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ON FILTERS IN WEAK BCC-ALGEBRAS

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Abstract. The concept of weak BCC algebra (shortly: BZ - algebra) is a much-researched logic-algebraic entity. Observing and analyzing the substructures of this algebraic structure is important since some of them can be related to the congruences in such algebras. In this article we introduce and discuss the concept of BZ-filters in BZ-algebras in a slightly different way than is present in the literature. Also, we establish connection between BZ-ideals and BZ-filters. In addition, we consider several additional conditions imposed on BZ-filters and establish links between them.

1. Introduction

BCC-algebras, introduced by Y. Komori (see [**5, 6**]), are an algebraic model of BIK^+-logic, i.e., implicational logic. Many authors have tried to construct some generalizations of this and similar algebras. One such an algebraic system have the same partial order as BCC-algebras and BCK-algebras but has no minimal element. Such obtained system is called a BZ-algebra [**1, 9**] or a weak BCC-algebra [**3, 8**]. From the mathematical point of view the last name is more corrected but more popular is the first ([**4**]).

Many mathematicians studied such algebras as BCI-algebras, B-algebras, difference algebras, implication algebras, G-algebras, Hilbert algebras, d-algebras and many others. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras and, in fact, are generalization or a special case of weak BCC-algebras. So, results obtained for weak BCC-algebras are in some sense fundamental for these algebras, especially for BCC/BCH/BCI/BCK-algebras.

A very important role in the theory of such algebras plays ideals. Many types of ideals in these algebras have been studied with various relations between them [**1, 2, 4, 8, 9, 10**].

We will hereinafter use the BZ mark instead of the wBCC mark. In this article our intention is introduction of the concept of BZ-filters in BZ-algebras. In addition,

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2. Preliminaries

DEFINITION 2.1. A non-empty set A with a binary operation $' \cdot '$ and a distinguished element 0 is called a BZ-algebra (or a weak BCC-algebra) if the following axioms:

(BZ-1) $(\forall x, y, z \in A)(((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = 0)$,
(BZ-2) $(\forall x \in A)(x \cdot 0 = x)$,
(BZ-3) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot z = 0) \implies x = y)$.

are satisfied.

Similarly as in BCI-algebras in any BZ-algebra A we can introduce a natural partial order $' \leqslant '$ putting

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

It is not difficult to see that in BZ-algebras the following hold
(i) $(\forall x \in A)(x \cdot x = 0)$,
(ii) $(\forall x, y, z \in A)(x \leqslant y \implies (x \cdot z \leqslant y \cdot z \ \wedge \ z \cdot y \leqslant z \cdot x))$.

DEFINITION 2.2. ([**1**], Definition 2.3) Determine $\varphi : A \longrightarrow A$ by

$$(\forall x \in A)(\varphi(x) = 0 \cdot x).$$

The following lemma is true.

LEMMA 2.1 ([**1**], Lemma 2.2). *In any BZ-algebra we have*
(a) $(\forall x, y \in A)(\varphi(x \cdot y) \leqslant y \cdot x)$,
(b) $(\forall x \in A)(\varphi^2(x) \leqslant x)$,
(c) $(\forall x, y \in A)(\varphi(x) \cdot (y \cdot x) = \varphi(y))$,
(d) $(\forall x, y \in A)(\varphi(x \cdot y) \cdot \varphi(x) = \varphi^2(y))$,
(e) $(\forall x \in A)(\varphi^3(x) = \varphi(x))$,
(f) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi^2(x) \cdot \varphi^2(y))$,
(g) $(\forall x, y \in A)(x \leqslant y \implies \varphi(x) = \varphi(y))$,
(h) $(\forall x, y \in A)(\varphi^2(x \cdot y) = \varphi(y \cdot x))$,
(i) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot \varphi(x) \leqslant x \cdot ((y \cdot z) \cdot \varphi(z)))$.

3. Some types of ideals in wBCC-algebras

DEFINITION 3.1. ([**1**]) A non-empty subset J of a BZ-algebra A is a BZ-ideal if

(1) $0 \in J$
(2) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$.

In the following statement is given a fundamental property of BZ-ideal

THEOREM 3.1. *Let J be a BZ-ideal of a BZ-algebra A. Then*
(3) $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ y \in J) \implies x \in J)$
(4) $(\forall x, y \in A)((\varphi(y) \in J \ \wedge \ x \in J) \implies x \cdot y \in J)$.

Proof. If put $z = 0$ in (2) we get (3).
If we put $y = x$ and $z = y$ in (2) have

$$(\forall x, y \in A)(((x \cdot x) \cdot y = 0 \cdot y = \varphi(y) \in J) \wedge y \in J) \Longrightarrow x \cdot y \in J).$$

So, the statement (4) is proven. □

Corollary 3.1. *Let J be a BZ-ideal in a BZ-algebra A. Then*
(5) $(\forall x, y \in A)((x \leqslant y \wedge y \in J) \Longrightarrow x \in J)$.

In what follows, we remind the reader on some types of BZ-ideals

Definition 3.2. ([**1**], Definition 3.4; [**8**], Definition 68) An ideal BZ-ideal J of BZ-algebra A is called
- *closed* if $\varphi(J) \subseteq J$;
- $(*)$-*BZ-ideal* if $(\forall x, y \in A)((x \in J \wedge y \in A \backslash J) \Longrightarrow x \cdot y \in J)$;
- *anti-grouped* if $(\forall x \in A)(\varphi^2(x) \in J \Longrightarrow x \in J)$;
- *strong* if $(\forall xy \in A)((x \in J \wedge y \in A \backslash J) \Longrightarrow x \cdot y \in A \backslash J)$;
- *regular* if $(\forall x, y \in A)((x \cdot y \in J \wedge x \in J) \Longrightarrow y \in J)$;
- *associative* if $(\forall x, y \in A)(x \cdot \varphi(y) \in J \Longrightarrow y \cdot x \in J)$;
- T-*ideal* if $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

For any ideal J of a BZ-algebra A we can define a binary relation $' \prec '$ on A putting:

$$(\forall x, y \in A)(x \prec y \Longleftrightarrow x \cdot y \in J).$$

Such defined relation is a quasi-order relation on A left-compatible and right anti-compatible with the internal operation in A. Then the relation $' \sim '$ defined in A by $\sim = \prec \cap \prec^{-1}$ is a congruence on A. This can be proven in an analogous way as it was done in the article [**7**]. The set $A/\sim = \{[x]_\sim : x \in A\}$ is a BZ-algebra with respect to the internal operation $[x]_\sim \circ [y]_\sim = [x \cdot y]_\sim$ (for any $x, y \in A$) ([**1**]). In this algebra the relation $' \preceq '$ defined by $[x]_\sim \preceq [y]_\sim \Longleftrightarrow x \prec y$ is an order relation on $A/\sim$.

4. The concept of BZ-filters in wBCC-algebras

First, we introduce the concept of BZ filters in a BZ-algebra by the following definition looking at the way we made it into the article [**7**].

Definition 4.1. A subset F of a BZ-algebra A is a BZ-filter of A if
(6) $\neg(0 \in F)$,
(7) $(\forall x, y, z \in A)((\neg((x \cdot y) \cdot z \in F) \wedge x \cdot z \in F) \Longrightarrow y \in F)$.

The BZ-filter defined on this way has the following properties.

Theorem 4.1. *Let A be BZ-algebra and F a BZ-filter of A. Then*
(8) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge x \in F) \Longrightarrow y \in F)$,
(9) $(\forall x, y \in A)((x \cdot y \in F \wedge \neg(\varphi(y) \in F)) \Longrightarrow x \in F)$.

Proof. Putting $z = 0$ in (7) we obtain (8).
If we put $y = x$ and $z = y$ in (7), we have

$$(\neg((x \cdot x) \cdot y = 0 \cdot y = \varphi(y) \in F) \wedge x \cdot y \in F) \Longrightarrow x \in F.$$

Therefore, (9) is proved. □

COROLLARY 4.1. *Let F be a BZ-filter of a BZ-algebra A. Then*
(10) $(\forall x, y \in F)((x \leqslant y \wedge x \in F) \implies y \in F)$

PROOF. Let $x, y \in A$ be arbitrary elements such that $x \leqslant y$ and $x \in F$. Thus $\neg(x \cdot y = 0 \in F)$ and $x \in F$. Then by (8) we have $y \in F$. □

THEOREM 4.2. *If F is a BZ-filter of BZ-algebra A, then the set $J = A \backslash F$ is a BZ-ideal. Opposite, if J is a BZ-ideal of BZ-algebra A, then the set $F = A \backslash J$ is a BZ-filter of A.*

PROOF. It is clear that $0 \in J$. Let $x, y, z \in A$ be arbitrary elements such that $(x \cdot y) \cdot z \in J$ and $y \in J$. Then we have $\neg((x \cdot y) \cdot z \in F)$ and $\neg(y \in F)$. If we suppose that $x \cdot z \in F$ by (7) we will have $y \in F$. So, must to be $\neg(x \cdot z \in F)$.

Opposite, let J be a BZ-ideal of A. It is that $\neg(0 \in A \backslash J)$. Let $x, y, z \in A$ be arbitrary elements such that $\neg((x \cdot y) \cdot z \in A \backslash J)$ and $x \cdot z \in A \backslash J$. Then $y \in A \backslash J$. Indeed. If it were $y \in J$ then $x \cdot z \in J$ follows from $(x \cdot y) \cdot z \in J$ and the $y \in J$ which is in a contradiction with the assumption $x \cdot z \in A \backslash J$. □

THEOREM 4.3. *The family $\mathfrak{F}_A$ of all BZ-filters in BZ-algebra A forms a completely lattice.*

PROOF. Let $\{F_{i \in I}\}$ be a family of BZ-filters in BZ-algebra A. It is clear that $\neg(0 \in \bigcap_{i \in I} F_i)$ and $\neg(0 \in \bigcup_{i \in I} F_i)$.

(a) Let $x, y, z \in A$ arbitrary elements such that $\neg((x \cdot y) \cdot z \in \bigcup_{i \in I} F_i)$ and $x \cdot z \in \bigcup_{i \in I} F_i$. Thus $\neg((x \cdot y) \cdot z \in F_i)$ for all $i \in I$ and there exists an index $j \in I$ such that $x \cdot z \in F_j$. Then $y \in F_j \subseteq \bigcup_{i \in I} F_i$.

(b) Let $\mathfrak{X}$ be the family of all BCC-filters which contained in the intersection $\bigcap_{i \in I} F_i$. The union $\bigcup \mathfrak{X}$ is the minimal BCC-filter contained in the intersection $\bigcap_{i \in I} F_i$.

(c) So, if we choose $\sqcap_{i \in I} F_i = \bigcup \mathfrak{X}$ and $\sqcup_{i \in I} F_i = \bigcup_{i \in I} F_i$, then $(\mathfrak{F}, \sqcap, \sqcup)$ is a completely lattice. □

5. Some types of BZ-filters

In the theory of algebras, important role play ideals. Filters in these algebras can also be important since they are highly related to the ideals. Our intent in this section is to determine the various types of filters in the BZ-algebras and establish the relationships between them.

In what follows, we analyze the conditions that a BZ-filter could satisfy. Marks for these conditions we borrowed from the corresponding ideals in BZ-algebras since we have been given them by modification of the logical atoms in the formulas that are determined these ideals.

(T) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in F) \wedge x \cdot z \in F) \implies y \in F)$.
(∗) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \implies x \in F$;
(R) $(\forall x, y \in A)((\neg(x \in F) \wedge y \in F) \implies xy \in F)$;
(S) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge y \in F) \implies x \in F)$;
(A) $(\forall x, y \in A)(x \cdot y \in F \implies y \cdot \varphi(x) \in F)$;

(C) $(\forall x \in A)(\varphi(x) \in F \implies x \in F)$;
(AG) $(\forall x \in A)(x \in F \implies \varphi^2(x) \in F)$.

Our first proposition refers to BZ-filter which satisfies condition (T).

Proposition 5.1. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (T). *Then*
(T1) $(\forall x, y \in A)(x \cdot \varphi(y) \in F \vee \neg(x \cdot y \in F))$.

Proof. If we put $y = 0$ and $z = y$ in formula (T), we get the following

$$(\forall x, y \in A)((\neg(x \cdot \varphi(y) \in F) \wedge x \cdot y \in F) \implies 0 \in F)$$

which is a contradiction. So, we have

$$(\forall x, y \in A)\neg(\neg(x \cdot \varphi(y) \in F) \wedge x \cdot y \in F).$$

Therefore, for any BZ-filter F satisfying condition (T) have to be

$$(\forall x, y \in A)(x \cdot \varphi(y) \in F \vee \neg(x \cdot y \in F))$$

□

Corollary 5.1. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (T). *Then*
(T2) $(\forall y \in A)(\varphi^2(y) \in F \vee \neg(\varphi(y) \in F))$.

Proof. Putting $x = 0$ in (T1) we get (T2). □

Our second proposition relates to a BZ-filter that satisfies the condition $(*)$

Proposition 5.2. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* $(*)$. *Then*
$(*1)$ $(\forall y \in A)(\neg(\varphi(y) \in F) \vee \neg(y \in F))$.

Proof. If we put $x = 0$ in formula $(*)$ we get the following

$$(\forall y \in A)((0 \cdot y \in F \wedge y \in F) \implies 0 \in F)$$

which is a contradiction. So, we have

$$(\forall y \in A)\neg(\varphi(y) \in F \wedge y \in F).$$

Therefore, for any BZ-filter F satisfying the condition $(*)$, the condition $(*1)$ is valid also. □

Proposition 5.3. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (R). *Then*
(R1) $(\forall y \in A)(y \in F \implies \varphi(y) \in F)$.

Proof. If we put $x = 0$ in the formula (R) we have $(\neg(0 \in F) \wedge y \in F) \implies \varphi(y) \in F$. Since the condition $\neg(0 \in F)$ is valid by (6), for any $y \in A$ we have $y \in F \implies \varphi(y) \in F$. So, for any BZ-filter in BZ-algebra A satisfying the condition (R), the formula (R1) is valid. □

Our third proposition refers to the BZ-filter that satisfies the condition (S).

PROPOSITION 5.4. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (S). *Then*
(S1) $(\forall y \in A)(\varphi(y) \in F \ \vee \ \neg(y \in F))$.

PROOF. Putting $x = 0$ in (S) we get the following

$$(\forall y \in A)((\neg(\varphi(y) \in F) \ \wedge \ y \in F) \implies 0 \in F).$$

We got a contradiction. This contradiction negates the hypothesis in the previous implication. So, have to be $\neg(\neg(\varphi(y) \in F) \ \wedge \ y \in F)$. Therefore, the following $\varphi(y) \in F \ \vee \ \neg(y \in F)$ is proven for any $y \in A$. □

Our next assertion refers to the BZ filter in the BZ algebra that satisfies condition (A).

PROPOSITION 5.5. *Let F be a BZ-filter in a BZ-algebra A satisfying the condition* (A). *Then*
(A1) $(\forall x \in A)(x \in F \implies \varphi^2(x) \in F)$, *and*
(A2) $(\forall y \in A)(\varphi(y) \in F \implies y \in F)$.

PROOF. Putting $y = 0$ in (A) we get (A1). If put $x = 0$ in (A) we will get (A2). □

With (mark) we mark any of the conditions mentioned above. We will write $F \in (mark)$ if we want to say that the BZ-filter F satisfies the condition $(mark)$. The obtained results in the preceding propositions allow us to make a summary of the interdependence of the conditions that we intend BZ-filters satisfies.

COROLLARY 5.2. *For any BZ-filter F of BZ-algebra A the following holds*

$$F \in (A) \Longrightarrow F \in (C) \cap (AG).$$

Before we expose the first theorem about the BZ-filters in the BZ-algebra, we recall the readers to the term 'consistent subset': For a subset X of a algebra A we say that it is a *consistent subset* in A if and only if the following is valid

$$(\forall x, y \in A)(x \cdot y \in X \implies (x \in X \ \vee \ y \in X)).$$

THEOREM 5.1. *A BZ-filter F of BZ-algebra A satisfies the condition (C) if and only if F is a consistent subset of A.*

PROOF. Assume that a BZ-filter F is a consistent subset in A. Let $x \in A$ be an arbitrary element such that $\varphi(x) \in F$. Thus from $0 \cdot x \in F$ follows $0 \in F$ or $x \in F$. Since the first option is impossible by (6), we have $x \in F$. So, $F \in (C)$.

Conversely, Let $F \in (C)$ be holds for a BZ-filter F of a BZ-algebra A and let $x, y \in A$ be arbitrary elements such that $x \cdot y \in F$. We have two options:

(i) Suppose $x \cdot y \in F$ and $\neg(y \in F)$. Thus $\neg(\varphi(y) \in F)$. Then $x \in F$ by (9).

(ii) Let $x \cdot y \in F$ and $\neg(x \in F)$ is valid. If we suppose $\neg(y \in F)$, thus $\neg(\varphi(y) \in F)$ is valid. Then from $x \cdot y \in F$ and $\neg(\varphi(y) \in F)$ would get $x \in F$ according to (9). We got a contradiction. Therefore, it must be $y \in F$.

Finally, the filter F is a consistent subset of A. □

THEOREM 5.2. *For a BZ-filter F of BZ-algebra A the following holds*

$$F \in (C) \cap (*) \text{ if and only if } (\forall x \in A)\neg(\varphi(x) \in F).$$

Proof. Suppose $(\forall x \in A)\neg(\varphi(x) \in F)$ holds. Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in F$ and $y \in F$. Then $\neg(\varphi(y) \in F)$ by hypothesis. Thus from $x \cdot y \in F$ and $\neg(\varphi(y) \in F)$follows $x \in F$ by (9). So, $F \in (*)$.

Further, suppose $\varphi(x) \in F$ and $\neg(x \in F)$. Thus $x \in A \backslash F$ and $\neg(\varphi(x) \in F)$. We got a contradiction. So, have to be $x \in F$. So, $F \in (C)$.

Opposite, suppose (C) and $(*)$ hold. Let $x \in A$ be an arbitrary element. If we suppose $\varphi(x) \in F$, thus $x \in F$ by the condition (C). On the other side, from $0 \cdot x \in F$ and $x \in F$ follows $0 \in F$. We got a contradiction. So, have to be $\neg(\varphi(x) \in F)$. Therefore, for a BZ-filter F satisfies conditions (C) and $(*)$ the following $(\forall x \in A)\neg(\varphi(x) \in F)$ holds. □

Theorem 5.3. *For a BZ-filter F of a BZ-algebra A the following are equivalent:*
(i) $F \in (AG)$;
(ii) $(\forall x, y \in A)((y \in F \wedge x \leqslant y) \implies x \in F)$;
(iii) $(\forall x, y, z \in A)((x \in F \wedge \neg(y \in F)) \implies (x \cdot z) \cdot (y \cdot z) \in F)$;
(iv) $(\forall x, z \in A)(x \in F \implies (x \cdot z) \cdot \varphi(z) \in F)$.

Proof. (i) $\implies$ (ii). L$y \in F$ and $x \leqslant y$. Thus $x \cdot y = 0$. Suppose $y \cdot x \in F$. Then $\varphi^2(y \cdot x) \in F$ because $F \in (AG)$. From this follows $0 = \varphi(0) = \varphi(x \cdot y) = \varphi^2(y \cdot x) \in F$ by (h). We got a contradiction. So, $\neg(y \cdot x \in F)$. Now, from $\neg(y \cdot x \in F)$ and $y \in F$ follows $x \in F$ by (8).

(ii) $\implies$ (iii). Suppose (ii) holds. Let $x, y \in A$ be arbitrary elements such that $x \in F$ and $\neg(y \in F)$. Thus $x \cdot y \in F$. Indeed. If there were $\neg(x \cdot y \in F)$ then from $\neg(x \cdot y \in F)$ and $x \in F$ would follow $y \in F$ by (8). It would have been contradictory. Therefore, it must be $x \cdot y \in F$. Since $(x \cdot z) \cdot (y \cdot z) \leqslant x \cdot y$ by (BZ-1) and $x \cdot y \in F$ we have $(x \cdot z) \cdot (y \cdot z) \in F$, by (ii).

(iii) $\implies$ (iv). Putting $y = 0$ we get (iv).

(iv) $\implies$ (i). Putting $z = x$ we get (i). □

Theorem 5.4. *A BZ-filter F of a BZ-algebra A satisfies condition (C) and (AG) if and only if for every $x \in A$ both x and $\varphi(x)$ belong or not belong to F.*

Proof. Suppose $F \in (C) \cap (AG)$. Let $x \in A$ be an arbitrary element such that $x \in F$. Thus $\varphi^2(x) \in F$ by (AG). Then $\varphi(x) \in F$ by (C). On the other hand, if $\varphi(x) \in F$ then $x \in F$ according to (c). Therefore, both x $\varphi(x)$ belong or not belong to F.

Conversely, let a BZ-filter F has the property that both x and $\varphi(x)$ belong or not belong to F for any $x \in A$. Suppose $x \in F$ and $\varphi(x) \in F$. Thus $\varphi^2(x) \in F$ again. So, F obviously satisfy the condition (C) and the condition (AG). Analogously, the second option can be demonstrated. □

Theorem 5.5. *For any BZ-filter F of a BZ-algebra A the following holds*

$$F \in (S) \iff F \in (C) \cap (AG).$$

Proof. Suppose $F \in (S)$.

Let $x \in A$ be arbitrary element such that $x \in F$. Since $\varphi^2(x) \leqslant x$ by (b), from $\neg(\varphi^2(x) \cdot x = 0 \in F)$ and $x \in F$ follows $\varphi^2(x) \in F$ by (S). So, $F \in (AC)$.

Let $x \in A$ be arbitrary element such that $\varphi(x) \in F$. Suppose $\neg(\varphi^2(x) \in F)$. Thus $0 \in F$ by the condition (S). So, it have to be $\varphi^2(x) \in F$. Since $\varphi^2(x) \leqslant x$, it follows $x \in F$ by Corollary 4.1. Therefore, $F \in (C)$.

Suppose $F \in (C)$ and $F \in (AG)$. Let $x, y \in A$ be arbitrary elements such that $\neg(x \cdot y \in F)$ and $y \in F$. Suppose that $\neg(x \in F)$. Thus $\varphi^2(y) \in F$ by (AG). Further on, then by (d) $\varphi^2(y) = \varphi(x \cdot y) \cdot \varphi(x) \in F$ and $\varphi(x \cdot y) \in F$ or $\varphi(x) \in F$ by Theorem 5.1 and $x \cdot y \in F$ or $x \in F$. Since both cases are in contradictions with hypothesis, then it must be $x \in F$. Finally, $F \in (S)$. □

THEOREM 5.6. *For any BZ-filter F of a BZ-algebra A the following holds*

$$F \in (R) \iff F \in (C) \cap (AG).$$

PROOF. Suppose $F \in (R)$.

Let $x \in A$ is an arbitrary element such that $x \in F$ and $\neg(\varphi^2(x) \in F)$. Thus $\varphi^2(x) \cdot x \in F$ by (R). On the other hand, according to (b), we have $\varphi^2(x) \cdot x = 0$ and $\neg(\varphi^2(x) \cdot x \in F)$ We got a contradiction. So, it must be $\varphi^2(x) \in F$. This means $F \in (AG)$.

Suppose $\varphi(x) \in F$. But, from $\neg(0 \in F)$ and $\varphi(x) \in F$ follows $\varphi^2(x) = 0 \cdot \varphi(x) \in F$ by (R) Further, from $\varphi^2(x) \leqslant x$ and $\varphi^2(x) \in F$ follows $x \in F$ by Corollary 4.1. So, $F \in (C)$.

Opposite, suppose $F \in (C) \cap (AG)$ holds. Let $x, y \in A$ be arbitrary elements such that $\neg(x \in F)$ and $y \in F$. Thus $\neg(\varphi(c) \in F)$ by (C) and $\neg(\varphi^2(x) \in F)$ by (C) again. Besides we have $\varphi^2(y) \in F$ from $y \in F$ by (AG). On the other hand we have $\varphi^2(y) = \varphi(x \cdot y) \cdot \varphi(x) \in F$ by (d). Now, from $\varphi(x \cdot y) \cdot \varphi(x) \in F$ and $\neg(\varphi^2(x) \in F)$ follows $\varphi(x \cdot y) \in F$ by (9). Finally, thus $x \cdot y \in F$ by (C). Therefore, $F \in (R)$. □

COROLLARY 5.3. *For any BZ-filter F of BZ-algebra A the following holds*

$$F \in (S) \iff F \in (R).$$

THEOREM 5.7. *For any BZ-filter of a BZ-algebra the following conditions are equivalent:*

(T) $(\forall x, y, z \in F)(\neg(x \cdot (y \cdot z) \in F) \wedge x \cdot z \in F) \implies y \in F)$,

(Ta) $(\forall x, z \in A)(x \cdot z \in F \implies x \cdot \varphi(z) \in F)$,

(Tb) $(\forall x, z \in A)(x \cdot \varphi^2(z) \in F \implies x \cdot \varphi(z) \in F)$,

(Tc) $(\forall x \in A)\neg(\varphi(x) \cdot x \in F)$.

PROOF. (T) $\implies$ (Ta). Putting $y = 0$ in (T) we obtain (T1). Thus (Ta).

(Ta) $\implies$ (T). Suppose (Ta). Let $x, z \in A$ be arbitrary elements such that $\neg(x \cdot (y \cdot z) \in F$ for some $y \in A$ and $x \cdot z \in F$. Thus $x \cdot \varphi(z) \in F$. On the other hand, if we put $x = y$ and $y = 0$ in (BZ-1) we get $(y \cdot z) \cdot (0 \cdot z) \leqslant y \cdot 0$. Again, if we put $y = y \cdot z$ and $z = \varphi(z)$ in (BZ-1) we get $(x \cdot \varphi(z)) \cdot ((y \cdot z)\varphi(z)) \leqslant x \cdot (y \cdot z)$. Since $\neg(x \cdot (y \cdot z) \in F)$ we have $\neg((x \cdot \varphi(z)) \cdot ((y \cdot z)\varphi(z)) \in F)$. Now, from the last formula and $x \cdot \varphi(z) \in F$ we get $(y \cdot z) \cdot \varphi(z) \in F$ by (8). Again, from $(y \cdot z) \cdot \varphi(z) \leqslant y$ we get $y \in F$ by Corollary 4.1.

(Ta) $\Longrightarrow$ (Tb). Putting $z = \varphi^2(z)$ in (Ta) we obtain $x \cdot \varphi^3(z) \in F$ and $x \cdot \varphi(z) \in F$ by (e).

(Tb) $\Longrightarrow$ (Ta). Let $x, z \in A$ be arbitrary elements such that $x \cdot z \in F$. Since $\varphi^2(z) \leqslant z$ by (b), we have $x \cdot z \leqslant x \cdot \varphi^2(z)$. Thus $x \cdot \varphi^2(z) \in F$ by Corollary 4.1 and $x \cdot \varphi(z) \in F$ by (Tb).

(T) $\Longrightarrow$ (Tc). By Proposition 5.1 we have (T) $\Longrightarrow$ (T1). Putting $x = \varphi(y)$ in (T1) we obtain $0 = \varphi(y) \cdot \varphi(y) \in F$ or $\neg(\varphi(y) \cdot y \in F)$. Since, the first option is impossible, we have $\neg(\varphi(y) \cdot y \in F)$.

(Tc) $\Longrightarrow$ (Ta). Let (Tc) be holds. Let $x, y, z \in A$ be arbitrary elements such that $x \cdot z \in F$. Suppose $\neg(x \cdot \varphi(z) \in F)$. On the other hand, from (BZ-a) with $y = \varphi 9z)$ we have $(x \cdot z) \cdot (\varphi(z) \cdot z) \leqslant x \cdot \varphi(z)$. Since $\neg(x \cdot \varphi(z) \in F)$, then $\neg((x \cdot z) \cdot (\varphi(z) \cdot z) \in F$ by Corollary 4.1. Now, from $\neg((x \cdot z) \cdot (\varphi(z) \cdot z) \in F$ and $x \cdot z \in F$ follows $\varphi(z) \cdot z \in F$ by (8). We got a contradiction. Therefore, it have to be $x \cdot \varphi(z) \in F$. So, we are proven (Ta). □

6. Final Observation

In this text, we try to develop the theory of filters in the BZ algebras. First, we introduce the concept of the BZ filters in the BZ algebra (Definition 4.1). In addition, we analyze some of the additional conditions that we assume that BZ-filters can satisfy them. In addition, we analyze some interrelations between these additional conditions (Theorems 5.1 - 5.7). The author is convinced that this analysis enriches our knowledge of BZ-algebras.

By the fluctuation of logic atoms in the formulas that determine the particular types of ideals in these algebra it can be obtained some other types of filters in the weak BCC-algebras. Some of these additional conditions that can be imposed on BZ-filters in BZ-algebra are shown below

(O) $(\forall x, y \in A)((\neg(x \cdot y \in F) \wedge y \in F) \Longrightarrow \neg(x \in F))$,

(I) $(\forall x, y, z \in A)(\neg((x \cdot y) \cdot z \in F) \wedge x \cdot z \in F) \Longrightarrow y \cdot z \in F)$

(As) $(\forall x, y, z \in A)((\neg((x \cdot y) \cdot z \in F) \wedge x \in F) \Longrightarrow y \cdot z \in F)$.

It is immediately apparent that if we put $z = 0$ in (I) and (As) we get (8) in both cases.

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DYNAMICAL SYSTEM ON A PARABOLIC AND ELLIPTIC GELFAND-TYPE EQUATION

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Abstract. We consider the parabolic equation with exponential nonlinearity and the corresponding elliptic equation. First, we study the set of stationary solution and its spectral property. Next we show that the solution of parabolic equation blows up in finite time for the initial value satisfying a positive integrand condition by the Kaplan method. Finally we find a global solution for the negative initial value by upper-lower solution method and for the two dimensional domain by the Trudinger-Moser inequality, respectively. By the global boundedness and the existence of Lyapunov function, we treat its dynamical properties of the omega limit set.

1 Introduction We consider the parabolic equation

$$\begin{cases} u_t = \Delta u + \lambda\,(e^u - 1) & x \in \Omega,\ \ t \in (0, T_{u_0}), \\ u(x,t) = 0 & x \in \partial\Omega,\ \ t \in (0, T_{u_0}), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{1}$$

and the associated elliptic equation

$$\begin{cases} \Delta v + \lambda\,(e^v - 1) = 0 & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$ and T_{u_0} denotes the maximal existing time for the solution of (1) with $u(x,0) = u_0(x)$. In the case of $n = 1$, we suppose that $\Omega = (0,1)$. The corresponding nonlocal parabolic and elliptic problems with the Neumann boundary condition are given by

$$\begin{cases} u_t = \Delta u + \lambda\Big(\frac{e^u}{\int_\Omega e^u\,dx} - \frac{1}{|\Omega|}\Big) & x \in \Omega,\ \ t \in (0, T_{u_0}), \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & x \in \partial\Omega,\ \ t \in (0, T_{u_0}), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{3}$$

and

$$\begin{cases} \Delta v + \lambda\Big(\frac{e^v}{\int_\Omega e^v\,dx} - \frac{1}{|\Omega|}\Big) = 0 & x \in \Omega, \\ \frac{\partial v}{\partial \nu}(x) = 0 & x \in \partial\Omega, \end{cases} \tag{4}$$

respectively, where $|\Omega|$ is the measure of Ω in $\mathbb{R}^n$ and $\nu(x)$ is the outer unit normal vector at $x \in \partial\Omega$. (3) and (4) for $n = 1$ are investigated in [12, 17]. We have already obtained the results of the elliptic properties such as the structure of set of stationary solutions and the monotonicity of the Morse index. In the parabolic problem, for any $\lambda > 0$ and $u_0 \in H$ with an appropriate space H, (3) admits a unique global solution. However, by the lack of comparison principle we do not know whether the Morse-Smale property holds or not for

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the nonlocal problem.
Nowadays, it seems that there are not enough studies for (1) and (2) except [1]. The aim of this paper is to study (1) and (2), respectively. The solution set of (2) has already been investigated in [1]. To introduce the known results obtained in [1] and state our theorems, we denote the m-th eigenvalue and eigenfunction of $-\Delta$ in Ω with the Dirichlet boundary condition by μ^m and ϕ^m normalized as $\|\phi^m\|_2 = 1$ for $m \in \mathbb{N}$, respectively, where $\|\cdot\|_p$ is the standard L^p norm in Ω with $p \in [1, \infty]$. For the sake of convenience, we set $\mu^0 = 0$. We define the solution set $\mathcal{C}$ by

$$\mathcal{C} \equiv \Big\{(\lambda, v) \in \mathbb{R}^+ \times \big(C^2(\Omega) \cap C_0(\overline{\Omega})\big) \;\big|\; v = v(x) \text{ solves (2) for } \lambda > 0\Big\},$$

where $\mathbb{R}^+ = \{x \mid x > 0\}$ and

$$C_0(\overline{\Omega}) \equiv \Big\{v \in C(\overline{\Omega}) \;\big|\; v(x) = 0 \text{ on } x \in \partial\Omega\Big\}$$

endowed with the L^∞ norm. In [1], they derived the necessary condition for the existence of positive classical solution of (2). Together with their results, we have the following proposition:

Proposition 1 (Cf. Proposition 1.2 in [1]) *Let $\Omega \subset \mathbb{R}^n$ be a star-shaped domain with respect to the origin with C^2- boundary $\partial\Omega$ for $n \in \mathbb{N}$. There exists λ_0 which depends only on Ω satisfying $\lambda_0 \geq 0$ for $n = 1, 2$ and $\lambda_0 > 0$ for $n \geq 3$. Then we have following:*
(i) *if $(\lambda, v) \in \mathcal{C}$ satisfies $v > 0$ in Ω, then $\lambda \in (\lambda_0, \mu^1)$,*
(ii) *if $(\lambda, v) \in \mathcal{C}$ satisfies $v < 0$ in Ω, then $\lambda > \mu^1$,*
(iii) *if $n \geq 3$ and $\lambda \in (0, \lambda_0)$, then $(\lambda, v) \in \mathcal{C}$ satisfies $v = 0$ in Ω.*

If $(\lambda, v) \in \mathcal{C}$ is a classical solution, the Morse index $i = i(\lambda, v)$ is defined by the number of negative eigenvalues ν of

$$\text{(5)} \qquad \begin{cases} \Delta\psi + \lambda e^v \psi = -\nu\psi & x \in \Omega, \\ \psi(x) = 0 & x \in \partial\Omega, \\ \|\psi\|_2 = 1. \end{cases}$$

First of all, we introduce results of the stationary solution. It is clear that (2) has a trivial solution $(\lambda, v) = (\lambda, 0)$ for any $\lambda > 0$. The second proposition is concerned with the bifurcation from the trivial solution and Morse index around the bifurcation point. The result for (2) is a little bit similar to that for (4). The difference is the value of the Morse index on the branch of the nontrivial solution set. We prove the existence of nontrivial solution by the bifurcation theory [2]. We compute Morse index by the exchange of eigenvalues [11, 12, 13].

Proposition 2 *Let $\Omega = (0, 1)$. Then we have $\mu^m = (m\pi)^2$ and $i(\lambda, 0) = m - 1$ for $\lambda \in (\mu^{m-1}, \mu^m]$ with $m \in \mathbb{N}$. Two continua $\mathcal{S}_m^{\pm} \subset \mathcal{C}$ of nontrivial solution bifurcate at $(\lambda, v) = (\mu^m, 0)$. Furthermore*

$$i(\lambda, v) = \begin{cases} 2k - 2 & \textit{for } (\lambda, v) \in \mathcal{S}_m^- \textit{ and } m = 2k - 1, \\ 2k - 1 & \textit{for } (\lambda, v) \in \mathcal{S}_m^+ \textit{ and } m = 2k - 1, \\ 2k - 1 & \textit{for } (\lambda, v) \in \mathcal{S}_m^{\pm} \textit{ and } m = 2k \end{cases}$$

holds for sufficiently close to the bifurcation point $(\mu^m, 0)$, where $k \in \mathbb{N}$.

In [1], they studied the bifurcation diagram and computed the bound for Morse index globally, not locally around a bifurcation point. If Ω is a unit ball and the solution is positive and radially symmetric, they establish the existence of singular solution, multiple existence of the regular solution and bound for its Morse index. Their results are similar to those of the well-known Gelfand problem

$$\Delta v + \lambda e^v = 0.$$

As mentioned in [6, 15], they also derived the bending result of the solution set for $n \in [3, 9]$. Next, we consider the case where the solution blows up in finite time. Thanks to the convexity of $e^v - 1$, we can apply the Kaplan method [8] to obtain following two blow-up conditions of u_0.

Proposition 3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. For $u_0 \in C_0(\overline{\Omega})$, the solution of (1) blows up in finite time on the condition that*
(i) $\int_\Omega u_0(x)\phi^1(x)\,dx > 2\left\|\phi^1\right\|_1\left(\mu^1 - \lambda\right)/\lambda$ *holds for* $0 < \lambda < \mu^1$,
(ii) $\int_\Omega u_0(x)\phi^1(x)\,dx > 0$ *holds for* $\lambda \geq \mu^1$.

The first global existence result is concerned with the nonpositive solution. For the nonpositive initial value, we establish the global solution by constructing a lower-upper solution pair.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. For $u_0 \in C_0(\overline{\Omega})$ with $u_0(x) \leq 0$ for any $x \in \Omega$, we have $T_{u_0} = +\infty$ and*

$$u \in C\big([0, +\infty); C_0(\overline{\Omega})\big) \cap C^1\big((0, +\infty); C_0(\overline{\Omega})\big)$$

satisfying $u \leq 0$ in $\Omega \times [0, +\infty)$. If $\lambda < \mu^1$, then

$$\|u(\cdot, t)\|_{H_0^1} \to 0$$

as $t \to +\infty$, where $\|w\|_{H_0^1} = \|\nabla w\|_2$ for $w \in H_0^1(\Omega)$.

We introduce the main theorems on the global existence for $\Omega \subset \mathbb{R}^2$ and $\Omega = (0, 1)$, respectively. For small parameter and initial value, we construct the global solution by the Lyapunov function

$$L_\lambda(u) \equiv \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega (e^u - u)\,dx, \tag{6}$$

Sobolev and Trudinger-Moser inequalities.

Theorem 2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\lambda > 0$ and $u_0 \in H_0^1(\Omega)$ satisfying*

$$\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + \|u_0\|_{H_0^1}^2 < 4\pi\left(\log 4\pi - 1\right), \tag{7}$$

we have $T_{u_0} = +\infty$ and

$$u \in C\big([0, +\infty); H_0^1(\Omega)\big) \cap C^1\big((0, +\infty); L^2(\Omega)\big),$$

where $C_{TM} > 0$ is a constant which depends only on Ω coming from the Trudinger-Moser inequality. Moreover, there is some $\lambda_1 > 0$ such that for any $\lambda < \lambda_1$, we have

$$\|u(\cdot, t)\|_{H_0^1} \to 0$$

as $t \to +\infty$.

Theorem 3 *Let $\Omega = (0,1)$. If we replace (7) by*

$$2\left(e^{2eC_S^2} + \frac{1}{\pi^2}\right)\lambda^2 + \|u_0\|_{H_0^1}^2 < e\log 2,$$

then the conclusion of Theorem 2 is still true, where $C_S > 0$ is an embedding constant which depends only on Ω coming from $H_0^1(\Omega) \subset C(\overline{\Omega})$.

In the last result, we derive the dynamical properties. The Lyapunov function (6) plays an important role in arguing the convergence problem.

Proposition 4 (Cf. Theorem 2.1 in [4]) *Under the same hypotheses as Theorems 2 or 3, $\omega(u_0)$ is invariant, non-empty, compact and connected in $H_0^1(\Omega)$. Moreover $\omega(u_0)$ is a single point in $H_0^1(\Omega)$.*

This paper is composed of 5 sections. In Section 2, we show Propositions 1 and 2. We obtain the stationary solution by a bifurcation theory and compute the Morse index. In Section 3, we obtain some differential inequalities by the energy method and solve them to show Proposition 3. In Section 4, we decompose a solution of (1) into that of the heat equation with the non-zero initial value and the nonlinear heat equation with the zero initial value. Then we construct a lower and upper solution, which leads us to the proof of Theorem 1. In Section 5, we use the Lyapunov function and Trudinger-Moser inequality. Then we derive the H^1 estimate, which gives us the proof of Theorems 2 and 3. Finally, we derive the compactness of the orbit, which prove that the global solution converges to a stationary solution. By the existence of the Lyaounov function, we can prove Proposition 4.

2 Stationary solution First, we consider the condition on a parameter when a positive, negative or trivial solution exists. We use the Kaplan method [8] and Pohožaev identity [16]. If Ω is a ball, similar results to those in Proposition 1 are obtained in [1]. Next, we apply a bifurcation theory in [2], obtain a curve of solution (λ, v) and parametrize the solution $(\lambda, v) = (\lambda(s), v(\cdot, s))$ and the eigenpair $(\nu, \psi) = (\nu(s), \psi(\cdot, s))$, respectively. To compute the Morse index, we consider the signs of $\nu'(s)$ and $\nu''(s)$ at bifurcation points in the same way as [11, 12, 13]. Owing to the boundary condition, the proof for (2) is more complicated than that for (4) as proven in [12]. However, for completeness we prove it.

We prepare the Pohožaev identity in [16].

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let $f(v) \in C(\mathbb{R})$. Suppose that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} \Delta v + f(v) = 0 & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega. \end{cases}$$

Then the identity

$$\frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)\left(\frac{\partial v}{\partial \nu}\right)^2 d\omega + \frac{n-2}{2}\int_\Omega f(v)v\,dx = n\int_\Omega F(v)\,dx$$

holds, where $d\omega$ is the area element of $\partial\Omega$ with standard metric, ν is the outer unit normal vector at x and

$$F(v) = \int_0^v f(p)\,dp.$$

Proof of Proposition 1 By μ and ϕ, we denote the first eigenvalue μ^1 and corresponding eigenfunction ϕ^1 of $-\Delta$ in a star-shaped Ω with the Dirichlet boundary condition. Then we have $\phi > 0$ in Ω. First of all, we obtain relations between μ and λ stated in (i) and (ii) of the proposition. Multiplying (2) by $\phi > 0$ and integrating it over Ω, we have

$$-\mu \int_\Omega \phi v \, dx + \lambda \int_\Omega \phi \left(e^v - 1\right) \, dx = 0$$

and then

$$(\lambda - \mu) \int_\Omega \phi v \, dx < 0$$

for $v \not\equiv 0$ by $\exp x - 1 \geq x$ for $x \in \mathbb{R}$. If a solution $v > 0$ in Ω, then we have $\lambda < \mu$. For $v < 0$, we have $\lambda > \mu$. Next we obtain a lower bound λ_0 given in (i) and (ii). In the case of $n = 1, 2$, since

$$\int_\Omega |\nabla v|^2 \, dx = \lambda \int_\Omega \left(e^v - 1\right) v \, dx$$

holds, we conclude that $\lambda_0 \geq 0$ for $v \not\equiv 0$ by $(\exp x - 1)\, x \geq 0$ for all $x \in \mathbb{R}$. Thus we concentrate on the case of $n \geq 3$. Applying Lemma 1 to $f(v) = \lambda \left(e^v - 1\right)$ and $F(v) = \lambda \left(e^v - v - 1\right)$ and noting that Ω is star-shaped, we find

$$\frac{n-2}{2} \lambda \int_\Omega \left(e^v - 1\right) v \, dx \leq n\lambda \int_\Omega \left(e^v - v - 1\right) \, dx$$

and then

$$\begin{aligned} \frac{n-2}{4} \int_\Omega |\nabla v|^2 \, dx &= \left(\frac{n-2}{2} - \frac{n-2}{4}\right) \lambda \int_\Omega \left(e^v - 1\right) v \, dx \\ &\leq n\lambda \int_\Omega \left(e^v - v - 1\right) \, dx - \frac{n-2}{4} \lambda \int_\Omega \left(e^v - 1\right) v \, dx. \end{aligned}$$

Putting

$$g(x) = n\left(e^x - x - 1\right) - \frac{n-2}{4}\left(e^x - 1\right) x = ne^x - \frac{3n+2}{4} x - n - \frac{n-2}{4} e^x x,$$

we have $g(1) > 0$,

$$g'(x) = \frac{3n+2}{4} e^x - \frac{3n+2}{4} - \frac{n-2}{4} e^x x$$

and $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} g'(x) = -\infty$. Hence there exists $\xi > 0$ which depends only on n such that $g(x) \leq 0$ for $x \geq \xi$. Since for $\varepsilon,\, \kappa \in [0, 1]$,

$$\begin{aligned} g(x) &\leq n\left(e^x - x - 1\right) \\ &\leq n \int_0^1 \frac{d}{d\varepsilon} \left(e^{\varepsilon x} - \varepsilon x\right) \, d\varepsilon \\ &= nx \int_0^1 \left(e^{\varepsilon x} - 1\right) \, d\varepsilon \\ &= nx \int_0^1 \int_0^1 \frac{d}{d\kappa} \left(e^{\kappa \varepsilon x}\right) \, d\kappa \, d\varepsilon \\ &= n\varepsilon x^2 \int_0^1 \int_0^1 e^{\kappa \varepsilon x} \, d\kappa \, d\varepsilon \\ &\leq nx^2 \int_0^1 \int_0^1 e^{\kappa \varepsilon x} \, d\kappa \, d\varepsilon \\ &\leq \begin{cases} nx^2 e^x & \text{for } x \geq 0, \\ nx^2 & \text{for } x \leq 0 \end{cases} \end{aligned}$$

holds for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{n-2}{4}\int_\Omega |\nabla v|^2\,dx \\
\le\ & \lambda\int_{\Omega\cap\{v\le 0\}} g(v)\,dx + \lambda\int_{\Omega\cap\{0\le v\le \xi\}} g(v)\,dx + \lambda\int_{\Omega\cap\{v\ge \xi\}} g(v)\,dx \\
\le\ & \lambda n\int_{\Omega\cap\{v\le 0\}} v^2\,dx + \lambda n\int_{\Omega\cap\{0\le v\le \xi\}} v^2 e^v\,dx \\
\le\ & \lambda n\left(e^\xi+1\right)\int_\Omega v^2\,dx \\
\le\ & \frac{\lambda n}{\mu^1}\left(e^\xi+1\right)\int_\Omega |\nabla v|^2\,dx
\end{aligned}$$

thanks to the Poincaré inequality

$$\|v\|_2 \le \frac{1}{\sqrt{\mu^1}}\|\nabla v\|_2 \tag{8}$$

for all $v \in H_0^1(\Omega)$, which implies that

$$\lambda_0 \equiv \frac{\mu^1(n-2)}{4n\left(e^\xi+1\right)} < \mu^1 \quad \text{and} \quad 0<\lambda_0\le\lambda$$

for a nontrivial solution v. Finally we show the last claim (iii). Suppose that $v(x)$ is a nontrivial solution of (2). Then for $0<\lambda<\lambda_0$, we have

$$\lambda_0\int_\Omega |\nabla v|^2\,dx \le \lambda\int_\Omega |\nabla v|^2\,dx < \lambda_0\int_\Omega |\nabla v|^2\,dx,$$

which implies that $v\equiv 0$. □

Proof of Proposition 2 An easy calculation yields

$$(\mu^m,\phi^m) = \left((m\pi)^2, \sqrt{2}\sin m\pi x\right)$$

for $m\in\mathbb{N}$. At $(\lambda,v)=(\lambda,0)$, (5) has the k-th eigenvalue $\nu^k=\mu^k-\lambda$ and the corresponding eigenfunction $\psi^k=\phi^k$ for $k\in\mathbb{N}$. Hence, we have a simple eigenvalue $\nu^m=0$ at $(\lambda,v)=(\mu^m,0)$ and $i(\lambda,v)=m-1$ for $(\lambda,v)=(\lambda,0)$ with $\mu^{m-1}<\lambda\le\mu^m$ with $m\in\mathbb{N}$. The first part of proposition is proved. We will show that the nontrivial solutions bifurcate from $(\lambda,v)=(\mu^m,0)$. We define $\mathcal{X}=C^2(\overline{\Omega})\cap C_0(\overline{\Omega})$, $\mathcal{Y}=C(\overline{\Omega})$ and a mapping $F:\mathbb{R}^+\times\mathcal{X}\to\mathcal{Y}$ by

$$F(\lambda,v)=\Delta v+\left(\lambda+\mu^m\right)\left(e^v-1\right)$$

for $m\in\mathbb{N}$. Then $F(\lambda,0)=0$ and the Fréchet derivative is given as

$$F_v(\lambda,v)[w]=\Delta w+\left(\lambda+\mu^m\right)e^v w$$

for $w\in\mathcal{X}$. Since

$$F_v(0,0)[w]=\Delta w+\mu^m w,$$

the kernel of $F_v(0,0)$ is spanned by $w_0=\phi^m$. We have

$$F_{\lambda v}(\lambda,v)[\Lambda,w]=\Lambda e^v w,$$

which implies that $F_{\lambda v}(0,0)[\Lambda, w_0]$ does not belong to the range of $F_v(0,0)$. Hence applying Theorem 1.7 in [2] to this setting, we obtain two continua $\mathcal{S}_m^{\pm}$ of solutions (λ, v) of (2) bifurcating from $(\lambda, v) = (\mu^m, 0)$ satisfying

$$\mathcal{S}_m^+ = \Big\{ \big(\lambda(s), v(\cdot, s)\big) \mid \lim_{s\to+0} \big(\lambda(s), v(\cdot, s)\big) = \big(\mu^m, 0\big) \text{ and } s \in (0, \alpha) \Big\}$$

and

$$\mathcal{S}_m^- = \Big\{ \big(\lambda(s), v(\cdot, s)\big) \mid \lim_{s\to-0} \big(\lambda(s), v(\cdot, s)\big) = \big(\mu^m, 0\big) \text{ and } s \in (-\alpha, 0) \Big\}$$

in $\mathbb{R}^+ \times \mathcal{X}$ with some $\alpha > 0$. Moreover the mapping

$$s \in (-\alpha, \alpha) \mapsto \big(\lambda(s), v(\cdot, s)\big) \in \mathbb{R}^+ \times \mathcal{X}$$

belongs to $C^2(-\alpha, \alpha)$ and $v(\cdot, s)$ is expressed as

$$v(\cdot, s) = s\phi^m(\cdot) + s\rho(\cdot, s)$$

for a function $\rho(\cdot, s) : (-\alpha, \alpha) \to \mathcal{Z}$ with C^2 dependence in s and $\rho(\cdot, 0) = 0$, where $\mathcal{Z}$ is a complement of the kernel of $F_v(0,0)$. We set

$$\mathcal{C}_m = \mathcal{S}_m^- \cup \big\{(\mu^m, 0)\big\} \cup \mathcal{S}_m^+.$$

The bifurcation result is established. Finally, we will compute the Morse index. At $\big(\lambda(s), v(\cdot, s)\big) \in \mathcal{C}_m$, it follows from a perturbation theory in [9] that the k-th eigenpair $\big(\nu_m^k, \psi_m^k\big) = \big(\nu_m^k(s), \psi_m^k(\cdot, s)\big)$ is C^2 dependence in s. A simple computation yields

$$\nu_m^k(0) = \big(k^2 - m^2\big)\pi^2 \qquad \text{and} \qquad \psi_m^k(x, 0) = \phi^k(x) = \sqrt{2}\sin k\pi x$$

for $\big(\lambda, v\big) = \big(\mu^m, 0\big) \in \mathcal{C}_m$. Under these notations, we have

$$\dot{v}(x, 0) = \phi^m(x) = \psi_m^m(x, 0) = \sqrt{2}\sin m\pi x,$$

where $\dot{v}$ stands for $(d/ds)v(\cdot, s)$ and $\dot{v}(\cdot, 0) = (d/ds)v(\cdot, s)|_{s=0}$. Now we have the following lemmas:

Lemma 2 *For $k \in \mathbb{N}$, we have the following*:

$$\begin{aligned}
I_{k,1} &\equiv \int_0^1 \psi_m^k(x, 0)\, dx = \frac{\sqrt{2}}{k\pi}\left\{1 - (-1)^k\right\}, \\
I_{k,2} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^2 dx = 1, \\
I_{k,3} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^3 dx = \frac{4\sqrt{2}}{3k\pi}\left\{1 - (-1)^k\right\}, \\
I_{k,4} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^4 dx = \frac{3}{2}.
\end{aligned}$$

Lemma 3 *If m is even and $\dot{\nu}_m^m(0) = \dot{\lambda}(0) = 0$, then we have*

$$J_m \equiv \int_0^1 \left\{\psi_m^m(x, 0)\right\}^2 \dot{\psi}_m^m(x, 0)\, dx = -\frac{5}{6}.$$

Proof Differentiating (5) with respect to s and putting $s = 0$ and $k = m$, then we have

$$\Delta\dot{\psi}_m^m(x,0) + \mu^m \dot{\psi}_m^m(x,0) = -\mu^m \left\{\psi_m^m(x,0)\right\}^2$$

because of $\nu_m^m(0) = \dot{\nu}_m^m(0) = \dot{\lambda}(0) = 0$ and

$$\Delta\dot{\psi}_m^m(x,0) + (m\pi)^2 \dot{\psi}_m^m(x,0) = (m\pi)^2 (\cos 2m\pi x - 1).$$

Solving this ordinary differential equation with $\dot{\psi}_m^m(0,0) = \dot{\psi}_m^m(1,0) = 0$ under the restriction

$$\frac{d}{ds}\int_0^1 \left\{\psi_m^m(x,s)\right\}^2 dx = 2\int_0^1 \psi_m^m(x,s)\dot{\psi}_m^m(x,s)\,dx = 0$$

at $s = 0$ and noting that m is even, we obtain

$$\dot{\psi}_m^m(x,0) = \frac{4}{3}\cos m\pi x - 1 - \frac{1}{3}\cos 2m\pi x = \frac{4}{3}\cos m\pi x - \frac{4}{3} + \frac{1}{3}\left\{\psi_m^m(x,0)\right\}^2,$$

which yields the conclusion along with $I_{m,2}$ and $I_{m,4}$ in Lemma 2. □

We return back to the proof of Proposition 2. Differentiating (2) twice and (5) once with respect to s, we have

$$\Delta\ddot{v} + \ddot{\lambda}(e^v - 1) + 2\dot{\lambda}e^v\dot{v} + \lambda e^v \dot{v}^2 + \lambda e^v \ddot{v} = 0$$

and

$$\Delta\dot{\psi} + \dot{\lambda}e^v\psi + \lambda e^v \dot{v}\psi + \lambda e^v \dot{\psi} = -\dot{\nu}\psi - \nu\dot{\psi}.$$

Putting $s = 0$ and $k = m$, multiplying by $\psi_m^m(x,0)$ and integrating them over $(0,1)$, we have

$$2\dot{\lambda}(0)\int_0^1 \left\{\psi_m^m(x,0)\right\}^2 dx + \mu^m \int_0^1 \left\{\psi_m^m(x,0)\right\}^3 dx = 0$$

and

$$\left(\dot{\lambda}(0) + \dot{\nu}_m^m(0)\right)\int_0^1 \left\{\psi_m^m(x,0)\right\}^2 dx + \mu^m \int_0^1 \left\{\psi_m^m(x,0)\right\}^3 dx = 0.$$

Hence, $I_{m,2}$ and $I_{m,3}$ in Lemma 2 yield

$$\dot{\lambda}(0) = \dot{\nu}_m^m(0) = -\frac{2\sqrt{2}}{3}m\pi\left\{1 - (-1)^m\right\} = \begin{cases} 0 & \text{for even } m, \\ -\frac{4\sqrt{2}}{3}m\pi & \text{for odd } m. \end{cases}$$

Henceforth, we assume that m is even. Differentiating (2) three times and (5) twice with respect to s, putting $s = 0$ and $k = m$, multiplying by $\psi_m^m(x,0)$, integrating them over $(0,1)$ and eliminating $\int_0^1 \left\{\psi_m^m(x,0)\right\}^2 \ddot{v}(x,0)\,dx$, we have

$$2(m\pi)^2 I_{m,4} + 6(m\pi)^2 J_m = -3\ddot{\nu}_m^m(0) I_{m,2}$$

and hence

$$\ddot{\nu}_m^m(0) = \frac{2(m\pi)^2}{3} > 0$$

by Lemmas 2 and 3. We conclude that

$$\begin{cases} \nu_m^m(0) = 0 \quad \text{and} \quad \dot{\nu}_m^m(0) < 0 & \text{for odd } m, \\ \nu_m^m(0) = \dot{\nu}_m^m(0) = 0 \quad \text{and} \quad \ddot{\nu}_m^m(0) > 0 & \text{for even } m. \end{cases}$$

In the case of $m = 2k - 1$, we have

$$\begin{cases} \nu_m^m(s) > 0 & \text{for sufficiently small } s < 0, \\ \nu_m^m(0) = 0, & \\ \nu_m^m(s) < 0 & \text{for sufficiently small } s > 0 \end{cases}$$

and $i\left(\lambda(0), 0\right) = i\left(\mu^{2k-1}, 0\right) = 2k - 2$. Hence we have

$$i(\lambda(s), v(\cdot, s)) = \begin{cases} 2k-2 & \text{for sufficiently small } s < 0, \\ 2k-1 & \text{for sufficiently small } s > 0. \end{cases}$$

In the case of $m = 2k$, we have $\nu_m^m(s) > 0$ for sufficiently small $|s| > 0$, $\nu_m^m(0) = 0$ and $i\left(\lambda(0), 0\right) = i\left(\mu^{2k}, 0\right) = 2k - 1$. Hence we have

$$i(\lambda(s), v(\cdot, s)) = 2k - 1$$

for sufficiently small $|s| > 0$, which completes the proof. □

Remark 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \geq 2$. If we assume that the eigenvalue μ^m is simple for all $m \in \mathbb{N}$, the conclusion of bifurcation in Proposition 2 is still true. However, the relation of Morse index is open. The bifurcation problems where the zero eigenvalue is double are also considered for the equations other than* (2)*. In* [3]*, the authors obtain four bifurcation curves and compute their Morse indices in a square in Theorem 1.1. For a disk with Neumann boundary condition, in* [10]*, the author gives a sufficient condition for a branch of non-trivial solution not to have a secondary bifurcation point in Theorem B and applies these results including the simple eigenvalue case in Theorem C.*

3 Blow-up Let $u_0 \in C_0(\overline{\Omega})$. We transform (1) into the integral equation

$$u(t) = e^{-At}u_0 + \lambda \int_0^t e^{-A(t-s)}\left(e^{u(s)} - 1\right) ds$$

and establish a time-local solution

$$u \in C\left([0, T_{u_0}); C_0(\overline{\Omega})\right) \cap C^1\left((0, T_{u_0}); C_0(\overline{\Omega})\right)$$

by an abstract theory of evolution equation. Here, we extend $A \equiv -\Delta$ to be a self-adjoint positive operator in $C_0(\overline{\Omega})$ with the domain

$$\mathcal{D}(A) = \left\{u \in C_0(\overline{\Omega}), \quad Au \in C_0(\overline{\Omega})\right\}$$

and write the semi-group generated by A as e^{-At}. In order to prove the proposition by Kaplan's method [8], we integrate the solution multiplied by the first eigenfunction ϕ^1 of A and differentiate it with respect to t. Then we get the differential inequalities and solve them.

Proof of Proposition 3 We set

$$k = \left\|\phi^1\right\|_1 \qquad \text{and} \qquad a(t) = \int_\Omega u(x,t)\phi^1(x)\, dx$$

for all $t \in (0, T_{u_0})$. We have

$$a' = -\mu^1 a + \lambda \int_\Omega (e^u - 1)\phi^1(x)\,dx > \left(\lambda - \mu^1\right) a$$

by $\exp x - 1 \geq x$ for $x \in \mathbb{R}$ and integrate this inequality to obtain

$$a > e^{\left(\lambda-\mu^1\right)t} a(0) = e^{\left(\lambda-\mu^1\right)t} \int_\Omega u_0(x)\phi^1(x)\,dx > 0$$

for all $t \in (0, T_{u_0})$. Next the Jensen inequality [20] and positivity of a imply that

$$\begin{aligned} a' &= -\mu^1 a + \lambda \int_\Omega e^u \phi^1\,dx - \lambda k \\ &> -\mu^1 a + \lambda k \Big(\exp \frac{a}{k} - 1 \Big) \\ &> \left(\lambda - \mu^1\right) a + \frac{\lambda}{2k} a^2 \end{aligned}$$

by $\exp x \geq 1 + x + x^2/2$ for $x \geq 0$. Hence putting $p(t) \equiv 1/a(t)$ for all $t \in (0, T_{u_0})$, we have

$$p' + \left(\lambda - \mu^1\right) p < -\frac{\lambda}{2k}.$$

Multiplying $\exp\left(\lambda - \mu^1\right) t$, and integrating this differential inequality with respect to t, we obtain

$$0 < p < \begin{cases} p_0 e^{-\left(\lambda-\mu^1\right)t} + \frac{\lambda}{2k(\lambda-\mu^1)} \left(e^{-\left(\lambda-\mu^1\right)t} - 1\right) & \text{for } \lambda \neq \mu^1, \\ p_0 - \frac{\lambda}{2k} t & \text{for } \lambda = \mu^1, \end{cases}$$

where $p_0 = 1/a(0)$. Let

$$T \equiv \begin{cases} \frac{-1}{\mu^1-\lambda} \log\left(1 - \frac{2p_0 k\left(\mu^1-\lambda\right)}{\lambda}\right) & \text{for } 0 < \lambda < \mu^1, \\ \frac{2k}{\lambda} p_0 & \text{for } \lambda = \mu^1, \\ \frac{1}{\lambda-\mu^1} \log\left(1 + \frac{2p_0 k\left(\lambda-\mu^1\right)}{\lambda}\right) & \text{for } \lambda > \mu^1. \end{cases}$$

Then since the assumption of (i) in Proposition 3 is equivalent to

$$\frac{1}{p_0} > \frac{2k\left(\mu^1 - \lambda\right)}{\lambda}$$

for $0 < \lambda < \mu^1$, we have

$$0 < 1 - \frac{2p_0 k\left(\mu^1 - \lambda\right)}{\lambda} < 1.$$

Hence T is well-defined and we find $0 < T < +\infty$ such that $p(t) \to +0$ as $t \to T$. The same is true of the case of $\lambda \geq \mu^1$. Finally, we have

$$+\infty = \lim_{t\to T} \frac{1}{p(t)} = \lim_{t\to T} a(t) = \lim_{t\to T} \int_\Omega u(x,t)\phi^1(x)\,dx \leq k \lim_{t\to T} \sup_{x\in\overline{\Omega}} u(x,t),$$

which leads us to the proof of proposition. □

4 Negative global solution We decompose (1) into the heat equation with $u_0 \le 0$ and the nonlinear equation with $w_0 \equiv 0$. First, we begin with the fundamental lemmas.

Lemma 4 *Let $u_0 \in C_0(\overline{\Omega})$ be $u_0(x) \le 0$ for any $x \in \Omega$. The function*

$$w_1(x,t) \equiv e^{-At}u_0 \in C\big([0,+\infty); C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty); C_0(\overline{\Omega})\big)$$

solves

$$\begin{cases} w_t = \Delta w & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = u_0(x) & x \in \Omega \end{cases}$$

and satisfies $w_1 \le 0$ in $\Omega \times [0,+\infty)$.

Lemma 5 *Let $\lambda > 0$. The function*

$$w_2(x,t) \equiv \int_0^t e^{-A(t-s)}\,(-\lambda)\ ds \in C\big([0,+\infty); C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty); C_0(\overline{\Omega})\big)$$

solves

$$\begin{cases} w_t = \Delta w - \lambda & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = 0 & x \in \Omega \end{cases}$$

and satisfies $w_2 \le 0$ in $\Omega \times [0,+\infty)$.

We solve the nonlinear equation by constructing the lower-upper solution pair.

Proposition 5 *Let w_1 be a solution obtained in Lemma 4. For any $\lambda > 0$, there exists a unique solution*

$$w_3 \in C\big([0,+\infty); C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty); C_0(\overline{\Omega})\big)$$

of

$$\begin{cases} w_t = \Delta w + \lambda\,(e^{w_1+w} - 1) & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = 0 & x \in \Omega \end{cases}$$

satisfying $w_3 \le 0$ in $\Omega \times [0,+\infty)$.

Proof Since $(0, w_2)$ is an upper and lower solution pair, the statement follows from [18].

Proof of Theorem 1 Setting $u = w_1 + w_3$, we have a unique solution in an element of the desired space satisfying $u \le 0$ in $\Omega \times \mathbb{R}^+$. Note that $u(\cdot,t) \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega}) \subset H^2(\Omega) \cap H_0^1(\Omega)$ for all $t \ge 1$. Applying the Poincaré inequality (8), we have

$$\mu^1 \int_\Omega |\nabla u|^2\ dx \le \int_\Omega (\Delta u)^2\ dx$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and finally

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\ dx &= -\int_\Omega (\Delta u)^2\ dx + \lambda \int_\Omega e^u\,|\nabla u|^2\ dx \\ &\le -\int_\Omega (\Delta u)^2\ dx + \lambda \int_\Omega |\nabla u|^2\ dx \\ &\le -\big(\mu^1 - \lambda\big)\int_\Omega |\nabla u|^2\ dx \end{aligned}$$

for $t \ge 1$, which yields

$$\|u(\cdot,t)\|_{H_0^1}^2 \le e^{-2\left(\mu^1-\lambda\right)(t-1)}\,\|u(\cdot,1)\|_{H_0^1}^2 \to 0 \tag{9}$$

as $t \to +\infty$. □

5 Global solution In this section, we concentrate on $\Omega \subset \mathbb{R}^2$ and apply the Trudinger-Moser inequality to our problem. We establish the global solution for sufficiently small parameter and initial value. To obtain the estimate in the H_0^1 space, we extend $B = -\Delta$ to be a self-adjoint positive operator in $X = L^2(\Omega)$ with the domain $\mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega)$ and write the semi-group generated by B as e^{-Bt}. For $n = 1$, we can also derive the similar estimates. We start with the Trudinger-Moser inequality and an easy lemma.

Proposition 6 ([14, 19]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Then there exists $C_{TM} > 0$ which depends only on Ω such that*

$$\int_\Omega e^u \, dx \le C_{TM} \exp\left(\frac{1}{16\pi} \int_\Omega |\nabla u|^2 \, dx \right)$$

holds for all $u \in H_0^1(\Omega)$.

Lemma 6 *Let $a, b, k > 0$. We define*

$$f(t) = ae^{kt} + b - t$$

for $t \ge 0$. If

$$b + \frac{1}{k} \log a < -\frac{1}{k} (\log k + 1)$$

holds, then there exist $t_1 \in (0, t_0)$ and $t_2 \in (t_0, +\infty)$ such that $f(t) \ge 0$ is equivalent to $0 \le t \le t_1$ or $t \ge t_2$, where t_0, t_1 and t_2 satisfy $t_0 = (1/k) \log(1/ak)$ and $f'(t_0) = f(t_1) = f(t_2) = 0$.

Proof First, we have $f(0) = a + b > 0$ and $f'(t) = ake^{kt} - 1$. Note that

$$t_0 = \frac{1}{k} \log \frac{1}{ak} > b + \frac{1}{k} > 0$$

from the assumption. We find $f'(t_0) = ake^{kt_0} - 1 = 0$ and

$$f(t_0) = ae^{kt_0} + b - t_0 = b + \frac{1}{k} \log a + \frac{1}{k} (\log k + 1) < 0,$$

which completes the proof. □

Proof of Theorem 2 Note that $e^u \in L^1(\Omega)$ for $u \in H_0^1(\Omega)$ by Proposition 6. Hence the Lyapunov function $L_\lambda(u)$ defined in (6) is well-defined for $u \in H_0^1(\Omega)$. Since

$$\frac{d}{dt} L_\lambda(u) = -\|u_t(t)\|_2^2 \le 0$$

holds, $L_\lambda(u)$ is the Lyapunov function for (1) and we have

$$L_\lambda(u) = L_\lambda(u_0) - \int_0^t \|u_t(s)\|_2^2 \, ds \le L_\lambda(u_0),$$

which yields

$$\frac{1}{2} \|\nabla u\|_2^2 \le \lambda \int_\Omega e^u \, dx - \lambda \int_\Omega u \, dx + \frac{1}{2} \|u_0\|_{H_0^1}^2 + \lambda \int_\Omega (u_0 - e^{u_0}) \, dx.$$

Since

$$\lambda \int_\Omega e^u\,dx = C_{TM}\lambda \times \frac{1}{C_{TM}} \int_\Omega e^u\,dx \le \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2C_{TM}^2}\left(\int_\Omega e^u\,dx\right)^2$$

holds by the Young inequality, we have

$$\begin{aligned} \frac{1}{2}\|\nabla u\|_2^2 &< \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2C_{TM}^2}\left(\int_\Omega e^u\,dx\right)^2 + \lambda\sqrt{|\Omega|}\,\|u\|_2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 \\ &\le \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2}\exp\left(\frac{1}{8\pi}\|\nabla u\|_2^2\right) + \frac{\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}}\|\nabla u\|_2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 \end{aligned}$$

owing to $\exp x > x$ for $x \in \mathbb{R}$, the Trudinger-Moser inequality (Proposition 6) and the Poincaré inequality (8). Again the Young inequality yields

$$\frac{\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}}\|\nabla u\|_2 = \frac{\sqrt{2}\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}} \times \frac{1}{\sqrt{2}}\|\nabla u\|_2 \le \frac{|\Omega|}{\mu^1}\lambda^2 + \frac{1}{4}\|\nabla u\|_2^2, \tag{10}$$

which implies that

$$\frac{1}{2}\|\nabla u\|_2^2 \le \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2}\exp\left(\frac{1}{8\pi}\|\nabla u\|_2^2\right) + \frac{|\Omega|}{\mu^1}\lambda^2 + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2$$

and that

$$\|u\|_{H_0^1}^2 \le 2\exp\left(\frac{1}{8\pi}\|u\|_{H_0^1}^2\right) + 2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2.$$

Hence for $f(t) = ae^{kt} + b - t$ with

$$a = 2, \quad b = 2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2 \quad \text{and} \quad k = \frac{1}{8\pi},$$

then we have $f(\|u\|_{H_0^1}^2) \ge 0$ for all $t \ge 0$. Then the assumption

$$b + \frac{1}{k}\log a < -\frac{1}{k}(\log k + 1)$$

in Lemma 6 is satisfied. In fact

$$2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2 + 8\pi\log 2 < 8\pi(\log 8\pi - 1)$$

holds by (7). We can apply Lemma 6 to obtain $\|u\|_{H_0^1}^2 \le t_1$ or $\|u\|_{H_0^1}^2 \ge t_2$. Now that we have

$$\|u_0\|_{H_0^1}^2 < 4\pi(\log 4\pi - 1) < 8\pi\log 4\pi = t_0$$

along with (7), we find

$$\|u\|_{H_0^1}^2 \le t_1 < t_0 = 8\pi\log 4\pi \tag{11}$$

so long as the local solution exists. Hence we have a global solution in $H_0^1(\Omega)$. Next, we define X^α as the domain of B^α for $\alpha \ge 0$ with graph norm $\|u\|_{X^\alpha} = \|B^\alpha u\|_2$ for $u \in X^\alpha$.

Then we derive the estimate in $X^{1/2+\varepsilon} = H^{1+2\varepsilon}(\Omega)$ for $\varepsilon \in (0, 1/2)$ by the smoothing effect of the heat equation. In fact, we have for $t \geq 1$

$$
\begin{aligned}
& \|u\|_{X^{1/2+\varepsilon}} \\
= \; & \left\| B^{\frac{1}{2}+\varepsilon} u \right\|_2 \\
\leq \; & \left\| B^{\varepsilon} e^{-Bt} B^{\frac{1}{2}} u_0 \right\|_2 + \lambda \int_0^t \left\| B^{\frac{1}{2}+\varepsilon} e^{-B(t-s)} \left(e^{u(s)} - 1 \right) \right\|_2 ds \\
\leq \; & C_1 t^{-\varepsilon} e^{-C_2 t} \|u_0\|_{H_0^1} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \|e^u - 1\|_2 \, ds \\
\leq \; & C_1 \sqrt{t_0} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \left(\left\| e^{2u} \right\|_1^{\frac{1}{2}} + \|1\|_2 \right) ds \\
\leq \; & C_1 \sqrt{t_0} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \left\{ C_{TM}^{\frac{1}{2}} \exp\left(\frac{1}{8\pi} \|\nabla u\|_2^2 \right) + |\Omega|^{\frac{1}{2}} \right\} ds \\
< \; & C_1 \sqrt{t_0} + \frac{\lambda C_3}{C_2^{\frac{1}{2}-\varepsilon}} \left\{ \sqrt{C_{TM}} \exp\left(\frac{t_0}{8\pi} \right) + \sqrt{|\Omega|} \right\} \Gamma\left(\frac{1}{2} - \varepsilon \right)
\end{aligned}
$$

by (11), where $\Gamma(p)$ is a gamma function defined by

$$
\Gamma(p) = \int_0^{+\infty} e^{-x} x^{p-1} dx
$$

for $p > 0$ and henceforth in this proof we will denote by C_i a positive constant which depends only on Ω and ε, where $i \in \mathbb{N}$. Since we have $H^{1+2\varepsilon}(\Omega) \subset C(\overline{\Omega})$ with the Sobolev embedding constant $C_4 > 0$, we find

$$
\|u\|_\infty \leq C_4 \|u\|_{X^{\frac{1}{2}+\varepsilon}} < C_5
$$

for $t \geq 1$. Hence the estimate similar to (9) is given as

$$
\|u(\cdot, t)\|_{H_0^1}^2 < e^{-2\left(\mu^1 - \lambda \exp C_5\right)(t-1)} \|u(\cdot, 1)\|_{H_0^1}^2 \to 0
$$

for $\lambda < \lambda_1 \equiv \mu^1 / \exp C_5$ as $t \to +\infty$. □

Proof of Theorem 3 First, we note that $\mu^1 = \pi^2$. In the same manner, Lyapunov function yields

$$
\begin{aligned}
\frac{1}{2} \|\nabla u\|_2^2 \; & \leq \; \lambda \int_0^1 e^{|u|} \, dx + \lambda \|1\|_2 \|u\|_2 + \frac{1}{2} \|u_0\|_{H_0^1}^2 \\
& \leq \; \lambda e^{C_S \|u\|_{H_0^1}} + \frac{\lambda}{\pi} \|\nabla u\|_2 + \frac{1}{2} \|u_0\|_{H_0^1}^2 \\
& \leq \; \lambda e^{C_S \|u\|_{H_0^1}} + \frac{1}{4} \|\nabla u\|_2^2 + \frac{\lambda^2}{\pi^2} + \frac{1}{2} \|u_0\|_{H_0^1}^2
\end{aligned}
$$

owing to $H^1(0,1) \subset C([0,1])$ with the Sobolev embedding constant $C_S > 0$, (8) and (10). Since we have

$$
C_S \|u\|_{H_0^1} = \sqrt{2e} C_S \times \frac{1}{\sqrt{2e}} \|u\|_{H_0^1} \leq e C_S^2 + \frac{1}{4e} \|u\|_{H_0^1}^2 ,
$$

we find

$$\begin{aligned}
\frac{1}{2}\|\nabla u\|_2^2 &\le \lambda e^{eC_S^2}\exp\left(\frac{1}{4e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2\\
&= \sqrt{2}\lambda e^{eC_S^2}\times\frac{1}{\sqrt{2}}\exp\left(\frac{1}{4e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2\\
&\le \lambda^2 e^{2eC_S^2}+\frac{1}{4}\exp\left(\frac{1}{2e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2
\end{aligned}$$

again by the Young inequality and

$$\exp\left(\frac{1}{2e}\|u\|_{H_0^1}^2\right)+4\left(e^{2eC_S^2}+\frac{1}{\pi^2}\right)\lambda^2+2\|u_0\|_{H_0^1}^2-\|u\|_{H_0^1}^2\ge 0,$$

which completes the proof by applying Lemma 6 for

$$a=1,\quad b=4\left(e^{2eC_S^2}+\frac{1}{\pi^2}\right)\lambda^2+2\|u_0\|_{H_0^1}^2\quad\text{and}\quad k=\frac{1}{2e}.$$

□

Proof of Proposition 4. Since the embedding $X^{1/2+\varepsilon}\subset H_0^1(\Omega)$ is compact, the orbit $\cup_{t\ge1}\{u(\cdot,t)\}$ is relatively compact in $H_0^1(\Omega)$. Hence the omega limit set $\omega(u_0)$ is invariant, non-empty, compact and connected in $H_0^1(\Omega)$ by Theorem 5.1.8 in [7]. Again by Corollary 7.2.2 in [7] and the existence of the Lyapunov function, we have $\omega(u_0)\subset\mathcal{C}$. Moreover, thanks to the regularity of nonlinear term (e^u-1) of (1), $\omega(u_0)$ is a single point by Theorems 1.2 in [5] or 11.4.3 in [7].

□

Remark 2 *In the proposition, we impose the same hypotheses as Theorems 2 or 3, which is not needed. If the global soluiton exists, then the conclusion is true by Theorem 2.1 in* [4].

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SOME TOPOLOGICAL STRUCTURES AND RELATED GROUPS ON DIGITAL PLANE

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Abstract. The aim of the present paper is devoted to discussing some more properties of β-irresolute mappings, contra β-irresolute mappings and two weak homeomorphisms such as βc-homeomorphisms and contra βc-homeomorphisms. Further, we investigate some new groups related to the mappings above and some examples of them on the digital plane and we construct the concept of $\beta_{(2)}$-open sets of the digital plane.

1 Introduction and preliminaries Abd El Monsef el al. [1] and Andrijević [3] introduced independently the concept of β-open sets [1] and semi-preopen sets [3], respectively. Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1.1 A subset A of a topological space (X,τ) is called a *β-open set* [1] (or *semi-preopen set* [3]) if $A \subseteq Cl(Int(Cl(A)))$ holds in (X,τ). The complement of a β-open (or semi-preopen) set is called *β-closed* (or *semi-preclosed*).

Throughout the present paper, we use the terminology due to [1] for the naming of the above set, that is, β-open sets, β-closed sets. The β-closure of a subset E of a topological space (X,τ) is defined by $\beta Cl(E) := \bigcap\{F : E \subseteq F, F$ is β-closed in $(X,\tau)\}$ and it is the smallest β-closed set containing E. And $\beta Cl(A) = A$ holds if and only if A is β-closed in (X,τ). We recall some importance properties of β-open sets in Section 4 (Theorem 4.1).

In the present paper, we use the following notation and other notation (cf. Notation 3.3, Notation 5.5, Definition 5.12, Remark 5.13, Proposition 5.16(i), Propositon 5.18(i)).

Definition 1.2 Let (X,τ) be a topological space.
$\beta O(X,\tau) = SPO(X,\tau) := \{B : B$ is β-open in $(X,\tau)\}$ (cf. [1, Defintion 1.1], [3]),
$\beta C(X,\tau) = SPC(X,\tau) := \{F : F$ is β-closed in (X,τ), i.e. $Int(Cl(Int(F))) \subseteq F\}$ [1], [3],
$SO(X,\tau) := \{G : G$ is semi-open in (X,τ), i.e. $G \subseteq Cl(Int(G))\}$ [26],
$SC(X,\tau) := \{F : F$ is semi-closed in (X,τ), i.e. $Int(Cl(F)) \subseteq F\}$ [8].

One of the purposes of this paper is to investigate some group structures of the new families of mappings, i.e., $\mathcal{G}(X, X\setminus H;\tau) :=$ *con-*$\beta ch(X, X\setminus H;\tau) \cup \beta ch(X, X\setminus H;\tau)$ and $\mathcal{G}_0(X, X\setminus H;\tau)$:=*con-*$\beta ch_0(X, X\setminus H;\tau) \cup \beta ch_0(X, X\setminus H;\tau)$, where $H \subset X$ with $H \neq \emptyset$ (cf. Notation 3.3, Theorems 3.5,3.6 and Theorem 4.7). If we assume $X = H$ (resp. *con-*$\beta ch(X, X\setminus H;\tau) = \emptyset$) in Theorem 3.5(i), then we have the property [4, Theorem 4.4(i)] due to S.C. Arora et al. (resp. [40, Theorem 2.2] due to Sanjay Tahiliani). And, if *con-*$\beta ch(X, X\setminus H;\tau) = \emptyset =$ *con-*$\beta ch_0(X, X\setminus H;\tau)$ are assumed in Theorem 4.7(i), then the properties [40, Theorem 2.7(ii)] etc are obtained.

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In the last Section 5, a characterization (cf. Corollary 5.8) of β-open sets of the digital plane $(\mathbb{Z}^2, \kappa^2)$ will be studied and we prove that $(*)$ $con\text{-}\beta ch(\mathbb{Z}^2; \kappa^2) = \emptyset$ (cf. Corollary 5.11(ii)$'$) and so $\mathcal{G}(\mathbb{Z}^2; \kappa^2) = \beta ch(\mathbb{Z}^2; \kappa^2)$ (cf. Corollary 5.11(iii)$'$). Therefore, we define and construct the new concept, say $\beta_{(2)}$-open sets in a set H, where $H \subseteq \mathbb{Z}^2$ with $|H| \geq 2$ (cf. Definition 5.15). And, using such $\beta_{(2)}$-open sets, we construct new groups, say $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ and $p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$ etc (cf. Definition 5.21, Theorem 5.25). As examples, it is obtained that some motions, say ρ_{45} and $(\rho_{45})^{-1}$, are elements of $con\text{-}p.\beta_{(2)}ch(U; \kappa^2|U)$ and so $con\text{-}p.\beta_{(2)}ch(U; \kappa^2|U) \neq \emptyset$, where U is the smallest open set containing the origin $(0, 0)$ in $(\mathbb{Z}^2, \kappa^2)$ (cf. $(*)$ above and Notation 5.26, Example 5.27; Definition 5.20, Definition 5.21).

2 Contra-β-irresolute mappings and β-irresolute mappings. Let (X, τ), (Y, σ) and (Z, η) be topological spaces.

Definition 2.1 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be
(i) *β-continuous* [1] if $f^{-1}(V)$ is a β-closed set of (X, τ) for each closed set V of (Y, σ),
(ii) *perfectly continuous* [37] if $f^{-1}(V)$ is clopen in (X, τ) for each open set V of (Y, σ),
(iii) *contra-continuous* [13] if $f^{-1}(V)$ is closed in (X, τ) for each open set V of (Y, σ),
(iv) *contra-β-continuous* ([7], [18]) if $f^{-1}(V) \in \beta C(X, \tau)$ for each open set V of (Y, σ),
(v) *strongly contra-β-continuous* if f is a contra-β-continuous mapping such that the inverse image of such nonempty open set of (Y, σ) has an interior point,
(vi) B-*continuous* [43] if $f^{-1}(V)$ is a B-set of (X, τ) for each nonempty open set V of (Y, σ), where the B-set is the intersection of an open set and a semi-closed set of (X, τ) (this is defined by [43]),
(vii) B*-*continuous* [12] (cf. (vi)) if $f^{-1}(V)$ contains a nonempty B-set of (X, τ) for each nonempty open set V of (Y, σ),
(viii) *strongly β-closed* [17] if $f(G)$ is β-closed in (Y, σ) for each β-closed set G of (X, τ).

Definition 2.2 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be
(i) *irresolute* [8, Definition 1.1] if $f^{-1}(U) \in SO(X, \tau)$ for every set $U \in SO(Y, \sigma)$,
(ii) *β-irresolute* [36] if $f^{-1}(U) \in \beta O(X, \tau)$ for every set $U \in \beta O(Y, \sigma)$,
(iii) *contra-β-irresolute* [5] if $f^{-1}(U) \in \beta C(X, \tau)$ for every set $U \in \beta O(Y, \sigma)$ (cf. Remark 2.9(ii)),
(iv) *perfectly contra-β-irresolute* if $f^{-1}(V)$ is β-clopen in (X, τ) for each set $V \in \beta O(Y, \sigma)$,
(v) *contra-irresolute* [5] if $f^{-1}(U) \in SC(X, \tau)$ for every set $U \in SO(Y, \sigma)$,
(vi) *perfectly contra-irresolute* [5] if $f^{-1}(U)$ is semi-open and semi-closed in (X, τ) for each set $U \in SO(Y, \sigma)$.

Theorem 2.3 *A mapping $f : (X, \tau) \to (Y, \sigma)$ is* B*-*continuous if one of the following conditions* (1) *and* (2) *is satisfied,*
(1) *f is a strongly contra-β-continuous mapping* (cf. Definition 2.1(v)).
(2) *f is an onto and* B-*continuous mapping* (cf. Definition 2.1(vi)).

Proof. Let V be a nonempty open set of (Y, σ). Under the case of the assumption (1), we have that $f^{-1}(V) \in \beta C(X, \tau)$ and $Int(f^{-1}(V)) \neq \emptyset$, and so $\emptyset \neq Int(f^{-1}(V)) = X \cap Int(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ contains a nonempty B-open set U, say $U := X \cap Int(f^{-1}(V))$. Indeed, $X \in SC(X, \tau)$, $Int(f^{-1}(V)) \in \tau$ and $U \neq \emptyset$. Thus, f is B*-continuous, under the assumption (1). Under the case of the assumption (2), we have that $\emptyset \neq f^{-1}(V)$ and $f^{-1}(V)$ is a B-set. And so $f^{-1}(V)$ contains a nonempty B-set $f^{-1}(V)$. Thus, f is B*-continuous, under the assumption (2). □

Remark 2.4 The following diagram shows implications among several mappings defined above, where $p \to q$ (resp. $p \nleftrightarrow q$) means that p implies q (resp. p and q are independent). The implications are not reversible and the independence is explained (cf. Remark 2.5 below).

$$\begin{array}{ccc} & \textbf{contra-continuity} & \\ \swarrow & & \searrow \\ \textbf{contra-}\beta\textbf{-continuity} \not\leftrightarrow \textbf{B-continuity} & & \textbf{onto and B-continuity} \\ \uparrow & & \downarrow \\ \textbf{strongly contra-}\beta\textbf{-continuity} & \longrightarrow & \textbf{B}^*\textbf{-continuity} \end{array}$$

Remark 2.5 (i) Let $(\mathbb{R}, E)$ be the real line with the Euclidean topology E. The mappings $f, 1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ of (i-1) below are seen in [14].
(i-1) Let $f : (\mathbb{R}, E) \to (\mathbb{R}, E)$ be a mapping defined by $f(x) = [x]$, where $[x]$ is the Gaussian symbol. Then f is contra-β-continuous (cf. Definition 2.1(iv)). However, f is not contra-continuous, because for an open interval $(\frac{1}{2}, \frac{3}{2})$, $f^{-1}((\frac{1}{2}, \frac{3}{2})) = [1, 2)$ is not closed in $(\mathbb{R}, E)$.
(i-2) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ is B-continuous (cf. Definition 2.1(vi)) but not contra-β-continuous, since the inverse image of each singleton is not β-open. Moreover, $1_{\mathbb{R}}$ is not contra-continuous.

(ii) The mapping $f : (X, \tau) \to (X, \tau)$ is contra-β-continuous, but f is not B-continuous. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$. Then we have $\beta C(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\},$-$\{a, c\}, X\}$ and $SC(X, \tau) = \{\emptyset, \{c\}, X\}$. We define the mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$.

(iii) The converse of Theorem 2.3 under the assumption (1) is not reversible. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then since $\beta C(X, \tau) = SC(X, \tau) = P(X)\backslash\{\{a, b\}\}$, we show f is B-continuous and onto. By Theorem 2.3 under the assumption (2), it is obtained that f is B*-continuous. This mapping f is contra-β-continuous, but $Int(f^{-1}(\{a\})) = Int(\{c\}) = \phi$ hold. And so f is not strongly contra-β-continuous.

(iv) The converse of Theorem 2.3 under the assumption (2) is not reversible. The mapping $f : (X, \tau) \to (X, \tau)$ defined in (ii) above is not B-continuous (cf. (ii)). But f is B*-continuous, because $\{c\}$ and X are the nonempty B-sets.

(v) The contra-β-continuous mapping $f : (X, \tau) \to (X, \tau)$ of (ii) above is not strongly contra-β-continuous (cf. Definition 2.1(v), because $Int(f^{-1}(\{a, b\})) = \emptyset$.

Remark 2.6 (i) Let $X = Y = \{a, b\}$ and $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}\}$. Then the identity mapping $1_X : (X, \tau) \to (Y, \sigma)$ is a contra-β-continuous mapping but not β-continuous.

(ii) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ of Remark 2.5(i)(i-2) is β-continuous but not contra-β-continuous.

Remark 2.7 The following properties are well known.

(i) If $f : (X, \tau) \to (Y, \sigma)$ is contra-β-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is β-continuous, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra-β-continuous (cf. [7, Theorem 2.18]).

(ii) ([4, Theorem 2.3(iv)]) Every homeomorphism is β-irresolute.

Remark 2.8 (i) By the following examples (i-1) and (i-2), it is shown that the contra-β-irresoluteness and β-irresoluteness are independent. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset,$ -$\{a\}, \{a, b\}\}$. Then

(i-1) The identity mapping on (X, τ) above is β-irresolute but not contra-β-irresolute.

(i-2) Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by: $f(a) = b = f(b)$, $f(c) = a$. Then f is contra-β-irresolute but not β-irresolute.

(ii) In general, for any topological space (X, τ), the identity mapping $1_X : (X, \tau) \to (X, \tau)$ is contra-β-irresolute if and only if $\beta O(X, \tau) = \beta C(X, \tau)$ holds. And, 1_X on any topological space (X, τ) is β-irresolute.

Remark 2.9 (i) Every contra-β-irresolute mapping is contra-β-continuous, but it is shown that its converse is not true, by the following example. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset,$ -$\{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) = c$, $f(b) = a$, $f(c) = b$. One can deduce that f is contra-β-continuous, but it is not contra-β-irresolute.

(ii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is contra-β-irresolute if and only if the inverse image $f^{-1}(F)$ of each β-closed set F of (Y, σ) is β-open in (X, τ).

(iii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is β-irresolute if and only if the inverse image $f^{-1}(F)$ of each β-closed set F of (Y, σ) is β-closed in (X, τ).

Remark 2.10 (i) The following diagram of implications is well known.

Contra-irresolute $\leftarrow$ Perfectly contra-irresolute $\rightarrow$ Irresolute.

We have also the following diagram of implications.

Contra-β-irresolute $\leftarrow$ Perfectly contra-β-irresolute $\rightarrow$ β-irresolute;

and they are not reversible (cf. Remark 2.8(i) above and Remark 2.11 below).

(ii) In the implications above, the irresoluteness (resp. contra-irresoluteness, perfectly contra-irresoluteness) and the β-irresoluteness (resp. contra-β-irresoluteness, perfectly contra-β-irresoluteness) are independent (cf. (a), (b), (c) below).

Let $X = \{a, b, c\}$. We consider the topologies on X: $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\tau_3 = \{X, \emptyset\}$. We have the following: $SO(X, \tau) = \beta O(X, \tau) = P(X) \setminus \{\{c\}\}$, $SO(X, \tau_1) = \beta O(X, \tau_1) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $SO(X, \tau_2) = \tau_2$, $\beta O(X, \tau_2) = P(X)$, $SO(X, \tau_3) = \{\emptyset, X\}$, $\beta O(X, \tau_3) = P(X)$.

(a)(a-1) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is irresolute; f is not β-irresolute.

(a-2) Let $f : (X, \tau_3) \to (X, \tau)$ be the identity mapping. Then f is β-irresolute; f is not irresolute.

(b)(b-1) Let $f : (X, \tau_2) \to (X, \tau_1)$ be the identity mapping. Then f is contra-β-irresolute, f is not contra-irresolute.

(b-2) Define a mapping $f : (X, \tau_1) \to (X, \tau_2)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = b$. Then f is contra-irresolute, f is not contra-β-irresolute.

(c)(c-1) Let $f : (X, \tau_3) \to (X, \tau_2)$ be the identity mapping. Then f is perfectly contra β-irresolute, f is not perfectly contra-irresolute.

(c-2) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is perfectly contra-irresolute, f is not perfectly contra-β-irresolute.

Remark 2.11 We have a decomposition of perfectly contra-β-irresolute mappings. The following conditions (1) and (2) are equivalent: (1) $f : (X, \tau) \to (Y, \sigma)$ is perfectly contra-β-irresolute; (2) $f : (X, \tau) \to (Y, \sigma)$ is contra-β-irresolute and β-irresolute.

3 Groups $\mathcal{G}(X, X \setminus H; \tau)$ and $\mathcal{G}_0(X, X \setminus H; \tau)$.

Main purposes of the present Section 3 are to prove Theorems 3.5 and Theorem 3.6 (cf. Notation 3.3).

Definition 3.1 Let (X, τ) and (Y, σ) be topological spaces.

(i) A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i-1) a *βc-homeomorphism* ([4, Definition 3.1(ii)]) if f is a β-irresolute bijection and f^{-1} is β-irresolute,

(i-2) a *contra-βc-homeomorphism* if f is a contra-β-irresolute ([6], [4, Definition 4.1]) bijection and f^{-1} is contra-β-irresolute (cf. Definition 2.2(iii)).

(ii) (ii-1) $\beta ch(X; \tau) := \{f |\ f : (X, \tau) \to (X; \tau)$ is a βc-homeomorphism$\}$ ([4, notation (3) after Definiton 3.1],

(ii-2) *con-βch*$(X; \tau)$:=$\{f |\ f : (X, \tau) \to (X, \tau)$ is a contra-βc-homeomorphism$\}$ ([4, Definition 4.3(1)]).

(iii) $h(X; \tau) := \{f |\ f : (X, \tau) \to (X, \tau)$ is a homeomorphism$\}$ (e.g., [4, notation (3) after Definiton 3.1]).

(iv) ([40, Definition 2.1]) Let H be a nonempty subset of X.

(iv-1) $\beta ch(X, X\setminus H;\tau) := \{a|\ a\in \beta ch(X;\tau)$ and $a(X\setminus H) = X\setminus H\}$.

(iv-1)$'$ $con\text{-}\beta ch(X, X\setminus H;\tau) := \{a|\ a\in con\text{-}\beta ch(X;\tau)$ and $a(X\setminus H) = X\setminus H\}$ (cf. (ii)(ii-2)).

(iv-2) $\beta ch_0(X, X\setminus H;\tau){:=}\{a|\ a\in \beta ch(X, X\setminus H;\tau)$ and $a(x) = x$ for every point $x\in X\setminus H\}$.

(iv-2)$'$ $con\text{-}\beta ch_0(X, X\setminus H;\tau) := \{a|\ a\in con\text{-}\beta ch(X, X\setminus H;\tau)$ and $a(x) = x$ for every point $x\in X\setminus H\}$, where $H\neq X$ (cf. (ii)(ii-2) above).

Remark 3.2 (i) In 2010, N. Arora et al. [4, Theorem 4.4(i)] proved that $\beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)$ forms a group under the composition of mappings.

(ii) In 2019, Sanjay Tahiliani [40, Theorem 2.2] proved that $\beta ch(X, X\setminus H;\tau)$ and $\beta ch_0(X, X\setminus H;\tau)$ form groups under the composition of mappings, where H is a subset of X, and Sanjay Tahiliani proved important propertes [40, Theorem 2.7].

Notation 3.3 Let (X,τ) be a topological space and $H\subseteq X$ with $H\neq\emptyset$.

(i) $\mathcal{G}(X;\tau) := \beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)$.

(ii) $\mathcal{G}(X, X\setminus H;\tau) := con\text{-}\beta ch(X, X\setminus H;\tau)\cup \beta ch(X, X\setminus H;\tau)$.

(ii)$'$ $\mathcal{G}_0(X, X\setminus H;\tau) := con\text{-}\beta ch_0(X, X\setminus H;\tau)\cup \beta ch_0(X, X\setminus H;\tau)$.

Remark 3.4 Let us consider especially the case where that $H = X$ in Notation 3.3(ii) above. Then, we have that $\mathcal{G}(X, X\setminus X;\tau) = \mathcal{G}(X;\tau)$ holds. (cf. Definition 3.1(ii),(iv)).

Theorem 3.5 *Let H be a nonempty subset of (X,τ) and $\mathcal{G}(X, X\setminus H;\tau)$, $\mathcal{G}_0(X, X\setminus H;\tau)$ and $\mathcal{G}(X;\tau)$ be the families defined in* Notation 3.3 *above, respectively. Then,*

(i) *$\mathcal{G}(X, X\setminus H;\tau)$ forms a group under the composition of mappings.*

(i)$'$*$\mathcal{G}_0(X, X\setminus H;\tau)$ forms a subgroup of $\mathcal{G}(X, X\setminus H;\tau)$, where $H\neq X$.*

(ii) *The group $\beta ch_0(X, X\setminus H;\tau)$ forms a subgroup of $\mathcal{G}_0(X, X\setminus H;\tau)$, where $H\neq X$.*

(iii) *The groups $\mathcal{G}(X, X\setminus H;\tau)$ and $\mathcal{G}_0(X, X\setminus H;\tau)$ (where $H\neq X$) are subgroups of* $\mathcal{G}(X;\tau)$ (cf. Notation 3.3, [4, Theorem 4.4(i)]).

Proof. Throghout the present proofs of (i),(i)$'$ and (ii), let us denote: $\mathcal{G} := \mathcal{G}(X, X\setminus H;\tau)$ and $\mathcal{G}_0 := \mathcal{G}_0(X, X\setminus H;\tau)$. **(i)** A binary operation $w: \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is well defined by $w(a,b) := b\circ a$, where $b\circ a$ is the composite function of the functions a and b. Indeed, it is shown by the following four cases.

Case 1 (resp. Case 1$'$) a (resp. b) $\in \beta ch(X, X\setminus H;\tau)$ and b (resp. a) $\in con\text{-}\beta ch(X, X\setminus H;\tau)$**.** For the present case, $b\circ a: (X,\tau)\to(X,\tau)$ is a *contra*-β-irresolute bijection such that $(b\circ a)^{-1}$ is also *contra*-β-irresolute and $(b\circ a)(X\setminus H) = X\setminus H$ (cf. [4, Lemma 4.2(i-2)]). And so, $w(a,b)\in con\text{-}\beta ch(X, X\setminus H)\subseteq\mathcal{G}$.

Case 2 (resp. Case 3) $a, b\in con\text{-}\beta ch(X, X\setminus H;\tau)$ (resp. $\beta ch(X, X\setminus H;\tau)$)**.** For the present case, $b\circ a: (X,\tau)\to(X,\tau)$ is a β-irresolute bijection such that $(b\circ a)^{-1}$ is also β-irresolute and $(b\circ a)(X\setminus H){=}X\setminus H$ (cf. [4, Lemma 4.2(i-1)]). And so, $w(a,b)\in\beta ch(X, X\setminus H)\subseteq\mathcal{G}$.

Thus, the binary operation $w: \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is well defined. For all elements $a, b, c\in\mathcal{G}, w(w(a,b),c) = w(a, w(b,c))$ holds. The identity element $e\in\mathcal{G}$ is well defined by the identity mapping $1_X: (X,\tau)\to(X,\tau)$, i.e., $e := 1_X\in\mathcal{G}$; and so $w(e,a) = a = w(a,e)$ hold for all element $a\in\mathcal{G}$. The inverse element of an element $a\in\mathcal{G}$ is well defined by the inverse mapping a^{-1} of $a: (X,\tau)\to(X,\tau)$ and so $w(a,a^{-1}) = e = w(a^{-1},a)$ hold for all element $a\in\mathcal{G}$. And hence $(\mathcal{G}, w)$ forms a group under the composition of mappings, i.e., $\mathcal{G}$ is a group. **(i)$'$** Since $1_X\in\beta ch_0(X, H;\tau)$, we have the following: $\mathcal{G}_0\neq\emptyset$. For any element $a, b\in\mathcal{G}_0$ and the binary operation $w: \mathcal{G}\times\mathcal{G}\to\mathcal{G}$, it is seen that $w(a,b^{-1}) = b^{-1}\circ a\in\mathcal{G}$ and $(b^{-1}\circ a)(x) = b^{-1}(x) = x$ for every point $x\in X\setminus H$; and so $w(a,b^{-1})\in\mathcal{G}_0$. **(ii)** By (i)$'$, the subgroup $\mathcal{G}_0$ has the binary operation $w|\mathcal{G}_0$. Let $a, b\in\beta ch_0(X, X\setminus H;\tau)$. Then, $1_X\in\beta ch_0(X, X\setminus H;\tau)\neq\emptyset$ and $(w|\mathcal{G}_0)(a, b^{-1}) = b^{-1}\circ a\in\beta ch_0(X, X\setminus H;\tau)\subseteq\mathcal{G}_0$; and so

$\beta ch_0(X, X \setminus H; \tau)$ is a subgroup of $\mathcal{G}_0$. **(iii)** The group $\mathcal{G}(X;\tau)$ (cf. Notation 3.3(i)) forms a group under the compositon of mappings ([4, Theorem 4.4(i)]). And, $\mathcal{G}$ forms a group by the composition of mappings (cf. (i)) such that $\mathcal{G} \subseteq \mathcal{G}(X;\tau)$ (cf. Notation 3.3). Thus, $\mathcal{G}$ is a subgroup of $\mathcal{G}(X,\tau)$. And using (i)$'$, $\mathcal{G}_0$ is a subgroup of $\mathcal{G}(X,\tau)$. □

Theorem 3.6 (i) *If* $f : (X,\tau) \to (Y,\sigma)$ *is a contra-*βc*-homeomorphism* (cf. Definition 3.1(i)(i-2))*, then* f *induces also an isomorphism* $f_* : \mathcal{G}(X, X \setminus H; \tau) \to \mathcal{G}(Y, Y \setminus f(H); \sigma)$*, where* f_* *is defined by* $f_*(a) := f \circ a \circ f^{-1}$.

(ii) *If* $f : (X,\tau) \to (Y,\sigma)$ *is a* βc*-homeomorphism* (cf. Definition 3.1(i))*, then* f *induces an isomorphism* $f_* : \mathcal{G}(X, X \setminus H; \tau) \to \mathcal{G}(Y, Y \setminus f(H); \sigma)$*, where* f_* *is defined in* (i) *above.*

(iii) *Suppose one of the following properties* (a), (b) *below on mappings* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\eta)$*:* (a) f *and* g *are contra-*βc*-homeomorphisms,* (b) f *and* g *are* βc*-homeomorphisms.*
Then, $(g \circ f)_* = g_* \circ f_* : \mathcal{G}(X, X \setminus H; \tau) \to \mathcal{G}(Z, Z \setminus g(f(H);\eta))$ *holds and* $(1_X)_* = 1 : \mathcal{G}(X, X \setminus H; \tau) \to \mathcal{G}(X, X \setminus H; \tau)$ *is the identity isomorphism, where* $1_X : (X,\tau) \to (X,\tau)$ *on the identity and* 1 *is the identity on* $\mathcal{G}(X, X \setminus H)$.

Proof. **(i)** Under the assumption that f is a contra-βc-homeomorphism, it is proved that f_* is an isomorphism between the groups. Indeed, we have the following properties: **(1)** mapping $f_* : \mathcal{G}(X, X \setminus H; \tau) \to \mathcal{G}(Y, Y \setminus f(H); \sigma)$ is well defined; **(2)** f_* is a homomorphism; **(3)** f_* is a bijection. **Proof of (1)** Let $a \in \mathcal{G}(X, X \setminus H; \tau)$. We first have the following: $f_*(a)(Y \setminus f(H)) = Y \setminus f(H)$ holds. And, we consider the following two cases.
Case 1 $a \in \beta ch(X, X \setminus H; \tau)$ **(resp. Case 2** $a \in con\text{-}\beta ch(X, X \setminus H; \tau)$**).** Let $B \in \beta O(Y,\sigma)$. Then, for the Case 1 (resp. Case 2), we have the following: $(f_*(a))^{-1}(B) = f(a^{-1}(f^{-1}(B))) \in \beta O(Y;\sigma)$ (resp. $\beta C(Y,\sigma)$) and so $f_*(a) : (Y,\sigma) \to (Y,\sigma)$ is β-irresolute (resp. contra-β-irresolute) bijection. Moreover, we have the following: $f_*(a)(B) \in \beta O(Y,\sigma)$ (resp. $\beta C(Y,\sigma)$). Then, $(f_*(a))^{-1} : (Y,\sigma) \to (Y,\sigma)$ is β-irresolute (resp. contra-β-irresolute). Thus, for the Case 1 (resp. Case 2), we prove that $f_*(a) : (Y,\sigma) \to (Y,\sigma)$ is a βc-homeomorhism (resp. *contra*-βc-homeomorphism) such that $f_*(a)(Y \setminus f(H)) = Y \setminus f(H)$. Namely, using Notation 3.3, we have that $f_*(a) \in \mathcal{G}(Y, Y \setminus f(H); \sigma)$.
Proof of (2) Let a and b be elements of $\mathcal{G}(X, X \setminus H; \tau)$. Then, we have the following: $f_*(w_X(a,b)) = (f \circ b \circ f^{-1}) \circ (f \circ a \circ f^{-1}) = w_Y(f_*(a), f_*(b))$.
Proof of (3) Let $a, b \in \mathcal{G}(X, X \setminus H; \tau)$ such that $f_*(a) = f_*(b)$. Then, $f \circ a \circ f^{-1} = f \circ b \circ f^{-1}$ and so $a = b$. Let $d \in \mathcal{G}(Y, Y \setminus f(H); \sigma)$. Then, it is proved that $f^{-1} \circ d \circ f \in \mathcal{G}(X, X \setminus H; \tau)$ and $f_*(f^{-1} \circ d \circ f) = d$. **(ii)** Under the assumption that f is a βc-homeomorphism, it is proved similarly that of (i) that f_* is isomorphism between the groups. **(iii)** By definitions and (i) (resp. (ii)), the present properties are shown. □

Corollary 3.7 (i) (resp. (ii)) *If* $f : (X,\tau) \to (Y,\sigma)$ *is a contra-*βc*-homeomorphism* (resp. βc*-homeomorphism*)*, then* f *induces an isomorphism* $f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Y;\sigma)$*, where* f_* *is defined by* $f_*(a) := f \circ a \circ f^{-1}$ *for any* $a \in \mathcal{G}(X;\tau)$.

(iii) *Suppose one of the following properties* (a), (b) *below on mappings* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\eta)$*,* (a) f *and* g *are contra-*βc*-homeomorphisms,* (b) f *and* g *are* βc*-homeomorphisms.*
Then, we have the following properties (1), (2) *and* (3) *on* f_*, g_*.
(1) $(g \circ f)_* = g_* \circ f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Z;\eta)$. (2) $(1_X)_* = 1 : \mathcal{G}(X;\tau) \to \mathcal{G}(X;\tau)$ *is the identity isomorphism, where* $1_X : (X,\tau) \to (X,\tau)$ *is the identity.*
(3) (3-1) $f_*(con\text{-}\beta ch(X;\tau)) = con\text{-}\beta ch(Y;\sigma)$ *holds.* (3-2) $f_*(\beta ch(X;\tau)) = \beta ch(Y;\sigma)$ *holds.* (3-3) $f_*(h(X;\tau)) \subseteq \beta c\text{-}h(Y;\sigma)$ *holds* (cf. Definition 3.1(iii)).

(iv) (cf. [4, Theorem 4.5(i)]) *Especially, if* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\eta)$ *are homeomorphisms, then the induced mappings* $f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Y;\sigma)$ *and* $g_* : \mathcal{G}(Y;\sigma) \to \mathcal{G}(Z;\eta)$ *are isomorphisms* (cf. (i))*. Moreover, they have the same property of* (1), (2) *and* (3)(3-1)(3-2) *in* (iii)*. We note that, in* (3)(3-3)*,* $f_*(h(X;\tau)) = h(Y;\sigma)$ *holds.*

Proof. **(i), (ii), (iii)(1)(2)** are obtained respectively, by setting that $H = X$ in Theorem 3.6 above (cf. Remark 3.4). **(iii)(3) Proof of (3-1) (resp. (3-2))** By setting the case where $H = X$ in the proof of Theorem 3.6(i) (resp. (ii)), it is obtained that $f_*(con\text{-}\beta ch(X;\tau)) \subseteq con\text{-}\beta ch(Y;\sigma)$ (resp. $f_*(\beta ch(X;\tau)) \subseteq \beta ch(Y;\sigma)$) holds, under the assumption (a) (resp. (b)) on f. Conversely, for each element $d \in con\text{-}\beta ch(Y,\sigma)$ (resp. $\beta ch(Y;\sigma)$), we take a mapping $f^{-1} \circ d \circ f : (X,\tau) \to (X,\tau)$. Then, it is shown that $f^{-1} \circ d \circ f \in con\text{-}\beta ch(X;\tau)$ (resp. $\beta ch(X;\tau)$) and $f_*(f^{-1} \circ d \circ f) = d$ and so $d \in f_*(con\text{-}\beta ch(X;\tau))$ (resp. $\beta ch(X;\tau)$).
Proof of (3-3) By [4, Theorems 3.2(iii), 3.3(vi)], it is well known that $h(X;\tau) \subseteq \beta ch(X,\tau)$; and so $f_*(h(X;\tau)) \subseteq f_*(\beta ch(X;\tau)) = \beta ch(Y;\sigma)$ (cf. (3)(3-1) above).

(iv) Since any homeomorphism is a βc-homeomorhism ([4, Theorems 3.2(iii), 3.3(vi)]), then by (ii) it is shown that f_* and g_* are isomorphisms. By (1),(2) of (iii), the same properties (1), (2) and (3)(3-1)(3-2) are obtained;the present property (3-3) is well known. □

Corollary 3.8 (cf. Notation 3.3, Corollary 3.7(i)(ii)) (i) *If* $\mathcal{G}(X;\tau) \ncong \mathcal{G}(Y;\sigma)$, *then*

(i-1) *there does not exist any contra-βc-homeomorphism between two topological spaces* (X,τ) *and* (Y,σ), *and* (i-2) *there is not any βc-homeomorphism between* (X,τ) *and* (Y,σ), *and hence* (i-3) $(X,\tau) \ncong (Y,\sigma)$ (i.e., (X,τ) *is not homeomorphic to* (Y,σ)).

(ii) *If* $\beta ch(X;\tau) \ncong \beta ch(Y;\sigma)$, *then there does not exist any βc-homeomorphism between* (X,τ) *and* (Y,σ). □

Example 3.9 Let $(X,\tau), (Y,\sigma), (Y(1),\sigma_1)$ and $(Y(2),\sigma_2)$ be four topological spaces, where $X = Y = Y(1) = Y(2) := \{a,b,c\}$, $\tau := \{\emptyset,\{a\},\{b,c\},X\}$, $\sigma := \{\emptyset,\{a\},\{b\},\{a,b\},Y\}$, $\sigma_1 := \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},Y(1)\}$ and $\sigma_2 := \{\emptyset,\{a\},Y(2)\}$. And, let $h_x : X \to X$ be a bijection such that $h_x(x) = x$ and $h_x \neq 1_X$ for a given point $x \in X$. Then we have the following properties.

(i) $\mathcal{G}(X;\tau) \ncong \mathcal{G}(Y;\sigma)$ (cf. Corollary 3.8(i) above). Indeed, it is shown that $\beta O(X,\tau) = P(X) = \beta C(X,\tau)$ hold and so $\beta ch(X;\tau) = con\text{-}\beta ch(X;\tau) \cong S_3$(=the symmetric group of degree 3) and hence $\mathcal{G}(X;\tau) \cong S_3$. And, it is shown that $\beta O(Y,\sigma) = P(Y) \setminus \{\{c\}\}$ and $\beta C(Y,\sigma) = P(Y) \setminus \{\{a,b\}\}$ hold; and so $con\text{-}\beta ch(Y;\sigma) = \emptyset$ and $\mathcal{G}(Y;\sigma) = \beta ch(Y;\sigma) = \{1_Y, h_c\}$.

(ii) $\mathcal{G}(X;\tau) \ncong \mathcal{G}(Y(1);\sigma_1)$. Indeed, it is shown that $\beta O(Y(1),\sigma_1) = \sigma_1$ and $\beta ch(Y(1);\sigma_1)$ $=\{1_{Y(1)}\}$ and $con\text{-}\beta ch(Y(1);\sigma_1) = \{h_b\}$; and so $\mathcal{G}(Y(1);\sigma_1) = \{1_{Y(1)}, h_b\} \ncong S_3$ (cf. (i) above).

(iii) $\mathcal{G}(Y(1);\sigma_1) \cong \mathcal{G}(Y(2);\sigma_2)$ and $\beta ch(Y(1);\sigma_1) \ncong \beta ch(Y(2);\sigma_2)$. Indeed, it is shown that $\beta O(Y(2),\sigma_2) = \{\emptyset,\{a\}, \{a,b\},\{a,c\},Y(2)\}$ and $\beta ch(Y(2);\sigma_2) = \{1_{Y(2)}, h_a\}$, $con\text{-}\beta ch(Y(2);\sigma_2) = \emptyset$.

4 Groups $\mathcal{G}(X, X \setminus H;\tau)/Ker((r_H)_*))$, $\mathcal{G}_0(X, X \setminus H;\tau)/Ker((r_H)_*)$ and $\mathcal{G}(H;\tau|H)$.

The purpose of the present section is to prove Theorem 4.7. We first recall the concept of α-open sets et al. due to [35], i.e., $(*)$ a subset H of a topological space (X,τ) is said to be *α-open* in (X,τ) if $H \subseteq Int(Cl(Int(H)))$ holds in (X,τ) and the compliment of an α-open set is called *α-closed.* The family of all α-open sets (resp. α-closed sets) of (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$).
And we recall some importante properties on β-open sets as follows.

Theorem 4.1 (i)([1], e.g., [32, Lemma 3.3(b)],[19, Lemma 4.1(2)]) *Let* $A \subseteq H \subseteq X$. *If* $A \in \beta O(H,\tau|H)$ *and* $H \in \beta O(X,\tau)$, *then* $A \in \beta O(X,\tau)$.

(ii)([1, Lemma 2.5 and its Proof], e.g. [32, Lemma 3.2(b)], [19, Lemma 4.1(1)]) *Let* $H \subseteq X$ *and* $A_1 \subseteq X$. *If* $H \in \alpha O(X,\tau)$ *and* $A_1 \in \beta O(X,\tau)$, *then* $A_1 \cap H \in \beta O(H,\tau|H)$.

(ii)′ (cf. (ii),(i) above, [2, Corollary 2.14(a)]) *Let* $H \subseteq X$ *and* $A_1 \subseteq X$. *If* $H \in \alpha O(X,\tau)$ *and* $A_1 \in \beta O(X,\tau)$, *then* $A_1 \cap H \in \beta O(X,\tau)$.

(iii)([1, Remark 1.1]) *Arbitrary union of β-open sets of* (X,τ) *is β-open in* (X,τ).

(iv)([19, Lemma 4.3(2)]) *If* $A \subseteq H \subseteq X$ *and* $H \in \alpha O(X,\tau)$, *then* $\beta Cl(A) \cap H = \beta Cl_H(A)$, *where* $\beta Cl_H(A)$ *denotes the* β*-closure of* A *in the subspace* $(H, \tau|H)$.

(iv-1) *Let* $F \subseteq H \subseteq X$. *If* $H \in \alpha O(X,\tau)$ *and* $F \in \beta C(X,\tau)$, *then* $F \in \beta C(H,\tau|H)$ (i.e., $\beta_H Cl(F) = F$ *holds*).

(iv-2) *Let* F_1 *and* H *be subsets of* X. *If* $H \in \alpha O(X,\tau)$ *and* $F_1 \in \beta C(X,\tau)$, *then* $F_1 \cap H \in \beta C(H,\tau|H)$ (i.e., $\beta_H Cl(F_1 \cap H) = F_1 \cap H$ *holds*).

(iv-3) *Let* $F \subseteq H \subseteq X$. *If* $H \in \alpha O(X,\tau) \cap \beta C(X,\tau)$ *and* $F \in \beta C(H,\tau|H)$, *then* $F \in \beta C(X,\tau)$.

Proof. **(iv-1) (resp. (iv-2))** By the assumptions and (iv), it is shown that $\beta_H Cl(F) = F$ (resp. $\beta_H Cl(F_1 \cap H) = \beta Cl(F_1 \cap H) \cap H \subseteq \beta Cl(F_1) \cap H = F_1 \cap H$ and so $\beta_H Cl(F_1 \cap H) = F_1 \cap H$). Therefore, we have the following: $F \in \beta C(H,\tau|H)$ (resp. $F_1 \cap H \in \beta C(H,\tau|H)$).
(iv-3) By the assumptions, (iv) and (iii), it is shown that $F = \beta_H Cl(F) = H \cap \beta Cl(F)$ and so $H \cap \beta Cl(F) \in \beta C(X,\tau)$. □

Remark 4.2 It follows from the following example that one of assumptions of Theorem 4.1(ii) above (i.e., $H \in \alpha O(X,\tau)$) is not removed. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then, a subset $H := \{b,c\} \notin \alpha O(X,\tau)$ and $H \in \beta O(X,\tau)$. And, for a set $A_1 := \{a,c\} \in \beta O(X,\tau), A_1 \cap H = \{c\} \notin \beta O(H,\tau|H)$. Indeed, we have that $\beta O(X,\tau) = P(X)\backslash\{\{c\}\}, \beta O(H,\tau|H) = \{\emptyset, \{b\}, H\}$ and $\alpha O(X,\tau) = \tau$ hold and $Cl_H(Int_H(Cl_H(A_1 \cap H))) = Cl_H(Int_H(\{c\})) = \emptyset \not\supseteq A_1 \cap H$.

Remark 4.3 (i) Let H and K be subsets of X and Y, respectively. For a mapping $f : X \to Y$ satisfying $K = f(H)$, we define the map $r_{H,K}(f) : H \to K$ by $(r_{H,K}(f))(x) := f(x)$ for every $x \in H$. Then, we have the following: $j_K \circ (r_{H,K}(f)) = f|H : H \to Y$, where $j_K : K \to Y$ is the inclusion defined by $j_K(y) := (1_Y|K)(y) = y$ for every $y \in K$ and $f|H : H \to Y$ is the restriction of f to H defined by $(f|H)(x) := f(x)$ for every $x \in H$. Especially, if $X = Y = H = K$, then $r_{H,H}(f) = f$ holds.

(ii) Especially, we suppose that $X = Y, H = K \subseteq X$ and $a(H) = H, b(H) = H$ for mappings $a, b : X \to X$. Then, $r_{H,H}(b \circ a) = (r_{H,H}(b)) \circ (r_{H,H}(a))$ holds.
Moreover, if $a : X \to X$ is a bijection such that $a(H) = H$, then $r_{H,H}(a) : H \to H$ is bijective and $r_{H,H}(a^{-1}) = (r_{H,H}(a))^{-1}$.

Theorem 4.4 (cf. [40, Lemma 2.8]) (i) *Let* $H \in \alpha O(X,\tau)$. *If* $f : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp.β*-irresolute*), *then the restriction of* f *to* H, *say* $f|H : (H,\tau|H) \to (Y,\sigma)$, *is contra-*$\beta$*-irresolute* (resp.$\beta$*-irresolute*).

(ii) *Let* $k : (X,\tau) \to (K,\sigma|K)$ *be a mapping and* $j_K : (K,\sigma|K) \to (Y,\sigma)$ *be the inclusion, where* $K \subseteq Y$. *If* K *is* α*-open in* (Y,σ), *then the following properties* (1), (2) *are equivalent:*

(1) $k : (X,\tau) \to (K,\sigma|K)$ *is contra-*β*-irresolute* (resp. β*-irresolute*)*;*

(2) $j_K \circ k : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*).

(iii) *Suppose that* $f : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*), $H \in \alpha O(X,\tau)$ *and* $f(H) \in \alpha O(Y,\sigma)$. *Then,* $r_{H,f(H)}(f) : (H,\tau|H) \to (f(H),\sigma|f(H))$ *is contra-*β*-irresolute* (resp.β*-irresolute*).

Proof. **(i)** Let $A \in \beta O(Y,\sigma)$ (resp. $\beta C(Y,\sigma)$). Then, we have the following: $f^{-1}(A) \in \beta C(X,\tau)$. By Theorem 4.1(iv-2), it is shown that $(f|H)^{-1}(A) = f^{-1}(A) \cap H \in \beta C(H,\tau|H)$ and so $f|H : (H,\tau|H) \to (Y,\sigma)$ is contra-β-irresolute (resp. β-irresolute).
(ii) (1)⇒(2) Let $A_1 \in \beta O(Y, \sigma)$. We have the followng: $(*1)$ $j_K^{-1}(A_1) = K \cap A_1 \in \beta O(K,\sigma|K)$ (cf. Theorem 4.1(ii)). Then, using $(*1)$ above and assumption (1), we have that $(j_K \circ k)^{-1}(A_1) = k^{-1}(j_K^{-1}(A_1)) \in \beta C(X,\tau)$ (resp. $\beta O(X,\tau)$). **(2)⇒(1)** Let $B \in \beta O(K,\sigma|K)$. Then, we have that $B \in \beta O(Y,\sigma)$ (cf. Theorem 4.1(i)). Then, using (2), we show that $k^{-1}(B) = (j_K \circ k)^{-1}(B) \in \beta C(X,\tau)$ (resp. $\beta O(X,\tau)$). **(iii)** By assumption and (i), it is obtained that $f|H : (H,\tau|H) \to (Y,\sigma)$ is contra-β-irresolute (resp. β-irresolute). Since $f|H = j_{f(H)} \circ (r_{H,f(H)}(f))$ (cf. Remark 4.3(i)), by (ii) it is shown that the mapping $r_{H,f(H)}(f)$ is contra-β-irresolute (resp. β-irresolute). □

Theorem 4.5 (cf. [40, Definition 2.5, Theorem 2.7(i)]) *Suppose that* $H \in \alpha O(X,\tau)$.

(i) *If* $f \in$ *con-*$\beta ch(X, X\backslash H;\tau)$ (resp. $\beta ch(X, X\backslash H;\tau)$), *then* $r_{H,H}(f) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$).

(ii) *The following mapping* $(r_H)_* : \mathcal{G}(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is well defined by* $(r_H)_*(f) := r_{H,H}(f)$ *for every* $f \in \mathcal{G}(X, X\setminus H;\tau)$.

(iii) *The following mapping* $(r_H)_{*,0} : \mathcal{G}_0(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is well defined by* $(r_H)_{*,0}(f) := r_{H,H}(f)$ *for every* $f \in \mathcal{G}_0(X, X\setminus H;\tau)$.

(iv) (cf. [4, Theorem 4.4(i)]), Notation 3.3, Theorem 3.5(i),(i)$'$)

(iv-1) $(r_H)_* : \mathcal{G}(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is a homomorphism of group.*

(iv-2) $(r_H)_{*,0} : \mathcal{G}_0(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is a homomorphism of group.*

(iv-3) $(r_H)_*|\mathcal{G}_0(X, X\setminus H;\tau) = (r_H)_{*,0}$.

Proof. **(i)** In Theorem 4.4(iii), let consider the case where $Y = X$ and $\tau = \sigma$. Since $f \in$ *con-*$\beta ch(X, X\setminus H;\tau)$ (resp. $\beta ch(X, X\setminus H;\tau)$), we have the following property that both f and f^{-1} are contra-β-irresolute (resp. β-irresolute) bijections from (X,τ) onto itself such that $f(X\setminus H) = X\setminus H = f^{-1}(X\setminus H)$) (cf. Definition 3.1), and so $f(H) = H = f^{-1}(H)$, $f(H)$ and $f^{-1}(H)$ are α-open in (X,τ). Then, by Theorem 4.4(iii), it is shown that $r_{H,H}(f)$ and $(r_{H,H}(f))^{-1} = r_{H,H}(f^{-1}) : (H,\tau|H) \to (H,\tau|H)$ are contra-β-irresolure (resp. β-irresolute) bijections (cf. Remark 4.3(ii)). Namely, we have the followng: $r_{H,H}(f) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$). **(ii)** Let $a \in \mathcal{G}(X, X\setminus H)$. For the case where that $a \in$ *con-*$\beta ch(X, X\setminus H;\tau)$ (resp. $\beta ch(X, X\setminus H;\tau)$), by using (i) it is shown that $r_{H,H}(a) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$) and so $r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$. Therefore, $(r_H)_*(a) := r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$ holds for any element $a \in \mathcal{G}(X, X\setminus H;\tau)$ and so $(r_H)_*$ is well defined.

(iii) We recall that $\mathcal{G}_0(X, X\setminus H;\tau) \subseteq \mathcal{G}(X, X\setminus H;\tau)$. Then, by the definition of $(r_H)_{*,0}$ and (ii), it is obtaned that $(r_H)_{*,0}(a) := r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$ for every $a \in \mathcal{G}_0(X, X\setminus H)$.

(iv) We denote $\mathcal{G} := \mathcal{G}(X, X\setminus H;\tau)$ and $\mathcal{G}_0 := \mathcal{G}_0(X, X\setminus H;\tau)$, throughout the present proof of (iv). **(iv-1)** Let $a, b \in \mathcal{G}$ and $w : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ be the binary operation of the group $\mathcal{G}$ (cf. Proof of Theorem 3.5). Then, by definition, $w(a,b) := b\circ a$ for $a, b \in \mathcal{G}$ and $(r_H)_*(w(a,b)) = r_{H,H}(b\circ a) = (r_{H,H}(b))\circ(r_{H,H}(a))$ hold (cf. (ii) above, Remark 4.3(ii)). Here, we recall that the group $\mathcal{G}(H;\tau|H) :=$ *con-*$\beta ch(H;\tau|H) \cup \beta ch(H;\tau|H)$ has the binary operation $w_H : \mathcal{G}(H;\tau|H)\times\mathcal{G}(H;\tau|H) \to \mathcal{G}(H;\tau|H)$ defined by the composite mapping: $w_H(f,g) := g\circ f$, where $f, g \in \mathcal{G}(H;\tau|H)$ (cf. [4, Theorem 4.4(i)]). Thus, we have the following: $(r_H)_*(w(a,b)) = (r_{H,H}(b))\circ(r_{H,H}(a)) = w_H(r_{H,H}(a), r_{H,H}(b)) = w_H((r_H)_*(a), (r_H)_*)(b))$ and hence $(r_H)_* : \mathcal{G} \to \mathcal{G}(H;\tau|H)$ is a homomorphism of group. **(iv-2)** Since $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$ (cf. Theorem 3.5(i)$'$), by an argument similar to that of (iv-1) it is shown that $(r_H)_{*,0} : \mathcal{G}_0 \to \mathcal{G}(H;\tau|H)$ is a homomorphism of group. **(iv-3)** For an element $a \in \mathcal{G}_0$, we have the following: $((r_H)_*|\mathcal{G}_0)(a) = (r_H)_*(a) = r_{H,H}(a)$, on the other hand, $(r_H)_{*,0}(a) = r_{H,H}(a)$ and hence $(r_H)_*|\mathcal{G}_0 = (r_H)_{*,0}$. □

Lemma 4.6 ([40, Lemma 2.6] *for the case where* β*-irresoluteness*) *Let* (X,τ) *and* (Y,σ) *be topological spaces such that* $X = U_1\cup U_2$ *with* $U_j \neq \emptyset$ $(j \in \{1,2\})$. *Let* $f_1 : (U_1,\tau|U_1) \to (Y,\sigma)$ *and* $f_2 : (U_2,\tau|U_2) \to (Y,\sigma)$ *be the two contra-*β*-irresolute* (resp. β*-irresolute*) *mappings with* $f_1(x) = f_2(x)$ *for every point* $x \in U_1\cap U_2$. *If* U_1 *and* U_2 *are* α*-open sets of* (X,τ), *then its combination* $f_1\nabla f_2 : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*), *where* $(f_1\nabla f_2)(z) = f_j(z)$ *for every* $z \in U_j$ $(j \in \{1,2\})$.

Proof. By using Theorem 4.1(i)(iii) and above definitions, this lemma is proved. □

Theorem 4.7 (cf. [40, Theorem 2.7 (ii),(iii)]) (i) *Suppose that* $H \in \alpha O(X,\tau)$. *Then, we have the following isomorphisms of groups* (cf. Theorem 4.5(ii),(iii),(iv)).

(i-1) $\mathcal{G}(X, X\setminus H;\tau)/Ker((r_H)_*) \cong Im((r_H)_*)$.

(i-2) $\mathcal{G}_0(X, X\setminus H;\tau) \cong Im((r_H)_{*,0})$, *where* $Ker((r_H)_*) := \{a \in \mathcal{G}(X, X\setminus H;\tau) | (r_H)_*(a) = 1_H\}$ *is a normal subgroup of* $\mathcal{G}(X, X\setminus H;\tau)$, *and* $Im((r_H)_*) := \{(r_H)_*(a) | a \in \mathcal{G}(X, X\setminus H;\tau)\}$ *and* $Im((r_H)_{*,0}) := \{(r_H)_{*,0}(b) | b \in \mathcal{G}_0(X, X\setminus H;\tau)\}$ *are subgroups of* $\mathcal{G}(H;\tau|H)$.

(ii) *Suppose that* $H \in \alpha O(X,\tau) \cap \alpha C(X,\tau)$ (cf. Lemma 4.6, the top of the present Section 4). *Then, under the assumption above, we have the following properties on the homomorphisms* $(r_H)_*$ *and* $(r_H)_{*,0}$ (cf. Theorem 4.5).
(ii-1) *If con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$*, then* $(r_H)_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is onto.*
(ii-2) *If con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$*, then* $(r_H)_{*,0} : \mathcal{G}_0(X, X \setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is onto.*

(iii) *Suppose that* $H \in \alpha O(X,\tau) \cap \alpha C(X,\tau)$*. Then, we have the following isomorphisms of groups.*
(iii-1) *If con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$*, then* $\mathcal{G}(X, X \setminus H;\tau)/Ker((r_H)_*) \cong \mathcal{G}(H;\tau|H)$.
(iii-2) *If con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$*, then* $\mathcal{G}_0(X, X \setminus H;\tau) \cong \mathcal{G}(H;\tau|H)$.

Proof. **(i)** Since $H \in \alpha O(X,\tau)$, the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are the well defined homomorphisms of groups (cf. Theorem 4.5, Remark 4.3(i)). Then, by using the first isomorphism theorem of group theory, it is obtained that there are group isomorphisms below, under the α-openness of H in (X,τ): **(i-1)** $\mathcal{G}(X, X \setminus H;\tau)/Ker((r_H)_*) \cong Im((r_H)_*)$ and **(i-2)**$_1$ $\mathcal{G}_0(X, X \setminus H;\tau)/Ker((r_H)_{*,0}) \cong Im((r_H)_{*,0})$. In (i-2)$_1$ above, it is shown that **(i-2)**$_2$ $Ker((r_H)_{*,0}) = \{1_X\}$. Indeed, let $u_0 \in Ker((r_H)_{*,0}) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$. Then, $(r_H)_{*,0}(u_0) = 1_H$ holds,where 1_H is the identity element of $\mathcal{G}(H;\tau|H)$, by definitions (cf. Theorem 4.5(iii),(ii) and Remark 4.3(i)), it is shown that, for any point $x \in H, 1_H(x) = ((r_H)_{*,0}(u_0))(x) = (r_{H,H}(u_0))(x) = u_0(x)$ and so $u_0(x) = x$ holds for any point $x \in H$. Moreover, for any point $x \in X \setminus H, u_0(x) = x$ holds, because of $u_0 \in \mathcal{G}_0(X, X \setminus H;\tau)$ (cf. Notation 3.3(ii)$'$, Definition 3.1(iv)) and hence we prove (i-2)$_2 Ker((r_H)_{*,0}) = \{1_X\}$. Thus, by using (i-2)$_1$ and (i-2)$_2$ above, the isomorphism (i-2) is proved.

(ii) (ii-1) Let $h \in \mathcal{G}(H;\tau|H)$. We find a mapping, say $h_1 \in \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h_1) = h$. Indeed, we consider the following two cases (because of $\mathcal{G}(H;\tau|H) := \beta ch(H;\tau|H) \cup$ *con-*$\beta ch(H;\tau|H)$).
Case 1 $h \in$ *con-*$\beta ch(H;\tau|H)$**.** For the present case, we select an element g belonging to *con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$ by one of assumptions. By Theorem 4.4(i), it is obtained that $g|(X \setminus H) : (X \setminus H, \tau|X \setminus H) \to (X,\tau)$ is a contra-β-irresolute mapping such that $(g|(X \setminus H))(X \setminus H)$=$X \setminus H$. Then, since $j_H \circ h : (H,\tau|H) \to (X,\tau)$ is a contra-β-irresolute mapping (cf. Theorem 4.4(ii)), by Lemma 4.6, it is shown that the combination, say $h_1 := (j_H \circ h)\nabla(g|(X \setminus H)) : (X,\tau) \to (X,\tau)$, is a contra-$\beta$-irresolute bijection. And, we have the followng:$h_1(X \setminus H) = X \setminus H$ and $h_1^{-1} = (j_H \circ h^{-1})\nabla(g^{-1}|g(X \setminus H))$. Using Theorem 4.4(ii) and (i) above, it is shown that $j_H \circ h^{-1} : (H,\tau|H) \to (X,\tau)$ and $g^{-1}|g(X \setminus H) : (X \setminus H, \tau|(X \setminus H)) \to (X,\tau)$ are contra-β-irresolute mappings; and so the mapping h_1^{-1} is contra-β-irresolute (cf. Lemma 4.6). Thus, we proved that $h_1 \in$ *con-*$\beta ch(X, X \setminus H;\tau)$ and $(r_H)_*(h_1)$=$r_{H,H}(h_1)$=$r_{H,H}((j_H \circ h)\nabla(g|(X \setminus H)))$= $r_{H,H}(j_H \circ h) = h$ (cf. Theorem 4.5(ii), Remark 4.3(i)). Namely, for the present case, there exists an element $h_1 \in$ *con-*$\beta ch(X, X \setminus H;\tau) \subseteq \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h_1)$= h.
Case 2 $h \in \beta ch(H;\tau|H)$**.** For the present case, using [40, Theorem 2.7 (i)(i-2)]. there is an element $h'_1 \in \beta r$-$h(X, X \setminus H;\tau) \subseteq \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h'_1)$= h, where $h'_1 := (j_H \circ h)\nabla(1_X|(X \setminus H)) : (X,\tau) \to (X,\tau)$. Therefore, using Case 1 and Case 2, we prove that $(r_H)_*$ is onto.

(ii-2) Let $h \in \mathcal{G}(H;\tau|H)$. We consider the following two cases.
Case 1 $h \in$ *con-*$\beta ch(H;\tau|H)$**.** For the present case, we select an element $g_0 \in$ *con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$. By an argument similar to that in the proof of (ii)(ii-1) Case 1, it is proved that $h_2 := (j_H \circ h)\nabla(g_0|(X \setminus H)) \in$ *con-*$\beta ch(X, X \setminus H) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$ and $(r_H)_{*,0}(h_2) = h$.
Case 2 $h \in \beta ch(H;\tau|H)$**.** For the present case, using [40, Theorem 2.7 (i)(i-2)], there exists an element $h'_2 \in \beta ch_0(X, X \setminus H;\tau) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$ such that $(r_H)_{*,0}(h'_2)$= h, where $h'_2 := (j_H \circ h)\nabla(1_X|(X \setminus H))$. Therefore, $(r_H)_{*,0}$ is onto. **(iii)** By (i) and (ii), the isomorphisms (iii-1) and (iii-2) are obtained. □

5 A characterization of β-open sets of the digital plane $(\mathbb{Z}^2, \kappa^2)$ and some new groups on $(\mathbb{Z}^2, \kappa^2)$. In the present Section 5, we have the following four subsections (I), (II), (III) and (IV).

(I) Introduction of some related notation. We recall the concept of the digital plane.

Definition 5.1 (E.D.Khalimsky, R.Koppermann,P.R.Meyer,T.Y.Kong, cf. [22, p.175,-Definition 4], [20, p.905,p.908], [29, Section 2], [30, Example 4 in Section 2]).

(i) The *digital line* or the *Khalimsky line* $(\mathbb{Z}, \kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase (e.g., [27, Section 3 (I)], [16], [38, Section 6 in p.6]).

(ii) The *digital plane* or the *Khalimsky plane* is the Cartesian product (=topological product) of 2-copies of the digital line $(\mathbb{Z}, \kappa)$. This topological space is denoted by $(\mathbb{Z}^2, \kappa^2)$, where $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 := \kappa \times \kappa$ (e.g., [16], [9, Section 6], [39, Section 5], [11, Section 7], [10], [33, Section 6], [27, Section 3(II) in p.322]).

($\bullet$) In $(\mathbb{Z}, \kappa)$, for each integer s, $\{2s\}$ is closed and it is not open, and $\{2s+1\}$ is open and it is not closed. And so $Cl(\{2s\}) = \{2s\}$, $Cl(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$, $Int(\{2s\}) = \emptyset$ and $Int(\{2s+1\}) = \{2s+1\}$.

($\bullet$) In $(\mathbb{Z}^2, \kappa^2)$, for each integers s and m, $\{(2s, 2m)\}$ is closed and it is not open, and $\{(2s+1, 2m+1)\}$ is open and it is not closed, and $\{(2s+1, 2m)\}$ and $\{(2s, 2m+1)\}$ are not open and they are not closed. And so we have the following properties:

$\cdot$ $Cl(\{(2s, 2m)\}) = \{(2s, 2m)\}$, $Cl(\{(2s+1, 2m+1)\}) = \{2s, 2s+1, 2s+2\} \times \{2m, 2m+1, 2m+2\}$, $Cl(\{(2s+1, 2m)\}$=$\{2s, 2s+1, 2s+2\} \times \{2m\}$, $Cl(\{(2s, 2m+1)\} = \{2s\} \times \{2m, 2m+1, 2m+2\}$, and

$\cdot$ $Int(\{(2s, 2m)\}) = \emptyset$, $Int(\{(2s+1, 2m+1)\}) = \{(2s+1, 2m+1)\}$, $Int(\{(2s+1, 2m)\}) = Int(\{(2s, 2m+1)\}) = \emptyset$.

Definition 5.2 (cf. Notation 5.5 below) Let A be a subset of $(\mathbb{Z}^2, \kappa^2)$.

(i) $A_{\kappa^2} := \{x |\ x \in A \text{ and } \{x\} \in \kappa^2\}$, (ii) $A_{\mathcal{F}^2}$:=$\{x |\ x \in A$ and $\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\}$,

(iii) A_{mix} :=$\{x |\ x \in A, x \notin A_{\kappa^2}$ and $x \notin A_{\mathcal{F}^2}\}$, and

(iv) for the set $A = \emptyset$, $A_{\kappa^2} := \emptyset$, $A_{\mathcal{F}^2} := \emptyset$, $A_{mix} := \emptyset$.

(v) Note that, sometimes, the set A_{κ^2} (resp. $A_{\mathcal{F}^2}, A_{mix}$) above is denoted by $(A)_{\kappa^2}$ (resp. $(A)_{\mathcal{F}^2}$, $(A)_{mix}$ (cf. Notation 5.5).

Definition 5.3 (i) For an open set E and a point $x \in E, E$ is said to be *the smallest open set containing* x, if $E \subseteq G$ holds for every open set G containng x (e.g., [31, Definition 2.5, Remark 2.6 (ii)], [27, Section 3], [25, p.6 of Section 1]).

(ii) The smallest open set containing a point x in $(\mathbb{Z}^2, \kappa^2)$ is denoted by $U(x)$ throughout the present section (cf. Remark 5.4(iv) below).

Remark 5.4 The following properties are well known. Let A be a subset of $(\mathbb{Z}^2, \kappa^2)$ in (i), (ii) and (iii).

(i) $(\mathbb{Z}^2)_{\kappa^2} = \{(2s+1, 2m+1) |\ s,\ m \in \mathbb{Z}\}$, $A_{\kappa^2} = A \cap (\mathbb{Z}^2)_{\kappa^2}$,

(ii) $(\mathbb{Z}^2)_{\mathcal{F}^2} = \{(2s, 2m) |\ s,\ m \in \mathbb{Z}\}$, $A_{\mathcal{F}^2} = A \cap (\mathbb{Z}^2)_{\mathcal{F}^2}$,

(iii) $(\mathbb{Z}^2)_{mix} = \{(2s+1, 2m) |\ s,\ m \in \mathbb{Z}\} \cup \{(2s', 2m'+1) |\ s', m' \in \mathbb{Z}\}$, $A_{mix} = A \cap (\mathbb{Z}^2)_{mix}$.

(iv) Moreover, we have the following properties:

(iv-1) if $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $x := (2s+1, 2m+1)$ for some $s, m \in \mathbb{Z}$ and $U((2s+1, 2m+1))$=$\{(2s+1, 2m+1)\}$ (cf. (i) above and Definition 5.3(ii) for the notation $U(\bullet)$),

(iv-2) if $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $x := (2s, 2m)$ for some $s, m \in \mathbb{Z}$ and $U((2s, 2m))$=$\{2s-1, 2s, 2s+1\} \times \{2m-1, 2m, 2m+1\}$ (cf. (ii) above),

(iv-3) if $x \in (\mathbb{Z}^2)_{mix}$, then $x = (2s+1, 2m)$ or $x = (2s, 2m+1)$ for some $s, m \in \mathbb{Z}$ and $U((2s+1, 2m)) = \{2s+1\} \times \{2m-1, 2m, 2m+1\}$, $U((2s, 2m+1)) = \{2s-1, 2s, 2s+1\} \times \{2m+1\}$ (cf. (iii) above).

(II) A characterization of β-open sets of $(\mathbb{Z}^2, \kappa^2)$. We prepare the following notation which are used in Theorem 5.7 and Corollary 5.8 below. And, we note $(X)_{\kappa^2}$:=$\{y|y \in X$ and $\{y\} \in \kappa^2\}$ for a subset X of $(\mathbb{Z}^2, \kappa^2)$.

Notation 5.5 (cf. Definition 5.2) Let A be a nonempty subset of $(\mathbb{Z}^2, \kappa^2)$.
$V(A_{\mathcal{F}^2}) := \bigcup\{\{x\} \cup (A \cap U(x))_{\kappa^2} |\ x \in A_{\mathcal{F}^2}$ and $(A \cap U(x))_{\kappa^2} \neq \emptyset\}$,
$V(A_{mix}) := \bigcup\{\{y\} \cup (A \cap U(y))_{\kappa^2} |\ y \in A_{mix}$ and $(A \cap U(y))_{\kappa^2} \neq \emptyset\}$. And, for the case where $A_{\mathcal{F}^2} = \emptyset$ (resp. $A_{mix} = \emptyset$), we set that $V(A_{\mathcal{F}^2}) := \emptyset$ (resp. $V(A_{mix}) := \emptyset$).

Example 5.6 Let $A := \{x, p_x, y, y^-, y', z\} \cup \{a\} \subset \mathbb{Z}^2$ and $B := A \setminus \{a\}$, where $x := (0, 0), p_x := (1, 1), y := (2, 1), y^- := (3, 1), y' := (3, 0)$, $z := (5, 1)$ and $a := (-2, 0)$. Then, we have the following properties: **(1)** the set A is not β-open and B is β-open,
(2) $V(A_{\mathcal{F}^2}) \cup V(A_{mix}) \cup A_{\kappa^2} = \{p_x, y^-, z, x, y, y'\}$=$A \setminus \{a\} \neq A$,
(2)′ $V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$=$\{x, p_x\} \cup \{p_x, y, y^-, y'\} \cup \{p_x, y^-, z\}$=$B$.
Proof of (1) We see that $Cl(Int(Cl(A))) = \{0, 1, 2, 3, 4, 5, 6\} \times \{0, 1, 2\} \not\ni (-2, 0) = a$ and so $Cl(Int(Cl(A))) \not\supseteq A$ holds. For the set B, we see that $Cl(Int(Cl(B))) = \{0, 1, 2, 3, 4, 5, 6\} \times \{0, 1, 2\} \supset B$. **Proof of (2)** Since $A_{\mathcal{F}^2} = \{a, x\}$, we see that $(A \cap U(a))_{\kappa^2} = \emptyset$ and $(A \cap U(x))_{\kappa^2} \neq \emptyset$, $V(A_{\mathcal{F}^2}) = \{x\} \cup (A \cap U(x))_{\kappa^2} = \{x, p_x\}$ hold. Since $A_{mix} = \{y, y'\}$, we see that $V(A_{mix}) = [\{y\} \cup (A \cap U(y))_{\kappa^2}] \cup [\{y'\} \cup (A \cap U(y'))_{\kappa^2}]$ =$\{y, p_x, y^-\} \cup \{y', y^-\}$=$\{y, p_x, y^-, y'\}$. Since $A_{\kappa^2} = \{p_x, y^-, z\}$, we prove (2). **Proof of (2)′** For this β-open set B, we are able to see that: $B_{\mathcal{F}^2} = \{x\}$, $B_{mix} = A_{mix}$, $B_{\kappa^2} = A_{\kappa^2}$ and so we have the following: $V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$=$\{x, p_x\} \cup \{p_x, y, y^-, y'\} \cup \{p_x, y^-, z\}$=$B$.

By investigating Example 5.6 above, we find one of the characterization of β-open sets of $(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 5.7(i)(i-2),(ii) and Corollary 5.8 below).

Theorem 5.7 (i) (i-1) *If B is a nonempty β-open subset of $(\mathbb{Z}^2, \kappa^2)$, then $(B \cap U(x))_{\kappa^2} \neq \emptyset$ holds for each point $x \in B_{\mathcal{F}^2} \cup B_{mix}$* (cf. Remark 5.4(i)(ii)).
(i-2) *If B is a β-open set of $(\mathbb{Z}^2, \kappa^2)$, then B is expressible as follows: $B = V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$* (cf. Notation 5.5, Remark 5.4(i)).
(ii) *If a subset B is expressible as $B = V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$, then B is β-open in $(\mathbb{Z}^2, \kappa^2)$.*

Proof. We note that, in general, for a point $w \in \mathbb{Z}^2$ and a subset B of $(\mathbb{Z}^2, \kappa^2)$,
(1) $(B \cap U(w))_{\kappa^2}$=$B \cap (U(w))_{\kappa^2}$ holds, where $(U(w))_{\kappa^2}$:=$\{z|z \in U(w)$ and $\{z\} \in \kappa^2\}$.
(i)(i-1) Let $x \in B_{\mathcal{F}^2} \cup B_{mix}$. Then, since $B \subseteq Cl(Int(Cl(B)))$, we have the following: $U(x) \cap Int(Cl(B)) \neq \emptyset$ holds and so there exists a point $z(x)$ such that $z(x) \in U(x) \cap Int(Cl(B))$. Then **(2)** $U(z(x)) \subseteq Cl(B)$ and $\emptyset \neq (U(z(x)) \subseteq U(x)$ (cf. Definition 5.3(ii)). Then, using (2), we see that $\emptyset \neq (U(z(x)))_{\kappa^2} \subseteq (U(x))_{\kappa^2}$ and we can take an open singleton $\{p(x)\}$ such that $p(x) \in U(z(x))$ and so $p(x) \in B$. Thus, we have the following:$p(x) \in B \cap (U(x))_{\kappa^2}$, i.e., $(B \cap U(x))_{\kappa^2} \neq \emptyset$ (cf. (1) above). **(i-2)** First we note that the sets $V(B_{\mathcal{F}^2})$ and $V(B_{mix})$ are well defined by (i-1) above, respectively, for a nonempty β-open set B. We prove that: **(3)** $V(B_{\mathcal{F}^2}) \cup V(B_{\mathcal{F}^2}) \cup B_{\kappa^2} \subseteq B$ holds and **(4)** $B \subseteq V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$ holds. **Proof of (3)** We see that $V(B_{\mathcal{F}^2}) \subseteq \bigcup\{\{x\} \cup B_{\kappa^2} | x \in B_{\mathcal{F}^2}\}$=$B_{\mathcal{F}^2} \cup B_{\kappa^2} \subseteq B$ and $V(B_{mix}) \subseteq \bigcup\{\{y\} \cup B_{mix} | y \in B_{mix}\}$=$B_{mix} \subseteq B$. And so we prove (3).
Proof of (4) Let $z \in B$. And we consider the following two cases.
Case 1 $z \in B_{\mathcal{F}^2} \cup B_{mix}$**.** For the present case, by (i-1), it is shown that $(B \cap U(z))_{\kappa^2} \neq \emptyset$ and $z \in V(B_{\mathcal{F}^2}) \cup V(B_{mix})$, and so **(5)** $z \in V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$.
Case 2 $z \in B_{\kappa^2}$**.** For the present case, it is seen clearly that (5) above holds. Thus, by Case 1 and Case 2 above, (4) is proved. Therefore, by (3), (4) and Notation 5.5, it is proved that $B = V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$ holds if B is β-open in $(\mathbb{Z}^2, \kappa^2)$.
(ii) Let $B \neq \emptyset$. By using the assumption of (ii), the definition of $V(B_{\mathcal{F}^2})$ and $V(B_{mix})$ (cf. Notation 5.5), it is shown that $(B \cap U(x))_{\kappa^2} \neq \emptyset$ holds for each point $x \in B_{\mathcal{F}^2} \cup B_{mix}$. We show firstly that: **(6)** $\{x\} \cup (B \cap U(x))_{\kappa^2}$ is β-open for each point $x \in B_{\mathcal{F}^2} \cup B_{mix}$. Indeed,

since $(B\cap U(x))_{\kappa^2} \neq \emptyset$, there exists a point, say $z(x)$, such that $z(x) \in B_{\kappa^2}\cap(U(x))_{\kappa^2}$. Then, it is shown that $x \in Cl(\{z(x)\})$, because $z(x) \in (U(x))_{\kappa^2} \subset U(x) \subseteq W$ hold for every open set W containing x. And we have the following: $Cl(Int(Cl(\{z(x)\}) \supset Cl(Int(\{z(x)\}))) = Cl(\{z(x)\}) \ni x$, and so $Cl(Int(Cl(\{x\} \cup (B \cap U(x))_{\kappa^2}))) \supseteq Cl(Int(Cl(\{z(x)\}))) \supset \{x\}$. Then, $Cl(Int(Cl(\{x\} \cup (B \cap U(x))_{\kappa^2}))) \supseteq \{x\} \cup Cl(Int(Cl((B \cap U(x))_{\kappa^2}))) \supseteq \{x\} \cup (B \cap U(x))_{\kappa^2}$, because $(B \cap U(x))_{\kappa^2} \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus we prove the property (6), i.e., $\{x\} \cup (B \cap U(x))_{\kappa^2} \in \beta O(\mathbb{Z}^2, \kappa^2)$ for each point $x \in B_{\mathcal{F}^2} \cup B_{mix}$.

Therefore, since any union of β-open sets is β-open (cf. Theorem 4.1(iii)), by using (6), Notation 5.5 and the assumption of (ii), it is proved that the set B is β-open in $(\mathbb{Z}^2, \kappa^2)$. □

Corollary 5.8 *A subset B of $(\mathbb{Z}^2, \kappa^2)$ is β-open if and only if B is expressible as $B = V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$.* □

(III) A proof of *con-$\beta ch(\mathbb{Z}^2;\kappa^2) = \emptyset$* (cf. Corollary 5.11(ii)′ below). We first prepare the following notation: **(III-1)** $U := \{-1, 0, 1\} \times \{-1, 0, 1\}$ (i.e., $U := U((0, 0))$: the smallest open set of $(\mathbb{Z}^2; \kappa^2)$ containing $(0, 0)$): **(III-2)** $O := (0, 0), p^{(1)} := (1, -1), p^{(2)} := (-1, -1), p^{(3)} := (-1, 1), p^{(4)} := (1, 1)$ and $y^{(1)} := (0, -1), y^{(2)} := (-1, 0), y^{(3)} := (0, 1)$,-$y^{(4)} := (1, 0)$, and so **(III-3)** $(\cdot)$ $U = \{O, p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$ and $(\cdot)$ $U_{\kappa^2} = \{p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}\}$, $(\cdot)$ $U_{\mathcal{F}^2} = \{O\}, (\cdot)$ $U_{mix} = \{y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$ and so $(\cdot)$ $U = U_{\kappa^2} \cup U_{mix} \cup U_{\mathcal{F}^2}$ (disjoint union) and $(\cdot)$ the smallest open sets $U(y^{(i)})$ of $(\mathbb{Z}^2; \kappa^2)$ containing the point $y^{(i)} (1 \leq i \leq 4)$ is defined as follows: $U(y^{(i)}) = \{p^{(i+1)}, y^{(i)}, p^{(i)}\} (1 \leq i \leq 4)$, where $p^{(5)} = p^{(1)}$ (cf. $(**)$ in the first part of the subsection (IV) below).

Proposition 5.9 *Let* $U := U((0, 0))$ (cf. (III-1) above) *and* $f : (U, \kappa^2|U) \to (U, \kappa^2|U)$ *be a mapping. If* f *is bijective, then*

(i) f^{-1} *is not contra-β-irresolute* (cf. Definition 2.2) *and*

(ii) f^{-1} *and* f *are not contra-βc-homeomorphisms* (cf. Definition 3.1).

Proof. **(i)** We select three points, say $p^{(1)} \in U_{\kappa^2}, p^{(2)} \in U_{\kappa^2}$ and $y^{(1)} \in U_{mix}$ with the smallest open set $U(y^{(1)}) = \{p^{(1)}, y^{(1)}, p^{(2)}\}$ (cf. (III-2) above). For the point $y^{(1)}$ there exists an only one point, say $z(y^{(1)})$, such that $z(y^{(1)}) \in U$ and $f(z(y^{(1)})) = y^{(1)}$. Then, (•) we take a set $B := U \setminus \{z(y^{(1)})\}$. Then, we claime that:
(1) the set B is β-open in $(U, \kappa^2|U)$ and
(2) $f(B)$ is not β-closed in $(U, \kappa^2|U)$. **Proof of (1)** We consider the following two cases, because of $z(y^{(1)}) \in U = U_{\kappa^2} \cup (U_{mix} \cup U_{\mathcal{F}^2})$ (cf. (III)-1, (III)-2, (III)-3 above).
Case 1 $z(y^{(1)}) \in U_{\kappa^2}$. For the present case, since $\{z(y^{(1)})\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and $z(y^{(1)}) \in U$, we see that $z(y^{(1)}) = p^{(j_0)}$ for some j_0 with $1 \leq j_0 \leq 4$. And, so we have the followng: $Cl(B) = \bigcup\{Cl(\{p^{(i)}\}|i \neq j_0$ with $1 \leq i \leq 4\}$ and $Int(Cl(B)) \supseteq \bigcup\{Int(Cl(\{p^{(i)}\}))|i \neq j_0$ with $1 \leq i \leq 4\}=\bigcup\{\{p^{(i)}\}|i \neq j_0$ with $1 \leq i \leq 4\}$ and so $Cl(Int(Cl(B))) \supseteq \bigcup\{Cl(\{p^{(i)}\})|i \neq j_0$ with $1 \leq i \leq 4\} = Cl(B) \supset B$, i.e., for the present Case 1, B is β-open in $(\mathbb{Z}^2, \kappa^2)$.
Case 2 $z(y^{(1)}) \in U_{mix} \cup U_{\mathcal{F}^2}$. For the present case, since $B = U \cap (\mathbb{Z}^2 \setminus \{z(y^{(1)})\})$, we have the following: $Cl(Int(Cl(B))) \supseteq Cl(Int(U \cap Cl(\mathbb{Z}^2 \setminus \{z(y^{(1)})\}))=Cl(Int(U \cap \mathbb{Z}^2)) = Cl(Int(U)) = Cl(U) \supset B$ and so $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus, for each case, $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and so, by using Theorem 4.1(ii), it is shown that $B = B \cap U$ is β-open in $(U, \kappa^2|U)$ (note: $U \in \kappa^2 \subset \alpha O(\mathbb{Z}^2, \kappa^2)$). **Proof of (2)** For the point $z(y^{(1)})$ with $y^{(1)} = f(z(y^{(1)}))$, $B := U \setminus \{z(y^{(1)}\}$ and the bijection $f : U \to U$, we see that $f(B) = U \setminus \{y^{(1)}\}$, where $U := U((0, 0))$. Using Theorem 4.1(iv), we have the following: $\beta Cl_U(f(B)) = U \cap \beta Cl(f(B))= U \cap [f(B) \cup Int(Cl(Int(f(B)))] = U \cap [(U \setminus \{y^{(1)}\}) \cup U] = U$ and so $\beta Cl_U(f(B)) = U \neq U \setminus \{y^{(1)}\} = f(B)$. Thus, we prove (2). Therefore, by (1), (2) and definitions, it is proved that $f^{-1} : (U, \kappa^2|U) \to (U, \kappa^2|U)$ is not contra-β-irresolute (cf. Definition 2.2(iii)). **(ii)** By (i) above and Definition 3.1(i)(i-2), it is obtained that f^{-1} and f are not contra-βc-homeomorphisms. □

Remark 5.10 Let $g : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be a bijective function. Then, the inverse $g^{-1} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is not necessarily contra-β-irresolute and so g and g^{-1} are not necessarily a contra-βr-homeomorphism (cf. Definitions 2.2, 3.1). Indeed, we take a mixed point, say $y \in (\mathbb{Z}^2)_{mix}$, and a set $B := \mathbb{Z}^2 \setminus \{g^{-1}(y)\}$. We prove that $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and $g(B) = \mathbb{Z}^2 \setminus \{y\}$ is not β-closed in $(\mathbb{Z}^2, \kappa^2)$ (cf. Proof of Proposition 5.9(i)).

Corollary 5.11 *Let $U := U((0,0))$, i.e., $U := \{-1, 0, 1\} \times \{-1, 0, 1\} (\subset \mathbb{Z}^2)$. Then, we have the following properties.*

(i) *Every homeomorphism $f : (U, \kappa^2|U) \to (U, \kappa^2|U)$ is not a contra-βr-homeomorphism.*

(i)′ *Every homeomorphism $g : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is not a contra-βr-homeomorphism.*

(ii) $h(U; \kappa^2|U) \not\subseteq$ *con-$\beta ch(U; \kappa^2|U)$ and con-$\beta ch(U; \kappa^2|U) = \emptyset$.*

(ii)′ $h(\mathbb{Z}^2; \kappa^2) \not\subseteq$ *con-$\beta ch(\mathbb{Z}^2; \kappa^2)$ and con-$\beta ch(\mathbb{Z}^2; \kappa^2) = \emptyset$.*

(iii) $\mathcal{G}(U; \kappa^2|U) = \beta ch(U; \kappa^2|U)$ (cf. Notation 3.3, Definition 3.1(ii)).

(iii)′ $\mathcal{G}(\mathbb{Z}^2; \kappa^2) = \beta ch(\mathbb{Z}^2; \kappa^2)$ (cf. Notation 3.3, Definition 3.1(ii)). □

(IV) New groups $\beta_{(2)}ch(H; \kappa^2|H)$, $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$, - $p.\beta_{(2)}ch(H; \kappa^2|H)$ and $p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$.

Our main results of (IV) are Theorems 5.23, 5.25 and Example 5.27 (cf. Definition 5.15). We introduce some new concepts $\beta_{(2)}$-open sets (cf. Definition 5.12, Definiton 5.15). We first recall the following notation **(∗)** et al. and we prepare new definitions (Definition 5.12, Remark 5.13).

(∗) Let $x = (x_1, x_2) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (i.e., x_1 and x_2 are even). For this point x, we denote the points belonging to the smallest open set $U(x)$ containg the point x as follows:
$U(x) := \{x_1 - 1, x_1, x_1 + 1\} \times \{x_2 - 1, x_2, x_2 + 1\} = \{x, p_x^{(1)}, p_x^{(2)}, p_x^{(3)}, p_x^{(4)}, y_x^{(1)}, y_x^{(2)}, y_x^{(3)}, y_x^{(4)}\}$, where $p_x^{(1)} := (x_1 + 1, x_2 - 1)$, $p_x^{(2)} := (x_1 - 1, x_2 - 1)$, $p_x^{(3)} := (x_1 - 1, x_2 + 1)$, $p_x^{(4)} := (x_1 + 1, x_2 + 1)$, $y_x^{(1)} := (x_1, x_2 - 1)$, $y_x^{(2)} := (x_1 - 1, x_2)$, $y_x^{(3)} := (x_1, x_2 + 1)$, $y_x^{(4)} := (x_1 + 1, x_2)$.

(∗∗) The following illustration shows the points belonging in $U(x)$, where $\{x\} = \{(x_1, x_2)\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ (i.e., $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$), $(U(x))_{\mathcal{F}^2} = \{x\}$, $(U(x))_{\kappa^2} = \{p_x^{(i)} | i \in \{1, 2, 3, 4\}\}$ and $(U(x))_{mix} = \{y_x^{(i)} | i \in \{1, 2, 3, 4\}\}$.

$\mathbb{Z}$
$\uparrow$
$\cdots \quad \circ p_x^{(3)} \quad \cdot y_x^{(3)} \quad \circ p_x^{(4)} \quad \cdots$

$\cdots \quad \cdot y_x^{(2)} \quad \bullet\, x \quad \cdot y_x^{(4)} \quad \cdots$

$\cdots \quad \circ p_x^{(2)} \quad \cdot y_x^{(1)} \quad \circ p_x^{(1)} \quad \cdots$

$\bullet\, O \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \to \quad \mathbb{Z}$

When $x = (0, 0)$ (i.e., $x =$"the origin O" of $\mathbb{Z}^2$), we simply denote $p_x^{(i)}$ and $y_x^{(i)}$ as $p^{(i)}$ and $y^{(i)}$, respectively, and so $U := U((0,0)) = \{(0,0), p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$.

Definition 5.12 Let $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $U(x)$ be the smallest open set containing x. We define the following two families, $\beta_{(2)}O(U(x))$ and $\beta_{(2)}C(U(x))$ as follow (cf. Propositions 5.16(i), 5.18(i)):

(i) $\beta_{(2)}O(U(x)) := \{B | B \in \beta O(\mathbb{Z}^2, \kappa^2), B \subset U(x) \text{ and } |B| = 2\}$ (cf. Remark 5.13(i)),

(ii) $\beta_{(2)}C(U(x)) := \{F | F \in \beta C(\mathbb{Z}^2, \kappa^2), F \subset U(x) \text{ and } |F| = 2\}$ (cf. the definition of $p.\beta_{(2)}C(U(x)) \subset \beta_{(2)}C(U(x))$ in Remark 5.13(ii) below).

Remark 5.13 For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and the smallest open set $U(x)$ containing x, we have the following precise form of families $\beta_{(2)}O(U(x))$ and $\beta_{(2)}C(U(x))$ above, respectively (cf. Propositions 5.16(i), 5.18(i)).

(i) $\beta_{(2)}O(U(x)) = \{\{x, p_x^{(1)}\}, \{x, p_x^{(2)}\}, \{x, p_x^{(3)}\}, \{x, p_x^{(4)}\}, \{y_x^{(1)}, p_x^{(1)}\}, \{y_x^{(2)}, p_x^{(2)}\},$-$\{y_x^{(3)}, p_x^{(3)}\}, \{y_x^{(4)}, p_x^{(4)}\}, \{p_x^{(2)}, y_x^{(1)}\}, \{p_x^{(3)}, y_x^{(2)}\}, \{p_x^{(4)}, y_x^{(3)}\}, \{p_x^{(1)}, y_x^{(4)}\}, \{p_x^{(2)}, p_x^{(1)}\},$-$\{p_x^{(3)}, p_x^{(2)}\}, \{p_x^{(4)}, p_x^{(3)}\}, \{p_x^{(1)}, p_x^{(4)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\}\ \}$. (We use the following abbreviated notation: $\beta_{(2)}O(U(x)) := \{\{x, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(i+1)}, p_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$.)

(ii) (ii-1) $\beta_{(2)}C(U(x))$=$\{\{x, y_x^{(i)}\}, \{y_x^{(i+1)}, y_x^{(i)}\}, \{y_x^{(1)}, y_x^{(3)}\}, \{y_x^{(2)}, y_x^{(4)}\}, \{y_x^{(i)}, p_x^{(i)}\},$ -$\{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\}, \{x, p_x^{(i)}\}, \{p_x^{(i+2)}, y_x^{(i)}\}, \{p_x^{(i+3)}, y_x^{(i)}\} | i \in \{1,2,3,4\}\}$, and

(ii-2) we introduce the following importante subfamily, say $p.\beta_{(2)}C(U(x))$, of $\beta_{(2)}C(U(x))$ above and this concept is used in Theorem 5.23, 5.25 and Example 5.27, $p.\beta_{(2)}C(U(x))$:=$\{\{x, y_x^{(i)}\}, \{y_x^{(i+1)}, y_x^{(i)}\}, \{y_x^{(1)}, y_x^{(3)}\}\{y_x^{(2)}, y_x^{(4)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}$ -$| i \in \{1,2,3,4\}\})$, where $y_x^{(5)} := y_x^{(1)}$ and $p_x^{(5)} := p_x^{(1)}, p_x^{(6)} := p_x^{(2)}, p_x^{(7)} := p_x^{(3)}$.

Remark 5.14 (cf. Definition 5.12) We have the following inclusions of families.

(i) $\beta_{(2)}O(U(x)) \subset \beta O(\mathbb{Z}^2, \kappa^2)$. (i)$'$ $\beta_{(2)}O(U(x)) \subset \beta O(U(x), \kappa^2|U(x)) \subset \beta O(\mathbb{Z}^2, \kappa^2)$ (cf. (i) above, Definition 5.12(i), Theorem 4.1(i),(ii)).

(ii) $p.\beta_{(2)}C(U(x)) \subset \beta_{(2)}C(U(x)) \subset \beta C(\mathbb{Z}^2, \kappa^2)$. (ii)$'$ $\beta_{(2)}C(U(x)) \subset \beta C(U(x), \kappa^2|$-$U(x)) \subset \beta C(\mathbb{Z}^2, \kappa^2)$ (cf. (ii) above, Definition 5.12(ii), Remark 5.13(ii) and, Theorem 4.1(iv-1),(iv-3)).

(Note) By definition, we say that: (1) $\emptyset \notin \beta_{(2)}O(U(x))$ and $\emptyset \notin \beta_{(2)}C(U(x))$ and (2) $\beta_{(2)}O(U(x)) \cap \beta_{(2)}C(U(x))$=$\{\{x, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$.

Definition 5.15 Let $H \subseteq \mathbb{Z}^2$ with $|H| \geq 2$. A subset B (resp. F, F_1) of $(\mathbb{Z}^2, \kappa^2)$ is said to be *a* $\beta_{(2)}$*-open* (resp. $\beta_{(2)}$*-closed,* $p.\beta_{(2)}$*-closed*) *set of* H, if $B \subseteq H$ (resp. $F \subseteq H$, $F_1 \subseteq H$) and there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $B \in \beta_{(2)}O(U(x))$ (resp. $F \in \beta_{(2)}C(U(x))$, $F_1 \in p.\beta_{(2)}C(U(x))$) (cf. Definition 5.12(i) and Remark 5.13(i) (resp. Definition 5.12(ii) and Remark 5.13(ii))). The family of all $\beta_{(2)}$-open (resp. $\beta_{(2)}$-closed, $p.\beta_{(2)}$-closed) sets of H is denoted by $\beta_{(2)}O(H)$ (resp. $\beta_{(2)}C(H)$, $p.\beta_{(2)}C(H)$). (Note: Proposition 5.16 (resp. 5.18(i), 5.18(i)) below.)

Proposition 5.16 *Let* $H \subseteq \mathbb{Z}^2$ *with* $|H| \geq 2$*.*

(i) $\beta_{(2)}O(H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ *holds* (cf. Remark 5.17(i) below).

(ii) $\beta_{(2)}O(H) \subseteq \beta O(H, \kappa^2|H)$ *holds* (cf. Remark 5.17(i) below).

(iii) If H is β-open in $(\mathbb{Z}^2, \kappa^2)$, then $\beta O(H, \kappa^2|H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 4.1(i), Remark 5.17(ii) below).

Proof **(i), (ii)** Let $B \in \beta_{(2)}O(H)$. By Definition 5.15 and Remark 5.14(i), it is obtained that $(*)B \in \beta_{(2)}O(U(x))$ and $B \subset H$ for some point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.

The proof of (i) By Definition 5.12(i) (or Remark 5.14(i)), it is obtained that $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and so **(i)**:$\beta_{(2)}O(H) \subset \beta O(\mathbb{Z}^2, \kappa^2)$.

The proof of (ii) is as follows. Let $B \in \beta_{(2)}O(H)$. We show that $B \in \beta O(H, \kappa^2|H)$. Indeed, by $(*)$ above in the top of the present proof, $B \subset H$ and $B \in \beta_{(2)}O(U(x))$ for some point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. We investigate the proof with the following cases: Case 1, Case 2 and Case 3 (cf. Remark 5.13(i) etc.).

Case 1 $B \in \{\{x,\ p_x^{(i)}\} | i \in \{1,2,3,4\}\}$ **(resp. Case 2** $B \in \{\{y_x^{(i)}, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i+1)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{5} := p_x^{(1)}$**.)** For the present case, we put $B = \{u, p\}$, where $u \in (U(x))_{\mathcal{F}^2}$ (i.e., $u = x$) and $p \in (U(x))_{\kappa^2}$ (resp. $u \in (U(x))_{mix}$ and $p \in (U(u))_{\kappa^2}$). Then, we have the following: **(1)** $Int_H(B) \supseteq \{p\}$ and **(2)** $Cl_H(\{p\}) \supseteq B$. **Proof of (1)** Since $U(p) \cap H = \{p\} \cap H = \{p\} \in \kappa^2|H$, the set $U(p) \cap H = \{p\}$ is the smallest open set of $(H, \kappa^2|H)$ containing p such that $p \in B$ and so $p \in Int_H(B)$. **Proof of (2)** Since $U(u) \cap H$ is the smallest open set of $(H, \kappa^2|H)$ containing u and $(U(u) \cap H) \cap \{p\} \neq \emptyset$, we have the

following: $u \in Cl_H(\{p\})$ and so $Cl_H(\{p\}) \supset \{u, p\} = B$.
Then, using (1) and (2), we see that $Cl_H(Int_H(Cl_H(B))) \supseteq Cl_H(Int_H(B)) \supseteq Cl_H(\{p\}) \supseteq B$ and so B is β-open in $(H, \kappa^2|H)$ for the present Case 1 (resp. Case 2).
Case 3 $B \in \{\{p_x^{(i)},\ p_x^{(i+1)}\}, \{p_x^{(i)},\ p_x^{(i+2)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$ and $p_x^{(6)} := p_x^{(2)}$.
For the present case, we put $B = \{p,\ p'\}$ where $p \in (U(x))_{\kappa^2}$ and $p' \in (U(x))_{\kappa^2}$ with $p \neq p'$. Then, we have **(3)** $Int_H(B) = B$. **Proof of (3)** Since $B = \{p, p'\}$, where $p \neq p'$ and $p, p' \in (U(x))_{\kappa^2}$, it is shown that $U(p) \cap H = \{p\} \cap H = \{p\} \subset B$ and $U(p') \cap H \subset B$, and so $U(p) \cap H$ (resp. $U(p') \cap H$) is the smallest open set of $(H, \kappa^2|H)$ containing p (resp. p'). Thus we have the following: $p \in Int_H(B)$ (resp. $p' \in Int_H(B)$) and so $B \subset Int_H(B)$, i.e., $B = Int_H(B)$. Then, the set B is β-open in $(H, \kappa^2|H)$ for the present Case 3. Therefore, by all cases above, it is shown that $\beta_{(2)}O(H) \subseteq \beta O(H, \kappa^2|H)$ (cf. Remark 5.17(i) below).
(iii) Suppose that $B \in \beta O(H, \kappa^2|H)$. Since $B \subseteq H \subseteq \mathbb{Z}^2$ and H is β-open in $(\mathbb{Z}^2, \kappa^2)$ (by assumptions), using Theorem 4.1(i), we have the followng: $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus we prove that $\beta O(H, \kappa^2|H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ if H is β-open in $(\mathbb{Z}^2, \kappa^2)$. □

Remark 5.17 (i) In Proposition 5.16(i)(ii)), $\beta_{(2)}O(H)$ is a proper subfamily of $\beta O(H, \kappa^2|H)$ and $\beta O(\mathbb{Z}^2, \kappa^2)$, respectively. Indeed, let $B := \{p, p'\}$ and $H := B \cup \{x\}$, where $p := (-1, -1)$, $p' := (3, -1)$ and $x := (0, 0)$. Then, $B \in \beta O(H; \kappa^2|H)$ and $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. However, $B \notin \beta_{(2)}O(H)$, because $B \notin \beta_{(2)}O(U(x))$ for any $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.

(ii) It follows from the following example that the assumption of Proposition 5.16(iii) is not removed. Let $H := \{p, p', y, x, y'\}$ and $B := \{p, p', y, y'\}$, where $p := (-1, -1), p' := (3, -1), y := (-1, 0), x := (0, 0), y' := (1, 0)$. Then, H is not β-open in $(\mathbb{Z}^2, \kappa^2)$, because $Cl(Int(Cl(H))) = Cl(\{p, p'\}) \not\ni y'$ and $y' \in H$. And, B is β-open in $(H, \kappa^2|H)$, because $Cl_H(Int_H(Cl_H(B))) = H \supset B$; however, $B \notin \beta O(\mathbb{Z}^2, \kappa^2)$, because $Cl(Int(Cl(B))) = Cl(\{p, p'\}) \not\ni y'$ and $y' \in B$.

Proposition 5.18 *Let $H \subset \mathbb{Z}^2$ with $|H| \geq 2$.*

(i) *$p.\beta_{(2)}C(H) \subseteq \beta_{(2)}C(H) \subseteq \beta C(\mathbb{Z}^2, \kappa^2)$ hold.*

(ii) *If H is α-open in $(\mathbb{Z}^2, \kappa^2)$, then $\beta_{(2)}C(H) \subseteq \beta C(H; \kappa^2|H)$* (cf. Theorem 4.1(iv-2), Remark 5.19(i) below).

(iii) *If H is α-open and β-closed in $(\mathbb{Z}^2, \kappa^2)$, then $\beta C(H, \kappa^2|H) \subseteq \beta C(\mathbb{Z}^2, \kappa^2)$* (cf. Theorem 4.1(iv-3), Remark 5.19(ii) below).

Proof. The proof is analogous to the case of $\beta_{(2)}O(H)$ (cf. Proposition 5.16 above) and so is omitted. □

Remark 5.19 (i) In Proposition 5.18(i), $\beta_{(2)}C(H)$ is a proper subfamily of $\beta C(\mathbb{Z}^2, \kappa^2)$. Indeed, let $H := U(x) \cup U(x')$, where $x := (0, 0)$ and $x' := (2, 0)$. Then, a β-closed set $F := \{x, x'\}$ of $(\mathbb{Z}^2, \kappa^2)$ is not $\beta_{(2)}$-closed in H, because $F \notin \beta_{(2)}C(U(z))$ for any point $z \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (cf. Remark 5.13(ii)).

(ii) It follows from the following example that the assumption of Proposition 5.18(iii) is not removed. Let $F := \{p, x, p'\}$ and $H := F \cup \{q\}$, where $x := (0, 0)$, $p := (-1, -1)$, $p' := (1, -1)$, $q := (3, -1)$. Then, $Int(Cl(Int(H))) = \{-1, 0, 1, 2, 3\} \times \{-1\} \not\supset H$ and so H is not α-open in $(\mathbb{Z}^2, \kappa^2)$. Since $Int_H(Cl_H(Int_H(F))) = F$ holds, we see that $F \in \beta C(H, \kappa^2|H)$. However, $F \notin \beta C(\mathbb{Z}^2, \kappa^2)$, because $Int(Cl(Int(F))) = \{-1, 0, 1\} \times \{-1\} \not\subset F$.

Definition 5.20 Let $(H, \kappa^2|H)$ be a subspace of $(\mathbb{Z}^2, \kappa^2)$ with $|H| \geq 2$ and $f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ be a mapping. And, let $\mathcal{A}_H$ and $\mathcal{B}_H$ be collections of subsets of H such that:- $\mathcal{A}_H, \mathcal{B}_H \in \{\beta_{(2)}O(H),\ \beta_{(2)}C(H), p.\beta_{(2)}C(H)\}$ (cf. Definition 5.15, Remark 5.13).

Then, f is said to be $(\mathcal{A}_H, \mathcal{B}_H)$*-irresolute*, if $f^{-1}(E) \in \mathcal{B}_H$ for every set $E \in \mathcal{A}_H$, where $(\mathcal{A}_H, \mathcal{B}_H)$ denotes the ordered pair of the collections $\mathcal{A}_H$ and $\mathcal{B}_H$.

(Note 1) Especially, if $\mathcal{A}_H = \mathcal{B}_H$, then the concept of the $(\mathcal{A}_H, \mathcal{A}_H)$-irresolute mapping is simply said to be $\mathcal{A}_H$*-irresolute*.

(Note 2) In the present definition, we are able to define the concepts of the following mappings: the *$\beta_{(2)}O(H)$-irresolute mappings, $\beta_{(2)}C(H)$-irresolute mappings, $p.\beta_{(2)}C(H)$-irresolute mappings, $(\beta_{(2)}O(H), \beta_{(2)}C(H))$-irresolute mappings, $(\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute mappings, $(\beta_{(2)}O(H), p.\beta_{(2)}C(H))$-irresolute mappings, $(p.\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute mappings.*

Definition 5.21 For a subspace $(H, \kappa^2|H)$ of $(\mathbb{Z}^2, \kappa^2)$, where $|H| \geq 2$, we define the following collections of mappings as follows (cf. Definitions 5.20, 5.15).

(i) $\beta_{(2)}ch(H; \kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $\beta_{(2)}O(H)$-irresolute and they are $\beta_{(2)}C(H)$-irresolute$\}$.

(i)′ $p.\beta_{(2)}ch(H; \kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute$\}$.

(ii) *con-*$\beta_{(2)}ch(H; \kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $(\beta_{(2)}O(H), \beta_{(2)}C(H))$-irresolute and they are $(\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute$\}$.

(ii)′ *con-*$p.\beta_{(2)}ch(H; \kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $(\beta_{(2)}O(H), p.\beta_{(2)}C(H))$-irresolute and they are $(p.\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute$\}$.

Lemma 5.22 *Let $(H, \kappa^2|H)$ be a subspace of $(\mathbb{Z}^2, \kappa^2)$ such that:*
$(*)$ $H = \bigcup\{U(z)|z \in A_{\mathcal{F}^2}\}$, where $A_{\mathcal{F}^2}$ is a nonempty subset of $\mathbb{Z}^2$. If $f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a homeomorphism, then for a point $x \in H_{\mathcal{F}^2}$,

(i) *$f(H_{\kappa^2}) = H_{\kappa^2}$, $f(H_{\mathcal{F}^2}) = H_{\mathcal{F}^2}$ and $f(H_{mix}) = H_{mix}$ hold in $(\mathbb{Z}^2, \kappa^2)$,*

(ii) *$f(U(p)) = U(f(p))$ holds for each point $p \in (U(x))_{\kappa^2}$,*

(iii) *$f(U(y)) = U(f(y))$ holds for each point $y \in (U(x))_{mix}$, and*

(iv) *$f(U(x)) = U(f(x))$.*

Proof. Since f is homeomorphic, we have the property (i) (cf. Definition 5.2, Remark 5.4). Then, by the standard method, the properties (ii), (iii) and (iv) are proved. □

Let us consider the case $H = \mathbb{Z}^2$, i.e., $H = \bigcup\{U(z)|z \in (\mathbb{Z}^2)_{\mathcal{F}^2}\}$. Then, we have the following properties on $(\mathbb{Z}^2, \kappa^2)$.

Theorem 5.23 *Let $h(\mathbb{Z}^2; \kappa^2)$ be the group of all homeomorphisms from $(\mathbb{Z}^2, \kappa^2)$ onto itself* (cf. Definition 3.1(iii)).

(i) *If $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is a homeomorphism, then* (cf. Definition 5.20, Note:1)
(1a) *f is $\beta_{(2)}C(\mathbb{Z}^2)$-irresolute,* (1b) *$f$ is $\beta_{(2)}O(\mathbb{Z}^2)$-irresolute, and*
(1c) *f is $p.\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.*

(ii) *$h(\mathbb{Z}^2; \kappa^2) \subseteq \beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$ holds* (cf. Theorem 5.25(iv)′ below).

(ii)′ *$h(\mathbb{Z}^2; \kappa^2) \subseteq p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$ holds* (cf. Theorem 5.25(iv)′ below).

Proof. **(i) (Proof of (1a))** Let $F \in \beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15). We claim that $f^{-1}(F) \in \beta_{(2)}C(\mathbb{Z}^2)$. Indeed, there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $F \in \beta_{(2)}C(U(x))$ and we note that $f^{-1}(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (cf. Lemma 5.22(i) above). Since $F \in \beta_{(2)}C(U(x))$, we have that $F \subset U(x), |F| = 2$ and $F \in \beta C(\mathbb{Z}^2, \kappa^2)$ (cf. Definition 5.12(ii), or Remark 5.14(ii)). By definitions., Remark 2.7(ii) and Lemma 5.22(i)(iv), it is shown that $f^{-1}(F) \subset U(f^{-1}(x)) \subset \mathbb{Z}^2, |f^{-1}(F)| = 2$, $f^{-1}(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $f^{-1}(F) \in \beta C(\mathbb{Z}^2, \kappa^2)$, and so $f^{-1}(F) \in \beta_{(2)}C(U(f^{-1}(x)))$ (cf. Definition 5.12(ii)). Then, we conclude that $f^{-1}(F) \in \beta_{(2)}C(\mathbb{Z}^2)$ holds (cf. Definition 5.15).
(Proof of (1b)) Let $B \in \beta_{(2)}O(\mathbb{Z}^2)$. We claim that $f^{-1}(B) \in \beta_{(2)}O(\mathbb{Z}^2)$. The proof is analogous to the proof of (1a) above using Definitions 5.15, 5.12(i), Remark 5.13(i) and Lemma 5.22. And so the proof is omitted.
(Proof of (1c)) Let $F_1 \in p.\beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15, Remark 5.13(ii)(ii-2)). Then, $F_1 \subset \mathbb{Z}^2$, $|F_1| = 2$ and there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $F_1 \in p.\beta_{(2)}C(U(x))$. Then, we have the following: $F_1 \in \{\{x, y_x^{(i)}\}, \{y_x^{(i)}, y_x^{(i+1)}\}, \{y_x^{(1)}, y_x^{(3)}\}, \{y_x^{(2)}, y_x^{(4)}\}, \{y_x^{(i)}, -$

$p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$, $y_x^{(5)} := y_x^{(1)}$. Using Lemma 5.22, we note that $f^{-1}(x) \in (U(f^{-1}(x)))_{\mathcal{F}^2}$, $f^{-1}(U(p_x^{(i)})) = U(f^{-1}(p_x^{(i)}))$ and $f^{-1}(U(y_x^{(i)})) = U(f^{-1}(y_x^{(i)}))$. Put $z := f^{-1}(x)$. Then, $f^{-1}(p_x^{(i)}) = p_z^{(k(i))}$ and $f^{-1}(y_x^{(i)}) = y_z^{(k'(i))}$ are well defined in $U(z)$ for some integers $k(i), k'(i) \in \{1,2,3,4\}$, where $i \in \{1,2,3,4\}$. We show that (••) $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$. Indeed, we show (••) above for the following precise cases.

Case 1 $F_1 := \{p_x^{(i)}, y_x^{(i)}\}$. For the present case, we see that: $f^{-1}(F_1) = \{p_z^{k(i)}, y_z^{k'(i)}\}$. Since $p_x^{(i)} \in U(y_x^{(i)})$, by Lemma 5.22, it is shown that $f^{-1}(p_x^{(i)}) = p_z^{(k(i))} \in U(y_z^{(k'(i))})$, and $|k(i) - k'(i)| \leq 1$ (cf. (∗∗) of the first part of the present (IV)) and so we have the following: $k'(i) = k(i)$ or $k'(i) = k(i) - 1$ because of $k'(i) \leq k(i)$. Thus, we show that $f^{-1}(F_1) = \{p_z^{(k(i))}, y_z^{(k(i))}\}$ or $f^{-1}(F_1) = \{p_z^{(k(i))}, y_z^{(k(i)-1)}\}$, where $y_z^{(0)} := y_z^{(4)}$ and so $f^{-1}(F_1) \in pure\beta_{(2)}C(U(z))$.

Case 1′ $F_1 := \{p_x^{(i+1)}, y_x^{(i)}\}$. For the present case, we see that: $f^{-1}(F_1) = \{p_z^{(k(i+1))}, y_z^{(k'(i))}\}$. We claim that $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$. The proof is analogous to the proof of Case 1 above, using Definition 5.12(ii), Remark 5.13(ii)(ii-2) and Lemma 5.22. And so the proof is omitted.

Case 2 $F_1 := \{x, y_x^{(i)}\}$. For the present case, we see that: $f^{-1}(F_1) = \{z, y_z^{(k(i))}\} \in pure\beta_{(2)}C(U(z))$.

Case 3 $F_1 = \{y_x^{(1)}, y_x^{(3)}\}$. For the present case, we see that: $f^{-1}(F_1) = \{y_z^{(k'(1))}, y_z^{(k'(3))}\}$, where $\{k'(1), k'(3)\} \subset \{1,2,3,4\}$. Since $U(y_z^{(k'(1))}) \cap U(y_z^{(k'(3))}) = \emptyset$ and $U(y_z^{(k'(i))}) \subset U(z)$ for each $i \in \{1,3\}$, we have the following: $0 < |k'(1) - k'(3)| \leq 2$. And, if $|k'(1) - k'(3)| = 1$ then $U(y_z^{(k'(1))}) \cap U(y_z^{(k'(3))}) \neq \emptyset$ and so $|k'(1) - k'(3)| = 2$. Thus, we see that $f^{-1}(F_1) = \{y_z^{(k'(1))}, y_z^{(k'(3))}\} \in p.\beta_{(2)}C(U(z))$.

Case 3′ $F_1 = \{y_x^{(2)}, y_x^{(4)}\}$. For the present case, we see that: $f^{-1}(F_1) = \{y_z^{(k'(2))}, y_z^{(k'(4))}\} \in p.\beta_{(2)}C(U(z))$, where $\{k'(2), k'(4)\} \subset \{1,2,3,4\}$. The proof is analogous to the proof of Case 3 above and so the proof is omitted.

Case 4 $F_1 = \{y_x^{(i)}, y_x^{(i+1)}\}$. For the present case, we see that:
$f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i+1))}\}$, where $\{k'(i), k'(i+1)\} \subset \{1,2,3,4\}$ and $i \in \{1,2,3,4\}$. Since $U(y_x^{(i)}) \cap U(y_x^{(i+1)}) = \{p_x^{(i+1)}\}$, we have the following that: $U(y_z^{(k'(i))}) \cap U(y_z^{(k'(i+1))}) = \{p_z^{(k(i+1))}\}$ (cf. Lemma 5.22) and $|k'(i) - k'(i+1)| = 1$, i.e., $k'(i) - k'(i+1) = 1$ (if $k'(i) > k'(i+1)$) or $k'(i+1) - k'(i) = 1$ (if $k'(i) < k'(i+1)$). Thus, we prove that $f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i+1))}\} = \{y_z^{(k'(i))}, y_z^{(k'(i)-1))}\}$ or $f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i)+1))}\}$, and so $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$ (cf. Remark 5.13(ii)(ii-2), where $y_z^{(5)} := y_z^{(1)}$ and $y_z^{(0)} := y_z^{(4)}$.

Thus, by all cases above, the property (••) is proved. And, it is shown that, for each set $F_1 \in p.\beta_{(2)}C(\mathbb{Z}^2)$, there exists a point $z \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$, $f^{-1}(F_1) \subset \mathbb{Z}^2$ with $|f^{-1}(F_1)| = 2$, i.e., $f^{-1}(F_1) \in p.\beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15). Therefore, the homeomorphism $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is $p.\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.

(ii) (resp. (ii)′) By Definition 5.21(i) (resp. (i)′) and (i)(1a) (1b) (resp. (i)(1b) (1c)) above, the present (ii) (resp. (ii)′) is proved. □

Remark 5.24 The properties (1a), (1b) and (1c) in Theorem 5.23(i) are not hold, in general. This can be shown in the following example. Let $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be a bijection defined by $f((x_1, x_2) := (x_1 + 1, x_2)$ for each point $(x_1, x_2) \in \mathcal{Z}^2$. Then, $f^{-1}(\{(1,1)\}) = \{(0,1)\} \notin \kappa^2$ for the set $\{(1,1)\} \in \kappa^2$ and so f is not a homeomorphsm. For the set $V := \{(1,1), (1,-1)\} \in \beta_{(2)}O(\mathbb{Z}^2)$, we have the following: $f^{-1}(V) = \{(0,1), (0,-1)\} \notin \beta_{(2)}O(\mathbb{Z}^2)$ and so f is not $\beta_{(2)}O(\mathbb{Z}^2)$-irresolute. And, for a set $F := \{(0,1), (0,-1)\} \in p.\beta_{(2)}C(\mathbb{Z}^2)$, we have the following: $f^{-1}(F) = \{(-1,1), (-1,-1)\} \notin p.\beta_{(2)}C(\mathbb{Z}^2)$ and so f is not p.$\beta_{(2)}C(\mathbb{Z}^2)$-irresolute. Moreover, for the above sets F and $f^{-1}(F)$, since $F \in \beta_{(2)}C(\mathbb{Z}^2)$ and $f^{-1}(F) \notin \beta_{(2)}C(\mathbb{Z}^2)$, the bijection f is not $\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.

Theorem 5.25 *Let $(H, \kappa^2|H)$ be a subspace of $(\mathbb{Z}^2, \kappa^2)$ where $|H| \geq 2$.*

(i) (resp. (i)′) *The collection $\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p.\beta_{(2)}ch(H; \kappa^2|H)$) forms a group under the composition of mappings* (cf. Definition 5.21(i) (resp. (i)′).

(ii) (resp. (ii)′) *The union of two collections: $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$* (resp. *$p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$) forms a group under the composition of mappings* (cf. Definition 5.21(i),(ii) (resp. (i)′, (ii)′).

(iii) (resp. (iii)′) *The group $\beta_{(2)}ch(H; \kappa^2|H)$* (resp. *$p.\beta_{(2)}ch(H; \kappa^2|H)$) is a non-empty subgroup of $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$* (resp. *$p.\beta_{(2)}ch(H; \kappa^2|H) \cup$ - $con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$).*

(iv) (resp. (iv)′) *The group $h(\mathbb{Z}^2; \kappa^2)$ is a subgroup of the group $\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$* (resp. *$p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$) and so $h(\mathbb{Z}^2; \kappa^2)$ is a subgroup of the group $\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2) \cup con\text{-}\beta_{(2)}ch$ - $(\mathbb{Z}^2; \kappa^2)$* (resp. *$p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2) \cup con\text{-}p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$).*

Proof. **(i) (resp. (i)′)** A binary operation $\eta_H : \beta_{(2)}ch(H; \kappa^2|H) \times \beta_{(2)}ch(H; \kappa^2|H) \to \beta_{(2)}ch(H; \kappa^2|H)$ (resp. $\eta'_H : p.\beta_{(2)}ch(H; \kappa^2|H) \times p.\beta_{(2)}ch(H; \kappa^2|H) \to p.\beta_{(2)}ch(H;$- $\kappa^2|H))$ is well defined by $\eta_H(g_1, g_2){:=}g_2 \circ g_1$ (resp. $\eta'_H(g_1, g_2){:=}g_2 \circ g_1$). Indeed, by using Definitions 5.20(Note:1),5.21(i) (resp. (i)′), it is shown that $g_2 \circ g_1$ and $(g_2 \circ g_1)^{-1}$ are both $\beta_2 O(H)$-irresolute and they are $\beta_2 C(H)$-irresolute (resp. $g_2 \circ g_1$ and $(g_2 \circ g_1)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute). Thus, we prove that $\eta_H(g_1, g_2) \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. $\eta'_H(g_1, g_2) \in p.\beta_{(2)}ch(H; \kappa^2|H)$) and the binary operation η_H (resp. η'_H) satisfies the axiom of group. Therefore, the pair $(\beta_{(2)}ch(H; \kappa^2|H), \eta_H)$ (resp. $(p.\beta_{(2)}ch(H; \kappa^2|H), \eta'_H)$) forms a group under compositions of mappings.

(ii) (resp. (ii)′) We first note on the following notation that: let $\mathcal{G}_H := \beta_{(2)}ch(H; \kappa^2|H) \cup$ $con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p\mathcal{G}_H := p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$) throughout the present proof of (ii) (resp. (ii)′). A binary operation $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ (resp. $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$) is well defined by $w_H(f, f') := f' \circ f$ (resp. $w'_H(f, f') := f' \circ f$). Indeed, let $(f, f') \in \mathcal{G}_H \times \mathcal{G}_H$.

Case 1 $f \in \beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 1′** $f \in p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, it is claimed that $w_H(f, f') \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $(\beta_{(2)}O(H), \beta_{(2)}C(H))$-irresolute and they are $(\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $(\beta_{(2)}O(H), p.\beta_{(2)}C(H))$-irresolute and they are $(p.\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute). And so, we have that $w_H(f, f') \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in con\text{-}p.\beta_{(2)}ch$ - $(H; \kappa^2|H) \subseteq p\mathcal{G}_H$).

Case 2 $f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 2′** $f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, by similar argument of that of Case 1 (resp. Case 1′) above, it is shown that $w_H(f, f') = f' \circ f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') = f' \circ f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$).

Case 3 $f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 3′** $f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, it is shown that $w_H(f, f') \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $\beta_{(2)}C(H)$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute).

Case 4 $f \in \beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 4′** $f \in p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, by definitions, it is shown that $w_H(f, f') \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w_H(f, f') \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $\beta_{(2)}C(H)$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-iresolute). Finally, the binary operation $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ (resp. $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$) satisfies the axiom of group (cf. the proof of (i)) the identity function on $H, 1_H \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $1_H \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$); and so the pair $(\mathcal{G}_H, w_H)$ (resp. $(p\mathcal{G}_H, w'_H)$ forms a group under compositions of mappings. **(iii)** We recall that $\mathcal{G}_H := \beta_{(2)}ch(H;\kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Proof of (ii) above). Since $1_H \in \beta_{(2)}ch(H;\kappa^2|H)$, we have the following: $(\cdot 1)$ $\beta_{(2)}ch(H;\kappa^2|H)$ is a nonempty subset of the group $(\mathcal{G}_H, w_H)$, where $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ is the binary operation (cf. (ii) above). Let $f, g \in \beta_{(2)}ch(H;\kappa^2|H)$. Then, we see that $(\cdot 2)$ $w_H(f, g^{-1}) = g^{-1} \circ f \in \beta_{(2)}ch(H;\kappa^2|H)$. Therefore, by $(\cdot 1)$ and $(\cdot 2)$, it is shown that $\beta_{(2)}ch(H;\kappa^2|H)$ is a subgroup of $\mathcal{G}_H$ (cf. (ii) above). **(iii)$'$** We recall that $p\mathcal{G}_H := p.\beta_{(2)}ch(H;\kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Proof of (ii)$'$ above). We see that $1_H \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Indeed, for a set $B \in \beta_{(2)}O(H)$ and $F_1 \in p.\beta_{(2)}C(H)$, we have the following: $1_H(B) = (1_H)^{-1}(B) = B \in \beta_{(2)}O(H)$ and $1_H(F_1) = (1_H)^{-1}(F_1) = F_1 \in p.\beta_{(2)}C(H)$; and so 1_H and $(1_H)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute. Thus, by Defintion 5.21(i)$'$, it is obtained that $1_H \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Then, we have the following: $(\cdot 1)'p.\beta_{(2)}ch(H;\kappa^2|H)$ is a nonempty subset of the group $(p\mathcal{G}_H, w'_H)$, where $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$ is the binary operation (cf. Proof of (ii)$'$ above). Next, we claim $(\cdot 2)'$ below. Let $f, g \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Then, since f, f^{-1}, g and g^{-1} are all $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute, $g^{-1} \circ f$ and $(g^{-1} \circ f)^{-1} = f^{-1} \circ g$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute bijections. Thus, we prove that: $(\cdot 2)'w'_H(f, g^{-1}) = g^{-1} \circ f \in p.\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Definition 5.21(i)$'$). Therefore, by $(\cdot 1)'$ and $(\cdot 2)'$ above, it is obtained that $p.\beta_{(2)}ch(H;\kappa^2|H)$ is a subgroup of $p\mathcal{G}_H$ (cf. (ii)$'$ above). **(iv) (resp. (iv)$'$)** We see that the identity function $1_{\mathbb{Z}^2} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is a homeomorphism and so $1_{\mathbb{Z}^2} \in h(\mathbb{Z}^2;\kappa^2) \neq \emptyset$. By (i) (resp. (i)$'$) above and its proof, it is known that $\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$) forms a group with the binary operation $\eta_{\mathbb{Z}^2}$ (resp. $\eta'_{\mathbb{Z}^2}$) defined by $\eta_{\mathbb{Z}^2}(a, b) = b \circ a$ (resp. $\eta'_{\mathbb{Z}^2}(a, b) = b \circ a$) for every $a, b \in \beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). And, using Theorem 5.23(ii) (resp. (ii)$'$), we recall that $h(\mathbb{Z}^2;\kappa^2) \subseteq \beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $h(\mathbb{Z}^2;\kappa^2) \subseteq p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). Then, we have the following: $\eta_{\mathbb{Z}^2}(f, g^{-1}) = g^{-1} \circ f$ (resp. $\eta'_{\mathbb{Z}^2}(f, g^{-1}) = g^{-1} \circ f) \in h(\mathbb{Z}^2;\kappa^2)$ for any $f, g \in h(\mathbb{Z}^2;\kappa^2)$. Therefore, it is proved that $h(\mathbb{Z}^2;\kappa^2)$ is a subgroup of $\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). And so, using (iii) (resp. (iii)$'$) above, it is obtained that $h(\mathbb{Z}^2;\kappa^2)$ is also a subgroup of $\mathcal{G}_{\mathbb{Z}^2}$ (resp. $p\mathcal{G}_{\mathbb{Z}^2}$) (cf. the proof of (ii) or (iii) (resp. (ii)$'$ or (iii)$'$) for the notation). □

Notation 5.26 The present notations are applied to Example 5.27 below. Let $H := U((0,0))$. And $U((0,0))$ is denoted abbreviately by U (i.e., $U := U((0,0))$). We define the following functions and two families of functions, **$(\cdot 1)$** $\rho_{45} : (U;\kappa^2|U) \to (U;\kappa^2|U)$ is defined by $\rho_{45}((0,0)) := (0,0)$, $\rho_{45}(p^{(i)}) := y^{(i)}$, $\rho_{45}(y^{(i)}) := p^{(i+1)}$ for each $i \in \{1,2,3,4\}$ with $p^{(5)} := p^{(1)}$ (cf. $(*)$ of line 5 from the top of the present subsection (IV), or (III-2) of the subsection (III)), **$(\cdot 2)$** $\rho_{0\times 90} := 1_U$ (the identity function on U) and $\rho_{k\times 90} := \rho_{(k-1)\times 90} \circ (\rho_{45} \circ \rho_{45})$ for each $k \in \{1,2,3\}$, $(\cdot$**3**$)\rho_{1\times 45} := \rho_{45}, \rho_{m\times 45} := \rho_{90} \circ \rho_{(m-2)\times 45}$ (for $m = 3,5,7$) and **$(\cdot 4)$** $\mathcal{R}_{45} := \{\rho_{m\times 45}, (\rho_{m\times 45})^{-1} | m \in \{1,3,5,7\}\}$, $\mathcal{R}_{90} := \{1_U, \rho_{k\times 90}, (\rho_{k\times 90})^{-1} | k \in \{1,2,3\}\}$.

Example 5.27 Let $H := U((0,0))$ and $U := U((0,0))$. We have the following examples.

(i) $\{\rho_{45}, (\rho_{45})^{-1}\} \subseteq con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Corollary 5.11(ii)).

(ii) (1) $\{\rho_{90}, (\rho_{90})^{-1}\} \subseteq \beta_{(2)}ch(U;\kappa^2|U)$,**(2)** $\{\rho_{90}, (\rho_{90})^{-1}\} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U)$.

In general, we have that:

(i)$'$ $\mathcal{R}_{45} \subseteq con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Corollary 5.11(ii)).

(ii)$'$ (1)$'$ $\mathcal{R}_{90} \subseteq \beta_{(2)}ch(U;\kappa^2|U)$,

(2)$'$ $\mathcal{R}_{90} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Notation 5.26). Hence we have the following:

(iii) (1) $\mathcal{R}_{45} \cup \mathcal{R}_{90} \subseteq \beta_{(2)}ch(U;\kappa^2|U) \cup con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$,

(2) $\mathcal{R}_{45} \cup \mathcal{R}_{90} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U) \cup con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$. The proofs are omitted on the present paper (cf. the detailed proofs are shown by the following pre-print; The detailed Example 5.27 [34]).

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A BEHAVIOR OF THE STRUCTURE TENSOR ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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Abstract. In the theory of real hypersurfaces in a nonflat complex space form, the behavior of the structure tensor ϕ is significant. In this paper, we investigate generalizations of the parallelism of the structure tensor ϕ.

1 Introduction Contact Riemannian geometry is one of the active field in Riemannian geometry. In particular, it is known that cosymplectic manifolds, Sasakian manifolds and Kenmotsu manifolds are characterized by using a behavior of the structure tensor ϕ on contact Riemannian manifolds (see [1]).

In a nonflat complex space form $\widetilde{M}_n(c)$ (namely, a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$), real hypersurfaces admit the almost contact metric structure (ϕ, ξ, η, g) induced from the ambient space. The structure tensor ϕ plays an important role not only contact Riemannian geometry but also the theory of real hypersurfaces in $\widetilde{M}_n(c)$. In this paper, we focus on the parallelism of the structure tensor ϕ of real hypersurfaces in $\widetilde{M}_n(c)$. It is known that *there exists no real hypersurface in $\widetilde{M}_n(c)$ whose the structure tensor ϕ is parallel* (see [4]).

The purpose of this paper is to generalize this fact and to investigate such real hypersurfaces. We first study the following three conditions:

(1.1) $\nabla_\xi \phi = 0$ (ξ-parallelism),

(1.2) $\nabla_X \phi = 0$ for $\forall X \in T^0M$ (T^0M-parallelism),

(1.3) $(\nabla_X \phi)Y - (\nabla_Y \phi)X = 0$ for $\forall X, Y \in TM$ (the Codazzi tensor),

where TM is the tangent bundle of M^{2n-1} and T^0M is the holomorphic distribution, that is, $T^0M = \{X \in TM : X \perp \xi\}$. These conditions are simple generalizations of the parallelism of the structure tensor ϕ. In particular, Conditions (1.1) and (1.2) give characterizations of *Hopf hypersurfaces* and *ruled real hypersurfaces* in $\widetilde{M}_n(c)$, respectively. These classes of real hypersurfaces are significant examples. On the other hand, there exists no real hypersurface satisfying Condition (1.3). So, it is natural to consider generalizations of Condition (1.3).

Secondly, we study the following condition which is a certain generalization of (1.3):

(1.4) $$\operatorname{div} \phi = 0.$$

This condition is inspired by Sharma's work (see [6]). By Condition (1.4), we obtain a characterization of Hopf hypersurfaces with constant mean curvature given by $\alpha/(2n-1)$, where $\alpha = g(A\xi, \xi)$ (Theorem 1).

Finally, we give applications of the discussion of Theorem 1. To do this, we focus on the following two classes of real hypersurfaces in $\widetilde{M}_n(c)$:

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(1) The class of minimal Hopf hypersurfaces in $\widetilde{M}_n(c)$;

(2) The class of real hypersurfaces in $\mathbb{C}H^n(c)$ which satisfies the following the condition:

$$\overline{Ric} = \frac{1}{4}(\text{Trace } A)^2 + \frac{c}{2}(n-1),$$

where $\overline{Ric}$ is the maximal Ricci curvature of real hypersurfaces in $\mathbb{C}H^n(c)$. The latter class was investigated by B. Y. Chen (see [2]). He showed that every real hypersurface in $\mathbb{C}H^n(c)$ satisfies the following inequality:

$$\overline{Ric} \leqq \frac{1}{4}(\text{Trace } A)^2 + \frac{c}{2}(n-1).$$

Moreover he also investigated the equality case of the above inequality.

In the latter of this paper, we characterize the above classes of real hypersurfaces by using the modification of Condition (1.4).

2 Preliminaries Let M^{2n-1} be a real hypersurface with a unit local vector field $\mathcal{N}$ of a complex n-dimensional nonflat complex space form $\widetilde{M}_n$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,\, Y)\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. The former is called *Gauss's formula*, and the latter is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} admits an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. *The characteristic vector field* ξ of M^{2n-1} is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \eta(X) = g(X, \xi),\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \\ &g(\phi X, Y) = -g(X, \phi Y) \quad \text{and} \quad g(\phi X,\, \phi Y) = g(X\,, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}. We call ϕ and η *the structure tensor* and *the contact form* of M^{2n-1}, respectively.

The following equation is a fundamental tool in the theory of real hypersurfaces in $\widetilde{M}_n(c)$:

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.2}$$

We usually call M^{2n-1} a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in $\widetilde{M}_n(c)$ (see [5]).

The following lemma clarifies a fundamental property which is a useful tool in the theory of Hopf hypersurfaces in $\widetilde{M}_n(c)$.

Lemma 1 ([5]). *For a Hopf hypersurface M^{2n-1} with the principal curvature α corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$, we have the following:*

(1) *α is locally constant on M^{2n-1};*

(2) *If X is a tangent vector of M^{2n-1} perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + (c/2))\phi X$.*

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius r around a complex hyper quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;

(C) A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n(\geqq 5)$ is odd;

(D) A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;

(E) A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [5]):

	(A_1)	(A_2)	(B)	(C), (D), (E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
α	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [5]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\alpha)$	1	1	1	1	1	1

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;

($A_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;

($A_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^\ell(c)\,(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;

(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), ($A_{1,0}$), ($A_{1,1}$), (A_2) and (B). Summing up, real hypersurfaces of types ($A_{1,0}$) and ($A_{1,1}$), we call them real hypersurfaces of type (A_1). The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (cf. [5]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
α	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

Finally, we define ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is called *a ruled real hypersurface* if the holomorphic distribution $T^0M = \{X \in TM : X \perp \xi\}$ is integrable and each of its leaves (the maximal integrable manifolds) is a totally geodesic submanifold $\widetilde{M}_{n-1}(c)$ in $\widetilde{M}_n(c)$. The following lemma is known as the characterization of ruled real hypersurfaces from the viewpoint of the shape operator A (cf. [5]).

Lemma 2 ([5]). *Let M^{2n-1} be a real hypersurface M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following three conditions are mutually equivalent*:

1. *M^{2n-1} is a ruled real hypersurface*;
2. *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} | \beta(x) \neq 0\}$ with a unit vector field U orthogonal to ξ :*
$$A\xi = \alpha\xi + \beta U,\ AU = \beta\xi,\ AX = 0$$
for an arbitrary tangent vector X orthogonal to ξ and U, where α, β are differentiable functions on M_1 by $\alpha = g(A\xi, \xi)$ and $\beta = \|A\xi - \alpha\xi\|$;
3. *The shape operator A of M^{2n-1} satisfies $g(AX, Y) = 0$ for arbitrary tangent vectors $X, Y \in T^0M$.*

3 The parallelism of the structure tensor ϕ and its generalizations In the theory of real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$, it is well-known that *there exists no real hypersurface whose structure tensor ϕ is parallel in $\widetilde{M}_n(c)$* (see [4]). This implies that there exists no *cosymplectic* real hypersurfaces in $\widetilde{M}_n(c)$ from the viewpoint of almost contact metric geometry (for detail, see [1]). In this section, we consider simple generalizations of the above fact.

Proposition 1. *Let M^{2n-1} be a real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following statements* (1), (2) *and* (3) *hold*:

(1) *M^{2n-1} satisfies the condition $\nabla_\xi\phi = 0$ if and only if M^{2n-1} is locally congruent to a Hopf hypersurfaces in $\widetilde{M}_n(c)$;*

(2) *M^{2n-1} satisfies the condition $\nabla_X\phi = 0$ for any $X \in T^0M$ if and only if M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$;*

(3) *There does not exist a real hypersurface* M^{2n-1} *in* $\widetilde{M}_n(c)$ *satisfying the condition* $(\nabla_X \phi)Y = (\nabla_Y \phi)X$ *for any vectors* X *and* Y *on* M^{2n-1}.

Proof. (1) Suppose that M^{2n-1} has condition $\nabla_\xi \phi = 0$. By (2.2), we have

$$(\nabla_\xi \phi)X = \eta(X)A\xi - g(A\xi, X)\xi = 0. \tag{3.1}$$

for any vector $X \in TM$. Putting $X = \xi$ in (3.1), then we get $A\xi = g(A\xi, \xi)\xi$. Hence M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Obviously, the converse holds.

(2) Suppose that M^{2n-1} has the condition $(\nabla_X \phi)Y = 0$ for any vector $X \in T^0M$ and $Y \in TM$. By (2.2), we get $g(AX, Y) = 0$ for any vectors $X, Y \in T^0M$. From Lemma 2, M^{2n-1} is a ruled real hypersurface in $\widetilde{M}_n(c)$.

(3) Suppose that M^{2n-1} satisfies the condition $(\nabla_X \phi)Y = (\nabla_Y \phi)X$ for any vectors $X, Y \in TM$. By (2.2), we have

$$\eta(Y)AX = \eta(X)AY \tag{3.2}$$

for any vectors $X, Y \in TM$.

Now we suppose that M^{2n-1} is a non-Hopf hypersurface in $\widetilde{M}_n(c)$. Then the shape operator A forms $A\xi = \alpha\xi + \beta U$, where the function $\beta \neq 0$ and a unit vector U is orthogonal to the characteristic vector field ξ. We put $X \perp \xi$ and $Y = \xi$ in (3.2). Then we have $AX = 0$ for any vector $X \in T^0M$. Hence we can see

$$0 = g(AU, \xi) = g(U, A\xi) = \beta,$$

which is a contradiction.

Next we suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_n(c)$. We take a unit tangent vector field $V(\perp \xi)$ such that $AV = \lambda V$. By using the equation (3.2), we have $\eta(Y)AV = 0$. Putting $Y = \xi$ in this equation, we can see that $AV = \lambda V = 0$. This implies that

$$\lambda = 0. \tag{3.3}$$

Setting $X = \phi V$ and $Y = \xi$ in (3.2), we get $A\phi V = 0$. From this equation, (3.3) and Lemma 1, we obtain

$$0 = (2\lambda - \alpha)A\phi V = (\alpha\lambda + (c/2))\phi V = (c/2)\phi V \neq 0,$$

which is a contradiction. □

Remark 1. *J. T. Cho studied the condition of transversally Killing of* ϕ *namely,* $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0$ *for any* $X, Y \in T^0M$ *(for details, see [3]). This condition give the characterization of ruled real hypersurfaces in* $\widetilde{M}_n(c)$.

4 Statements of results Motivated by (3) of the above proposition, we investigate *the divergence of the structure tensor* ϕ. If the structure tensor ϕ is a Codazzi tensor, then we have

$$\begin{aligned}(\operatorname{div} \phi)X &= \sum_{i=1}^{2n-1} g((\nabla_{e_i}\phi)X, e_i) = \sum_{i=1}^{2n-1} g((\nabla_X \phi)e_i, e_i) \\ &= \operatorname{Trace}(\nabla_X \phi) = \nabla_X(\operatorname{Trace} \phi)\end{aligned}$$

for any tangent vector field $X \in TM$. Note that $\operatorname{Trace} \phi = 0$, we obtain the condition $\operatorname{div} \phi = 0$. Namely, this condition is a generalization of the condition (1.3).

Next we investigate real hypersurfaces in $\widetilde{M}_n(c)$ satisfying $\operatorname{div} \phi = 0$.

Theorem 1. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)\,(n \geqq 2)$. Then M^{2n-1} satisfies the condition* $\mathrm{div}\,\phi = 0$ *if and only if M^{2n-1} is locally congruent to a Hopf hypersurface with constant mean curvature given by $\alpha/(2n-1)$, where $\alpha = g(A\xi, \xi)$. If M^{2n-1} has constant principal curvatures then M^{2n-1} is locally congruent to one of the following*:

(i) *A tube of radius r around a totally geodesic $\mathbb{C}P^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/\sqrt{c}$ and $\cot(\sqrt{c}\,r/2) = \sqrt{\ell/(n-\ell-1)}$*;

(ii) *A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n(\geqq 5)$ is odd and $\cot(\sqrt{c}\,r/2) = (\sqrt{n-1} + \sqrt{2}\,)/\sqrt{n-3}$*;

(iii) *A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n = 9$ and $\cot(\sqrt{c}\,r/2) = 1 + \sqrt{2}$*;

(iv) *A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n = 15$ and $\cot(\sqrt{c}\,r/2) = \sqrt{5+\sqrt{21}}/\sqrt{2}$.*

Proof. Suppose M^{2n-1} satisfies $\mathrm{div}\,\phi = 0$. Then we have

$$
\begin{aligned}
(4.1)\qquad (\mathrm{div}\,\phi)X &= \sum_{i=1}^{2n-1} g((\nabla_{e_i}\phi)X, e_i)\\
&= \sum_{i=1}^{2n-1} g(\eta(X)Ae_i - g(Ae_i, X)\xi, e_i) \quad \text{(from (2.2))}\\
&= \eta(X)(\mathrm{Trace}\,A) - \eta(AX) = 0
\end{aligned}
$$

for any vector $X \in TM$. Hence $g(A\xi, X) = 0$ for any vector $X \in T^0M$. This implies that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Putting $X = \xi$ in (4.1), then we have

$$(4.2)\qquad \mathrm{Trace}\,A = \alpha.$$

This, together with Lemma 1, yields $\mathrm{Trace}\,A = \alpha =$ constant.

Conversely, we suppose that M^{2n-1} is a Hopf hypersurface with $\mathrm{Trace}\,A = g(A\xi, \xi) = \alpha$. Then we can easily check that M^{2n-1} satisfies $\mathrm{div}\,\phi = 0$ (see the relation (4.1)).

Next we suppose that M^{2n-1} has constant principal curvatures. Namely, M^{2n-1} is a Hopf hypersurface with constant principal curvatures. Hence we shall check that M^{2n-1} satisfies the condition (4.2) one by one. Obviously real hypersurfaces of type (A_1) in $\mathbb{C}P^n(c)$, types (A) and (B) in $\mathbb{C}H^n(c)$ do not fulfill the condition (4.2) (see the tables of Section 2).

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/\sqrt{c}$. From (4.2), we have $(2n - 2\ell - 2)x - 2\ell(1/x) = 0$. This implies

$$x^2 = \frac{\ell}{n-\ell-1}.$$

Since $x > 0$, we obtain

$$x = \sqrt{\frac{\ell}{n-\ell-1}}.$$

Hence we have the case (i) of our theorem.

Let M^{2n-1} be a real hypersurface of type (B) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/(2\sqrt{c})$. From (4.2), we have

$$\frac{1+x}{1-x} - \frac{1-x}{1+x} = 0.$$

This means $x = 0$. However, since $x > 1$, M^{2n-1} does not satisfy (4.2).

Let M^{2n-1} be a real hypersurface of type (C) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/(2\sqrt{c})$. From (4.2), we have

$$\frac{2(1+x)}{1-x} - \frac{2(1-x)}{1+x} + (n-3)x - (n-3)\frac{1}{x} = 0.$$

This implies that $(n-3)x^4 - 2(n+1)x^2 + n - 3 = 0$. Hence we obtain

$$x^2 = \frac{n+1 \pm 2\sqrt{2n-2}}{n-3}.$$

Since $x > 1$, we have

$$x = \frac{\sqrt{n-1} + \sqrt{2}}{\sqrt{n-3}}.$$

Hence we have the case (ii) of our theorem. Similarly, we also obtain the cases (iii) and (iv) of our theorem. □

Next we consider the case of 3-dimensional real hypersurfaces in $\widetilde{M}_2(c)$.

Theorem 2. *There does not exist a real hypersurface M^3 in $\widetilde{M}_2(c)$ satisfying the condition* $\mathrm{div}\,\phi = 0$ *in $\widetilde{M}_2(c)$.*

Proof. We suppose that M^3 satisfies the condition $\mathrm{div}\phi = 0$. By Theorem 1, M^3 is locally congruent to a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_2(c)$ and M^3 fulfills Trace $A = \alpha$. We take a unit tangent vector field $V(\perp \xi)$ such that $AV = \lambda V$. When $(2\lambda - \alpha)(p) \neq 0$ at some point $p \in M^{2n-1}$, there exists a neighborhood $\mathcal{U}$ of p such that $2\lambda - \alpha \neq 0$ on $\mathcal{U}$. By using Lemma 1, we have

$$\lambda + \frac{\alpha\lambda + (c/2)}{2\lambda - \alpha} = 0.$$

This equation implies that λ is locally constant.

Next we consider the case $2\lambda - \alpha = 0$ at $q \in M^3$. Then there exists a neighborhood $\mathcal{V}$ of the point q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Indeed, we suppose that there exists no neighborhood $\mathcal{V}$ of q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Then there exists a sequence $\{q_n\}$ on M^3 such that

$$\lim_{n\to\infty} q_n = q \quad \text{and} \quad (2\lambda - \alpha)(q_n) \neq 0 \text{ for each } n.$$

The above discussion in the case $(2\lambda - \alpha)(p) \neq 0$ implies that the continuous function $2\lambda - \alpha$ is constant on some small neighborhood $\mathcal{V}_{q_n}$ of q_n for each n. Then we have $(2\lambda - \alpha)(q) \neq 0$, which is a contradiction. Hence there exists a neighborhood $\mathcal{V}$ of the point q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Thus the function λ is locally constant. Therefore M^3 is locally congruent to a Hopf hypersurface with constant principal curvatures. We know that M^3 is one of the real hypersurfaces of types (A_1), (A_2) or (B) in $\widetilde{M}_2(c)$. However these real hypersurfaces do not satisfy the condition Trace $A = \alpha$ (see the table in Section 2). Therefore we obtain the non-existence of real hypersurfaces M^3 satisfying the condition $\mathrm{div}\,\phi = 0$. □

5 Applications of the discussion in Theorem 1 As a immediate consequence of Theorem 1, if both of the divergence of the structure tensor ϕ and the principal curvature α corresponding to the principal vector ξ vanish identically, then M^{2n-1} is a minimal Hopf hypersurface in $\widetilde{M}_n(c)$. However the converse does not hold. Indeed, a minimal real hypersurface of type (A_1) in $\mathbb{C}P^n(c)$ does not satisfy two conditions $\mathrm{div}\,\phi = 0$ and $\alpha = 0$.

In this section, we first characterize the class of minimal Hopf hypersurfaces in $\widetilde{M}_n(c)$ by the modification of the condition (1.4). This is an application of the discussion in Theorem 1.

Proposition 2. *Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$ ($n \geqq 2$). Then M^{2n-1} is locally congruent to a minimal Hopf hypersurface in $\widetilde{M}_n(c)$ if and only if M^{2n-1} satisfies the condition*

$$(\mathrm{div}\,\phi)X = -\alpha\eta(X) \tag{5.1}$$

for any vector $X \in TM$.

Proof. Suppose that M^{2n-1} satisfies the condition (5.1). By the calculation (4.1), we have

$$\eta(X)(\mathrm{Trace}\,A) - \eta(AX) = -\alpha\eta(X) \tag{5.2}$$

for any $X \in TM$. This means that $g(A\xi, X) = 0$ for any $X \in T^0M$. Hence M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Putting $X = \xi$ in (5.2) we get $\mathrm{Trace}\,A = 0$. So we can see that M^{2n-1} is a minimal Hopf hypersurface in $\widetilde{M}_n(c)$. Clearly, the converse holds by the calculation (4.1). □

Remark 2. *By a direct calculation, real hypersurfaces of types* (A_1), (A_2), (B), (C), (D) *and* (E) *in $\mathbb{C}P^n(c)$ whose radius r satisfies the following table are known as minimal Hopf hypersurfaces with constant principal curvatures in $\widetilde{M}_n(c)$.*

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$\cot \frac{\sqrt{c}r}{2}$	$\frac{1}{\sqrt{2n-1}}$	$\sqrt{\frac{(2\ell+1)}{(2n-2\ell-1)}}$	$\sqrt{n}+\sqrt{n-1}$	$\frac{\sqrt{n}+\sqrt{2}}{\sqrt{n-2}}$	$\sqrt{5}$	$\frac{\sqrt{15}+\sqrt{6}}{3}$

B. Y. Chen studied the maximal Ricci curvature of real hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$ (see [2]). Now we denote by $\overline{Ric}$ the maximal Ricci curvature function on M^{2n-1}, namely

$$\overline{Ric}(p) = \mathrm{Max}\{S(X,X) : X \in T_pM^{2n-1}, \|X\| = 1\}, \quad p \in M^{2n-1},$$

where S is the Ricci tensor of M^{2n-1}. In [2], he showed that every real hypersurface in $\mathbb{C}H^n(c)$ satisfies the following inequality:

$$\overline{Ric} \leqq \frac{1}{4}(\mathrm{Trace}\,A)^2 + \frac{c}{2}(n-1). \tag{5.3}$$

In particular, we can characterize real hypersurfaces which satisfy the equality case of (5.3).

Proposition 3. *Let M^{2n-1} be a real hypersurface in $\mathbb{C}H^n(c)$ ($n \geqq 2$). Then the following three conditions are mutually equivalent:*

(1) *M^{2n-1} satisfies the condition $(\mathrm{div}\,\phi)X = \eta(AX)$ for any tangent vector field X on M^{2n-1};*

(2) *M^{2n-1} satisfies the condition $\overline{Ric} = (1/4)(\mathrm{Trace}\,A)^2 + (c/2)(n-1)$;*

(3) *M^{2n-1} is locally congruent to a Hopf hypersurface with constant mean curvature is given $2\alpha/(2n-1)$.*

Proof. (2) ⇔ (3). See [2].
(1) ⇔ (3). We can prove it by using the same discussion of Theorem 1. □

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CAUSALITY FOR CHARN MODELS

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Abstract. In this study, we consider Granger causality with a highly flexible nonlinear time series model, the conditional heteroscedastic autoregressive nonlinear (CHARN) model. We show that the causality of the CHARN models can be examined by a Portmanteau test based on a constrained maximum likelihood estimator of the parameters, and the test statistic has an approximate asymptotic Chi-square distribution. We describe the Chi-square asymptotics of the Portmanteau test for a CHARN model, provide calculations of the test statistic and investigate the performance of the Portmanteau test using a simulation. This idea is also illustrated using a real data set.

1 Introduction Causality is a relationship between a cause and an effect. The cause is considered to occur not later than the effect and it can help in predictions of the effect. Granger causality, defined by [8], is not necessarily a true causality, but a contributory factor in prediction. That is, for two random variables, X and Y, Granger causality does not clarify whether X causes Y, but focuses on whether X forecasts Y.

Granger causality was proposed in a vector autoregressive (VAR) processes, that is, a linear combination form of random vectors of stationary time series. A standard way to examine Granger causality is the Wald test for the coefficients of VAR model with a limiting χ^2-distribution ([14]). It tests whether the coefficients of the elements from distinct sequences in the VAR system are zero or not. Since the asymptotic χ^2 distribution is often a poor approximation when sample size is small, an F-version of the Wald test is often used instead. The test statistic is obtained by dividing the χ^2-statistic by its degrees of freedom, and is considered from an F-distribution. Likelihood ratio test, the Lagrange multiplier test ([16]) and the other test methods for Granger causality are discussed and compared in [7]. These classical tests give pairwise diagnoses for fixed time lag.

For multiple testing, Portmanteau test is popular. It can test overall significance of the serial correlations over various time lags. Portmanteau test was first proposed by Box and Pierce [2] for model diagnostics of autoregressive and moving average processes. For an autoregressive moving average model of order (p, q), ARMA(p, q), the Box and Pierce test statistic is defined as $n\sum_{k=1}^{h}\hat{r}_k^2$, where n is the sample size, $\hat{r}_k$ is the residual empirical autocorrelation at lag k. For moderately large n and h, the Box-Pierce test statistic is considered approximately χ^2 distributed with degrees of freedom $h-p-q$. A modified version of the Box-Pierce test, i.e. Ljung-Box test ([12]), substantially improves the approximation of the $\chi^2(h-p-q)$ distribution and is frequently applied in a variety of fields.

Many other modifications have been suggested. Among them, Taniguchi and Amano [17] pointed out that both the Box-Pierce statistic and the Ljung-Box statistic never converge to $\chi^2(h-p-q)$ for finite h. Instead, they proposed a modified Whittle likelihood ratio test which is asymptotically chi-square distributed for any finite h under ARMA(p, q) models and Bloomfield's exponential spectral density assumption. Recently, Chen and Lee [4] developed a Bayesian procedure for Granger causality test based on the generalized auto-regressive conditional heteroscedasticity (GARCH) type of integer-valued models and applied it to

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testing causality relationships between temperature and crime data, as well as air pollution and human influenza data ([5]). Moreover, Akashi et al [1] proposed a new likelihood ratio based Portmanteau test which is applicable in more general situations. They also showed application examples for linear models.

Ideas for test of nonlinearity are also available. For example, Tsay [20] generalized Keenan's ([11]) Tukey nonadditivity-type test, improved its power, and proposed a test for concurrent nonlinearity as a diagnostic tool to examine the linearity assumption of time series models. Castle and Hendry [3] provided a Portmanteau test based on polynomial and exponential functions. It focuses on the nonlinearity in the conditional mean. Chen et al [6] proposed a hysteretic vector autoregressive (HVAR) model to test nonlinear Granger causality between two target time series and using posterior odds ratios for multiple testing.

To investigate Granger causality for nonlinear time series models, we consider a more flexible model, the conditional heteroscedastic autoregressive nonlinear (CHARN) model, where both the conditional mean and the residual are functions of the past. The CHARN model was introduced by [9] for financial data analysis. Because of its non-normality, non-linearity and the blindingly general form, it has come into use in various fields of time series ([10], [19]).

We are interested in whether the Portmanteau test proposed by [1] can be used to detect the Granger causality for the CHARN model. In this paper, we examine nonlinear causality with this method. To show the feasibility and the performance of the method, we provide an example with the calculation of the test statistic for a specified CHARN model and conduct a simulation to confirm its capability for different sample sizes and different parameter settings of the CHARN model. We also demonstrate that the Portmanteau test can be used in practice if the the normality of the residuals of the CHARN model is satisfied.

The paper is organized as follows. Section 2 sets up the high dimensional stochastic process of the CHARN model, provides assumptions for stationarity of the process and requirements for the asymptotic optimal estimation theory of the parameters in the model, and formulates the nonlinear Portmanteau test for the CHARN model. Section 3 discusses the asymptotic distribution of the Portmanteau test and calculates the test statistic for a given CHARN model. In Section 4, we investigate the performance of the test by simulation. Finally, in Section 5, supposing that data follow CHARN models, we test whether the infection number of COVID-19 in Tokyo Granger causes the infection numbers in two of surrounding prefectures of Tokyo in Japan.

2 Assumptions and the Portmanteau Test

Let

$$\boldsymbol{X}(t) = \begin{pmatrix} \boldsymbol{X}_1(t) \\ \boldsymbol{X}_2(t) \end{pmatrix} = \begin{pmatrix} X_{1,1}(t) \\ \vdots \\ X_{1,m_1}(t) \\ X_{2,1}(t) \\ \vdots \\ X_{2,m_2}(t) \end{pmatrix} \tag{1}$$

be a $(m_1 + m_2 =)m$-dimensional stochastic process generated by

$$\boldsymbol{X}(t) = \boldsymbol{F}_{\boldsymbol{\theta}}\{\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-p)\} + \boldsymbol{H}_{\boldsymbol{\theta}}\{\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-q)\}\, \boldsymbol{U}(t), \tag{2}$$

where $\boldsymbol{F}_{\boldsymbol{\theta}} : \mathbb{R}^{mp} \to \mathbb{R}^m$ is a vector-valued measurable function, $\boldsymbol{H}_{\boldsymbol{\theta}} : \mathbb{R}^{mq} \to \mathbb{R}^{m\times m}$ is a positive definite matrix-valued measurable function, and $\boldsymbol{U}(t) = (\boldsymbol{U}_1(t), \boldsymbol{U}_2(t))'$ combining an m_1-vector $\boldsymbol{U}_1(t)$ and an m_2-vector $\boldsymbol{U}_2(t)$, is a sequence of m i.i.d. random variables with $E\{\boldsymbol{U}(t)\} = \boldsymbol{0}$, $E|\boldsymbol{U}(t)| < \infty$, and $\boldsymbol{U}(t)$ is independent of $\{\boldsymbol{X}(s),\ s < t\}$. Here, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)' \in \Theta \subset \mathbb{R}^r$ is a vector of unknown parameters.

We write $\boldsymbol{x} = (x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}, \ldots, x_{p1}, \ldots, x_{pm})'$ as an (mp)-vector, and $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)' = (u_1, u_2, \ldots, u_{m_1}, u_{m_1+1}, \ldots, u_{m2})'$ an m-vector. From now on, without loss of generality, we assume $p = q$ and make the following assumptions.

Assumption 1 ([13]) (A.1) *$\boldsymbol{U}(t)$ has a probability density $p(\boldsymbol{u}) > 0$ on $\mathbb{R}^m$.*

(A.2) *There exist constants $a_{ij} \geq 0$, $b_{ij} \geq 0$, $1 \leq i \leq p$, $1 \leq j \leq m$, such that as $|\boldsymbol{x}| \to \infty$,*

$$|\boldsymbol{F}_{\boldsymbol{\theta}}(\boldsymbol{x})| \leq \sum_{i=1}^{p} \sum_{j=1}^{m} a_{ij} |x_{ij}| + o(|\boldsymbol{x}|),$$

$$|\boldsymbol{H}_{\boldsymbol{\theta}}(\boldsymbol{x})| \leq \sum_{i=1}^{p} \sum_{j=1}^{m} b_{ij} |x_{ij}| + o(|\boldsymbol{x}|),$$

where $|\boldsymbol{A}|$ denotes the sum of the absolute values of all entries of $\boldsymbol{A}$.

(A.3) *$\boldsymbol{H}_{\boldsymbol{\theta}}(\boldsymbol{x})$ is continuous and symmetric on $\mathbb{R}^{mp}$, and there exists a positive constant λ such that*

$$\lambda_m\{\boldsymbol{H}_{\boldsymbol{\theta}}(\boldsymbol{x})\} \geq \lambda \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^{mp},$$

where $\lambda_m\{(\cdot)\}$ is the minimum eigenvalue of $(\cdot)$.

(A.4)

$$\max_{1 \leq j \leq m} \left\{ \sum_{i=1}^{p} a_{ij} + E|\boldsymbol{U}(t)| \sum_{i=1}^{p} b_{ij} \right\} < 1.$$

Assumption 1 guarantees that $\{X_t\}$ is strictly stationary.

Assumption 2 ([18]) (B.1)

$$E_{\boldsymbol{\theta}}||\boldsymbol{F}_{\boldsymbol{\theta}}(\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-p))||^2 < \infty,$$
$$E_{\boldsymbol{\theta}}||\boldsymbol{H}_{\boldsymbol{\theta}}(\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-p))||^2 < \infty, \quad \text{for all } \boldsymbol{\theta} \in \boldsymbol{\Theta},$$

where $||\boldsymbol{A}||$ indicates the Euclidian norm of a vector $\boldsymbol{A}$ or a matrix $\boldsymbol{A}$.

(B.2) *There exists $c > 0$ such that*

$$c \leq ||\boldsymbol{H}_{\boldsymbol{\theta}'}^{-1/2}(\boldsymbol{x}) \boldsymbol{H}_{\boldsymbol{\theta}}(\boldsymbol{x}) \boldsymbol{H}_{\boldsymbol{\theta}'}^{-1/2}(\boldsymbol{x})|| < \infty,$$

for all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \boldsymbol{\Theta}$, and for all $\boldsymbol{x} \in \mathbb{R}^{mp}$.

(B.3) *$\boldsymbol{H}_{\boldsymbol{\theta}}$ and $\boldsymbol{F}_{\boldsymbol{\theta}}$ are continuously differentiable with respect to $\boldsymbol{\theta}$, and their derivatives $\partial_j \boldsymbol{H}_{\boldsymbol{\theta}}$ and $\partial_j \boldsymbol{F}_{\boldsymbol{\theta}}$ ($\partial_j = \partial/\partial\theta_j$), $j = 1, \ldots, r$, satisfy the condition that there exist square-integrable functions A_j and B_j such that $||\partial_j \boldsymbol{H}_{\boldsymbol{\theta}}|| \leq A_j$, and $||\partial_j \boldsymbol{F}_{\boldsymbol{\theta}}|| \leq B_j$ ($j = 1, \ldots, r$), for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.*

(B.4) *Density $p(\cdot)$ satisfies*

$$\lim_{||\boldsymbol{u}|| \to \infty} ||\boldsymbol{u}|| p(\boldsymbol{u}) = 0 \quad \text{and} \quad \int \boldsymbol{u}\boldsymbol{u}' p(\boldsymbol{u}) d\boldsymbol{u} = \boldsymbol{I}_m,$$

where $\boldsymbol{I}_m$ is the $m \times m$ identity matrix.

(B.5) *The continuous derivative* $D_p = D_p(\boldsymbol{u}) \equiv \left(\frac{d}{du_1}p(\boldsymbol{u}), \ldots, \frac{d}{du_m}p(\boldsymbol{u})\right)'$ *exists on* $\mathbb{R}^m$ *and*

$$\int ||p^{-1}D_p||^4 p(\boldsymbol{u})d\boldsymbol{u} < \infty,$$
$$\int ||\boldsymbol{u}||^2 ||p^{-1}D_p||^2 p(\boldsymbol{u})d\boldsymbol{u} < \infty.$$

Assumption 2 is necessary in construction of the asymptotic optimal estimation theory for $\boldsymbol{\theta}$.

For given time series data in (1), to consider the Granger causality from $\boldsymbol{X}_2(t)$ to $\boldsymbol{X}_1(t)$, we focus on the prediction of $\boldsymbol{X}_1(t)$, $t = 1, \ldots, n$. Similar to (2), assume that $\boldsymbol{X}_1(t)$ is observed from the following CHARN model

$$\boldsymbol{X}_1(t) = \boldsymbol{F}_{1,\boldsymbol{\theta}}\{\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-p)\} + \boldsymbol{H}_{1,\boldsymbol{\theta}}\{\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-q)\}\, \boldsymbol{U}_1(t), \tag{3}$$

where $\boldsymbol{\theta}$ is a vector of unknown parameters, $\boldsymbol{F}_{1,\boldsymbol{\theta}} : \mathbb{R}^{mp} \to \mathbb{R}^{m_1}$ is a vector-valued measurable function, $\boldsymbol{H}_{1,\boldsymbol{\theta}} : \mathbb{R}^{mq} \to \mathbb{R}^{m_1 \times m_1}$ is a positive definite matrix-valued measurable function, and the model satisfies Assumptions 1 and 2 for $\boldsymbol{U}_1(t)$, $\boldsymbol{F}_{1,\boldsymbol{\theta}}$ and $\boldsymbol{H}_{1,\boldsymbol{\theta}}$ instead of $\boldsymbol{U}(t)$, $\boldsymbol{F}_{\boldsymbol{\theta}}$ and $\boldsymbol{H}_{\boldsymbol{\theta}}$, respectively.

Suppose that the dynamic part of (3) can be expressed as

$$\begin{aligned}\boldsymbol{F}_{1,\boldsymbol{\theta}} &= \boldsymbol{F}_{1,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}\{\boldsymbol{X}(t-1), \ldots, \boldsymbol{X}(t-p)\} \\ &= \boldsymbol{F}_{1,\boldsymbol{\theta}_1}\{\boldsymbol{X}_1(t-1), \ldots, \boldsymbol{X}_1(t-p)\} + \boldsymbol{\theta}^*_{r_1+1}\boldsymbol{X}_2(t-1) + \ldots + \boldsymbol{\theta}^*_{r_1+r_1'}\boldsymbol{X}_2(t-p) \\ &\quad + \sum_{\ell=1}^{p} \exp\left\{-\frac{1}{2}\mathrm{tr}\{\boldsymbol{X}_1(t-\ell)\boldsymbol{X}_1(t-\ell)'\}\right\}\boldsymbol{\theta}^*_{r_1+r_1'+\ell}\boldsymbol{X}_2(t-\ell),\end{aligned} \tag{4}$$

where $\boldsymbol{F}_{1,\boldsymbol{\theta}}$ is the prediction of $\boldsymbol{X}_1(t)$ using information of both $\boldsymbol{X}_1(t-\ell)$ and $\boldsymbol{X}_2(t-\ell)$, $\boldsymbol{F}_{1,\boldsymbol{\theta}_1}$ is the prediction of $\boldsymbol{X}_1(t)$ with only information of $\boldsymbol{X}_1(t-\ell)$, for $\ell = 1, \ldots, p$, and the subscript of $\boldsymbol{\theta}^*_{r_1+r_1'+\ell}$ changes from $(r_1 + r_1' + 1)$ to $(r_1 + r_1' + p) = (r_1 + r_2)$.

In the vector of the unknown parameters

$$\boldsymbol{\theta} = (\mathrm{vec}(\boldsymbol{\theta}_1), \boldsymbol{\theta}_2)' = \left(\mathrm{vec}(\boldsymbol{\theta}_1), \mathrm{vec}(\boldsymbol{\theta}^*_{r_1+1}), \cdots, \mathrm{vec}(\boldsymbol{\theta}^*_{r_1+r_1'}), \cdots, \mathrm{vec}(\boldsymbol{\theta}^*_{r_1+r_2})\right)',$$

$\boldsymbol{\theta}_1$ is an $m_1 \times r_1$ matrix in function $\boldsymbol{F}_{1,\boldsymbol{\theta}_1}$ of $\boldsymbol{X}_1(t-\ell)$; $\boldsymbol{\theta}^*_{r_1+1}, \ldots, \boldsymbol{\theta}^*_{r_1+r_1'}, \ldots$, and $\boldsymbol{\theta}^*_{r_1+r_2}$ are $m_1 \times m_2$ matrices for terms containing $\boldsymbol{X}_2(t-\ell)$, $\ell = 1, \ldots, p$.

Then the prediction error of $\boldsymbol{X}_1(t)$ by $\boldsymbol{F}_{1,\boldsymbol{\theta}_1}$ becomes

$$P_1 = E[\mathrm{tr}\{(\boldsymbol{X}_1(t) - \boldsymbol{F}_{1,\boldsymbol{\theta}_1})(\boldsymbol{X}_1(t) - \boldsymbol{F}_{1,\boldsymbol{\theta}_1})'\}]$$

and that by $\boldsymbol{F}_{1,\boldsymbol{\theta}}$ is

$$P_2 = E[\mathrm{tr}\{(\boldsymbol{X}_1(t) - \boldsymbol{F}_{1,\boldsymbol{\theta}})(\boldsymbol{X}_1(t) - \boldsymbol{F}_{1,\boldsymbol{\theta}})'\}].$$

Letting $V \equiv P_1 - P_2$, we can introduce a nonlinear causality from $\{\boldsymbol{X}_2(t)\}$ to $\{\boldsymbol{X}_1(t)\}$ by V, i.e., if $V = 0$, then we say that $\{\boldsymbol{X}_2(t)\}$ does not cause $\{\boldsymbol{X}_1(t)\}$ in our CHARN setting (for short, $\boldsymbol{X}_2(t) \nrightarrow \boldsymbol{X}_1(t)$). We can understand that $\boldsymbol{X}_2(t) \nrightarrow \boldsymbol{X}_1(t)$ is grasped by the testing problem:

$$H : \boldsymbol{\theta}_2 = \boldsymbol{0}, \quad v.s. \quad A : \boldsymbol{\theta}_2 \neq \boldsymbol{0}. \tag{5}$$

Let $\boldsymbol{X}(1), \ldots, \boldsymbol{X}(n)$ be an observed stretch from (2), and let

$$\begin{aligned}\ell(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) &= \log\{\text{likelihood function based on } \boldsymbol{X}(1), \ldots, \boldsymbol{X}(n)\} \\ &= \sum_{t=p}^{n} \log \left[p\{\boldsymbol{H}_{\boldsymbol{\theta}}^{-1}(\boldsymbol{X}(t) - \boldsymbol{F}_{\boldsymbol{\theta}})\}\{\det \boldsymbol{H}_{\boldsymbol{\theta}}\}^{-1} \right].\end{aligned}$$

In what follows we deal with the following marginal log-likelihood function:

$$\ell_1(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \sum_{t=p}^{n} \log \left[p_1 \left\{ \boldsymbol{H}_{1,\boldsymbol{\theta}}^{-1} \left(\boldsymbol{X}_1(t) - \boldsymbol{F}_{1,\boldsymbol{\theta}} \right) \right\} \left(\det \boldsymbol{H}_{1,\boldsymbol{\theta}} \right)^{-1} \right], \tag{6}$$

where $p_1(\cdot)$ is the marginal pdf of $\boldsymbol{U}_1(t)$.

Define

$$\widehat{\boldsymbol{\theta}}_1 = \arg\max_{\boldsymbol{\theta}_1} \ell_1(\boldsymbol{\theta}_1, \boldsymbol{0}), \quad \widehat{\boldsymbol{\theta}}_2 = \arg\max_{\boldsymbol{\theta}_2} \ell_1(\widehat{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2).$$

For the problem of testing (5), we introduce the following test of Portmanteau type:

$$PT = 2[\ell_1(\widehat{\boldsymbol{\theta}}_1, \widehat{\boldsymbol{\theta}}_2) - \ell_1(\widehat{\boldsymbol{\theta}}_1, \boldsymbol{0})]. \tag{7}$$

The Fisher information matrix for the general model is given by

$$\begin{aligned}\mathcal{F}(\boldsymbol{\theta}) &\equiv \lim_{n\to\infty} \frac{1}{n} E\left[\frac{\partial}{\partial \boldsymbol{\theta}} \ell_1(\boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}'} \ell_1(\boldsymbol{\theta}) \right] \\ &= \begin{bmatrix} \boldsymbol{F}_{11} & \boldsymbol{F}_{12} \\ \boldsymbol{F}_{21} & \boldsymbol{F}_{22} \end{bmatrix}, \quad \text{(say)}.\end{aligned}$$

The following two lemmas follow from [1].

Lemma 1 *Under H,*

$$PT = \boldsymbol{N}_{R_2}' \boldsymbol{F}_{22\cdot1}^{1/2} \boldsymbol{F}_{22}^{-1} \boldsymbol{F}_{22\cdot1}^{1/2} \boldsymbol{N}_{R_2} + o_p(1),$$

where $\boldsymbol{N}_{R_2}$ is the $R_2(= m_1 m_2 r_2)$-dimensional standard normal random vector, and $\boldsymbol{F}_{22\cdot1} = \boldsymbol{F}_{22} - \boldsymbol{F}_{21}\boldsymbol{F}_{11}^{-1}\boldsymbol{F}_{12}$.

Lemma 2 **(i)** *Let $R_1 = m_1 r_1$. If $R_1 < R_2$, $\boldsymbol{F}_{22} = \boldsymbol{I}_{R_2}$ and $\boldsymbol{F}_{21}\boldsymbol{F}_{11}^{-1}\boldsymbol{F}_{12}$ is idempotent with rank $\bar{r}$, then*

$$PT \xrightarrow{d} \chi^2_{R_2 - \bar{r}} \qquad \text{under } H.$$

(ii) *If $\boldsymbol{F}_{22} \neq \boldsymbol{I}_{R_2}$ and $\boldsymbol{F}_{12} = \boldsymbol{0}$, then*

$$PT \xrightarrow{d} \chi^2_{R_2} \qquad \text{under } H.$$

3 Asymptotic Distribution of Portmanteau Test In this section, we describe the χ^2-asymptotics of the Portmanteau test PT for the simplest case of (1), where $m_1 = m_2 = 1$, that is, $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$.

Let

$$\begin{aligned}\boldsymbol{Z}(t) = \Bigg(& X_2(t-1), \ldots, X_2(t-p), \exp\left(-\frac{1}{2}X_1(t-1)^2\right) X_2(t-1), \ldots, \\ & \exp\left(-\frac{1}{2}X_1(t-p)^2\right) X_2(t-p) \Bigg)',\end{aligned}$$

whose dimension is r_2. Then we can write $\boldsymbol{F}_{1,\boldsymbol{\theta}}$ of (4) as

$$\begin{aligned}\boldsymbol{F}_{1,\boldsymbol{\theta}} &= \boldsymbol{F}_{1,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}\{\boldsymbol{X}(t-1),\dots,\boldsymbol{X}(t-p)\}\\ &= \boldsymbol{F}_{1,\boldsymbol{\theta}_1}\{X_1(t-1),\dots,X_1(t-p)\} + \boldsymbol{\theta}_2'\boldsymbol{Z}(t),\end{aligned}$$

where $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are r_1 and r_2-vectors, respectively.

If $p_1(\cdot)$ is Gaussian, it is not difficult to show

$$\boldsymbol{F}_{12} = \lim_{n\to\infty}\frac{-1}{n}E\left[\frac{\partial^2}{\partial\boldsymbol{\theta}_1\partial\boldsymbol{\theta}_2'}\ell_1(\boldsymbol{\theta})\right] = \mathbf{0}.$$

Then we have

Proposition 1 *If $p_1(\cdot)$ is a Gaussian probability density, under H,*

$$PT \xrightarrow{d} \chi^2_{(r_2)}.$$

Example 1. In (4), let

$$\begin{aligned}\boldsymbol{F}_{1,\boldsymbol{\theta}} &= \theta_1 X_1(t-1) + \theta_2\exp\left(-\frac{1}{2}X_1(t-1)^2\right) + \theta_3 X_2(t-1) + \theta_4\exp\left(-\frac{1}{2}X_1(t-1)^2\right)X_2(t-1)\\ &= \boldsymbol{Y}'(t-1)\boldsymbol{\theta},\end{aligned}$$

where

(8)

$$\begin{aligned}\boldsymbol{Y}(t-1) &:= (Y_1(t-1), Y_2(t-1), Y_3(t-1), Y_4(t-1))'\\ &:= \left(X_1(t-1),\ \exp\left(-\frac{1}{2}X_1(t-1)^2\right),\ X_2(t-1),\ \exp\left(-\frac{1}{2}X_1(t-1)^2\right)X_2(t-1)\right)',\end{aligned}$$

and $\boldsymbol{\theta} = (\theta_1,\theta_2,\theta_3,\theta_4)'$. In (6), let $\boldsymbol{H}_{1,\boldsymbol{\theta}} = \sqrt{0.1 + \varepsilon X_1(t-1)^2 + \delta X_2(t-1)^2} =: \sqrt{W(t-1)}$, where ε and δ are small positive values providing minor effects on the residual part of the CHARN model. Assume that $p_1(\cdot)$ is the pdf of $N(0,1)$. We see that $\boldsymbol{\theta}_1 = (\theta_1,\theta_2)'$, $\boldsymbol{\theta}_2 = (\theta_3,\theta_4)'$, $r_1 = 2$, $r_2 = 2$. Suppose that $\sum_{j=1}^4|\theta_j| < 1$. Then we can find PT in the following way.

Since

$$p_1\left\{\frac{X_1(t)-\boldsymbol{F}_{1,\boldsymbol{\theta}}}{\boldsymbol{H}_{1,\boldsymbol{\theta}}}\right\}\times\frac{1}{\boldsymbol{H}_{1,\boldsymbol{\theta}}} = \frac{1}{\sqrt{2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}}\exp\left\{-\frac{\left(X_1(t)-\boldsymbol{F}_{1,\boldsymbol{\theta}}\right)^2}{2\left(\boldsymbol{H}_{1,\boldsymbol{\theta}}\right)^2}\right\}$$

$$\begin{aligned}=&\frac{1}{\sqrt{2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}}\\ &\exp\left(-\frac{[X_1(t)-\theta_1X_1(t-1)-\theta_2\exp\{-\frac{1}{2}X_1(t-1)^2\}-\theta_3X_2(t-1)-\theta_4\exp\{-\frac{1}{2}X_1(t-1)^2\}X_2(t-1)]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}\right),\end{aligned}$$

and

$$\begin{aligned}&\log\left[p_1\left\{\frac{X_1(t)-\boldsymbol{F}_{1,\boldsymbol{\theta}}}{\boldsymbol{H}_{1,\boldsymbol{\theta}}}\right\}\times\frac{1}{\boldsymbol{H}_{1,\boldsymbol{\theta}}}\right]\\ &= -\frac{1}{2}\log(2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\})\\ &-\frac{[X_1(t)-\theta_1X_1(t-1)-\theta_2\exp\{-\frac{1}{2}X_1(t-1)^2\}-\theta_3X_2(t-1)-\theta_4\exp\{-\frac{1}{2}X_1(t-1)^2\}X_2(t-1)]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}},\end{aligned}$$

Equ. (6) becomes

(9)

$$\ell_1(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=\sum_{t=2}^{n}\log\left[p_1\left\{\frac{X_1(t)-\boldsymbol{F}_{1,\boldsymbol{\theta}}}{\boldsymbol{H}_{1,\boldsymbol{\theta}}}\right\}\times\frac{1}{\boldsymbol{H}_{1,\boldsymbol{\theta}}}\right]$$

$$=-\frac{1}{2}\sum_{t=2}^{n}\log(2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\})$$

$$-\sum_{t=2}^{n}\frac{[X_1(t)-\theta_1X_1(t-1)-\theta_2\exp\{-\frac{1}{2}X_1(t-1)^2\}-\theta_3X_2(t-1)-\theta_4\exp\{-\frac{1}{2}X_1(t-1)^2\}X_2(t-1)]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}.$$

Under the null hypothesis H, the log-likelihood function becomes

$$\ell_1(\boldsymbol{\theta}_1,\mathbf{0})=-\frac{1}{2}\sum_{t=2}^{n}\log(2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\})$$
$$-\sum_{t=2}^{n}\frac{[X_1(t)-\theta_1X_1(t-1)-\theta_2\exp\{-\frac{1}{2}X_1(t-1)^2\}]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}.$$

Setting $\partial\ell_1/\partial\theta_i=0,\ i\in\{1,2\}$, with notations $Y_1(t-1)$, $Y_2(t-1)$ and $W(t-1)$, we see that

$$\begin{pmatrix}\sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_1(t-1)\}/W(t-1)\\ \sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_2(t-1)\}/W(t-1)\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$
$$=\begin{pmatrix}\sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_1(t-1)\}/W(t-1)\\ \sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_2(t-1)\}/W(t-1)\end{pmatrix}\boldsymbol{\theta}_1$$
$$=\begin{pmatrix}\sum_{t=2}^{n}\{X_1(t)Y_1(t-1)\}/W(t-1)\\ \sum_{t=2}^{n}\{X_1(t)Y_2(t-1)\}/W(t-1)\end{pmatrix}$$

and obtain

$$\widehat{\boldsymbol{\theta}}_1=\begin{pmatrix}\sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_1(t-1)\}/W(t-1)\\ \sum_{t=2}^{n}\{(Y_1(t-1),Y_2(t-1))Y_2(t-1)\}/W(t-1)\end{pmatrix}^{-1}\begin{pmatrix}\sum_{t=2}^{n}\{X_1(t)Y_1(t-1)\}/W(t-1)\\ \sum_{t=2}^{n}\{X_1(t)Y_2(t-1)\}/W(t-1)\end{pmatrix}.$$

Substitute $\widehat{\boldsymbol{\theta}}_1=(\hat{\theta}_1,\hat{\theta}_2)'$ into Equ. (9) and let $Y_0(t):=X_1(t)-\hat{\theta}_1X_1(t-1)-\hat{\theta}_2\exp\{-\frac{1}{2}X_1(t-1)^2\}$, $t=2,3,\ldots$, we maximize

$$\ell_1(\widehat{\boldsymbol{\theta}}_1,\boldsymbol{\theta}_2)=-\frac{1}{2}\sum_{t=2}^{n}\log(2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\})$$
$$-\sum_{t=2}^{n}\frac{[X_1(t)-\hat{\theta}_1X_1(t-1)-\hat{\theta}_2\exp\{-\frac{1}{2}X_1(t-1)^2\}-\theta_3X_2(t-1)-\theta_4\exp\{-\frac{1}{2}X_1(t-1)^2\}X_2(t-1)]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}$$
$$=-\frac{1}{2}\sum_{t=2}^{n}\log(2\pi\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\})$$
$$-\sum_{t=2}^{n}\frac{[Y_0(t)-\theta_3X_2(t-1)-\theta_4\exp\{-\frac{1}{2}X_1(t-1)^2\}X_2(t-1)]^2}{2\times\{0.1+\varepsilon X_1(t-1)^2+\delta X_2(t-1)^2\}}.$$

Set $\partial\ell_1/\partial\theta_i=0,\ i\in\{3,4\}$, we have

$$\begin{pmatrix}\sum_{t=2}^{n}[\{(Y_3(t-1),Y_4(t-1))Y_3(t-1)\}/W(t-1)]\\ \sum_{t=2}^{n}[\{(Y_3(t-1),Y_4(t-1))Y_4(t-1)\}/W(t-1)]\end{pmatrix}\begin{pmatrix}\theta_3\\ \theta_4\end{pmatrix}=\begin{pmatrix}\sum_{t=2}^{n}[Y_0(t)Y_3(t-1)/W(t-1)]\\ \sum_{t=2}^{n}[Y_0(t)Y_4(t-1)/W(t-1)]\end{pmatrix},$$

then $\boldsymbol{\theta}_2$ can be estimated as follows

$$\widehat{\boldsymbol{\theta}}_2 = \begin{pmatrix} \hat{\theta}_3 \\ \hat{\theta}_4 \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{t=2}^{n} \left[\{ (Y_3(t-1), Y_4(t-1)) Y_3(t-1) \} / W(t-1) \right] \\ \sum_{t=2}^{n} \left[\{ (Y_3(t-1), Y_4(t-1)) Y_4(t-1) \} / W(t-1) \right] \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=2}^{n} \left[Y_0(t) Y_3(t-1)/W(t-1) \right] \\ \sum_{t=2}^{n} \left[Y_0(t) Y_4(t-1)/W(t-1) \right] \end{pmatrix}.$$

By substituting the obtained estimates $\widehat{\boldsymbol{\theta}}_1$ and $\widehat{\boldsymbol{\theta}}_2$ into (7), we then can calculate the Portmanteau test statistic PT. That is, with $Y_0(t) = X_1(t) - (Y_1(t-1), Y_2(t-1))\, \widehat{\boldsymbol{\theta}}_1$,

$$\begin{aligned} PT &= -\sum_{t=2}^{n} \log(2\pi W(t-1)) - \frac{2}{2} \sum_{t=2}^{n} \frac{\left\{ Y_0(t) - (Y_3(t-1), Y_4(t-1))\ \widehat{\boldsymbol{\theta}}_2 \right\}^2}{W(t-1)} \\ &\quad + \sum_{t=2}^{n} \log(2\pi W(t-1)) + \frac{2}{2} \sum_{t=2}^{n} \frac{Y_0^2(t)}{W(t-1)} \\ &= \sum_{t=2}^{n} \frac{Y_0^2(t)}{W(t-1)} - \sum_{t=2}^{n} \frac{\left\{ Y_0(t) - (Y_3(t-1), Y_4(t-1))\ \widehat{\boldsymbol{\theta}}_2 \right\}^2}{W(t-1)}. \end{aligned}$$

4 Simulation Study To evaluate the availability of the Portmanteau test for Granger causality, we carry out the following simulation. We generate data from the two dimensional stochastic process below in which $X_2(t)$ is AR(1) and $X_1(t)$ is a CHARN model:

$$\boldsymbol{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}'(t-1)\boldsymbol{\theta} + \sqrt{0.1 + \varepsilon X_1(t-1)^2 + \delta X_2(t-1)^2}\, U_1(t) \\ \theta_{21} X_2(t-1) + U_2(t) \end{pmatrix},$$

where $\boldsymbol{Y}(t-1)$ is defined in (8), $\boldsymbol{\theta} = (\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14})'$, and $U_i,\ i = 1, 2$ are i.i.d. $N(0, 1)$. For eight models, the parameters are set as in the table below. We generate data for each model with three different lengths, $n = 50, 300, 1000$. For each model and each length, 3000 replications are made. We then test the non-linear causality $H_0 :\ \theta_{13} = \theta_{14} = 0$, and calculate the empirical rejection ratios for each situation. The last three columns in Table 1 present the empirical rejection ratios of non-causality $X_2(t) \nrightarrow X_1(t)$. Nominal significance level is 0.05.

Table 1: Empirical rejection ratio of the null hypothesis of non-causality for eight models

Model	ε	δ	$\boldsymbol{\theta}$	θ_{21}	$n = 50$	$n = 300$	$n = 1000$
i	0	0	$(0.1, 0, 0, 0)'$	0.2	0.035	0.049	0.048
ii	0.01	0	$(0.1, 0, 0, 0)'$	0.2	0.036	0.043	0.049
iii	0.01	0.01	$(0.1, 0, 0, 0)'$	0.2	0.036	0.045	0.055
iv	0.01	0.05	$(0.1, 0, 0, 0)'$	0.2	0.041	0.051	0.055
v	0.01	0	$(0.1, 0, 0.1, 0)'$	0.2	0.462	0.999	1.000
vi	0.01	0.01	$(0.1, 0, 0.1, 0)'$	0.2	0.362	0.995	1.000
vii	0.01	0.01	$(0.1, 0.1, 0.1, 0)'$	0.2	0.376	0.996	1.000
viii	0.01	0.01	$(0.1, 0.1, 0.1, 0.1)'$	0.2	0.916	1.000	1.000

From this result, we see that when the sample size is large or moderately large, the Portmanteau test works well, although there is a need to improve the power when the sample size is small. When there is no Granger causality (Models i – iv), the empirical rejection ratio is close to the significance level; when the Granger causality exists (Models v – viii), the empirical rejection ratio is close to one.

5 Data Analysis We examine Granger causality between the numbers of infected people with COVID-19 in Tokyo and its two neighboring prefectures. The data are taken from the website of NHK `https://www3.nhk.or.jp/news/special/coronavirus/data-widget/#mokuji1`. We focus on the data of Kanagawa Prefecture, Yamanashi Prefecture and the Tokyo metropolitan area, from January 16 to December 17, 2020. The three time series as well as their cross-autocorrelation functions (CCFs) are plotted in Figure 1. A clear seven day period can be seen from the original data and the CCFs.

Since the variance increases substantially when the number of infections grows and there are clear trends in the sequences, we set the zero values in the data set as 0.5, take logarithm for all the data and take the first difference to remove the trends. The detrended data and their CCFs are given in Figure 2. We also plot the autocorrelation functions (ACFs) and the partial autocorrelation functions (PACFs) for Kanagawa and Yamanashi Prefectures in Figure 3.

For the detrended data, we test whether the number of infections in Tokyo causes the numbers of infections in Kanagawa and Yamanashi in the Granger sense. We denote the detrended time series of Kanagawa, Yamanashi and Tokyo in Figure 2 as $\{X_1(t)\}, \{X_2(t)\}$ and $\{X_3(t)\}$, respectively. According to the CCFs of Kanagawa and Tokyo in Figure 2 and the ACFs and PACFs of Kanagawa in Figure 3, we try time lags $P = 7, 14, \ldots, 20$. We also prepare the nine possible terms in Table 2 for model selection for $p = 1, \ldots, P$. The use of $\exp\{-0.5(X_i(t-p))^2\}$ and $\exp\{-0.5(X_i(t-p))^2\}X_j(t),\ \ i, j \in \{1, 2, 3\}$, is due to Assumption (B.1), and the exponential function can moderate sharp fluctuations in the time series.

Table 2: Possible terms for model selection

$X_1(t-p)$	$X_2(t-p)$	$X_3(t-p)$
$\exp\{-0.5(X_1(t-p))^2\}$	$\exp\{-0.5(X_2(t-p))^2\}$	$\exp\{-0.5(X_3(t-p))^2\}$
$\exp\{-0.5(X_1(t-p))^2\}X_2(t)$	$\exp\{-0.5(X_1(t-p))^2\}X_3(t-p)$	$\exp\{-0.5(X_2(t-p))^2\}X_3(t-p)$

The data analysis below is carried out in a two-step procedure. We first use AIC to select models for $X_1(t)$ and $X_2(t)$, respectively; then for the models containing effects from $X_3(t)$, we do the Portmanteau test to examine the Granger causality. Since the Portmanteau test requires that the distribution of the residuals under H_0 be normal, we also take this into account in the model selection. That is, the selected model should minimize AIC, and if there are several models having similar small values of AIC, we choose the model whose residuals under H_0 are closest to the normal distribution.

For Kanagawa Prefecture, the following model with $P = 18$ is selected.

$$X_1(t) = \sum_{p=1}^{P} \alpha_p X_1(t-p) + \sum_{p=1}^{P} \beta_p \exp\{-0.5(X_1(t-p))^2\} + \sum_{p=1}^{P} \theta_p \exp\{-0.5(X_3(t-p))^2\} + \varepsilon_1(t), \tag{10}$$

where $\varepsilon_1(t)$ is assumed normal distributed $N(0, \sigma_1^2)$.

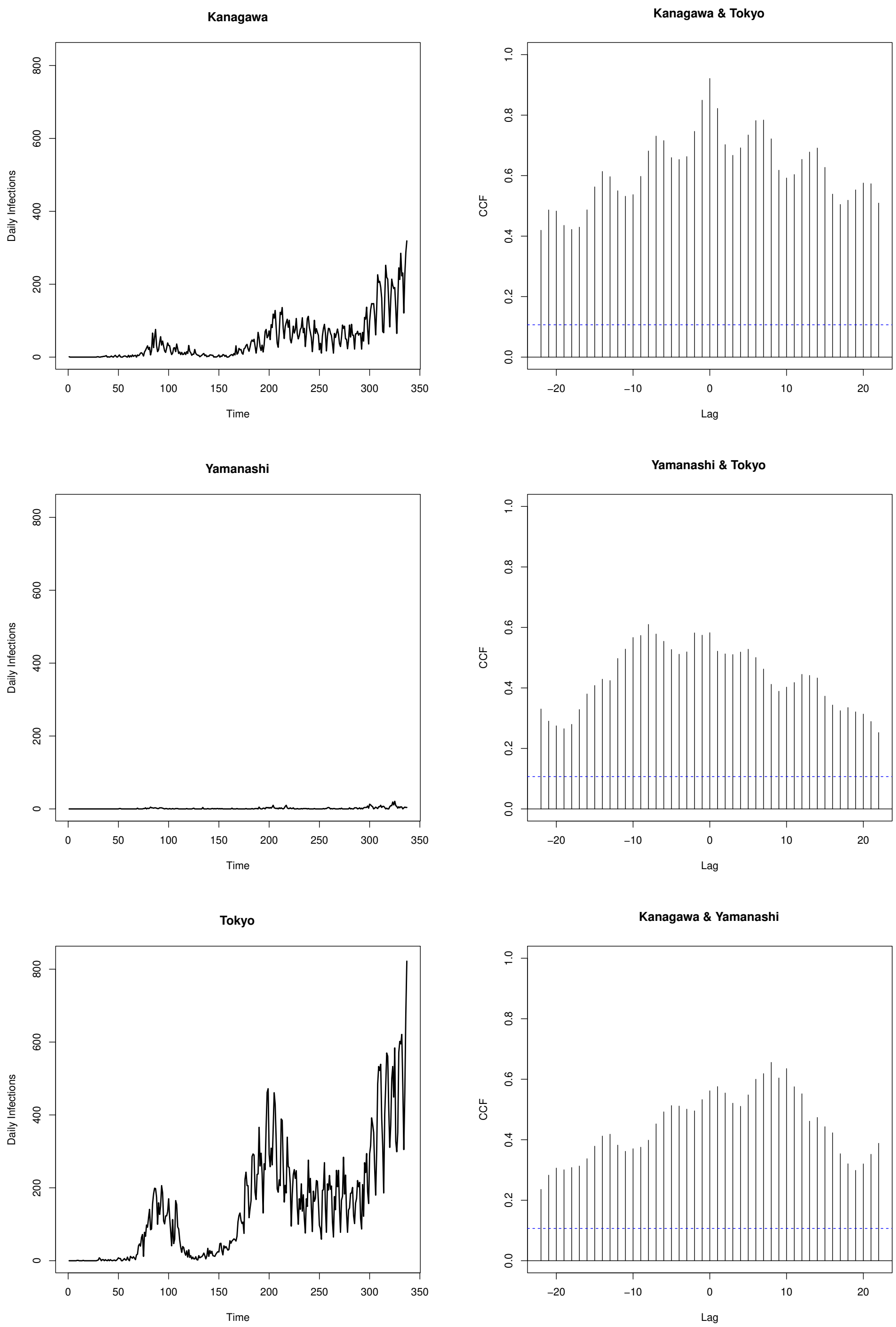

Figure 1: Left: Infection numbers of COVID-19 in Kanagawa, Yamanashi and Tokyo; Right: CCFs of Kanagawa, Yamanashi and Tokyo.

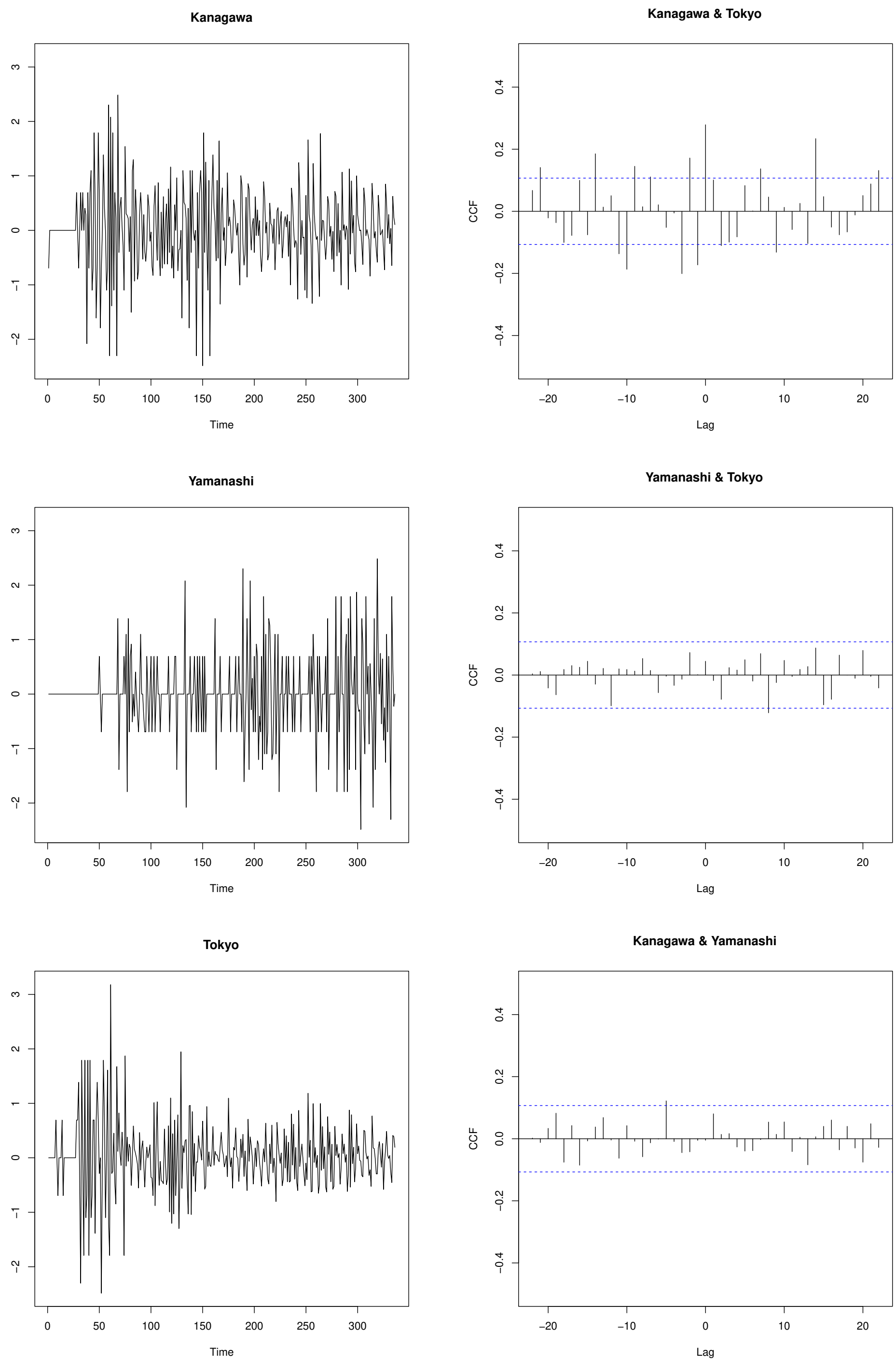

Figure 2: The detrended time series and their CCFs.

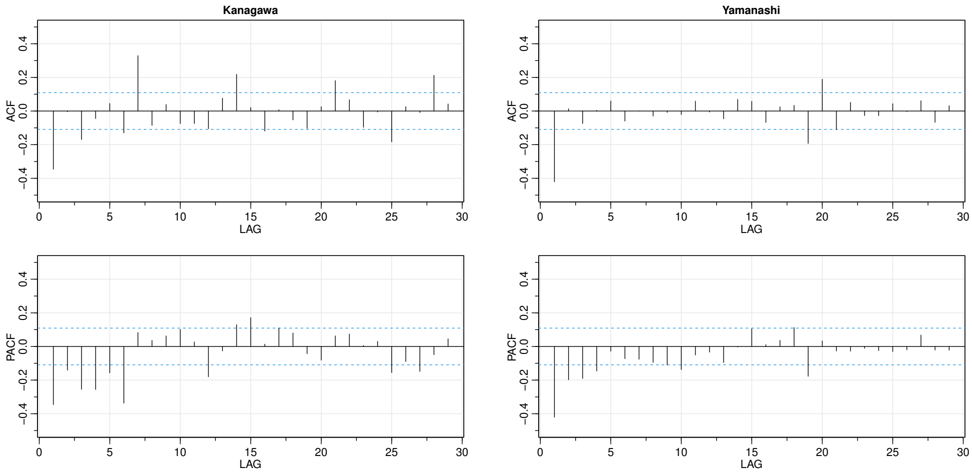

Figure 3: The ACFs and PACFs of Kanagawa and Yamanashi Prefectures.

Testing the Granger causality for this model is equivalent to test

$$H_0: \quad \theta_1 = \cdots = \theta_P = 0.$$

In the calculation of PT, we first compute $\widehat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_P)'$ and $\widehat{\boldsymbol{\beta}} = (\hat{\beta}_1, \cdots, \hat{\beta}_P)'$ under H_0, then substitute them into model (10) and find $\widehat{\boldsymbol{\theta}} = (\hat{\theta}_1, \cdots, \hat{\theta}_P)'$ by maximizing the log-likelihood function.

The residuals from the model under the hypothesis have mean 0.012, standard deviation 0.546; the residuals obtained from the model under the alternative have mean 0.003, and standard deviation 0.502. The two sequences of residuals are shown in the first row of Figure 4. The ACFs and PACFs of the sequences of residuals in the second row of the figure show that there is almost no correlation in the sequences. Corresponding Q-Q plots of the standardized residuals are shown in the last row of Figure 4. We see that their distributions are close to the standard normal distribution.

As the value of the test statistic, $PT = 14.514$, is smaller than the critical point $\chi^2_{0.95}(18) = 28.869$, we cannot reject the hypothesis H_0 at an $\alpha = 0.05$ significance level, and cannot conclude that the number of infections in Kanagawa is Granger caused by the number of infections in Tokyo.

For Yamanashi Prefecture, we try time lags $P \in \{7, \ldots, 20\}$ for the model selection. From the possible terms given in Table 2 and their linear combinations, AIC selects the simple AR(P) model

$$X_2(t) = \sum_{p=1}^{P} \gamma_p X_2(t-p) + \varepsilon_2(t), \tag{11}$$

with $P = 19$. According to the ACFs and PACFs of Yamanashi in Figure 3, we see this model is suitable, because the ACF tails off and the PACF cuts off after lag 19.

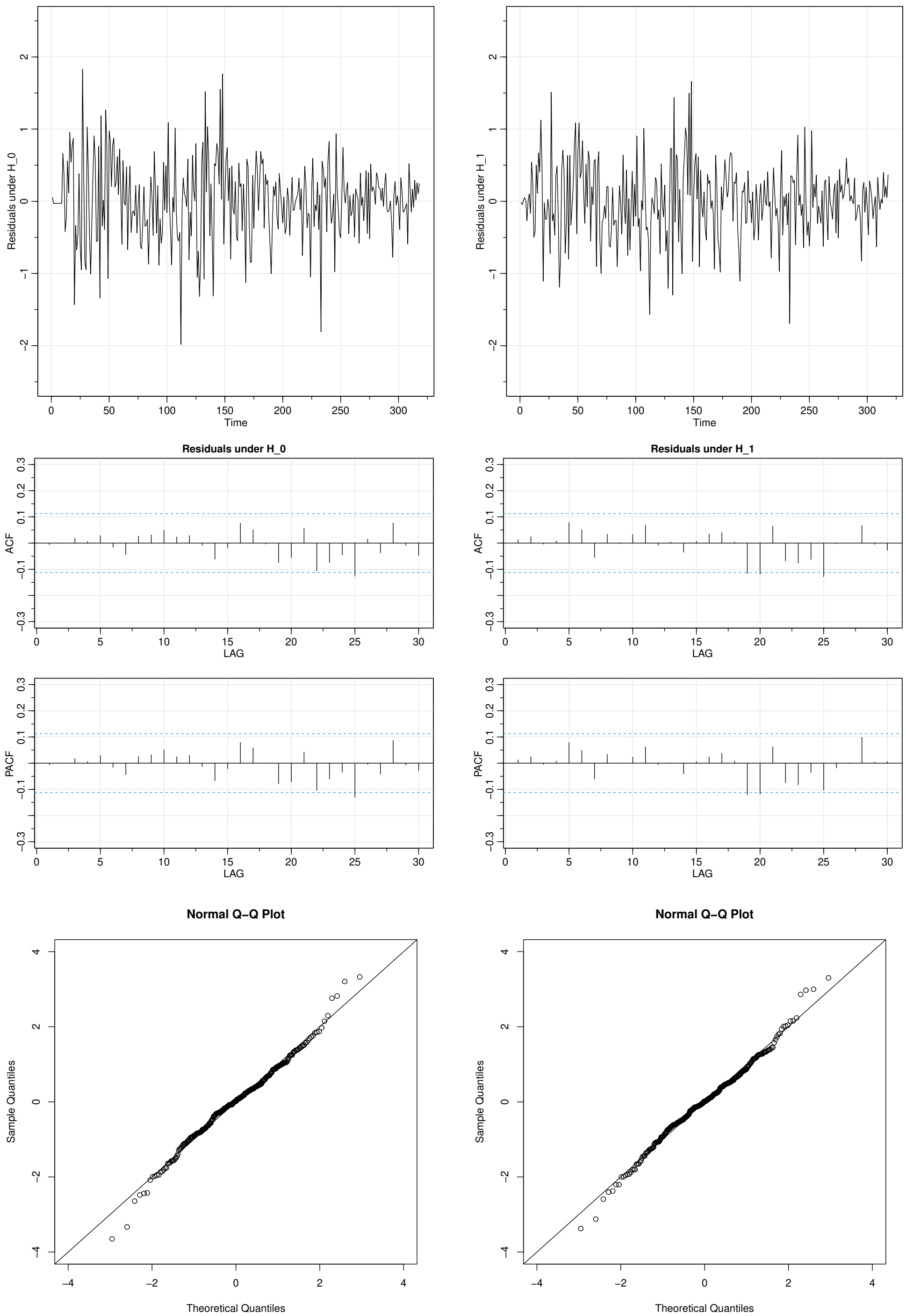

Figure 4: The left and right panels are the sequences of residuals, their ACFs, PACFs, and their Q-Q plots under H_0 and H_1, respectively.

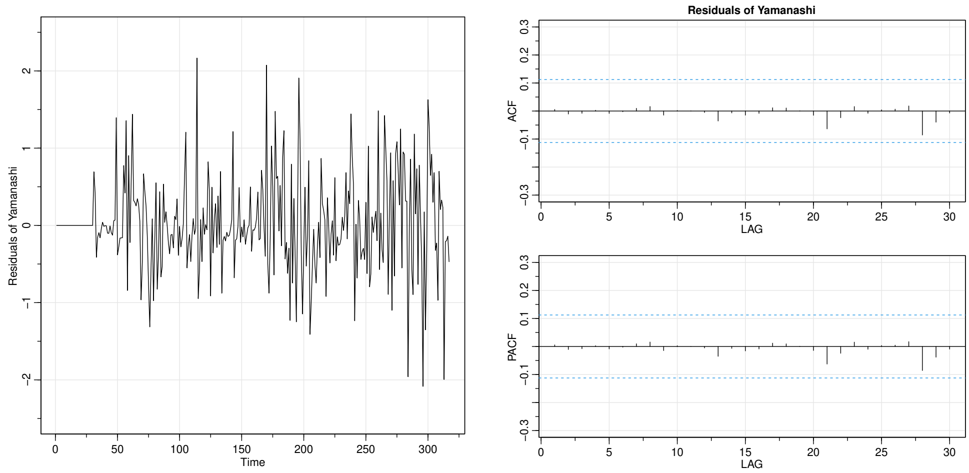

Figure 5: The sequence of residuals of Yamanashi Prefecture and the ACFs and PACFs.

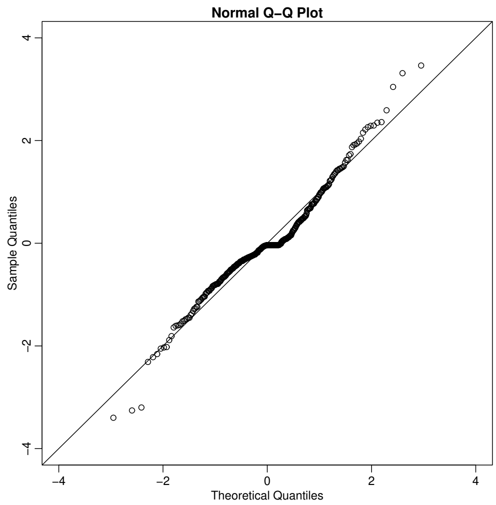

Figure 6: Q-Q plot of the standardized residuals of Yamanashi Prefecture.

The residuals of Yamanashi Prefecture obtained from model (11) have mean 0.022 and standard deviation 0.619. The sequence of the residuals, their ACFs and PACFs are plotted in Figure 5. This model does not contain any information of $X_3(t)$ and we cannot perform the Portmanteau test. However, since AIC can be used in Granger causality detection by the selection of the orders of bivariate autoregressive models when the sample size is large ([15]), we can conclude that there is no Granger causality of Tokyo to Yamanashi Prefecture.

For a simultaneous test of the Granger causality from Tokyo to both of the prefectures of Kanagawa and Yamanashi, a possible solution is to take $\boldsymbol{X}_1(t) = (X_1(t), X_2(t))'$ together as Kanagawa and Yamanashi, and investigate the effect of Tokyo $X_3(t)$ to $\boldsymbol{X}_1(t)$ using the

Portmanteau test. In the multivariate case, we also need a precondition that the distribution of the residuals under H_0 should be a two-dimensional normal distribution. However, the Q-Q plot of the standardized residuals of $X_2(t)$ obtained from (11) in Figure 6 shows a significant departure from the standard normal distribution. This means that the selected models (10) and (11) are not able to make the precondition satisfied and the Portmanteau test cannot be applied directly to this data set. Besides, in our models (10) and (11), different time lags P are used: $P = 18$ for $X_1(t)$, and $P = 19$ for $X_2(t)$. This results in different lengths of residuals of the two time series and makes it difficult to construct a two-dimensional normal distribution. Additionally, because of the different patterns of the ACFs and PACFs for Kanagawa and Yamanashi Prefectures in Figure 3, it is unlikely to obtain the same or similar models for $X_1(t)$ and $X_2(t)$ even if other model selection methods are used. For these reasons, we decide to cease the simultaneous Granger causality test for this data set.

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Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

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ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
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Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	¥8,000	US$80， Euro75	US$50, Euro47
1-year Students member	¥4,000	US$50， Euro47	US$30， Euro28
Life member*	Calculated as below*	US$750， Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.
(73－age) × ¥3,000
Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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