immediately that the *similar*-factor $(X, \circ, 0)$ is equal to $(X, \bullet, 0)$ and the *signature*-factor $(X, *, 0)$ is equal to $id_{Bin(X)}$. Thus this decomposition is basically a trivial factorization, i.e.,

$$(X, \bullet, 0) = (X, *, 0) \diamond (X, \circ, 0) = id_{Bin(X)} \diamond (X, \bullet, 0)$$

and

$$(X, \bullet, 0) = (X, \circ, 0) \diamond (X, *, 0) = (X, \bullet, 0) \diamond id_{Bin(X)}.$$

3.1. UA**-Factorization.** In this subsection, we explore a UA-factorization of a given groupoid $(X, \bullet)$ in $Bin(X)$. In the next subsection, a AU-factorization is considered, where the order of the product of the two factors is "reversed". We emphasize that such factorization is unique and not necessarily reversible. Then, we classify a given groupoid as UA- and/or *AU-composite*, *u-composite* or *u-normal;* and as *signature*- or *similar-prime.*

Example 3.1.1 Let $X = \mathbb{Z}$ be the set of all integers and let "$-$" be the usual subtraction on $\mathbb{Z}$. Then $(\mathbb{Z}, -)$ is a BH-algebra since it satisfies axioms **B1**, **B2** and **BH** as seen from its partial table below:

$-$	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
-2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
-1	2	1	0	-1	-2	-3	-4	-5	$\cdots$
0	3	2	1	0	-1	-2	-3	-4	$\cdots$
1	4	3	2	1	0	-1	-2	-3	$\cdots$
2	5	4	3	2	1	0	-1	-2	$\cdots$
3	6	5	4	3	2	1	0	-1	$\cdots$
4	7	6	5	4	3	2	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Define two binary operations "$*$" and "$\circ$" on $\mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$,

$$x * y = \begin{cases} x & \text{if } x = y, \\ x - y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Then it is easy to check that $(\mathbb{Z}, -) = (\mathbb{Z}, *) \diamond (\mathbb{Z}, \circ)$ and $(\mathbb{Z}, *) = U(\mathbb{Z}, -)$ and $(\mathbb{Z}, \circ) = A(\mathbb{Z}, -)$. Thus we have a UA-factorization of $(\mathbb{Z}, -)$.

A groupoid $(X, \bullet)$ is said to be *signature-prime* if $U(X, \bullet) = id_{Bin(X)}$, and is said to be *similar-prime* if $A(X, \bullet) = id_{Bin(X)}$. Alternatively, if $(X, \bullet)$ is neither *signature*- nor *similar*-prime, then $(X, \bullet)$ is said to be

(1) *UA-composite* if $(X, \bullet) = U(X, \bullet) \diamond A(X, \bullet)$;
(2) *AU-composite* if $(X, \bullet) = A(X, \bullet) \diamond U(X, \bullet)$.

Consequently, $(X, \bullet)$ is said to be *u-composite* if both (1) and (2) hold.

Example 3.1.2 Let $(X, \bullet) = (\mathbb{Z}_5, \bullet)$ where the product "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4
0	3	2	2	1	1
1	1	3	3	2	3
2	3	3	0	3	0
3	1	0	1	1	2
4	1	1	2	4	2

If we derive its *signature*- and *similar*- factors $(\mathbb{Z}_5, *) = U(\mathbb{Z}_5, \bullet)$ and $A(\mathbb{Z}_5, \bullet) = (\mathbb{Z}_5, \circ)$ as in (3.1), then we have their $\diamond$ product as follows:

$*$	0	1	2	3	4
0	0	2	2	1	1
1	1	1	3	2	3
2	3	3	2	3	0
3	1	0	1	3	2
4	1	1	2	4	4

$\diamond$

$\circ$	0	1	2	3	4
0	3	0	0	0	0
1	1	3	1	1	1
2	2	2	0	2	2
3	3	3	3	1	3
4	4	4	4	4	2

$=$

∇	0	1	2	3	4
0	3	2	2	3	3
1	1	3	1	2	3
2	3	1	0	3	0
3	3	0	1	1	2
4	3	1	2	4	2

We can clearly conclude that $U(\mathbb{Z}_5, \bullet) \diamond A(\mathbb{Z}_5, \bullet) \neq (\mathbb{Z}_5, \bullet)$ since $(\mathbb{Z}_5, \bullet) \neq (\mathbb{Z}_5, \nabla)$ and hence such a groupoid does not have a UA-factorization. Moreover, $(\mathbb{Z}_5, \bullet)$ is not a strong groupoid since $0 \bullet 4 = 4 \bullet 0$. In turn, we have the next theorem.

Theorem 3.1.3 *A strong groupoid has a UA-factorization.*

Proof. Let $(X, \bullet) \in Str\,(X)$, the collection of all strong groupoids defined on X, and let $(X, \odot) = (X, *) \diamond (X, \circ)$ where $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$. Then $x \odot y = (x * y) \circ (y * x)$ for all $x, y \in X$. It follows that $x * x = x$, $x * y = x \bullet y$ when $x \neq y$; and $x \circ x = x \bullet x$, $x \circ y = x$ when $x \neq y$.
Next, we show that $(X, \bullet) = (X, \odot)$. Given $x, y \in X$, if $x = y$, then $x \odot x = (x * x) \circ (x * x) = x \circ x = x \bullet x$. Assume $x \neq y$, we claim that $x * y = y * x$ is not possible:
(i) If $x * y = y * x$, then $x \bullet y = x * y = y * x = y \bullet x$. Since $(X, \bullet)$ is strong, we obtain $x = y$, a contradiction.
(ii) If $x * y \neq y * x$, then $x * y = x \bullet y$, $y * x = y \bullet x$, since $x \neq y$.
Therefore $x \odot y = (x * y) \circ (y * x) = (x \bullet y) \circ (y \bullet x) = x \bullet y$, since $x \bullet y \neq y \bullet x$. This proves that $(X, \odot) = (X, \bullet)$.

■

Corollary 3.1.4 *The factorization in Theorem 3.1.3 is unique.*

Proof. Let $(X, \bullet)$ be a strong groupoid with a UA-factorization such that $(X, \bullet) = (X, *) \diamond (X, \circ)$ where $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$. Let $(X, \bullet) = (X, \triangledown) \diamond (X, \triangle)$ where $(X, \triangledown) = U(X, \bullet)$ and $(X, \triangle) = A(X, \bullet)$. For any $x \in X$, we have $x * x = x = x \triangledown x$, and $x * y = x \triangledown y$ when $x \neq y$. Hence $(X, *) = (X, \triangledown)$. Similarly, if $x \in X$, then $x \circ x = x \bullet x = x \triangle x$. When $x \neq y$, we have $x \circ y = x = x \triangle y$, proving that $(X, \circ) = (X, \triangle)$.

■

Example 3.1.5 [32] Consider the d-algebra $(X, \bullet, 0)$ from Example 2.10. Observe that $(X, \bullet, 0)$ is a strong d-algebra. Let $(X, *, 0) := U(X, \bullet, 0)$ and $(X, \circ, 0) := A\,(X, \bullet, 0)$, such that $U(X, \bullet, 0)$ and $A(X, \bullet, 0)$ are its derived *signature*- and *similar*-factors, respectively, as in (3.1). Next, verify that $(X, *, 0) \diamond (X, \circ, 0) = (X, \bullet, 0)$:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	1	1	0	1
2	2	2	2	3	0
3	3	3	2	3	3
4	4	4	1	1	4

$\diamond$

$\circ$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$=$

$\bullet$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Indeed we can see that $x \bullet y = (x * y) \circ (y * x)$ for any $x, y \in X$. For instance:

$$\begin{aligned}(1 * 0) \circ (0 * 1) &= 1 \circ 0 = 1 = 1 \bullet 0,\\ (3 * 4) \circ (4 * 3) &= 3 \circ 4 = 3 = 3 \bullet 4.\end{aligned}$$

Moreover, since $U(X, \bullet, 0) \neq id_{Bin(X)}$ and $A(X, \bullet, 0) \neq id_{Bin(X)}$, then $(X, \bullet, 0)$ is UA-composite.

3.2. *AU*-**Factorization.** In this subsection we reverse the order of the *signature*- and *similar*-factors of any groupoid $(X, \bullet)$ in $Bin(X)$. We conclude that an arbitrary groupoid $(X, \bullet)$ will always have an AU-factorization. However, this factorization might be trivial and hence the groupoid is either noted as *signature*- or *similar*-prime. Otherwise, if the decomposition is not trivial, we say the groupoid is AU-composite.

Example 3.2.1 Let $(X, \bullet, 0)$ be the strong d-algebra defined in Examples 2.10 and 3.1.5 in which we determined that $(X, \bullet, 0)$ is UA-composite. Similarly, we can take the product of $A(X, \bullet, 0)$ and $U(X, \bullet, 0)$ as follows:

$\circ$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$\diamond$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	1	1	0	1
2	2	2	2	3	0
3	3	3	2	3	3
4	4	4	1	1	4

$=$

$\bullet$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

By routine checking of $(x \circ y) * (y \circ x) = x \bullet y$ for any $x, y \in X$, we conclude that $(X, \bullet, 0)$ has an AU-factorization. Moreover, we can see that this particular groupoid has both, a non-trivial UA- and AU-factorization. Therefore, $(X, \bullet, 0)$ is u-composite.

Remark 3.2.2 Note that $A(X, \bullet) \diamond U(X, \bullet) = U(X, \bullet) \diamond A(X, \bullet)$ does not imply that $(X, \bullet)$ is u-composite. It simply implies that the factors of $(X, \bullet)$ commute. This motivates the next definition.

A groupoid $(X, \bullet)$ is said to be *u-normal* if it admits a UA- and an AU-factorization, i.e., if

(i) $(X, \bullet) = U(X, \bullet) \diamond A(X, \bullet)$, and
(ii) $(X, \bullet) = A(X, \bullet) \diamond U(X, \bullet)$.

Theorem 3.2.3 *Any given groupoid has an AU-factorization, i.e., if* $(X, \bullet) \in Bin(X)$, *then*

$$(X, \bullet) = A(X, \bullet) \diamond U(X, \bullet).$$

Proof. Let $(X, \bullet) \in Bin(X)$ and let $(X, \odot) = (X, \circ) \diamond (X, *)$ where $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$. Then $x \odot y = (x \circ y) * (y \circ x)$ for all $x, y \in X$. It follows that $x * x = x$, $x * y = x \bullet y$ when $x \neq y$, and $x \circ x = x \bullet x$, $x \circ y = x$ when $x \neq y$. Given $x, y \in X$, if $x = y$, then $x \odot x = (x \circ x) * (x \circ x) = (x \bullet x) * (x \bullet x) = x \bullet x$. Assume $x \neq y$, then $x \odot y = (x \circ y) * (y \circ x) = x * y = x \bullet y$. This proves that $(X, \odot) = (X, \bullet)$.

■

Corollary 3.2.4 *The factorization in Theorem 3.2.3 is unique.*

Proof. The proof is similar to that of Corollary 3.1.4.

■

Corollary 3.2.5 *A strong groupoid is u-normal.*

Proof. The proof follows directly from Theorems 3.1.3, 3.2.3 and the definition.

■

Example 3.2.6 Let $(X,\bullet) = (\{0,1,2\},+)$ be the cyclic group of order 3. Observe that $(\{0,1,2\},+)$ has an AU-factorization but fails to have a UA-factorization. Take $(\{0,1,2\},*) = U(\{0,1,2\},+)$ and $(\{0,1,2\},\circ) = A(\{0,1,2\},+)$ such that:

$$x \circ y = \begin{cases} (x+x) \bmod 3 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases} \quad \text{and} \quad x * y = \begin{cases} x; & \text{if } x = y, \\ (x+y) \bmod 3 & \text{otherwise.} \end{cases}$$

Routine checking of the product $A(\{0,1,2\},+) \diamond U(\{0,1,2\},+)$ gives $(\{0,1,2\},+)$:

$\circ$	0	1	2
0	0	0	0
1	1	2	1
2	2	2	1

$\diamond$

$*$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	2

$=$

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

But, the product $U(\{0,1,2\},+) \diamond A(\{0,1,2\},+)$ does not give $(\{0,1,2\},+)$:

$*$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	2

$\diamond$

$\circ$	0	1	2
0	0	0	0
1	1	2	1
2	2	2	1

$=$

∇	0	1	2
0	0	2	1
1	2	2	0
2	1	0	1

Therefore, $(\{0,1,2\},+)$ is not u-normal, it is simply AU-composite.

Proposition 3.2.7 *Any signature- or similar-prime groupoid is u-normal.*

Proof. The proof is straightforward and we omit it.

■

Proposition 3.2.8 *The right-zero-semigroup on X is similar-prime.*

Proof. Let $(X,\bullet)$ be the right-zero-semigroup on X. Then $x \bullet y = y$ for all $x, y \in X$. Let $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$, thus

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y = y & \text{otherwise} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = x & \text{if } x = y \\ x \circ y = x & \text{otherwise} \end{cases}$$

Hence for all $x, y \in X$, $(X,\bullet) = (X,\bullet) \diamond id_{Bin(X)}$.

■

Example 3.2.9 Let $(X,\bullet) = (\{a,b,c\},\bullet)$ be the right-zero-semigroup on $\{a,b,c\}$. Its Cayley table together with its associated *signature-similar*-product tables, respectively, are:

$\bullet$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

$$\begin{array}{c|ccc} * & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & a & b & c \\ \hline a & a & a & a \\ b & b & b & b \\ c & c & c & c \end{array} \quad = \quad \begin{array}{c|ccc} \bullet & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array}$$

Therefore, the right-zero-semigroup of order 3 is *similar*-prime since its *similar*-factor $A(\{a,b,c\},\bullet)$ is $id_{Bin(X)}$, i.e., the left-zero-semigroup for $\{a,b,c\}$.

Proposition 3.2.10 *A non-locally-zero strong groupoid is u-composite.*

Proof. Let $(X,\bullet) \in Bin(X) - ZBin(X)$, then $x \bullet y \neq \{x,y\}$ for any $x,y \in X$. Meaning, $(X,\bullet)$ cannot be the left- nor the right-zero-semigroup on X. By Proposition 3.2.5, $(X,\bullet)$ is u-normal. Let $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$, then

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y & \text{otherwise} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x & \text{if } x = y \\ x \circ y = x & \text{otherwise} \end{cases}$$

Hence, for all $x,y \in X$, $(X,*) \neq (X,\bullet) \neq (X,\circ)$ and $(X,*) \neq id_{Bin(X)} \neq (X,\circ)$. Therefore, $(X,\bullet)$ is u-composite.

3.3. **Factoring U $(X,\bullet)$ and A $(X,\bullet)$.** Let $Str(X)$ be the collection of all strong groupoids on a non-empty set X. Consider a groupoid $(X,\bullet) \in Str(X)$, we classify the *signature*- and *similar*-factors of $(X,\bullet)$ as UA-composite, *signature*- or *similar*-prime. We conclude that $U(X,\bullet)$ and $A(X,\bullet)$ are *similar*- and *signature*-prime, respectively.

Theorem 3.3.1 *The signature-factor of a strong groupoid is similar-prime, and the similar-factor is signature-prime.*

Proof. Let $(X,\bullet) \in Str(X)$. Suppose that $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$. Let $(X,\circledast) = U(X,*)$ and $(X,\odot) = A(X,*)$, then "$\circledast$" and "$\odot$" are defined as:

$$x \circledast y = \begin{cases} x; & \text{if } x = y, \\ x * y = x \bullet y & \text{otherwise} \end{cases} \quad \text{and} \quad x \odot y = \begin{cases} x * x = x & \text{if } x = y, \\ x; & \text{otherwise.} \end{cases}$$

Hence $A(X,*) = id_{Bin(X)}$, and therefore $U(X,\bullet)$ is *similar*-prime. Similarly, if we let $(X,\boxtimes) = U(X,\circ)$ and $(X,\boxdot) = A(X,\circ)$, then "$\boxtimes$" and "$\boxdot$" are defined as:

$$x \boxtimes y = \begin{cases} x & \text{if } x = y, \\ x \circ y = x & \text{otherwise} \end{cases} \quad \text{and} \quad x \boxdot y = \begin{cases} x \circ x = x \bullet x & \text{if } x = y, \\ x; & \text{otherwise.} \end{cases}$$

Therefore, $U(X,\circ) = id_{Bin(X)}$, and hence $A(X,\bullet)$ is *signature*-prime.

■

Corollary 3.3.2. *Let $(X,\bullet)$ be any groupoid and let $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$. If $(X,\bullet)$ has a UA-factorization, i.e., if $(X,\bullet) = (X,*) \diamond (X,\circ)$, then*

$$(X,\bullet) = U(X,*) \diamond A(X,\circ).$$

Proof. This follows immediately from the previous theorem. In fact, suppose $(X,\bullet)$ has a UA-factorization, then

$$\begin{aligned} (X,\bullet) &= (X,*) \diamond (X,\circ) \\ &= (U(X,*) \diamond A(X,*)) \diamond (U(X,\circ) \diamond A(X,\circ)) \\ &= (U(X,*) \diamond id_{Bin(X)}) \diamond \left(id_{Bin(X)} \diamond A(X,\circ)\right) \\ &= U(X,*) \diamond A(X,\circ). \end{aligned}$$

■

Corollary 3.3.3. *Let $(X,\bullet)$ be a groupoid and let $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$. If $(X,\bullet)$ has a AU-factorization then*

$$(X,\bullet) = A(X,\circ) \diamond U(X,*) .$$

Proof. The proof is very similar to that of the previous Corollary.

■

Corollary 3.3.4. *Let $(X,\bullet)$ be a strong groupoid and let $(X,*) = U(X,\bullet)$ and $(X,\circ) = A(X,\bullet)$, then*

$$(X,\bullet) = A(X,\circ) \diamond U(X,*) = U(X,*) \diamond A(X,\circ) .$$

Proof. This is a direct result of Theorem 3.1.3 and the previous two Corollaries.

■

As a final observation, a groupoid is *similar*-prime if it is similar to the left-zero-semigroup or a locally-zero-groupoid, in other words, if it is idempotent. Hence, we need another method of factorization for idempotent groupoids.

4. Orient-Skew Factorization

We say a groupoid $(X,*)$ has the *orientation property OP* [33] if $x * y \in \{x, y\}$ for all $x, y \in X$. Moreover, $(X,*)$ has the *twisted orientation property TOP* if $x * y = x$ implies $y * x = x$ for all $x, y \in X$. In this section, we introduce a unique factorization which can be applied to groupoids with *OP*. This type of groupoids has proven to be useful in graph theory, where in a directed graph $x*y = x$ can mean there is a path from vertex x to vertex y, i.e. $x \rightarrow y$; while $x * y = y$ can mean there is no path from x to y, i.e. $x \nrightarrow y$. In fact, if $\Gamma_{(X,*)}$ is the directed graph on vertex set X and $(X,*) \in TOP(X)$, then $\Gamma_{(X,*)}$ is a simple graph [1]. For more details on groupoids associated with directed and simple graphs we refer to [1, 35].

Example 4.1 Let $X = \{0, 1\}$ and $(X, \leq)$ be a linearly ordered set. Define a binary operation "$\bullet$" on X such that:

$$x \bullet y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then the binary system $(X,\bullet)$ has the orientation property.

Example 4.2 Let $X = \{a, b, c\}$. Define a binary operation "$\bullet$" on X by the following table:

$\bullet$	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

Then $(X,\bullet)$ has the twisted orientation property.

We consider three functions to represent operations on the main diagonal and on the anti-diagonal of the associated Cayley table of a binary operation on a finite set.

Let $(X,*)$ be a groupoid of finite order n and binary operation "$*$", i.e., $|X| = n$ and $* : X^2 \longrightarrow X$. Then for all x_i, $x_j \in X$, $i, j = 1, 2, ..., n$, and $i + j = n + 1$, we call:

- **diag-1:** $\overline{d^*}$ the *anti-diagonal function* of $(X,*)$ such that $\overline{d^*} : \mathbb{N} \longrightarrow X$, defined by $\overline{d^*}(i) = x_i * x_j$.
- **diag-2:** $\widehat{d^*}$ the *reverse-diagonal function* of $(X,*)$ such that $\widehat{d^*} : \mathbb{N} \longrightarrow X$, defined by $\widehat{d^*}(i) = x_j * x_j$.

diag-3: $\widetilde{d^*}$ the *skew-diagonal function* of $(X,*)$ such that $\widetilde{d^*}:\mathbb{N}\longrightarrow X$, defined by $\widetilde{d^*}(i)=\widehat{\overline{d^*}}(i)=x_j*x_i$.

Example 4.3 Consider the groupoid $(\{0,1,2,3\},*)$ where "$*$" is given by the following table:

$*$	0	1	2	3
0	0	1	0	3
1	1	1	1	0
2	2	2	2	3
3	0	3	2	3

Observe that $n=4$ and the main diagonal $d^*=\{0,1,2,3\}$. For instance, $d^*(2)=2*2=2$. Also, the anti-diagonal $\overline{d^*}=\{3,1,2,0\}$. For example, $\overline{d^*}(1)=x_1*x_4=0*3=3$. Moreover, the reverse of the diagonal is $\widehat{d^*}=\{3,2,1,0\}$. For instance, $\widehat{d^*}(4)=x_1*x_1=0*0=0$. So the skew-diagonal defined here is the reverse of the anti-diagonal, hence, $\widetilde{d^*}=\{0,2,1,3\}$. For example, $\widetilde{d^*}(3)=\widehat{\overline{d^*}}(3)=x_2*x_3=1*2=1$.

Given these definitions, we can *derive* the *orient*-factor of a groupoid from $id_{Bin(X)}$, such that all its elements are the same as those of the left-zero-semigroup except elements belonging to the anti-diagonal, which we construct from the skew-diagonal of $id_{Bin(X)}$. Similarly, the *skew*-factor is *derived* from the parent groupoid by letting its anti-diagonal be that of the skew-diagonal of the parent groupoid, otherwise all other elements are kept the same as the parent groupoid.

Let $(X,\bullet)$ be a groupoid. Let $D^\diamond$ denote the main diagonal of $id_{Bin(X)}$. *Derive* groupoids $(X,*)$ and $(X,\circ)$ from $id_{Bin(X)}$ and $(X,\bullet)$, respectively, as follows:
For all $x,y\in X$,

(4.1) (i) $\overline{d^*}=\widetilde{D^\diamond}$, (ii) $x*y=x$; otherwise. and (i) $\overline{d^\circ}=\widetilde{d^\bullet}$, (ii) $x\circ y=x\bullet y$; otherwise.

Groupoids $(X,*)$ and $(X,\circ)$ are said to be the *orient-* and *skew-factor* of $(X,\bullet)$, respectively, denoted by $O(X,\bullet)$ and $J(X,\bullet)$. As previously mentioned, the product "$\diamond$" is not commutative. Hence, for $(X,\bullet)\in Bin(X)$, we may have an *OJ-factorization* such that

$$(X,\bullet)=O(X,\bullet)\diamond J(X,\bullet) \tag{4.2}$$

or a *JO-factorization* such that

$$(X,\bullet)=J(X,\bullet)\diamond O(X,\bullet). \tag{4.3}$$

Proposition 4.4 *The orient-factor of a given groupoid is locally-zero.*

Proof. Given $(X,\bullet)\in Bin(X)$, let $(X,*)=O(X,\bullet)$. Then, $d^*=D^\diamond$, i.e. $x*x=x$, and $x*y=x$ for all $x,y\in X$ except when $x,y\in\overline{d^*}$. In fact, for any $x\neq y$ in X, $(\{x,y\},\bullet)$ is either a left-zero-semigroup or a right-zero-semigroup. Moreover, $x\bullet x=x$ for all $x\in X$ which implies that $O(X,\bullet)$ is locally-zero.

∎

Corollary 4.5 *The orient-factor of a given groupoid is a unit in $Bin(X)$.*

Proof. This follows immediately from Propositions 2.8 and 4.4.

∎

Example 4.6 Let $X = \{e, a, b, c\}$. Define a binary operation "$\bullet$" by the following table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Then, clearly $(X, \bullet, e)$ is a group. Derive its *orient*-factor $(X, *, e) = U(X, \bullet, e)$ as in 4.1 to obtain:

$*$	e	a	b	c
e	e	e	e	c
a	a	a	b	a
b	b	a	b	b
c	e	c	c	c

Hence, $(X, *, e)$ is locally-zero.

4.1. OJ**-Factorization.** In this subsection, we explore an OJ-factorization of any groupoid $(X, \bullet)$ in $Bin\,(X)$, i.e., into its *orient*- and *skew*-factors, respectively. The next subsection discusses a JO-factorization where the product of the two factors is "reversed". Then, we classify $(X, \bullet)$ as OJ- and/or JO-*composite*, j-*composite* or j-*normal;* and as *orient*- or *skew-prime.*

A groupoid $(X, \bullet)$ is *bi-diagonal* if its anti-diagonal is symmetric, meaning if $\overline{d^\bullet} = \widetilde{d^\bullet}$.

Example 4.1.1. Let $(\mathbb{Z}, <)$ be a linearly ordered set. Consider groupoid $(\mathbb{Z}, \bullet)$ where $x \bullet y = \max\{x, y\}$ for all $x, y \in \mathbb{Z}$. Define two binary operations on $\mathbb{Z}$ such that:

$$x * y = \begin{cases} x & \text{if } x < y, \\ y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Then clearly $(X, *) \diamond (X, \circ)$ is an OJ-factorization of $(X, \bullet)$, where $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$. Moreover, $(\mathbb{Z}, \bullet)$ is bi-diagonal.

A groupoid $(X, \bullet)$ is said to be *orient-prime* if $O\,(X, \bullet) = id_{Bin(X)}$, and is said to be *skew-prime* if $J\,(X, \bullet) = id_{Bin(X)}$. Alternatively, if $(X, \bullet)$ is neither *orient*- nor *skew*-prime, then $(X, \bullet)$ is said to be

(1) OJ-*composite* if $(X, \bullet) = O\,(X, \bullet) \diamond J\,(X, \bullet)$;
(2) JO-*composite* if $(X, \bullet) = J\,(X, \bullet) \diamond O\,(X, \bullet)$.

Consequently, $(X, \bullet)$ is said to be j-*composite* if both (1) and (2) hold.

Just as with UA-factorization, not every groupoid will have a JO-factorization. But it is possible to *derive* an OJ-factorization of any given groupoid.

Theorem 4.1.2 *Any given groupoid has an OJ-factorization, i.e., if $(X, \bullet) \in Bin(X)$, then*

$$(X, \bullet) = O(X, \bullet) \diamond J(X, \bullet).$$

Proof. Let $(X, \bullet) \in Bin\,(X)$ such that $O(X, \bullet)$ and $J(X, \bullet)$ are defined as in 4.1. Let $(X, \odot) = (X, *) \diamond (X, \circ)$ where $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$. Then $x \odot y = (x * y) \circ (y * x)$ for all $x, y \in X$. It follows that

(i) If $x = y$, $x * x = x$ and $x \circ x = x \bullet x$.
(ii) If $x \neq y$, then if $x * y \in \overline{d^*}$, $x * y \in \widetilde{D^\diamond}$, and for $x \circ y \in \overline{d^\circ}$, then $x \circ y \in \widetilde{d^\bullet}$. Otherwise, $x * y = x$, and $x \circ y = x \bullet y$.

Next, we show that $(X, \bullet) = (X, \odot)$. Given $x, y \in X$,

(i) If $x = y$, $x \odot x = (x \circ x) * (x \circ x) = x \circ x = x \bullet x$.

(ii) If $x \neq y$, then if $x * y = y * x$, then $x \odot y = (x * y) \circ (y * x) = x \circ y = x \bullet y$ and $y \odot x = (y * x) \circ (x * y) = y \circ x = x \bullet y$. If $x * y \neq y * x$, then $x \odot y = (x * y) \circ (y * x) = (x * y) \bullet (y * x) \in \{x \bullet y,\ y \bullet x\}$.

Thus, $x \odot y = x \bullet y$ for all $x, y \in X$. This proves that $(X, \odot) = (X, \bullet)$.

■

Corollary 4.1.3 *The factorization in Theorem 4.1.2 is unique.*

Proof. Let $(X, \bullet) \in Bin(X)$ with an OJ-factorization such that $(X, \bullet) = (X, *) \diamond (X, \circ)$ where $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$. Let $(X, \bullet) = (X, \bigtriangledown) \diamond (X, \bigtriangleup)$ where $(X, \bigtriangledown) = O(X, \bullet)$ and $(X, \bigtriangleup) = J(X, \bullet)$. For any $x \in X$, we have $x * x = x = x \bigtriangledown x$, and $x * y = x \bigtriangledown y$ when $x \neq y$. Hence $(X, *) = (X, \bigtriangledown)$. Similarly, if $x \in X$, then $x \circ x = x \bullet x = x \bigtriangleup x$. When $x \neq y$, we have $x \circ y = x \bullet y = x \bigtriangleup y$, proving that $(X, \circ) = (X, \bigtriangleup)$.

■

Example 4.1.4 [32] Consider the groupoid $(X, \bullet) = (\{1, 2, 3, 4\}, \bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

By *deriving* its *orient*- and *skew*-factors $O(X, \bullet)$ and $J(X, \bullet)$, respectively, and by letting $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$ shows that $(X, *) \diamond (X, \circ) = (X, \bullet)$.
Indeed, $(X, \bullet)$ has an OJ-factorization:

$*$	1	2	3	4
1	1	1	1	4
2	2	2	3	2
3	3	2	3	3
4	1	4	4	4

$\diamond$

$\circ$	1	2	3	4
1	1	1	3	4
2	2	2	2	2
3	1	3	3	4
4	1	4	3	4

$=$

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

Also, since $O(X, \bullet) \neq id_{Bin(X)} \neq J(X, \bullet)$, then $(X, \bullet)$ is OJ-composite.

4.2. ***JO*-Factorization.** In this subsection, we reverse the product of the *orient*- and *skew*-factors of a given groupoid $(X, \bullet) \in Bin(X)$. We find that an arbitrary groupoid admits a JO-factorization if it has the orientation property.

Example 4.2.1 Consider the groupoid $(X, \bullet) = (\{1, 2, 3, 4\}, \bullet)$ defined as in Example 4.1.4:

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

Through routine calculations, we find that $(X, \bullet)$ admits a JO-factorization since $J(X, \bullet) \diamond O(X, \bullet) = (X, \circ) \diamond (X, *) = (X, \bullet)$. In addition, $(X, \bullet) \in OP(X)$.

A groupoid $(X, \bullet)$ is said to be *j-normal* if it admits an OJ- and a JO-factorization, i.e., if

(i) $(X, \bullet) = O(X, \bullet) \diamond J(X, \bullet)$ and
(ii) $(X, \bullet) = J(X, \bullet) \diamond O(X, \bullet)$.

Theorem 4.2.3 *A groupoid $(X, \bullet)$ with the orientation property has a JO-factorization.*

Proof. Let $(X,\bullet) \in OP(X)$. Define $(X,\odot) = (X,*) \diamond (X,\circ)$ where $(X,*) = O(X,\bullet)$ and $(X,\circ) = J(X,\bullet)$. Then $x \odot y = (x * y) \circ (y * x)$ for all $x, y \in X$. It follows that

(i) If $x = y$, then $x * x = x$ and $x \circ x = x \bullet x$.
(ii) If $x \neq y$, the two cases arise: if $x * y \in \overline{d^*}$ and $x \circ y \in \overline{d^\circ}$, then $x * y \in \widetilde{D^\diamond}$ and $x \circ y \in \widetilde{d^\bullet}$ which also $\in \{x,\ y\}$. Otherwise, $x * y = x$,and $x \circ y = x \bullet y$.

Next, we show that $(X,\bullet) = (X,\odot)$. Given $x, y \in X$,

(i) If $x = y$, then $x \odot x = (x \circ x) * (x \circ x) = x * x = x \bullet x$.
(ii) If $x \neq y$, then $x \odot y = (x \circ y) * (y \circ x)$. If $x \circ y = y \circ x$, then $x \odot y = (x \circ y) * (x \circ y) = x \circ y = x \bullet y$. If $x \circ y \neq y \circ x$, then $x \odot y = (x * y) \bullet (y * x) \in \{x \bullet y,\ y \bullet x\}$.

Thus $x \odot y = x \bullet y$ for all $x, y \in X$. This proves that $(X,\odot) = (X,\bullet)$.

■

Corollary 4.2.4 *The factorization in Theorem 4.2.3 is unique.*

Proof. The proof is very similar to that of Corollary 4.1.3 so we omit it.

■

Proposition 4.2.5 *A groupoid with the orientation property is j-normal.*

Proof. The result follows from Theorems 4.1.2, 4.2.3 and the definition.

■

Example 4.2.6 Let $(X,\bullet)$ be defined as in Example 4.2.1 where we determined that $(X,\bullet)$ admits an OJ-factorization. It can be verified that $J(X,\bullet) \diamond O(X,\bullet) = (X,\bullet)$, which shows that $(X,\bullet)$ admits a JO-factorization as well. Therefore, $(X,\bullet)$ is j-normal in $(Bin(X), \diamond)$. Additionally, $J(X,\bullet) \neq id_{Bin(X)} \neq O(X,\bullet)$ implies that $(X,\bullet)$ is j-composite.

4.3. **Factoring $\mathbf{O}(X,\bullet)$ and $\mathbf{J}(X,\bullet)$.** In this subsection, the *orient*- and *skew*-factors of $(X,\bullet) \in OP(X)$ are factored to deduce that $O(X,\bullet)$ is *skew*-prime while $J(X,\bullet)$ is *binary*-equivalent to $(X,\bullet)$.

Let $(X,\bullet)$ and $(X,\circ)$ be groupoids in $Bin(X)$. We say that $(X,\circ)$ is *binary-equivalent* to $(X,\bullet)$ if there exists $(X,*) \in Bin(X)$ such that
(i) $(X,\bullet) = (X,*) \diamond (X,\circ)$; and
(ii) $(X,\circ) = (X,*) \diamond (X,\bullet)$.

Theorem 4.3.1 *Given a groupoid $(X,\bullet)$ with the orientation property. Its orient-factor is skew-prime, and its skew-factor is binary-equivalent to $(X,\bullet)$.*

Proof. Let $(X,\bullet) \in OP(X)$. Suppose that $(X,*) = O(X,\bullet)$ and $(X,\circ) = J(X,\bullet)$. Then by Theorem 4.1.2 $(X,\bullet) = O(X,\bullet) \diamond J(X,\bullet) = (X,*) \diamond (X,\circ)$. Let $(X,\circledast) = O(X,*)$ and $(X,\odot) = J(X,*)$, then for $\circledast$: (i) $\overline{d^\circledast} = \widetilde{D^\diamond}$, (ii) $x \circledast y = x$, otherwise; and for $\odot$: (i) $\overline{d^\odot} = \widetilde{d^*} = D^\diamond$, (ii) $x \odot y = x * y = x$, otherwise. Hence,

$$(X,*) = (X,*) \diamond id_{Bin(X)}$$

and $O(X,\bullet)$ is *skew*-prime. Similarly, if we let $(X,\boxtimes) = O(X,\circ)$ and $(X,\boxdot) = J(X,\circ)$, then for $\boxtimes$: (i) $\overline{d^\boxtimes} = \widetilde{D^\diamond}$, (ii) $x \boxtimes y = x$, otherwise; and for $\boxdot$: (i) $\overline{d^\boxdot} = \widetilde{d^\circ} = \overline{d^\bullet}$, (ii) $x \boxdot y = x \circ y = x \bullet y$, otherwise. Thus,

$$(X,\circ) = (X,*) \diamond (X,\bullet)$$

and the final result follows.

■

Example 4.3.2 Consider the locally-zero groupoid $(X, \bullet) = (\{0,1,2,3,4,5\}, \bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4	5
0	0	1	0	0	4	0
1	0	1	2	3	1	5
2	2	1	2	3	4	2
3	3	1	2	3	3	3
4	0	4	2	4	4	4
5	5	1	5	5	5	5

Since $(X, \bullet)$ has the orientation property, then $(X, \bullet)$ is j-normal by Proposition 4.2.5. Factoring its *orient*- and *skew*-factors $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$ into their respective *orient*- and *skew*-factors, $O(X, *)$, $J(X, *)$ and $O(X, \circ)$, $J(X, \circ)$, is observed through their respective product tables:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$=$

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$\diamond$

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5

$\circ$	0	1	2	3	4	5
0	0	1	0	0	4	5
1	0	1	2	3	4	5
2	2	1	2	2	4	2
3	3	1	3	3	3	3
4	0	1	2	4	4	4
5	0	1	5	5	5	5

$=$

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$\diamond$

$\bullet$	0	1	2	3	4	5
0	0	1	0	0	4	0
1	0	1	2	3	1	5
2	2	1	2	3	4	2
3	3	1	2	3	3	3
4	0	4	2	4	4	4
5	5	1	5	5	5	5

Indeed, $(X, *) = O(X, *) \diamond J(X, *) = (X, *) \diamond id_{Bin(X)}$ and $(X, \circ) = O(X, \circ) \diamond J(X, \circ) = (X, *) \diamond (X, \bullet)$. This clearly shows the results of Theorem 4.3.1.

Theorem 4.3.3 *The right-zero-semigroup on X is j-composite.*

Proof. Let $(X, \bullet)$ be the right-zero-semigroup on X. Suppose that $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$. By applying Proposition 4.2.5, $\underline{(X, \bullet)}$ is j-normal. Thus, $(X, \bullet) = (X, *) \diamond (X, \circ) = (X, \circ) \diamond (X, *)$. Consider $(X, *)$: (i) $\overline{d^*} = \widetilde{D^\diamond}$, (ii) $x * y = x$, otherwise; and for $(X, \circ)$: (i) $\overline{d^\circ} = \widetilde{d^\bullet}$, (ii) $x \circ y = x \bullet y = y$, otherwise. Since neither one of the factors is the left-zero-semigroup for $Bin(X)$, $(X, \bullet)$ is j-composite.

■

Example 4.3.4 Let $(X, \bullet)$ be the right-zero-semigroup as in Example 3.2.9 where $X = \{a, b, c\}$. Let $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$, we can check that $(X, \bullet)$ is in fact OJ- and JO-composite. Hence, $(X, \bullet)$ is j-composite:

$*$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

$\diamond$

$\circ$	a	b	c
a	a	b	a
b	a	b	c
c	c	b	c

$=$

$\bullet$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Moreover, its *orient*-factor $(X,*)$ has the following subtables:

$*$	a	b
a	a	a
b	b	b

$*$	a	c
a	a	c
c	a	c

$*$	b	c
b	b	b
c	c	c

which implies that $(X,*)$ is locally-zero.

Given two distinct groupoids $(X,\triangleright)$ and $(X,\triangleleft)$ in $Bin\,(X)$. Suppose that $(X,\triangleright)\neq id_{Bin(X)}$ and $(X,\triangleleft)\neq id_{Bin(X)}$. Let $(X,\bullet)$ be a groupoid such that $(X,\triangleright)\neq(X,\bullet)\neq(X,\triangleleft)$. Then $(X,\bullet)$ is said to be:

(i) *partially-right-prime,* ∂_r-prime, if $(X,\bullet)=(X,\bullet)\diamond(X,\triangleright)$;

(ii) *partially-left-prime,* ∂_l-prime, if $(X,\bullet)=(X,\triangleleft)\diamond(X,\bullet)$.

Whence $(X,\triangleright)$ and $(X,\triangleleft)$ behave like *right-* and *left*-identities respectively. Here, $(X,\triangleright)$ and $(X,\triangleleft)$ could be either $O\,(X,\bullet)$, $J\,(X,\bullet)$, $U\,(X,\bullet)$, $A\,(X,\bullet)$ or any other factor of $(X,\bullet)$. The next proposition demonstrates one such case.

Proposition 4.3.5 *A bi-diagonal groupoid is partially-left-prime.*

Proof. Given a bi-diagonal groupoid $(X,\bullet)$, then its *skew*-factor $J(X,\bullet)=(X,\bullet)$ since $\overline{d^\circ}=\widetilde{d^\bullet}=\overline{d^\bullet}$ and $x\circ y=x\bullet y$ otherwise. Meanwhile, its *orient*-factor $O(X,\bullet)$ is not affected by the bi-diagonal property. By Theorem 4.1.2, $(X,\bullet)$ has an OJ-factorization,

$$\begin{aligned}(X,\bullet)&=O(X,\bullet)\diamond J(X,\bullet)\\&=O(X,\bullet)\diamond(X,\bullet).\end{aligned}$$

Therefore, $O(X,\bullet)$ is a left-identity in $(Bin\,(X)\,,\diamond)$ and the result follows.

■

Example 4.3.6 Consider the group $(X,\bullet,e)$ as defined in Example 4.5. Then clearly $(X,\bullet,e)$ is bi-diagonal. Recall its *orient*-factor $(X,*,e)=O(X,\bullet,e)$ and derive its *skew*-factor $(X,\circ,e)=J(X,\bullet,e)$ to obtain:

$*$	e	a	b	c
e	e	e	e	c
a	a	a	b	a
b	b	a	b	b
c	e	c	c	c

and

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Then $(X,\bullet,e)=O(X,\bullet,e)\diamond(X,\bullet,e)$ and therefore the group $(X,\bullet,e)$ is ∂_l-prime.

5. Application

Recall some of the algebras described in "**Figure 1**" of Section 2.

We shall say an algebra $(X,\bullet,0)$ of type $(2,0)$ is a *strong* $B1$*-algebra* if it satisfies (B1) and equation 2.1. Meaning, if for all $x,y\in X$,

(i) $x\bullet x=0$,

(ii) $x\bullet y=y\bullet x$ implies $x=y$.

A groupoid $(X,\bullet,0)$ is *semi-neutral* if for all $x,y\in X$,

(i) $x\bullet x=0$,

(ii) $x\bullet y=x$.

A $B1$-algebra $(X,\bullet,0)$ is *semi-neutral* if for $x\neq y$, $x\bullet y=x$ for all $x,y\in X$.

A normal/composite groupoid is *semi-normal (resp., semi-composite)* if only one of its factors is semi-neutral.

Proposition 5.1 *A semi-neutral groupoid is signature-prime and OJ-composite.*

Proof. Let $(X, \bullet, 0)$ be the semi-neutral groupoid on X. Then $x \bullet y = x$ for all $x, y \in X$ and $x \bullet x = 0$. Let $(X, *, 0) = U(X, \bullet, 0)$ and $(X, \circ, 0) = A(X, \bullet, 0)$, its *signature*- and *similar*-factors, respectively. Deriving them according to 3.1 gives:

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y = x & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Hence for all $x, y \in X$, $(X, \bullet, 0) = id_{Bin(X)} \diamond (X, \bullet, 0)$.
By Theorem 4.1.2, $(X, \bullet, 0)$ has an OJ-factorization. Let $(X, \circledast, 0) = O(X, \bullet, 0)$ and $(X, \odot, 0) = J(X, \bullet, 0)$, its *orient*- and *skew*-factors, respectively. Deriving them according to 4.1 gives: for $\circledast$: (i) $\overline{d^{\circledast}} = \widetilde{D^{\diamond}}$, (ii) $x \circledast y = x$, otherwise; and for $\odot$: (i) $\overline{d^{\odot}} = \widetilde{d^{\bullet}} \neq D^{\diamond}$, (ii) $x \odot y = x \bullet y$, otherwise. Thus, $(X, *, 0) \neq id_{Bin(X)} \neq (X, \circ, 0)$ and $(X, *, 0) \neq (X, \bullet, 0) \neq (X, \circ, 0)$.

■

Corollary 5.2 *A semi-neutral groupoid is semi-normal.*

Proof. This is a direct result of Proposition 5.1 and the definition of a semi-normal groupoid.

■

Proposition 5.3 *The product of semi-neutral groupoids is semi-neutral.*

Proof. Consider semi-neutral groupoids $(X, *, 0)$ and $(X, \circ, 0)$. Let $(X, *, 0) \diamond (X, \circ, 0) = (X, \bullet, 0)$ such that $x \bullet y = (x * y) \circ (y * x)$. Then, $x \bullet x = (x * x) \circ (x * x) = 0$. If $x \neq y$,.$x \bullet y = x \circ y$. It follows that $(X, \bullet, 0) = (X, \circ, 0)$ and therefore is semi-neutral.

■

Proposition 5.4 *The similar-factor of a $B1$-algebra is semi-neutral.*

Proof. Let $(X, \bullet, 0)$ be a $B1$-algebra. Consider the AU-factorization $(X, \bullet, 0) = A(X, \bullet, 0) \diamond U(X, \bullet, 0)$. Let $(X, *, 0) := U(X, \bullet, 0)$ and $(X, \circ, 0) := A(X, \bullet, 0)$, its *signature*- and *similar*-factors, respectively. Deriving them according to 3.1 gives:

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $(X, \circ, 0)$ is semi-neutral.

■

Corollary 5.5 *A strong $B1$-algebra is semi-normal.*

Proof. This is a direct result of Corollary 3.2.5, Proposition 5.4 and the definition of a semi-normal algebra.

■

Corollary 5.6 *A strong $B1$-algebra $(X, \bullet, 0)$ is semi-composite if it is not semi-neutral, i.e., if $x \bullet y \neq x$* for all x, $y \in X$.

Proof. Let $(X, \bullet, 0)$ be a strong $B1$-algebra. Let $(X, *, 0) := U(X, \bullet, 0)$ and $(X, \circ, 0) := A(X, \bullet, 0)$, its *signature*- and *similar*-factors respectively. Deriving them according to 3.1. Assume that $x \bullet y = x$. Then $x * y = x$ for all $x, y \in X$. Thus, $(X, \bullet, 0) = id_{Bin(X)} \diamond (X, \bullet, 0)$ which makes it signature-prime and not u-composite.

■

Example 5.7 Let $(X, \bullet, 0) = (\{0, 1, 2\}, \bullet)$ be a strong BCK-algebra of order 3 where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Let $(\{0,1,2\},*) = U(\{0,1,2\},\bullet)$ and $(\{0,1,2\},\circ) = A(\{0,1,2\},\bullet)$. Its UA.-factorization is:

$*$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$\diamond$

$\circ$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

$=$

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Therefore, $(\{0,1,2\},\bullet)$ is *signature*-prime and u-normal. Moreover, $(\{0,1,2\},\bullet)$ as defined is semi-neutral. Next, *derive* its *orient*- and *skew*-factors $O(X,\bullet,0)$ and $J(X,\bullet,o)$, respectively. Let $(X,\circledast,0) = O(X,\bullet,0)$ and $(X,\odot,0) = J(X,\bullet,0)$. We have the following product:

$*$	0	1	2
0	0	0	2
1	1	1	1
2	0	2	2

$\diamond$

$\circ$	0	1	2
0	0	0	2
1	1	0	1
2	0	2	0

$=$

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Hence, $O(X,\bullet,0) \neq id_{Bin(X)} \neq J(X,\bullet,0)$ implies that $(X,\bullet,0)$ is OJ-composite.

Example 5.8 Let $(X,\bullet,0) = \{(0,1,2),\bullet\}$ be a strong Q-algebra of order 3 where "$\bullet$" is given by the following Cayley table:

$\bullet$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let $(\{0,1,2\},*) = U(\{0,1,2\},\bullet)$ and $(\{0,1,2\},\circ) = A(\{0,1,2\},\bullet)$. Its UA.-factorization is:

$*$	0	1	2
0	0	2	1
1	1	1	2
2	2	1	2

$\diamond$

$\circ$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

$=$

$\bullet$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Since $(\{0,1,2\},*,0) \neq Id_{Bin(X)}$ and $(\{0,1,2\},\circ,0)$ is semi-neutral, we can conclude that $(\{0,1,2\},\bullet,0)$ is semi-composite.

P.J. Allen, H.S. Kim and Neggers in [4] introduced the notion of Smarandache disjointness in algebras. Two groupoids (algebras) $(X,\bullet)$ and $(X,*)$ are said to be Smarandache disjoint if we add some axioms of an algebra $(X,\bullet)$ to an algebra $(X,*)$, then the algebra $(X,*)$ becomes a trivial algebra, i.e., $|X| = 1$.

Proposition 5.9 *The class of abelian groupoids and the class of u-normal groupoids are Smarandache disjoint.*

Proof. Let $(X,\bullet) \in Ab(X)$, the collection of all abelian groupoids defined on X. Suppose that $(X,\circ) = A(X,\bullet)$ and $(X,*) = U\,(X,\bullet)$. By Theorem 3.2.3, $(X\bullet)$ admits an *AU*-factorization. Consider $(X,*)\diamond(X,\circ)$, then for $x = y$,

$$\begin{aligned} x \diamond x &= (x * x) \circ (x * x) \\ &= x \circ x \\ &= x \bullet x. \end{aligned}$$

If $x \neq y$,

$$\begin{aligned} x \diamond y &= (x * y) \circ (y * x) \\ &= (x \bullet y) \circ (y \bullet x) \\ &= (x \bullet y) \bullet (x \bullet y) . \end{aligned}$$

Hence, $(X\bullet)$ admits a UA-factorization only if $(x \bullet y) \bullet (x \bullet y) = x \bullet y$. This means that $(X, \bullet)$ is u-normal if it is either the left- or right-zero-semigroup. Since both such groupoids are not abelian, then X must only have one element and the conclusion follows.

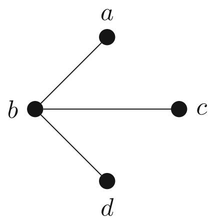
■

Suppose that in $Bin(X)$ we consider all those groupoids $(X, *)$ with the orientation property. Thus, $x * x = x$ as a consequence. If $(X, *)$ and $(X, \circ)$ both have the orientation property, then for $x \diamond\ y = (x * y) \circ (y * x)$ we have the possibilities: $x * x = x$, $y * y = y$, $x * y \in \{x, y\}$ and $y * x \in \{x, y\}$, so that $x \diamond y \in \{x, y\}$. It follows that if $OP(X)$ denotes this collection of groupoids, then $(OP(X), \diamond)$ is a subsemigroup [33] of $(Bin(X), \diamond)$.

In a sequence of papers Nebeský ([27], [28], [29]) associated with graphs (V, E) groupoids $(V, *)$ with various properties and conversely. He defined a *travel groupoid* $(X, *)$ as a groupoid satisfying the axioms: $(u * v) * u = u$ and $(u * v) * v = u$ implies $u = v$. If one adds these two laws to the orientation property, then $(X, *)$ is an OP-travel-groupoid. In this case $u * v = v$ implies $v * u = u$, i.e., $uv \in E$ implies $vu \in E$, i.e., the digraph (X, E) is a (simple) graph if $uu \notin E$, with $u * u = u$. Also, if $u \neq v$, then $u * v = u$ implies $(u * v) * v = u * v = u$ is impossible, whence $u * v = v$ and $uv \in E$, so that (X, E) is a complete (simple) graph.

In a recent paper, Ahn, Kim and Neggers [1] related graphs with binary systems in the center of $Bin\,(X)$. Given an element of $ZBin(X)$, say $(X, \bullet)$, they constructed a graph, Γ_X by letting $V(\Gamma_X) = X$ and $(x, y) \in E(\Gamma_X)$, the edge set of Γ_X, such that $x \neq y$, $y \bullet x = y$ and $x \bullet y = x$. Thus, if $(x, y) \in E(\Gamma_X)$, then $(y, x) \in E(\Gamma_X)$ as well and they identify $(x, y) = (y, x)$ as an undirected edge of Γ_X. Then they concluded that if $(X, \bullet)$ is the left-zero-semigroup, then Γ_X is the complete graph on X. Also, if $(X, \bullet)$ is the right-zero-semigroup, then Γ_X is the null graph on X, since $E(\Gamma_X) = \varnothing$.

Example 5.10 Let $X = \{a, b, c, d\}$ and consider the simple graph on X:



Then the associated groupoid table with binary operation "•" is:

•	a	b	c	d
a	a	a	c	d
b	b	b	b	b
c	a	c	c	d
d	a	d	c	d

By applying Proposition 4.2.5 to $(X, \bullet)$, we have the product of $O\,(X, \bullet)$ and $J\,(X, \bullet)$ given by their respective tables:

$*$	a	b	c	d
a	a	a	a	d
b	b	b	c	b
c	c	b	c	c
d	a	d	d	d

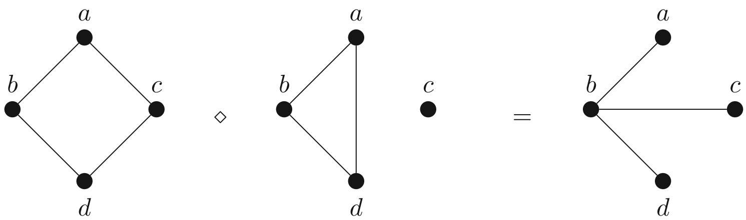
$\diamond$

$\circ$	a	b	c	d
a	a	a	c	a
b	b	b	c	b
c	a	b	c	d
d	d	d	c	d

$=$

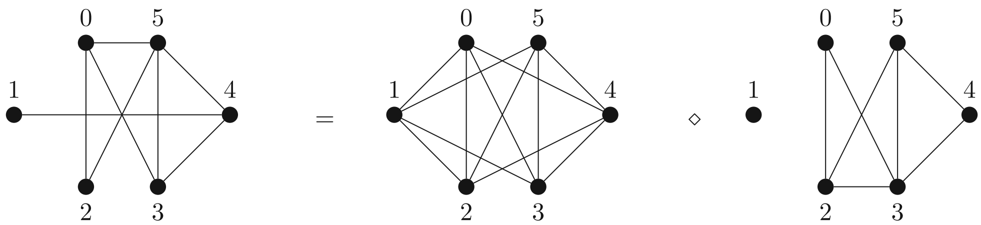
$\bullet$	a	b	c	d
a	a	a	c	d
b	b	b	b	b
c	a	c	c	d
d	a	d	c	d

We can visualize this product with the associated graphs of groupoids $(X, *)$ and $(X, \circ)$:



Thus, any simple graph constructed in this manner can be decomposed into two or more other factors with the binary product "$\diamond$". This fact is further illustrated in the next example.

Example 5.11 Let $(X, \bullet) = (\{0, 1, 2, 3, 4, 5\}, \bullet)$ be the locally-zero groupoid defined as in Example 4.3.2. Then its associated graph decomposes into its factors $(X, *)$ and $(X, \circ)$:



6. Generalization and Summary

In this final note, we discuss two generalizations which can serve as grounds for future exploration of groupoid factorizations or algebra decompositions via the groupoid product "$\diamond$".

6.1. ***Ψ-type-Factorization.*** Let Ψ be a groupoid operation that interchanges elements of any two given groupoids and produces two other (possibly identical) groupoids. Given groupoid $(X, \bullet) \in Bin(X)$ and the left-zero-semigroup as $id_{Bin(X)}$, define $\Psi : Bin(X) \times Bin(X) \to Bin(X) \times Bin(X)$. A Ψ-type-factorization of $(X, \bullet)$ gives a pair of groupoid factors as follows:

$$\Psi((X, \bullet), id_{Bin(X)}) = ((X, \bullet)_L, (X, \bullet)_R)$$

where $(X, \bullet)_L = \Psi_\alpha((X, \bullet), id_{Bin(X)})$ and $(X, \bullet)_R = \Psi_\alpha(id_{Bin(X)}, (X, \bullet))$, the *left-* and *right-Ψ-factors* of $(X, \bullet)$, respectively, such that the maps Ψ_α and α are defined as $\Psi_\alpha : Bin(X) \times Bin(X) \to Bin(X)$ and $\alpha : Bin(X) \to Bin(X)$.

Let $(X,*) := (X,\bullet)_L$ and $(X,\circ) := (X,\bullet)_R$, then $(X,\bullet)$ can be represented as a product of the groupoid pair, i.e.,

$$\begin{aligned}(X,\bullet) &= (X,*)\diamond(X,\circ) \text{ and/or}\\ (X,\bullet) &= (X,\circ)\diamond(X,*)\end{aligned}$$

thus rendering $(X,\bullet)$ as:

(i) Ψ-*prime*, if $(X,\bullet)_L = id_{Bin(X)}$ or $(X,\bullet)_R = id_{Bin(X)}$; or
(ii) Ψ-*normal* if $(X,*)\diamond(X,\circ) = (X,\circ)\diamond(X,*)$; or
(iii) Ψ-*composite* if $(X,\bullet)$ is Ψ-normal but not Ψ-prime.

An example of this Ψ-type-factorization is our first method of *similar-signature*-factorization where

$$\Psi_d((X,\bullet), id_{Bin(X)}) = \{(X,\bullet)\,|d(X,\bullet) = d\left(id_{Bin(X)}\right)\}$$

and

$$\Psi_d(id_{Bin(X)}, (X,\bullet)) = \{id_{Bin(X)}|d\left(id_{Bin(X)}\right) = d(X,\bullet)\}$$

The Ψ in that case switched the diagonal d of the parent groupoid $(X,\bullet)$ with that of the left-zero-semigroup, $id_{Bin(X)}$, to obtain the *signature*- and *similar*-factors $(X,\circ)$ and $(X,*)$, respectively. Hence, the *signature*- and *similar*-factors of a groupoid are Ψ*-type-factors.*

6.2. ***τ-type-Factorization.*** Let τ be a groupoid operation that manipulates elements of any given pair of groupoid in the same fashion. Given groupoid $(X,\bullet) \in Bin(X)$ and the left-zero-semigroup as $id_{Bin(X)}$, define $\tau : Bin(X)\times Bin(X) \to Bin(X)\times Bin(X)$. A *$\tau$-type-factorization* of $(X,\bullet)$ is given as follows:

$$\tau\left((X,\bullet), id_{Bin(X)}\right) = ((X,\bullet)_L, (X,\bullet)_R)$$

where $(X,\bullet)_L = \theta(id_{Bin(X)})$ and and $(X,\bullet)_R = \theta(X,\bullet)$ such that the map $\theta : Bin(X) \to Bin(X)$, the *left*- and *right-τ-factors* of $(X,\bullet)$, respectively. Let $(X,*) := (X,\bullet)_L$ and $(X,\circ) := (X,\bullet)_R$, then $(X,\bullet)$ could factor into a product of the groupoid pair, i.e.,

$$\begin{aligned}(X,\bullet) &= (X,*)\diamond(X,\circ) \text{ and/or}\\ (X,\bullet) &= (X,\circ)\diamond(X,*).\end{aligned}$$

Once again rendering $(X,\bullet)$ as:

(i) τ-*prime*, if $(X,\bullet)_L = id_{Bin(X)}$ or $(X,\bullet)_R = id_{Bin(X)}$; or
(ii) τ-*normal* if $(X,*)\diamond(X,\circ) = (X,\circ)\diamond(X,*)$; or
(iii) τ-*composite* if $(X,\bullet)$ is τ-normal but not τ-prime.

An example of this τ-type-factorization is our second method of *orient-skew*-factorization where $O(X,\bullet) := (X,\bullet)_L$ and $J(X,\bullet) := (X,\bullet)_R$. The τ (indeed, θ) in that scenario reversed the anti-diagonal of a given groupoid. Hence, applying τ to the left-zero-semigroup $id_{Bin(X)}$ and to the parent groupoid $(X,\bullet)$ results in the *orient*- and *skew*-factors $(X,*)$ and $(X,\circ)$, respectively. In conclusion, the *orient*- and *skew*-factors of a groupoid are *τ-type-factors.*

6.3. **Summary.** The goal of this paper was to gain more insight about the dynamics of binary systems, namely groupoids or algebras equipped with a single binary operation. We have shown that a strong groupoid can be represented as a "composite" groupoid of its *similar*- and *signature*- derived factors. Moreover, we concluded that an idempotent groupoid with the orientation property, can be decomposed into a product of its *orient*- and *skew*- factors. An application into the fields of logic-algebras and graph theory were briefly introduced. We found that a *semi*-neutral $B1$-algebra is *signature*-prime, OJ-composite and *semi*-normal. Meanwhile, a strong $B1$-algebra is then *semi*-composite if it is not *semi*-neutral. We finished our note with generalizations of our two methods in hopes that other factorizations can be discovered in the near future. It may be interesting to find other conditions for a groupoid to have such decompositions. As a final reminder, factorization can be useful in various applications such as algebraic cryptography and DNA code theory. We intend to extend our investigation in the future to hypergroupoid, semigroups as well as determine other factorizations and explore their applications.

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KURATSUBO PHENOMENON OF THE FOURIER SERIES OF SOME RADIAL FUNCTIONS IN FOUR DIMENSIONS

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Abstract. On the Fourier series the Gibbs-Wilbraham phenomenon is well known. In 1993, Pinsky, Stanton and Trapa discovered the so called Pinsky phenomenon on the spherical partial sum for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 3$. In 2010, Kuratsubo discovered the third phenomenon in dimension $d \geq 5$. Recently, Taylor found that the Pinsky phenomenon arises even in two dimension. In this paper we prove that the Kuratsubo phenomenon arises even in four dimension.

1 Introduction For the Fourier series of piecewise continuous functions, the Gibbs-Wilbraham phenomenon is well known. For example, let

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a, \end{cases} \quad x \in \mathbb{R}^d, \quad a > 0.$$

Let $d = 1$. Then the partial sums overshoot the jump at $x = \pm a$ by approx. 9% of the jump, while its partial sum $S_\lambda(\chi_a)(x)$ converges $\chi_a(x)$ as $\lambda \to \infty$ at $x \neq \pm a$. This phenomenon can

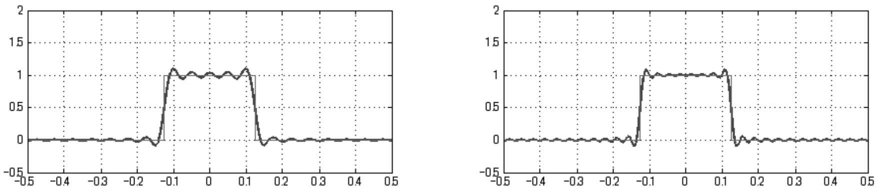

Figure 1: Gibbs-Wilbraham phenomenon $S_\lambda(\chi_{1/8})$ $(\lambda = 20, 30)$ [2]

be seen not only in one dimension but also in higher dimensions (see for example [1, 8, 12]).

In one dimension, it is also well known as the localization property that, if the function is zero on an interval, then the Fourier series converges to zero there. However, in higher dimensions this property is no longer valid. In 1993, Pinsky, Stanton and Trapa [10] showed that, for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 3$, the spherical partial sum diverges at the center of the ball. This phenomenon is called the Pinsky phenomenon.

In 1996 Kuratsubo [3] conjectured that, if $d \geq 5$, then the third phenomenon would arise, see also [4]. After the numerical calculation by [7] (2006) he proved that his conjecture is true in [5] (2010). Namely, for the Fourier series of the indicator function of a d-dimensional ball with $d \geq 5$, the spherical partial sum diverges at all rational points, while it converges almost everywhere, see Figures 3–5. Figure 4 is the expansion of Figure 3 to the direction of the vertical axis for the interval $[0.2, 0.5]$. Figure 5 is created using 3D graphics.

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Key words and phrases. multiple Fourier series, spherical partial sum, Pinsky phenomenon, Kuratsubo phenomenon, lattice point problem.

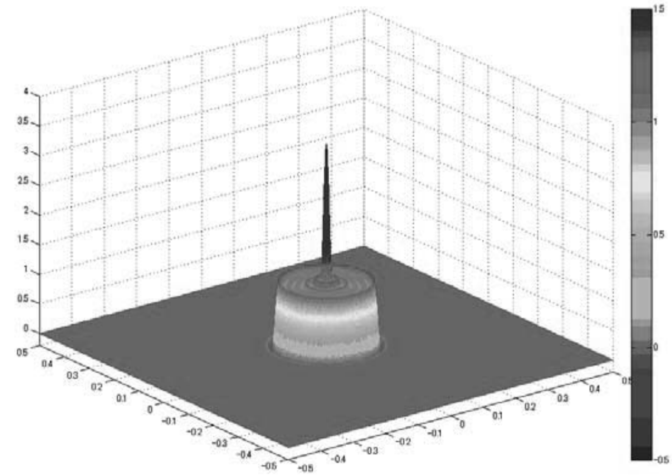

Figure 2: Pinsky phenomenon in 4 dim. $S_\lambda(\chi_{1/8})(x_1, x_2, 0, 0)$ $(\lambda = 47)$ [2]

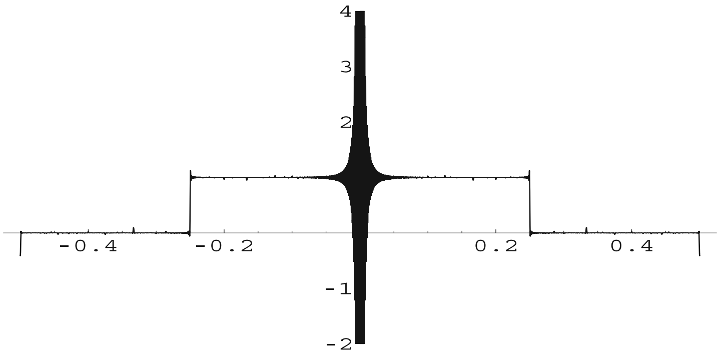

Figure 3: Kuratsubo phenomenon in 6 dim. $S_\lambda(\chi_{1/4})(x, 0, 0, 0, 0, 0)$ $(\lambda = 800)$ [7]

Recently, Taylor [13, 14] found that the Pinsky phenomenon arises even in two dimensions. He treated the radial function

$$U_a(x) = \begin{cases} 1/\sqrt{a^2 - |x|^2}, & |x| < a, \\ 0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^2, \ a > 0.$$

See Figure 6.

Our aim in this paper is to prove the Kuratsubo phenomenon in four dimensions. We consider the Fourier series of the function

$$(1.1) \qquad U_{\beta,a}(x) = \begin{cases} (a^2 - |x|^2)^\beta, & |x| < a \\ 0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^4, \ a > 0, \ \beta > -1.$$

If $\beta = 0$, then $U_{\beta,a}(x)$ is the same as the indicator function of the ball centered at the origin and of radius a. If $\beta = -1/2$, then $U_{\beta,a}(x)$ is the function considered by Taylor [13, 14]. We consider the case $-1 < \beta < -1/2$.

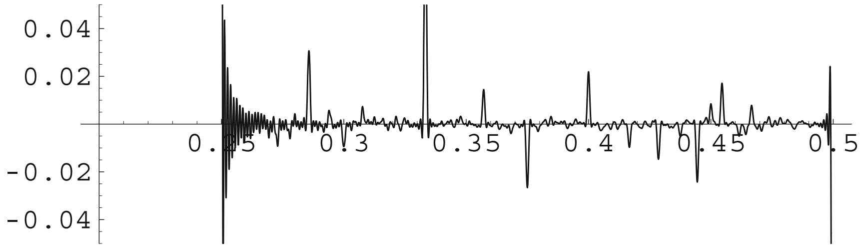

Figure 4: Kuratsubo phenomenon (expansion of Figure 3) [7]

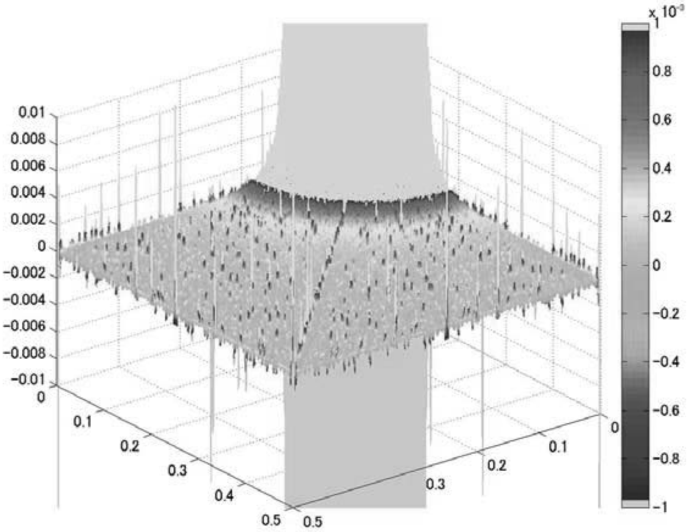

Figure 5: Kuratsubo phenomenon $S_\lambda(\chi_{1/8})(x_1, x_2, 0, 0, 0, 0)$ ($\lambda = 407$) [2]

In the next section we give the definitions of the Fourier spherical partial sum and the Fourier spherical partial integral and state some known results on them. Then we state our main result in Section 3 and prove it in Section 4.

At the end of this section we note the sources of the figures. Figures 1, 2 and 5 were made by MATLAB in [2]. Figures 3 and 4 were made by Mathematica in [7]. In this time we made Figure 6 by Mathematica and Figures 7 and 8 by gnuplot with Java.

2 Definitions and known results By $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ we denote the d-dimensional Euclidean space, integer lattice and torus, respectively. In this paper, however, we always identify $\mathbb{T}^d$ with $(-1/2, 1/2]^d$, that is, $x \in \mathbb{T}^d$ means $x \in (-1/2, 1/2]^d$ and $\mathbb{T}^d \subset \mathbb{R}^d$. Let $\mathbb{Q}$ be the set of all rational numbers, and let $\mathbb{Q}^d = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{Q}\}$.

For an integrable function $F(x)$ on $\mathbb{R}^d$, its Fourier transform $\hat{F}(\xi)$ and its Fourier spher-

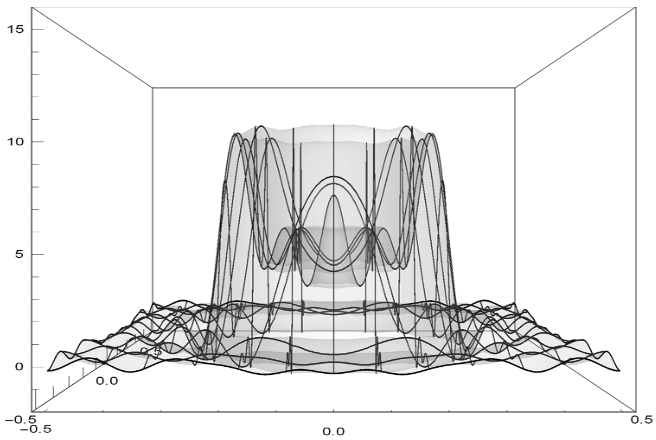

Figure 6: Pinsky phenomenon in 2 dim, $S_{10}(U_{1/4})$

ical partial integral $\sigma_\lambda(F)(x)$ of order $\lambda \geq 0$ are defined by

$$\hat{F}(\xi) = \int_{\mathbb{R}^d} F(x) e^{-2\pi i \xi x}\, dx, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{2.1}$$

$$\sigma_\lambda(F)(x) = \int_{|\xi|<\lambda} \hat{F}(\xi) e^{2\pi i \xi x} d\,\xi, \quad |\xi| = \sqrt{\sum_{k=1}^{d} {\xi_k}^2}, \ x \in \mathbb{R}^d, \tag{2.2}$$

respectively, where ξx is the inner product $\sum_{k=1}^{d} \xi_k x_k$. Also, for an integrable function $f(x)$ on $\mathbb{T}^d$, its Fourier coefficients $\hat{f}(m)$ and its Fourier spherical partial sum $S_\lambda(f)(x)$ of order $\lambda \geq 0$ are defined by

$$\hat{f}(m) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i m x}\, dx, \quad m = (m_1, \cdots, m_d) \in \mathbb{Z}^d, \tag{2.3}$$

$$S_\lambda(f)(x) = \sum_{|m|<\lambda} \hat{f}(m) e^{2\pi i m x}, \quad |m| = \sqrt{\sum_{k=1}^{d} {m_k}^2}, \ x \in \mathbb{T}^d, \tag{2.4}$$

respectively.

For an integrable function $F(x)$ on $\mathbb{R}^d$, we consider its periodization

$$f(x) = \sum_{m \in \mathbb{Z}^d} F(x+m), \quad x \in \mathbb{T}^d. \tag{2.5}$$

Note that in (2.5) the series converges with respect to the L^1-norm on $\mathbb{T}^d$ and then f is an integrable function on $\mathbb{T}^d$. Then it is known as the Poisson summation formula that the equation

$$\hat{f}(m) = \hat{F}(m), \quad m \in \mathbb{Z}^d \tag{2.6}$$

holds, see for example [11, Theorem 2.4 (page 251)]. The left hand side of (2.6) is defined by (2.3) and the right hand side of (2.6) is defined by (2.1) with $\xi = m$.

In particular, we denote by $u_{\beta,a}(x)$ the periodization of $U_{\beta,a}(x)$. That is,

$$u_{\beta,a}(x) = \sum_{m \in \mathbb{Z}^d} U_{\beta,a}(x+m), \quad x \in \mathbb{T}^d. \tag{2.7}$$

In this paper we always assume that $0 < a < 1/2$. Then

$$u_{\beta,a}(x) = U_{\beta,a}(x), \quad x \in \mathbb{T}^d. \tag{2.8}$$

The behavior of $\sigma_\lambda(U_{\beta,a})(x)$ as $\lambda \to \infty$ is known by [6]. Let Γ be the Gamma function and J_ν the Bessel function of order ν. Then the following theorem is known:

Theorem 2.1 ([6, Theorem 4.1]). *Let $d \geq 1$, $a > 0$ and $\beta > -1$. Then*

$$\sigma_\lambda(U_{\beta,a})(x) = 2^\beta \Gamma(\beta+1) a^{2\beta} \int_0^{2\pi a\lambda} \frac{J_{\frac{d}{2}-1}(\frac{|x|}{a}s) J_{\frac{d}{2}+\beta}(s)}{\left(\frac{|x|}{a}\right)^{\frac{d}{2}-1} s^\beta}\, ds, \tag{2.9}$$

for all $x \in \mathbb{R}^d$ and $\lambda > 0$. Moreover, $\sigma_\lambda(U_{\beta,a})$ has the following properties:

1. *At $x = 0$,*
   (a) *if $\beta > (d-3)/2$, then $\sigma_\lambda(U_{\beta,a})(0)$ converges to $U_{\beta,a}(0)$ as $\lambda \to \infty$,*
   (b) *if $-1 < \beta \leq (d-3)/2$, then $\sigma_\lambda(U_{\beta,a})$ reveals the Pinsky phenomenon.*
2. *For near $|x| = a$,*
   (a) *if $\beta > 0$, then $\sigma_\lambda(U_{\beta,a})(x)$ converges to $U_{\beta,a}(x)$ as $\lambda \to \infty$,*
   (b) *$-1 < \beta \leq 0$, then $\sigma_\lambda(U_{\beta,a})$ reveals the Gibbs-Wilbraham phenomenon.*
3. *If $x \neq 0$ and $|x| \neq a$, then $\sigma_\lambda(U_{\beta,a})(x)$ converges to $U_{\beta,a}(x)$ as $\lambda \to \infty$ and the convergence is uniform on any compact subset of $\mathbb{R}^d \setminus \{x \neq 0, |x| \neq a\}$.*

The difference between $\sigma_\lambda(U_{\beta,a})$ and $\sigma_\lambda(u_{\beta,a})$ is also known by [6]. For $j = 0, 1, 2, \ldots$, let

$$D_j(s:x) = \frac{1}{\Gamma(j+1)} \sum_{|m|^2<s} (s - |m|^2)^j e^{2\pi i m x}, \quad s > 0, \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_j(s:x) = \frac{1}{\Gamma(j+1)} \int_{|\xi|^2<s} (s - |\xi|^2)^j e^{2\pi i x\xi}\, d\xi, \quad s > 0, \quad x \in \mathbb{R}^d,$$

and

$$\Delta_j(s:x) = D_j(s:x) - \mathcal{D}_i(s:x), \quad s > 0,\ x \in \mathbb{R}^d. \tag{2.10}$$

Further, for $j = 0, 1, 2, \ldots$, $\beta > -1$ and $a > 0$, let

$$A^{(j)}_{\beta,a}(s) = (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j}} a^{\frac{d}{2}+\beta+j} \frac{J_{\frac{d}{2}+\beta+j}(2\pi a\sqrt{s})}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j)}}, \quad s > 0, \tag{2.11}$$

and let

$$\mathcal{K}_{\beta,a}(s:x) = \sum_{j=0}^{d_\sharp} (-1)^j \Delta_j(s:x) A^{(j)}_{\beta,a}(s), \tag{2.12}$$

where $d_\sharp$ is the integer part of $(d+1)/2$. Then the following theorem is known:

Theorem 2.2 ([6, Corollary 6.2]). *Let $d \geq 1, \beta > -1$ and $0 < a < 1/2$. Then*

$$S_\lambda(u_{\beta,a})(x) = \sigma_\lambda(U_{\beta,a})(x) + \mathcal{K}_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta-1}) \quad as \quad \lambda \to \infty \tag{2.13}$$

for all $x \in \mathbb{T}^d$.

In the above O is Landau's symbol, that is, $f(s) = O(g(s))$ as $s \to \infty$ means that $\limsup_{s\to\infty} |f(s)|/g(s) < \infty$ for the positive valued function g. Similarly, $f(s) = o(g(s))$ as $s \to \infty$ means that $\lim_{s\to\infty} f(s)/g(s) = 0$.

Therefore, to investigate the behavior of $S_\lambda(u_{\beta,a})(x)$ as $\lambda \to \infty$ we need to estimate $\mathcal{K}_{\beta,a}(\lambda^2 : x)$.

3 Main result

Recall that

$$u_{\beta,a}(x) = U_{\beta,a}(x), \quad x \in \mathbb{T}^d, \tag{3.1}$$

under the assumption $0 < a < 1/2$. Let

$$E_a = \{x \in \mathbb{T}^d : x \neq 0, |x| \neq a\}.$$

Our main result is the following:

Theorem 3.1. *Let $d = 4$ and $0 < a < 1/2$. Fix a point $x \in E_a \cap \mathbb{Q}^4$ arbitrarily. If $S_\lambda(u_{\beta,a})(x)$ converges $u_{\beta,a}(x)$ as $\lambda \to \infty$ for some $\beta \in (-1, -1/2)$, then $S_\lambda(u_{\beta,a})(x)$ diverges for all other $\beta \in (-1, -1/2)$.*

This theorem shows that the Kuratsubo phenomenon arises even if $d = 4$. On the other hand, it is known by [6, Theorem 1.3] that, if $d = 4$ and $\beta > -1/10$, then $S_\lambda(u_{\beta,a})(x)$ converges to $u_{\beta,a}(x)$ as $\lambda \to \infty$ for all $x \in E_a$. Therefore, the case of $\beta \in [-1/2, -1/10]$ is an open problem. Note also that, if $\beta > -1/2$, then $S_\lambda(u_{\beta,a})(x)$ converges to $u_{\beta,a}(x)$ as $\lambda \to \infty$ a.e. $x \in \mathbb{T}^4$, see [6, Theorem 1.5].

Figures 7 and 8 are graphs of $S_\lambda(u_{\beta,a})(x, 0, 0, 0)$ for $\beta = -9/10$ and $a = 1/8$ in four dimensions. We can observe the Kuratsubo phenomenon in Figure 8.

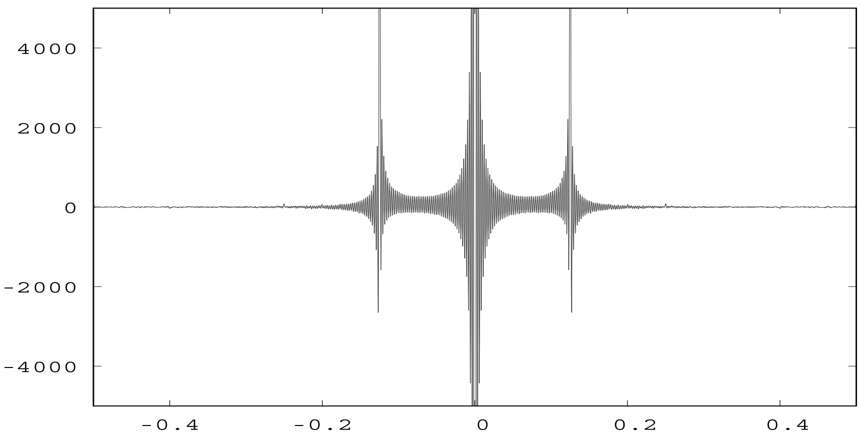

Figure 7: Kuratsubo phenomenon in 4 dim. $S_\lambda(u_{\beta,a})(x, 0, 0, 0)$ ($\lambda = 400$)

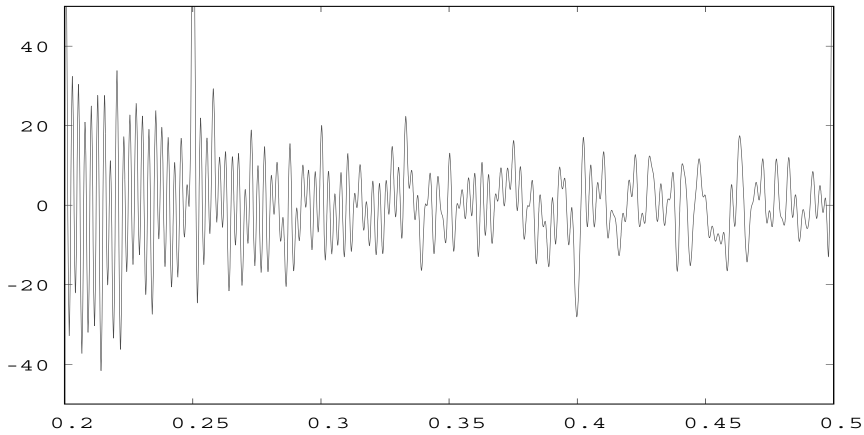

Figure 8: Kuratsubo phenomenon in 4 dim. (expansion of Figure 7)

4 Proof Let $x \in E_a$. By Theorems 2.1, 2.2 and (3.1), we see that $S_\lambda(u_{\beta,a})(x) \to u_{\beta,a}(x)$ if and only if $\mathcal{K}_{\beta,a}(\lambda^2 : x) \to 0$. To estimate

$$\mathcal{K}_{\beta,a}(\lambda^2 : x) = \sum_{j=0}^{d_\sharp} (-1)^j \Delta_j(\lambda^2 : x) A_{\beta,a}^{(j)}(\lambda^2),$$

we combine the estimates of $\Delta_j(\lambda^2 : x)$ and $A_{\beta,a}^{(j)}(\lambda^2)$.

Firstly, by the asymptotic behavior of Bessel functions

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{2\nu+1}{4}\pi\right) + O(s^{-3/2}) \quad \text{as} \quad s \to \infty, \tag{4.1}$$

we see that

$$\begin{aligned} A_{\beta,a}^{(j)}(s) &= (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j}} a^{\frac{d}{2}+\beta+j} \frac{J_{\frac{d}{2}+\beta+j}(2\pi a\sqrt{s})}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j)}} \\ &= (-1)^j \frac{\Gamma(\beta+1)}{\pi^{\beta-j+1}} \frac{a^{\frac{d}{2}+\beta+j-\frac{1}{2}}}{s^{\frac{1}{2}(\frac{d}{2}+\beta+j+\frac{1}{2})}} \cos\left(2\pi a\sqrt{s} - \frac{d+2\beta+2j+1}{4}\pi\right) \\ &\qquad + O(s^{-\frac{1}{2}(\frac{d}{2}+\beta+j+\frac{3}{2})}) \quad \text{as } s \to \infty, \end{aligned} \tag{4.2}$$

which shows

$$A_{\beta,a}^{(j)}(\lambda^2) = O(\lambda^{-(\frac{d}{2}+\beta+j+\frac{1}{2})}) \quad \text{as} \quad \lambda \to \infty. \tag{4.3}$$

In the above, for the asymptotic behavior (4.1) of Bessel functions, see [11, Lemma 3.11 on page 158] for example.

For the terms $\Delta_j(s : x)$, we use known results related to the lattice point problem.

Lemma 4.1 ([6, Lemma 5.1])**.** *Let $d \geq 1$. Then, as $s \to \infty$,*

$$\Delta_\alpha(s : x) = \begin{cases} O(s^{\frac{d}{2}-\frac{d}{d+1}}), & \text{if } \alpha = 0, \\ O(s^{\frac{d}{2}-\frac{d}{d+1}+\frac{\alpha}{d+1}+\varepsilon}) \text{ for every } \varepsilon > 0, & \text{if } 0 < \alpha \leq \frac{d-1}{2}, \\ O(s^{\frac{d-1}{4}+\frac{\alpha}{2}}), & \text{if } \alpha > \frac{d-1}{2}, \end{cases} \tag{4.4}$$

uniformly with respected to $x \in \mathbb{T}^d$.

For $\alpha \geq 0$, let

$$P_\alpha(s:x) = D_\alpha(s:x) - \frac{\pi^{\frac{d}{2}} s^{\frac{d}{2}+\alpha}}{\Gamma(\frac{d}{2}+\alpha+1)} \delta(x), \quad x \in \mathbb{R}^n, \ s \geq 0, \tag{4.5}$$

where $\delta(x)$ is the indicator function of $\mathbb{Z}^d$.

Theorem 4.2 (Novák [9]). *Let $d \geq 3$. Then, for all $x \in \mathbb{Q}^d$, there exists a positive constant $K_d(x)$ such that*

$$\int_0^s |P_0(t:x)|^2 \, dt = \begin{cases} K_d(x) s^2 \log s + O(s^2 \log^{1/2} s), & \textit{if } d = 3, \\ K_d(x) s^3 + O(s^{5/2} \log s), & \textit{if } d = 4, \\ K_d(x) s^4 + O(s^3 \log^2 s), & \textit{if } d = 5, \\ K_d(x) s^{d-1} + O(s^{d-2}), & \textit{if } d \geq 6. \end{cases} \tag{4.6}$$

Remark 4.1. In Theorem 4.2 the positive constant $K_d(x)$ is given explicitly for each $x \in \mathbb{Q}^d$, see [5].

Remark 4.2. Theorem 4.2 valids for $\Delta_\alpha(s:x)$ instead of $P_\alpha(s:x)$, if $x \in \mathbb{T}^d \cap \mathbb{Q}^d$. Actually,

$$\Delta_\alpha(s:x) - P_\alpha(s:x) = \begin{cases} 0, & \text{if } x = 0, \\ -\mathcal{D}_\alpha(s:x) = O(s^{\frac{d-1}{4}+\frac{\alpha}{2}}), & \text{if } x \in \mathbb{T}^d \setminus \{0\}, \end{cases}$$

see [6, Remark 5.2].

Lemma 4.3. *Let $d = 4$. Then, for all $x \in \mathbb{Q}^d$ and all $\mu > 0$, there exists a positive constant $K(x)$ and a sequence $\{\lambda_k\}_k$ such that $\lambda_k \to \infty$ as $k \to \infty$ and*

$$|\Delta_0({\lambda_k}^2, x)| \geq K(x) {\lambda_k}^{2-\mu}. \tag{4.7}$$

Proof. By Theorem 4.2 and Remark 4.2 we have

$$\int_0^s |\Delta_0(t,x)|^2 \, dt = K_4(x) s^3 + O(s^{5/2} \log s) \quad \text{as} \quad s \to \infty. \tag{4.8}$$

We may assume that $0 < \mu < 1$. If there exists a positive constant t_0 such that, for all $t > t_0$,

$$|\Delta_0(t,x)| \leq K_4(x) t^{1-\mu/2}, \quad \text{i.e.} \quad |\Delta_0(t^2,x)| \leq K_4(x) t^{2-\mu},$$

then

$$\int_0^s |\Delta_0(t,x)|^2 \, dt \leq {K_4}^2(x) s^{3-\mu} \quad \text{for large } s,$$

which contradicts (4.8). □

Proof of Theorem 3.1. Let $d = 4$ and $0 < a < 1/2$. Then $d_\sharp = 2$ and

$$\mathcal{K}_{\beta,a}(\lambda^2 : x) = \sum_{j=0}^{2} (-1)^j \Delta_j(\lambda^2 : x) A^{(j)}_{\beta,a}(\lambda^2).$$

By Lemma 4.1 and (4.3) we have

$$\begin{cases} \Delta_0(\lambda^2, x) = O(\lambda^{12/5}), \\ \Delta_1(\lambda^2, x) = O(\lambda^{14/5+\epsilon}), \\ \Delta_2(\lambda^2, x) = O(\lambda^{7/2}), \end{cases} \qquad \begin{cases} A^{(0)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-5/2}), \\ A^{(1)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-7/2}), \\ A^{(2)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-9/2}), \end{cases} \quad \text{as} \quad \lambda \to \infty,$$

which imply

$$\text{(4.9)} \qquad \begin{cases} \Delta_0(\lambda^2 : x)A^{(0)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-1/10}), \\ \Delta_1(\lambda^2 : x)A^{(1)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-7/10+\varepsilon}), \\ \Delta_2(\lambda^2 : x)A^{(2)}_{\beta,a}(\lambda^2) = O(\lambda^{-\beta-1}), \end{cases} \quad \text{as} \quad \lambda \to \infty.$$

It follows that, if $\beta > -1/10$, then $\mathcal{K}_{\beta,a}(\lambda^2 : x) \to 0$ as $\lambda \to \infty$, that is, $S_\lambda(u_{\beta,a})(x) \to u_{\beta,a}(x)$ for all $x \in E_a$, which is a known result as mentioned after Theorem 3.1.

We shall consider the main term $\Delta_0(\lambda^2 : x)A^{(0)}_{\beta,a}(\lambda^2)$ more precisely. From (4.2) it follows that

$$\text{(4.10)} \qquad A^{(0)}_{\beta,a}(\lambda^2) = C_0\lambda^{-\beta-5/2}\cos\left(2\pi a\lambda - \frac{2\beta+5}{4}\pi\right) + O(\lambda^{-\beta-7/2}),$$

where $C_0 = \Gamma(\beta+1)a^{3/2+\beta}/\pi^{\beta+1}$. Let $x \in E_a \cap \mathbb{Q}^4$. For any small $\mu > 0$, take $\{\lambda_k\}_k$ as in Lemma 4.3. If there exists $\beta_0 \in (-1, -1/2-\mu)$ such that $S_\lambda(u_{\beta_0,a})(x) \to u_{\beta_0,a}(x)$ as $\lambda \to \infty$, then $\mathcal{K}_{\beta_0,a}(\lambda^2 : x) \to 0$ as $\lambda \to \infty$, which implies

$$\text{(4.11)} \qquad \lim_{k\to\infty} \cos\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right) = 0.$$

Actually, if

$$\limsup_{k\to\infty}\left|\cos\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right)\right| = 2\delta > 0,$$

then by (4.7) and (4.10) we have

$$|\Delta_0({\lambda_k}^2 : x)A^{(0)}_{\beta_0,a}({\lambda_k}^2)| \geq C_0K(x){\lambda_k}^{-\beta_0-1/2-\mu}\delta,$$

for infinitely many k, which means that $\mathcal{K}_{\beta_0,a}({\lambda_k}^2 : x)$ diverges, since the other terms are smaller, see (4.9).

Now, (4.11) is equivalent to

$$\lim_{k\to\infty}\left(2\pi a\lambda_k - \frac{2\beta_0+5}{4}\pi\right) = \frac{\pi}{2} \mod. \pi.$$

In this case, for all $\beta \in (-1, -1/2-\mu) \setminus \{\beta_0\}$,

$$\lim_{k\to\infty}\left(2\pi a\lambda_k - \frac{2\beta+5}{4}\pi\right) = \frac{\pi}{2} - \frac{(\beta-\beta_0)\pi}{2} \mod. \pi,$$

which shows

$$\lim_{k\to\infty}\left|\cos\left(2\pi a\lambda_k - \frac{2\beta+5}{4}\pi\right)\right| > 0.$$

This means that $\mathcal{K}_{\beta,a}({\lambda_k}^2 : x)$ diverges as seen before. Since $\mu > 0$ is arbitrary, we have the desired conclusion. □

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KURATSUBO PHENOMENON OF THE FOURIER SERIES OF SOME RADIAL FUNCTIONS IN FOUR DIMENSIONS

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FUZZY SCHWARZ INEQUALITY

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Abstract. In the present paper, the fuzzy Schwarz inequality in inner product spaces is derived. It is an extension of the Schwarz inequality, and is described by using a fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors. On the other hand, the fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

1 Introduction The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [18]. Then, fuzzy set theory has been applied in various areas such as economics, management science, engineering, optimization theory, operations research, etc. [6, 10, 14, 15, 16, 17]. Zadeh's extension principle [4, 18] provides a natural way for extending the domain of a mapping. It is an important tool in the development of fuzzy arithmetic and other areas. Let $f : X \times Y \to Z$ be a mapping, and let $\widetilde{a}$ and $\widetilde{b}$ be fuzzy sets on X and Y, respectively. In addition, let $f(\widetilde{a}, \widetilde{b})$ be the fuzzy set on Z obtained from $\widetilde{a}$ and $\widetilde{b}$ by Zadeh's extension principle. In [12], relationships between $f([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha)$ and $[f(\widetilde{a}, \widetilde{b})]_\alpha$ are investigated, where $[\widetilde{a}]_\alpha$, $[\widetilde{b}]_\alpha$, and $[f(\widetilde{a}, \widetilde{b})]_\alpha$ are the α-level sets of $\widetilde{a}$, $\widetilde{b}$, and $f(\widetilde{a}, \widetilde{b})$, respectively. A fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle are proposed, and their properties are investigated in [9] and [8], respectively. We adopt them. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set.

Fuzzy normed spaces and fuzzy inner product spaces have been discussed in several papers; see, for example [13] and references therein. Fuzzy norms and fuzzy inner products in most of papers are based on axioms rather than Zadeh's extension principle, and their values are fuzzy sets for norms and inner products of crisp vectors rather than of fuzzy sets. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors, and it has a long history; see, for example [3]. We consider the Schwarz inequality in fuzzy settings by using our adopted fuzzy norm and fuzzy inner product. The Schwarz inequality is derived for fuzzy matrices by using a fuzzy norm and a fuzzy inner product based on axioms rather than Zadeh's extension principle in [5], and the Schwarz inequality is derived for fuzzy integrals in [2]. Our settings such as the fuzzy norm and the fuzzy inner product are different from the previous works on the Schwarz inequality in fuzzy settings.

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In the present paper, the fuzzy Schwarz inequality in inner product spaces is derived. It is an extension of the Schwarz inequality, and is described by using the fuzzy norm and the fuzzy inner product. The fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

The remainder of the present paper is organized as follows. In Section 2, some notations are presented. In Section 3, we investigate relationships between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle, and the fuzzy norm and the fuzzy inner product defined by Zadeh's extension principle are presented. In Section 4, the fuzzy Schwarz inequality is derived by using the fuzzy norm and the fuzzy inner product as an extension of the Schwarz inequality in inner product spaces. Finally, conclusions are presented in Section 5.

2 Preliminaries In this section, some notations are presented.

Let $\mathbb{R}$ and $\mathbb{C}$ be the set of all real numbers and the set of all complex numbers, respectively. We set $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$. For $A \subset \mathbb{R}$, we denote the interior of A by $\mathrm{int}(A)$. For $a, b \in \mathbb{R}$, we set $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a,b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a,b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a,b[= \{x \in \mathbb{R} : a < x < b\}$.

Let X be a set. Then, $\widetilde{a} : X \to [0,1]$ is called a fuzzy set on X, and let $\mathcal{F}(X)$ be the set of all fuzzy sets on X. For $\widetilde{a} \in \mathcal{F}(X)$ and $\alpha \in]0,1]$, the *α-level set* of $\widetilde{a}$ is defined as

$$[\widetilde{a}]_\alpha = \{x \in X : \widetilde{a}(x) \geq \alpha\}. \tag{1}$$

For a crisp set $S \subset X$, the indicator function of S is a function $c_S : X \to \{0, 1\}$ defined as $c_S(x) = 1$ if $x \in S$, and $c_S(x) = 0$ if $x \notin S$ for each $x \in X$. A fuzzy set $\widetilde{a} \in \mathcal{F}(X)$ can be represented as

$$\widetilde{a} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{a}]_\alpha} \tag{2}$$

which is well-known as the *decomposition theorem* or the *representation theorem*; see, for example [4].

We consider fuzzy sets on a topological space. Let $(X, \mathbb{T})$ be a topological space. Let $\mathcal{C}(X)$ and $\mathcal{K}(X)$ be the set of all closed subsets of X and the set of all compact subsets of X, respectively. Let $\widetilde{a} \in \mathcal{F}(X)$. The fuzzy set $\widetilde{a}$ is called a *closed fuzzy set* (on X) if $[\widetilde{a}]_\alpha \in \mathcal{C}(X)$ for any $\alpha \in]0,1]$. The fuzzy set $\widetilde{a}$ is a closed fuzzy set on X if and only if $\widetilde{a}$ is an upper semicontinuous function on X. The fuzzy set $\widetilde{a}$ is called a *compact fuzzy set* (on X) if $[\widetilde{a}]_\alpha \in \mathcal{K}(X)$ for any $\alpha \in]0,1]$. Let $\mathcal{FC}(X)$ and $\mathcal{FK}(X)$ be the set of all closed fuzzy sets on X and the set of all compact fuzzy sets on X, respectively.

In $\mathbb{R}$, we define an order relation for crisp sets, and then define an order relation for fuzzy sets by using the order relation for crisp sets. Let $A, B \subset \mathbb{R}$. We write $A \leq B$ if $B \subset A + \mathbb{R}_+$ and $A \subset B + \mathbb{R}_-$, and write $A < B$ if $B \subset A + \mathrm{int}(\mathbb{R}_+)$ and $A \subset B + \mathrm{int}(\mathbb{R}_-)$. Then, $\leq$ is a pseudo order on $2^{\mathbb{R}}$. $B \subset A + \mathbb{R}_+$ if and only if for any $b \in B$, there exists $a \in A$ such that $a \leq b$. $A \subset B + \mathbb{R}_-$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \leq b$. $B \subset A + \mathrm{int}(\mathbb{R}_+)$ if and only if for any $b \in B$, there exists $a \in A$ such that $a < b$. $A \subset B + \mathrm{int}(\mathbb{R}_-)$ if and only if for any $a \in A$, there exists $b \in B$ such that $a < b$. Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R})$. We write $\widetilde{a} \preceq \widetilde{b}$ if $[\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha$ for any $\alpha \in]0,1]$, and write $\widetilde{a} \prec \widetilde{b}$ if $[\widetilde{a}]_\alpha < [\widetilde{b}]_\alpha$ for any $\alpha \in]0,1]$. Then, $\preceq$ and $\prec$ are called the *fuzzy max order* and the *strict fuzzy max order*, respectively, and $\preceq$ is a pseudo order on $\mathcal{F}(\mathbb{R})$; see [7, 11].

3 Images of fuzzy sets by Zadeh's extension principle In this section, we investigate relationships between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle, and the fuzzy norm and the fuzzy inner product defined by Zadeh's extension principle are presented.

Definition 1. Let X_i, $i = 1, 2, \cdots, n$ be sets, and let $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$. Then, $\prod_{i=1}^n \widetilde{a}_i \in \mathcal{F}\left(\prod_{i=1}^n X_i\right)$ is defined as

$$\left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) = \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i)$$

for each $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n X_i$, and is called the *fuzzy product set* of $\widetilde{a}_i$, $i = 1, 2, \cdots, n$. The fuzzy product set $\prod_{i=1}^n \widetilde{a}_i$ is also represented as $\widetilde{a}_1 \times \widetilde{a}_2 \times \cdots \times \widetilde{a}_n$ or $(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

The following definition provides images of fuzzy sets under a crisp mapping by Zadeh's extension principle; see [4, 18] for Zadeh's extension principle.

Definitin 2. Let X_i, $i = 1, 2, \cdots, n$ and Y be sets, and let $f : \prod_{i=1}^n X_i \to Y$. Then, for $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$, $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}(Y)$ is defined as

$$f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)(y) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i)$$

for each $y \in Y$, where $\sup \emptyset = 0$.

Let X be a real or complex normed space equipped with a norm $\|\cdot\|$, and set $f : X \to \mathbb{R}$ as $f(x) = \|x\|$ for each $x \in X$. For $\widetilde{a} \in \mathcal{F}(X)$, it follows that

$$f(\widetilde{a})(y) = \|\widetilde{a}\|(y) = \sup_{x\in f^{-1}(y)} \widetilde{a}(x), \quad y \in \mathbb{R} \tag{3}$$

from Definition 2. Then, $\|\widetilde{a}\| \in \mathcal{F}(\mathbb{R})$ is called the *fuzzy norm* of $\widetilde{a}$, and some properties of fuzzy norms are investigated in [9].

Let X be a real or complex inner product space equipped with an inner product $\langle\cdot,\cdot\rangle$, and set $f : X \times X \to \mathbb{K}$ as $f(x, y) = \langle x, y\rangle$ for each $x, y \in X$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(X)$, it follows that

$$f(\widetilde{a}, \widetilde{b})(z) = \langle\widetilde{a}, \widetilde{b}\rangle(z) = \sup_{(x,y)\in f^{-1}(z)} \min\{\widetilde{a}(x), \widetilde{b}(y)\}, \quad z \in \mathbb{K} \tag{4}$$

from Definition 2. Then, $\langle\widetilde{a}, \widetilde{b}\rangle \in \mathcal{F}(\mathbb{K})$ is called the *fuzzy inner product* between $\widetilde{a}$ and $\widetilde{b}$, and some properties of fuzzy inner products are investigated in [8].

The following theorem provides a relationship between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle.

Theorem 1. *[12] Let X_i, $i = 1, 2, \cdots, n$ and Y be sets, and let $f : \prod_{i=1}^m X_i \to Y$. In addition, let $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$. Then,*

$$[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) \tag{5}$$

for any $\alpha \in]0, 1]$ if and only if $y \in Y$ and $f^{-1}(y) \neq \emptyset$ imply the existence of $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y)$ such that

$$\min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i^*) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \min_{i=1,2,\cdots,m} \widetilde{a}_i(x_i).$$

The following theorem gives sufficient conditions for (5) in Theorem 1 to hold.

Theorem 2. *Let $(X_i, \mathbb{T}_i)$, $i = 1, 2, \cdots, n$ be Hausdorff spaces, and let Y be a T_1-space. Assume that $f : \prod_{i=1}^n X_i \to Y$ is continuous. In addition, let $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. Then,*

$$[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$$

for any $\alpha \in]0, 1]$.

Proof. Fix any $y \in Y$, and suppose that $f^{-1}(y) \neq \emptyset$. Then, it is sufficient to show the existence of $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y)$ such that

$$\min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i^*) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n)$$

from Theorem 1. For any $\alpha \in]0, 1]$, since $[\widetilde{a}_i]_\alpha \in \mathcal{K}(X_i)$, $i = 1, 2, \cdots, n$, it follows that

$$\left[\prod_{i=1}^n \widetilde{a}_i\right]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}\left(\prod_{i=1}^n X_i\right)$$

from Tychonoff's theorem; see [1] for Tychonoff's theorem. Thus, it follows that

$$\prod_{i=1}^n \widetilde{a}_i \in \mathcal{FK}\left(\prod_{i=1}^n X_i\right) \subset \mathcal{FC}\left(\prod_{i=1}^n X_i\right),$$

and that $\prod_{i=1}^n \widetilde{a}_i$ is an upper semicontinuous function on $\prod_{i=1}^n X_i$. If

$$\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) = 0,$$

then

$$\left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1', x_2', \cdots, x_n') = 0 = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n)$$

for any $(x_1', x_2', \cdots, x_n') \in f^{-1}(y)$. Suppose that

$$\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) > 0.$$

Then, there exists $(x_1'', x_2'', \cdots, x_n'') \in f^{-1}(y)$ such that

$$\left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1'', x_2'', \cdots, x_n'') > 0.$$

We set

$$\beta = \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1'', x_2'', \cdots, x_n'') > 0.$$

Then, since $x_i'' \in [\widetilde{a}_i]_\beta$, $i = 1, 2, \cdots, n$, it follows that

$$(x_1'', x_2'', \cdots, x_n'') \in f^{-1}(y) \cap \left(\prod_{i=1}^n [\widetilde{a}_i]_\beta\right).$$

Since

$$\prod_{i=1}^{n}[\widetilde{a}_i]_\beta \in \mathcal{K}\left(\prod_{i=1}^{n} X_i\right)$$

and

$$f^{-1}(y) \in \mathcal{C}\left(\prod_{i=1}^{n} X_i\right)$$

by the continuity of f, it follows that

$$f^{-1}(y) \cap \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right) \in \mathcal{K}\left(\prod_{i=1}^{n} X_i\right).$$

Since

$$\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) \geq \beta$$

for any

$$(x_1, x_2, \cdots, x_n) \in f^{-1}(y) \cap \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right),$$

and

$$\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) < \beta$$

for any

$$(x_1, x_2, \cdots, x_n) \in f^{-1}(y) \setminus \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right),$$

we have

$$\begin{aligned}
&\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) \\
&\quad = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)\cap\left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n).
\end{aligned}$$

By the compactness of $f^{-1}(y) \cap (\prod_{i=1}^{n}[\widetilde{a}_i]_\beta) \neq \emptyset$ and the upper semicontinuity of $\prod_{i=1}^{n}\widetilde{a}_i$, there exists $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y) \cap (\prod_{i=1}^{n}[\widetilde{a}_i]_\beta)$ such that

$$\begin{aligned}
&\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1^*, x_2^*, \cdots, x_n^*) \\
&\quad = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)\cap\left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n).
\end{aligned}$$

□

The following theorem gives sufficient conditions for the fuzzy set obtained by Zadeh's extension principle from other fuzzy sets to be a compact fuzzy set.

Theorem 3. *Let $(X_i, \mathbb{T}_i), i = 1, 2, \cdots, n$ be Hausdorff spaces, and let Y be a T_1-space. Assume that $f : \prod_{i=1}^n X_i \to Y$ is continuous. In addition, let $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. Then, $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{FK}(Y)$.*

Proof. Fix any $\alpha \in]0, 1]$. Since $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$, it follows that $[\widetilde{a}_i]_\alpha \in \mathcal{K}(X_i)$, $i = 1, 2, \cdots, n$, and that $\prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}\left(\prod_{i=1}^n X_i\right)$ from Tychonoff's theorem. From Theorem 2 and the continuity of f, if follows that $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$ $\in \mathcal{K}(Y)$. Therefore, we have $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{FK}(Y)$. □

The following theorem shows that order relations of functions imply order relations of fuzzy sets obtained by Zadeh's extension principle using the functions.

Theorem 4. *Let $(X_i, \mathbb{T}_i)$, $i = 1, 2, \cdots, n$ be Hausdorff spaces, and let $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. Assume that $f, g : \prod_{i=1}^n X_i \to \mathbb{R}$ are continuous.*

(i) If $f \leq g$, then $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \preceq g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

(ii) If $f < g$, then $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \prec g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

Proof. We shall show only (i). (ii) can be shown in the similar way to (i).

From Theorem 2, it follows that $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$ and $[g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$ for any $\alpha \in]0, 1]$. Fix any $\alpha \in]0, 1]$.

First, let $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$. Then, there exists $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n [\widetilde{a}_i]_\alpha$ such that $z = g(x_1, x_2, \cdots, x_n)$. Set $y = f(x_1, x_2, \cdots, x_n) \in f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) = [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, then it follows that $y = f(x_1, x_2, \cdots, x_n) \leq g(x_1, x_2, \cdots, x_n) = z$ from the assumption. Thus, for any $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, there exists $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ such that $y \leq z$.

Next, let $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$. Then, there exists $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n [\widetilde{a}_i]_\alpha$ such that $y = f(x_1, x_2, \cdots, x_n)$. Set $z = g(x_1, x_2, \cdots, x_n) \in g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) = [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, then it follows that $y = f(x_1, x_2, \cdots, x_n) \leq g(x_1, x_2, \cdots, x_n) = z$ from the assumption. Thus, for any $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, there exists $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ such that $y \leq z$.

Therefore, we have $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \preceq g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$ since $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha \leq [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ for any $\alpha \in]0, 1]$. □

4 Fuzzy Schwarz inequality In this section, the fuzzy Schwarz inequality is derived by using the fuzzy norm and the fuzzy inner product as an extension of the Schwarz inequality in inner product spaces.

Throughout this section, let X be a real or complex inner product space equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For each $x \in X$, the norm of x is defined as

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{6}$$

The same notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are used when some inner product spaces are considered.

The inner product on $X \times X$ is defined as

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \tag{7}$$

for each $(x_1, y_1), (x_2, y_2) \in X \times X$, and the norm on $X \times X$ is defined as

$$\|(x, y)\| = \sqrt{\langle (x, y), (x, y) \rangle} = \sqrt{\|x\|^2 + \|y\|^2} \tag{8}$$

for each $(x, y) \in X \times X$.

The following theorem provides an inequality which evaluates the inner product between two vectors in an inner product space by norms of the two vectors; see, for example [3].

Theorem 5. (Schwarz Inequality) *For any* $x, y \in X$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Moreover, equality holds in this inequality if and only if x *and* y *are linearly dependent.*

In order to derive fuzzy Schwarz inequality, we present the following lemma.

Lemma 1. *Define* $f_1 : X \times X \to \mathbb{K}$ *as* $f_1(x, y) = \langle x, y\rangle$ *for each* $(x, y) \in X \times X$, $f_2 : \mathbb{K} \to \mathbb{R}$ *as* $f_2(z) = |z|$ *for each* $z \in \mathbb{K}$, *and* $f : X \times X \to \mathbb{R}$ *as* $f(x, y) = f_2(f_1(x, y)) = |\langle x, y\rangle|$ *for each* $(x, y) \in X \times X$. *In addition, define* $g_1 : X \to \mathbb{R}$ *as* $g_1(x) = \|x\|$ *for each* $x \in X$, $g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *as* $g_2(u, v) = uv$ *for each* $(u, v) \in \mathbb{R} \times \mathbb{R}$, *and* $g : X \times X \to \mathbb{R}$ *as* $g(x, y) = g_2(g_1(x), g_1(y)) = \|x\|\|y\|$ *for each* $(x, y) \in X \times X$. *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$. *Then,*

$$f(\widetilde{a}, \widetilde{b}) = f_2(f_1(\widetilde{a}, \widetilde{b})) = |\langle \widetilde{a}, \widetilde{b}\rangle|, \tag{9}$$

$$g(\widetilde{a}, \widetilde{b}) = g_2(g_1(\widetilde{a}), g_1(\widetilde{b})) = \|\widetilde{a}\|\|\widetilde{b}\| \tag{10}$$

where the second equalities in (9) and (10) are definitions.

Proof. For $A, B \subset X$, it can be shown easily that

$$f(A, B) = f_2(f_1(A, B)) = |\langle A, B\rangle|, \tag{11}$$

$$g(A, B) = g_2(g_1(A), g_1(B)) = \|A\|\|B\| \tag{12}$$

where the second equalities in (11) and (12) are definitions.

Since f_1, f_2, and f are continuous, it follows that $[f(\widetilde{a}, \widetilde{b})]_\alpha = f([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha) = f_2(f_1([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha)) = f_2([f_1(\widetilde{a}, \widetilde{b})]_\alpha) = [f_2(f_1(\widetilde{a}, \widetilde{b}))]_\alpha$ for any $\alpha \in]0, 1]$ from Theorems 2, 3, and (11). Therefore, we have $f(\widetilde{a}, \widetilde{b}) = f_2(f_1(\widetilde{a}, \widetilde{b}))$ from the decomposition theorem (2).

Since g_1, g_2, and g are continuous, it follows that $[g(\widetilde{a}, \widetilde{b})]_\alpha = g([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha) = g_2(g_1([\widetilde{a}]_\alpha), g_1([\widetilde{b}]_\alpha)) = g_2([g_1(\widetilde{a})]_\alpha, [g_1(\widetilde{b})]_\alpha) = [g_2(g_1(\widetilde{a}), g_1(\widetilde{b}))]_\alpha$ for any $\alpha \in]0, 1]$ from Theorems 2, 3, and (12). Therefore, we have $g(\widetilde{a}, \widetilde{b}) = g_2(g_1(\widetilde{a}),\ g_1(\widetilde{b}))$ from the decomposition theorem (2). □

The following theorem provides an inequality which evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

Theorem 6. (Fuzzy Schwarz Inequality) *For any* $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$,

$$|\langle \widetilde{a}, \widetilde{b}\rangle| \preceq \|\widetilde{a}\|\|\widetilde{b}\|.$$

Proof. Define $f, g : X \times X \to \mathbb{R}$ as $f(x, y) = |\langle x, y\rangle|$ and $g(x, y) = \|x\|\|y\|$ for each $(x, y) \in X \times X$. From Theorem 5, it follows that $f(x, y) = |\langle x, y\rangle| \leq \|x\|\|y\| = g(x, y)$ for any $x, y \in X$.

Let $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$. Then, it follows that $f(\widetilde{a}, \widetilde{b}) = |\langle \widetilde{a}, \widetilde{b}\rangle|$ and $g(\widetilde{a}, \widetilde{b}) = \|\widetilde{a}\|\|\widetilde{b}\|$ from Lemma 1. Since f, g are continuous, we have $|\langle \widetilde{a}, \widetilde{b}\rangle| = f(\widetilde{a}, \widetilde{b}) \preceq g(\widetilde{a}, \widetilde{b}) = \|\widetilde{a}\|\|\widetilde{b}\|$ from Theorem 4 (i). □

5 Conclusions In the present paper, the fuzzy Schwarz inequality in inner product spaces was derived. It was an extension of the Schwarz inequality, and was described by using a fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors. On the other hand, the fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

First, the fuzzy norm and the fuzzy inner product were defined by Zadeh's extension principle. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set. Next, sufficient conditions for the image of level sets of fuzzy sets to coincide with level sets of another fuzzy set obtained by Zadeh's extension principle were given as Theorem 2. Next, sufficient conditions for the fuzzy set obtained by Zadeh's extension principle from other fuzzy sets to be a compact fuzzy set were given as Theorem 3. Next, it was shown that order relations of functions implied order relations of fuzzy sets obtained by Zadeh's extension principle using the functions as Theorem 4. Finally, based on these results, the fuzzy Schwarz inequality was derived as Theorem 6.

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L^p-BOUNDEDNESS OF A HAUSDORFF OPERATOR ASSOCIATED WITH CHANGE OF VARIABLES AND WEIGHTS

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Abstract

Multivariate Hausdorff operators associated with linear transformations on $L^p(\mathbb{R}^n)$ are investigated by Brown and Moricz. We replace the linear transformation with a general change of variables and introduce modified Hausdorff operators $\mathcal{H}_\psi$ associated with a change of variables and weights. We obtain a condition of ψ under which the operator is bounded from L^p to L^p. The modified Hausdorff operators cover the Hausdorff operators defined on the Euclidean space, the Dunkl hypergroup and the Jacobi hypergroup. In each case, we give conditions of ψ under which the operators are bounded from L^p to L^p.

1 The modified Hausdorff operator

Let $\mu(t)$ be a Borel measure on $\mathbb{R}^n$ and $A(t)$ a $n \times n$ matrix whose entries $a_{ij}(t)$ are functions on $\mathbb{R}^n$. Brown and Moricz [2] introduce the multivariate Hausdorff operator H_ψ acting on functions on $\mathbb{R}^n$ as

$$H_\psi(f)(x) = \int_{\mathbb{R}^n} \psi(t) f(A(t)x) d\mu(t)$$

provided that the integral on the right-hand side exists. For $1 \leq p \leq \infty$ they obtain a condition of ψ under which H_ψ is bounded from L^p to L^p (see §3.1). Moreover, the boundedness on H^p, BMO, Herz-type spaces, Morrey-type spaces, and so on are investigated by many authors (see [1]

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and references therein). The Hausdorff operators are generalized on abstract groups. For example, on the Heisenberg groups, Guo, Sun and Zhao [4] obtain the sharp L^p estimates of high-dimensional and multilinear Hausdorff operators. Then the operators on other function spaces are investigated (see [7] and references therein). In this paper we introduce a modified Hausdorff operator $\mathcal{H}_\psi$ by replacing $A(t)x$ with a general change of variable $F_t(x)$ and $d\mu(t)$ with a weight function $\omega(t)dt$. In particular, treating the cases that the weight functions $\omega(t)$ corresponds to the Dunkl and the Jacobi hypergroups respectively, we can obtain some conditions of ψ under which $\mathcal{H}_\psi$ for the Dunkl and the Jacobi hypergroups are bounded from L^p to L^p (see §3.2 and §3.3).

Let $U \subset \mathbb{R}^n$ be an open subset and let $F : U \to \mathbb{R}^n$ be a C^1 function. We suppose that F is one-to-one and that, for all $x \in U$, the derivative $DF(x)$ is invertible. Hence $V = F(U) \subset \mathbb{R}^n$ is open and $F : U \to V$ is a diffeomorphism. Then for a suitable function f on V,

$$\int_V f(v)dv = \int_U f(F(u))|\det DF(u)|du,$$

where dv and du are Lebesgue measures on $\mathbb{R}^n$. Let ω_U and ω_V are positive functions on U and V respectively. We denote by $L^p(U, \omega_U)$ (resp. $L^p(V, \omega_V)$) the space of L^p functions on U with respect to $\omega_U(u)du$ (resp. on V with respect to $\omega_V(v)dv$). For $g \in L^p(U, \omega_U)$, we put

$$g_F^*(v) = g(F^{-1}(v))\frac{\omega_U(F^{-1}(v))}{\omega_V(v)}|\det DF(F^{-1}(v))|^{-1}.$$

If $g \in L^1(U, \omega_U)$, then it follows from the change of variables formula that

$$\int_V g_F^*(v)\omega_V(v)dv = \int_U g(u)\omega_U(u)du. \tag{1}$$

We now suppose that F depends on a parameter $t \in U$, and write $F = F_t$. Let ψ be a positive function on U. We define the Hausdorff operator $\mathcal{H}_\psi$ acting on functions on V and its dual $\mathcal{H}_\psi^*$ acting on functions on U as follows.

$$(\mathcal{H}_\psi f)(u) = \int_U \psi(t)f(F_t(u))\omega_U(t)dt,$$
$$(\mathcal{H}_\psi^* g)(v) = \int_U \psi(t)g_{F_t}^*(v)\omega_U(t)dt.$$

Actually, they satisfy the following relation.

$$
\begin{aligned}
&\int_U (\mathcal{H}_\psi f)(u)\overline{g(u)}\omega_U(u)du \\
&= \int_U \psi(t)\omega_U(t)(\int_U f(F_t(u))\overline{g(u)}\omega_U(u)du)dt \\
&= \int_U \psi(t)\omega_U(t)(\int_V f(v)\overline{g(F_t^{-1}(v))}\omega_U(F_t^{-1}(v))|\det DF_t(F_t^{-1}(v))|^{-1}dv)dt \\
&= \int_V f(v)(\int_U \psi(t)\overline{g^*_{F_t}(v)}\omega_U(t)dt)\omega_V(v)dv \qquad (2)\\
&= \int_V f(v)\overline{(\mathcal{H}^*_\psi g)(v)}\omega_V(v)dv.
\end{aligned}
$$

Lemma 1.1. *We suppose that $\psi \in L^1(U,\omega_U)$. Then for all f in $L^\infty(V,\omega_V)$,*

$$\|\mathcal{H}_\psi f\|_{L^\infty(U,\omega_U)} \le \|\psi\|_{L^1(U,\omega_U)}\|f\|_{L^\infty(V,\omega_V)}.$$

Proof. This is obvious from the definition of $\mathcal{H}_\psi$. □

Lemma 1.2. *We suppose that*

$$d_\psi = \sup_{v\in V}\int_U \psi(t)\frac{\omega_U(F_t^{-1}(v))}{\omega_V(v)}|\det DF_t(F_t^{-1}(v))|^{-1}\omega_U(t)dt < \infty. \qquad (3)$$

Then for all f in $L^1(V,\omega_V)$,

$$\|\mathcal{H}_\psi f\|_{L^1(U,\omega_U)} \le d_\psi\|f\|_{L^1(V,\omega_V)}.$$

Proof. By letting $g = 1$, the inequality follows from (2). □

Therefore, by using the interpolation and the duality, we can deduce the following.

Theorem 1.3. *Let $1 \le p \le \infty$ and $\frac{1}{p}+\frac{1}{p^*} = 1$. Then for all f in $L^p(V,\omega_V)$ and all g in $L^p(U,\omega_U)$,*

$$
\begin{aligned}
\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} &\le (d_\psi)^{\frac{1}{p}}\|\psi\|_{L^1(U,\omega_U)}^{\frac{1}{p^*}}\|f\|_{L^p(V,\omega_V)},\\
\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} &\le (d_\psi)^{\frac{1}{p^*}}\|\psi\|_{L^1(U,\omega_U)}^{\frac{1}{p}}\|g\|_{L^p(U,\omega_U)}.
\end{aligned}
$$

2 Another L^p boundedness

To obtain the L^p boundedness of $\mathcal{H}_\psi$ in Theorem 1.3 we use the interpolation. Here we shall calculate the L^p norm of $\mathcal{H}_\psi$ directly. We put

$$\rho(t) = \inf_{u\in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}$$

and for $1 \le p \le \infty$,

$$C^p_{\psi,\rho} = \int_U \psi(t)\rho(t)^{-\frac{1}{p}}\omega_U(t)dt. \tag{4}$$

Theorem 2.1. *Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then for all f in $L^p(V, \omega_V)$ and all g in $L^p(U, \omega_U)$,*

$$\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} \le C^p_{\psi,\rho}\|f\|_{L^p(V,\omega_V)},$$
$$\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} \le C^{p^*}_{\psi,\rho}\|g\|_{L^p(U,\omega_U)}$$

provided that $C^p_{\psi,\rho} < \infty$ and $C^{p^*}_{\psi,\rho} < \infty$ respectively.

Proof. We shall prove the second inequality. Then for $g \in L^p(U, \omega_U)$ and $1 \le p < \infty$, we see that

$$\begin{aligned}\|g^*_{F_t}\|^p_{L^p(V,\omega_V)} &= \int_U |g(u)|^p\Big(\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}\Big)^{1-p}\omega_U(u)du \\ &\le \rho(t)^{1-p}\|g\|^p_{L^p(U,\omega_U)}.\end{aligned} \tag{5}$$

Hence, by the definition of $\mathcal{H}^*_\psi g$ and (5), we see that

$$\begin{aligned}\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} &\le \int_U \psi(t)\omega_U(t)\|g^*_{F_t}\|_{L^p(V,\omega_V)}dt \\ &\le \int_U \psi(t)\rho(t)^{\frac{1}{p}-1}\omega_U(t)dt \cdot \|g\|_{L^p(U,\omega_U)}.\end{aligned}$$

The case $p = \infty$ is obvious. The first inequality follows by the duality. Here we give a direct proof. We suppose $p < \infty$. We see that

$$\begin{aligned}&\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} \\ &\le \int_U \psi(t)\omega_U(t)\|f(F_t(\cdot))\|_{L^p(U,\omega_U)}dt \\ &\le \int_U \psi(t)\omega_U(t)\Big(\int_V |f(v)|^p\Big(\frac{|\det DF_t(F_t^{-1}(v))|\omega_V(v)}{\omega_U(F_t^{-1}(v))}\Big)^{-1}\omega_V(v)dv\Big)^{\frac{1}{p}}dt \quad (6) \\ &\le \int_U \psi(t)\rho(t)^{-\frac{1}{p}}\omega_U(t)dt \cdot \|f\|_{L^p(V,\omega_V)}.\end{aligned}$$

The case $p = \infty$ is obvious. □

Remark 2.2. We note that $d_\psi \leq C^1_{\psi,\rho}$. Moreover, by using the Hölder inequality, it follows that $C^p_{\psi,\rho} \leq (C^1_{\psi,\rho})^{\frac{1}{p}} \|\psi\|^{\frac{1}{p*}}_{L^1(U,\omega_U)}$ and thus, $C^1_{\psi,\rho} \geq (C^p_{\psi,\rho})^p$ $\|\psi\|^{1-p}_{L^1(U,\omega_U)}$. Therefore, if

$$\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}$$

does not depend on u (see §3), then it follows that $d_\psi = C^1_{\psi,\rho}$ and thus,

$$(d_\psi)^{\frac{1}{p}} \|\psi\|^{\frac{1}{p^*}}_{L^1(U,\omega_U)} \geq C^p_{\psi,\rho}.$$

3 Variants of weights

Our modified Hausdorff operator $\mathcal{H}_\psi$ depends on a weight function ω_U. Therefore, by changing the weight, we can treat the Hausdorff operators for the Euclidean space, the Dunkl hypergroup, and the Jacobi hypergroup similarly

3.1 Euclidean space

Let $A(u) = (a_{ij}(u))^n_{i,j=1}$ be an $n \times n$ matrix, where coefficients $a_{ij}(u)$ are measurable functions of u and $A(u)$ may be singular on a set of measure zero. We take $U = V = \mathbb{R}^n$,

$$\begin{array}{cccc} F_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \Cup & & \Cup \\ & x & \longmapsto & xA(t), \end{array}$$

and $w_U(x) = w_V(x) = 1$. Here $xA(t)$ is the multiplication of the row vector x and the matrix $A(t)$. Then

$$g^*_{F_t}(x) = g\left(xA^{-1}(t)\right) |\det(A(t))|^{-1}.$$

Hence the Hausdorff type operator and its dual are given as follows.

$$(\mathcal{H}_\psi f)(u) = \int_{\mathbb{R}^n} \psi(t) f(uA(t))dt,$$
$$\left(\mathcal{H}^*_\psi g\right)(v) = \int_{\mathbb{R}^n} \psi(t) g(vA^{-1}(t))|\det(A(t))|^{-1}dt.$$

Moreover, it follows that

$$\rho(t) = \inf_{u\in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = |\det A(t)|.$$

Then the following corollary is obtained (see [2]).

Corollary 3.1. *Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix and may be singular on a set of measure zero in $\mathbb{R}^n$. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. We put*

$$C^p_{\psi,A} = \int_{\mathbb{R}^n} \psi(t) |\det A(t)|^{-\frac{1}{p}} dt.$$

Then for all f in $L^p(\mathbb{R}^n)$,

$$\|\mathcal{H}_\psi f\|_{L^p(\mathbb{R}^n)} \leq C^p_{\psi,A} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\mathcal{H}^*_\psi f\|_{L^p(\mathbb{R}^n)} \leq C^{p^*}_{\psi,A} \|f\|_{L^p(\mathbb{R}^n)}$$

provided that $C^p_{\psi,A} < \infty$ and $C^{p^*}_{\psi,A} < \infty$ respectively.

Remark 3.2. Let $A(t)$ be a diagonal matrix $\operatorname{diag}(t_1, t_2, \cdots, t_n)$. Then $g^*_{F_t}(x)$ is a kind of dilation of g. Actually, when $n = 1$, $g^*_{F_t}(x)$ coincides with the dilation of g and $\mathcal{H}_\psi$ is the classical one-dimensional Hausdorff operator. However, there are various kinds of extension of the classical Hausforff operators. For example, in [5], the case that $U = V = \mathbb{R}^n$, $\omega_U(x) = \omega_V(x) = \|x\|^\alpha$ and $F_t(x) = \frac{x}{\|t\|}$ is investigated.

3.2 Dunkl hypergroup

As an extension of one-dimensional Hausdorff operator, we shall consider a modified Hausdorff operator related with the Dunkl hypergroup (see [3]). Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ where each κ_j is a nonnegative real number. Let $d\mu_\kappa$ denote the associated measure given for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ by

$$d\mu_\kappa(x) = h^2_\kappa(x) dx,$$

where h_κ is the $\mathbb{Z}^n_2$-invariant function defined by

$$h_\kappa(x) = \prod_{j=1}^n |x_j|^{\kappa_j}.$$

Let $A(s) = \operatorname{diag}(a_1(s), \cdots, a_n(s))$ be a diagonal matrix, where coefficients $a_j(s)$ are measurable functions of s and $A(s)$ may be singular on a set of measure zero. We take $U = V = \mathbb{R}^n$,

$$\begin{array}{rccc} F_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \cup & & \cup \\ & x & \longmapsto & xA(t), \end{array}$$

and $w_U(x) = w_V(x) = h_\kappa^2(x)$. Then

$$g^*_{F_t}(x) = g(xA^{-1}(t))\frac{|\det(A(t))|^{-1}}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j}} = g(xA^{-1}(t))\frac{1}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}}.$$

Hence the modified Hausdorff operator and its dual are given as follows.

$$(\mathcal{H}_{\kappa,\psi}f)(x) = \int_{\mathbb{R}^n} \psi(s)f(xA(s))d\mu_\kappa(s),$$
$$(\mathcal{H}^*_{\kappa,\psi}g)(v) = \int_{\mathbb{R}^n} \psi(t)g(vA^{-1}(t))\frac{d\mu_\kappa(t)}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}}.$$

Moreover, it follows that

$$\rho(t) = \inf_{u\in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = \prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}.$$

Corollary 3.3. *Let* $A(t) = \operatorname{diag}(a_1(t), \cdots, a_n(t))$ *be a diagonal matrix and may be singular on a set of measure zero in* $\mathbb{R}^n$. *Let* $1 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{p^*} = 1$. *We put*

$$C^p_{\psi,A,\kappa} = \int_{\mathbb{R}^d} \psi(t)\Big(\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}\Big)^{-\frac{1}{p}} d\mu_\kappa(t).$$

Then for all f *in* $L^p(\mathbb{R}^n, d\mu_\kappa)$,

$$\|\mathcal{H}_{\kappa,\psi}f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)} \le C^p_{\psi,A,\kappa}\|f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)},$$
$$\|\mathcal{H}^*_{\kappa,\psi}f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)} \le C^{p^*}_{\psi,A,\kappa}\|f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)}$$

provided that $C^p_{\psi,A,\kappa} < \infty$ *and* $C^{p^*}_{\psi,A,\kappa} < \infty$ *respectively.*

Next let us consider the case that $A(s) = (a_{ij}(s))$ is a non-singular upper triangular matrix with $a_{ij}(s) \ge 0$ for all $j \ge i$. Then for $u = (x_1, x_2, \cdots, x_n)$,

$$\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = |\det(A(t)|\frac{\prod_{j=1}^n |\sum_{i=1}^j a_{ij}(t)x_i|^{2\kappa_j}}{\prod_{j=1}^d |x_j|^{2\kappa_j}}$$
$$= |\det(A(t)|\prod_{j=1}^n |a_{jj}(t) + \sum_{i<j} a_{ij}(t)\frac{x_i}{x_j}|^{2\kappa_j}.$$

Hence, by taking the infimum of the above over $u \in \mathbb{R}^d_+$, then the infimum $\rho(t)$ is given by $\prod_{j=1}^n |a_{jj}(t)|^{2\kappa_j+1}$. Moreover, $\{xA(t) \mid x \in \mathbb{R}^n_+\} \subset \mathbb{R}^n_+$. Then, noting (5) and (6), we can obtain the following.

Corollary 3.4. *Let $A(s) = (a_{ij}(s))$ be a non-singular upper triangular matrix with $a_{ij}(s) \geq 0$ for all $j \geq i$. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then for all f in $L^p(\mathbb{R}^n_+, d\mu_\kappa)$,*

$$\|\mathcal{H}_{\kappa,\psi} f\|_{L^p(\mathbb{R}^n, d\mu_\kappa)} \leq C^p_{\psi,A,\kappa} \|f\|_{L^p(\mathbb{R}^n_+, d\mu_\kappa)},$$
$$\|\mathcal{H}^*_{\kappa,\psi} f\|_{L^p(\mathbb{R}^n, d\mu_\kappa)} \leq C^{p^*}_{\psi,A,\kappa} \|f\|_{L^p(\mathbb{R}^n_+, d\mu_\kappa)}$$

provided that $C^p_{\psi,A,\kappa} < \infty$ and $C^{p^*}_{\psi,A,\kappa} < \infty$ respectively.

3.3 Jacobi hypergroup

We shall consider a modified Hausdorff operator related with the Jacobi hypergroup $(\mathbb{R}_+, *, \Delta)$ (see [6]). Let $\alpha \geq \beta \geq -\frac{1}{2}$, $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$ and put $\Delta(x) = (\sinh x)^{2\alpha+1} (\cosh x)^{2\beta+1}$ on $\mathbb{R}_+$. We define the L^p-norm of a function f on $\mathbb{R}_+$ by

$$\|f\|_{L^p(\Delta)} = \Big(\int_0^\infty |f(x)| \Delta(x) dx \Big)^{\frac{1}{p}}.$$

Let $L^p(\Delta)$ denote the space of functions on $\mathbb{R}_+$ with finite L^p-norm. For $\phi \in L^1(\Delta)$ we define the dilation $\phi_t, t > 0$ of ϕ as

$$\phi_t(x) = \frac{1}{t} \frac{1}{\Delta(x)} \phi\left(\frac{x}{t}\right) \Delta\left(\frac{x}{t}\right).$$

for $x \in \mathbb{R}_+$. We see that $\|\phi_t\|_{L^1(\Delta)} = \|\phi\|_{L^1(\Delta)}$. We take $U = V = \mathbb{R}_+$,

$$\begin{array}{rccc} F_t : & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ & \cup & & \cup \\ & x & \longmapsto & xt, \end{array}$$

and $w_U(x) = w_V(x) = \Delta(x)$. Then $L^1(U, \omega_U) = L^1(V, \omega_V) = L^1(\Delta)$ and

$$\begin{aligned} g^*_{F_t}(x) =& g(F_t^{-1}(x)) \frac{\omega_U(F_t^{-1}(x))}{\omega_V(x)} |\det DF(F_t^{-1}(x))|^{-1} \\ =& \frac{1}{t} \frac{1}{\Delta(x)} g\left(\frac{x}{t}\right) \Delta\left(\frac{x}{t}\right) = g_t(x). \end{aligned}$$

Hence the modified Hausdorff operator and its dual are given as follows.

$$(\mathcal{H}_\psi f)(u) = \int_0^\infty f(ut) \psi(t) \Delta(t) dt,$$
$$(\mathcal{H}^*_\psi g)(v) = \int_0^\infty g_t(v) \psi(t) \Delta(t) dt.$$

Corollary 3.5. *We suppose that* $\psi \in L^1(\Delta)$. *Then for all* $f \in L^\infty(\Delta)$,

$$\|\mathcal{H}_\psi f\|_{L^\infty(\Delta)} \leq \|\psi\|_{L^1(\Delta)} \|f\|_{L^\infty(\Delta)}$$

and for all $g \in L^1(\Delta)$,

$$\|\mathcal{H}^*_\psi g\|_{L^1(\Delta)} \leq \|\psi\|_{L^1(\Delta)} \|g\|_{L^1(\Delta)}.$$

We note that if $t < 1$, then

$$\rho(t) = \inf_{0 \leq u < \infty} \frac{t\Delta(tu)}{\Delta(u)} = 0$$

and if $t \geq 1$, then $\rho(t) = t^{2\alpha+2}$, because $t \sinh u \leq \sinh(tu)$. Therefore, if $\psi \in L^1(\Delta)$ is supported on $[1, \infty)$, then $C^p_{\psi,\rho}$ equals

$$C^p_{\psi,\Delta} = \int_1^\infty \psi(t) t^{-\frac{2\alpha+2}{p}} \Delta(t) dt \leq \|\psi\|_{L^1(\Delta)}$$

and also, $C^{p^*}_{\psi,\Delta} \leq \|\psi\|_{L^1(\Delta)}$ for $\frac{1}{p} + \frac{1}{p^*} = 1$. Therefore, we can obtain the following.

Corollary 3.6. *Let* $1 \leq p \leq \infty$. *We suppose that* $\psi \in L^1(\Delta)$ *and is supported on* $[1, \infty)$. *Then for all* f *in* $L^p(\Delta)$,

$$\|\mathcal{H}_\psi f\|_{L^p(\Delta)} \leq C^p_{\psi,\Delta} \|f\|_{L^p(\Delta)},$$
$$\|\mathcal{H}^*_\psi f\|_{L^p(\Delta)} \leq C^{p^*}_{\psi,\Delta} \|f\|_{L^p(\Delta)}.$$

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UNCONVENTIONAL PROVING TECHNIQUES IN CYBER - PHYSICAL SYSTEMS

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Abstract

Cyber - Physical Systems [CPS] are "Engineered systems that are built from, and depend upon, the seamless integration of computational algorithms and physical components". CPS have the potential to provide much richer functionality - including efficiency, flexibility, autonomy, and reliability – than systems that are loosely coupled, discrete, or manually operated. CPS also can create vulnerability related to protection, security and reliability. This can result in a chaotic collapse around the many new complex and powerful technological systems we rely on. The very complexity and interconnectedness of such CPS warrants unconventional proofing to unravel. Moreover, CPS is diffused across the social fabric. The sociology of mathematics is quite elusive for the construction of formal proofing in CPS.

The gap between rigorous argument and formal proof in the sense of mathematical logic is one that will close in CPS.

The generic characteristics of CPS are:

- Self-organization
- Interdependence
- Feedback
- Far from equilibrium
- Exploration of the space of possibilities
- History and path dependence
- Creation of new order

Cyber risk is an increasing concern in the complex, connected world of CPS.The complexity of the ecosystem, the connectivity of devices and the criticality of devices and services all increase risk, and the necessary formal proofs are elusive to take an effective action. 'Fake People" is the Case Study presented in this paper to illustrate unconventional proofing in Humane Security Engineering of CPS. Adapting the Cynefin Framework with the inclusion of Neurotheology, Complexity Science and Indic Studies in Consciousness enables the construction of unconventional proofing systems that transcend the software design limits of CPS.

1. Introduction

"a mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof …….. is more often than not a way of making sure that our minds are not playing tricks."**- Gian-Carlo Rota, Introduction to the Book "The Mathematical Experience" by Philip Davis and Reuben Hersh, Mariner Books [Reprint], 1999.**

Aristotle observed that within the universe there are three natural languages which perfectly describe the supreme science - music, color, and numbers. Ancient cultures had no conception of computing beyond simple arithmetic. Modeling the Human Brain in the form of Computers began with Numbers. However, the human brain in itself is an enigma.

1.1 Basics of the Human Brain

Human brain is a collection of large networks of nerve cells. A nerve cell or neuron is the basic unit of neural networks, which can be said to perform computation. Natural neural networks are complex arrangements and connections of a usually large number of nerve cells. Natural Neural Networks are also loosely referred to as Biological Neural Networks.

The nervous system in human beings is classified into
1. Central Nervous System, and
2. Peripheral Nervous System

The central nervous system is further divided into two parts namely
1. The Brain
2. The Spinal Cord

In the average adult human, the brain weighs 1.3 to 1.4kg (about 3 pounds). The brain contains about 100 billion nerve cells and trillions of "support cells" called glia. There are over 1011 neurons. There are over 1014 connections.

Human Brain is organised into regions and the various regions are organised as layers. Cortex and cerebellum are good examples of layered parts. There are more than 100 different types of neurons as well as associated glial (neuroglial) cells. There are a number of different transmitter substances. The cerebral hemispheres are split in right and left and only joined by the corpus callosum.

The spinal cord is about 43 cm long in adult women and 45 cm long in adult men and weighs about 35-40 gm. The vertebral column, the collection of bones (back bone) that houses the spinal cord, is about 70 cm long. So the spinal cord is much shorter than the vertebral column. The various divisions are shown in Figure 1.

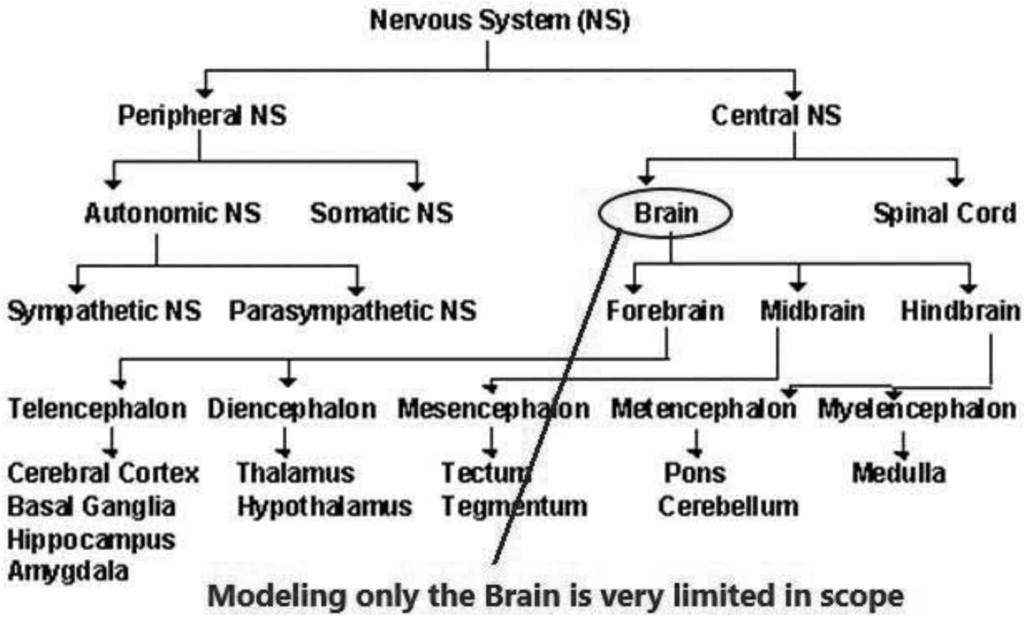

Figure 1: Divisions of Human Nervous System

The human body is made up of billions of cells. The central nervous system [CNS] is composed entirely of two kinds of specialized cells: neurons and glia. Neurons, are specialized to carry "messages" through an electrochemical process. The human brain has about 100 billion neurons. Neurons (nerve cells) come in many different shapes and sizes. Some of the smallest neurons have cell bodies that are only 4 microns wide, while some of the biggest neurons have cell bodies that are 100 microns wide.Glia (or glial cells) are the cells that provide support to the neurons. There are as many as 50 times more glia than neurons in the CNS. Alan Turing accentuated the Neurons and all Computational Models functionally based on the Neurons with support systems necessary for information processing structures.

Neurons are similar to other cells in the body in some ways such as:

- Neurons are surrounded by a cell membrane.
- Neurons have a nucleus that contains genes.
- Neurons carry out basic cellular processes like protein synthesis and energy production.

Neurons differ from the other body cells in some ways such as:

- Neurons have specialized extensions called dendrites and axons. Dendrites bring information to the cell body and axons take information away from the cell body.
- Neurons communicate with each other through an electrochemical process.
- Neurons contain some specialized structures (for example, synapses) and chemicals (for example, neurotransmitters).

A neuron typically has many dendrites and one axon. The dendrites branch and terminate in the vicinity of the cell body. In contrast, axons can extend to distant targets, more than a meter away in some instances. Dendrites are rarely more than about a millimeter long and often much shorter. Neurons communicate through specialized junctions called 'synapses'. There are as many as 10,000 specific types of neurons in the human brain.The most widely used types of neurons based on their structure are shown in Figure 2.

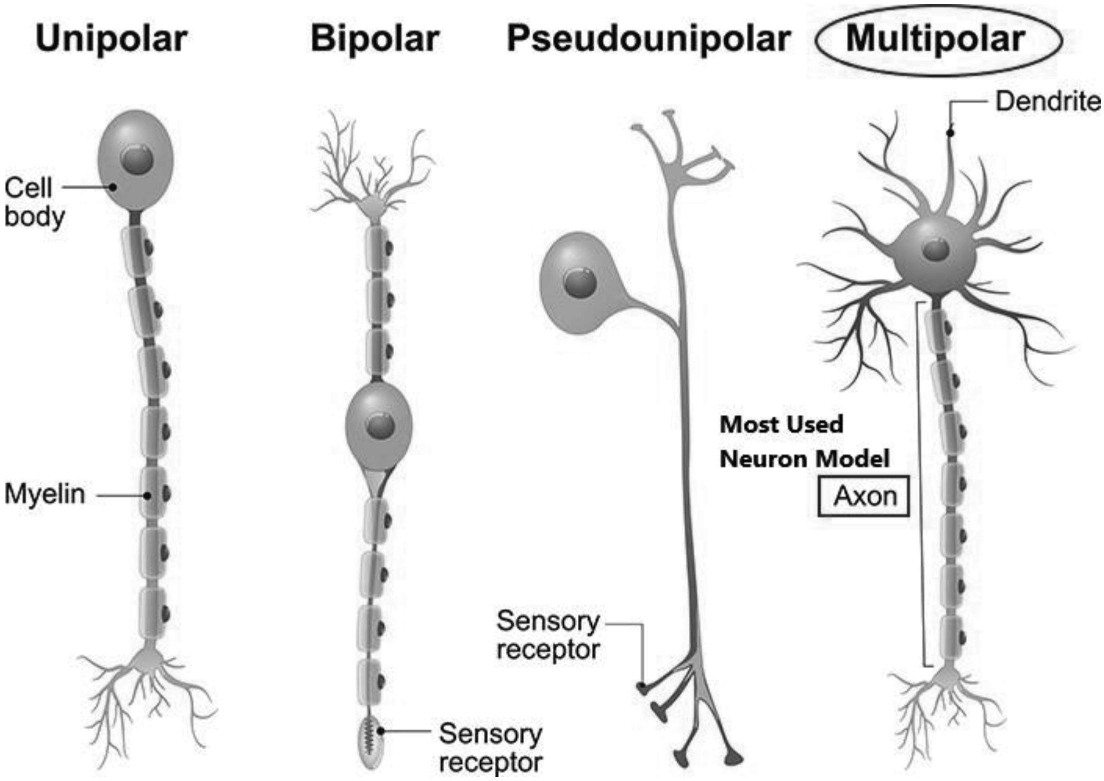

Figure 2: Most Used Types of Neurons based on Structure that is amenable to Mathematical Modeling

Neuron model represents a mathematical structure that incorporates its biophysical and geometrical characteristics. There are a few typical types of Neurons based on the functionality such as Motor, Sensory and those that connect these types called Interneurons.

Human Brain is an electro-, chemical- and biological- organ. Hardly 1% of brain and its functioning are understood by neuroscientists. New research suggests that genius can be nurtured as well. Expertise in calculation is not due to increased activity of processes that exist in non-experts, but they are due to the usage of different brain areas. Musical training at an early age may lead to the increased growth of certain brain regions. These results have made modeling the Human Brain even more difficult.

Alan Turing showed that under certain conditions, random heterogeneities in chemically interacting diffusible substances could generate patterns without a pre-existing organization. In other words self-organization happens. These ideas were very difficult for the biologists.

The notion of Intelligence which is associated with the Human Brain began to appear more attractive than modeling the Human Brain.

"A computer would deserve to be called intelligent if it could deceive a human into believing that it was human" – Alan M Turing

Turing devised a test, which he called "the imitation game," to herald the advent of computers that were indistinguishable from human minds.

"I believe that in about 50 years' time it will be possible to program computers … so well that an average interrogator will not have more than a 70% chance of making the right identification after five minutes of questioning." - Alan M Turing

1.2 The Turing Brain

"Some of the feats that will be able to be performed by Britain's new electronic brain, which is being developed at the N.P.L., Teddington, were described to the SURREY COMET yesterday by Dr. A. M. Turing, 34-year-old mathematics expert, who is the pioneer of the scheme in this country.The machine is to be an improvement on the American ENIAC, and it was in the brain of Dr Turing that the more efficient model was developed.... " - Surrey Comet, 9 November 1946

Turing gives two examples of artificial unorganized machines, which he claims are about the simplest possible models of the nervous system.

The first type are A-type machines – these are randomly connected networks of NAND gates (where every node has two states representing 0 or 1, two inputs and any number of outputs).

The second type Turing calls B-type machines – these are derived from any A-type network by intersecting every inter-node connection with a construction of three further A-type nodes which form a connection modifier. B-type networks with their propensity to form loops of various lengths may be well suited to model the kind of massive, widespread feedback and interacting waves of activity.

Turing also proposed P-type unorganised machines, which are **not neuron-like** and have only two interfering inputs, one for **"pleasure" or "reward"** ... and the other for **"pain" or "punishment".** Turing studied P-types in the hope of discovering training procedures "analogous to the kind of process by which a child would really be taught"'.Since this type is non neuron-like and is a modified Turing Machine. The Turing Machine model for computing has enabled the progress of computation as seen in the Figure 3.

Artificial Neuron is a barely functional model for the biological neuron. However, the Artificial Neural Networks have been found applicable in modeling learning. It is another puzzle that the P-type i.e non-neuron like unorganized machines proposed by Turing gave an application domain for the use of Artificial Neural Networks in their present form. Please see the Figure 4.

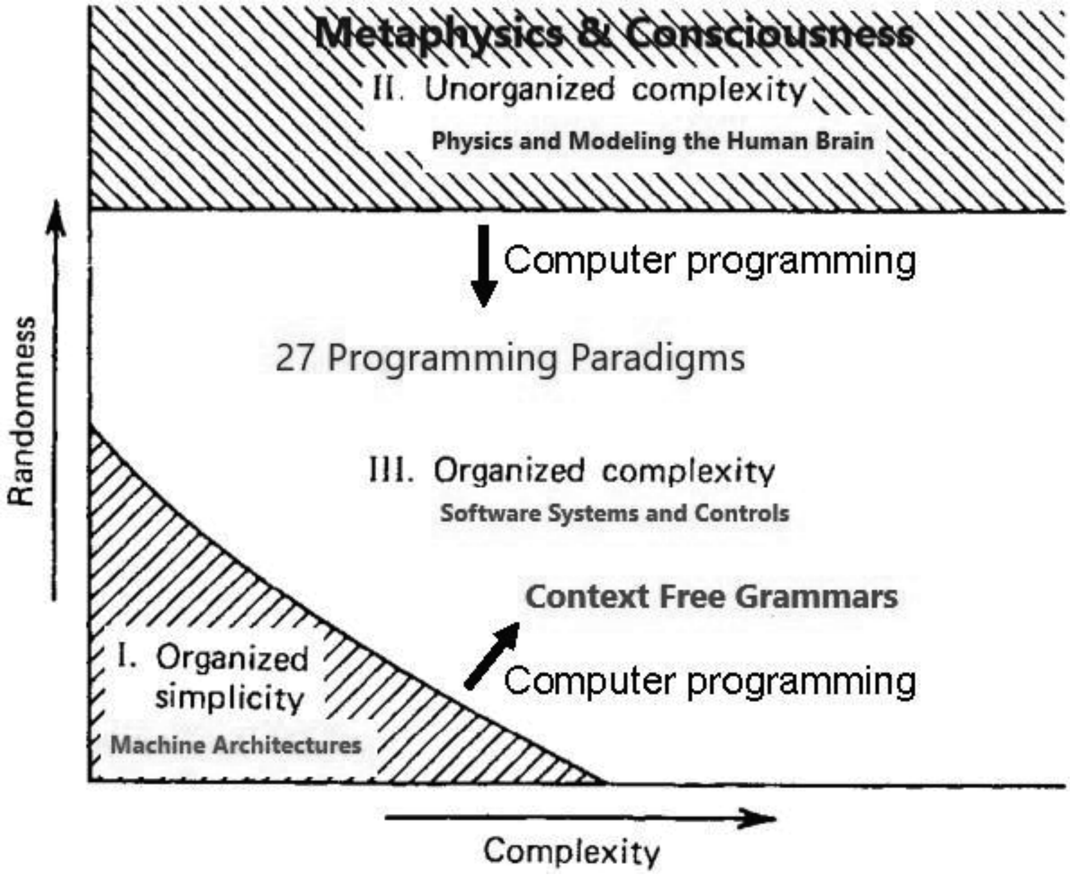

Figure 3: Progress in Computation with the Turing Machine Model

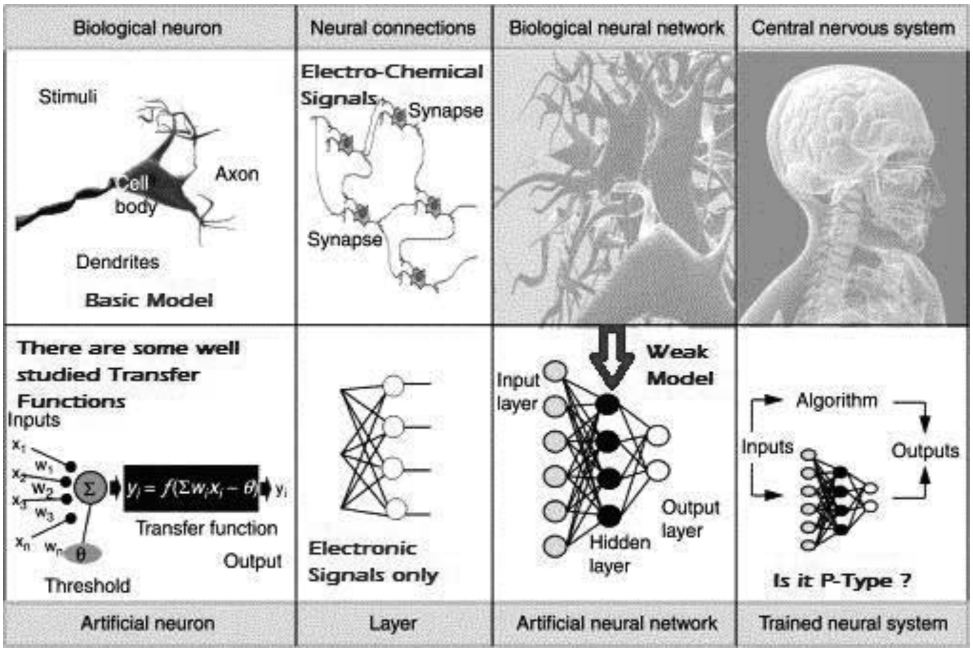

Figure 4: The puzzle of P-Type Unorganized Machine proposed by Alan Turing

"Restrained by our organic constitution and by our different emotions in the lower sphere of our daily occupations, we also feel ourselves urged on by the appeal of the ideal, by more or less precise aspiration towards spiritual values, and from those sentiments even the worst amongst us do not entirely escape." - Louis deBroglie, 'Physics and Microphysics', Pantheon 1955 Quantum Questions, Mystical Writings of the World's Great Physicists [Ken Wilber, Editor]

Turing was way ahead of Artificial Neural Networks. Turing described his idea as a "Universal Computing Machine".In October 1945, Turing joined the National Physical Laboratory [NPL] where he worked on developing the ACE (Automatic Computing Engine). By 1946 he had a finished proposal for the computer, but unfortunately NPL did not have the resources to turn it into reality.

The author opines that Type A is the Reality Layer and Type B is the Spiritual Layer. This is the foundation for the Unconventional computing model for constructing the unconventional proof.

To develop Turing's idea of building a brain-like B-type machine we need to mirror the brain's own development. The proliferation of neurones during the brain's formation involves a substantial random element and only later is this growth fine-tuned by killing off the cells that have grown in the wrong places. This process of weeding out is called programmed cell death, is essential to the development of intelligence, and means that we start off with many more brain cells than we actually need to function as a normal adult.

A B-type cortex would begin with a very large number of nodes and follow a developmental path with the same delicate mix of the random and determined as a living brain. At a magnification where individual nodes and connections could be seen, the resulting very large B-type network would typically look much like a bowl of spaghetti. Such a disorderly structure is prone to forming feedback loops of varying lengths which take varying times to traverse, thus forming possible delay or memory circuits.

In a large network these loops can lead to greatly varying patterns of activity, regardless of input, since activity can be perpetually recycled in a complex manner. The activity in many conventional neural networks stops when the output layer settles into a stable pattern; the equivalent of a Turing Machine halting, its computation over. But just as the brain does not halt, large B-type networks will tend not to either.

The Universal Turing Machine is an excellent mathematical model to establish computations as an activity of the human brain. It is an essential element for explaining how the mind works. However, mind is not so obvious in the present understanding of the work proposed by Alan Turing. This is the premise for the "Spiritual Layer" which fosters Mystic Visions.

In the new field of "Neurotheology," scientists seek the biological basis of spirituality. Is God all in our heads?

The American Psychological Association published "Varieties of Anomalous Experience," covering enigmas from near-death experiences to mystical ones. Some of the early results in from the field of Neurotheology include:

- **Attention:** Linked to concentration, the frontal lobe lights up during meditation.
- **Religious Emotions:** The middle temporal lobe is linked to emotional aspects of religious experience, such as joy and awe.
- **Sacred Images:** The lower temporal lobe is involved in the process by which images, such as candles or crosses, facilitate prayer and meditation.
- **Response to Religious Words:** At the juncture of three lobes, this region governs response to language.
- **Cosmic Unity:** When the parietal lobes quiet down, a person can feel at one with the universe.

"It is certain that thought may be transmitted from one individual to another, even if they are separated by long distance. These facts, which belong to the new science of metaphysics, must be accepted just as they are"…."They express a rare and almost unknown aspect of ourselves"…"What extraordinary penetration would result from the union of disciplined intelligence and of the telepathic aptitude"… - Dr. Alexis Carrel, Man the Unknown

"The catalogue of our ignorance must also include the understanding of the human brain, which is incomplete in one conspicuous way: nobody understands how decisions are made or how imagination is set free. What consciousness consists of (or how it should be defined) is equally a puzzle. Despite the marvelous successes of neuroscience in the past century (not to mention the disputed relevance of artificial intelligence), we seem as far from understanding cognitive process as we were a century ago." - Sir John Maddox, "Consciousness – The Unexpected Science to Come", Former Editor-in-Chief, Nature

2. Conventional Computing and Constructing Proofs

Describing the human brain in mathematical terms is coveted ambition of neuroscience research. The challenges remain considerable. It was Alan Turing who first demonstrated how time-consuming such an undertaking would be. Through the analogy of computer program, Turing argued that a complete mathematical description of the mind would take well over a thousand years.

Computing is essentially a combination of theoretical, scientific, and engineering traditions. Programming is a process of mapping the computing problem into a form that can be executed on an automaton. The resulting software implementations are representations (models) of real-world conceptual systems. The engineering processes move a concept from the Realm of Actions (concepts) to the Realm of Representations (technology). Modeling the application in terms of "well-defined structures and algorithms" is the most important step towards evolving a solution. It is becoming increasingly difficult to decide on a correct solution while building complex evolving software.

The Turing Machine [TM] was invented by Alan Turing in 1936 and it is used to accept Recursive Enumerable Languages [generated by Type-0 Grammar].

A turing machine consists of a tape of infinite length on which read and writes operation can be performed. The tape consists of infinite cells on which each cell either contains input symbol or a special symbol called blank. It also consists of a head pointer which points to cell currently being read and it can move in both directions. A TM is expressed as a 7-tuple **(Q, T, B, ∑, δ, q0, F)** where:

Q is a finite set of states
T is the tape alphabet (symbols which can be written on Tape)
B is blank symbol (every cell is filled with B except input alphabet initially)
∑ is the input alphabet (symbols which are part of input alphabet)
δ is a transition function which maps $Q \times T \rightarrow Q \times T \times \{L,R\}$. Depending on its present state and present tape alphabet (pointed by head pointer), it will move to new state, change the tape symbol (may or may not) and move head pointer to either left or right.
q0 is the initial state
F is the set of final states. If any state of F is reached, input string is accepted.

Formal proof of correctness is tedious, time-consuming, and outlandishly expensive. Also, it is not necessarily effective. People commit errors when attempting a formal proof. There is no way of determining if a proof is correct. "Clean Room Approach" with informal techniques of proving programs correct is in vogue. The code is never run by the programmers in this approach. It is typically not formal proof of correctness. It is acceptable as a pragmatic practice.

The computational complexity of a problem is the amount of resources, such as time or space, required by a turing machine that solves the problem. The descriptive complexity of problems is the complexity of describing problems in some logical formalism over finite structures. One of the exciting developments in complexity theory is the discovery of a very intimate connection between computational and descriptive complexity.

Computational complexity theory classifies the computational problems according to their inherent difficulty, and relates these classes to each other. A computational problem is a task solved by a computer. It is solvable by mechanical application of mathematical steps, such as an algorithm. The following classes of problems and their inter-relations are well studied.

- EXPSPACE Solvable with exponential space
- EXPTIME Solvable in exponential time
- IP Solvable in polynomial time by an interactive proof system
- NP "YES" answers checkable in polynomial time
- co-NP "NO" answers checkable in polynomial time by a non-deterministic machine
- RP Solvable in polynomial time by randomized algorithms (NO answer is probably right, YES is certainly right)
- ZPP Solvable by randomized algorithms (answer is always right, average running time is polynomial)
- P Solvable in polynomial time
- NL "YES" answers checkable with logarithmic space
- L Solvable with logarithmic (small) space
- BPP Solvable in polynomial time by randomized algorithms (answer is probably right)

The relations among these complexity classes open research problems. There are some standard and startling results based on the Turing Machine Model. A Turing Machine is essentially a neural framework for mental programs.

Proof must begin from axioms that are not themselves proved. To prove a proposition, one starts from some first principles, derive some results from those axioms, then, using those axioms and results, push on to prove other results. This is to avoid mistaken "theorems", based on fallible intuitions, of which many instances have occurred in the history of the subject Axioms in traditional thought were "self-evident truths", but that conception is problematic. At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an axiomatic system.

Computational proof offers only the probability - not the certainty - of truth, a statement.The complexity of the Turing machine is limited to serve this purpose. The success of the Turing machine model broke the ideal of axiomatization of mathematics. It paved way for a theoretical computational machine for scoping the capabilities and limitations of an algorithm. Variants of the Turing Machine such as **“Multiple Track Turing Machine”** and **“Two-way Infinite Tape Turing Machine” are well studied.**

The odd symbols and scattered numerals look like a strange language, and yet to read them, neurologists tell us, we would have to use parts of our brains that have nothing to do with what we normally think of as reading and writing. Mathematics and physics researchers are the interpreters of this unconventional language. The subject matter confounds even mathematicians and physicists, as they use mathematics to calculate the inconceivable, undetectable, nonexistent and impossible.Our brains have the ability to compute the abstract mathematics they created to construct theories about reality, and yet they may never be smart enough to comprehend those theories, let alone explain them.

Cyber – Physical Systems need the interplay between software, control and social systems.The existing interplay as shown in the Figure 4 isfar from comprehensive. Neither the Control theories nor the social systems seldom play a major role in the quality assurance of software analysis and design. The core of software engineering and that of control theory / engineering have been developed independently of each other.

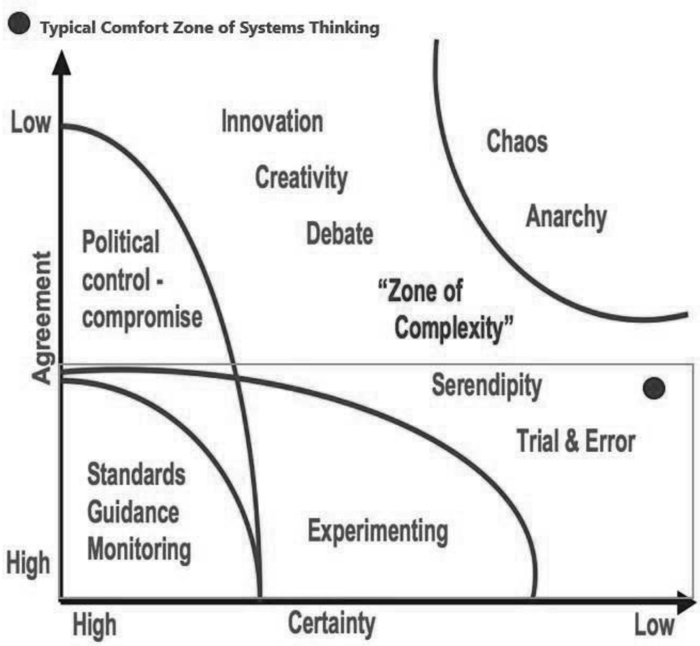

Figure 5: Zone of Complexity and the Present Comfort Zone

The quest for unconventional models of computation and proving techniques for CPS is an important area of research. It is a challenge to attempt the "Zone of Complexity" as shown in Figure 5 and factor as much of the zone not labeled the Comfort Zone.

3. Unconventional Computing

The methods of developing software for the CPS have severe limits such as:

- The Laws of Physics
- The Principles and Concepts of Software Engineering
- The Challenge of Algorithms and Expressing the Solutions for the Machine to Execute
- The Difficulty of Distribution, Decentralization, Centralization, Localization

- The Lack of Design Rules for Software
- The Difficulty of Factoring the Organization – Structure and Behavior
- The Economics of Development i.e Cost and Time
- The Influence of Politics
- The Limits of Human Imagination to work with Incomplete Information and Unstable Requirements

The crux of the proposed Unconventional proofing is to bring back the mind in the Turing machine model. There have been Physical, Chemical, Biochemical, Biological and Mathematical approaches to specify the Unconventional Computing Model with severe constraints. The generic model for Unconventional computing is elusive. The Cynefin Framework, is an interpretative model of the different levels of the systems complexity, ranging from order to disorder. The author uses this model to support the unconventional proofing as shown in the Figure 6.

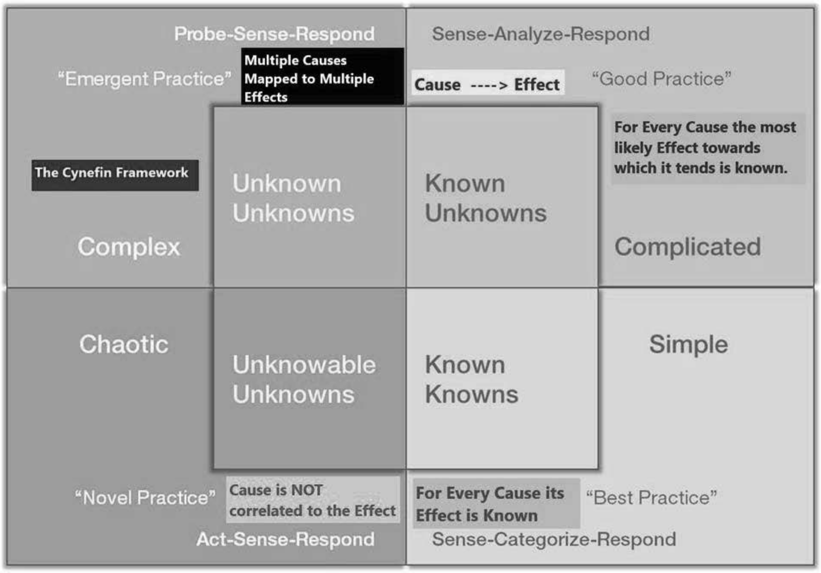

Figure 6: The Cynefin Framework for Unconventional Proofing

The framework is independent of the specifics of the technology to Sense, Choose, Categorize, Probe, Respond and Act that works with this.

3.1 The Need for Unconventional Proofs

The Latin phrase "quod erat demonstrandum [Q.E.D / QED]" placed at the end of a mathematical proof or a philosophical argument indicates that the proof or the argument is complete. In a Cyber – Physical System, the following three standards of proof are necessary in the increasing level of standardization. Please see the Figure 7.

1. Preponderance of the Evidence [50% Proven]
2. Clear and Convincing Evidence [> 70% Proven]

3. Proof Beyond a Reasonable Doubt [> 95% Proven]

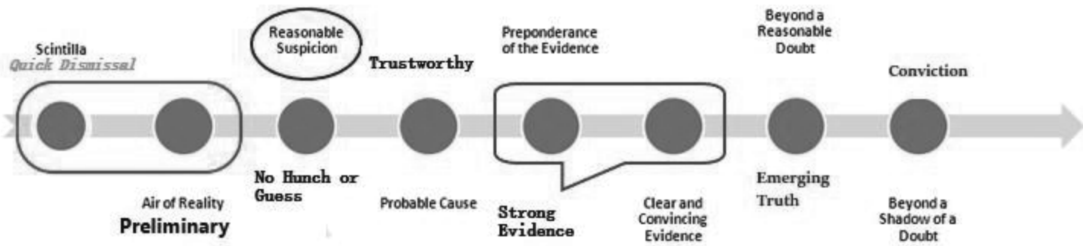

Figure 7: Levels of Standards of Proof

It is a fact that the term "point" is often left undefined in geometrytexts. It is easy for us to conceptualize a point, but it is quitedifficult to define exactly."The Chemical Basis of Morphogenesis" by Alan Turing describes the way in which natural patterns such as stripes, spots and spirals may arise out of a homogeneous, uniform state. Turing' theory that can be called a reaction–diffusion theory of morphogenesis, has served as a basic model in theoretical biology. Please see the Figure 8. Principles of Natural Selection or Artificial Selection govern the inclusion and a given CPS can evolve very slowly in this manner.

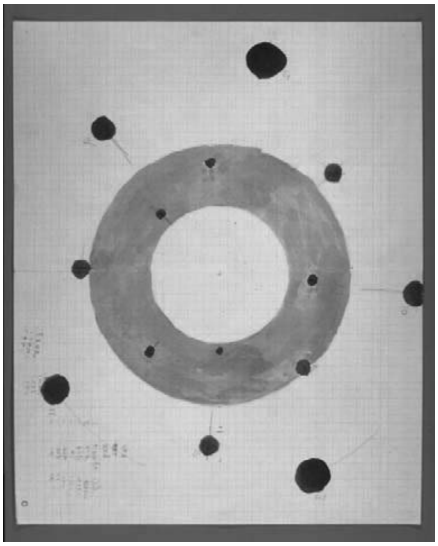

Figure 8: Turing's hand-drawn, hand-colored Chemical Basis of Morphogenesis Diagram

In a CPS, it is important to understand that a point is not a thing, but a place or a computational node. If a set of points all lie in a straight line, they are called 'collinear'. If a set of points all lie on the same plane, they are called 'coplanar'. For coplanar points, we need mythical rules of inclusion. Please see the Figure 9. These rules of inclusion can also be based on the theory of neural systems or fuzzy systems. The interactions among the points is a challenging computational problem. Please see Figure 10 for potential models of Virtual Organizations made from interactions.

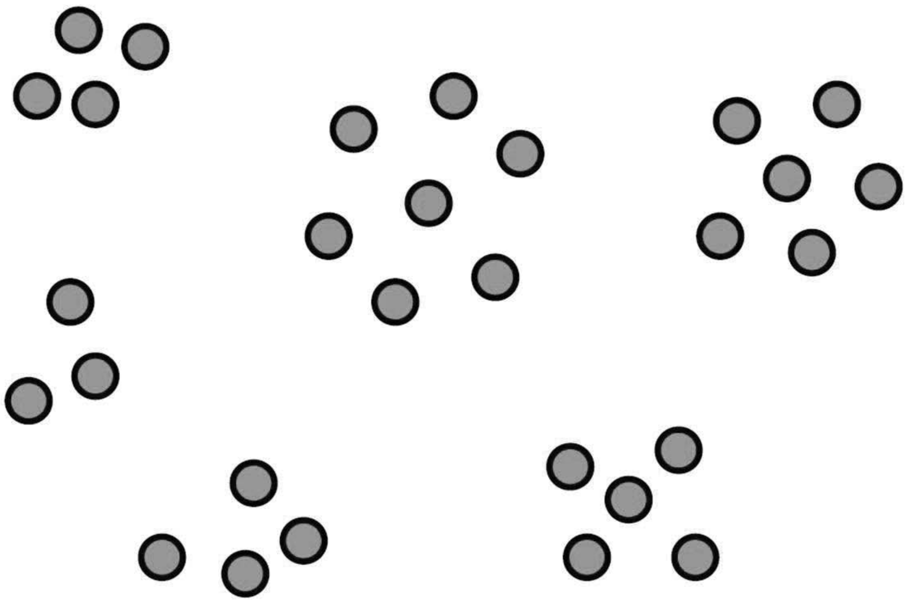

Figure 9: Mythical Rules of Inclusion

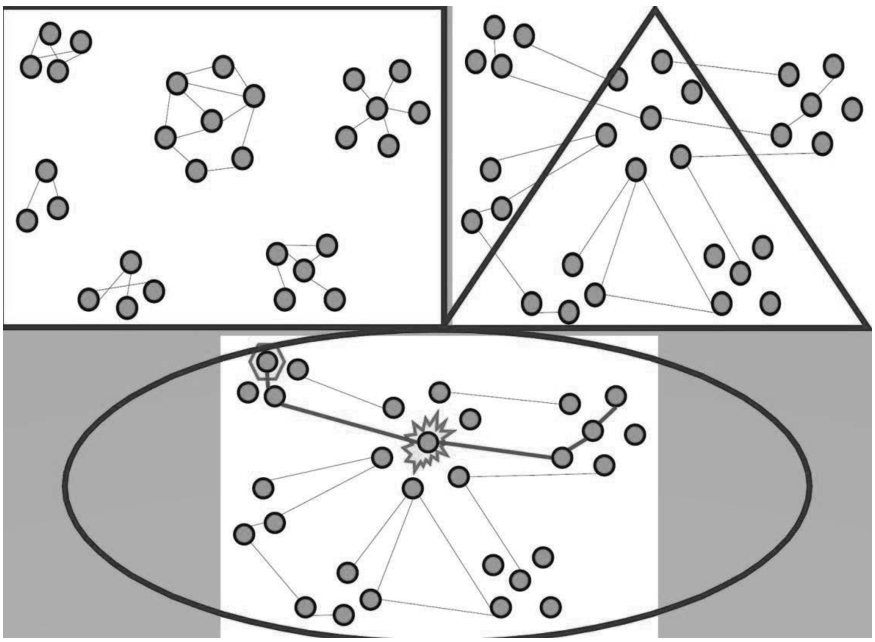

Figure 10: Finding the Optimal Interactions is Computationally Challenging

Bringing the context into the design of CPS is vital as seen in the Figure 11.

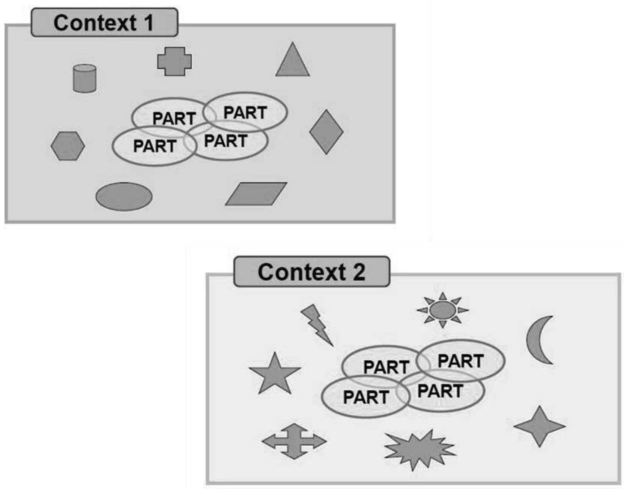

Figure 11: Bringing the Context into Design

Theoretically multiple – realities are possible, A model of CPS for geometric processing emerges with various standards of proof indicated in Figure 7 becoming apparent.

4. Physics and Metaphysics of CPS

The mathematical model of computation has always been challenged. The turing machine provided the algorithmic method of computation that can be formally verified and validated within the scope of the Universal Turing Machine. Problems can be classified based on the complexity classes. As Software Engineering progressed, the Conceptual Metaphor Theory was found useful to prepare the requirements and analyze them. Metaphors and more shallow concepts called "Similes" were soon found to be limited in scope for the purpose of design and development of Software. It is the Natural Language basis of Metaphors and Similies that was promising when a majority of the stakeholders providing the requirements were not from the Computing domain of specialization. The success rate for the software projects was not high.

These models are seldom conclusive when we are not Bayesian. The concept of "Rational Agent" is the cornerstone of classical decision theory. Completeness and Transitivity of Choice are difficult to establish. Desirability of a choice is the same as Utility. A study of Cause – Effect relationships shown in Figure 6 provides the basis for Physics into the proofing. A preliminary mathematical representation of the software development process is shown in Figure 12. Such a representation provide a very limited scope for using the Laws of Motion in the context of evolving complex software systems in CPS..

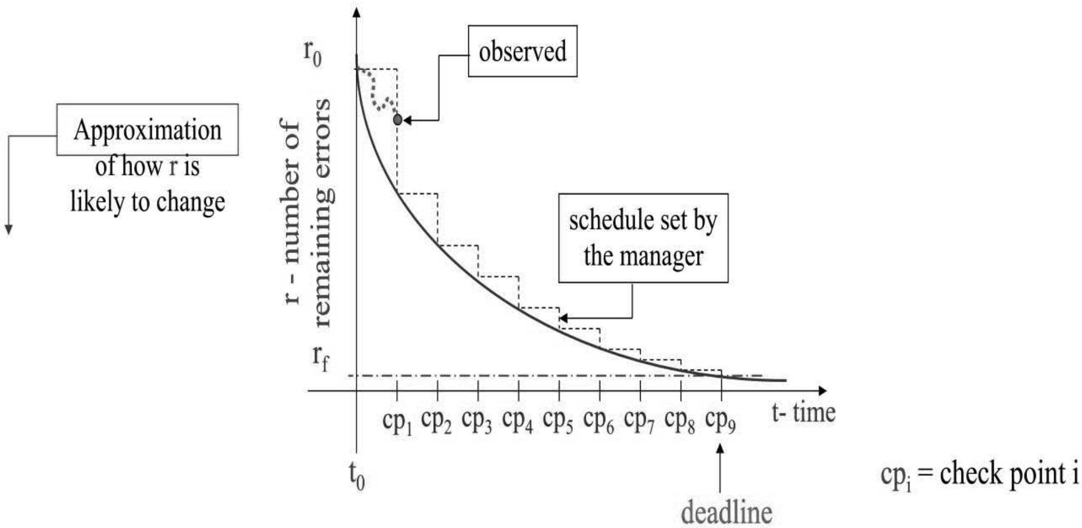

Figure 12: A simple mathematical representation of the Software Development process

Many mathematical objects, such as sets of numbers and functions, exhibit internal structure. The structural properties of these objects are investigated in the study of groups, rings, fields and other abstract systems, which are themselves such objects. This is the field of abstract algebra. Differential Geometry, Representation Theory, Algebraic Topology, and Algebraic Geometry are some computationally effective models for the virtual organizations and contexts of CPS indicated in Figure 10 and Figure 11. This is an unconventional proofing proposed by the author to mark the various standards of proof indicated in Figure 7.

Hempel's dilemma is the classical exploration of naturalism and physicalism. Physicalism avers that everything is physical. "Once every physical aspect of the world is settled or modeled, every other aspect will follow" forms the basis of proofing in Physics. This is far from being a satisfactory metaphysical conception of Physicalism. Even today, certain natural events that involve immaterial entities such as gods, angels and magical creatures in general are perceived to occur.

The mind-body problem is often described as the problem of explaining how the mind fits into the physical world. More generally, it is the problem of explaining the relationship between physical properties and mental properties. Is the "State" in the "Universal Turing Machine" the same as "State of Mind"? Is the computational model same for both? The answers to these questions have to resolve the Hempel's dilemma on the current physics will be discarded in the future, and the not yet known nature of the future physics. The present notion of State in a the Turing machine very limited in scope. It does not factor the human mind that can routinely solve the Towers of Hanoi problem while the mathematical answer is that it takes several times more the age of the universe to solve. The strange geometry of thought is the crux. The author opines that the model in Figure 13 is more suited for the purpose. The concept of geophilosophy, or to be more precise geo-metaphysics, is an enduring bond between the philosophical thought and its terrestial support for contextualization.

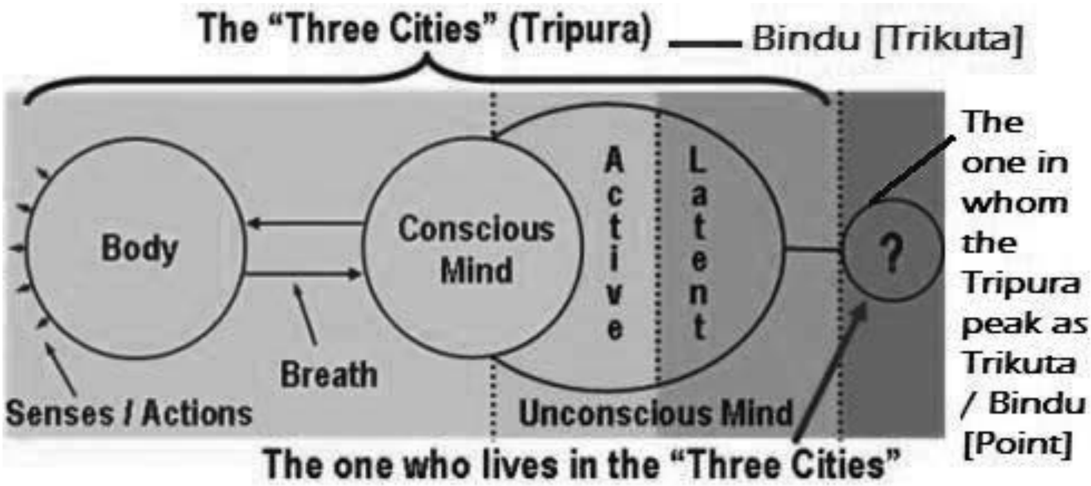

Figure 13: The Model for the Mind – Body

A Voronoi diagram is a partition of a plane into regions close to each of a given set of objects. In the simplest case, these objects are just finitely many points in the plane. For each seed there is a corresponding region consisting of all points of the plane closer to that seed than to any other. A point of view or a thought can be a **"Voronoi Tessellation".** A point in the CPS can be depicted as shown in Figure 14. In theory this can be the Bindu or Trikuta shown in Figure 13. It needs a simulator to study the resulting complex mathematical model for the CPS.

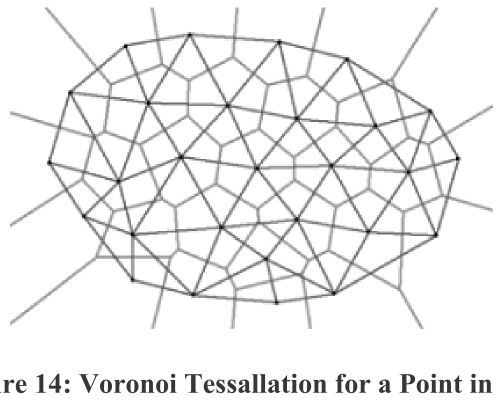

Figure 14: Voronoi Tessallation for a Point in CPS

5. Conclusions

There have been several attempts at specifying unconventional computing such as Reservoir Computing, Tangible Computing, Spintronics , Atomtronics, Fluidics and Chaos Computing. None of these have been a generic model for computation. In the context of CPS they even more restrictive. In this paper, the author proposes unconventional proofing in a CPS using

Neurotheology, Geometry, Physics and Metaphysics based of Indic studies in Consciousness. If the metaphysical dominates, the proof tends to be more experiential than expressive. These are the difficult questions related to Consciousness and modeling to thought and its seamless transmission to other receptive brains.

6. Acknowledgements

The author places on record sincere thanks to Anna University, various committees for the series of conferences such as Theory and Applications of Models of Computation [TAMC], Norbert Wiener in the 21st Century, Unconventional Modelling, Simulation & Optimization [UMSO] and International Symposium on Management Engineering [ISME].

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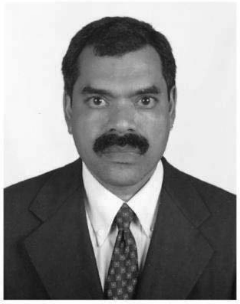

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1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting**: Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

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(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.
(73 − age) × ￥3,000
Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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