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CHARACTERIZATION OF CLOSED BALLS VIA METRIC PROJECTIONS

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Abstract. Consider the following property (P) for a bounded closed convex set C in a Banach space X (P) : For every $x \in X$, a positive-scalar multiple of x gives a nearest point in C to x. Then it is clear that a closed ball with its center at the origin has this property. The converse of this assertion is the subject of this paper, and it is proved that a bounded closed convex set $C \subset X$ with $0 \in \operatorname{Int} C$ possessing property (P) is a closed ball with center 0, provided $\dim X > 1$. The proof is achieved by reducing the general case to that of 2-dimensional spaces.

1 Introduction

Let X denote a real Banach space with norm $\|\cdot\|$ and let A be a subset of X. Then, $x_0 \in A$ is called a nearest point in A to $x \in X$ if $\|x-x_0\| = \min\{\, \|x-y\| \mid y \in A\}$ holds. In this sense, for a closed ball C in X with center 0 (the origin) and radius r, it is clear that a nearest point Px in C to $x \in X$ is given by the following formula:

$$Px = \begin{cases} \dfrac{r}{\|x\|} \cdot x & (x \notin C), \\ x & (x \in C). \end{cases}$$

In other words, Px is a positive-scalar multiple of x for every $x \in X$.

The authors happened to wonder if the converse of this fact holds or not. That is to say, suppose that $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x, then should C be a closed ball centered at the origin or not?

It is clear that this question is answered in the negative for the extreme case of 1-dimensional spaces: Consider $C = [-1, 2]$ in $\mathbb{R}$ with the usual norm. However, the present authors could firstly answer affirmatively for the case of Hilbert spaces with dimension greater than 1. Their proof depends heavily on the neat geometric property of Hilbert spaces. Namely, for every non-empty closed convex set C in a Hilbert space X, there exists so-called metric projection $P_C\colon X \to C$ that maps $x \in X$ to the unique nearest point in C to x, and the proof utilized the characterization of P_C by inner products, and also the contractivity of P_C.

As an extension of the case of Hilbert spaces, the authors found that the question is affirmatively answered for Lebesgue's L^p spaces with $p \in [2, 4)$. This result is based on the following theorem in Li–Wang–Yang [5] saying that in case of p-uniformly convex and simultaneously q-uniformly smooth Banach spaces, metric projections onto a closed convex set is locally Hölder continuous of order q/p.

Moreover, no example of a Banach space with dimension greater than 1 was found which answers the question in the negative. So, the authors formulated the question into the following conjecture and began to investigate its validity in earnest:

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Conjecture: For every real Banach space X with $\dim X > 1$ the following assertion (A) holds:

(A) If $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x, then C should be a closed ball with center 0.

Note that the uniqueness of nearest points is not assumed in the conjecture.

As a result of investigation, the authors have managed to prove that the conjecture is right (Theorem 15). However, in the real process of investigation, they first found that the conjecture is right when it is restricted to the category of *smooth* Banach spaces (Theorem 4).

In Section 2, we give a proof of Theorem 4 since this proof contains the essence of that for general case (Theorem 15) and easier to understand. Moreover, it seems interesting that Theorem 4 is proved by making the best use of undergraduate calculus. In Section 3, the proof of the full conjecture is given, with a detailed description of facts necessary for the line of the proof in Section 2 to work.

Throughout this paper, conventional notations concerning general topology are freely used. For example, $\operatorname{Int} A$ denotes the interior of A, ∂A the boundary of A and $\overline{A}$ the closure of A.

2 Preliminaries and the result for smooth Banach spaces The following Proposition shows that 2-dimensional case is essential for our problem.

Proposition 1. *Let X be a real Banach space with* $\dim X > 1$. *Then the following assertions hold.*

(1) *If $C \subset X$ is a bounded closed convex set with $0 \in \operatorname{Int} C$, then for every 2-dimensional subspace E of X, $C \cap E$ is a bounded closed convex set with $0 \in \operatorname{Int}(C \cap E)$ in E. Moreover, provided that a nearest point in C to $x \in X$ is always given by a positive-scalar multiple of x, a nearest point in $C \cap E$ to $x \in E$ is given by a positive-scalar multiple of x.*

(2) *Assertion (A) holds for X if (A) holds for every 2-dimensional subspace of X.*

Proof. Since assertion (1) is trivial, we only give a proof of (2). So, suppose that $C \subset X$ satisfies the assumption in assertion (A). Then, for every 2-dimensional subspace E of X, $C \cap E$ satisfies the assumption in (A) as a subset of E (*cf.* assertion (1) in the present proposition). Therefore if (A) holds for every 2-dimensional subspace of X, $C \cap E$ is a ball in E with center 0. So, take a fixed element $e_0 \in \partial C$, and choose an arbitrary $e_1 \in \partial C$ that is linearly independent of e_0. Then, for the 2-dimensional subspace E generated by $\{e_0, e_1\}$, $e_0, e_1 \in \partial(C \cap E)$ holds and hence $\|e_1\| = \|e_0\|$. Moreover, if $e_1 \in \partial C$ is linearly dependent of e_0, by taking another $e_2 \in \partial C$ that is linearly independent of e_0 and setting E the linear span of $\{e_0, e_2\}$, it is shown that $\|e_0\| = \|e_1\| = \|e_2\|$ since $e_0, e_1, e_2 \in \partial(C \cap E)$. Thus the norm of every element of ∂C is equal, which shows that C is a ball centered at the origin. □

Here let us record the following immediate

Corollary 2. *The conjecture above holds if its restriction to 2-dimensional Banach spaces holds.*

Next we recall the definition of the smoothness of Banach spaces (see e.g. [1]). A Banach space $(X, \|\cdot\|)$ is said to be smooth if there exists a unique bounded linear functional $f_x \in X^*$ such that

$$f_x(x) = \|x\| \text{ and } \|f_x\| = 1$$

for each $x \in X \setminus \{0\}$. It is known that X is smooth if and only if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for every $x \in X \setminus \{0\}$ and $y \in X$. Moreover it is also known that

$$f_x(y) = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

holds for every $x \in X \setminus \{0\}$ and $y \in X$, and the mapping $x \longmapsto f_x$ from $X \setminus \{0\}$ to X^* is norm to weak* continuous. For these facts, see [1, pp. 20–23].

Although the following lemma is an easy consequence of the definition of smoothness itself, characterization by the existence of the limit in (1) makes the proof completely trivial.

Lemma 3. *Let X denote a real Banach space with* $\dim X > 1$. *Then X is smooth if and only if every 2-dimensional subspace E of X is smooth.*

Now we prove the following

Theorem 4. *For every smooth Banach space X with* $\dim X > 1$, *assertion (A) in our Conjecture holds.*

Proof. **Step 1.** (Preliminaries)

Because of Proposition 1 and Lemma 3, it suffices to show that assertion (A) holds for 2-dimensional smooth Banach spaces. So, hereafter in this proof we assume that $(X, \|\cdot\|)$ is a 2-dimensional smooth Banach space and $C \subset X$ is a bounded closed convex set containing the origin in its interior, and a positive-scalar multiple of $x \in X$ gives a nearest point in C to x.

Now let $e_1, e_2 \in X$ be linearly independent vectors and define

$$p(s,t) = \|se_1 + te_2\|$$

for $(s,t) \in \mathbb{R}^2$. Then p is a norm on $\mathbb{R}^2$. Since the mapping ι from $(\mathbb{R}^2, p)$ to $(X, \|\cdot\|)$ defined by $\iota(s,t) := se_1 + te_2$ is an isometric isomorphism, we may identify $(\mathbb{R}^2, p)$ with $(X, \|\cdot\|)$. Hence we prove the theorem in $(\mathbb{R}^2, p)$ instead of $(X, \|\cdot\|)$. By equation (2), we obtain the following equality for the gradient ∇p:

$$\nabla p(s,t) = (f_{se_1+te_2}(e_1), f_{se_1+te_2}(e_2)),$$

which implies that $p(s,t)$ is of class C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$. Moreover,

$$\begin{aligned} \nabla p(\lambda s, \lambda t) &= \nabla p(s,t) \quad (\lambda > 0), \\ (\nabla p)(-s,-t) &= -\nabla p(s,t) \end{aligned} \tag{3}$$

hold by equation (2).

In addition to the Cartesian coordinates (s,t), we also use polar coordinates (r, θ) defined through $s = r\cos\theta$, $t = r\sin\theta$. Then, the boundary of the unit disk with respect to the norm p is described by a polar equation $r = g(\theta)$. Note that $g(-\theta) = g(\theta)$ holds for any θ.

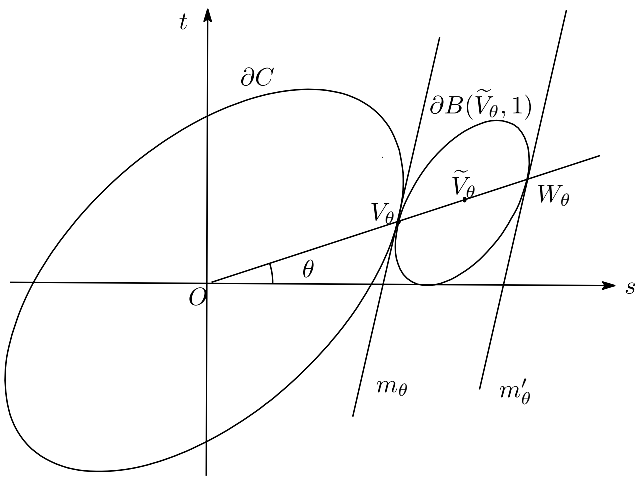

Figure 1: For specification of a supporting line

Note also that the function g is of class C^1 by our assumption of smoothness. Moreover, the boundary ∂C of the closed convex set C is also described by another polar equation $r = l(\theta)$. Lastly, the open disk with center P and radius ρ with respect to the norm p will be denoted by $B(P, \rho)$.

Step 2. (Specification of a tangent-like line at a point of ∂C)

By the well-known Hahn–Banach separation theorem, it is proved that for every point $U \in \partial C$, there exists a line m that supports C and passes U, i.e., $U \in m$ and $\operatorname{Int} C$ is contained in one of the half-spaces separated by m. Since the smoothness of ∂C is not assumed, m might not be uniquely defined. However, an appropriate m could be determined under the assumption that a positive-scalar multiple of x gives a nearest point in C to x. Namely, let $V_\theta \in \partial C$ be the point that is specified as $(l(\theta), \theta)$ in polar coordinates. Then, our assumption about nearest points implies that V_θ becomes a nearest point in C to the following point $\widetilde{V}_\theta$ (see Figure 1):

$$\widetilde{V}_\theta := \left(1 + \frac{g(\theta)}{l(\theta)}\right) \cdot V_\theta.$$

It is easy to see that the distance of $\widetilde{V}_\theta$ and V_θ with respect to p is equal to 1. Hence, $C \cap B(\widetilde{V}_\theta, 1) = \emptyset$ and $V_\theta \in C \cap \overline{B(\widetilde{V}_\theta, 1)}$. Therefore the Hahn–Banach separation theorem yields a line m_θ that passes V_θ and separates C and $B(\widetilde{V}_\theta, 1)$. This time m_θ is uniquely determined since the boundary of $B(\widetilde{V}_\theta, 1)$ is smooth. To obtain the precise description of m_θ, note that because of the symmetry $-B(O, 1) = B(O, 1)$ it is parallel to the tangent m'_θ to $\partial B(\widetilde{V}_\theta, 1)$ at W_θ in Figure 1 (analytically speaking, this is proved by (3)). And m'_θ is further parallel to the tangent m''_θ to $\partial B(O, 1)$ (O denotes the origin) at the point U_θ with angular coordinate θ (see Figure 2). Since m''_θ is not radial, there exists a point T on m''_θ for which the angular coordinate is greater than that of U_θ. Then an angle $\phi(\theta)$ is introduced by the following formula:

$$\angle OU_\theta T = \frac{\pi}{2} + \phi(\theta) \quad \left(-\frac{\pi}{2} < \phi(\theta) < \frac{\pi}{2}\right).$$

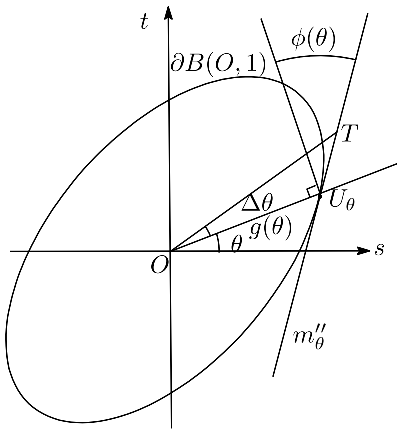

Figure 2: tangent to $\partial B(0,1)$

As to $\phi(\theta)$, an elementary argument in calculus gives

$$\frac{g'(\theta)}{g(\theta)} = \tan\phi(\theta), \tag{4}$$

where g' means the derivative of g. Therefore, the continuity of g' implies the existence of a constant $M > 0$ such that

$$|\tan\phi(\theta)| \leq M \quad (\theta \in [0, 2\pi]). \tag{5}$$

Step 3. (Estimate of $l(\theta)$)

Now, we have seen that a supporting line m_θ for C at V_θ is parallel to m''_θ. Hence the angle $\phi(\theta) \in (-\pi/2, \pi/2)$ in Figure 3 is the same as in Figure 2 and so determined by (4). Note also that because of (5), there exists an $\varepsilon > 0$ such that the radial half-line with angular coordinate $\theta + \Delta\theta$ intersect with m_θ for any $\theta \in [0, 2\pi]$ and $\Delta\theta \in (0, \varepsilon)$. So, let $0 < \Delta\theta < \varepsilon$ and let T' be the intersection point of radial half-line with angular coordinate $\theta + \Delta\theta$ and m_θ. Further let $\hat{l}$ be the radial coordinate of T' as in Figure 3, while we have already denoted the radial coordinate of V_θ by $l(\theta)$. Then we obtain

$$\frac{l(\theta)}{\sin\left(\frac{\pi}{2} - \phi(\theta) - \Delta\theta\right)} = \frac{\hat{l}}{\sin\left(\frac{\pi}{2} + \phi(\theta)\right)}$$

by applying the sine rule to $\triangle OV_\theta T'$. Hence

$$l(\theta + \Delta\theta) \leq \hat{l} = l(\theta) \cdot \frac{\cos\phi(\theta)}{\cos(\phi(\theta) + \Delta\theta)}$$

and so

$$\frac{l(\theta)}{l(\theta + \Delta\theta)} \geq \frac{\cos(\phi(\theta) + \Delta\theta)}{\cos\phi(\theta)}. \tag{6}$$

Now take $\theta_1, \theta_2 \in [0, 2\pi]$ with $\theta_1 < \theta_2$. Then $\Delta_n\theta := (\theta_2 - \theta_1)/n < \varepsilon$ for sufficiently large $n \in \mathbb{N}$ ($n > 2\pi/\varepsilon$ will do). For such n, inequality (6) yields

$$\begin{aligned}
\frac{l(\theta_1)}{l(\theta_2)} &= \frac{l(\theta_1)}{l(\theta_1+\Delta_n\theta)} \cdot \frac{l(\theta_1+\Delta_n\theta)}{l(\theta_1+2\Delta_n\theta)} \cdot \cdots \cdot \frac{l\,(\theta_1+(n-1)\Delta_n\theta)}{l(\theta_2)} \\
&\geq \prod_{k=0}^{n-1} \frac{\cos(\phi(\theta_1+k\Delta_n\theta)+\Delta_n\theta)}{\cos\phi(\theta_1+k\Delta_n\theta)} \\
&= \cos^n(\Delta_n\theta) \prod_{k=0}^{n-1} \{1 - \tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}\,. \qquad (7)
\end{aligned}$$

Noting the estimates (5) and $0 < \Delta_n\theta \leq 2\pi/n$, we see that for sufficiently large $n \in \mathbb{N}$,

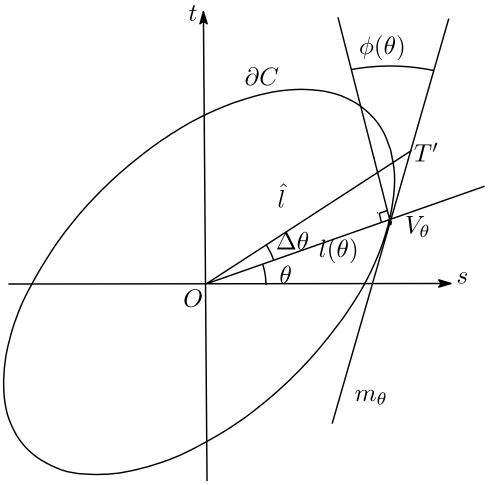

Figure 3: m_θ and ∂C

$$\begin{aligned}
&\log\{1 - \tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\} \\
&= -\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta) + \frac{\{\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}^2}{2\{1-\eta\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}^2}
\end{aligned}$$

and

$$\tan(\Delta_n\theta) = \Delta_n\theta + \frac{\sin(\eta'\Delta_n\theta)}{\cos^3(\eta'\Delta_n\theta)} \cdot (\Delta_n\theta)^2$$

hold for some $\eta, \eta' \in (0,1)$ by applying Taylor's theorem to $\log(1-x)$ and $\tan x$. Hence

$$\log\{1 - \tan\phi(\theta_1+k\Delta_n\theta)\tan(\Delta_n\theta)\} = -\tan\phi(\theta_1+k\Delta_n\theta)\Delta_n\theta + O\left(\frac{1}{n^2}\right) \qquad (8)$$

holds by (5), where $O\left(1/n^2\right)$ is Landau's big O notation. Precisely speaking, the absolute value of this remainder term is estimated by K/n^2 from above where K is independent of

θ_1, θ_2 and sufficiently large n. Therefore we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1} \log\{1 - \tan\phi(\theta_1 + k\Delta_n\theta)\tan(\Delta_n\theta)\} \\
&= -\sum_{k=0}^{n-1} \frac{g'(\theta_1 + k\Delta_n\theta)}{g(\theta_1 + k\Delta_n\theta)} \cdot \Delta_n\theta + O\left(\frac{1}{n}\right) \\
&\to -\int_{\theta_1}^{\theta_2} \frac{g'(\theta)}{g(\theta)} d\theta = \log\frac{g(\theta_1)}{g(\theta_2)} \quad (n \longrightarrow \infty)
\end{aligned}$$

by (4) and (8). Hence by taking the logarithm of both sides in (7) and letting $n \longrightarrow \infty$, we obtain

$$\log\frac{l(\theta_1)}{l(\theta_2)} \geq \log\frac{g(\theta_1)}{g(\theta_2)},$$

since $\lim_{n\to\infty} \cos^n(\Delta_n\theta) = 1$.

Therefore $l(\theta)/g(\theta)$ is a decreasing function of θ. However $l(0)/g(0) = l(2\pi)/g(2\pi)$ and so $l(\theta)/g(\theta)$ is a constant, which implies $\partial C = \partial B(O, r)$ for some $r > 0$. Thus C is a closed ball with center 0. $\square$

3 Result without the assumption of smoothness

3.1 Preliminaries This section is devoted to a proof that the conclusion in Theorem 4 holds without the assumption of smoothness. For this purpose, it suffices to prove that assertion (A) (in the *Conjecture* in Section 1) holds for general 2-dimensional real Banach space X, by virtue of Corollary 2.

So, hereafter let X denote a 2-dimensional Banach space and X is identified with $\mathbb{R}^2$ as in Step 1 of the proof of Theorem 4 via a basis of X. We use freely the standard Cartesian coordinate (s, t) and the polar coordinate (r, θ) of $\mathbb{R}^2$ that are defined there. We also adopt the notation $B(P, \rho)$ to denote the open disk with center P and radius ρ with respect to the norm on $\mathbb{R}^2$ that is induced from that of X through the identification.

The boundary $\partial B(O, 1)$ is described by a polar equation $r = g(\theta)$. Firstly, just to be sure, we record the fact that g is continuous without the assumption of smoothness of the norm.

Lemma 5. *$\partial B(O, 1)$ is described by a polar equation $r = g(\theta)$ with a continuous periodic function g.*

This continuity of g could be proved quite easily, e.g., by using the fact that equalities $1 = p(g(\theta)\cos(\theta), g(\theta)\sin(\theta)) = g(\theta)p(\cos(\theta), \sin(\theta))$ hold and p is continuous.

Next we recall well-known fundamental properties of convex functions, which will be crucial to treat the present non-smooth case. For a proof, see e.g., Tiel [4, Chapter 1] or Godement [2, Chapitre V, Théorème 15].

Proposition 6. *Let I be an open interval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Then f enjoys the following properties.*

(i) For every $t \in I$, the left [resp. right] derivative $f'_-(t)$ [resp. $f'_+(t)$] exists and the inequality $f'_-(t) \leq f'_+(t)$ holds.

(ii) f'_- and f'_+ are increasing functions and are continuous except for points of an at most countable set. Here, the term "increasing" is used in its wider sense, i.e., $t \leq s$ implies $f'_-(t) \leq f'_-(s)$ [resp. $f'_+(t) \leq f'_+(s)$].

(*iii*) *For every subinterval* $[a,b]$ *of* I, $|f'_-(t)|$ *and* $|f'_+(t)|$ *are bounded by* $\max\{\,|f'_-(a)|, |f'_+(a)|, |f'_-(b)|, |f'_+(b)|\,\}$. *Hence they are locally bounded on* $(-\rho,\rho)$.

(*iv*) *Except for at most countably many points,* $f'_-(t) = f'_+(t)$ *holds and hence* f *is differentiable there. Moreover,* f'_- *and* f'_+ *are continuous at points where* f *is differentiable.*

For later use we record the following fact.

Lemma 7. *Let* $\rho > 0$ *and* $h\colon (-\rho,\rho) \to \mathbb{R}$ *be a negative valued convex function. Then the following inequality holds where* $h'_\pm$ *denotes either of the one-sided derivatives* h'_- *and* h'_+*:*

$$uh'_\pm(u) - h(u) > 0 \quad (\forall u \in (-\rho,\rho)) \tag{9}$$

Proof. $uh'_\pm(u) - h(u) > 0$ clearly holds for $u = 0$. Suppose now $0 < u < \rho$. Then,

$$\frac{h(u)}{u} < \frac{h(u) - h(0)}{u} \le h'_-(u) \le h'_+(u)$$

holds, and consequently we obtain $uh'_\pm(u) - h(u) > 0$.

In the case of $-\rho < u < 0$,

$$\frac{h(u)}{u} > \frac{h(u) - h(0)}{u} = \frac{h(0) - h(u)}{0 - u} \ge h'_+(u) \ge h'_-(u)$$

imply the desired inequality. □

3.2 Introduction of a collection of coordinate systems A key to our proof is to show that the function g above has the same level of differentiability property as that of convex functions. To do so, it is necessary to introduce a collection of coordinate systems that is fitted to make well use of the convexity of $B(O,1)$. The need for such a collection would be understood by the fact that g itself is convex if and only if it is constant, hence the polar coordinate does not immediately lead to useful knowledge that compensates for the lack of smoothness.

Set $\theta = \theta_1$ in Figure 2 and let the coordinate system (frame) $\mathcal{F}$ for the Cartesian coordinates (s,t) be rotated around the origin by the angle $\frac{\pi}{2} + \theta_1$ to form a new coordinate system $\mathcal{F}_{\theta_1}$ (see Figure 4). In the sequel, $(u,v)_{\mathcal{F}_{\theta_1}}$ denotes the geometric point for which the coordinate with respect to $\mathcal{F}_{\theta_1}$ is (u,v).

The merit of introducing the coordinate system $\mathcal{F}_{\theta_1}$ is well explained by the following

Lemma 8. *There exists a positive constant* ρ_0 *such that for every angle* θ_1 *there exists a unique* convex *function* $h_{\theta_1}(u) < 0$ *of* $u \in (-\rho_0,\rho_0)$ *for which* $(u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} \in \partial B(O,1)$.

Proof. Since the origin is an interior point of $B(O,1)$, there exists a $\rho_0 > 0$ such that $s^2 + t^2 < {\rho_0}^2$ implies $(s,t) \in B(O,1)$.

Now, take an arbitrary angle θ_1. Then, $(u,v)_{\mathcal{F}_{\theta_1}} \in B(O,1)$ holds provided $u^2 + v^2 < {\rho_0}^2$. This means that $(u,0)_{\mathcal{F}_{\theta_1}} \in B(O,1)$ holds for each u with $|u| < \rho_0$. For such u, it is clear that the half-line $\{\,(u,v)_{\mathcal{F}_{\theta_1}} \mid v \le 0\,\}$ intersects $\partial B(O,1)$ at a single point $(u, v_u)_{\mathcal{F}_{\theta_1}}$ for some $v_u < 0$. Then $h_{\theta_1}(u) := v_u$ clearly yields the desired convex function. □

3.3 Differentiability of $g(\theta)$

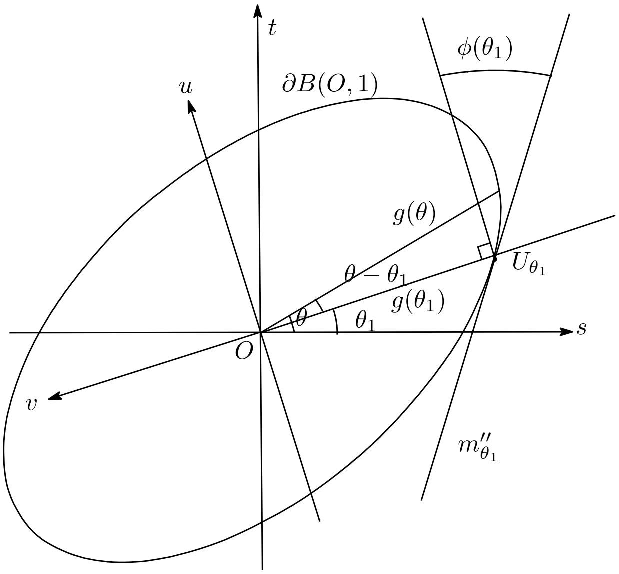

Figure 4: a new coordinate system $(u,v)_{\mathcal{F}_{\theta_1}}$

3.3.1 θ as an independent variable and as a function Lemma 8 implies that $u \in (-\rho_0, \rho_0)$ gives a *local* coordinate for the curve $\partial B(O,1)$ by the correspondence $u \mapsto (u, h_{\theta_1})_{\mathcal{F}_{\theta_1}}$. On the other hand, the angular variable θ could also be used as a local coordinate for $\partial B(O,1)$ by the mapping $\theta \mapsto (g(\theta), \theta)_{rad}$, where $(g(\theta), \theta)_{rad}$ denotes the geometric point with the polar coordinates $(g(\theta), \theta)$, i.e., $(g(\theta)\cos\theta, g(\theta)\sin\theta) \in \mathbb{R}^2$.

We can see that θ is determined (mod 2π) as a continuous function of $u \in (-\rho_0, \rho_0)$ by $(u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} = (g(\theta), \theta)_{rad}$, since $\{\, (u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} \mid u \in (-\rho_0, \rho_0) \,\}$ is contained in the half-space $\{\, (u, v)_{\mathcal{F}_{\theta_1}} \mid u \in \mathbb{R},\ v \leq 0 \,\}$. Therefore, once a coordinate system $\mathcal{F}_{\theta_1}$ is designated, local coordinate θ for $\partial B(O,1)$ might be considered as a continuous function of $u \in (-\rho_0, \rho_0)$, where u denotes the first component of the coordinate with respect to $\mathcal{F}_{\theta_1}$. In the sequel, θ considered as a function in this way will be denoted simply by $\theta(u)$, avoiding more accurate but rather awkward expression such as $\theta_{\theta_1}(u)$. In addition, intuitively speaking, it is clear that the function $\theta(u)$ defined above is a strictly increasing continuous function of $u \in (-\rho_0, \rho_0)$. To prove these facts analytically, we give an explicit expression of $\theta(u)$.

Lemma 9. *Let θ_1 be arbitrarily fixed and let $h_{\theta_1}(u)$ be the convex function described in Lemma 8. Then, the relation $(u, h_{\theta_1})_{\mathcal{F}_{\theta_1}} = (r, \theta)_{rad}$ $(-\rho_0 < u < \rho_0)$ is satisfied if and only if*

$$\theta = \arcsin\left(\frac{u}{\sqrt{u^2 + h_{\theta_1}(u)^2}}\right) + \theta_1. \tag{10}$$

Therefore, by taking the principal branch of arcsin, *a continuous function $\theta = \theta(u)$ of $u \in (-\rho_0, \rho_0)$ is obtained. Then, the left derivative $\theta'_-(u)$ and the right derivative $\theta'_+(u)$ everywhere exist and are positive, and hence $\theta(u)$ is a strictly increasing function of u. Moreover, the set $\mathcal{N}$ of points where $\theta(u)$ is not differentiable is at most countable and θ'_- and θ'_+ are continuous at points in $(-\rho_0, \rho_0) \setminus \mathcal{N}$.*

Proof. $(u, h_{\theta_1})_{\mathcal{F}_{\theta_1}} = (r, \theta)_{rad}$ is nothing but

$$u = g(\theta)\sin(\theta - \theta_1),$$
$$h_{\theta_1}(u) = -g(\theta)\cos(\theta - \theta_1).$$

Hence

$$g(\theta)^2 = u^2 + h_{\theta_1}(u)^2, \tag{11}$$

$$u = \sqrt{u^2 + h_{\theta_1}(u)^2}\ \sin(\theta - \theta_1), \tag{12}$$

whence follows (10). Since Lemma 8 and Proposition 6 yield one-sided differentiability of h_{θ_1}, (10) shows that $\theta = \theta(u)$ is one-sided differentiable at every $u \in (-\rho_0, \rho_0)$ and

$$\theta'_{\pm}(u) = -\frac{h_{\theta_1}(u) - u(h_{\theta_1})'_{\pm}(u)}{u^2 + h_{\theta_1}(u)^2} \tag{13}$$

holds. Hence we obtain $\theta'_{\pm}(u) > 0$ by virtue of Lemma 7.

The assertions on the differentiability of $\theta(u)$ and the continuity of $\theta'_{\pm}$ follow immediately from (13) and Lemma 8 applied for h_{θ_1}. □

3.3.2 Proof of the differentiability of $g(\theta)$ As a consequence of the results proved so far, the following assertion concerning the differentiability of $g(\theta)$ is obtained.

Lemma 10. *Let θ_1 be arbitrarily fixed and let $\theta = \theta(u)$ is defined by* (10) *as a function of $u \in (-\rho_0, \rho_0)$ (*arcsin *is construed to mean its principal branch). Then, by virtue of Lemma 9, the range of $\theta(u)$ is an open interval I_{θ_1} containing θ_1, and one-sided derivatives of $g(\theta)$ exist at every $\theta \in I_{\theta_1}$. Moreover, there exists an at most countable set $\mathcal{N}$ for which $g'_{\pm}(\theta)$ is continuous and $g(\theta)$ is differentiable at every point in $I_{\theta_1} \setminus \mathcal{N}$. In addition, $g'_{\pm}(\theta)$ is bounded in a neighbourhood of θ_1.*

Proof. Let $|u| < \rho_0$ and $k \neq 0$ is sufficiently close to 0, then the following algebraic transformation is valid (note that $\theta(u)$ is 1 to 1 by Lemma 9):

$$\begin{aligned}
&\frac{g(\theta(u+k)) - g(\theta(u))}{\theta(u+k) - \theta(u)} \cdot \frac{\theta(u+k) - \theta(u)}{k} \\
&= \frac{g(\theta(u+k)) - g(\theta(u))}{k} \\
&= \frac{\sqrt{(u+k)^2 + h_{\theta_1}(u+k)^2} - \sqrt{u^2 + h_{\theta_1}(u)^2}}{k}.
\end{aligned} \tag{14}$$

Here, $k \to +0$ [resp. $k \to -0$] implies $\theta(u+k) \to \theta(u) + 0$ [resp. $\theta(u+k) \to \theta(u) - 0$ by Lemma 9. Therefore, by letting $k \to \pm 0$ in (14), the one-sided differentiability of h_{θ_1} and $\theta(u)$ (Lemma 9) with $\theta'_{\pm}(u) > 0$ yield the one-sided differentiability of $g(\theta)$. Explicitly speaking, calculation of the limit of (14) as $k \to \pm 0$ yields

$$g'_{\pm}(\theta)\big|_{\theta=\theta(u)} = \frac{u + h_{\theta_1}(u)(h_{\theta_1}(u))'_{\pm}}{\sqrt{u^2 + h_{\theta_1}(u)^2}} \cdot \frac{1}{\theta'_{\pm}(u)}. \tag{15}$$

The existence of an at most countable set $\mathcal{N}$ as stated in the Lemma follows from this expression, Proposition 6, Lemma 8, Lemma 9 and the fact that $\theta(u)$ is an order preserving homeomorphism from $(-\rho_0, \rho_0)$ to I_{θ_1}.

The boundedness of $g'_{\pm}$ in a neighbourhood of θ_1 is clear from (15) and (13) since $(h_{\theta_1})'_{\pm}$ is locally bounded on $(-\rho_0, \rho_0)$ by Proposition 6 and Lemma 8.

□

Remark 11. *By a detailed analysis, it can be shown that* $g'_\pm$ *is locally bounded on* I_{θ_1}.

Remark 12. *For later use, note that* (15), (13) *and equality* $h_{\theta_1}(0) = -g(\theta_1)$ *imply*

$$(h_{\theta_1})'_\pm(0) = -\frac{g'_\pm(\theta_1)}{g(\theta_1)} \tag{16}$$

for every θ_1.

Arbitrariness of θ_1 in the previous Lemma and the compactness of $[0, 2\pi]$ readily lead to the following

Lemma 13. *One-sided derivatives* $g'_\pm$ *exist everywhere and are bounded on* $[0, 2\pi]$. *In addition there exists an at most countable set* $\mathcal{N} \subset [0, 2\pi]$ *for which* g *is differentiable and* $g'_\pm$ *are continuous at each point of* $[0, 2\pi] \setminus \mathcal{N}$.

Although the next lemma might be well known, we state it with a proof since it is crucial to our purpose.

Lemma 14. *The function* $\log g(\theta)$ *is uniformly Lipschitz continuous on* $[0, 2\pi]$.

Proof. Set $\psi(\theta) := \log g(\theta)$. Then it is clear from Lemma 13 that ψ has left and right derivative at every point and that $\psi'_\pm = g'_\pm/g$ is bounded on $[0, 2\pi]$. So let us take a constant K such that $K \geq |\psi'_\pm(\theta)|$ for every $\theta \in [0, 2\pi]$.

Suppose that $\eta, \tilde{\eta} \in [0, 2\pi]$, $\eta < \tilde{\eta}$ and set

$$F(\theta) := \psi(\theta) - \psi(\eta) - \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta}(\theta - \eta)$$

for $\theta \in [\eta, \tilde{\eta}]$. Then

$$F'_\pm(\theta) = \psi'_\pm(\theta) - \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta}.$$

Since $F(\theta)$ is continuous, $F(\theta)$ attains its maximum value and minimum value on $[\eta, \tilde{\eta}]$. Since $F(\eta) = F(\tilde{\eta}) = 0$, $F(\theta)$ attains at least either of the maximum value or the minimum value at some $\xi \in (\eta, \tilde{\eta})$. If $F(\xi)$ is the maximum value, then

$$\begin{aligned} 0 \leq F'_-(\xi) &= \psi'_-(\xi) - \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta} \\ 0 \geq F'_+(\xi) &= \psi'_+(\xi) - \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta}, \end{aligned}$$

and so

$$\psi'_+(\xi) \leq \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta} \leq \psi'_-(\xi).$$

Similarly, if $F(\xi)$ is the minimum value,

$$\psi'_-(\xi) \leq \frac{\psi(\tilde{\eta}) - \psi(\eta)}{\tilde{\eta} - \eta} \leq \psi'_+(\xi).$$

Thus, in either case we obtain

$$|\psi(\tilde{\eta}) - \psi(\eta)| \leq K|\tilde{\eta} - \eta|$$

and hence ψ is uniformly Lipschitz continuous. □

3.4 Completion of the proof In this last subsection we give a proof of the following theorem by completing the proof for the special case of dim $X = 2$.

Theorem 15. *For every Banach space X with* dim $X > 1$, *assertion* (A) *in our Conjecture holds.*

So, as stated at the beginning of this Section 3, let X denote a Banach space with dim $X = 2$ identified with $\mathbb{R}^2$ and let the unit "sphere" $\partial B(O,1)$ be described by a polar equation $r = g(\theta)$. Note that we have shown Lemmas 10 to 14 concerning analytic properties of g.

Now, suppose that $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x. Our task is to show that on this supposition C is indeed a closed ball. The proof proceeds along the line of that in Section 2 (the case of smooth Banach spaces).

Firstly, note that for every angle θ our supposition yields the existence of a line m_θ enjoying the following properties: m_θ supports C at V_θ in Fig. 1 and is parallel to a supporting line m''_θ of $B(O,1)$ at $(g(\theta),\theta)_{rad}$ (see Fig. 2). This is a consequence of the Hahn–Banach separation theorem that does not require smoothness. Although m_θ is not uniquely determined in general, consider that one of such line is assigned for every θ and named m_θ. Then, one can see that the angle $\phi(\theta)$ in Fig. 2 satisfies the following estimate by locally considering the curve $\partial B(O,1)$ as the graph $\{ (u, h_\theta(u)) \mid |u| < \rho_0 \}$ in the coordinate system $\mathcal{F}_\theta$:

$$(h_\theta)'_-(0) \le -\tan\phi(\theta) \le (h_\theta)'_+(0).$$

Hence (16) yields

$$\frac{g'_+(\theta)}{g(\theta)} \le \tan\phi(\theta) \le \frac{g'_-(\theta)}{g(\theta)} \tag{17}$$

for every θ, and Lemma 10 shows that $|\tan\phi(\theta)|$ is bounded on $[0, 2\pi]$.

Now, prior to going into the heart of the proof, note that the boundedness of various quantities can be readily obtained from Lemma 13: There exists a constant $K > 0$ such that

$$|g'_\pm(\theta)|,\ \frac{|g'_\pm(\theta)|}{g(\theta)},\ |\tan\phi(\theta)| \le K \quad (\forall\theta \in [0, 2\pi]). \tag{18}$$

As in the previous section, let the curve ∂C be described by a polar equation $r = \ell(\theta)$ and let us return to Fig. 3. Because of estimate (18), there exists an $\varepsilon > 0$ for which the ray with angle $\theta + \Delta\theta$ intersects the line m_θ, which separates C and a translation of the unit ball (see Fig. 1), provided $0 < \Delta\theta < \varepsilon$. Now take $\theta_1, \theta_2 \in [0, 2\pi]$ with $\theta_1 < \theta_2$ and $n \in \mathbb{N}$ with $n > 2\pi/\varepsilon$ and set $\Delta_n\theta := (\theta_2 - \theta_1)/n$. Then the argument leading to (7) is also valid in the present case and we obtain

$$\frac{l(\theta_1)}{l(\theta_2)} \ge \cos^n(\Delta_n\theta) \prod_{k=0}^{n-1} \{1 - \tan(\phi(\theta_1 + k\Delta_n\theta)) \tan(\Delta_n\theta)\}. \tag{19}$$

From (17) and (18), we obtain the following asymptotic formula as $n \to \infty$:

$$\begin{aligned}
&\sum_{k=0}^{n-1} \log\left(1 - \{\tan\phi(\theta + k\Delta_n\theta)\}\tan(\Delta_n\theta)\right) \\
&= -\sum_{k=0}^{n-1} \tan\phi(\theta_1 + k\Delta_n\theta)\Delta_n\theta + O\left(\frac{1}{n}\right) \\
&\geq -\sum_{k=0}^{n-1} \frac{g'_-(\theta + k\Delta_n\theta)}{g(\theta + k\Delta_n\theta)}\Delta_n\theta + O\left(\frac{1}{n}\right). \qquad (20)
\end{aligned}$$

By Lemma 13, $g'_-(\theta)/g(\theta)$ is Riemann integrable on $[\theta_1, \theta_2]$, and so

$$\begin{aligned}
&-\sum_{k=0}^{n-1} \frac{g'_-(\theta + k\Delta_n\theta)}{g(\theta + k\Delta_n\theta)}\Delta_n\theta + O\left(\frac{1}{n}\right) \\
&\longrightarrow -\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \quad (n \longrightarrow \infty).
\end{aligned}$$

Taking the logarithm of (19) and using the asymptotic formula above, we obtain

$$\log\frac{l(\theta_1)}{l(\theta_2)} \geq -\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \qquad (21)$$

since $\cos^n(\Delta_n\theta) \to 1$ as $n \to \infty$. Moreover, Lemma 14 shows that $\log(g(\theta))$ is absolutely continuous on $[\theta_1, \theta_2]$. Hence the Fundamental Theorem of Calculus for Lebesgue Integral ([3, p. 148]) implies

$$-(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta = \log\frac{g(\theta_1)}{g(\theta_2)}, \qquad (22)$$

where $(L)\int$ means the Lebesgue integral. In addition, Lemma 13 shows that

$$\frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta) = \frac{g'_-(\theta)}{g(\theta)}$$

holds except for θ in some at most countable set. Hence

$$(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta = (L)\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \qquad (23)$$

holds. Since Lebesgue integral and Riemann integral coincide for Riemann integrable functions, (21), (23) and (22) yield

$$\begin{aligned}
\log\frac{l(\theta_1)}{l(\theta_2)} &\geq -(L)\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \\
&= -(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta \\
&= \log\frac{g(\theta_1)}{g(\theta_2)}.
\end{aligned}$$

Therefore

$$\log \frac{\ell(\theta_1)}{g(\theta_1)} \geq \log \frac{\ell(\theta_2)}{g(\theta_2)}$$

holds and hence $\ell(\theta)/g(\theta)$ is a decreasing function (in the wider sense) of $\theta \in [0, 2\pi]$. This in turn implies that $\ell(\theta)/g(\theta)$ is a constant function since the values at 0 and 2π coincide, and the proof is thus completed.

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PUTTING QUANTUM MV ALGEBRAS ON THE "MAP"

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In memoriam William W. McCune

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Abstract

In this paper, we clarify mainly some aspects concerning the quantum MV (QMV) algebras as non-lattice generalizations of MV algebras. We redefine the QMV algebras as involutive m-BE algebras and we introduce three generalizations: the pre-MV (PreMV), the metha-MV (MMV) and the orthomodular (OM) algebras. We prove that the antisymmetric QMV algebras - but also the antisymmetric PreMV and antisymmetric MMV algebras - coincide with the MV algebras, while the antisymmetric OM algebras are generalizations of the MV algebras. We introduce also the transitive QMV, PreMV, MMV, OM algebras and finally we put the QMV and the transitive QMV algebras on the same "map" with the MV algebras. The *transitive antisymmetric orthomodular algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras, is pointed out. Many examples are provided.

Keywords: m-BE algebra, m-aBE algebra, m-pre-BCK algebra, m-BCK algebra, MV algebra, quantum MV algebra, orthomodular lattice, orthomodular algebra, pre-MV algebra, metha-MV algebra

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1 Introduction

The *algebraists* work usually with the commutative additive groups and with the positive (right) cone of a partially-ordered commutative group $(G, \leq, +, -, 0)$, where there are essentially a sum $\oplus = +$ and an element 0. Sometimes, the negative (left) cone is needed also, where there are essentially a product $\odot = +$ and an element $1 = 0$. They work with algebras that have associated an (pre-order) order relation, which usually does not appear explicitely in the definitions. The presence of the (pre-order) order relation implies the presence of the (generalized) duality principle. Thus, each algebra has a dual one, the (pre-order) order relation has a dual one. We have given names to the dual algebras [15], [17], [19]: "left" algebra and "right" algebra, names connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. Hence, the algebraists usually work with the commutative *right-unital magmas*.

By contrary, the *logicians* work with the logic of *truth*, where the *truth* is represented by 1, and there is essentially one implication; we could name this logic "left-logic". One can imagine also a "right-logic", as a logic of *false*, where the *false* is represented by 0. Hence, the logicians usually work with the commutative *left-algebras of logic* (or the *algebras of left-logic*).

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Summarizing, for algebraists, the appropriate algebras are the unital magmas, not the algebras of logic, and among the unital magmas, the appropriate algebras are the right-algebras. For logicians, by contrary, the appropriate algebras are the algebras of logic, not the unital magmas, and among the algebras of logic, the appropriate algebras are the left-algebras. This explains why, for examples, the MV algebras were initially introduced as right-unital magmas, while the Wajsberg algebras were initially introduced as left-algebras of logic.

In this paper, *regarding from (algebras of) logic side, we shall work with left-algebras (left-unital magmas), in principal*, therefore, the unital magmas will be defined multiplicatively, in principal.

Thus, the commutative algebraic structures connected directly or indirectly with classical/non-classical logics belong to two parallel "worlds":

1. the "world" of *left-algebras of logic*, where there are essentially one implication, $\to$ (two, in the non-commutative case), and an element 1 (that can be the last element); the algebras $(A, \to, 1)$, verifying the basic property (M): $1 \to x = x$, are called *left-M algebras* [17], [19]; among the M algebras with additional operations, there are the algebras $(A, \to, 0, 1)$ (where a negation can be defined by: $x^- = x \to 0$), or $(A, \to, ^-, 1)$, with $1^- = 0$, where 1 is the *last element*, verifying (or not) (Ex) (Exchange): $x \to (y \to z) = y \to (x \to z)$; an internal binary relation can be defined by: $x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$ ($\leq$ can be a pre-order, an order, or even a lattice order); algebras belonging to this "world" are [17], [19]: the bounded MEL, BE and aBE, pre-BCK algebras, BCK algebras, bounded BCK algebras, BCK(P) algebras, Hilbert algebras, Wajsberg algebras, implicative-Boolean algebras, etc. A "Big map" (hierarchy of algebras of logic) is presented in ([19], Figure 1).

2. the "world" of *left-algebras*, where there are essentially a product, $\odot$, and an element 1 (that can be the last element); the algebras $(A, \odot, 1)$, verifying the corresponding basic properties (PU): $1 \odot x = x$ and (Pcomm): $x \odot y = y \odot x$, are called *commutative left-unital magmas*; among the commutative left-unital magmas with additional operations, there are the algebras [19] $(A, \odot, ^-, 1)$, with $1^- = 0$, where 1 is the *last element*, verifying (or not) (Pass) (associativity of product): $x \odot (y \odot z) = (x \odot y) \odot z$; an internal binary relation can be defined by: $x \leq_m y \Longleftrightarrow x \odot y^- = 0$ ($\leq_m$ can be a pre-order, an order, or even a lattice order), where 'm' comes from 'magma'; algebras belonging to this "world" are [17], [19]: the m-MEL, m-BE and m-aBE, m-pre-BCK algebras, m-BCK algebras, pocrims, (bounded) lattices, residuated lattices, BL algebras, MTL algebras, NM algebras, MV algebras, Boolean algebras, etc. A corresponding "Big map" (hierarchy of algebras) is presented in ([19], Figure 10) - see Figure 1.

MV algebras were introduced in 1958 by C.C. Chang [4], as a model of $\aleph_0$-valued Łukasiewicz logic. Chang's definition of MV algebras has 17 axioms. There is a huge literature concerning the MV algebras; we mention only a reference book, [3].

Between the two parallel " worlds" there are some connections, as for examples: the equivalence between BCK(P) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$). In [19], the two general Theorems 9.1 and 9.3 connect the two 'worlds' in the involutive case, by the inverse maps Φ ($x \odot y \overset{def.}{=} (x \to y^-)^-$) and Ψ ($x \to y \overset{def.}{=} (x \odot y^-)^-$) (Theorem 9.1 is for algebras *with last element*, while Theorem 9.3 is for algebras *without last element*); recall, for examples, that $\leq \Longleftrightarrow \leq_m$, that (M) $\Longleftrightarrow$ (PU) + (Pcomm), (Ex) $\Longleftrightarrow$ (Pass) etc. These theorems can be used to prove the *definitionally equivalence* (d.e.) between the analogous involutive (left-) algebras from the two "worlds" simply by choosing appropriate definitions of these algebras; for examples, one can prove the d.e. between implicative-Boolean algebras and Boolean algebras, between involutive BCK algebras and (involutive) m-BCK algebras etc..

Beside the classical and non-classical logics, there exist the quantum logics. Examples of algebraic structures connected with quantum logics (= quantum structures/algebras) are the bounded implicative (implication) lattices, the De Morgan algebras, the ortholattices, the orthomodular lattices, the quantum MV algebras (a better name is perhaps *quantum-MV algebras*, because they are generalizations of MV algebras, and not particular cases of MV algebras, i.e. they are not MV algebras that are 'quantum'), etc.

Quantum-MV algebras (or QMV algebras) were introduced by Roberto Giuntini in [7] (see also [8], [9], [10], [11], [12], [13], [6]), as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices. Cf. [6], from an algebraic point of view, MV algebras and QMV algebras share a common set of axioms, which S. Gudder [14] has called *supplement algebra* (*S algebra*). An MV algebra is an S-algebra verifying the axiom (MV), while an QMV algebra is an S algebra verifying the axiom (QMV), which is weaker than (MV).

Orthomodular lattices (particular ortholattices) generalize the Boolean algebras. They have arisen, cf. [28], "in the study of quantum logic, that is, the logic which supports quantum mechanics and which does not conform to classical logic. As noted by Birkhoff and von Neumann in 1936 [2], the calculus of propositions in quantum logic "is formally indistinguishable from the calculus of linear subspaces [of a Hilbert space] with respect to set products, linear sums and orthogonal complements" in the role of *and*, *or* and *not*, respectively. This has led to the study of the closed subspaces of a Hilbert space, which form an orthomodular lattice in contemporary terminology. As often happens in algebraic logic, the study of orthomodular lattices has tremendously developed, both for their interest in logic and for their own sake, see Kalmbach [26]".

The connections between algebras of logic/algebras and quantum algebras were not very clear. But, in papers [19], [23], we established important connections, by redefining equivalently the bounded involutive lattices and De Morgan algebras as involutive *m-MEL algebras* and the ortholattices, the MV and the Boolean algebras as involutive *m-BE algebras*, verifying some properties, and then putting all of them on the involutive "Big map"; thus, we have proved that the quantum algebras: bounded involutive lattices, De Morgan algebras and the ortholattices belong, in fact, to the "world" of *left-algebras* (involutive unital magmas).

In this paper, we clarify, mainly, some aspects concerning the quantum-MV algebras as non-lattice generalizations of MV algebras. We put MV algebras and quantum-MV algebras on the involutive "Big map", thus proving that quantum-MV algebras also belong, in fact, to the "world" of *left-algebras* (involutive left-unital magmas). We continue here the research from [23], [24], based on [19], in the "world" of involutive *left-algebras* of the form $(A, \odot, ^-, 1)$ verifying (Pass), with $1^- = 0$, 1 *being the last element*. This paper, like [19], [23], [24], presents the facts in the same *unifying way*, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different algebras and for obtaining results.

The paper is organized as follows. In Section 2 (**Preliminaries**), we recall the original definitions of quantum MV (QMV) algebras and of orthomodular lattices and also the definitions of MV algebras, m-MEL, m-BE, m-pre-BCK, m-BCK algebras and some results from [19]. In Section 3 (**Redefining the QMV algebras**), we introduce the operation $\wedge_m^B$ (beside $\wedge_m^M$) and the binary relation $\leq_m^B$ (beside $\leq_m^M$) and prove that $\leq_m \Longleftrightarrow \leq_m^B$. We redefine the QMV algebras as involutive m-BE algebras verifying the property (Pqmv), just as we have redefined in [19] the MV algebras as involutive m-BE algebras verifying the property ($\wedge_m$-comm). We prove that (Pqmv) is equivalent with only two properties, (Pmv) and (Pq) (Theorem 3.19); we prove that (Pq) is equivalent with (Pom), the property characterizing the orthomodular lattices among the ortholattices - this being the core of the paper (Theorem 3.26), by its difficulty. We also prove that, if (Pom) holds, then (Pmv) is equivalent with (Δ_m), the largest non-antisymmetric generalization of ($\wedge_m$-comm) (Theorem 3.32); thus, finally, (Pqmv) is equivalent with (Δ_m) and (Pom). In Section 4 (**Three generalizations of QMV algebras**), we introduce three generalizations of QMV algebras: the *pre-MV (PreMV) algebras*, the *metha-MV (MMV) algebras* and the *orthomodular (OM) algebras*, and study their transitive and/or antisymmetric members. We clarify the connection between QMV and MV algebras, by proving that MV algebras coincide with the antisymmetric QMV algebras - but also with the antisymmetric preMV and with the antisymmetric MMV algebras - (Corollary 4.10), result enabling us to put QMV algebras and MV algebras on the same "map". We point out the *transitive QMV (tQMV) algebra*, a particular QMV algebra, and the *transitive antisymmetric orthomodular (taOM) algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras. Finally, we put MV algebras and QMV, tQMV algebras on the same "map" (the involutive "Big map"); there is one open problem. In Section 5 (**Concluding remarks and future work**), we present conclusions and future work. In Section 6 (**Examples**), we present nine examples of the involved algebras.

Some of the proofs and of the examples were found by the powerful computer program *Prover9/Mace4* (version dec. 2007) created by William W. McCune (1953 - 2011). Therefore, we dedicate this paper to his memory.

2 Preliminaries

2.1 The original definition of quantum-MV algebras. The property (Wom)

Definition 2.1 ([6], Definition 2.1.1) [14]

A *supplement algebra*, or an *S algebra* for short, is an algebra $\mathcal{M} = (M, \oplus, ^-, 0, 1)$ consisting of a nonempty set M, two constant elements $0, 1$ in M, a unary operation $^-$ and a binary operation $\oplus$ on M satisfying the following axioms: for all $x, y, z \in M$,
(S1) $x \oplus y = y \oplus x$, (S2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (S3) $x \oplus x^- = 1$, (S4) $x \oplus 0 = x$, (S5) $(x^-)^- = x$ (or $x^= = x$), (S6) $0^- = 1$, (S7) $x \oplus 1 = 1$.

On every S algebra, the following operations can be introduced: $x \odot y := (x^- \oplus y^-)^-$, $\quad x \sqcap y := (x \oplus y^-) \odot y$, $\quad x \sqcup y := (x \odot y^-) \oplus y$.

QMV algebras were introduced by Roberto Giuntini [7], as S algebras satisfying additionally five axioms. The equivalence of the five axioms with the next axiom (QMV) was proved in [12].

Definition 2.2 ([6], Definition 2.3.1) A *quantum MV algebra*, or a *QMV algebra* for short, is an S algebra $\mathcal{M} = (M, \oplus, ^-, 0, 1)$ satisfying: for all $x, y, z \in M$,
(QMV) $x \oplus ((x^- \sqcap y) \sqcap (z \sqcap x^-)) = (x \oplus y) \sqcap (x \oplus z)$.

Note that QMV algebras were originally defined as right-algebras (see more on left- and right-algebras in [15], [17], [19]).

The most used definition of MV algebras is the following:

Definition 2.3 [3] An *MV algebra* is an algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ satisfying: for all $x, y, y \in A$,
(MV1) $x \oplus 0 = x$, (MV2) $x \oplus y = y \oplus x$, (MV3) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (MV4) $x \oplus 1 = 1$, where $1 \overset{def.}{=} 0^-$, (MV5) $(x^-)^- = x$ (or $x^= = x$), (MV6) = (MV) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Note that MV algebras were defined as right-algebras.

Definition 2.4 (See [28], [5]) An *ortholattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$ such that the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and the unary operation $^-$ satisfies: for all $x, y \in A$,
(DN) $(x^-)^- = x$ (or $x^= = x$) (Double Negation),
(DeM1) $(x \vee y)^- = x^- \wedge y^-$ (De Morgan law 1) and, dually,
(DeM2) $(x \wedge y)^- = x^- \vee y^-$ (De Morgan law 2),
and the complementation laws:
(m-WRe) $x \wedge x^- = 0$ (noncontradiction principle) and, dually,
(m-VRe) $x \vee x^- = 1$ (excluded middle principle).

Definition 2.5 An *orthomodular lattice* is an ortholattice $(A, \wedge, \vee, ^-, 0, 1)$ which satisfies the *orthomodular law*: for all $x, y \in A$,
(OML) $x \leq y \Longrightarrow x \vee (x^- \wedge y) = y$.

Note that property (OML) is not an identity, but there are many identities equivalent to (OML) within the class of ortholattices [28], as for example:

Proposition 2.6 *([28], Corollary 4.10.3) The following identity characterizes orthomodular lattices among ortholattices:*
(Wom) $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Note that orthomodular lattices were originally defined as left-algebras.

The dual of (Wom) is:
(Vom) $(x \vee y) \wedge ((x \vee y)^- \vee x) = x$,
where 'W' comes from 'wedge' (the LaTeX command for the meet, $\wedge$) and 'V' comes from 'vee' (the LaTeX command for the join, $\vee$).

2.2 The "Big map" of algebras. The involutive m-BE algebras

Recall from [19] the following:

Let $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ be an algebra of type $(2, 1, 0)$ and define $0 \overset{def.}{=} 1^-$. Define an *internal* binary relation $\leq_m$ on A^L by: for all $x, y \in A^L$,
(m-dfrelP) $\quad x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$.

Consider the following list **m-A** of basic properties that can be satisfied by $\mathcal{A}^L$ [19]:

(PU) $1 \odot x = x = x \odot 1$ (unit element of product, the *identity*),
(Pcomm) $x \odot y = y \odot x$ (commutativity of product),
(Pass) $x \odot (y \odot z) = (x \odot y) \odot z$ (associativity of product);
(Neg1-0) $1^- = 0$,
(Neg0-1) $0^- = 1$;
(m-An) $(x \odot y^- = 0$ and $y \odot x^- = 0) \Longrightarrow x = y$ (antisymmetry),
(m-B) $[(x \odot y^-)^- \odot (x \odot z)] \odot (y \odot z)^- = 0$,
(m-BB) $[(z \odot x)^- \odot (y \odot x)] \odot (y \odot z^-)^- = 0$,
(m-*) $x \odot y^- = 0 \Longrightarrow (z \odot y^-) \odot (z \odot x^-)^- = 0$,
(m-**) $x \odot y^- = 0 \Longrightarrow (x \odot z) \odot (y \odot z)^- = 0$,
(m-L) $x \odot 0 = 0$ (last element),
(m-Re) $x \odot x^- = 0$ (reflexivity),
(m-Tr) $(x \odot y^- = 0$ and $y \odot z^- = 0) \Longrightarrow x \odot z^- = 0$ (transitivity),
etc.,
where 'm' comes from 'magma'.

Dually, let $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ be an algebra of type $(2, 1, 0)$ and define $1 \stackrel{def.}{=} 0^-$. Define an *internal* binary relation $\geq_m$ on A^R by: for all $x, y \in A^R$,
(m-dfrelS) $x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1$.
The list of dual properties is omitted.

Recall from [19] the definitions of the algebras needed in this paper (the dual ones are omitted):

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an algebra of type $(2, 1, 0)$ through this paper. Define $0 \stackrel{def.}{=} 1^-$ (hence (Neg1-0) holds) and suppose that $0^- = 1$ (hence (Neg0-1) holds too). We say that $\mathcal{A}^L$ is a [19]:
- *left-m-MEL algebra*, if (PU), (Pcomm), (Pass), (m-L) hold;
- *left-m-BE algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) hold;
- *left-m-pre-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) and (m-BB) hold;
- *left-m-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An) and (m-BB) hold.

Denote by **m-MEL**, **m-BE**, ..., **m-BCK** these classes of left-algebras, respectively.

We say that $\mathcal{A}^L$ is [19] *reflexive*, if $\leq_m$ is reflexive (i.e. (m-Re) holds); *transitive*, if $\leq_m$ is transitive (i.e. (m-Tr) holds); *antisymmetric*, if $\leq_m$ is antisymmetric (i.e. (m-An) holds). If **X** is a class of algebras, we shall denote by **tX** (**aX**, **atX**=**taX**) the subclass of all transitive (antisymmetric, transitive and antisymmetric, respectively) algebras of **X**.

A hierarchy of classes of such algebras will be represented by a kind of Hasse-type diagram, where the algebras are represented as follows:

- *reflexive* algebras by ◯
- *antisymmetric* algebras by ∘
- *transitive* algebras by •
- *reflexive* and *antisymmetric* algebras by ◎
- *reflexive* and *transitive* algebras by ⊙
- *ordered* algebras by ⬤

and a class of algebras which does not verify (m-Re), (m-An), (m-Tr), by □.

In ([19], Figure 10) - see next Figure 1 - the "Big map", connecting the commutative left-unital magmas, including these new algebras, was drawn.

We say that an algebra is *involutive*, if it verifies (DN). If **X** is a class of algebras, we shall denote by $\mathbf{X}_{(DN)}$ the subclass of all involutive algebras of **X**. By ([19], Theorem 6.12), in any involutive m-BE algebra we have the equivalences: (m-BB) ⇔ (m-B) ⇔ (m-**) ⇔ (m-*) ⇔ (m-Tr).

Note that: $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)} = \mathbf{pre\text{-}m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}tBE}_{(DN)})$.

Any left-m-BCK algebra is involutive, by ([19], Theorem 6.13). We write: **m-BCK**= $\mathbf{m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}taBE}_{(DN)})$. Note that an (involutive) m-BCK algebra satisfies all the properties in the list **m-A** of properties and, additionally, (DN) and other properties.

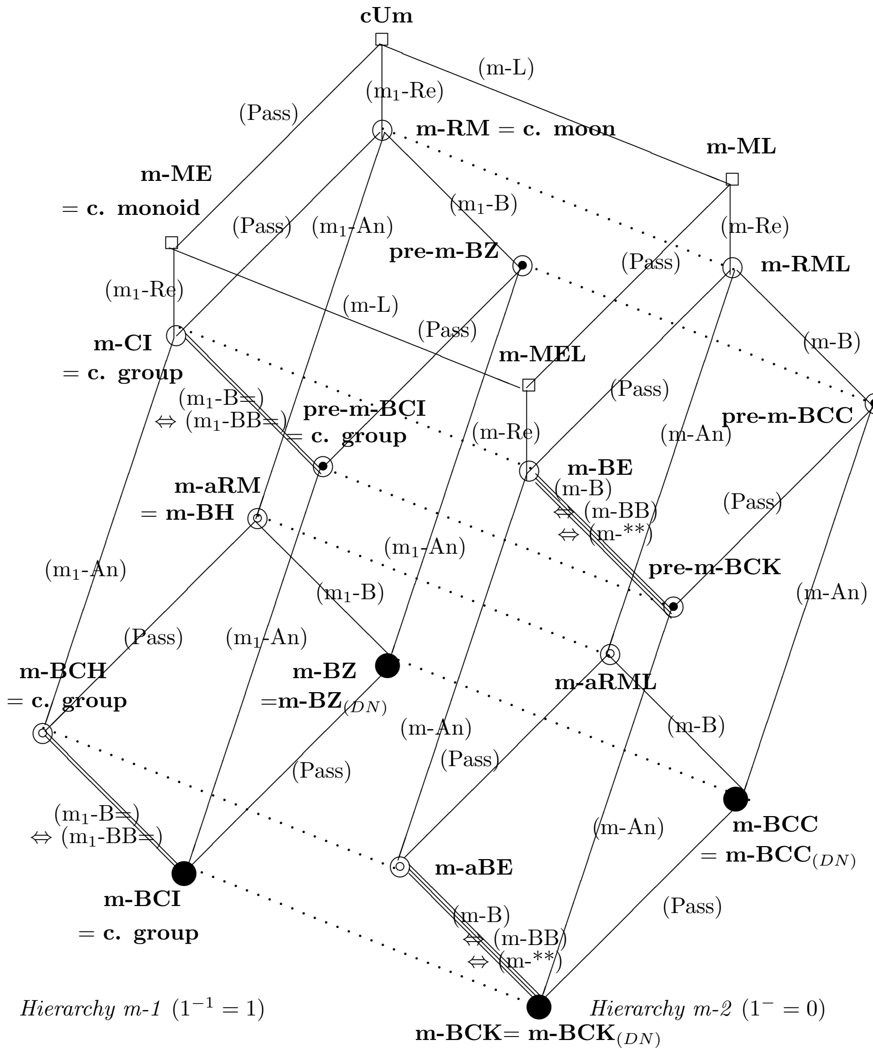


Figure 1: The "Big map": hierarchies m-1 and m-2 of commutative unital magmas (**cUm**)

Note that the binary relation $\leq_m$ is only **reflexive** in **m-BE**$_{(DN)}$, it is a **pre-order** in **m-pre-BCK**$_{(DN)}$ and it is an **order** in **m-BCK**.

2.2.1 Involutive m-MEL algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Because of the axiom (DN), we have introduced in [23] the new operation sum, $\oplus$, the dual of product, $\odot$, by: for all $x, y \in A^L$,

$$x \oplus y \stackrel{def.}{=} (x^- \odot y^-)^-. \tag{1}$$

Hence,

$$x \odot y = (x^- \oplus y^-)^-. \tag{2}$$

Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-MEL algebra. We have: $x \leq_m y \Longleftrightarrow y \geq_m x$.

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [23] a new binary relation:
(m-dfP) $x \ \leq_m^P \ y \stackrel{def.}{\Longleftrightarrow} x \odot y = x$ and, dually,
(m-dfS) $x \ \geq_m^S \ y \stackrel{def.}{\Longleftrightarrow} x \oplus y = x$.
By ([23], Proposition 3.11), $\leq_m^P$ is antisymmetric and transitive and $0 \leq_m^P x \leq_m^P 1$, for any x.

With the notations from this subsection, Definition 2.3 of MV algebras becomes [19]:

Definition 2.7
(i) A *left-MV algebra* is an algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ of type $(2, 1, 0)$ verifying (PU), (Pcomm), (Pass), (m-L), (DN) and:
($\wedge_m$-comm) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.
(i') Dually, a *right-MV algebra* is an algebra $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ of type $(2, 1, 0)$ verifying (SU), (Scomm), (Sass), (m-L^R), (DN) and:
($\vee_m$-comm) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

We recall the following important remark, which was the motivation of paper [19]:
(i) The left-MV algebra is just the involutive left-m-MEL algebra verifying ($\wedge_m$-comm).
(i') Dually, the right-MV algebra is just the involutive right-m-MEL algebra verifying ($\vee_m$-comm).

Denote by **MV** the class of all left-MV algebras and by **MV**^{R} the class of all right-MV algebras.

Proposition 2.8 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Then:*
$(\wedge_m - comm) \iff (\vee_m - comm)$.

Proof. Routine. □

2.2.2 Involutive m-BE algebras. The property (Pom)

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-BE algebra.

Remark 2.9 An *S algebra* (Definition 2.1) is just an *involutive right-m-BE algebra.*

Remarks 2.10 *(The dual one is omitted)*
(i) Since ($\wedge_m$-comm) implies (m-Re), by ([19], (mB1)), it follows that **any left-MV algebra is in fact an involutive left-m-BE algebra verifying ($\wedge_m$-comm)**.
(ii) Since ($\wedge_m$-comm) implies also (m-An) and (m-BB) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-Tr)), by ([19], (mB2), (mCBN1)), respectively, i.e. we have:

$$(\wedge_m - comm) \implies (m-An) \ + \ (m-Tr), \tag{3}$$

it follows that **any left-MV algebra is in fact a left-m-BCK algebra**, *i.e. we have:*

$$\mathbf{MV} \subset \mathbf{m-BCK} = \mathbf{m-BCK_{(DN)}}(= \mathbf{m-taBE_{(DN)}}).$$

(iii) Moreover, by ([19], Theorem 6.21), the class of left-MV algebras is d.e. with the class of $\wedge_m$-commutative (involutive) left-m-BCK algebras (i.e. left-m-BCK algebras verifying ($\wedge_m$-comm)).

In ([19], Figure 8), the connections between m-BE algebras, m-BCK algebras, MV algebras, ortholattices and Boolean algebras were established, thus putting MV algebras, ortholattices and Boolean algebras on the "map" (the right side of the involutive "Big map").

We have redefined equivalently the ortholattices (Definition 2.4) in [19], [23] as follows (Definition 2):

(i) A *left-ortholattice* is a (involutive) left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying:
(m-Pimpl) $[(x \odot y^-)^- \odot x^-]^- = x$.

(i') Dually, a *right-ortholattice* is a (involutive) right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ verifying:
(m-Simpl) $[(x \oplus y^-)^- \oplus x^-]^- = x$.

It follows that an orthomodular lattice (Definition 2.5) can be redefined equivalently as follows (Definition 2):

(i) An *orthomodular left-lattice* is a (involutive) left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pom) ((Wom) becomes (Pom)), where:
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$.

(i') Dually, an *orthomodular right-lattice* is a (involutive) right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ verifying (m-Simpl) and (Som) ((Vom) becomes (Som)), where:
(Som) $(x \oplus y) \odot ((x \oplus y)^- \oplus x) = x$.

Thus, **(Pom) is the property characterizing the orthomodular left-lattices among the left-ortholattices** (Definitions 2). It will play a major role in QMV algebras.

3 Redefining the QMV algebras

Remark 3.1 *Starting from the equality from the property*

$$(\wedge_m - comm) \qquad (x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x,$$

verified by a left-MV algebra, we could introduce two different 'twin' operations, $\wedge_m^M$ ('M' comes from 'MV algebra') and $\wedge_m^B$ ('B' comes from 'Boolean algebra'), by: for all x, y:
$x \wedge_m^M y = (x^- \odot y)^- \odot y$ *and* $x \wedge_m^B y = (y^- \odot x)^- \odot x$.
Then, ($\wedge_m$-comm) would mean: either
($\wedge_m^M$-comm) $x \wedge_m^M y = y \wedge_m^M x$ *or*
($\wedge_m^B$-comm) $x \wedge_m^B y = y \wedge_m^B x$.

In left-MV algebras, the two operations $\wedge_m^M$ and $\wedge_m^B$ are equal, but in general, in an involutive left-m-MEL algebra, they are different; but note that: $x \wedge_m^M y = y \wedge_m^B x$, which means, in the finite case, that the matrix of $\wedge_m^B$ is the transposed matrix of that of $\wedge_m^M$ and vice-versa.

Following the above Remark 3.1, we shall introduce in an involutive left-m-MEL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ the following operations:

(4) $$x \wedge_m^M y \overset{def.}{=} (x^- \odot y)^- \odot y \overset{(Pcomm)}{=} y \odot (y \odot x^-)^- \quad \textit{and, dually,}$$

(5) $$x \vee_m^M y \overset{def.}{=} (x^- \wedge_m^M y^-)^- = [(x \odot y^-)^- \odot y^-]^- = (x \odot y^-) \oplus y = (x^- \oplus y)^- \oplus y = y \oplus (y \oplus x^-)^-$$

and

(6) $$x \wedge_m^B y \overset{def.}{=} (y^- \odot x)^- \odot x \overset{(Pcomm)}{=} x \odot (x \odot y^-)^- = x \odot (x \to y) = y \wedge_m^M x \quad \textit{and, dually,}$$

(7) $$x \vee_m^B y \overset{def.}{=} (x^- \wedge_m^B y^-)^- = [(y \odot x^-)^- \odot x^-]^- = (y \odot x^-) \oplus x = (y^- \oplus x)^- \oplus x = x \oplus (x \oplus y^-)^- = y \vee_m^M x.$$

Note that the dual operations $\wedge_m^M$, $\vee_m^M$ are just $\sqcap$, $\sqcup$, respectively, recalled in subsection 2.1.
In what follows, we shall present only the properties of $\wedge_m^M$ and $\vee_m^M$.

Beside the old, natural binary relation $\leq_m$, and its dual $\geq_m$, and the binary relation $\leq_m^P$, and its dual $\geq_m^S$, we introduce two binary relations, the old $\leq_m^M$ (see [6]) and the new $\leq_m^B$: for all $x, y \in A^L$,
(m-dfWM) $x \ \leq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^M y = x$ and, dually,
(m-dfVM) $x \ \geq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^M y = x$,
and
(m-dfWB) $x \ \leq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^B y = x$ ($\Longleftrightarrow y \wedge_m^M x = x$) and, dually,
(m-dfVB) $x \ \geq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^B y = x$ ($\Longleftrightarrow y \vee_m^M x = x$).

Lemma 3.2 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*
(1) $\quad x \odot y^- = 0 \Longleftrightarrow x \odot (x \odot y^-)^- = x$ *and, dually,*
(1') $\quad x \oplus y^- = 1 \Longleftrightarrow x \oplus (x \oplus y^-)^- = x$.

Proof. (1): Suppose that $x \odot y^- = 0$; then, $(x \odot y^-)^- = 1$, hence $x \odot (x \odot y^-)^- = x \odot 1 = x$.
Conversely, suppose that $x \odot (x \odot y^-)^- = x$; then,
$x \odot y^- = (x \odot (x \odot y^-)^-) \odot y^- \stackrel{(Pcomm),(Pass)}{=} (x \odot y^-) \odot (x \odot y^-)^- \stackrel{(m-Re)}{=} 0$.
(1'): Suppose that $x \oplus y^- = 1$; then, $(x \oplus y^-)^- = 0$, hence $x \oplus (x \oplus y^-)^- = x \oplus 0 = x$.
Conversely, suppose that $x \oplus (x \oplus y^-)^- = x$; then,
$x \oplus y^- = (x \oplus (x \oplus y^-)^-) \oplus y^- \stackrel{(Scomm),(Sass)}{=} (x \oplus y^-) \oplus (x \oplus y^-)^- \stackrel{(m-Re^R)}{=} 1$. □

Proposition 3.3 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*
(1) $\quad x \leq_m y \Longleftrightarrow x \leq_m^B y$ *and, dually*
(1') $\quad x \geq_m y \Longleftrightarrow x \geq_m^B y$.
(2) If ($\wedge_m$-comm) holds (i.e. $x \wedge_m^M y = y \wedge_m^M x$), then

$$x \leq_m y \ (\Longleftrightarrow x \leq_m^B y) \ \Longleftrightarrow x \leq_m^M y.$$

(2') If ($\wedge_m$-comm) holds, then ($\vee_m$-comm) holds (i.e. $x \vee_m^M y = y \vee_m^M x$) and

$$x \geq_m y \ (\Longleftrightarrow x \geq_m^B y) \ \Longleftrightarrow x \geq_m^M y.$$

Proof. (1): By above Lemma 3.2 (1),
$x \leq_m y \stackrel{def.}{\Longleftrightarrow} x \odot y^- = 0 \Longleftrightarrow x \odot (x \odot y^-)^- = x \Longleftrightarrow x \wedge_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \leq_m^B y$.
(1'): By above Lemma 3.2 (1'),
$x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1 \Longleftrightarrow x \oplus (x \oplus y^-)^- = x \Longleftrightarrow x \vee_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \geq_m^B y$.
(2): By above (1),
$x \leq_m^M y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^M y = x \stackrel{(\wedge_m-comm)}{\Longleftrightarrow} y \wedge_m^M x = x \Longleftrightarrow x \wedge_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \leq_m^B y \Longleftrightarrow x \leq_m y$.
(2'): By Proposition 2.8 and above (1'). □

Remark 3.4 *The equivalence $\leq_m \Longleftrightarrow \leq_m^B$ implies that $\leq_m$ is an order relation if and only if $\leq_m^B$ is an order relation. But, it does not imply that if $\leq_m$ is a lattice order w.r. to say $\wedge, \vee$, then $\leq_m^B$ is a lattice order too with respect to $\wedge_m^B, \vee_m^B$.*

Proposition 3.5 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(x \leq_m^B y \Longleftrightarrow) \ x \leq_m y \Longleftrightarrow y \geq_m x \ (\Longleftrightarrow y \geq_m^B x).$$

Proof. By Proposition 3.3 and ([23], Proposition 3.10). □

Proposition 3.6 *(See ([6], Proposition 2.1.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:

(8) $$x \wedge_m^M 1 = x = 1 \wedge_m^M x, \quad x \wedge_m^M 0 = 0,$$

$$x \vee_m^M 0 = x = 0 \vee_m^M x, \quad x \vee_m^M 1 = 1, \tag{9}$$

$$(x \vee_m^M y)^- = x^- \wedge_m^M y^- \quad (De\ Morgan\ law\ 1), \tag{10}$$

$$(x \wedge_m^M y)^- = x^- \vee_m^M y^- \quad (De\ Morgan\ law\ 2)\ and,\ hence, \tag{11}$$

$$x \wedge_m^M y = (x^- \vee_m^M y^-)^-. \tag{12}$$

Proposition 3.7 *(See ([6], Proposition 2.1.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

$$if \quad x \odot y = 1, \quad then \quad x = y = 1, \tag{13}$$

$$if \quad x \wedge_m^M y = 1, \quad then \quad x = y = 1, \tag{14}$$

$$0 \wedge_m^M x = 0, \tag{15}$$

$$1 \vee_m^M x = 1, \tag{16}$$

$$x \wedge_m^M x = x, \quad x \vee_m^M x = x, \tag{17}$$

$$if \quad x \ \leq_m^M \ y, \quad then \quad y \wedge_m^M x = x, \tag{18}$$

$$if \quad x \ \leq_m^M \ y, \quad then \quad x \leq_m y. \tag{19}$$

Proposition 3.8 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*

$$x \oplus x^- = 1, \quad i.e. \quad (m - Re^R) \quad holds; \tag{20}$$

$$x \odot (y \wedge_m^M x^-) = 0, \tag{21}$$

$$x \odot (x^- \wedge_m^M y) = 0, \tag{22}$$

$$(y \vee_m^M x) \wedge_m^M x = x, \tag{23}$$

$$(y \wedge_m^M x) \vee_m^M x = x, \tag{24}$$

$$if \quad x \ \leq_m^M \ y, \quad then \quad x \vee_m^M y = y, \tag{25}$$

$$x \vee_m^M y = y \iff x \odot y^- = 0 \quad (\iff x \leq_m y), \tag{26}$$

$$(x \odot y) \vee_m^M x = x, \tag{27}$$

$$x \wedge_m^M (x \odot y) = x \odot y, \tag{28}$$

$$x \wedge_m^M (y \wedge_m^M x) = y \wedge_m^M x. \tag{29}$$

Proof. (20): $x \oplus x^- \overset{(1)}{=} (x^- \odot x)^- \overset{(Pcomm)}{=} (x \odot x^-)^- \overset{(m-Re)}{=} 0^- \overset{(Neg0-1)}{=} 1.$

(21): $x \odot (y \wedge_m^M x^-) = x \odot [(y^- \odot x^-)^- \odot x^-]$
$\overset{(Pcomm)}{=} x \odot [x^- \odot (y^- \odot x^-)^-] \overset{(Pass)}{=} (x \odot x^-) \odot (y^- \odot x^-)^-$
$\overset{(m-Re)}{=} 0 \odot (y^- \odot x^-)^- \overset{(Pcomm)}{=} (y^- \odot x^-)^- \odot 0 \overset{(m-L)}{=} 0.$

(22): $x \odot (x^- \wedge_m^M y) = x \odot [(x \odot y)^- \odot y] = x \odot [y \odot (x \odot y)^-] = (x \odot y) \odot (x \odot y)^- \overset{(m-Re)}{=} 0.$

(23): $(y \vee_m^M x) \wedge_m^M x \overset{(5)}{=} [(y \odot x^-)^- \odot x^-]^- \wedge_m^M x \overset{(4),(DN)}{=} (((y \odot x^-)^- \odot x^-) \odot x)^- \odot x$
$\overset{(Pass),(Pcomm)}{=} ((y \odot x^-)^- \odot (x \odot x^-))^- \odot x \overset{(m-Re)}{=} ((y \odot x^-)^- \odot 0)^- \odot x \overset{(m-L)}{=} 0^- \odot x = 1 \odot x = x.$

(24): $(y \wedge_m^M x) \vee_m^M x \overset{(4)}{=} [(y^- \odot x)^- \odot x] \vee_m^M x \overset{(5)}{=} ([(y^- \odot x)^- \odot x] \odot x^-) \oplus x$
$\overset{(Pass)}{=} ((y^- \odot x)^- \odot (x \odot x^-)) \oplus x \overset{(m-Re)}{=} ((y^- \odot x)^- \odot 0) \oplus x \overset{(m-L)}{=} 0 \oplus x \overset{(SU)}{=} x.$

(25): Suppose $x \leq_m^M y$, i.e. $x \wedge_m^M y = x$; then, $x \vee_m^M y = (x \wedge_m^M y) \vee_m^M y \overset{(24)}{=} y.$

(26): If $y = x \vee_m^M y \overset{(5)}{=} (x \odot y^-) \oplus y$, then $y^- = ((x \odot y^-) \oplus y)^- = (x \odot y^-)^- \odot y^-$. Then,
$x \odot y^- = x \odot [(x \odot y^-)^- \odot y^-] \overset{(Pcomm),(Pass)}{=} (x \odot y^-) \odot (x \odot y^-)^- \overset{(m-Re)}{=} 0.$
Conversely, if $x \odot y^- = 0$, then $x \vee_m^M y \overset{(5)}{=} (x \odot y^-) \oplus y = 0 \oplus y \overset{(SU)}{=} y.$

(27: $(x \odot y) \vee_m^M x = [[(x \odot y) \odot x^-]^- n \odot x^-]^- \overset{(m-Re),(m-L)}{=} [0^- \odot x^-]^- = (x^-)^- = x.$

(28): $x \wedge_m^M (x \odot y) = [x^- \odot (x \odot y)]^- \odot (x \odot y) \overset{(m-Re),(m-L)}{=} 0^- \odot (x \odot y) = 1 \odot (x \odot y) = x \odot y.$

(29): $x \wedge_m^M (y \wedge_m^M x) = x \wedge_m^M (x \odot (x \odot y^-)^-) = (x \odot (x \odot y^-)^-) \odot (x^- \odot (x \odot (x \odot y^-)^-))^-$
$\overset{(Pass)}{=} (x \odot (x \odot y^-)^-) \odot ((x^- \odot x) \odot (x \odot y^-)^-)^-$
$\overset{(Pcomm),(m-Re),(m-L)}{=} (x \odot (x \odot y^-)^-) \odot 0^- \overset{(Neg0-1),(PU)}{=} x \odot (x \odot y^-)^- = y \wedge_m^M x.$ □

Corollary 3.9 *(See ([6], Corollary 2.1.3))*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, the binary relation $\leq_m^M$ is reflexive and antisymmetric and $0 \leq_m^M x \leq_m^M 1$, for all $x \in A^L$, where $0 \overset{def.}{=} 1^-$.

3.1 Redefining the QMV algebras as involutive m-BE algebras

Following the original definition (Definition 2.2) of QMV algebras, the definition of involutive m-BE algebras and Remark 2.9, we obtain the following redefinition of QMV algebras as involutive m-BE algebras, which helps us to put them on the “map” (the involutive “Big map”):

Definitions 3.10
(i) A *left-quantum-MV algebra*, or a *left-QMV algebra* for short, is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-_L}, 1)$ verifying the following axiom: for all $x, y, z \in A^L$,
(Pqmv) $\quad x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z).$
(i') A *right-quantum-MV algebra*, or a *right-QMV algebra* for short, is an involutive right-m-BE algebra (= S algebra) $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-_R}, 0)$ verifying the following dual axiom: for all $x, y, z \in A^R$,
(Sqmv) = (QMV) $\quad x \oplus [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)] = (x \oplus y) \wedge_m^M (x \oplus z).$

We shall denote by **QMV** the class of all left-QMV algebras and by $\mathbf{QMV}^R$ the class of all right-QMV algebras.

Proposition 3.11 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then:*

$$(Pqmv) \Longleftrightarrow (Sqmv).$$

Proof. Suppose (Pqmv) holds; then, $x \oplus [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)]$
$\overset{(1)}{=} (x^- \odot [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)])^- \overset{(11)}{=} (x^- \odot [(x^- \wedge_m^M y)^- \vee_m^M (z \wedge_m^M x^-)^-])^-$
$\overset{(11)}{=} (x^- \odot [(x \vee_m^M y^-) \vee_m^M (z^- \vee_m^M x)])^- \overset{(Pqmv)}{=} ((x^- \odot y^-) \vee_m^M (x^- \odot z^-))^-$
$\overset{(10)}{=} (x^- \odot y^-)^- \wedge_m^M (x^- \odot z^-)^- \overset{(1)}{=} (x \oplus y) \wedge_m^M (x \oplus z)$, i.e. (Sqmv) holds.
Suppose (Sqmv) holds; then, $x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)]$

$\overset{(2)}{=} (x^- \oplus [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)])^- \overset{(10)}{=} (x^- \oplus [(x^- \vee_m^M y)^- \wedge_m^M (z \vee_m^M x^-)^-])^-$
$\overset{(10)}{=} (x^- \oplus [(x \wedge_m^M y^-) \wedge_m^M (z^- \wedge_m^M x)])^- \overset{(Sqmv)}{=} ((x^- \oplus y^-) \wedge_m^M (x^- \oplus z^-))^-$
$\overset{(11)}{=} (x^- \oplus y^-)^- \vee_m^M (x^- \oplus z^-)^- \overset{(2)}{=} (x \odot y) \vee_m^M (x \odot z)$, i.e. (Pqmv) holds. $\square$

Corollary 3.12 Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, $(A^L, \oplus, ^-, 0)$ is a right-QMV algebra.

Proof. By ([23], Corollary 4.3) and Proposition 3.11. $\square$

Proposition 3.13 *(See ([6], Proposition 2.3.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:

$$x \odot (y \vee_m^M x^-) = x \odot y, \tag{30}$$

$$(Pmv) \quad x \odot (x^- \vee_m^M y) = x \odot y,$$

$$(Pq) \quad x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z);$$

$$x \odot y \ \leq_m^M \ x, \quad i.e. \quad (x \odot y) \wedge_m^M x = x \odot y, \tag{31}$$

$$x \ \leq_m^M \ x \oplus y, \quad i.e. \quad x \wedge_m^M (x \oplus y) = x, \tag{32}$$

$$x \wedge_m^M y \ \leq_m^M \ y, \quad i.e. \quad (x \wedge_m^M y) \wedge_m^M y = x \wedge_m^M y, \tag{33}$$

$$y \ \leq_m^M \ x \vee_m^M y, \quad i.e. \quad y \wedge_m^M (x \vee_m^M y) = y, \tag{34}$$

$$x \odot [y \vee_m^M (x \odot z)^-] = (x \odot y) \vee_m^M (x \odot (x \odot z)^-), \tag{35}$$

$$x \vee_m^M (y \wedge_m^M x) = x, \tag{36}$$

$$x \ \leq_m^M \ y \Longrightarrow y \vee_m^M x = y, \tag{37}$$

$$x \ \leq_m^M \ y \Longrightarrow y^- \ \leq_m^M \ x^- \quad (order - reversibility\ of\ ^-), \tag{38}$$

$$x \ \leq_m^M \ y \Longrightarrow x \oplus z \ \leq_m^M \ y \oplus z \quad (monotonicity\ of\ \oplus), \tag{39}$$

$$x \ \leq_m^M \ y \Longrightarrow x \odot z \ \leq_m^M \ y \odot z \quad (monotonicity\ of\ \odot), \tag{40}$$

$$(x \wedge_m^M y) \wedge_m^M z = (x \wedge_m^M y) \wedge_m^M (y \wedge_m^M z), \tag{41}$$

$$(x \vee_m^M y) \vee_m^M z = (x \vee_m^M y) \vee_m^M (y \vee_m^M z), \tag{42}$$

$$x \odot y = x \odot y \odot (x \oplus y), \tag{43}$$

$$(x^- \odot y) \wedge_m^M (y^- \odot x) = 0. \tag{44}$$

Remarks 3.14 *(i) Concerning (33), note that $x \wedge_m^M y \ \not\leq_m^M \ x$. For example, in the left-QMV algebra from Example 6.4, $a \wedge_m^M c \ \not\leq_m^M \ a$. Indeed, $a \wedge_m^M c = c$, while $(a \wedge_m^M c) \wedge_m^M a = c \wedge_m^M a = a \neq c$.*

(ii) Concerning (34), note that $x \ \not\leq_m^M \ x \vee_m^M y$. For example, in the left-QMV algebra from Example 6.4, $a \ \not\leq_m^M \ a \vee_m^M c$. Indeed, $a \vee_m^M c = c$, while $a \wedge_m^M (a \vee_m^M c) = a \wedge_m^M c = c \neq a$.

Recall now the following well known property (prel) (prelinearity) from a bounded residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ [15]:
(prel) $\quad (x \to y) \vee (y \to x) = 1$.

Here, we shall consider that $\vee \stackrel{def.}{=} \vee_m^B$, which is no more a lattice operation, therefore the property will be denoted by (prel$_m$):
(prel$_m$) $\quad (x \to y) \vee_m^B (y \to x) = 1$,
where $x \to y \stackrel{def.}{=} (x \odot y^-)^-$ (see the map Ψ from [19]).

Note that, in MV algebras, (prel$_m$) and (prel) coincide.

Then, we have the following result:

Corollary 3.15 *Any left-QMV algebra verifies the property (prel$_m$).*

Proof. $(x \to y) \vee_m^B (y \to x) = (x \odot y^-)^- \vee_m^B (y \odot x^-)^-$
$= (y \odot x^-)^- \vee_m^M (x \odot y^-)^- = ((y \odot x^-) \wedge_m^M (x \odot y^-))^- \stackrel{(Pcomm),(44)}{=} 0^- = 1$. $\square$

Proposition 3.16 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:*

$$x \vee_m^M y \ \leq_m^M x \oplus y, \tag{45}$$

$$x \odot y \ \leq_m^M x \wedge_m^M y. \tag{46}$$

Proof. (45): Since $x \odot y^- \ \leq_m^M \ x$, by (31), then $x \vee_m^M y = (x \odot y^-) \oplus y \ \leq_m^M \ x \oplus y$, by (39).

(46): Since $x^- \odot y \ \leq_m^M \ x^-$, by (31), then $x \ \leq_m^M \ (x^- \odot y)^-$, by (38) and (DN); hence, $x \odot y \ \leq_m^M (x^- \odot y)^- \odot y = x \wedge_m^M y$, by (40). $\square$

Proposition 3.17 *(See ([6], Proposition 2.3.5), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:

$$x \wedge_m^M ((x \oplus y) \wedge_m^M z) = x \wedge_m^M z \quad (\textit{absorption law } 1), \tag{47}$$

$$x \vee_m^M ((x \odot y) \vee_m^M z) = x \vee_m^M z \quad (\textit{absorption law } 2), \tag{48}$$

$$x \ \leq_m^M \ z^-, \ y \ \leq_m^M \ z^-, \ x \oplus z = y \oplus z \implies x = y \quad (\textit{cancellation law } 1), \tag{49}$$

$$z^- \ \leq_m^M \ x, \ z^- \ \leq_m^M \ y, \ x \odot z = y \odot z \implies x = y \quad (\textit{cancellation law } 2), \tag{50}$$

$$x \ \leq_m^M \ y \implies x \wedge_m^M z \ \leq_m^M \ y \wedge_m^M z \quad (\textit{monotonicity of } \wedge_m^M), \tag{51}$$

$$x \ \leq_m^M \ y \implies x \vee_m^M z \ \leq_m^M \ y \vee_m^M z \quad (\textit{monotonicity of } \vee_m^M), \tag{52}$$

$$x \ \leq_m^M \ y, \ y \ \leq_m^M \ z \implies x \ \leq_m^M \ z \quad (\textit{transitivity of } \ \leq_m^M). \tag{53}$$

Corollary 3.18 *(See ([6], page 157))*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. The binary relation $\leq_m^M$ is an order relation.

We shall prove next **the first very important result of this paper**, Theorem 3.19, saying that axiom (Pqmv) is equivalent to only two properties, the properties (Pmv) and (Pq) from Proposition 3.13:
(Pmv) $\quad x \odot (x^- \vee_m^M y) = x \odot y$,
(Pq) $\quad x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$.
Recall that, cf. ([6], Proposition 2.3.4), Giuntini proved that axiom (Pqmv) is equivalent to the properties (Pmv), (35), (36), (41) and (44).

Theorem 3.19 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (Pmv) + (Pq).$$

Proof. By Proposition 3.13, the axioms of $\mathcal{A}^L$ and (Pqmv) imply (Pmv) and (Pq). To prove the converse, assume that (Pmv) and (Pq) are satisfied by $\mathcal{A}^L$. Then:
$x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)]$
$\overset{(5)}{=} x \odot [((x^- \vee_m^M y) \odot (z \vee_m^M x^-)^-) \oplus (z \vee_m^M x^-)]$
$\overset{(10),(DN)}{=} x \odot [((x^- \vee_m^M y) \odot (z^- \wedge_m^M x)) \oplus (z \vee_m^M x^-)]$
$\overset{(4),(DN)}{=} x \odot [((x^- \vee_m^M y) \odot ((z \odot x)^- \odot x)) \oplus (z \vee_m^M x^-)]$
$\overset{(Pass),(Pcomm)}{=} x \odot [((x \odot (x^- \vee y)) \odot (z \odot x)^-) \oplus (z \vee_m^M x^-)]$
$\overset{(Pmv)}{=} x \odot [((x \odot y) \odot (z \odot x)^-) \oplus (z \vee_m^M x^-)]$
$\overset{(Pcomm),(Pass)}{=} x \odot [(y \odot ((z \odot x)^- \odot x)) \oplus (z \vee_m^M x^-)]$
$\overset{(4),(DN)}{=} x \odot [(y \odot (z^- \wedge_m^M x)) \oplus (z \vee_m^M x^-)]$
$\overset{(10),(DN)}{=} x \odot [(y \odot (z \vee_m^M x^-)^-) \oplus (z \vee_m^M x^-)]$
$\overset{(5)}{=} x \odot [y \vee_m^M (z \vee_m^M x^-)]$
$\overset{(Pq)}{=} (x \odot y) \vee_m^M (x \odot z)$; thus, (Pqmv) holds. □

Proposition 3.20 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, (Pom) holds.*

Proof. Take $y = 1$ in (Pqmv). □

Proposition 3.21 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (G) ($x \odot x = x$). Then:*
(1) $\leq_m^P$ is reflexive also, hence it is an **order relation***.*
(2) We have the equivalence:

$$(x \odot y = x \Longleftrightarrow)\ x \leq_m^P y \Longleftrightarrow x \leq_m^M y\ (\Longleftrightarrow x \wedge_m^M y = x).$$

Proof. (1): $x \leq_m^P x \Longleftrightarrow x \odot x = x$, that is true by (G).
(2): Suppose $x \leq_m^P y$, i.e. $x \odot y = x$. Then, by (31), $x = x \odot y \leq_m^M y$. Conversely, suppose $x \leq_m^M y$. Then, by (40), we have: $x \overset{(G)}{=} x \odot x \leq_m^M y \odot x \overset{(Pcomm)}{=} x \odot y$, and since we also have, by (31), that $x \odot y \leq_m^M x$, we obtain, by antisymmetry of $\leq_m^M$ (by Corollary 3.9), that $x \odot y = x$, i.e. $x \leq_m^P y$. □

Remarks 3.22 *In a left-QMV algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:*
- the initial binary relation, $\leq_m$ ($x \leq_m y \iff x \odot y^- = 0$) ($\leq_m \Longleftrightarrow \leq_m^B$), is **only reflexive** *((m-Re) holds, by definition of m-BE algebra);*
- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an **order***, by Corollary 3.18, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$;*
- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is **only antisymmetric and transitive***, by ([23], Proposition 3.11).*
In a left-QMV algebra verifying (G), $\leq_m^M$ and $\leq_m^P$ are order relations and $\leq_m^M \Longleftrightarrow \leq_m^P$.

3.2 The equivalence between (Pq) and (Pom)

Consider now the properties:
(Pq) $x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$ and
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$, which characterizes the orthomodular lattices among ortholattices.

Proposition 3.23 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \Longrightarrow (Pom).$$

Proof. In (Pq) take $y := 1$ to obtain: $x = x \vee_m^M (x \odot z)$, i.e. (Pom). $\Box$

The converse result, the next Proposition 3.25 (saying that (Pom) implies (Pq)), was proved by *Prover9* **in about an hour, only after changing the basic Prover9 options** *order* **from 'lpo' to 'kbo' and** *eq-defs* **from 'unfold' to 'fold' and after removing those axioms of the algebra containing** 0, 1. **The proof by** *Prover9* **had the length 54 (i.e. there were 54 steps); after proving the 54 steps from the chain of length 54, we have gouped the steps into the following Lemma 3.24 and Proposition 3.25.**

Lemma 3.24 *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-MEL algebra. We have:*

$$x^- \oplus (y \odot x) = y \vee_m^M x^-, \tag{54}$$

$$x \oplus (y \odot (z \odot x^-)) = (y \odot z) \vee_m^M x, \tag{55}$$

$$x^- \oplus ((y \odot x) \vee_m^M z) = y \vee_m^M (x^- \oplus z), \tag{56}$$

$$(x \odot y) \oplus (z \odot (x^- \wedge_m^M y)) = (z \odot y) \vee_m^M (x \odot y), \tag{57}$$

Proof. (54): $y \vee_m^M x^- = x^- \oplus (y \odot x)$, by definition and (DN).

(55): $x \oplus (y \odot (z \odot x^-)) \stackrel{(Pass)}{=} x \oplus ((y \odot z) \odot x^-) = (y \odot z) \vee_m^M x$.

(56): The left side: $x^- \oplus ((y \odot x) \vee_m^M z) \stackrel{(55)}{=} x^- \oplus (z \oplus (y \odot (x \odot z^-))) \stackrel{(Pass)}{=} x^- \oplus [z \oplus ((y \odot x) \odot z^-)]$. The right side: $y \vee_m^M (x^- \oplus z) = (x^- \oplus z) \oplus (y \odot (x^- \oplus z)^-) = (x^- \oplus z) \oplus (y \odot (x \odot z^-)) \stackrel{(Pass),(Sass)}{=} x^- \oplus [z \oplus ((y \odot x) \odot z^-)]$. Hence, (56) holds.

(57): $(x \odot y) \oplus (z \odot (x^- \wedge_m^M y)) = (x \odot y) \oplus (z \odot (y \odot (x \odot y)^-)) \stackrel{(55)}{=} (z \odot y) \vee_m^M (x \odot y)$, for $X := x \odot y$, $Y := z$, $Z := y$ in (55). $\Box$

Proposition 3.25 *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies (Pq).$$

Proof. The proof has 13 steps:

$$x \wedge_m^M (y \oplus x) = x. \tag{58}$$

Indeed, $x^- \vee_m^M (y \oplus x)^- = x^- \vee_m^M (y^- \odot x^-) \stackrel{(Pom)}{=} x^-$; then, $x \wedge_m^M (y \oplus x) = (x^- \vee_m^M (y \oplus x)^-)^- = x^= \stackrel{(DN)}{=} x$.

$$x \vee_m^M (y \wedge_m^M x) = x, \tag{59}$$

$$x \wedge_m^M (y \vee_m^M x) = x. \tag{60}$$

Indeed, $x \vee_m^M (y \wedge_m^M x) = x \vee_m^M (x \odot (y \oplus x^-)) \stackrel{(Pom)}{=} x$; thus, (59) holds. (60) follows by duality.

$$(x \odot y) \wedge_m^M y = x \odot y, \tag{61}$$

$$(x \oplus y) \vee_m^M y = x \oplus y. \tag{62}$$

Indeed, $(x \odot y) \wedge_m^M y \stackrel{(Pom)}{=} (x \odot y) \wedge_m^M (y \vee_m^M (x \odot y)) \stackrel{(60)}{=} x \odot y$, with $X := x \odot y$; thus, (61) holds. (62) follows by duality.

• Now, we prove

$$(x \vee_m^M y) \odot (x^- \oplus y) = y, \tag{63}$$

(64) $$(x \wedge_m^M y) \oplus (x^- \odot y) = y.$$

Indeed, $(x \vee_m^M y) \odot (x^- \oplus y) = ((x^- \oplus y)^- \oplus y) \odot (x^- \oplus y) = y \wedge_m^M (x^- \oplus y) \stackrel{(58)}{=} y$; thus, (63) holds. (64) follows by duality.

(65) $$(x \odot y^-) \oplus (z \oplus (y \wedge_m^M x)) = z \oplus x.$$

Indeed, $(x \odot y^-) \oplus (z \oplus (y \wedge_m^M x)) \stackrel{(Scomm)}{=} (z \oplus (y \wedge_m^M x)) \oplus (x \odot y^-) \stackrel{(Sass)}{=} z \oplus ((y \wedge_m^M x) \oplus (x \odot y^-)) \stackrel{(Pcomm)}{=} z \oplus ((y \wedge_m^M x) \oplus (y^- \odot x)) \stackrel{(64)}{=} z \oplus x$.

(66) $$(x \oplus y) \vee_m^M (x \oplus (z \wedge_m^M y)) = x \oplus y.$$

Indeed, by (65), we have $(y \odot z^-) \oplus (x \oplus (z \wedge_m^M y)) = x \oplus y$; put $X := y \odot z^-$, $Y := x \oplus (z \wedge_m^M y)$; hence, we have $X \oplus Y = x \oplus y$; then, $x \oplus y = X \oplus Y \stackrel{(62)}{=} (X \oplus Y) \vee_m^M Y = (x \oplus y) \vee_m^M (x \oplus (z \wedge_m^M y))$.

(67) $$(x \oplus y) \vee_m^M (x \oplus (z \odot y)) = x \oplus y.$$

Indeed, $(x \oplus y) \vee_m^M (x \oplus (z \odot y)) \stackrel{(61)}{=} (x \oplus y) \vee_m^M (x \oplus ((z \odot y) \wedge_m^M y)) \stackrel{(66)}{=} x \oplus y$, where $Z := z \odot y$ in (66).

(68) $$x \vee_m^M ((y \odot x) \oplus (z \odot (y^- \wedge_m^M x))) = x.$$

Indeed, first, by (64), we have $(y \odot x) \oplus (y^- \wedge_m^M x) = x$; put $X := y \odot x$ and $Y := y^- \wedge_m^M x$, hence we have $X \oplus Y = x$; now, $x = X \oplus Y \stackrel{(67)}{=} (X \oplus Y) \vee_m^M (X \oplus (z \odot Y)) = x \vee_m^M ((y \odot x) \oplus (z \odot (y^- \wedge_m^M x)))$.

• Now, we prove

(69) $$x \vee_m^M ((y \odot x) \vee_m^M (z \odot x)) = x.$$

Indeed, $x \vee_m^M ((y \odot x) \vee_m^M (z \odot x)) \stackrel{(57)}{=} x \vee_m^M [(z \odot x) \oplus (y \odot (z^- \wedge_m^M x))] \stackrel{(68)}{=} x$, with $Y := z$ and $Z := y$.

• Finally, we prove (Pq), i.e. $x \odot [y \vee_m^M (z \vee_m^M x^-)] = (y \odot x) \vee_m^M (z \odot x)$.

Indeed, $x \odot [y \vee_m^M (z \vee_m^M x^-)]$
$\stackrel{(54)}{=} x \odot [y \vee_m^M (x^- \oplus (z \odot x))]$
$\stackrel{(56)}{=} x \odot [x^- \oplus ((y \odot x) \vee_m^M (z \odot x))]$
$\stackrel{(69)}{=} (x \vee_m^M [(y \odot x) \vee_m^M (z \odot x)]) \odot (x^- \oplus [(y \odot x) \vee_m^M (z \odot x)])$
$\stackrel{(63)}{=} (y \odot x) \vee_m^M (z \odot x) \stackrel{(Pcomm)}{=} (x \odot y) \vee_m^M (x \odot z)$. □

By Propositions 3.23 and 3.25, we obtain **the second very important result, the core of this paper**, by its difficulty:

Theorem 3.26 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \iff (Pom).$$

Consequently, by Theorems 3.19 and 3.26, we obtain:

Theorem 3.27 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (Pmv) + (Pom).$$

3.3 The property (Δ_m)

Consider now the property introduced in ([24] 5.2.1) (the dual one is omitted):
(Δ_m) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$.

Note that (Δ_m) is the largest non-antisymmetric generalization of ($\wedge_m$-comm). It is equivalent to: $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$.

Proposition 3.28 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmv) \implies (\Delta_m).$$

Proof. First, note that, by replacing y with y^-, (Pmv) becomes, by (DN):
(a) $x \odot ((x^- \odot y)^- \odot y)^- = x \odot y^-$.

Now, consider (Δ_m), i.e. $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$, and by interchanging x with y, we obtain:
(b) $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^- = 0$. We shall prove (b).
Indeed, $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^-$
$\overset{(Pcomm),(Pass)}{=} (x \odot ((x^- \odot y)^- \odot y)^-) \odot (x \odot y^-)^-$
$\overset{(a)}{=} (x \odot y^-) \odot (x \odot y^-)^- \overset{(m-Re)}{=} 0$. Thus, (Δ_m) holds. □

The converse of Proposition 3.28 does not hold, in general; there are examples of involutive m-BE algebras verifying (Δ_m) and not verifying (Pmv), see Example 6.3.

But, in particular, we have the following Proposition 3.31 (saying that if the involutive m-BE algebra verifies (Pom), then (Δ_m) implies (Pmv)), proved by *Prover9* in 2453 seconds, the length of the proof being 33; the proof by *Prover9* generated the proofs of the following Lemmas 3.29, 3.30 and Proposition 3.31.

Lemma 3.29 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra verifying (Pom). Then,*

$$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-, \tag{70}$$

$$(x \odot (y \odot z))^- \odot [x \odot (y \odot (x \odot (y \odot z))^-)]^- = (x \odot y)^-, \tag{71}$$

$$(x \odot y^-)^- \odot [x \odot ((y \odot z)^- \odot (x \odot y^-)^-)]^- = (x \odot (y \odot z)^-)^-. \tag{72}$$

Proof. (70): From (Pom), by (Pcomm).

(71: In (70), take $X := x \odot y$ and $Y := z$ to obtain:
$((x \odot y) \odot z)^- \odot ((x \odot y) \odot ((x \odot y) \odot z)^-)^- = (x \odot y)^-$; then, by (Pass), we obtain (71).

(72): In (71), take $X := x$, $Y := (y \odot z)^-$, $Z := (y \odot (y \odot z)^-)^-$ to obtain:
(a) $(X \odot (Y \odot Z))^- \odot [X \odot (Y \odot (X \odot (Y \odot Z))^-)]^- = (X \odot Y)^-$; but,
$X \odot (Y \odot Z) = x \odot ((y \odot z)^- \odot (y \odot (y \odot z)^-)^-) \overset{(70)}{=} x \odot y^-$; hence, (a) becomes:
$(x \odot y^-)^- \odot [x \odot ((y \odot z)^- \odot (x \odot y^-)^-)]^- = (x \odot (y \odot z)^-)^-$, that is (72). □

Lemma 3.30 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra verifying (Δ_m). Then,*

$$x \odot ((y^- \odot x)^- \odot (y \odot (x^- \odot y)^-)^-) = 0, \tag{73}$$

$$x \odot ((x \odot y^-)^- \odot (y \odot (y \odot x^-)^-)^-) = 0, \tag{74}$$

$$x^- \odot ((x^- \odot y^-)^- \odot (y \odot (y \odot x)^-)^-) = 0. \tag{75}$$

Proof. (73): Since $x \wedge_m y = y \odot (x^- \odot y)^-$, then (Δ_m) $((x \wedge_m y) \odot (y \wedge_m x)^- = 0)$ becomes:
(a) $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$; then, interchanging x with y in (a), we obtain:
(b) $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^- = 0$; then, by (Pass), we obtain (73).

(74): From (73), by (Pcomm).

(75): From (74), by taking $X := x^-$ and by (DN). □

Proposition 3.31 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) + (\Delta_m) \implies (Pmv).$$

Proof. First, we prove:

$$(x^- \odot (y \odot (y \odot x)^-)^-)^- = (x^- \odot y^-)^-. \tag{76}$$

Indeed, in (72), take $X := x^-$, $Y := y$ and $Z := (y \odot x)^-$ to obtain:
(a) $(x^- \odot y^-)^- \odot [x^- \odot ((y \odot (y \odot x)^-)^- \odot (x^- \odot y^-)^-)]^- = (x^- \odot (y \odot (y \odot x)^-)^-)^-$;
but, $x^- \odot ((y \odot (y \odot x)^-)^- \odot (x^- \odot y^-)^-) \overset{(75)}{=} 0$; hence, (a) becomes:
$(x^- \odot y^-)^- \odot [0]^- = (x^- \odot (y \odot (y \odot x)^-)^-)^-$, i.e. (76) holds, by (Neg0-1), (PU).
Next, from (76), it follows, by (DN) and (Pcomm):

$$x^- \odot ((x \odot y)^- \odot y)^- = x^- \odot y^-. \tag{77}$$

Finally, from (77), by taking $X := x^-$ and $Y := y^-$, we obtain, by (DN):
$x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$, that is (Pmv). □

Resuming, by Propositions 3.28, 3.31, we obtain **the third very important result of this paper**:

Theorem 3.32 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies ((Pmv) \Leftrightarrow (\Delta_m)).$$

Consequently, by Theorems 3.27 and 3.32, we obtain:

Theorem 3.33 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (\Delta_m) + (Pom).$$

4 Three generalizations of QMV algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra throughout this section.

4.1 The three algebras

Consider the properties:
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$ and, dually,
(Som) $(x \oplus y) \odot ((x \oplus y)^- \oplus x) = x$ or, equivalently, $x \wedge_m^M (x \oplus y) = x$;
(Pmv) $x \odot (x^- \vee_m^M y) = x \odot y$ and, dually,
(Smv) $x \oplus (x^- \wedge_m^M y) = x \oplus y$;
(Δ_m) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$ and, dually,
(∇_m) $(x \vee_m^M y) \oplus (y \vee_m^M x)^- = 1$.
We introduce the following notions:

Definitions 4.1
(i) An involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ is:
- a *left-orthomodular algebra*, or a *left-OM algebra* for short, if it verifies (Pom),
- a *left-pre-MV algebra*, or a *left-PreMV algebra* for short, if it verifies (Pmv),
- a *left-metha-MV algebra*, or a *left-MMV algebra* for short, if it verifies (Δ_m).

(i') Dually, an involutive right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ is:
- a *right-orthomodular algebra*, or a *right-OM algebra* for short, if it verifies (Som),
- a *right-pre-MV algebra*, or a *right-PreMV algebra* for short, if it verifies (Smv),
- a *right-metha-MV algebra*, or a *right-MMV algebra* for short, if it verifies (∇_m).

We shall denote by **OM**, **PreMV**, **MMV** the classes of the corresponding left-algebras and by $\mathbf{OM}^R$, $\mathbf{PreMV}^R$, $\mathbf{MMV}^R$ the classes of the corresponding right-algebras. See Examples 6.1, 6.2, 6.3 of left-OM, left-PreMV, left-MMV algebras, respectively, and Example 6.4 of left-QMV algebra.

By Propositions 3.20, 3.13, 3.28 and Theorems 3.27, 3.33, we obtain:

Corollary 4.2 *We have:*
QMV $\subset$ **OM**, **QMV** $\subset$ **PreMV** $\subset$ **MMV**
and
QMV $=$ **PreMV** $\cap$ **OM** $=$ **MMV** $\cap$ **OM**.

Note that we can say that QMV algebras are *orthomodular PreMV algebras*, or *orthomodular MMV algebras.*

Hence, we have the situation from the Figure 2.

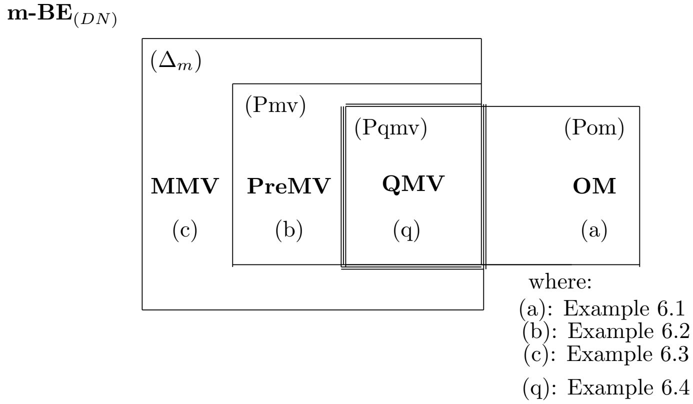


Figure 2: Resuming connections between **OM**, **PreMV**, **MMV** and **QMV**

4.2 The transitive and/or antisymmetric algebras

We shall denote by **tOM**, **tPreMV**, **tMMV**, **tQMV** the classes of the corresponding transitive left-algebras. Note that these classes of algebras are contained in the class **m-pre-BCK**$_{(DN)}$ = **m-tBE**$_{(DN)}$. See Examples 6.5, 6.6, 6.7, 6.8 of left-tOM, left–tPreMV, left-MMV, left-tQMV algebras, respectively.

By the previous Corollary 4.2, we obtain:

Corollary 4.3 *We have:*

$$\begin{array}{ccccccccc} \mathbf{QMV} & \subset & \mathbf{OM} & \quad & \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tOM}, & & \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \end{array}$$

and

tQMV $=$ **tPreMV** $\cap$ **tOM** $=$ **tMMV** $\cap$ **tOM**.

Hence, we have the situation from the Figure 3.

We shall denote by **aOM**, **aPreMV**, **aMMV**, **aQMV** the classes of the corresponding antisymmetric left-algebras. Note that these classes of algebras are contained in the class **m-aBE**$_{(DN)}$.

By Corollary 4.2 again, we obtain the analogous of Corollary 4.3, which by lack of space is omitted.

We shall denote by **taOM**, **taPreMV**, **taMMV**, **taQMV** the classes of the corresponding transitive and antisymmetric left-algebras. Note that these classes of algebras are contained in the class **m-BCK** = **m-taBE**$_{(DN)}$.

By Corollary 4.3 and its analogous, we then obtain:

Corollary 4.4 *We have:*

$$\begin{array}{ccccccccc} \mathbf{QMV} & \subset & \mathbf{OM} & \quad & \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tOM} & & \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathbf{taQMV} & \subset & \mathbf{taOM}, & & \mathbf{taQMV} & \subseteq & \mathbf{taPreMV} & \subseteq & \mathbf{taMMV} \end{array}$$

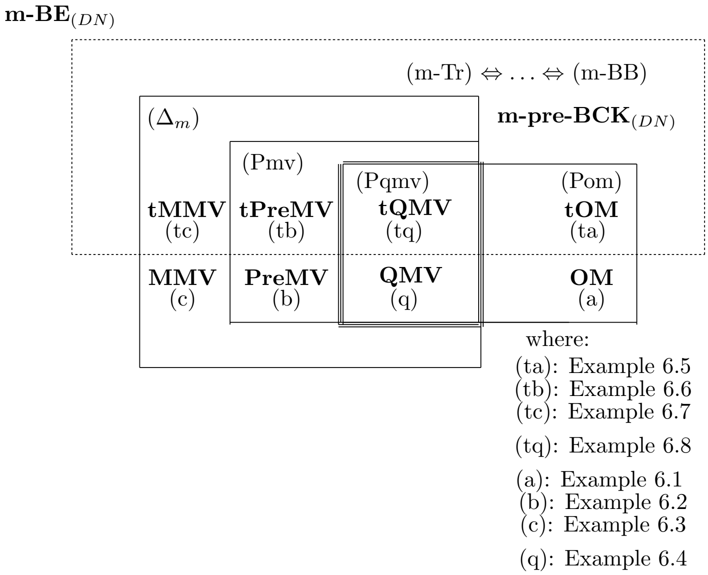


Figure 3: Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and (m-Tr)

and

$$
\begin{array}{ccc}
\mathbf{QMV} & \subset & \mathbf{OM} \\
\cup & & \cup \\
\mathbf{aQMV} & \subset & \mathbf{aOM} \\
\cup & & \cup \\
\mathbf{taQMV} & \subset & \mathbf{taOM},
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\
\cup & & \cup & & \cup \\
\mathbf{aQMV} & \subseteq & \mathbf{aPreMV} & \subseteq & \mathbf{aMMV} \\
\cup & & \cup & & \cup \\
\mathbf{taQMV} & \subseteq & \mathbf{taPreMV} & \subseteq & \mathbf{taMMV}
\end{array}
$$

and

$\mathbf{aQMV} = \mathbf{aPreMV} \cap \mathbf{aOM} = \mathbf{aMMV} \cap \mathbf{aOM}$,
$\mathbf{taQMV} = \mathbf{taPreMV} \cap \mathbf{taOM} = \mathbf{taMMV} \cap \mathbf{taOM}$.

4.3 The connections with the MV algebras

We know (see ([6], Example 2.3.14)) that any MV algebra is a QMV algebra: **MV** $\subset$ **QMV**, since:

(78) $$(\wedge_m - comm) \implies (Pqmv).$$

Consequently, we have:

(79) $$(\wedge_m - comm) \implies (Pom) + (Pmv) + (\Delta_m).$$

The next Theorems 4.6, 4.7 and 4.9 say that ($\wedge_m$-comm) is equivalent with some properties.

Proposition 4.5 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmv) + (m - An) \implies (\wedge_m - comm).$$

Proof. Suppose (m-An) holds, i.e. $X \leq_m Y$ and $Y \leq_m X$ imply $X = Y$, which mean $X \odot Y^- = 0$ and $Y \odot X^- = 0$ imply $X = Y$.

Take $X \overset{notation}{=} x \wedge_m^M y \overset{(4)}{=} (x^- \odot y)^- \odot y$ and $Y \overset{notation}{=} y \wedge_m^M x \overset{(4)}{=} (y^- \odot x)^- \odot x$.

We have: $X \odot Y^- = [(x^- \odot y)^- \odot y] \odot [y \wedge_m^M x]^-$
$\stackrel{(11)}{=} [(x^- \odot y)^- \odot y] \odot [y^- \vee_m^M x^-] \stackrel{(Pass)}{=} (x^- \odot y)^- \odot (y \odot (y^- \vee_m^M x^-))$
$\stackrel{(Pmv)}{=} (x^- \odot y)^- \odot (y \odot x^-) \stackrel{(Pcomm)}{=} (y \odot x^-) \odot (y \odot x^-)^- \stackrel{(m-Re)}{=} 0.$

Similarly, we have: $Y \odot X^- = (y^- \odot x)^- \odot x \odot [x \wedge_m^M y]^-$
$(y^- \odot x)^- \odot x \odot (x^- \vee_m^M y^-) \stackrel{(Pmv)}{=} (y^- \odot x)^- \odot (x \odot y^-) = 0.$

By (m-An), we obtain $X = Y$, i.e. ($\wedge_m$-comm) holds. □

By Proposition 4.5 and (3), (79), we obtain:

Theorem 4.6
$$(Pmv) \ + \ (m - An) \iff (\wedge_m - comm).$$

Recall again ([24] 5.2.1) saying that:

Theorem 4.7
$$(\Delta_m) \ + \ (m - An) \iff (\wedge_m - comm).$$

Proposition 4.8
$$(Pqmv) \ + \ (m - An) \implies (\wedge_m - comm).$$

Proof. By Propositions 3.13, 4.5, we obtain:
(Pqmv) + (m-An) $\implies$ (Pmv) + (m-An) $\implies$ ($\wedge_m$-comm). □

By Proposition 4.8 and by (3), (78), we obtain:

Theorem 4.9
$$(Pqmv) \ + \ (m - An) \iff (\wedge_m - comm).$$

By previous Theorems 4.6, 4.7, 4.9, we obtain **the fourth very important result of this paper**:

Corollary 4.10 *We have:*

PreMV	+	*(m-An)*	=	**MV**,	*i.e.*	**aPreMV** = **MV**,
MMV	+	*(m-An)*	=	**MV**,	*i.e.*	**aMMV** = **MV**,
QMV	+	*(m-An)*	=	**MV**,	*i.e.*	**aQMV** = **MV**.

Remark 4.11 *By (3), we have:*
MV = **aMV** = **tMV** = **taMV**, *hence* **taPreMV** = **taMMV** = **taQMV** = **MV**.

By Corollaries 4.4, 4.10 and by Remark 4.11, we obtain:

Corollary 4.12 *We have:*

$$\begin{array}{ccc} \mathbf{QMV} & \subset & \mathbf{OM} \\ \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tOM} \\ \cup & & \cup \\ \mathbf{MV} & \subset & \mathbf{taOM}, \end{array} \qquad \begin{array}{ccccc} \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \\ \cup & & \cup & & \cup \\ \mathbf{MV} & \subseteq & \mathbf{MV} & \subseteq & \mathbf{MV} \end{array}$$

and

$$\begin{array}{ccc} \mathbf{QMV} & \subset & \mathbf{OM} \\ \cup & & \cup \\ \mathbf{MV} & \subset & \mathbf{aOM} \\ \cup & & \cup \\ \mathbf{MV} & \subset & \mathbf{taOM} \end{array} \qquad \begin{array}{ccccc} \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup \\ \mathbf{MV} & \subseteq & \mathbf{MV} & \subseteq & \mathbf{MV} \end{array}$$

and

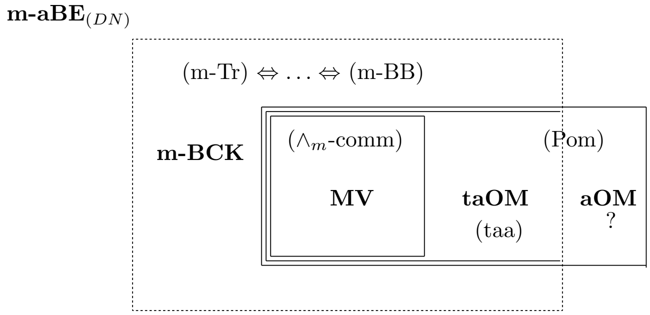
MV = **MV** ∩ **aOM**,
MV = **MV** ∩ **taOM**.

Note that *taOM algebras* are proper generalizations of MV algebras inside the class of m-BCK algebras. See Example 6.9 of left-taOM algebra.

A problem we have not been able to resolve is the following.

Open problem 4.13 Find an example of antisymmetric orthomodular algebra (aOM) which does not verify (m-Tr) ($\Longleftrightarrow \dots$ (m-BB)), i.e. a proper element of **aOM** (using Mace4, we have searched exhaustively for an example up through and including size 20), or prove that an involutive left-m-aBE algebra satisfying (Pom) satisfies also (m-Tr) (i.e. **aOM** = **taOM**) (we have also tried to find a proof using Prover9, but despite letting it run for several days, it was unable to find one).

Hence, we have the situation from the Figure 4.



where:
(taa): Example 6.9

Figure 4: Resuming connections between **MV**, **taOM** and **aOM**, where ? means that there is an open problem concerning **aOM**

Note that, by Theorems 4.6, 4.7, 4.9 again, we obtain:

Theorem 4.14 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-aBE algebra. Then,*

$$(\wedge_m - comm) \Longleftrightarrow (Pmv) \Longleftrightarrow (\Delta_m) \Longleftrightarrow (Pqmv).$$

Remarks 4.15 *(See Remarks 3.22) In a left-MV algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:*
- the initial binary relation, $\leq_m$ ($\Longleftrightarrow \leq_m^B$), is a **lattice order relation** *w.r.t. to $\wedge_m^B = \wedge_m^M, \vee_m^B = \vee_m^M$, since (m-Re), (m-An), (m-Tr) hold and since $\wedge_m^B$ is commutative,*
- the binary relation $\leq_m^M$ is a **lattice order relation** *w.r.t. to $\wedge_m^M, \vee_m^M$, by Corollary 3.18 and since $\wedge_m^M$ is commutative,*
- $\leq_m$ and $\leq_m^M$ are equivalent: $\leq_m$ ($\Longleftrightarrow \leq_m^B$) $\Longleftrightarrow \leq_m^M$, by Proposition 3.3; the lattice is distributive;
- the binary relation $\leq_m^P$ is **only antisymmetric and transitive**, *by ([23], Proposition 3.11).*

In a left-MV algebra verifying (G) ($x \odot x = x$), i.e. in a left-Boolean algebra,

$$\leq_m \ (\Longleftrightarrow \leq_m^B) \Longleftrightarrow \leq_m^M \Longleftrightarrow \leq_m^P .$$

4.4 Putting QMV and tQMV algebras on the "map"

By the previous results, we are now able to put QMV algebras and tQMV algebras (and MV algebras) on the involutive "Big map" (and, hence, on the "map") - see the Figure 5.

5 Concluding remarks and future work

In this paper, we have dug arround the structure of QMV algebras and we have obtained a decomposition of (Pqmv) into only two properties: (Pmv) and (Pq), at the begining, (Δ_m) and (Pom), at the end, where (Δ_m) is the largest non-antisymmetric generalization of ($\wedge_m$-comm) and (Pom) is the property

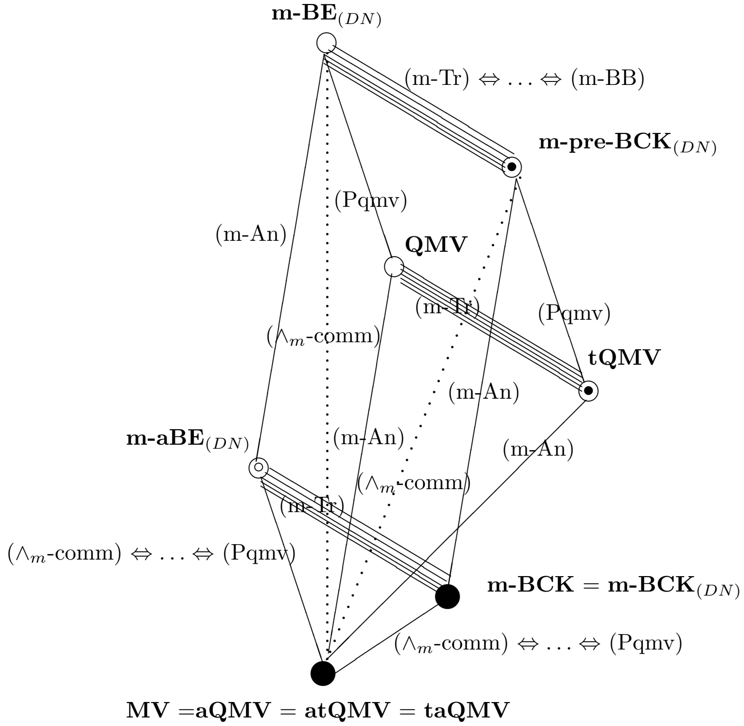


Figure 5: Putting QMV and tQMV algebras on the "map"

characterizing the orthomodular lattices among the ortholattices (Definitions 2). We have thus introduced three generalizations of the QMV algebras: two new non-antisymmetric generalizations of MV algebras, the *pre-MV (PreMV) algebras* and the *metha-MV (MMV) algebras*, and the *orthomodular (OM) algebras*. The QMV algebra is then just an *orthomodular PreMV algebra* or an *orthomodular MMV algebra*; in other words, the QMV algebra is that non-antisymmetric generalization of MV algebra that is an orthomodular algebra. We have also introduced and studied the *transitive QMV (tQMV) algebras*, the *transitive PreMV (tPreMV) algebras*, the *transitive MMV (tMMV) algebras* and the *transitive OM (tOM) algebras*. It was known that any MV algebra is a QMV algebra, but the exact connection between MV and QMV algebras it was not known. We have clarified this problem, by proving that MV algebras coincide with the antisymmetric QMV (aQMV) algebras - but also with the antisymmetric preMV (aPreMV) and with the antisymmetric MMV (aMMV) algebras. Consequently, MV algebras and QMV algebras, and also tQMV algebras, were put on the same "map" (involutive "Big map"). The *taOM algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras, is put in evidence.

By putting QMV (and tQMV) algebras on the "map", we have proved again (see [19], [23]) the deep connections existing between the algebraic structures connected to the classical and non-classical logics and the algebraic structures connected to the quantum logics: they exist on the same "map", but at different levels (parallels), i.e. the QMV (and tQMV) algebras also belong to the "world" of left-algebras (involutive left-unital magmas).

The 'story' of the algebras involved in this paper is connected to the 'story' of the three/four binary relations that can be defined in such algebras:

$x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$, with $x \leq_m y \iff x \leq_m^B y$, $\quad x \leq_m^B y \overset{def}{\Longleftrightarrow} x \wedge_m^B y = x$,

$x \leq_m^M y \overset{def}{\Longleftrightarrow} x \wedge_m^M y = x$, with $x \wedge_m^M y = y \wedge_m^B x$, and

$x \leq_m^P y \overset{def}{\Longleftrightarrow} x \odot y = x$.

Note that the central role is played by the binary relation $\leq_m$, that determines the "parallels" and the "meridians" of the "map".

By the inverse maps Φ $(x \odot y \overset{def.}{=} (x \to y^-)^-)$ and Ψ $(x \to y \overset{def.}{=} (x \odot y^-)^-)$ ([19], Theorem 9.1) that connect the "world" of *algebras of logic* of the form $(A, \to, ^-, 1)$ and the "world" of *algebras* of the form $(A, \odot, ^-, 1)$, in the involutive case, one can obtain simply, by choosing the appropriate definitions of the algebras, the definitionally equivalent *involutive algebras of logic* corresponding to the *involutive algebras* from this paper and the corresponding examples and results. Note that, in ([19], Definition 3.29), the *implicative-ortholattices* were already introduced as involutive *BE algebras* verifying (impl) $((x \to y) \to x = x)$ and their d.e. with the ortholattices was proved in Theorem 9.2. Similarly, one can introduce now the *quantum-Wajsberg algebras*, as algebras of logic (involutive *BE algebras* verifying, say, (qw)) d. e. with the quantum-MV algebras, etc. Note that the *BE algebras* were introduced in 2006 by H.S. Kim and Y.H. Kim [25] and are intensively studied.

This research is continued by the first author: in [20], we clarify some more aspects concerning the QMV algebras as non-lattice generalizations of MV algebras by studying more deeply the OM algebras; we prove that all the properties of QMV algebras, excepting (43) and (44), are verified by the OM algebras; we study in some details the taOM algebras. In [21], we clarify some aspects concerning the QMV algebras as non-idempotent generalizations of orthomodular lattices; we introduce and study two generalizations of orthomodular lattices, the orthomodular softlattices and the orthomodular widelattices, following [24]. Finally, in [22], we study the properties (m-Pabs-i) and (WNM_m), introduced in [24], in MV algebras and in tQMV algebras.

6 Examples

We introduce the following definition: an X algebra is said to be *proper*, if it verifies the properties from its definition and does not verify the other properties from this paper, except (prel_m).

Example 6.1 Proper orthomodular algebra: OM

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom) and (prel_m) and not verifying (m-B) for (a, b, a), (m-BB) for (a, a, b), (m-*) for (b, d, a), (m-**) for (a, b, a), (m-Tr) for (a, b, d), (m-An) for (a, b), (Pqmv) for $(d, d, 0)$, (Pmv) for (d, d), (Δ_m) for (a, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	1	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.2 Proper PreMV algebra: PreMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and (prel_m) and not verifying (m-B) for (a, d, c), (m-BB) for (a, c, a), (m-*) for (a, d, c), (m-**) for (a, d, c), (m-Tr) for (a, d, b), (m-An) for (a, c), (Pqmv) for $(a, 1, c)$, (Pom) for (a, c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	b	0	a
b	0	0	0	0	0	b
c	0	b	0	b	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	c	1	c	1
c	c	1	1	1	1	1
d	d	1	c	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.3 Proper MMV algebra: MMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and not verifying (m-B) for (a, b, a), (m-BB) for (a, a, b), (m-*) for (a, b, c), (m-**) for (a, b, a), (m-Tr) for (a, b, d), (m-An) for (a, b), (Pqmv) for $(d, 0, d)$, (Pom) for (d, d), (Pmv) for (d, d), (prel_m) for (a, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	b	0	c
d	0	0	0	0	b	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	1	1	1	1
b	b	1	c	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.4 Proper QMV algebra: QMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper left-QMV algebra, i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (Pqmv) (hence (Pom), (Pmv), (Δ_m), (prel_m)), (DN) hold and it does not verify (m-B) for (a, d, c), (m-BB) for (a, c, a), (m-*) for (a, d, c), (m-**) for (a, d, c), (m-Tr) for (a, d, b), (m-An) for (a, c), $(\wedge_m\text{-comm})$ for (a, c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	b	0	a
b	0	0	0	0	0	b
c	0	b	0	d	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	a	1	c	1
c	c	1	1	1	1	1
d	d	1	c	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.18, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.5 Proper transitive OM algebra : tOM

By MACE4 program, we found that the algebra $\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), and also (prel_m), and not verifying (m-An) for (c, d), (Pqmv) for $(b, b, 0)$, (Pmv) for (b, b), (Δ_m) for (a, b).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	c	d	c	d	a
b	0	0	0	0	0	0	0	b
c	0	c	0	0	0	0	0	c
d	0	d	0	0	0	0	0	d
e	0	c	0	0	0	b	b	e
f	0	d	0	0	0	b	b	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	e
d	f
e	c
f	d
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	1	1	1	1	1	1	1
b	b	1	b	e	f	e	f	1
c	c	1	e	a	a	1	1	1
d	d	1	f	a	a	1	1	1
e	e	1	e	1	1	1	1	1
f	f	1	f	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.6 Proper transitive pre-MV algebra: tPreMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), (prel_m) and not verifying (m-An) for (b, c), (Pqmv) for $(b, 1, d)$, (Pom) for (b, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	0	a	c
d	0	0	a	a	b	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	d	1	1
b	b	d	1	1	1	1
c	c	d	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.7 Proper transitive MMV algebra : tMMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), and also (prel$_m$), and not verifying (m-An) for (a, b), (Pqmv) for $(b, 0, a)$, (Pom) for (b, a), (Pmv) for (b, a).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	0	0	a
b	0	a	a	0	0	b
c	0	0	0	c	c	c
d	0	0	0	c	d	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	b	1	1	1
b	b	b	b	1	1	1
c	c	1	1	d	d	1
d	d	1	1	d	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.8 Proper transitive QMV algebra: tQMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper left-tQMV algebra, i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (Pqmv) (hence (Pom), (Pmv), (Δ_m), (prel$_m$)), (DN), (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB) hold and it does not verify (m-An) for (a, b).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	a	b	c
d	0	0	0	b	a	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	1	1	1
b	b	c	d	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.18, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.9 Transitive, antisymmetric OM algebra: taOM

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_4 = \{0, a, b, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a transitive, antisymmetric left-orthomodular algebra (= m-BCK algebra verifying (Pom)), i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An), (DN), (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), (Pom), but also (prel$_m$) hold and it does not verify ($\wedge_m$-comm) for (a, b), (Pqmv) for $(a, a, 0)$, (Pmv) for (a, a), (Δ_m) for (b, a).

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

and

x	x^-
0	1
a	b
b	a
1	0

, with

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	1	1
1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Note that this algebra is the NM (Nilpotent Minimum) algebra $\mathcal{F}_4$ from [15]; it will be reviewed in [20].

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ON IDEALS AND (θ, θ) HIGHER LEFT DERIVATIONS

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Abstract. Let R be a 2-torsion free prime ring and I be a non-zero ideal of R. Let θ be an automorphism of R. We will show that the existence of a non-zero Jordan $(\theta,\ \theta)$higher left derivation on an ideal of R implies that $I \subseteq Z(R)$.

1. INTRODUCTION

In this paper, R denotes to a prime ring and we write $[x,\ y]$ for the commutator $xy - yx$. Recall that R is prime if $xRy = \{0\}$ implies that $x = 0$ or $y = 0$. Higher derivations as a generalisation of derivations have been studied (see [3], [4]). The aim of this paper is to introduce Jordan $(\theta,\ \theta)$ higher left derivations on an ideal of R to improve the result of Bresar and Vukman that the existence of a non-zero Jordan left derivation on a 2-torsion free and 3-torsion free left R-module M, implies that R is commutative.

2. PRELIMINARIES

In this section we will introduce some definitions that are necessary for the main result of this paper.

Definition 2 .1. An additive mapping $D : I \longrightarrow I$ is said to be a $(\theta,\ \theta)$ derivation on an ideal I of R (resp. a Jordan $(\theta,\ \theta)$ derivation) on I if $D(xy) = \theta(x)D(y) + D(y)\theta(x)$ (resp. $D(x^2) = \theta(x)D(x) + D(x)\theta(x)$) for all $x,\ y$ in I.

Definition 2 .2. A $(\theta,\ \theta)$ left derivation on an ideal I of R (r esp. a Jordan $(\theta,\ \theta)$ left derivation on I) is an additive mapping $D : I \longrightarrow I$ satisfying $D(xy) = \theta(x)D(y) + \theta(y)D(x)$ (resp. $D(x^2) = 2\theta(x)D(x)$) for all $x,\ y$ in I.

In the following definitions we let $\mathbb{N}_0$ be the set of all non-negative integers and $c_{ij} \in \{0, 1\}$ such that $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq 1$.

Definition 2 .3. Let $D = \{D_1, D_2, ..., D_k\}$ be a sequence of additive mappings on an ideal I of R. D is said to be a $(\theta,\ \theta)$ higher left derivation on I of rank k if for each $n \in \mathbb{N}$ and $i,\ j \in \mathbb{N}_0$, $D_n(xy) = \sum_{\substack{i+j=n \\ i\leqslant j}} [\theta(D_i(x))D_j(y) + c_{ij}\theta(D_i(x))D_j(y)]$ for all $x,\ y \in I$, where $D_0(x) = 0$, for all $x \in I$.

Definition 2.4. Let $D = \{D_1,\ D_2, ...,\ D_k\}$ be a sequence of additive mappings on an ideal I of R. D is said to be a Jordan $(\theta,\ \theta)$ higher left derivation on I of rank k if for each $n \in \mathbb{N}$ and $i,\ j \in \mathbb{N}_0$, $D_n(x^2) = \sum_{\substack{i+j=n \\ i\leqslant j}} [(c_{ij}+1)\theta(D_i(x))D_j(x)]$ for all $x \in I$, where $D_0(x) = 0$, for all $x \in I$.

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3. MAIN RESULT

In order to prove the main result of this paper, we begin with the following lemma:

Lemma 3.1. Let R be a 2-torsion free ring and I be an ideal of R. Suppose that θ is an endomorphism of R. If D is a Jordan $(\theta,\ \theta)$ left derivation on I then

(1) $D(xy+yx) = 2\theta(x)D(y) + 2\theta D(x)$ for all $x,\ y \in I$.

(2) $\theta(x),\ \theta(y)]\theta(x)D(x) = \theta(x)[\theta(x),\ \theta(y)]D(x)$ for all $x,\ y \in I$.

(3) $D(xyx) = \theta(x^2)D(y) + 3\theta(x)\theta(y)D(x) - \theta(y)\theta(x)D(x)$ for all $x,\ y \in I$.

(4) $[\theta(x),\ \theta(y)](D(xy) - \theta(x)D(y) - \theta(y)D(x))$ for all $x,\ y \in I$.

Proof (1). Since D is a Jordan $(\theta,\ \theta)$ left derivation, we have $D(x^2) = 2\theta(x)D(x)$ for all $x,\ y \in I$. Replacing x by $x+y$, we get

$$D(x^2+xy+yx+y^2) = 2\theta(x+y)D(x+y).$$

Then

$$D(xy+yx) = 2\theta(x)D(y) + 2\theta(y)D(x),$$

for all $x,\ y \in I$.

(2). From (1), we have

$$D(xy+yx) = 2\theta(x)D(y) + 2\theta(y)D(x), \tag{1}$$

for all $x,\ y \in I$. Replacing y by $xy+yx$ in (1), we get

$$D(x(xy+yx)+(xy+yx)x) = 4\theta(x^2)D(y) + 6\theta(x)\theta(y)D(x) + 2\theta(y)\theta(x)D(x). \tag{2}$$

On the other hand,

$$\begin{aligned} D(x(xy+yx)+(xy+yx)x) &= D(x^2y+yx^2) + 2D(xyx) \\ &= 2\theta(x^2)D(y) + 4\theta(y)\theta(x)D(x) + 2D(xyx). \end{aligned} \tag{3}$$

Combining (2) with (3), we get

$$D(xyx) = \theta(x^2)D(y) + 3\theta(x)\theta(y)D(x) - \theta(y)\theta(x)D(x),$$

for all $x,\ y \in I$.

(3) From (2), we have

$$\begin{aligned} D((x+z)y(x+z)) = {} & \theta(x^2)D(y) + \theta(z^2)D(y) + (\theta(x)\theta(z) + \theta(z)\theta(x))D(y) + \\ & 3\theta(x)\theta(y)D(z) + 3\theta(x)\theta(y)D(x) + 3\theta(z)\theta(y)D(z) + 3\theta(z)\theta(y)D(x) \\ & -\theta(y)\theta(x)D(x) - \theta(y)\theta(x)D(z) - \theta(y)\theta(z)D(x) - \theta(y)\theta(z)D(z) \end{aligned} \tag{4}$$

for all $x,\ y,\ z \in I$. On the other hand,

$$\begin{aligned} D((x+z)y(x+z)) = D(xyx) + D(zyx) + D(xyz+zyx) = {} & \theta(x^2)D(y) \\ & +3\theta(x)\theta(y)D(x) - \theta(y)\theta(x)D(x) + \theta(z^2)D(y) \\ & +3\theta(z)\theta(y)D(z) - \theta(y)\theta(z)D(z) + D(xyz+zyx). \end{aligned} \tag{5}$$

Combining (1) and (2), we obtain

$$D(xyz+zyx) = (\theta(x)\theta(z)+\theta(z)\theta(x))D(y))D(y)+3\theta(x)\theta(y)D(z) + 3\theta(z)\theta(y)D(x)-\theta(y)\theta(x)D(z)-\theta(y)\theta(z)D(x) \tag{6}$$

for all $x,\ y,\ z\in I$. Since $D(x^2)=2\theta(x)D(x)$, we find that

$$D((xy)^2)=2\theta(xy)D(xy)=2\theta(x)\theta(y)D(xy).$$

Replacing z by xy in (3), we get

$$D(xy(xy)+(xy)yx) = (\theta(x^2)\theta(y)+\theta(x)\theta(y)\theta(x))D(y)+3\theta(x)\theta(y)D(xy) + 3\theta(x)\theta(y^2)D(x)-\theta(y)\theta(x)D(xy)-\theta(y)\theta(x)\theta(y)D(x). \tag{7}$$

On the other hand,

$$D((xy)^2+xy^2x) = 2\theta(x)\theta(y)D(xy)+2\theta(x^2)\theta(y)D(y) + 3\theta(x)\theta(y^2)D(x)-\theta(y^2)\theta(x)D(x). \tag{8}$$

Combining (4) and (5), we get

$$[\theta(x),\ \theta(y)]D(xy)=\theta(x)[\theta(x),\ \theta(y)]D(y)+\theta(y)[\theta(x),\ \theta(y)]D(x). \tag{9}$$

Replacing y by $x+y$ in (6), we have

$$2[\theta(x),\ \theta(y)]\theta(x)D(x)+[\theta(x),\ \theta(x),\ \theta(y)]D(xy)= 2\theta(x)[\theta(x),\ \theta(y)]D(x)+\theta(x)[\theta(x),\ \theta(y)]D(y)+\theta(y)[\theta(x),\ \theta(y)]D(x).$$

By using (6), we get the result.

(4). From (3), we have

$$[\theta(x),\theta(y)]\theta(x)D(x)=\theta(x)[\theta(x),\theta(y)]D(x), \tag{10}$$

for all $x,\ y\in I$. Replacing x by $x+y$ in (1), we get

$$\begin{aligned}
&[\theta(x),\ \theta(y)]\theta(x)D(y)+[\theta(x),\ \theta(y)]\theta(y)D(y)\\
&+[\theta(x),\ \theta(y)]\theta(x)D(y)+[\theta(x),\ \theta(y)]\theta(y)D(x)\\
&=\theta(x)[\theta(x),\ \theta(y)]D(x)+\theta(x)[\theta(x),\theta(y)]D(y)\\
&+\theta(y)[\theta(x),\ \theta(y)]D(x)+\theta(y)[\theta(x),\ \theta(y)]D(y).
\end{aligned}$$

Now application of (6) and (iii) yields that

$$[\theta(x),\ \theta(y)]\theta(x)D(y)+[\theta(x),\ \theta(y)]\theta(y)D(x)=[\theta(x),\ \theta(y)]D(xy).$$

Then

$$[\theta(x),\ \theta(y)](D(xy)-\theta(x)D(y)-\theta(y)D(x))=0,$$

for all $x,\ y\in I$.

Theorem 3.2. Let R be a 2-torsion free prime ring and I be a non-zero ideal of R. Let θ be an automorphism of R and $D=(D_n)_{n\in\mathbb{N}}$ be a Jordan higher $(\theta,\ \theta)$ left derivation on I. Then $D=0$ or $I\subseteq Z(R)$. Moreover, if $I=R$, then $D=0$ or R is commutative.

Proof. We prove by induction. Let $n=1$ and D_1 be a Jordan $(\theta,\ \theta)$ left derivation on I. Assume that $I\nsubseteq Z(R)$. From the above lemma (iii), we have

$$(\theta(x)[\theta(x),\ \theta(y)]-[\theta(x),\ \theta(y)]\theta(x)])D_1(x)=0,$$

for all $x,\ y \in I$. Replacing x by $[z,\ w] \in I$. Then

$$\begin{aligned} &[\theta(z),\ \theta(w)]^2\theta(y)D_1([\theta(z),\ \theta(w)]) \\ &-2[\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]D_1([\theta(z),\ \theta(w)]) \\ &+\theta(y)[\theta(z),\ \theta(w)]^2D_1([\theta(z),\ \theta(w)]) = 0. \end{aligned} \tag{11}$$

By using the above lemma (iv), we have

$$[\theta(z),\ \theta(w)](D_1(zw) - \theta(z)D_1(w) - \theta(w)D_1(z)) = 0, \tag{12}$$

for all $z,\ w \in I$. Also, we have

$$[\theta(w),\ \theta(z)](D_1(wz) - \theta(w)D_1(z) - \theta(z)D_1(w)) = 0, \tag{13}$$

for all $z,\ w \in I$. From (12) and (13), we get

$$[\theta(z),\ \theta(w)]D_1([z,\ w]) = 0, \tag{14}$$

for all $z,\ w \in I$. By using (11) and (14), we have

$$[\theta(z),\ \theta(w)]^2\theta(y)D_1([z,\ w]) = 0$$

for all $y,\ z,\ w \in I$. This implies that

$$\theta^{-1}([\theta(z),\ \theta(w)]^2)y\theta^{-1}(D_1([z,\ w])) = 0$$

for all $y,\ w,\ z \in I$. Replacing y by $yr \in I,\ r \in R$. Then

$$\theta^{-1}([\theta(z),\ \theta(w)]^2)yR\theta^{-1}(D_1([z,\ w])) = \{0\}$$

for all $y,\ w,\ z \in I$. This gives $[\theta(z),\ \theta(w)]^2\theta(y) = 0$ or $D_1([z,\ w]) = 0$. Suppose that $[\theta(z),\ \theta(w)]^2$ $\theta(y) = 0$. Replacing y by $ry \in I$, we have

$$[\theta(z),\ \theta(w)]^2\theta(r)\theta(y) = 0$$

for all $y,\ z,\ w \in I$ and $r \in R$. This implies that $\theta^{-1}([\theta(z),\ \theta(w)]^2)Ry = \{0\}$. Since $I \neq 0$, we get $[\theta(z),\ \theta(w)]^2 = 0$. By using the lemma and (14), we have

$$\begin{aligned} &D_1(([z,\ w]x)([z,\ w]y[z,\ w]) + ([z,\ w]y[z,\ w])([z,\ w]x)) \\ &= 6([\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]\theta(y))D_1([z,\ w]) \\ &+2([\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)])D_1([z,\ w]x), \end{aligned} \tag{15}$$

for all $x,\ y,\ z,\ w \in I$. On the other hand,

$$\begin{aligned} &D_1(([z,\ w]x)([z,\ w]y[z,\ w]) + ([z,\ w]y[z,\ w])([z,\ w]x)) \\ &= D_1([z,\ w](x[z,\ w]y)[z,\ w]) \\ &= 3[\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]D_1([z,\ w], \end{aligned} \tag{16}$$

for all $x,\ y,\ z,\ w \in I$. Compare (15) and (16), we get

$$\begin{aligned} &3([\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]\theta(y))D_1([z,\ w]) + \\ &2([\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)])D_1([z,\ w]x) = 0, \end{aligned}$$

for all $x,\ y,\ z,\ w \in I$. From the lemma, we have

$$\begin{aligned} &D_1(([z,\ w]x)([z,\ w]y[z,\ w]) + ([z,\ w]y[z,\ w])([z,\ w]x)) \\ &= 2[\theta(z),\ \theta(w)]D_1(x[z,\ w]). \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} &D_1(([z,\ w]x)([z,\ w]y[z,\ w]) + ([z,\ w]y[z,\ w])([z,\ w]x)) \\ &= D_1([z,\ w]x[z,\ w]) = 3[\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]). \end{aligned} \tag{18}$$

Combining (17) with (18), we get

$$3[\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w] = 2[\theta(z),\ \theta(w)]D_1(x[z,\ w]), \tag{19}$$

for all $x,\ z,\ w \in I$. By using the lemma, we have

$$[\theta(z),\ \theta(w)]D_1(x[z,\ w] + [z,\ w]x) = 2[\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]). \tag{20}$$

From (19) and (20), we have

$$\begin{aligned} 3[\theta(z),\ \theta(w)]D_1(x[z,\ w]) + 3[\theta(z),\ \theta(w)]D_1([z,\ w]x) \\ = 4[\theta(z),\ \theta(w)]D_1(x[z,\ w]). \end{aligned}$$

This implies that

$$[\theta(z),\ \theta(w)]D_1(x[z,\ w]) = 3[\theta(z),\ \theta(w)]D_1([z,\ w]x). \tag{21}$$

By applying the lemma, we get

$$[\theta(z),\ \theta(w)]D_1(x[z,\ w] + [z,\ w]x) = 2[\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]). \tag{22}$$

From (20), we have

$$[\theta(z),\ \theta(w)]D_1(x[z,\ w] + [z,\ w]x) = 4[\theta(z),\ \theta(w)]D_1([z,\ w]x). \tag{23}$$

From (21) and (22), we get

$$2(2[\theta(z),\ \theta(w)]D_1([z,\ w]x) - [\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]))$$

Since R is 2-torsion free, we have

$$2[\theta(z),\ \theta(w)]D_1([z,\ w]x) = [\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]). \tag{24}$$

Replacing x by $y[z,\ w]y$ in (23), we get

$$2[\theta(z),\ \theta(w)]D_1(([z,\ w]y)^2) = [\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]).$$

This implies that

$$4[\theta(z),\ \theta(w)]^2\theta(y)D_1([z,\ w]) = [\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w])$$

and so

$$[\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]) = 0. \tag{25}$$

Replacing y $x+y$ in (24), we get

$$\begin{aligned} [\theta(z),\ \theta(w)][\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]) \\ +[\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]\theta(x)D_1([z,\ w]) = 0. \end{aligned} \tag{26}$$

From (15) and (23), we obtain

$$\begin{aligned} 3[\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]) \\ +[\theta(z),\ \theta(w)]\theta(y)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]) = 0. \end{aligned} \tag{27}$$

From (25) and (26), we have

$$[\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]\theta(y)D_1([z,\ w]) = 0,$$

for all $x,\ y,\ z,\ w \in I$. Replacing y by yr, we get

$$[\theta(z),\ \theta(w)]\theta(x)[\theta(z),\ \theta(w)]\theta(y)\theta(r)D_1([z,\ w]) = 0.$$

This implies that

$$\theta^{-1}([\theta(z),\ \theta(w)])x\theta^{-1}([\theta(z),\ \theta(w)])yR\theta^{-1}(D_1([z,\ w])) = 0.$$

This gives $\theta^{-1}([\theta(z),\ \theta(w)])x\theta^{-1}([\theta(z),\ \theta(w)])y = 0$ or $\theta^{-1}(D_1([z,\ w])) = 0$. Since $I \neq 0$, we have $[\theta(z),\ \theta(w)] = 0$ or $D_1([z,\ w]) = 0$. This implies that $[z,\ w] = 0$ or $D_1([z,\ w]) = 0$. Suppose that $[z,\ w] = 0$ for all $z,\ w \in I$. Replacing w by $rw \in I,\ r \in R$. Then

$$[z,\ r] = 0$$

for all $z \in I$ and $r \in R$. This implies that $I \subseteq Z(R)$ a contradiction. Thus, we have $D_1([z,\ w]) = 0$ holds for all $z,\ w \in I$. This gives $D_1(zw) = D_1(wz)$ for all $z,\ w \in I$. This yields that $D_1((zw)z) = D_1(z(wz))$. By applying our lemma (i), we find that

$$\begin{aligned}
2D_1((wz)z) &= D_1((zw)z + z(wz)) \\
&= 2\theta(w)\theta(z)D_1(z) + 2\theta(z)D_1(wz) \\
&= 2\theta(w)\theta(z)D_1(z) + \theta(z)D_1(zw + wz) \\
&= 2\theta(w)\theta(z)D_1(z) + 2\theta(z)\theta(w)D_1(z) + 2\theta(z^2)D_1(w) \\
&= 2\{\theta(w)\theta(z)D_1(z) + 2\theta(z)\theta(w)D_1(z) + 2\theta(z^2)D_1(w)\}.
\end{aligned}$$

Since R is a 2-torsion free, we obtain

$$D_1((wz)z) = \theta(w)\theta(z)D_1(z) + \theta(z)\theta(w)D_1(z) + \theta(z^2)D_1(w).$$

Since $D_1([z,\ w]) = 0$ for all $z,\ w \in I$, using the lemma and (15), we get

$$2[\theta(z),\ \theta(w)]D_1(z) = 0.$$

Since R is a 2-torsion free, we have

$$[\theta(z),\ \theta(w)]D_1(z) = 0.$$

Replacing w by wx, we get

$$0 = [\theta(z),\ \theta(w)]\theta(x)D_1(z) = \theta^{-1}([\theta(z),\ \theta(w)])x\theta^{-1}(D_1(z))$$

for all $z,\ w,\ x \in I$. Replacing x by xr, we get

$$\theta^{-1}([\theta(z),\ \theta(w)])xr\theta^{-1}(D_1(z)) = 0$$

for all $z,\ w,\ x \in I$ and $r \in R$. Since R is a prime ring and $I \neq 0$, we get $[z,\ w] = 0$ or $D_1(z) = 0$. Now let $Z_1 = \{z \in Z | [z,\ w] = 0\}$ and $Z_2 = \{z \in Z | D_1(z) = 0\}$. Clearly Z_1 and Z_2 are additive subgroups of Z whose union is Z. But a group can not be written as a union of its two proper subgroups and hence by Brauer's trick either $Z = Z_1$ or $Z = Z_2$. If $Z = Z_1$, then $[z,\ w] = 0$ for all $z,\ w \in I$. Replacing w by rw, we have

$$[z,\ r] = 0$$

for all $z \in I$ and $r \in R$. This implies that $I \subseteq Z(R)$, a contradiction, we arrive at $D_1(z) = 0$ for all $z \in I$. Replacing z by $zr + rz \in I$, then $D_1(zr + rz) = 0$. This gives $2\theta(z)D_1(r) + 2\theta(r)D_1(z) = 0$, for all $z \in I$ and $r \in R$. Since R is a 2-torsion free, we have

$$\theta(z)D_1(r) = 0.$$

Replacing z by zs. Then $\theta(z)\theta(s)D_1(r) = 0$ for $z \in I$ and $r,\ s \in R$. This implies that

$$zR\theta^{-1}(D_1(r)) = \{0\}.$$

Since R is a prime ring and $I \neq 0$, we get $D_1 = 0$ on R. Thus, $D_1(I) = 0$, for all $I \subseteq R$.

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HARDY'S INEQUALITIES WITH NON-DOUBLING WEIGHTS AND SHARP REMAINDERS.

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ABSTRACT. In the present paper we shall establish N-dimensional Hardy's inequalities with non-doubling weight functions of the distance $\delta(x)$ to the boundary $\partial\Omega$, where Ω is a C^2 class bounded domain of $\mathbf{R}^N$ $(N \geq 1)$. This work is essentially based on one-dimensional weighted Hardy's inequalities with one-sided boundary condition and sharp remainders. As weights we admit rather general ones that may vanish or blow up in infinite order such as $e^{-1/t}$ or $e^{1/t}$ at $t=0$ in one-dimensional case. [1]

1 Introduction In the present paper, we shall begin with introducing one-dimensional weighted Hardy's inequalities with sharp remainders under one-sided boundary condition. As weights we shall deal with the so-called non-doubling weights in addition to usual doubling ones. Then we shall establish N-dimensional weighted Hardy's inequalities with non-doubling weights of the distance $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ to the boundary $\partial\Omega$, where Ω is a C^2 class bounded domain of $\mathbf{R}^N$ $(N \geq 2)$. A positive continuous function $w(t)$ on $(0,\infty)$ is said to be a doubling weight if there exists a positive number C such that we have

$$C^{-1}w(t) \leq w(2t) \leq Cw(t) \quad (0 < t < \infty), \tag{1.1}$$

where C is independent of each $t \in (0,\infty)$. When $w(t)$ does not possess this property, $w(t)$ is said to be a non-doubling weight in the present paper. In one-dimensional case we typically treat a weight function $w(t)$ that may vanish or blow up in infinite order such as $e^{-1/t}$ or $e^{1/t}$ at $t=0$. In such cases the limit of ratio $w(t)/w(2t)$ as $t \to +0$ may become 0 or $+\infty$, and hence they are regarded as non-doubling weights according to our notion.

Definition 1.1. *Let $1 < p < \infty$ and set $p' = p/(p-1)$. Λ_p denotes the one-dimensional Hardy best constant defined by*

$$\Lambda_p = \frac{1}{(p')^p} = \left(1 - \frac{1}{p}\right)^p.$$

Let $\eta > 0$. By $C_c^1((0,\eta])$ we denote the set of all C^1 functions with compact supports in $(0,\eta]$. Then one-dimensional Hardy's inequalities with one-sided boundary condition in this paper are typically represented by the followings.

Proposition 1.1. *Assume that $1 < p < \infty$, $\mu > 0$ and $\eta > 0$.*
1. For every $u \in C_c^1((0,\eta])$ we have

$$\int_0^\eta |u'(t)|^p e^{-(p-1)/t}\,dt + \frac{(\Lambda_p)^{1/p'}}{\mu^{p-1}}|u(\eta)|^p \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p e^{-(p-1)/t}\,dt}{\left(e^{-1/t}\left(\int_t^\eta e^{1/s}\,ds + \mu\right)\right)^p}. \tag{1.2}$$

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2. For every $u \in C_c^1((0,\eta])$ we have

$$\int_0^\eta |u'(t)|^p e^{(p-1)/t}\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p e^{(p-1)/t}\,dt}{\left(e^{1/t}\int_0^t e^{-1/s}\,ds\right)^p} + \frac{(\Lambda_p)^{1/p'}}{\left(\int_0^\eta e^{-1/s}\,ds\right)^{p-1}}|u(\eta)|^p. \tag{1.3}$$

Proposition 1.1 will be established as Corollary 3.1 to Theorem 3.1. When $u(\eta) = 0$, these inequalities are a variant of classical weighted Hardy's inequalities (see [9, 10, 13]). It is interesting that both coefficients Λ_p and $(\Lambda_p)^{1/p'}$ appearing in (1.2) and (1.3) are best, even though the inequalities contain two parameters η and μ. To see the sharpness of (1.2) and (1.3), by the density argument it suffices to employ $u_\varepsilon(t) = \left(\mu + \int_t^\eta e^{1/s}\,ds\right)^{1/p'-\varepsilon}$ and $u_\varepsilon(t) = \left(\int_0^t e^{-1/s}\,ds\right)^{1/p'+\varepsilon}$ for test functions ($\varepsilon \to +0$) respectively. (For the detail see Subsection 4.2 (Part 1).)

Our first purpose in this paper is not only to establish a general version of Proposition 1.1 but also improve it by adding sharp remainder terms. By $W(\mathbf{R}_+)$ we denote a class of functions

$$\{w \in C^1(\mathbf{R}_+) : w > 0, \lim_{t\to+0} w(t) = a \text{ for some } a \in [0,\infty]\}$$

with $\mathbf{R}_+ = (0,\infty)$. As weights we adopt functions $W_p(t) = w(t)^{p-1}$ with $w(t) \in P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$, where

$$\begin{cases} P(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \notin L^1((0,\eta)) \text{ for some } \eta > 0\}, \\ Q(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \in L^1((0,\eta)) \text{ for any } \eta > 0\}. \end{cases} \tag{1.4}$$

Clearly $W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$, $e^{-1/t} \in P(\mathbf{R}_+)$ and $e^{1/t} \in Q(\mathbf{R}_+)$ for $t > 0$ (For the precise definitions see Section 2). By virtue of Proposition 1.1, it is clearly seen that our results on this matter essentially depend on whether w belongs to $P(\mathbf{R}_+)$ or $Q(\mathbf{R}_+)$. In particular when $w(t) \in P(\mathbf{R}_+)$, it follows from Proposition 3.1 that

$$\inf_{u\in V_\eta} \int_0^\eta |u'(t)|^p W_p(t)\,dt = 0, \tag{1.5}$$

where $V_\eta = \{u \in C^1([0,\eta]) : u(0) = 0, u(\eta) = 1\}$. Nevertheless we have sharp Hardy type inequalities (3.4) and (3.5) in Theorem 3.1.

As an important application, in Theorem 3.3 we shall establish N-dimensional Hardy's inequalities with weights being functions of the distance $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ to the boundary $\partial\Omega$. In this task it is crucial to establish sharp weighted Hardy's inequalities in the tubler neighborhood Ω_η of Ω, which are reduced to the one dimensional inequalities in Theorem 3.1. To this end, Ω is assumed to be a bounded domain of $\mathbf{R}^N$ ($N \geq 2$) whose boundary $\partial\Omega$ is a C^2 compact manifolds in the present paper. We prepare more notations to describe our results. For $W_p(t) = w(t)^{p-1}$ with $w(t) \in W(\mathbf{R}_+)$, we define a weight function $W_p(\delta(x))$ on Ω by

$$W_p(\delta(x)) = (W_p \circ \delta)(x).$$

By $L^p(\Omega; W_p(\delta))$ we denote the space of Lebesgue measurable functions with weight $W_p(\delta(x))$, for which

$$\|u\|_{L^p(\Omega;W_p(\delta))} = \left(\int_\Omega |u(x)|^p W_p(\delta(x))\,dx\right)^{1/p} < +\infty. \tag{1.6}$$

By $C_c^\infty(\Omega)$ we denote the set of all C^∞ functions with compact supports in Ω. $W_0^{1,p}(\Omega, W_p(\delta))$ is given by the completion of $C_c^\infty(\Omega)$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}(\Omega;W_p(\delta))} = \||\nabla u|\|_{L^p(\Omega;W_p(\delta))} + \|u\|_{L^p(\Omega;W_p(\delta))}. \tag{1.7}$$

Then $W_0^{1,p}(\Omega; W_p(\delta))$ becomes a Banach space with the norm $\|\cdot\|_{W_0^{1,p}(\Omega;W_p(\delta))}$. Under these preparation we will establish N-dimensional weighted Hardy's inequality as Theorem 3.3, which is the counter-part to Theorem 3.1. In particular for $w(t) \in Q(\mathbf{R}_+)$, as in Corollary 3.3 we have a simple inequality which is a generalization of classical Hardy's inequality:

$$\int_\Omega |\nabla u|^p W_p(\delta(x))\,dx \geq \gamma \int_\Omega \frac{|u|^p W_p(\delta(x))}{F_\eta(\delta(x))^p}\,dx, \quad \forall u \in W_0^{1,p}(\Omega; W_p(\delta)), \tag{1.8}$$

where η is a sufficiently small positive number, γ is some positive constant and $F_\eta(\delta(x)) = (F_\eta \circ \delta)(x)$ is a nonnegative function defined in Definition 2.3. If $w = 1$, then $F_\eta(t) = t\ (t \leq \eta);\ \eta\ (t \geq \eta)$ and (1.8) is a well-known Hardy's inequality having $(\min(\delta(x), \eta))^{-p}$ as the Hardy potential, which is valid for a bounded domain Ω of $\mathbf{R}^N$ with Lipschitz boundary (cf. [5, 6, 11]). Further if Ω is convex, then $\gamma = \Lambda_p$ holds for arbitrary $1 < p < \infty$ (cf. [12, 14]).

It is worthy to remark that (1.8) is never valid in the case that $w(t) \in P(\mathbf{R}_+)$ by (1.5) (see also Proposition 3.1 and Proposition 3.2). Nevertheless, in this case we shall establish weighted Hardy's inequalities with a switching function in Theorem 3.3 and Corollary 3.3, which correspond to Theorem 3.1 and its corollaries. We remark that these Hardy's inequalities with a compact perturbation are closely relating to the so-called weak Hardy property of Ω. In fact, if $w(t) \in P(\mathbf{R}_+)$, then a constant γ^{-1} in (3.19) concerns the weak Hardy constant, but in this case the strong Hardy constant is $+\infty$ (see [6] for the detail). In [10], we have improved Hardy's inequalities adopting $\delta(x)^{\alpha p}$ (powers of the distance $\delta(x)$ to the boundary $\partial\Omega$) as weight functions instead of $W_p(\delta)$. In the present paper, some inequalities of Hardy type in [2] and [10] will be employed with minor modifications, especially when $1 < p < 2$ (see also [1]).

We remark that our results will be applicable to variational problems with critical Hardy potentials in a coming paper [4] (c.f. [3]) and also applicable to the Caffarelli-Kohn-Nirenberg type inequalities with non-doubling weights in the coming paper [8].

This paper is organized in the following way: In Section 2 we introduce a class of weight functions $W(\mathbf{R}_+)$ and two subclasses $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ together with functions such as $F_\eta(t)$ and $G_\eta(t)$, which are crucial in this paper. Further a notion of admissibilities for $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ is introduced. In Section 3, the main results are described. The results are divided into two cases (one-dimensional case and N-dimensional case), which are described in Subsection 3.1 and Subsection 3.2, respectively. Theorem 3.1 and Theorem 3.2 are established in Section 4. Theorem 3.3 together with Corollary 3.3 are proved in Section 5. The proof of Theorem 3.4 is given in Section 6 and the proofs of Proposition 3.1 and Proposition 3.2 are given in Section 7. In Appendix the proof of Lemma 4.6 is provided. Some auxiliary inequalities are also given as Lemma 8.2.

2 Preliminaries First we introduce a class of weight functions which is crucial in this paper.

Definition 2.1. *Let us set* $\mathbf{R}_+ = (0, \infty)$ *and*

$$W(\mathbf{R}_+) = \{w(t) \in C^1(\mathbf{R}_+) : w(t) > 0, \lim_{t \to +0} w(t) = a \textit{ for some } a \in [0, \infty]\}. \tag{2.1}$$

In the next we define two subclasses of this rather large space.

Definition 2.2. *Let us set*

$$P(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \notin L^1((0, \eta)) \textit{ for some } \eta > 0\}. \tag{2.2}$$

$$Q(\mathbf{R}_+) = \{w(t) \in W(\mathbf{R}_+) : w(t)^{-1} \in L^1((0, \eta)) \textit{ for any } \eta > 0\}. \tag{2.3}$$

Remark 2.1. *1. From Definition 2.1 and Definition 2.2 it follows that* $W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$ *and* $P(\mathbf{R}_+) \cap Q(\mathbf{R}_+) = \phi$.

2. *If $w^{-1} \notin L^1((0,\eta))$ for some $\eta > 0$, then $w^{-1} \notin L^1((0,\eta))$ for any $\eta > 0$. Similarly if $w^{-1} \in L^1((0,\eta))$ for some $\eta > 0$, then $w^{-1} \in L^1((0,\eta))$ for any $\eta > 0$.*

3. *If $w \in P(\mathbf{R}_+)$, then $\lim_{t \to +0} w(t) = 0$. Hence by setting $w(0) = 0$, w is uniquely extended to a continuous function on $[0,\infty)$. On the other hand if $w \in Q(\mathbf{R}_+)$, then possibly $\lim_{t \to +0} w(t) = +\infty$.*

Here we give some fundamental examples:

Example 2.1. *1. $e^{-1/t} \in P(\mathbf{R}_+)$ and $e^{1/t} \in Q(\mathbf{R}_+)$. (See Corollary 3.1.)*

2. *For $p' = p/(p-1)$, $t^{\alpha p'} \in P(\mathbf{R}_+)$ if $\alpha \geq 1/p'$ and $t^{\alpha p'} \in Q(\mathbf{R}_+)$ if $\alpha < 1/p'$. (See Corollary 3.2 with $W_p(t) = w(t)^{p-1} = t^{\alpha p}$.)*

3. *For $\alpha \in \mathbf{R}$, $t^{\alpha} e^{-1/t} \in P(\mathbf{R}_+)$ and $t^{\alpha} e^{1/t} \in Q(\mathbf{R}_+)$.*

In the next we define functions such as $F_\eta(t)$ and $G_\eta(t)$, which are crucial in considering variants of the Hardy potential like $F_\eta(\delta(x))^{-p}$ in (1.8).

Definition 2.3. *Let $\mu > 0$ and $\eta > 0$. For $w \in W(\mathbf{R}_+)$, we define the followings:*

1. *When $w \in P(\mathbf{R}_+)$,*

$$F_\eta(t;w,\mu) = \begin{cases} w(t)\left(\mu + \int_t^\eta w(s)^{-1}\,ds\right), & \text{if } t \in (0,\eta], \\ w(\eta)\mu, & \text{if } t \geq \eta. \end{cases} \tag{2.4}$$

$$G_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta F_\eta(s;w,\mu)^{-1}\,ds, & \text{if } t \in (0,\eta], \\ \mu, & \text{if } t \geq \eta. \end{cases} \tag{2.5}$$

2. *When $w \in Q(\mathbf{R}_+)$,*

$$F_\eta(t;w) = \begin{cases} w(t)\int_0^t w(s)^{-1}\,ds, & \text{if } t \in (0,\eta], \\ w(\eta)\int_0^\eta w(s)^{-1}\,ds, & \text{if } t \geq \eta. \end{cases} \tag{2.6}$$

$$G_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta F_\eta(s;w)^{-1}\,ds, & \text{if } t \in (0,\eta], \\ \mu, & \text{if } t \geq \eta. \end{cases} \tag{2.7}$$

3. *$F_\eta(t;w,\mu)$ and $F_\eta(t;w)$ are abbreviated as $F_\eta(t)$. $G_\eta(t;w,\mu)$ is abbreviated as $G_\eta(t)$.*

4. *For $w \in W(\mathbf{R}_+)$, we define*

$$W_p(t) = w(t)^{p-1}. \tag{2.8}$$

Remark 2.2. *In the definition (2.5), one can replace $G_\eta(t;w,\mu)$ with the more general $G_\eta(t;w,\mu,\mu')$ $= \mu' + \int_t^\eta F_\eta(s;w,\mu)^{-1}\,ds$, $t \in (0,\eta]$; μ', $t \geq \eta$ for $\mu' > 0$. However, for simplicity this paper uses (2.5).*

Example 2.2. *Let $w(t) = t^{\alpha p'}$ for $\alpha \in \mathbf{R}$, $1 < p < \infty$ and $p' = p/(p-1)$.*

1. *When $\alpha > 1/p'$, $F_\eta(t) = t/(\alpha p' - 1)$ and $G_\eta(t) = \mu + (\alpha p' - 1)\log(\eta/t)$ provided that $\mu = \eta^{1-\alpha p'}/(\alpha p' - 1)$.*

2. *When $\alpha = 1/p'$, $F_\eta(t) = t(\mu + \log(\eta/t))$ and $G_\eta(t) = \mu - \log\mu + \log(\mu + \log(\eta/t))$.*

3. *When $\alpha < 1/p'$, $F_\eta(t) = t/(1 - \alpha p')$ and $G_\eta(t) = \mu + (1 - \alpha p')\log(\eta/t)$.*

By using integration by parts we see the followings:

Example 2.3. 1. *If either* $w(t) = e^{-1/t} \in P(\mathbf{R}_+)$ *or* $w(t) = e^{1/t} \in Q(\mathbf{R}_+)$, *then* $F_\eta(t) = O(t^2)$ *as* $t \to +0$.

2. *Moreover, if* $w(t) = \exp(\pm t^{-\alpha})$ *with* $\alpha > 0$, *then* $F_\eta(t) = O(t^{\alpha+1})$ *as* $t \to +0$. *In fact, it holds that* $\lim_{t\to+0} F_\eta(t)/t^{\alpha+1} = 1/\alpha$.

In a similar way we define the following:

Definition 2.4. *Let* $p' = p/(p-1)$, $\mu > 0$ *and* $\eta > 0$. *For* $w \in W(\mathbf{R}_+)$ *and* $t \in (0,\eta)$, *we define the followings:*

1. *When* $w \in P(\mathbf{R}_+)$,

$$f_\eta(t;w,\mu) = \begin{cases} \mu + \int_t^\eta w(s)^{-1}\,ds, & \text{if } t \in (0,\eta], \\ \mu, & \text{if } t \geq \eta. \end{cases} \tag{2.9}$$

$$g_\eta(t;w,\mu) = (p' f_\eta(t;w,\mu))^{1/p'}, \qquad \text{if } t > 0. \tag{2.10}$$

2. *When* $w \in Q(\mathbf{R}_+)$,

$$f_\eta(t;w) = \begin{cases} \int_0^t w(s)^{-1}\,ds, & \text{if } t \in (0,\eta], \\ \int_0^\eta w(s)^{-1}\,ds, & \text{if } t \geq \eta. \end{cases} \tag{2.11}$$

$$g_\eta(t;w) = (p' f_\eta(t;w))^{1/p'}, \qquad \text{if } t > 0. \tag{2.12}$$

3. $f_\eta(t;w,\mu)$ *and* $f_\eta(t;w)$ *are abbreviated as* $f_\eta(t)$. $g_\eta(t;w,\mu)$ *and* $g_\eta(t;w)$ *are abbreviated as* $g_\eta(t)$.

Remark 2.3. 1. *We note that for* $0 < t < \eta$

$$\begin{cases} \frac{d}{dt}\log f_\eta(t) = -F_\eta(t)^{-1}, & \text{if } w \in P(\mathbf{R}_+), \\ \frac{d}{dt}\log f_\eta(t) = F_\eta(t)^{-1}, & \text{if } w \in Q(\mathbf{R}_+), \\ \frac{d}{dt}\log G_\eta(t) = -(F_\eta(t)G_\eta(t))^{-1}, & \text{if } w \in W(\mathbf{R}_+), \\ \frac{d}{dt}G_\eta(t)^{-1} = (F_\eta(t)G_\eta(t)^2)^{-1}, & \text{if } w \in W(\mathbf{R}_+). \end{cases} \tag{2.13}$$

By Definition 2.2 and (2.13), we see that $F_\eta(t)^{-1} \notin L^1((0,\eta))$, $(F_\eta(t)G_\eta(t))^{-1} \notin L^1((0,\eta))$ *but* $(F_\eta(t)G_\eta(t)^2)^{-1} \in L^1((0,\eta))$.

2. *If* $w \in W(\mathbf{R}_+)$, *then immediately we have* $\liminf_{t\to+0} F_\eta(t) = 0$ *from 1.*

Example 2.4. *If either* $w(t) = t^2 e^{-1/t} \in P(\mathbf{R}_+)$ *or* $w(t) = t^2 e^{1/t} \in Q(\mathbf{R}_+)$, *then* $F_\eta(t) = O(t^2)$ *and* $G_\eta(t) = O(1/t)$ *as* $t \to 0$.

Now we introduce two admissibilities for $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$.

Definition 2.5. 1. *A function* $w \in P(\mathbf{R}_+)$ *is said to be admissible if there exist positive numbers* η *and* K *such that we have*

$$\int_t^\eta w(s)^{-1}\,ds \leq e^{K/\sqrt{t}} \qquad (0 < t < \eta). \tag{2.14}$$

2. *A function $w \in Q(\mathbf{R}_+)$ is said to be admissible if there exist positive numbers η and K such that we have*

$$\int_0^t w(s)^{-1}\,ds \geq e^{-K/\sqrt{t}} \qquad (0 < t < \eta). \tag{2.15}$$

Definition 2.6. *By $P_A(\mathbf{R}_+)$ and $Q_A(\mathbf{R}_+)$ we denote the set of all admissible functions in $P(\mathbf{R}_+)$ and $Q(\mathbf{R}_+)$ respectively. We set*

$$W_A(\mathbf{R}_+) = P_A(\mathbf{R}_+) \cup Q_A(\mathbf{R}_+). \tag{2.16}$$

Here we give typical examples:

Example 2.5. $e^{-1/t} \notin P_A(\mathbf{R}_+)$, $e^{1/t} \notin Q_A(\mathbf{R}_+)$ *but* $e^{-1/\sqrt{t}} \in P_A(\mathbf{R}_+)$, $e^{1/\sqrt{t}} \in Q_A(\mathbf{R}_+)$.

Verifications:
$e^{-1/t} \notin P_A(\mathbf{R}_+)$: *For small $t > 0$, we have $\int_t^\eta e^{1/s}\,ds \geq \int_t^{2t} e^{1/s}\,ds \geq te^{1/(2t)}$. But this contradicts to (2.14) for any $K > 0$.*
$e^{-1/\sqrt{t}} \in P_A(\mathbf{R}_+)$: *Since $e^{1/\sqrt{s}} \leq e^{1/\sqrt{t}}$ $(t < s < \eta)$, we have $\int_t^\eta e^{1/\sqrt{s}}\,ds \leq \eta e^{1/\sqrt{t}} \leq e^{K/\sqrt{t}}$ for some $K > 1$.*
$e^{-1/t} \notin Q_A(\mathbf{R}_+)$: *For $0 < s \leq t$, we have $\int_0^t e^{-1/s}\,ds \leq te^{-1/t}$. But this contradicts to (2.15) for any $K > 0$.*
$e^{-1/\sqrt{t}} \in Q_A(\mathbf{R}_+)$: *For $t/2 < s < t$, we have $\int_0^t e^{-1/\sqrt{s}}\,ds \geq \int_{t/2}^t e^{-1/\sqrt{s}}\,ds \geq (t/2)e^{-\sqrt{2}/\sqrt{s}} \geq e^{-K/\sqrt{t}}$ for some $K > \sqrt{2}$.*

Proposition 2.1. 1. *Assume that either $w \in P_A(\mathbf{R}_+)$ or $w \in Q_A(\mathbf{R}_+)$. Then there exist positive numbers η and K such that we have*

$$\sqrt{t}\,G_\eta(t) \leq K \qquad t \in (0, \eta). \tag{2.17}$$

2. *Assume that $w \in W(\mathbf{R}_+)$ and w satisfies (2.17) for some positive numbers η and K. Then w is admissible in the sense of Definition 2.6, that is, w belongs to $W_A(\mathbf{R}_+)$.*

Proof:
By integrating (2.13), we have for $0 < t < \eta$

$$\begin{cases} G_\eta(t) = \log\left(\int_t^\eta w(s)^{-1}\,ds + \mu\right) - \log\mu + \mu, & \text{if } w \in P(\mathbf{R}_+), \\[2ex] G_\eta(t) = \log\left(\int_0^\eta w(s)^{-1}\,ds\right) - \log\left(\int_0^t w(s)^{-1}\,ds\right) + \mu, & \text{if } w \in Q(\mathbf{R}_+). \end{cases} \tag{2.18}$$

Hence the inequality (2.17) with positive numbers η and K is equivalent to

$$\begin{cases} \int_t^\eta w(s)^{-1}\,ds \leq \mu(e^{K/\sqrt{t}-\mu} - 1) \quad (0 < t < \eta), & \text{if } w \in P_A(\mathbf{R}_+), \\[2ex] \int_0^t w(s)^{-1}\,ds \geq e^\mu \int_0^\mu w(s)^{-1}\,ds\, e^{-K/\sqrt{t}} \quad (0 < t < \eta), & \text{if } w \in Q_A(\mathbf{R}_+). \end{cases} \tag{2.19}$$

Here we note that for each $\mu > 0$, $\eta > 0$ and $w \in Q(\mathbf{R}_+)$ there exist some positive numbers K_1, K_2 and K_3 such that $K_1 \leq K_2 \leq K_3$ and we have

$$\begin{cases} e^{K_1/\sqrt{t}} \leq \mu(e^{K_2/\sqrt{t}-\mu} - 1) \leq e^{K_3/\sqrt{t}} \quad (0 < t < \eta) \\[2ex] e^{-K_3/\sqrt{t}} \leq e^\mu \int_0^\mu w(s)^{-1}\,ds\, e^{-K_2/\sqrt{t}} \leq e^{-K_1/\sqrt{t}} \quad (0 < t < \eta). \end{cases} \tag{2.20}$$

Therefore the assertion is now clear. □

3 Main results

3.1 Results in the one dimensional case We introduce function spaces to state the results in one dimensional case:
By $L^p((0,\eta];W_p)$ we denote the space of Lebesgue measurable functions with weight $W_p(t)$, for which

$$\|u\|_{L^p((0,\eta];W_p)} = \left(\int_0^\eta |u(t)|^p W_p(t)\,dt\right)^{1/p} < +\infty. \tag{3.1}$$

By $C_c^\infty((0,\eta])$ we denote the set of all C^∞ functions with compact supports in $(0,\eta]$. $W_0^{1,p}((0,\eta];W_p)$ is given by the completion of $C_c^\infty((0,\eta])$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}((0,\eta];W_p)} = \|u'\|_{L^p((0,\eta];W_p)} + \|u\|_{L^p((0,\eta];W_p)}. \tag{3.2}$$

Then $W_0^{1,p}((0,\eta];W_p)$ becomes a Banach space with the norm $\|\cdot\|_{W_0^{1,p}((0,\eta];W_p)}$. We also define a switching function:

Definition 3.1. (Switching function) For $w \in W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$ we set

$$s(w) = \begin{cases} -1 & \text{if} \quad w \in P(\mathbf{R}_+), \\ 1 & \text{if} \quad w \in Q(\mathbf{R}_+). \end{cases} \tag{3.3}$$

We state one-dimensional Hardy's inequalities with one-sided boundary condition.

Theorem 3.1. ($w \in W(\mathbf{R}_+)$)
Assume that $1 < p < \infty$, $\eta > 0$, $\mu > 0$ and $w \in W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$. Then we have the followings:

1. *For every $u \in W_0^{1,p}((0,\eta];W_p) \cap C((0,\eta])$, we have*

$$\int_0^\eta |u'(t)|^p W_p(t)\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p. \tag{3.4}$$

   Moreover the coefficients Λ_p and $s(w)(\Lambda_p)^{1/p'}$ are optimal.

2. *There exist positive numbers $C = C(w,p,\eta,\mu)$ and $L = L(w,p,\eta,\mu)$ such that for every $u \in W_0^{1,p}((0,\eta];W_p) \cap C((0,\eta])$, we have*

$$\int_0^\eta |u'(t)|^p W_p(t)\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + C\int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p G_\eta(t)^2} + s(w)L|u(\eta)|^p, \tag{3.5}$$

   where C and L can be taken independent of each u.

The following is a direct consequence from this theorem. We remark that $C_c^1((0,\eta])$ is densely contained in $W_0^{1,p}((0,\eta];W_p)$.

Corollary 3.1. *1. Let $1 < p < \infty$, $\eta > 0$ and $\mu > 0$. Then, for every $u \in C_c^1((0,\eta])$ we have*

$$\int_0^\eta |u'(t)|^p e^{-(p-1)/t}\,dt + \frac{(\Lambda_p)^{1/p'}}{\mu^{p-1}}|u(\eta)|^p \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p e^{-(p-1)/t}\,dt}{\left(e^{-1/t}\left(\int_t^\eta e^{1/s}\,ds + \mu\right)\right)^p}. \tag{3.6}$$

2. *Let $1 < p < \infty$ and $\eta > 0$. Then, for every $u \in C_c^1((0,\eta])$ we have*

$$\int_0^\eta |u'(t)|^p e^{(p-1)/t}\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p e^{(p-1)/t}\,dt}{\left(e^{1/t}\int_0^t e^{-1/s}\,ds\right)^p} + \frac{(\Lambda_p)^{1/p'}}{\left(\int_0^\eta e^{-1/s}\,ds\right)^{p-1}}|u(\eta)|^p. \tag{3.7}$$

Definition 3.2. *For $1 < p < +\infty$ and $\alpha \in \mathbf{R}$ we set*

$$\Lambda_{\alpha,p} = \left|\frac{1}{p'} - \alpha\right|^p = \left|\frac{p-1-\alpha p}{p}\right|^p. \tag{3.8}$$

Corollary 3.2. *1. If $\alpha > 1/p'$, $1 < p < \infty$ and $\eta > 0$, then for every $u \in C_c^1((0,\eta])$ we have*

$$\int_0^\eta |u'(t)|^p t^{\alpha p}\,dt + \frac{(\Lambda_{\alpha,p})^{1/p'}}{\eta^{p-1-\alpha p}}|u(\eta)|^p \geq \Lambda_{\alpha,p}\int_0^\eta |u(t)|^p t^{(\alpha-1)p}\,dt. \tag{3.9}$$

2. *If $\alpha = 1/p'$, $1 < p < \infty$, $\eta > 0$ and $R > e$, then for every $u \in C_c^1((0,\eta])$ we have*

$$\int_0^\eta |u'(t)|^p t^{p-1}\,dt + \frac{(\Lambda_p)^{1/p'}}{(\log R)^{p-1}}|u(\eta)|^p \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p}{tA_1(t/\eta)^p}\,dt, \tag{3.10}$$

where $A_1(t) = \log(R/t)$ with $R > e$.

3. *If $\alpha < 1/p'$, $1 < p < \infty$ and $\eta > 0$, then for every $u \in C_c^1((0,\eta])$ we have*

$$\int_0^\eta |u'(t)|^p t^{\alpha p}\,dt \geq \Lambda_{\alpha,p}\int_0^\eta |u(t)|^p t^{(\alpha-1)p}\,dt + \frac{(\Lambda_{\alpha,p})^{1/p'}}{\eta^{p-1-\alpha p}}|u(\eta)|^p. \tag{3.11}$$

Proof of Corollary 3.2: In Theorem 3.1 we set $W_p(t) = w(t)^{p-1} = t^{\alpha p}$. Then $w(t) \in P(\mathbf{R}_+)$ for $\alpha \geq 1/p'$, and $w(t) \in Q(\mathbf{R}_+)$ for $\alpha < 1/p'$. Noting that $R > e$ and $(\Lambda_p/\Lambda_{\alpha,p})^{1/p} = |\alpha p' - 1|^{-1}$, we set

$$\mu = \begin{cases} (\Lambda_p/\Lambda_{\alpha,p})^{1/p}\eta^{(p-1-\alpha p)/(p-1)}, & \text{if } \alpha \neq 1/p', \\ \log R, & \text{if } \alpha = 1/p'. \end{cases} \tag{3.12}$$

When $\alpha \neq 1/p'$ we have

$$F_\eta(t) = (\Lambda_p/\Lambda_{\alpha,p})^{1/p}\,t \quad \text{and} \quad f_\eta(\eta) = \mu. \tag{3.13}$$

When $\alpha = 1/p'$ we have

$$F_\eta(t) = t\log(R\eta/t) \quad \text{and} \quad f_\eta(\eta) = \mu. \tag{3.14}$$

Then the assertions (3.9), (3.10) and (3.11) follow from (3.4). □

Remark 3.1. *1. The sharpness of coefficients in theses corollaries 3.1, 3.2 will be seen in Section 4 (4.2, Part 1).*

2. *If $u \in C_c^1((0,\infty))$, the inequalities (3.7), (3.9) and (3.11) remain valid for $\eta = +\infty$.*

In order to establish Hardy's inequalities in a bounded domain of $\mathbf{R}^N$ as an application, we need a further refinement of the previous results.

Theorem 3.2. $(w \in W_A(\mathbf{R}_+) = P_A(\mathbf{R}_+) \cup Q_A(\mathbf{R}_+))$
Assume that $1 < p < \infty$, $\mu > 0$ *and* $w \in W_A(\mathbf{R}_+)$. *If* $\eta > 0$ *is sufficiently small, then there exist positive numbers* $C_0 = C_0(w, p, \eta, \mu)$, $C_1 = C_1(w, p, \eta, \mu)$ *and* $L = L(w, p, \eta, \mu)$ *such that for every* $u \in W_0^{1,p}((0, \eta]; W_p) \cap C((0, \eta])$, *we have*

$$\begin{aligned}\int_0^\eta \left(|u'(t)|^p - \frac{|u(t)|^p}{F_\eta(t)^p} \left(\Lambda_p + \frac{C_0}{G_\eta(t)^2} \right) \right) W_p(t)\, dt \\ \geq C_1 \int_0^\eta \left(|u'(t)|^p + \frac{|u(t)|^p}{F_\eta(t)^p} \left(\Lambda_p + \frac{C_0}{G_\eta(t)^2} \right) \right) W_p(t)\, t\, dt + s(w) L |u(\eta)|^p,\end{aligned} \tag{3.15}$$

where C_0, C_1 *and* L *can be taken independent of each* u.

Lastly we state a fundamental result which will be useful in the subsequent.

Proposition 3.1. $(w \in P(\mathbf{R}_+))$
Assume that $w \in P(\mathbf{R}_+)$ *and* $1 < p < \infty$. *Then we have*

$$\inf_{u \in V_\eta} \int_0^\eta |u'(t)|^p W_p(t)\, dt = 0, \tag{3.16}$$

where $V_\eta = \{u \in C^1([0, \eta]) : u(0) = 0, u(\eta) = 1\}$.

Remark 3.2. *The proof will be given in Section 6.*

3.2 Results in a domain of $\mathbf{R}^N$ $(N \geq 2)$ As an important application of one dimensional Hardy's inequalities in the previous section, we describe Hardy's inequalities in a bounded domain Ω of $\mathbf{R}^N$ $(N \geq 2)$. Let $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. For each small $\eta > 0$, Ω_η denotes a tubular neighborhood of $\partial\Omega$ and Σ_η denotes the boundary of $\Omega \setminus \Omega_\eta$, namely

$$\Omega_\eta = \{x \in \Omega : \delta(x) < \eta\} \quad \text{and} \quad \Sigma_\eta = \{x \in \Omega : \delta(x) = \eta\}. \tag{3.17}$$

For the sake of simplicity, by $W_p(\delta)$, $F_\eta(\delta)$ and $G_\eta(\delta)$ we denote $W_p(\delta(x))$, $F_\eta(\delta(x))$ and $G_\eta(\delta(x))$ respectively.

The proofs of Theorem 3.3 and Corollary 3.3 will be given in Section 5. Theorem 3.4 will be proved in Section 6.

Theorem 3.3. ($w \in W_A(\mathbf{R}_+)$) *Assume that* Ω *is a bounded domain of class* C^2 *in* $\mathbf{R}^N$. *Assume that* $1 < p < \infty$ *and* $w \in W_A(\mathbf{R}_+)$. *Assume that* $\mu > 0$ *and* η *is a sufficiently small positive number. Then, there exist positive numbers* $C = C(w, p, \eta, \mu)$ *and* $L' = L'(w, p, \eta, \mu)$ *such that for every* $u \in W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$, *we have*

$$\begin{aligned}\int_{\Omega_\eta} \left(|\nabla u|^p - \Lambda_p \frac{|u|^p}{F_\eta(\delta)^p} \right) W_p(\delta)\, dx \geq C \int_{\Omega_\eta} \frac{|u|^p W_p(\delta)}{F_\eta(\delta)^p G_\eta(\delta)^2}\, dx \\ + s(w) L' \int_{\Sigma_\eta} |u|^p W_p(\delta)\, d\sigma_\eta,\end{aligned} \tag{3.18}$$

where $d\sigma_\eta$ *denotes surface elements on* Σ_η, *and* C, L' *are independent of each* u.

Remark 3.3. *We remark that the assumpsion* $w \in W_A(\mathbf{R}_+)$ *is needed even if we do not have the first term involving* $G_\eta(t)$ *in the right-hand side. (See also Corollary 3.3).*

Corollary 3.3. *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $w \in W_A(\mathbf{R}_+)$. Assume that $\mu > 0$ and η is a sufficiently small positive number. Then, there exist positive numbers $\gamma = \gamma(w, p, \eta, \mu)$ and $L' = L'(w, p, \eta, \mu)$ such that for every $u \in W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$, we have*

$$\int_\Omega \left(|\nabla u|^p - \gamma \frac{|u|^p}{F_\eta(\delta)^p} \right) W_p(\delta)\, dx \ge s(w) L' \int_{\Sigma_\eta} |u|^p W_p(\delta)\, d\sigma_\eta, \tag{3.19}$$

where $d\sigma_\eta$ denotes surface elements on Σ_η, and γ, L' are independent of each u.

Moreover we have the followings:

Theorem 3.4. ($w \in W_A(\mathbf{R}_+)$) *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$, $\mu > 0$ and $w \in W_A(\mathbf{R}_+)$. Then, the followings are equivalent to each other.*

1. *There exist positive numbers γ, η and L' such that the inequality (3.19) is valid for every $u \in W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$.*
2. *For a sufficiently small $\eta > 0$, there exist positive numbers κ, C and L' such that the inequality (3.18) with Λ_p replaced by κ is valid for every $u \in W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$.*

Proposition 3.2. ($w \in P(\mathbf{R}_+)$) *Assume that Ω is a bounded domain of class C^2 in $\mathbf{R}^N$. Assume that $1 < p < \infty$ and $w \in P(\mathbf{R}_+)$. Then, for an arbitrary $\eta \in (0, \sup_{x \in \Omega} \delta(x))$ we have*

$$\inf \left\{ \int_\Omega |\nabla u|^p W_p(\delta)\, dx : u \in C_c^1(\Omega), u = 1 \text{ on } \{\delta(x) = \eta\} \right\} = 0. \tag{3.20}$$

Remark 3.4.

1. *The proof will be given in Section 6 together with Proposition 3.1*
2. *From this we see that constant functions belong to $W_0^{1,p}(\Omega; W_p(\delta))$, and hence Hardy's inequality (1.8) never holds when $w \in P(\mathbf{R}_+)$.*

4 Proofs of Theorem 3.1 and Theorem 3.2

4.1 Lemmas First we prepare the following fundamental inequalities which are established in [1] as Lemma 2.1 for $X > -1$.

Lemma 4.1.

1. *For $p \ge 2$ we have*

$$|1 + X|^p - 1 - pX \ge c(p) |X|^q, \quad \text{for any } q \in [2, p] \text{ and } X \in \mathbf{R}. \tag{4.1}$$

2. *For $1 < p \le 2$ and $M \ge 1$, we have*

$$|1 + X|^p - 1 - pX \ge c(p) \begin{cases} M^{p-2} X^2, & |X| \le M, \\ |X|^p, & |X| \ge M. \end{cases} \tag{4.2}$$

Here $c(p)$ is a positive number independent of each X, $M \ge 1$ and $q \in [2, p]$.

Proof. By Taylar expansion we have (4.1) with $q = 2$. For $p > 1$, we note that

$$\lim_{X \to 0} \frac{|1 + X|^p - 1 - pX}{X^2} = \frac{p(p-1)}{2}, \quad \lim_{|X| \to \infty} \frac{|1 + X|^p - 1 - pX}{|X|^p} = 1. \tag{4.3}$$

Therefore (4.1) is valid for any $q \in [2, p]$ for a small $c(p) > 0$. If $X > -1$, then (4.2) also follows from Taylar expansion and (4.3). If we choose $c(p)$ sufficiently small, then it remains valid for $X \le -1$. □

Remark 4.1. *$C_c^1((0,\eta])$ is densely contained in $W_0^{1,p}((0,\eta];W_p)$. If $u \in W_0^{1,p}((0,\eta];W_p)$, then $|u| \in W_0^{1,p}((0,\eta];W_p)$ and $\|u\|_{W_0^{1,p}((0,\eta];W_p)} = \||u|\|_{W_0^{1,p}((0,\eta];W_p)}$. Therefore, in the proofs of Theorem 3.1 and Theorem 3.2, without the loss of generality we may assume that $u \in C_c^1((0,\eta])$ and $u \geq 0$ in $(0,\eta)$.*

For $u \in C_c^1((0,\eta])$ let us set

$$u(t) = g_\eta(t)v(t), \tag{4.4}$$

where $g_\eta(t)$ is given by Definition 2.4. Clearly $v(t) \in C_c^1((0,\eta])$ and $u(t)' = g_\eta'(t)v(t) + g_\eta(t)v'(t)$. We define

$$X(t) = \begin{cases} \frac{g_\eta(t)}{g_\eta'(t)}\frac{v'(t)}{v(t)} = p'F_\eta(t)\frac{v'(t)}{v(t)}s(w) & \text{if } v(t) \neq 0, \\ 0 & \text{if } v(t) = 0. \end{cases} \tag{4.5}$$

Then we have

$$u'(t) = g_\eta'(t)v(t)\left(1 + X(t)\right), \quad \text{if } v(t) \neq 0. \tag{4.6}$$

Then it follows immediately from Lemma 4.1 that we have the following.

Lemma 4.2. *Assume that $\eta > 0$, $\mu > 0$, $X = X(t)$ and $w \in W(\mathbf{R}_+)$. Then, there exists a positive number $c(p)$ such that we have the followings:*

1. *If $p \geq 2$, then for any $u \in C_c^1((0,\eta])$ and $q \in [2,p]$ we have*

$$\int_0^\eta |u'(t)|^p W_p(t)\,dt \geq \int_0^\eta |v(t)|^p |g_\eta'|^p \left(1 + pX + c(p)|X|^q\right) W_p(t)\,dt. \tag{4.7}$$

2. *If $1 < p < 2$, then for any $u \in C_c^1((0,\eta])$ and $M \geq 1$ we have*

$$\begin{aligned} \int_0^\eta |u'(t)|^p W_p(t)\,dt &\geq \int_0^\eta |v(t)|^p |g_\eta'|^p \left(1 + pX\right) W_p\,dt \\ &+ c(p)\int_0^\eta |v(t)|^p |g_\eta'|^p \left(M^{p-2}X^2\chi_{\{|X|\leq M\}} + |X|^p\chi_{\{|X|>M\}}\right) W_p(t)\,dt, \end{aligned} \tag{4.8}$$

where $\chi_S(t)$ denotes a characteristic function of a set S.

Now we prepare the following lemma, recalling a switching function;

$$s(w) = \begin{cases} -1 & \text{if} \quad w \in P(\mathbf{R}_+), \\ 1 & \text{if} \quad w \in Q(\mathbf{R}_+). \end{cases} \tag{4.9}$$

Lemma 4.3. *Assume that $\eta > 0$, $\mu > 0$ and $w \in W(\mathbf{R}_+)$. Assume that $u \in C_c^1((0,\eta])$, $X = X(t)$ and $u(t) = g_\eta(t)v(t)$. Then we have:*

1. $$|g_\eta'(t)|^{p-1} g_\eta(t) W_p(t) = 1. \tag{4.10}$$

2. $$p|v(t)|^p |g_\eta'(t)|^p X(t) W_p(t) = s(w)(|v(t)|^p)'. \tag{4.11}$$

3. $$|v(t)|^p |g_\eta'(t)|^p W_p(t) = \Lambda_p \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p}. \tag{4.12}$$

4.
$$|v(t)|^p|g'_\eta(t)|^p|X(t)|^2W_p(t) = \frac{4p'}{p^2}|(v(t)^{p/2})'|^2F_\eta(t). \tag{4.13}$$

5.
$$|v(t)|^p|g'_\eta(t)|^p|X(t)|^pW_p(t) = (p')^{p-1}|v'(t)|^pF_\eta(t)^{p-1}. \tag{4.14}$$

Proof of Lemma 4.3:
Proof of (4.10): By Definition 2.4 (2) we see
$$g'_\eta = (p')^{-1/p}(f_\eta)^{-1/p}w^{-1}s(w). \tag{4.15}$$
Then we have (4.10).
Proof of (4.11): Using (4.10) together with
$$p|v|^p|g'_\eta|^pX = s(w)(|v|^p)'|g'_\eta|^{p-1}g_\eta, \tag{4.16}$$
we have (4.11).
Proof of (4.12): Noting that $p+p' = pp'$ and $(p')^{-p} = \Lambda_p$, we have
$$|v|^p|g'_\eta|^p = \frac{|u|^p}{g_\eta^{p+p'}w^p} = \frac{|u|^p}{g_\eta^{pp'}w^p} = \Lambda_p\frac{|u|^p}{F_\eta^p}.$$
Proof of (4.13): Using (4.15) and $2-(p-2)/(p-1) = p'$, we have
$$|v|^p|g'_\eta|^p|X|^2W_p = |v|^{p-2}(v')^2|g'_\eta|^{p-2}g_\eta^2W_p = \frac{4p'}{p^2}|(v^{p/2})'|^2F_\eta.$$
Proof of (4.14): Using (4.15) and $g_\eta^p = (p'f_\eta)^{p-1}$ we have
$$|v|^p|g'_\eta|^p|X|^pW_p = |v'|^pg_\eta^pW_p = (p')^{p-1}|v'|^pF_\eta^{p-1}.$$
□

By the definition of Λ_p we have the following.

Lemma 4.4. *Assume that $\eta > 0$ and $w \in W(\mathbf{R}_+)$. Then we have*
$$\int_0^\eta (|v(t)|^p)'\,dt = \frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p, \tag{4.17}$$
where $v = u/g_\eta$ and $f_\eta(\eta) = \mu$, if $w \in P(\mathbf{R}_+)$; $\int_0^\eta w(s)^{-1}\,ds$, if $w \in Q(\mathbf{R}_+)$.

By virtue of Lemma 4.2, Lemma 4.3 and Lemma 4.4, we have the following:

Lemma 4.5. *Assume that $\eta > 0$, $\mu > 0$, $X = X(t)$ and $w \in W(\mathbf{R}_+)$. Then, there exists a positive number $c(p)$ such that we have the followings:*

1. *If $p \ge 2$, then for any $u \in C_c^1((0,\eta])$ we have*
$$\int_0^\eta |u'(t)|^pW_p(t)\,dt \ge \Lambda_p\int_0^\eta \frac{|u(t)|^pW_p(t)\,dt}{F_\eta(t)^p} + s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p + d(p)\int_0^\eta |(|v(t)|^{p/2})'|^2F_\eta(t)\,dt, \tag{4.18}$$
where $d(p) = c(p)4p'/p^2$.
Moreover the last term can be replaced by $c(p)(p')^{p-1}\int_0^\eta |v'(t)|^pF_\eta(t)^{p-1}\,dt$.

2. *If $1 < p < 2$, then for any $u \in C_c^1((0,\eta])$ and $M \geq 1$ we have*

$$\begin{aligned}\int_0^\eta |u'|^p W_p(t)\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p \\ + c(p)(p')^{p-1}\int_0^\eta |v'(t)|^p F_\eta(t)^{p-1}\chi_{\{|X|>M\}}\,dt \\ + M^{p-2}d(p)\int_0^\eta |(|v(t)|^{p/2})'|^2 F_\eta(t)\chi_{\{|X|\leq M\}}\,dt,\end{aligned} \tag{4.19}$$

where $d(p) = c(p)4p'/p^2$ and $\chi_S(t)$ denotes a characteristic function of a set S.

Proof of Lemma 4.5:
Proof of (4.18): This follows from (4.7) of Lemma 4.2, (4.10), (4.11), (4.12), (4.13) of Lemma 4.3 and Lemma 4.4. By using (4.14) instead of (4.13), the alternative inequality follows.
Proof of (4.19): This follows from (4.8) of Lemma 4.2, (4.10), (4.11), (4.12), (4.13), (4.14) of Lemma 4.3 and Lemma 4.4. □

Remark 4.2. *From (4.18) with the last term replaced by $c(p)(p')^{p-1}\int_0^\eta |v'|^p F_\eta^{p-1}\,dt$, we have the following particular inequality for $p \geq 2$:*

$$\begin{aligned}\int_0^\eta |u'|^p W_p\,dt \geq \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p \\ + \frac{d(p)}{2}\int_0^\eta |(|v(t)|^{p/2})'|^2 F_\eta(t)\,dt + \frac{c(p)(p')^{p-1}}{2}\int_0^\eta |v'(t)|^p F_\eta(t)^{p-1}\,dt.\end{aligned} \tag{4.20}$$

4.2 Proof of Theorem 3.1 Part 1: Proof of the inequality (3.4) and its sharpness

Assume that $w \in W(\mathbf{R}_+) = P(\mathbf{R}_+) \cup Q(\mathbf{R}_+)$. It follows from Lemma 4.5, we clearly have the inequality (3.4) in Theorem 3.1. Hence we proceed to the proof of optimality of the coefficients Λ_p and $(\Lambda_p)^{1/p'}$ in the right-hand side of (3.4). By the density argument, we can adopt as test functions

$$u_\varepsilon(t) = f_\eta(t)^{1/p'+s(w)\varepsilon} \qquad (0 < \varepsilon). \tag{4.21}$$

We note that

$$u_\varepsilon'(t) = \left(1/p' + s(w)\varepsilon\right) f_\eta(t)^{s(w)\varepsilon - 1/p} s(w) w^{-1}. \tag{4.22}$$

Then the left-hand side becomes

$$\begin{aligned}\int_0^\eta |u_\varepsilon'(t)|^p W_p(t)\,dt &= \left(1/p' + s(w)\varepsilon\right)^p \int_0^\eta f_\eta(t)^{s(w)\varepsilon p - 1} w(t)^{-1}\,dt \\ &= \left(1/p' + s(w)\varepsilon\right)^p f_\eta(\eta)^{s(w)\varepsilon p}(p\varepsilon)^{-1}.\end{aligned}$$

In a similar way the right-hand side becomes

$$\begin{aligned}&\Lambda_p \int_0^\eta \frac{|u_\varepsilon(t)|^p W_p(t)}{F_\eta(t)^p}\,dt + s(w)\Lambda_p^{1/p'}\frac{|u_\varepsilon(\eta)|^p}{f_\eta(\eta)^{p-1}} \\ &= \left(1/p'\right)^p \int_0^\eta f_\eta(t)^{s(w)\varepsilon p - 1} w(t)^{-1}\,dt + s(w)\Lambda_p^{1/p'}\frac{|u_\varepsilon(\eta)|^p}{f_\eta(\eta)^{p-1}} \\ &= \left(1/p'\right)^p f_\eta(\eta)^{s(w)\varepsilon p}(p\varepsilon)^{-1} + s(w)\Lambda_p^{1/p'}\frac{|u_\varepsilon(\eta)|^p}{f_\eta(\eta)^{p-1}}.\end{aligned}$$

Finally we reach to

$$\big(1/p' + s(w)\varepsilon\big)^p - \big(1/p'\big)^p \geq s(w)\Lambda_p^{1/p'} p\varepsilon \qquad (0 < \varepsilon), \tag{4.23}$$

and this inequality is clear from the convexity. □

Part 2: Proof of the inequality (3.5)

In order to establish the inequality (3.5) in Theorem 3.1 we shall estimate the term involving $|(|v|^{p/2})'|^2 F_\eta$ from below and use a positive term involving $|v'|^p F_\eta^{p-1}$ to absorb negative error terms. To this end we need more notations:

For $u, v \in C_c^1((0,\eta])$, we retain

$$u(t) = g_\eta(t)v(t) \quad \text{and} \quad X(t) = \frac{g_\eta(t)}{g'_\eta(t)}\frac{v'(t)}{v(t)} = p' F_\eta(t)\frac{v'(t)}{v(t)} s(w) \quad (v(t) \neq 0).$$

By Remark 4.1, it suffices to assume in the proof that u and v belong to the class $G((0,\eta])$ defined by the following:

Definition 4.1.

$$G((0,\eta]) = \{v \in C_c^1((0,\eta]) : v(t) \geq 0 \ \mathit{in}\ (0,\eta]\}. \tag{4.24}$$

Definition 4.2. *For $v \in G((0,\eta])$ and $M > 1$ we define two subsets of $[0,\eta]$ as follows:*

$$\begin{cases} A(v,M) = \{t \in [0,\eta] : \ p' F_\eta(t)|v'(t)| \leq M v(t)\}, \\ B(v,M) = \{t \in [0,\eta] : \ p' F_\eta(t)|v'(t)| > M v(t)\}. \end{cases} \tag{4.25}$$

$A(v,M)$ and $B(v,M)$ are sometimes abbreviated as A and B respectively.

Remark 4.3. *From Definition 4.1 and Definition 4.2 we see $A \cup B = [0,\eta]$. If $v(t) \in G((0,\eta])$ and $v(s) = 0$ for some $s \in [0,\eta)$, then $v'(s) = 0$, and hence $s \in A(v,M)$ for any $M > 0$. We note that the set $C(v,M) = \{t \in [0,\eta] : \ p' F_\eta(t)|v'(t)| = M|v(t)|\}$ coincides with the set of critical points of $\Psi(t) := \log(|v| f_\eta^{\pm M/p'})$. Namely, $\{t \in [0,\eta] : \Psi'(t) = 0\}$.*

The following lemma will be established in Appendix.

Lemma 4.6. *Assume that $\eta > 0$, $\mu > 0$, $M > 1$ and $w \in W(\mathbf{R}_+)$. For any $v \in G((0,\eta])$ we have the followings:*

1. *For $A = A(v,M)$ and $B = B(v,M)$,*

$$\begin{aligned} \int_A |(v(t)^{p/2})'|^2 F_\eta(t)\,dt &\geq -\frac{(\Lambda_p)^{1/p'}}{2\mu f_\eta(\eta)^{p-1}}|u(\eta)|^p + \frac{1}{4(p')^{p-1}}\int_A \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt \\ &+ \frac{1}{2(p')^{p-1}}\int_B \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt - \frac{p}{2\mu}\left(\frac{p'}{M}\right)^{p-1}\int_B |v'(t)|^p F_\eta(t)^{p-1}\,dt. \end{aligned} \tag{4.26}$$

2.

$$\begin{aligned} \int_0^\eta |(v(t)^{p/2})'|^2 F_\eta(t)\,dt \geq &-\frac{(\Lambda_p)^{1/p'}}{2\mu f_\eta(\eta)^{p-1}}|u(\eta)|^p \\ &+ \frac{1}{4(p')^{p-1}}\int_0^\eta \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt. \end{aligned} \tag{4.27}$$

Remark 4.4. *The assertion 2 follows from Lemma 8.1 with $S = [0,\eta]$. The inequality (3.5) will follow from this lemma together with Lemma 4.5.*

End of the proof of (3.5): If $p \geq 2$, then for any $u \in G((0,\eta])$ we have by using (4.27) of Lemma 4.6

$$\begin{aligned}\int_0^\eta |u'(t)|^p W_p(t)\,dt &\geq s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p + \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} \\ &+ d(p)\left(-\frac{(\Lambda_p)^{1/p'}}{2\mu f_\eta(\eta)^{p-1}}|u(\eta)|^p + \frac{1}{4(p')^{p-1}}\int_0^\eta \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt\right) \\ &= s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}\left(1 - s(w)\frac{d(p)}{2\mu}\right)|u(\eta)|^p + \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p}\,dt \\ &+ \frac{d(p)}{4(p')^{p-1}}\int_0^\eta \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt.\end{aligned}$$

Here we assume that $d(p)$ is so small that $d(p)/\mu < 1$ if $w \in Q(\mathbf{R}_+)$. Then we get (3.5) with

$$\begin{cases} L = (\Lambda_p)^{1/p'} f_\eta(\eta)^{1-p}\left(1 - s(w)d(p)/(2\mu)\right), \\ \\ C = d(p)(\Lambda_p)^{1/p'}/4. \end{cases} \tag{4.28}$$

If $1 < p < 2$, then for any $u \in G((0,\eta])$ and $M \geq 1$ we have

$$\begin{aligned}\int_0^\eta |u'|^p W_p\,dt &\geq s(w)\frac{(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}|u(\eta)|^p + \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} \\ &+ c(p)(p')^{p-1}\int_{B(v,M)} |v'(t)|^p F_\eta(t)^{p-1}\,dt + M^{p-2} d(p)\int_{A(v,M)} |(|v(t)|^{p/2})'|^2 F_\eta\,dt.\end{aligned}$$

Here $c(p)$ is a positive number independent of each u and $d(p) = c(p)4p'/p^2$. For the last term we use (4.26) of Lemma 4.6, then we have

$$\begin{aligned}\int_0^\eta |u'(t)|^p W_p(t)\,dt &\geq \frac{s(w)(\Lambda_p)^{1/p'}}{f_\eta(\eta)^{p-1}}\left(1 - s(w)\frac{d(p)M^{p-2}}{2\mu}\right)|u(\eta)|^p \\ &+ \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + c(p)(p')^{p-1}\left(1 - \frac{2}{\mu(p-1)M}\right)\int_{B(v,M)} |v'(t)|^p F_\eta(t)^{p-1}\,dt \\ &+ \frac{d(p)M^{p-2}(\Lambda_p)^{1/p'}}{2}\left(\frac{1}{2}\int_{A(v,M)} \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt + \int_{B(v,M)} \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt\right).\end{aligned}$$

Then we take a sufficiently large M so that we have $1 - 2/(\mu(p-1)M) > 0$ and $d(p)M^{p-2}/(2\mu) < 1$ if $w \in Q(\mathbf{R}_+)$. Then

$$\int_0^\eta |u'(t)|^p W_p(t)\,dt \geq s(w)L|u(\eta)|^p + \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} + C\int_0^\eta \frac{|u(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\,dt,$$

where

$$\begin{cases} L = (\Lambda_p)^{1/p'} f_\eta(\eta)^{1-p}\left(1 - s(w)d(p)M^{p-2}/(2\mu)\right), \\ \\ C = d(p)M^{p-2}(\Lambda_p)^{1/p'}/4. \end{cases} \tag{4.29}$$

$\square$

4.3 Proof of Theorem 3.2 Let $v \in G((0,\eta])$. Recall that $u(t) = g_\eta(t)v(t)$, $u'(t) = g'_\eta(t)v(t)\,(1+X(t))$ $(v \neq 0)$ and

$$X(t) = \frac{g_\eta(t)}{g'_\eta(t)}\frac{v'(t)}{v(t)} = p'F_\eta(t)\frac{v'(t)}{v(t)}s(w) \quad (v(t) \neq 0); 0 \quad (v(t) = 0).$$

Then we have two elementary lemmas.

Lemma 4.7. *For any $X \in \mathbf{R}$ and any $M > 1$,*

$$|1+X|^p \leq \begin{cases} (1+M)^p, & |X| \leq M, \\ 2^p|X|^p, & |X| > M. \end{cases} \tag{4.30}$$

Lemma 4.8. *For any $M > 1$,*

$$|u'(t)|^p W_p(t) \leq \begin{cases} \Lambda_p(1+M)^p u(t)^p W_p(t) F_\eta(t)^{-p}, & |X(t)| \leq M, \\ 2^p \Lambda_p^{-1/p'} |v'(t)|^p F_\eta(t)^{p-1}, & |X(t)| > M. \end{cases} \tag{4.31}$$

Proof: Since $u'(t) = g'_\eta(t)v(t)\,(1+X(t))$, we get

$$|u'(t)|^p W_p(t) \leq |g'_\eta(t)|^p v(t)^p |1+X(t)|^p W_p(t).$$

If $|X(t)| \leq M$, then from Lemma 4.7 and (4.12) we get the desired estimate. If $|X(t)| \geq M$, then from Lemma 4.7 and (4.14) we get the desired one. □

By Proposition 2.1 we have the following.

Lemma 4.9. *Assume that w is admissible in the sense of Definition 2.6. Then, there are some positive numbers K and η such that we have for any $u \in C_c^1(0,\eta])$*

$$\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} t\,dt \leq K^2 \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p G_\eta(t)^2}. \tag{4.32}$$

Proof of Theorem 3.2: For a sufficiently small $\eta > 0$, temporally we set

$$Q_L(u) = \int_0^\eta |u'(t)|^p W_p(t)\,dt - \Lambda_p \int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p} - s(w)L|u(\eta)|^p, \tag{4.33}$$

where L is a positive constant independent of each u, which will be specified later. In order to prove Theorem 3.2, it suffices to control three terms below by $Q_L(u)$:

$$\int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p G_\eta(t)^2}, \quad \int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} t\,dt \quad \text{and} \quad \int_0^\eta |u'(t)|^p W_p(t)\,t\,dt.$$

It follows form Theorem 3.1 (3.5) that we have for $D = C^{-1}$

$$\int_0^\eta \frac{|u(t)|^p W_p(t)\,dt}{F_\eta(t)^p G_\eta(t)^2} \leq D \cdot Q_L(u),$$

where $L = L(w,p,\eta,\mu)$ is the same constant in (3.5). From this and Lemma 4.9 we also have

$$\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} t\,dt \leq K^2 D \cdot Q_L(u).$$

By using Lemma 4.8 we see that

$$\int_0^\eta |u'(t)|^p W_p(t)\,t\,dt \le \Lambda_p(1+M)^p \int_{|X|\le M} \frac{u(t)^p W_p(t)}{F_\eta(t)^p}\,t\,dt \\ +2^p \Lambda_p^{1/p-1} \int_{|X|>M} |v'(t)|^p F_\eta(t)^{p-1}\,t\,dt.$$

Then, the first term in the right-hand side can be controlled by the same $Q_L(u)$. Now we claim that for some positive number K'

$$\int_{|X|>M} |v'(t)|^p F_\eta(t)^{p-1} t\,dt \le K' \cdot Q_L(u), \tag{4.34}$$

where $L = (\Lambda_p)^{1/p'} f_\eta(\eta)^{1-p}$.
First we assume that $p \ge 2$. By (4.18) of Lemma 4.5 with the last term replaced by

$$c(p)(p')^{p-1} \int_0^\eta |v'(t)|^p F_\eta(t)^{p-1}\,dt,$$

we see that (4.34) is valid, provided that η is sufficiently small (See also Remark 4.2). Secondly we assume that $1 < p < 2$. Then in a similar way the assertion follows from (4.19) of Lemma 4.5. As a result we have the desired inequality (3.15) for some positive numbers C_0, C_1 and L which are independent of each u. □

5 Proofs of Theorem 3.3 and Corollary 3.3 We establish Theorem 3.3 and Corollary 3.3 using Theorem 3.2.

Proof of Theorem 3.3: Let us prepare some notations and fundamental facts. Define $\Sigma = \partial\Omega$ and $\Sigma_t = \{x \in \Omega : \delta(x) = t\}$. Since Σ is is of class C^2, there exists an $\eta_0 > 0$ such that we have a C^2 diffeomorphism $G : \Omega_\eta \mapsto (0,\eta)\times\Sigma$ for any $\eta \in (0,\eta_0)$. By $G^{-1}(t,\sigma)\,((t,\sigma) \in (0,\eta_0)\times\Sigma)$ we denote the inverse of G. Let H_t denote the mapping $G^{-1}(t,\cdot)$ of Σ onto Σ_t. This mapping is also a C^2 diffeomorphism and its Jacobian is close to 1 in $(0,\eta_0)\times\Sigma$. Therefore, for every non-negative continuous function u on $\overline{\Omega_\eta}$ with $\eta \in (0,\eta_0)$ we have

$$\int_{\Omega_\eta} u = \int_0^\eta dt \int_{\Sigma_t} u\,d\sigma_t = \int_0^\eta dt \int_\Sigma u(H_t(\sigma))(\mathrm{Jac\,H_t})\,\mathrm{d}\sigma, \tag{5.1}$$

$$|\mathrm{Jac\,H_t}(\sigma) - 1| \le \mathrm{ct}, \quad \text{for every } (\mathrm{t},\sigma) \in (0,\eta_0)\times\Sigma, \tag{5.2}$$

where c is a positive constant independent of each (t,σ), $d\sigma$ and $d\sigma_t$ denote surface elements on Σ and Σ_t respectively. Then we have

$$\int_\Sigma u(H_\eta(\sigma))(1-c\eta)\,d\sigma \le \int_{\Sigma_\eta} u\,d\sigma_\eta \le \int_\Sigma u(H_\eta(\sigma))(1+c\eta)\,d\sigma. \tag{5.3}$$

Again for the sake of simplicity, we denote $W_p(\delta(x))$ and $F_\eta(\delta(x))$ by $W_p(\delta)$ and $F_\eta(\delta)$ respectively, and the symbol dx is often abbreviated. Then we immediately have for $v = u(H_t(\sigma))$

$$\int_\Sigma d\sigma \int_0^\eta \left|\frac{\partial v}{\partial t}\right|^p (1-ct)W_p(t)\,dt \le \int_{\Omega_\eta} |\nabla u|^p W_p(\delta)\,dx,$$

$$\int_\Sigma d\sigma \int_0^\eta \frac{|v|^p W_p(t)}{F_\eta(t)^p}(1-ct)\,dt \le \int_{\Omega_\eta} \frac{|u|^p W_p(\delta)\,dx}{F_\eta(\delta)^p} \le \int_\Sigma d\sigma \int_0^\eta \frac{|v|^p W_p(t)}{F_\eta(t)^p}(1+ct)\,dt.$$

Proof of (3.18): Under these consideration, (3.18) is reduced to the following one dimensional Hardy's inequality. Setting $v(t) = u(H_t(\sigma))$ and $v' = \partial v/\partial t$ we have

$$\begin{aligned} \int_0^\eta |v'(t)|^p W_p(t)(1-ct)\,dt &\geq \Lambda_p \int_0^\eta \frac{|v(t)|^p W_p(t)}{F_\eta(t)^p}(1+ct)\,dt \\ &+ C\int_0^\eta \frac{|v(t)|^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}(1+ct)\,dt + s(w)L'|v(\eta)|^p W_p(\eta)(1+s(w)c\eta). \end{aligned} \tag{5.4}$$

Equivalently we have

$$\begin{aligned} &\int_0^\eta \left(|v'(t)|^p - \frac{|v(t)|^p}{F_\eta(t)^p}\left(\Lambda_p + \frac{C}{G_\eta(t)^2}\right)\right) W_p(t)\,dt \\ &\geq c\int_0^\eta \left(|v'(t)|^p + \frac{|v(t)|^p}{F_\eta(t)^p}\left(\Lambda_p + \frac{C}{G_\eta(t)^2}\right)\right) W_p(t)\,t\,dt \\ &+ s(w)L'|v(\eta)|^p W_p(\eta)(1+s(w)c\eta). \end{aligned} \tag{5.5}$$

Assume that $0 < \eta \leq 1/(2c)$. Then $1 - c\eta \geq 1/2$ and (5.5) clearly follows from Theorem 3.2 with $C_0 = C$, $C_1 = c$ and $L = L'W_p(\eta)(1+s(w)c\eta)$. □

Proof of Corollary 3.3: First we treat the case that $w \in P_A(\mathbf{R}_+)$. Assume that Hardy's inequality (3.19) does not hold. Then there exists a sequence of functions $\{u_k\} \subset W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$ such that

$$\begin{cases} \lim_{k\to\infty}\left(\int_\Omega |\nabla u_k|^p W_p(\delta)\,dx + \int_{\Sigma_\eta} |u_k|^p W_p(\delta)\,d\sigma_\eta\right) = 0, \\ \int_\Omega |u_k|^p W_p(\delta)/F_\eta^p(\delta)\,dx = 1 \quad (k = 1,2,\cdots). \end{cases} \tag{5.6}$$

For a sufficiently small $\eta > 0$, let $W^{1,p}(\Omega\setminus\overline{\Omega_\eta}; W_p(\delta))$ be given by the completion of $C^\infty(\Omega\setminus\overline{\Omega_\eta})$ with respect to the norm defined by

$$\|u\|_{W^{1,p}(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))} = \||\nabla u|\|_{L^p(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))} + \|u\|_{L^p(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))}.$$

Since $W_p(\delta) > 0$ in $\overline{\Omega\setminus\Omega_\eta}$, $W^{1,p}(\Omega\setminus\overline{\Omega_\eta}; W_p(\delta))$ is well-defined and becomes a Banach space with the norm $\|\cdot\|_{W^{1,p}(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))}$. We note that $\int_{\Sigma_\eta} |u_k|^p W_p(\delta)\,d\sigma_\eta$ $(k = 1,2,3\dots)$ is bounded, because the trace operator $T : W^{1,p}(\Omega\setminus\overline{\Omega_\eta}; W_p(\delta)) \mapsto L^p(\Sigma_\eta; W_p(\eta))$ is continuous. By Theorem 3.3 we have

$$\begin{aligned} &\int_\Omega |\nabla u_k|^p W_p(\delta)\,dx = \int_{\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx \\ &\geq \Lambda_p\left(1 - \int_{\Omega\setminus\Omega_\eta} \frac{|u_k|^p W_p(\delta)}{F_\eta^p(\delta)}\,dx\right) + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx - L'\int_{\Sigma_\eta} |u_k|^p W_p(\delta)\,d\sigma_\eta. \end{aligned}$$

Since $\delta \geq \eta$ in $\Omega\setminus\Omega_\eta$, by the standard argument we have $u_k \to C\,(constant)$ in $W^{1,p}(\Omega\setminus\overline{\Omega_\eta}; W_p(\delta))$ as $k \to \infty$. By (5.6) we have $C = 0$. Hence we see $0 \geq \Lambda_p$, and we reach to a contradiction.

Secondly we treat the case that $w \in Q(\mathbf{R}_+)$. In this case we assume that

$$\lim_{k\to\infty}\int_\Omega |\nabla u_k|^p W_p(\delta)\,dx = 0, \qquad \int_\Omega \frac{|u_k|^p W_p(\delta)}{F_\eta^p(\delta)}\,dx = 1 \quad (k = 1,2,\cdots). \tag{5.7}$$

Then, by Theorem 3.3 we have

$$\begin{aligned} &\int_\Omega |\nabla u_k|^p W_p(\delta)\,dx = \int_{\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx \\ &\geq \Lambda_p\left(1 - \int_{\Omega\setminus\Omega_\eta} \frac{|u_k|^p W_p(\delta)}{F_\eta^p(\delta)}\,dx\right) + \int_{\Omega\setminus\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx + L'\int_{\Sigma_\eta} |u_k|^p W_p(\delta)\,d\sigma_\eta. \end{aligned} \tag{5.8}$$

Again we have $u_k \to C\,(constant)$ in $W^{1,p}(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))$ as $k\to\infty$. By (5.7) and (5.8) we have $C=0$. Hence we see $0\geq\Lambda_p$, and this is a contradiction.

$\square$

6 Proof of Theorem 3.4 By virtue of Corollary 3.3 and its proof, it suffices to show the implication $1\to 2$. Since $F_\eta^{-1}\notin L^1((0,\eta))$, for an arbitrary $\varepsilon>0$ we have $G_\eta(\delta)^{-1}<\varepsilon$ in Ω_η provided that η is sufficiently small. Therefore we assume that $C=0$ without the loss of generality.

Here we introduce a Banach space $W_0^{1,p}(\Omega_\eta;W_p(\delta))$ which corresponds to $W_0^{1,p}((0,\eta];W_p)$ defined in Subsection 3.1. Let $\tilde{C}_c^\infty(\Omega_\eta)=\{u|_{\Omega_\eta}: u\in C_c^\infty(\Omega)\}$, where by $u|_{\Omega_\eta}$ we denote the restriction of u to Ω_η. Let $W_0^{1,p}(\Omega_\eta;W_p(\delta))$ be given by the completion of $\tilde{C}_c^\infty(\Omega_\eta)$ with respect to the norm defined by

$$\|u\|_{W_0^{1,p}(\Omega_\eta;W_p(\delta))}=\||\nabla u|\|_{L^p(\Omega_\eta;W_p(\delta))}+\|u\|_{L^p(\Omega_\eta;W_p(\delta))}.$$

Then, $W_0^{1,p}(\Omega_\eta;W_p(\delta))$ is a Banach space with the norm $\|\cdot\|_{W_0^{1,p}(\Omega_\eta;W_p(\delta))}$. Then we prepare a lemma on extension:

Lemma 6.1. (Extension) *Assume that η_0 is a sufficiently small positive number. Then for any $\eta\in(0,\eta_0)$ there exists an extension operator $E=E(\eta): W_0^{1,p}(\Omega_\eta;W_p(\delta))\mapsto W_0^{1,p}(\Omega;W_p(\delta))$ such that:*

1. *$E(u)=u$ a.e. in Ω_η*
2. *There exists some positive number $C=C(\eta)$ such that for any $u\in W_0^{1,p}(\Omega_\eta;W_p(\delta))$*

$$\||\nabla E(u)|\|_{L^p(\Omega;W_p(\delta))}\leq C\Big(\||\nabla u|\|_{L^p(\Omega_{\eta/2};W_p(\delta))}+\|u\|_{W^{1,p}(\Omega_\eta\setminus\overline{\Omega_{\eta/2}};W_p(\delta))}\Big).$$

Admitting this for the moment, we prove Theorem 3.4. First we treat the case that $w\in P(\mathbf{R}_+)$. Then we see that $W_p/F_\eta^p\in L^1(0,\eta)$. In fact we have $\int_0^\eta W_p(t)/F_\eta(t)^p\,dt=\mu^{1-p}/(p-1)$. Now we assume that Hardy's inequality (3.18) with $C=0$ does not hold. Then there exists a sequence of functions $\{u_k\}\subset W_0^{1,p}(\Omega,W_p(\delta))\cap C(\Omega)$ satisfying

$$\begin{cases} \lim_{k\to\infty}\Big(\int_{\Omega_\eta}|\nabla u_k|^pW_p(\delta)\,dx+\int_{\Sigma_\eta}|u_k|^pW_p(\delta)\,d\sigma_\eta\Big)=0, \\ \int_{\Omega_\eta}|u_k|^pW_p(\delta)/F_\eta^p(\delta)\,dx=1 \quad (k=1,2,\cdots). \end{cases} \tag{6.1}$$

By $w_k=u_k|_{\Omega_\eta}$ we denote the restriction of u_k to Ω_η. Then $E(w_k)\in W_0^{1,p}(\Omega;W_p(\delta))$ for $k=1,2,\ldots$. On the other hand, it follows from (6.1) that $w_k=u_k\to C$ a.e. in Ω_η for some constant C as $k\to\infty$. From (6.1) we have $C=0$. Then, by the assumption 1 and the continuity of the trace operator $T: W^{1,p}(\Omega\setminus\overline{\Omega_\eta};W_p(\delta))\mapsto L^p(\Sigma_\eta;W_p(\delta))$ for a small $\eta>0$, we have

$$\begin{aligned} 1&\leq\int_\Omega\frac{|E(w_k)|^pW_p(\delta)}{F_\eta^p(\delta)}\,dx \\ &\leq\gamma^{-1}\left(\int_\Omega|\nabla E(w_k)|^pW_p(\delta)\,dx+L'\int_{\Sigma_\eta}|w_k|^pW_p(\delta)\,d\sigma_\eta\right) \\ &\leq\gamma^{-1}C'\Big(\||\nabla u_k|\|_{L^p(\Omega_{\eta/2};W_p(\delta))}+\|u_k\|_{W^{1,p}(\Omega_\eta\setminus\overline{\Omega_{\eta/2}};W_p(\delta))}\Big)^p\to 0, \quad \text{as } k\to\infty, \end{aligned} \tag{6.2}$$

where C' is some positive number independent of each u_k. But this is a contradiction.

Secondly we treat the case that $w \in Q(\mathbf{R}_+)$. Since $\int_0^\eta W_p(t)/F_\eta(t)^p\,dt = +\infty$, we see that $W_p/F_\eta^p \notin L^1(0,\eta)$. There is a sequence of functions $\{u_k\} \subset W_0^{1,p}(\Omega; W_p(\delta)) \cap C(\Omega)$ satisfying

$$\lim_{k\to\infty} \int_{\Omega_\eta} |\nabla u_k|^p W_p(\delta)\,dx = 0, \quad \int_{\Omega_\eta} \frac{|u_k|^p W_p(\delta)}{F_\eta^p(\delta)}\,dx = 1 \quad (k = 1,2,\cdots). \tag{6.3}$$

As before we see that $u_k \to C$ (constant) in Ω_η. From (6.3) we have $C = 0$. Then, by the assumption 1 and the continuity of the trace operator T, we have

$$1 \le \int_\Omega \frac{|E(w_k)|^p W_p(\delta)}{F_\eta^p(\delta)}\,dx \tag{6.4}$$

$$\le \gamma^{-1}\left(\int_\Omega |\nabla E(w_k)|^p W_p(\delta)\,dx - L' \int_{\Sigma_\eta} |w_k|^p W_p(\delta)\,d\sigma_\eta\right) \tag{6.5}$$

$$\le \gamma^{-1} C' \left(\| |\nabla u_k| \|_{L^p(\Omega_{\eta/2}; W_p(\delta))} + \|u_k\|_{W^{1,p}(\Omega_\eta \setminus \overline{\Omega_{\eta/2}}; W_p(\delta))} \right)^p \to 0, \quad \text{as } k \to \infty,$$

where C' is some positive number. But this is a contradiction. □

Proof of Lemma 6.1: Since η is small and δ is Lipschitz continuous, $\partial\Omega_\eta$ and $\partial\Omega_{\eta/2}$ are Lipschitz compact manifolds. By the standard theory (Theorem 1 in Section 5.4 of [7] for example) we have an extension operator $\tilde{E} : W^{1,p}(\Omega_\eta \setminus \overline{\Omega_{\eta/2}}; W_p(\delta)) \mapsto W^{1,p}(\Omega \setminus \overline{\Omega_{\eta/2}}; W_p(\delta))$ such that $\tilde{E}(u) = u$ a.e. in $\Omega_\eta \setminus \overline{\Omega_{\eta/2}}$, and

$$\| |\nabla \tilde{E}(u)| \|_{L^p(\Omega \setminus \overline{\Omega_{\eta/2}}; W_p(\delta))} \le C(\eta) \|u\|_{W^{1,p}(\Omega_\eta \setminus \overline{\Omega_{\eta/2}}; W_p(\delta))}.$$

Define for $u \in W_0^{1,p}(\Omega_\eta; W_p(\delta))$

$$E(u) = u, \, (x \in \overline{\Omega_{\eta/2}},); \quad \tilde{E}(u), \, (x \in \Omega \setminus \overline{\Omega_{\eta/2}}). \tag{6.6}$$

Then the assertion follows. □

7 Proofs of Propositions 3.1 and 3.2 Proposition 3.1 is known in a more general fashion. In fact a variant is seen in Maz'ya [13] (Lemma 2, p144). For the sake of reader's convenience we give an elementary verification. We note that Proposition 3.2 is a direct consequence of Proposition 3.1.

Proof of Proposition 3.1: For $w \in P(\mathbf{R}_+)$ and $\overline{\varepsilon} \in (0, \eta/2)$, define

$$\varphi_{\overline{\varepsilon}} = \ 0 \ (0 \le t \le \overline{\varepsilon}); \ \frac{f_\eta(\overline{\varepsilon}) - f_\eta(t)}{f_\eta(\overline{\varepsilon}) - f_\eta(\eta/2)} \ (\overline{\varepsilon} \le t \le \eta/2); \ 1 \ (\eta/2 \le t \le \eta). \tag{7.1}$$

Noting that $f_\eta(\overline{\varepsilon}) = \mu + \int_{\overline{\varepsilon}}^\eta 1/w(s)\,ds \to +\infty$ as $\overline{\varepsilon} \to +0$, we see that $\int_0^\eta |\varphi_{\overline{\varepsilon}}'|^p W_p(t)\,dt = (f_\eta(\overline{\varepsilon}) - f_\eta(\eta/2))^{1-p} \to 0$ as $\overline{\varepsilon} \to +0$. On the other hand we have $\varphi_{\overline{\varepsilon}}(0) = 0, \varphi_{\overline{\varepsilon}}(1) = 1$ and hence the assertion is now clear. Further we note that

$$\int_{\overline{\varepsilon}}^\eta |\varphi_{\overline{\varepsilon}}|^p \frac{W_p(t)}{F_\eta(t)^p}\,dt \ge \int_{\eta/2}^\eta \frac{W_p(t)}{F_\eta(t)^p}\,dt = \frac{f_\eta(\eta)^{1-p} - f_\eta(\eta/2)^{1-p}}{p-1} > 0 \quad \text{as } \overline{\varepsilon} \to +0.$$

□

Proof of Proposition 3.2: If a positive number η_0 is sufficiently small, then one can assume that $\delta \in C^2(\Omega_{\eta_0})$, $|\nabla\delta| = 1$ in Ω_{η_0} and a manifolds $\{x \in \Omega; \delta = \eta\}$ is of C^2 class for $\eta \in (0, \eta_0]$. Let $\varphi_{\overline{\varepsilon}}$ be defined by (7.1). By virtue of (5.3) we have

$$\int_{\Omega_\eta} |\nabla \varphi_{\overline{\varepsilon}}(\delta(x))|^p W_p(\delta(x))\,dx \le \int_\Sigma d\sigma \int_0^\eta \left|\varphi_{\overline{\varepsilon}}'(t)\right|^p W_p(t)(1+ct)\,dt,$$

hence the assertion follows from Proposition 3.1. □

8 Appendix

8.1 Proof of Lemma 4.6

Let $v \in G((0,\eta])$. We recall that

$$g_\eta = (p' f_\eta)^{1/p'}, \quad F_\eta = w f_\eta, \quad G_\eta = \mu + \int_t^\eta \frac{ds}{F_\eta(s)} \quad \text{and} \quad u = g_\eta v.$$

Let us prepare more notations. Let us set

$$v^{p/2} = z, \quad z = a\varphi \quad \text{and} \quad a = (G_\eta)^{1/2}, \tag{8.1}$$

where φ is some function in $C^1((0,\eta])$ with $\varphi(0) = 0$.
Then we have

$$z' = \varphi' a + \varphi a' \quad \text{and} \quad |z'|^2 = (\varphi')^2 a^2 + \frac{1}{2}(\varphi^2)'(a^2)' + \varphi^2 (a')^2.$$

Noting that $a^2 = G_\eta$, $(a^2)' = -1/F_\eta$ and $a' = -1/(2F_\eta a)$, we have

$$|z'|^2 F_\eta = (\varphi')^2 F_\eta G_\eta - \frac{1}{2}(\varphi^2)' + \frac{\varphi^2}{4F_\eta G_\eta} \geq -\frac{1}{2}(\varphi^2)' + \frac{\varphi^2}{4F_\eta G_\eta},$$

and we also have

$$\varphi^2 = \frac{v^p}{a^2} = \frac{u^p}{g_\eta^p G_\eta} = \frac{1}{(p')^{p-1}} \frac{u^p W_p}{F_\eta^{p-1} G_\eta}.$$

As a result, we have the following that is valid for any measurable set $S \subset [0,\eta]$.

Lemma 8.1. *Let S be a measurable set contained in $[0,\eta]$. Assume that $\eta > 0$, $\mu > 0$ and $w \in W(\mathbf{R}_+)$. For any nonnegative $v \in G([0,\eta])$*

$$\int_S |(v(t)^{p/2})'|^2 F_\eta(t)\, dt \geq -\frac{1}{2}\int_S (\varphi(t)^2)'\, dt + \frac{1}{4(p')^{p-1}} \int_S \frac{u(t)^p W_p}{F_\eta(t)^p G_\eta(t)^2}\, dt. \tag{8.2}$$

Proof of Lemma 4.6: By using Lemma 8.1 with $S = [0,\eta]$ we immediately have (4.27). Then we proceed to (4.26). We assume that $A = A(v,M)$ and $B = B(v,M)$ are defined by Definition 4.2. As for the first term of the right-hand side, we have

$$\begin{aligned} -\frac{1}{2}\int_A (\varphi(t)^2)'\, dt &= -\frac{1}{2}\int_0^\eta (\varphi(t)^2)'\, dt + \frac{1}{2}\int_B (\varphi(t)^2)'\, dt \\ &= -\frac{1}{2}\varphi(\eta)^2 - \int_B \frac{z(t)^2 a'(t)}{a(t)^3} + \int_B \frac{z(t) z'(t)}{a(t)^2}. \end{aligned}$$

Moreover we see that:

1. $$\varphi(\eta)^2 = \frac{z(\eta)^2}{a(\eta)^2} = \frac{1}{\mu}\frac{|u(\eta)|^p}{g_\eta(\eta)^p} = \frac{|u(\eta)|^p}{\mu (p')^{p-1} f_\eta(\eta)^{p-1}}. \tag{8.3}$$

2. $$-\int_B \frac{z(t)^2 a(t)'}{a(t)^3}\, dt = \frac{1}{2}\int_B \frac{z(t)^2}{a(t)^4 F_\eta(t)}\, dt = \frac{1}{2(p')^{p-1}} \int_B \frac{u(t)^p W_p(t)}{F_\eta(t)^p G_\eta(t)^2}\, dt. \tag{8.4}$$

3. Noting that $zz' = pv^{p-1}v'/2$, $|v| \le p'|v'|F_\eta/M$ in B and $G_\eta \ge \mu$, we have

$$\left|\int_B \frac{z(t)z'(t)}{a(t)^2}\,dt\right| \le \frac{p}{2}\int_B \frac{\left(\frac{p'}{M}|v'(t)|F_\eta(t)\right)^{p-1}|v'|}{G_\eta(t)}\,dt$$
$$\le \frac{p}{2\mu}\left(\frac{p'}{M}\right)^{p-1}\int_B |v'(t)|^p F_\eta(t)^{p-1}\,dt.$$

Finally we get the desired inequality (4.26) which proves Lemma 4.6. □

8.2 Auxiliary inequalities in the noncritical case If we restrict ourselves to the case that $w \in Q(\mathbf{R}_+)$, then Hardy's inequality (8.6) follows from the next simple lemma which is provided in [5] for $p = 2$ and $W_p = w = 1$.

Lemma 8.2. ($w \in Q(\mathbf{R}_+)$) *Assume that* $f \in C([0,\eta]) \cap C^1((0,\eta])$ *is a monotone nondecreasing function such that* $f(\eta) \le 1$. *Assume that* $1 < p < \infty$ *and* $w \in Q(\mathbf{R}_+)$. *Then for every* $u \in C_c^1((0,\eta])$, *we have*

$$\int_0^\eta \left(|u'(t)|^p - \Lambda_p \frac{|u(t)|^p}{F_\eta(t)^p}\right) W_p(t)\,dt \ge \int_0^\eta \left(|u'(t)|^p - \Lambda_p \frac{|u(t)|^p}{F_\eta(t)^p}\right) W_p(t) f(t)\,dt. \tag{8.5}$$

In particular we have

$$\int_0^\eta |u'(t)|^p W_p(t)\,dt \ge \Lambda_p \int_0^\eta \frac{|u(t)|^p}{F_\eta(t)^p} W_p(t)\,dt. \tag{8.6}$$

Proof of Lemma 8.2: Without loss of generality we assume that $f \ge 0$, $f(\eta) = 1$, and $u \ge 0$. Define $g = 1 - f$. Then $g \ge 0$ and $g' \le 0$. Noting that $u \in C_c^1((0,\eta])$ and

$$\frac{d}{dt}\left(\int_0^t \frac{1}{w(s)}\,ds\right)^{1-p} = (1-p)\frac{W_p(t)}{F_\eta(t)^p},$$

by integration by parts we have

$$(p-1)\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} g\,dt$$
$$= \int_0^\eta u(t)^p \left(\int_0^t \frac{1}{w(s)}\,ds\right)^{1-p} g'(t)\,dt + p\int_0^\eta u(t)^{p-1}u'(t)g(t)\left(\int_0^t \frac{1}{w(s)}\,ds\right)^{1-p} dt.$$

Since $g' = -f' \le 0$ and $g \ge 0$,

$$\frac{1}{p'}\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta^p(t)} g(t)\,dt \le \int_0^\eta u(t)^{p-1}u'(t)g(t)\left(\int_0^t \frac{1}{w(s)}\,ds\right)^{1-p} dt.$$

By Hölder's inequality, we have

$$\int_0^\eta u(t)^{p-1}u'(t)g(t)\left(\int_0^t \frac{1}{w(s)}\,ds\right)^{1-p} dt$$
$$\le \left(\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} g(t)\,dt\right)^{1/p'} \left(\int_0^\eta |u'(t)|^p W_p(t) g(t)\,dt\right)^{1/p}.$$

Hence we have

$$\frac{1}{p'}\left(\int_0^\eta \frac{u(t)^p W_p(t)}{F_\eta(t)^p} g(t)\,dt\right)^{1/p} \le \left(\int_0^\eta |u'(t)|^p W_p(t) g(t)\,dt\right)^{1/p}.$$

Using $g = 1 - f$ and the definition of Λ_p, we have (8.5). □

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(WEAK) PRECOMPACTNESS AND LOCAL (WEAK) PRECOMPACTNESS IN PREUNIFORM CONVERGENCE SPACES

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ABSTRACT. In this article, we introduce and study (weak) precompactness and local (weak) precompactness within the construct of preuniform convergence spaces and uniformly continuous maps. In this construct, the concepts of (weak) compactness and local (weak) compactness were previously investigated by Preuß (2010). We provide a comprehensive implication scheme between all beforementioned concepts of compactness in the realm of preuniform convergence spaces. We also establish the relationship between the concept of (weak) precompactness when applied to quasiuniform spaces and existing concepts for quasiuniform spaces, such as totally boundedness and Cauchy-boundedness. We prove Tychonoffs theorem with respect to (weakly) precompact and locally (weakly) precompact preuniform convergence spaces. Moreover, it turns out that the construct of all locally precompact preuniform convergence spaces (and uniformly continuous maps) is a topological universe. Finally, we prove that one needs not to distinguish between locally (weakly) compact and (weakly) compactly generated resp. locally (weakly) precompact and (weakly) precompactly generated preuniform convergence spaces. Our findings can be considered as generalizations of known theorems in the realm of quasiuniform spaces and semiuniform convergences spaces.

Dedicated to the memory of my great teacher Gerhard Preuß (1940 – 2011).

1. INTRODUCTION

Common concepts such as compactness, precompactness, its localizations and (pre) compact generation were already introduced and studied in the construct **SUConv** of semiuniform convergence spaces and uniformly continuous maps by Preuß [5, 4, 3]. In contrast to the construct **SUConv**, the construct **PUConv** of preuniform convergence spaces and uniformly continuous maps also covers non-symmetric topological constructs (see [6] for an overview). Preuniform convergence spaces do not only have an underlying symmetric Kent-convergence space but also an underlying generalized convergence space. Moreover, there are two kinds of Cauchy-filters. For this reason, additional concepts of compactness can be obtained: weak compactness, weak precompactness, its localizations and weak precompact generation.

(Weak) compactness and local (weak) compactness were already investigated by Preuß [7] in the realm of (fuzzy) preuniform convergence spaces. In this article, however, we focus on (weak) precompactness, local (weak) precompactness, (weak) compact generation and (weak) precompact generation in preuniform convergence spaces. We provide a comprehensive implication scheme between all concepts of compactness mentioned before. Moreover, we prove Tychonoffs theorem with respect to (weakly) precompact and locally (weakly) precompact preuniform convergence spaces. It also turns out that the construct **LPC-PUConv** of all locally precompact preuniform convergence spaces and uniformly continuous maps is a topological universe. Finally, we prove that one needs not to distinguish between locally (weakly) compact and (weakly) compactly generated resp. locally (weakly) precompact and (weakly) precompactly generated preuniform convergence spaces.

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All findings can be considered as generalizations of respective theorems by Preuß [4, 5] in the realm of **SUConv** which is a (bireflective and bicoreflective) subconstruct of **PUConv** (cf. [6]).

Since the construct **QUnif** of quasiuniform spaces and uniformly continuous maps is a (bireflective) subconstruct of **PUConv** (cf. [6]), it is possible to consider a quasiuniform space as a preuniform convergence space. Beside totally boundedness, Cauchy-boundedness (in the sense of Koppermann [2]) and precompactness (in the sense of Fletcher and Lindgren [1]) are well known concepts in the realm of quasiuniform spaces. Therefore, we also establish the relationship between the concept of (weak) precompactness when applied to quasiuniform spaces and these existing concepts. It turns out that some previous findings in the realm of quasiuniform spaces can be considered as special cases of the respective theorems proved in this article.

The terminology of this article corresponds to [5].

2. Preliminaries

Definition 2.1. 1. A *preuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X \subset F(X \times X)$, i.e., $\mathcal{J}_X$ is a set of filters on $X \times X$, such that the following conditions are satisfied:

UC_1 $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for all $x \in X$

UC_2 $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \supset \mathcal{F}$

The elements of $\mathcal{J}_X$ are called *uniform filters*. A map $f : (X, \mathcal{J}_X) \to (X', \mathcal{J}_{X'})$ is called *uniformly continuous* provided that $(f \times f)(\mathcal{F}) \in \mathcal{J}_{X'}$ for all $\mathcal{F} \in \mathcal{J}_X$. The construct of preuniform convergence spaces and uniformly continuous maps is denoted by **PUConv**.

2. A preuniform convergence space is called *uniform limit space* provided that the following is satisfied:

UC_3 $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} \in \mathcal{J}_X$

UC_4 $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ implies $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$

UC_5 $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ implies $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$, provided that $F \circ G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Definition 2.2. Let X be a set and $\mathcal{W}$ a filter on $X \times X$. An *uniform space* is a pair $(X, \mathcal{W})$ provided that the following is satisfied:

U_1 $\Delta \subset W$ for all $W \in \mathcal{W}$

U_2 $W^{-1} \in \mathcal{W}$ for all $W \in \mathcal{W}$

U_3 For all $W \in \mathcal{W}$ there is a $V \in \mathcal{W}$ such that $V^2 \subset W$.

A pair $(X, \mathcal{W})$ which only satisfies U_1 and U_3 is called a *quasiuniform space*. The elements of $\mathcal{W}$ are called *neighborhoods*. A map $f : (X, \mathcal{W}) \longrightarrow (X', \mathcal{W}')$ between (quasi-) uniform spaces is called *uniformly continuous* provided that $(f \times f)^{-1}[W'] \in \mathcal{W}$ for all $W' \in \mathcal{W}'$.

Remark 2.1. A quasiuniform space $(X, \mathcal{W})$ can be considered as a preuniform convergence space $(x, [\mathcal{W}])$ with $[\mathcal{W}] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{W}\}$.

Definition 2.3. 1. Let X be a set. A *generalized convergence space* is a pair (X, q), where $q \subset F(X) \times X$, such that the following is satisfied:

C_1 $(\dot{x}, x) \in q$ for all $x \in X$

C_2 $(\mathcal{F}, x) \in q$ and $\mathcal{G} \supset \mathcal{F}$ imply $(\mathcal{G}, x) \in q$.

$(\mathcal{F}, x) \in q$ is also denoted by $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{F}$ is said to *converge* to x. A map $f : (X, q) \longrightarrow (X', q')$ between generalized convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for all $(\mathcal{F}, x) \in q$.

2. Let $(X, \mathcal{J}_X)$ be a preuniform convergence space. Then $(X, q_{\mathcal{J}_X})$ is called its *underlying generalized convergence space*, where $q_{\mathcal{J}_X} = \{(\mathcal{F}, x) \in F(X) \times X : \dot{x} \times \mathcal{F} \in \mathcal{J}_X\}$. A filter $\mathcal{F}$ on X *preconverges* to x iff $(\mathcal{F}, x) \in q_{\mathcal{J}_X}$.

3. A generalized convergence space (X,q) is called *Kent-convergence space* provided that the following is satisfied

C_3 $(\mathcal{F}\cap\dot{x},x)\in q$ iff $(\mathcal{F},x)\in q$.

A Kent-convergence space is called *symmetric*, provided that from $(\mathcal{F},x)\in q$ and $y\in\bigcap\mathcal{F}$ it follows that $(\mathcal{F},y)\in q$.

4. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space. Then $(X,q_{\gamma_{\mathcal{J}_X}})$ is called its *underlying symmetric Kent-convergence space*, where $q_{\gamma_{\mathcal{J}_X}}=\{(\mathcal{F},x)\in F(X)\times X:\mathcal{F}\cap\dot{x}\times\mathcal{F}\cap\dot{x}\in\mathcal{J}_X\}$. A filter $\mathcal{F}$ on X *converges* to x iff $(\mathcal{F},x)\in q_{\gamma_{\mathcal{J}_X}}$.

Definition 2.4. 1. Let X be a set. A *filter space* is a pair (X,γ), where $\gamma\subset F(X)$, such that the following is satisfied:

F_1 $\dot{x}\in\gamma$ for all $x\in X$

F_2 $\mathcal{F}\in\gamma$ and $\mathcal{G}\supset\mathcal{F}$ imply $\mathcal{G}\in\gamma$

The elements of γ are called *Cauchy-filters*. **Fil** denotes the construct of filter spaces (and Cauchy-continuous maps), where a map $f:(X,\gamma)\longrightarrow(X',\gamma')$ between filter spaces is called *Cauchy-continuous* provided that $f(\mathcal{F})\in\gamma'$ for all $\mathcal{F}\in\gamma$.

2. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space. Then $(X,\gamma_{\mathcal{J}_X})$ is called its *underlying filter space*, where $\gamma_{\mathcal{J}_X}=\{\mathcal{F}\in F(X):\mathcal{F}\times\mathcal{F}\in\mathcal{J}_X\}$. Moreover $(X,\Gamma_{\mathcal{J}_X})$ is called the *induced filter space* (by $\mathcal{J}_X$), where $\Gamma_{\mathcal{J}_X}=\{\mathcal{F}\in F(X):$ it exists $\mathcal{G}\in F(X)$ such that $\mathcal{G}\times\mathcal{F}\in\mathcal{J}_X\}$.

3. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space. It is called **Fil**-*determined* provided that $\mathcal{J}_X=\{\mathcal{F}\in F(X\times X):$ there is some $\mathcal{G}\in\gamma_{\mathcal{J}_X}$ such that $\mathcal{G}\times\mathcal{G}\subset\mathcal{F}\}$.

4. Let (X,γ) be a filter space. Then $(X,\mathcal{J}_\gamma)$ is called the *corresponding preuniform convergence space*, where $\mathcal{J}_\gamma=\{\mathcal{F}\in F(X\times X):$ it exists $\mathcal{G}\in\gamma$ such that $\mathcal{G}\times\mathcal{G}\subset\mathcal{F}\}$.

Definition 2.5. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space.

1. $(X,\mathcal{J}_X)$ is called *complete*, provided that every filter in $\gamma_{\mathcal{J}_X}$ converges.
2. $(X,\mathcal{J}_X)$ is called *precomplete*, provided that every filter in $\Gamma_{\mathcal{J}_X}$ preconverges.
3. $(X,\mathcal{J}_X)$ is called *weakly complete*, provided that every ultrafilter in $\gamma_{\mathcal{J}_X}$ converges.
4. $(X,\mathcal{J}_X)$ is called *weakly precomplete*, provided that every ultrafilter in $\Gamma_{\mathcal{J}_X}$ preconverges.
5. $(X,\mathcal{J}_X)$ is called *compact* provided that its underlying symmetric Kent convergence space $(X,q_{\gamma_{\mathcal{J}_X}})$ is compact, i.e., provided that every ultrafilter on X converges.
6. $(X,\mathcal{J}_X)$ is called *weakly compact* provided that its underlying generalized convergence space $(X,q_{\mathcal{J}_X})$ is compact, i.e., provided that every ultrafilter on X preconverges.

Proposition 2.1. 1. Every compact preuniform convergence space $(X,\mathcal{J}_X)$ is weakly compact.

2. Every (weakly) compact preuniform convergence space is weakly (pre-)complete.

Proof. 1. The proposition follows immediately from $q_{\gamma_{\mathcal{J}_X}}\subset q_{\mathcal{J}_X}$ (because $(\mathcal{F}\cap\dot{x})\times(\mathcal{F}\cap\dot{x})\subset\dot{x}\cap\mathcal{F}$ for every Filter $\mathcal{F}$ on X and $x\in X$).

2. i. Let $\mathcal{U}$ be an ultrafilter in $\gamma_{\mathcal{J}_X}$. It exists $x\in X$ such that $(\mathcal{U},x)\in q_{\gamma_{\mathcal{J}_X}}$. Consequently, $\mathcal{U}\supset\mathcal{U}\cap\dot{x}\in\gamma_{\mathcal{J}_X}$. ii. Let $\mathcal{U}$ be an ultrafilter in $\Gamma_{\mathcal{J}_X}$. It exists $x\in X$ such that $(\mathcal{U},x)\in q_{\mathcal{J}_X}$, i.e., $\dot{x}\times\mathcal{U}\in\mathcal{J}_X$. □

Remark 2.2. 1. $q_{\gamma_{\mathcal{J}_X}}\supset q_{\mathcal{J}_X}$ does not generally apply [6, Remark 2.10]).

2. However, not every compact preuniform convergence space is complete [5, Remark 4.3.2.12]) and not every weakly compact preuniform convergence space is precomplete: Let X be an uncountable infinite set. We define $\mathcal{J}_X=\{\mathcal{F}\in F(X\times X):$ it exists an ultrafilter $\mathcal{U}$ on X and $x\in X$ such that $\mathcal{F}\supset\dot{x}\times\mathcal{U}\}\cup\{\mathcal{F}\in F(X\times X):\mathcal{F}\supset\mathcal{G}\times\mathcal{G}\}$, where $\mathcal{G}=\{U\subset X:X\setminus U$ is finite$\}$. Then $(X,\mathcal{J}_X)$ is weakly compact but the $\Gamma_{\mathcal{J}_X}$-Cauchyfilter $\mathcal{G}$ does not converge in $(X,q_{\mathcal{J}_X})$.

3. Weak compactness and compactness are equivalent concepts in **Fil**-determined preuniform convergence spaces and in uniform limit spaces.

Definition 2.6. 1. A preuniform convergence space $(X,\mathcal{J}_X)$ is called *locally (weakly) compact* provided that every filter $\mathcal{F}\in\mathcal{J}_X$ contains a (weakly) compact subset of the product space $(X,\mathcal{J}_X)\times(X,\mathcal{J}_X)$.

2. A preuniform convergence space is called *diagonal* provided that the filter (Δ_X) generated by the diagonal $\Delta_X=\{(x,x):x\in X\}$ of $X\times X$ is contained in $\mathcal{J}_X$ whenever the filter (Δ_X) exists.

Note that every quasiuniform space $(X,\mathcal{W})$ is diagonal since $\Delta_X\subset W$ for all $W\in\mathcal{W}$, i.e., $(\Delta_X)\supset\mathcal{W}$.

Proposition 2.2. Let $(X,\mathcal{J}_X)$ be a locally (weakly) compact preuniform convergence space. Then its underlying symmetric Kent-convergence space $(X,q_{\gamma_{\mathcal{J}_X}})$ (resp. its underlying generalized convergence space $(X,q_{\mathcal{J}_X})$) is locally compact.

Proof. [7, Proposition 3.7]. □

However, the underlying topological space of a locally compact preuniform convergence space needs not to be locally compact. For instance, the preuniform convergence space belonging to $\mathbb{R}_t^{\mathbb{N}}$ is locally compact but the topological space $\mathbb{R}_t^{\mathbb{N}}$ is not locally compact.

Local compactness of a convergence space (a preuniform convergence space that is generated by its convergent filters, i.e. $\mathcal{J}_X=\{\mathcal{F}\in F(X\times X):$ it exists $(\mathcal{G},x)\in q_{\gamma_{\mathcal{J}_X}}$ such that $\mathcal{F}\supset\mathcal{G}\times\mathcal{G}\}$) is equivalent to local compactness of its underlying symmetric Kent-convergence space [5, Proposition 6.2.1.6]). Moreover, local weak compactness of a preconvergence space (a preuniform convergence space that is generated by its preconvergent filters, i.e. $\mathcal{J}_X=\{\mathcal{F}\in F(X\times X):$ it exists $(\mathcal{G},x)\in q_{\mathcal{J}_X}$ such that $\mathcal{F}\supset\dot{x}\times\mathcal{G}\}$) is equivalent to local compactness of its underlying generalized convergence space [7, Proposition 3.12].

The inverse implication in the Proposition above does not generally apply. For instance, the uniform space $\mathbb{R}_u$ of all real numbers is not locally compact but its underlying symmetric Kent-convergence space is locally compact.

Proposition 2.3. 1. Every (weakly) compact preuniform convergence space is locally (weakly) compact.

2. A diagonal preuniform convergence space is (weakly) compact iff it is locally (weakly) compact.

Proof. 1. Let $\mathcal{F}\in\mathcal{J}_X$. X is a subspace of $(X,\mathcal{J}_X)$ and therefore (weakly) compact. Then the subspace $X\times X$ of $(X,\mathcal{J}_X)\times(X,\mathcal{J}_X)$ is also (weakly) compact because of [7, Corollary 2.12]).

2. Let $(X,\mathcal{J}_X)$ be diagonal and locally (weakly) compact. Because $(\Delta_X)\in\mathcal{J}_X$, it holds by assumption that there is a compact subset K of $(X,\mathcal{J}_X)\times(X,\mathcal{J}_X)$ such that $K\supset\Delta_X$. Since $X=p_1[\Delta_X]\subset p_1[M]\subset X$ and [7, Corollary 2.15], $(X,\mathcal{J}_X)$ is (weakly) compact, where $p_1:(X,\mathcal{J}_X)\times(X,\mathcal{J}_X)\to(X,\mathcal{J}_X)$ is the projection in the first component. □

This means that (weak) compactness and locally (weak) compactness are equivalent concepts in the realm of quasiuniform spaces (considered as preuniform convergence spaces).

3. (Weak) Precompactness

Definition 3.1. A preuniform convergence space is called *precompact* provided that its underlying filter space $(X,\gamma_{\mathcal{J}_X})$ is precompact, i.e., provided that every ultrafilter on X belongs to $\gamma_{\mathcal{J}_X}$. It is called *weakly precompact* provided that the induced filter space $(X,\Gamma_{\mathcal{J}_X})$ is precompact, i.e., provided that every ultrafilter on X belongs to $\Gamma_{\mathcal{J}_X}$.

Proposition 3.1. 1. Every precompact preuniform convergence space is weakly precompact.

2. Every (weakly) compact preuniform convergence space is (weakly) precompact.

3. A preuniform convergence space is (weakly) compact iff it is (weakly) precompact and weakly (pre-) complete. Since every (pre-) complete preuniform convergence space is weakly (pre-) complete, it holds that every (pre-) complete precompact preuniform convergence space is (weakly) compact.

Proof. 1. Let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U} \in \gamma_{\mathcal{J}_X}$, i.e., $\mathcal{U} \times \mathcal{U} \in \mathcal{J}_X$, and consequently $\mathcal{U} \in \Gamma_{\mathcal{J}_X}$.

2. Let $\mathcal{U}$ be an ultrafilter on X. It exists $x \in X$ such that $(\mathcal{U}, x) \in q_{\gamma_{\mathcal{J}_X}}$ (resp. $(\mathcal{U}, x) \in q_{\mathcal{J}_X}$). Consequently $\mathcal{U} \cap \dot{x} \in \gamma_{\mathcal{J}_X}$ (resp. $\dot{x} \times \mathcal{U} \in \mathcal{J}_X$). It follows $\mathcal{U} \in \gamma_{\mathcal{J}_X}$ (resp. $\mathcal{U} \in \Gamma_{\mathcal{J}_X}$).

3. "$\Rightarrow$". This follows from 2. and Proposition 2.1. "$\Leftarrow$". Let $\mathcal{U}$ be an ultrafilter on X and $(X, \mathcal{J}_X)$ a (weakly) precompact and weakly (pre-) complete preuniform convergence space. It is $\mathcal{U} \in \gamma_{\mathcal{J}_X}$ (resp. $\mathcal{U} \in \Gamma_{\mathcal{J}_X}$) because of (weak) precompactness. Since $(X, \mathcal{J}_X)$ is weakly (pre-) complete, it follows that $\mathcal{U}$ (pre-) converges. □

The inverse implication of assertions 1 and 2 are not generally true. With respect to assertion 1 we provide a counterexample in Remark 3.1. Note that precompactness and weak precompactness are equivalent concepts in uniform limit spaces and **Fil**-determined preuniform convergence spaces.

In quasiuniform spaces, totally boundedness, precompactness and Cauchy-boundedness are known concepts, for instance see [1] and [2]. In the following, we establish the relationship between these concepts and the concepts introduced in this article.

Theorem 3.1. Let $(X, \mathcal{W})$ be a quasiuniform space. Then the following is equivalent:

1. $(X, [\mathcal{W}])$ is precompact.

2. $(X, \mathcal{W})$ is totally bounded, i.e., for every ultrafilter $\mathcal{U}$ on X and every neighborhood $W \in \mathcal{W}$ it exists $U \in \mathcal{U}$ such that $U \times U \subset W$.

3. The bicoreflection $(X, \mathcal{W}^*)$ of $(X, \mathcal{W})$ with respect to **Unif** is totally bounded.

4. The bicoreflection $(X, \mathcal{W}^*)$ of $(X, \mathcal{W})$ with respect to **Unif** is precompact in the sense of Fletcher and Lindgren, i.e., for all $W \in \mathcal{W}^*$ it exists a finite subset F of X such that $W[F] = X$.

5. Every ultrafilter $\mathcal{U}$ on X is a $\mathcal{W}^*$-Cauchy-filter, i.e., for all $W^* \in \mathcal{W}^*$ exists a $x \in X$ such that $W^*(x) \in \mathcal{U}$.

6. For all $W \in \mathcal{W}$ it exists a finite subcover $\mathcal{A}$ of X such that $A \times A \subset W$ for all $A \in \mathcal{A}$.

7. For all $\mathcal{C} \in \mu_{\mathcal{W}} = \{\mathcal{C} \subset P(X) : (\bigcup \mathcal{C} = X \text{ and } \exists W \in \mathcal{W} : \forall x \in X\colon \exists C \in \mathcal{C} \text{ such that } W(x) \subset C)\}$ it exists a finite subset $\mathcal{B} \subset \mathcal{C}$, such that $\mathcal{B} \in \mu_{\mathcal{W}}$, i.e., every quasiuniform cover of X has a finite quasiuniform subcover.

Proof. The logical scheme of the proof is as follows:

$$\begin{tikzcd} (1) \arrow[r, Leftrightarrow] & (2) \arrow[d, Rightarrow] & (3) \arrow[l, Rightarrow] & (4) \arrow[l, Rightarrow] \\ & (5) \arrow[r, Rightarrow] & (6) \arrow[r, Rightarrow] & (7) \arrow[u, Rightarrow] \end{tikzcd}$$

(1) $\Leftrightarrow$ (2). Both says that $\mathcal{U} \times \mathcal{U} \supset \mathcal{W}$ for every ultrafilter $\mathcal{U}$ on X.

(2) $\Rightarrow$ (5). Let $\mathcal{U}$ be an ultrafilter on X and $W^* \in \mathcal{W}^*$, i.e., it exists $W \in \mathcal{W}$ such that $W^* \supset (W \cap W^{-1})$. By assumption, it exists $U \in \mathcal{U}$ such that $U \times U \subset W$. Then it holds $U \subset (W \cap W^{-1})(x) \subset W^*(x)$ for all $x \in U$. Hence, $W^*(x) \in \mathcal{U}$.

(5) $\Rightarrow$ (6). Let $W \in \mathcal{W}$. We can choose a symmetrical neighborhood $V \in \mathcal{W}^*$ such that $V \circ V \subset W$. First case: It exists a finite subset $F \subset X$ with $W[F] = X$. Then $\mathcal{A} = \{V(x) : x \in X\}$ is a finite subcover such that $V(x) \times V(x) \subset W$ for all $V(x) \in \mathcal{A}$. Second case: For every finite subset $F \subset X$

holds $W[F] \neq X$. Then $\mathcal{B} = \{X \setminus W[F] : F$ is a finite subset of $X\}$ is the base of a filter on X. Let $\mathcal{U}$ be an ultrafilter that is finer then $\mathcal{B}$. Then it is $X \setminus W(x) \in \mathcal{U}$ for all $x \in X$. This contradicts the premise and, hence, the second case can be excluded.

(6) $\Rightarrow$ (7). Let $\mathcal{C}$ be a quasiuniform cover, i.e., particularly it exists $W \in \mathcal{W}$ such that there is a $C \in \mathcal{C}$ for all $x \in X$ with $W(x) \subset C$. As required, for every W there is a finite cover $\mathcal{A}$ of X with $A \times A \subset W$ for all $A \in \mathcal{A}$. For every $A \in \mathcal{A}$ choose $x_A \in A$. Let C_A denote an element of cover $\mathcal{C}$ such that $A \subset W(x_A) \subset C_A$. Then $\mathcal{B} = \{C_A : A \in \mathcal{A}\}$ is a finite subcover of $\mathcal{C}$. This subcover is also quasiuniform: Let $x \in X$. It exists $A \in \mathcal{A}$ with $x \in A$. Then it is $W(x) \subset W^2(x_A) \subset C_A$ (note that, on the one hand, for $(x,y) \in W$ and $(x_A,x) \in W$ it holds $(x_A,y) \in W^2$ and, on the other hand, it is $A \times A \subset W \subset W^2 \subset (C_A \times C_A)^2 = C_A \times C_A$).

(7) $\Rightarrow$ (4). Let $W \in \mathcal{W}^*$. Then $\{W(x) : x \in X\}$ is a cover of X which is quasiuniform ($W(x) \subset W(x)$ for all $x \in X$). As required, it exists a finite quasiuniform subcover $\{W(x_i) : x_i \in X, i \leq n\}$. It follows $X = \bigcup_{i=1}^n W(x_i) = W[\bigcup_{i=1}^n \{x_i\}] = W[F]$ where $F := \bigcup_{i=1}^n \{x_i\}$.

(4) $\Rightarrow$ (3). Let $W \in W^*$ and $\mathcal{U}$ be an ultrafilter on X. We choose $V \in \mathcal{W}^*$ such that $V^2 \subset W$. As required, it exists a finite subset F of X with $V[F] = X$. From $X = V[F] = \bigcup_{y \in F} V(y) \in \mathcal{U}$ it follows that there is $x \in F$ with $V(x) \in \mathcal{U}$ (because $\mathcal{U}$ is an ultrafilter). Moreover, it is $V(x) \times V(x) \subset V^2 \subset W$ from which follows the assertion.

(3) $\Rightarrow$ (2). Let $(X, \mathcal{W}^*)$ be totally bounded. The assertion follows easily from $\mathcal{W} \subset \mathcal{W}^*$. □

Proposition 3.2. Let $(X, \mathcal{W})$ be a quasiuniform space. Then $(X, [\mathcal{W}])$ is weakly precompact iff $(X, \mathcal{W})$ is Cauchy-bounded (in the sense of R. D. Koppermann), i.e., every ultrafilter is D-Cauchy (which means that for every ultrafilter $\mathcal{U}$ there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \times \mathcal{U} \supset \mathcal{W}$).

This means that the concept of (weak) precompactness is a reasonable generalization of the concept of totally boundedness (Cauchy-boundedness).

Proposition 3.3. Every weakly precompact quasiuniform space is precompact in the sense of Fletcher and Lindgren.

Proof. Let $(X, \mathcal{W})$ be a weakly precompact quasiuniform space and let $\mathcal{U}$ be an ultrafilter on X. By assumption, it is $\mathcal{U} \in \Gamma_{[\mathcal{W}]}$, i.e., there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \times \mathcal{U} \supset \mathcal{W}$. Consequently, for every $W \in \mathcal{W}$ there is $F \in \mathcal{F}$ and $U \in \mathcal{U}$ with $F \times U \subset W$. Now we choose $x \in F$ (F is non-empty). Then it is $(F \times U)(x) = U \subset W(x)$, i.e., $W(x) \in \mathcal{U}$. Therefore, $(X, \mathcal{W})$ is precompact in the sense of Fletcher and Lindgren: Let us assume there is a $W \in \mathcal{W}$ such that $W(F) \neq X$ for all finite subsets $F \subset X$. Then $\mathcal{B} = \{X \setminus W[F] : F$ is a finite subset of $X\}$ is base of a filter on X. Let $\mathcal{U}$ be an ultrafilter which is finer than $\mathcal{B}$. Now we obtain $(X \setminus W(y)) \in \mathcal{U}$ for all $y \in X$. This is a contradiction: Following the argumentation above we can find $x \in X$ for every $\mathcal{U}$ such that $W(x) \in \mathcal{U}$. □

Remark 3.1. 1. The inverse implication is not generally true. It is possible to construct a precompact quasiuniform space in the sense of Fletcher and Lindgren which is not Cauchy-bounded [2, Example 8]).

2. A weakly precompact preuniform convergence space is not generally precompact: Let $X = \mathbb{R}$, $x \in X$ fix, $W_x = \Delta \cup \{(x,y) : y \in \mathbb{R}\}$ and $\mathcal{W}_x = (W_x)$ the quasiuniformity induced by W_x. Since $W(x) = \mathbb{R}$ for all $W \in \mathcal{W}_x$, $(X, \mathcal{W}_x)$ is precompact in the sense of Fletcher and Lindgren. Let $\mathcal{U}$ be an ultrafilter on X. Then $W(x) \in \mathcal{U}$ for all $W \in \mathcal{W}_x$. Define $\mathcal{F} = \dot{x}$. It follows $\mathcal{F} \times \mathcal{U} \supset \mathcal{W}_x$. Therefore, $(X, \mathcal{W}_x)$ is weakly precompact. However, the quasiuniform space $(X, \mathcal{W}_x)$ is not precompact because the bicoreflection of $\mathcal{W}_x$ is given by (Δ) which is not precompact in the sense of Fletcher and Lindgren.

3. The subsequent diagram summarizes the implications of the concepts for quasiuniform spaces as discussed above (FL-precompact is the abbreviation for precompact in the sense of Fletcher and Lindgren):

$$\begin{tikzcd}
\text{precompact} \arrow[rr, Rightarrow] \arrow[d, Leftrightarrow] & & \text{weakly precompact} \arrow[d, Leftrightarrow] \\
\text{totally bounded} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \text{Cauchy-bounded} \arrow[dl, Rightarrow] \\
& \text{FL-precompact} &
\end{tikzcd}$$

The inverse implications are not generally true (unless the concepts are restricted to uniform spaces).

Proposition 3.4. The construct **PC-PUConv** of precompact preuniform convergence spaces and uniformly continuous maps is a bireflective subconstruct of **PUConv**.

Proof. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space. Then $1_X : (X,\mathcal{J}_X) \longrightarrow (X,\mathcal{J}_X')$ is the desired bireflection with respect to **PC-PUConv**, where $\mathcal{J}_X' = \mathcal{J}_X \cup \{\mathcal{G} \in F(X\times X) :$ it exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{G} \supset \mathcal{U}\times\mathcal{U}\}$. □

Proposition 3.5. A preuniform convergence space $(X,\mathcal{J}_X)$ is (weakly) precompact iff for every filter $\mathcal{F}$ on X there is a filter $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ (resp. $\mathcal{G} \in \Gamma_{\mathcal{J}_X}$) such that $\mathcal{F} \subset \mathcal{G}$.

Proposition 3.6. Let $(X,\mathcal{J}_X)$ and $(X',\mathcal{J}_{X'})$ be preuniform convergence spaces. Moreover, let $f : (X,\mathcal{J}_X) \longrightarrow (X',\mathcal{J}_{X'})$ be a surjective uniformly continuous map. Then $(X,\mathcal{J}_X)$ (weakly) precompact implies $(X',\mathcal{J}_{X'})$ (weakly) precompact.

Proof. Let $\mathcal{F} \in F(X')$. Then $f^{-1}(\mathcal{F}) \in F(X)$ (f surjective) and, as required, there is a $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ (resp. a $\mathcal{G} \in \Gamma_{\mathcal{J}_X}$) such that $\mathcal{G} \supset f^{-1}(\mathcal{F})$. Since $f : (X,\gamma_{\mathcal{J}_X}) \longrightarrow (X',\gamma_{\mathcal{J}_{X'}})$ is Cauchy-continuous (resp. $f : (X,\Gamma_{\mathcal{J}_X}) \longrightarrow (X',\Gamma_{\mathcal{J}_{X'}})$ Cauchy-continuous) it holds $f(\mathcal{G}) \in \gamma_{\mathcal{J}_{X'}}$ (resp. $f(\mathcal{G}) \in \Gamma_{\mathcal{J}_{X'}}$). Moreover, $f(\mathcal{G}) \supset f(f^{-1}(\mathcal{F})) = \mathcal{F}$. Now the assertion follows from Proposition 3.5. □

Theorem 3.2 (Tychonoff)**.** Let $((X_i,\mathcal{J}_{X_i}))_{i\in I}$ a family of non-empty preuniform convergence spaces. Then the product $\prod_{i\in I}(X_i,\mathcal{J}_{X_i})$ is (weakly) precompact iff $(X_i,\mathcal{J}_{X_i})$ is (weakly) precompact for all $i \in I$.

Proof. 1. Precompactness: The assertion follows from Proposition 3.4 and 3.6.

2. Weak Precompactness: "$\Leftarrow$". Proposition 3.6. "$\Rightarrow$". Let $(X_i,\mathcal{J}_{X_i})$ be weakly precompact for every $i \in I$. Denote $\prod_{i\in I}(X_i,\mathcal{J}_{X_i})$ by $(X,\mathcal{J}_X)$. We shall prove that for every ultrafilter $\mathcal{U}$ on X there is a filter $\mathcal{F}$ on X such that $\mathcal{F}\times\mathcal{U} \in \mathcal{J}_X$. Now let $\mathcal{U}$ be an ultrafilter. By assumption, for all $i \in I$ there is a filter $\mathcal{F}_i$ on X_i such that $\mathcal{F}_i \times p_i(\mathcal{U}) \in \mathcal{J}_{X_i}$, where $p_i : \prod_{i\in I} X_i \to X_i$ denote the projection in the i-th component. Denote $\mathcal{F} = \prod_{i\in I}\mathcal{F}_i$. Then it is $\mathcal{F}\times\mathcal{U} \in \mathcal{J}_X$ because $p_i \times p_i(\mathcal{F}\times\mathcal{U}) = p_i(\prod_{i\in I}\mathcal{F}_i) \times p_i(\mathcal{U}) = \mathcal{F}_i \times p_i(\mathcal{U}) \in \mathcal{J}_{X_i}$ for all $i \in I$ (note $\mathcal{J}_X$ initial). □

Because (weak) precompactness is a generalization of the concept of totally boundedness (Cauchy-boundedness), Theorem 3.2 can be considered as a generalisation of respective theorems by Fletcher & Lindgren [1] and Koppermann [2, Theorem 10]. Moreover, it can be considered as a generalization of the respective proposition by Preuß with respect to semiuniform convergence spaces [5, Proposition 6.2.1.9].

4. Local (Weak) Precompactness

The following definition is similar to the respective definition for filter spaces.

Definition 4.1. 1. A preuniform convergence space $(X,\mathcal{J}_X)$ is called *locally (weakly) precompact* provided that every filter $\mathcal{F} \in \mathcal{J}_X$ contains a (weakly) precompact subset of the product space $(X,\mathcal{J}_X)\times(X,\mathcal{J}_X)$.

2. A preuniform convergence space $(X, \mathcal{J}_X)$ is called *t-regular* provided that for all uniform filters $\mathcal{F} \in \mathcal{J}_X$ it holds that the filter $\overline{\mathcal{F}}^t$ generated by the filter base of all closed elements of $\mathcal{F}$ is contained in $\mathcal{J}_X$.

Proposition 4.1. Let $(X, \mathcal{J}_X)$ be a locally (weakly) precompact preuniform convergence space. Then its underlying filter space $(X, \gamma_{\mathcal{J}_X})$ (resp. the induced filter space $(X, \Gamma_{\mathcal{J}_X})$) is locally precompact.

Proof. 1. Let $\mathcal{F} \in \gamma_{\mathcal{J}_X}$. It follows $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$ and, by assumption, there is a precompact set P such that there is a $F \in \mathcal{F}$ with $P \supset F \times F$. Due to Proposition 3.6 $p_1[P] \in \mathcal{F}$ is precompact in $(X, \mathcal{J}_X)$ and, hence, also in $(X, \gamma_{\mathcal{J}_X})$, where p_1 denotes the projection in the first component.

2. Let $\mathcal{F} \in \Gamma_{\mathcal{J}_X}$. Then there is a filter $\mathcal{G}$ on X such that $\mathcal{G} \times \mathcal{F} \in \mathcal{J}_X$. By assumption, there is a weakly precompact set P such that there is a $G \in \mathcal{G}$ and a $F \in \mathcal{F}$ with $P \supset G \times F$. Due to Proposition 3.6 $p_2[P] \in \mathcal{F}$ is weakly precompact in $(X, \mathcal{J}_X)$ and, hence, precompact in $(X, \Gamma_{\mathcal{J}_X})$, where p_2 denotes the projection in the second component. □

This means that local (weak) compactness is well-defined with respect to the underlying (the induced) filter space.

Proposition 4.2. Let (X, γ) be a filter space and $(X, \mathcal{J}_\gamma)$ the corresponding preuniform convergence space. Then it holds that $(X, \mathcal{J}_\gamma)$ is locally precompact iff (X, γ) is locally precompact.

Proof. (1) $\Rightarrow$ (2): See Proposition 4.1. (2) $\Rightarrow$ (1): Let $\mathcal{G} \in \mathcal{J}_\gamma$. Then there is a $\mathcal{F} \in \gamma$ with $\mathcal{F} \times \mathcal{F} \subset \mathcal{G}$. By assumption, there is a precompact set $P \in \mathcal{F}$ in (X, γ) and, hence, in $(X, \mathcal{J}_\gamma)$. This implies $P \times P$ is precompact in $(X, \mathcal{J}_\gamma) \times (X, \mathcal{J}_\gamma)$ (Theorem 3.2) and belongs to $\mathcal{G}$. □

Proposition 4.3. 1. Every (weakly) precompact preuniform convergence space is locally (weakly) precompact.

2. Every diagonal preuniform convergence space is (weakly) precompact iff it is locally (weakly) precompact.

3. Every locally precompact preuniform convergence space is locally weakly precompact.

4. Every locally (weakly) compact preuniform covergence space $(X, \mathcal{J}_X)$ is locally (weakly) precompact. The inverse implication holds, if $(X, \mathcal{J}_X)$ is weakly (pre)complete.

5. Let $(X, \mathcal{J}_X)$ be a t-regular and weakly complete preuniform convergence space. Then $(X, \mathcal{J}_X)$ is locally compact iff it is locally precompact.

Proof. 1. This can be verified similarly to Proposition 2.3 1.

2. This can be verified similarly to Proposition 2.3 2.

3. This follows from Proposition 3.1 1.

4. The first assertion follows from Proposition 3.1 2. and the second assertion from 3.1 3.

5. Due to 4. it remains to prove that every t-regular and weakly complete locally precompact space $(X, \mathcal{J}_X)$ is locally compact: Let $\mathcal{F} \in \mathcal{J}_X$. Due to t-regularity it is $\overline{\mathcal{F}}^t \in \mathcal{J}_X$. By assumption, there is a precompact subset H of $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$ such that $H \in \overline{\mathcal{F}}^t$, i.e., $H \supset F$ for a closed $F \in \overline{\mathcal{F}}^t$. It follows from Proposition 3.4 that F is precompact. Because closed subspaces of weakly complete preuniform convergence spaces are also weakly complete, it follows that F is also weakly complete and therefore compact. □

The subsequent diagrams summarize the implications between (weak) compactness, (weak) precompactness and its localizations:

$$
\begin{tikzcd}
 & \text{loc. comp.} \arrow[rr, Rightarrow] \arrow[dd, Rightarrow] & & \text{loc. precomp.} \arrow[dd, Rightarrow] \\
\text{comp.} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] \arrow[dd, Rightarrow] & & \text{precomp.} \arrow[ur, Rightarrow] \arrow[dd, Rightarrow] & \\
 & \text{loc. weakly comp.} \arrow[rr, Rightarrow] & & \text{loc. weakly precomp.} \\
\text{weakly comp.} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \text{weakly precomp.} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Under the following conditions the inverse implications apply (where $*$ stands for "**Fil**-determined" or for "uniform limit space" and Δ stands for "diagonal").

$$
\begin{tikzcd}
 & \text{loc. comp.} \arrow[rr, Leftarrow, "\text{t-regular and weakly complete}"] \arrow[dd, Leftarrow, "*"] & & \text{loc. precom.} \arrow[dd, Leftarrow, "*"] \\
\text{comp.} \arrow[ur, Leftarrow, "\Delta"] \arrow[rr, Leftarrow, "\text{weakly complete}"] \arrow[dd, Leftarrow, "*"] & & \text{precomp.} \arrow[ur, Leftarrow, "\Delta"] \arrow[dd, Leftarrow, "*"] & \\
 & \text{loc. weakly comp.} \arrow[rr, Leftarrow, "\Delta\text{ and weakly precomplete}"] & & \text{loc. weakly precomp.} \\
\text{weakly comp.} \arrow[ur, Leftarrow, "\Delta"] \arrow[rr, Leftarrow, "\text{weakly precomplete}"] & & \text{weakly precomp.} \arrow[ur, Leftarrow, "\Delta"] &
\end{tikzcd}
$$

Proposition 4.4. The construct **LPC-PUConv** (of locally precompact preuniform convergene spaces and uniformly continuous maps) and the construct **LWPC-PUConv** (of all locally weakly precompact preuniform convergence spaces and uniformly continuous maps) are bicoreflective subconstructs of **PUConv**.

Proof. Let $(X,\mathcal{J}_X)$ be a preuniform convergence space. Then $1_X : (X,\mathcal{J}'_X) \to (X,\mathcal{J}_X)$ is the desired bicoreflection, where $\mathcal{J}'_X = \{\mathcal{F} \in \mathcal{J}_X :$ it exists a (weakly) precompact subset K of $(X,\mathcal{J}_X) \times (X,\mathcal{J}_X)$ such that $K \in \mathcal{F}\}$. □

Theorem 4.1 (Tychonoff)**.** Let $((X_i,\mathcal{J}_{X_i}))_{i\in I}$ a non-empty family of preuniform convergence spaces. The product $\prod_{i\in I}(X_i,\mathcal{J}_{X_i})$ is locally (weakly) precompact iff $(X_i,\mathcal{J}_{X_i})$ is locally (weakly) precompact for all $i \in I$ and (weakly) precompact for all but finitely many $i \in I$.

Proof. "$\Leftarrow$". Denote $\prod_{i\in I}(X_i,\mathcal{J}_{X_i})$ by $(\prod_{i\in I} X_i,\mathcal{J}_X)$. Let $\mathcal{F} \in \mathcal{J}_X$. It follows $(p_i \times p_i)(\mathcal{F}) \in \mathcal{J}_{X_i}$, where $p_i : \prod_{i\in I} X_i \to X_i$ denotes the projection in the i-th component. By assumption, there exist finitely many elements $i_1,\dots,i_n$ in I such that $(X_i,\mathcal{J}_{X_i})$ is (weakly) precompact for all $i \in I\setminus\{i_1,\dots i_n\}$ and such that there is a (weakly) precompact set $H_i \in (p_i \times p_i)(\mathcal{F})$ for all $i \in \{i_1,\dots,i_n\}$. We denote $H_i = X_i \times X_i$ for all $i \in I\setminus\{i_1,\dots i_n\}$. It follows that $j(\prod_{i\in I} H_i) \in j(\prod_{i\in I}(p_i \times p_i)(\mathcal{F})) \subset \mathcal{F}$ is (weakly) precompact, where $j : \prod_{i\in I}(X_i \times X_i) \to (\prod_{i\in I} X_i) \times (\prod_{i\in I} X_i)$ denotes the canonical isomorphism (use Proposition 3.6 and Theorem 3.2).

"$\Rightarrow$". Let $\mathcal{F}_i \in \mathcal{J}_{X_i}$ for all $i \in I$. Then $\prod_{i\in I}\mathcal{F}_i$ is a filter on $\prod_{i\in I}(X_i \times X_i)$ and, hence, $j(\prod_{i\in I}\mathcal{F}_i)$ is a filter on $(\prod_{i\in I} X_i) \times (\prod_{i\in I} X_i)$. Because $(p_i \times p_i)(j(\prod_{i\in I}\mathcal{F}_i)) = p'_i(\prod_{i\in I}\mathcal{F}_i) = \mathcal{F}_i \in \mathcal{J}_{X_i}$ for all $i \in I$, it holds that $j(\prod_{i\in I}\mathcal{F}_i) \in \mathcal{J}_X$ ($\mathcal{J}_X$ initial), where $p'_i : \prod_{i\in I}(X_i \times X_i) \to X_i \times X_i$ denotes the projection in the i-th component for all $i \in I$. By assumption, there is a (weakly) precompact set $K \in j(\prod_{i\in I}\mathcal{F}_i)$. Due to Proposition 3.6 $(p_i \times p_i)(K) \in (p_i \times p_i)(j(\prod_{i\in I}\mathcal{F}_i)) = p'_i(\prod_{i\in I}\mathcal{F}_i) = \mathcal{F}_i$ is (weakly) precompact for all $i \in I$. Consequently, $(X_i,\mathcal{J}_{X_i})$ is locally (weakly) precompact for all $i \in I$. Moreover, it holds $K \supset j(\prod_{i\in I} F_i)$, where $F_i \in \mathcal{F}_i$ for all $i \in I$ and $F_i = X_i \times X_i$ for all but finitely many $i \in I$. Therefore $X_i \times X_i = F_i = p_i'(\prod_{i\in I}\mathcal{F}_i) = (p_i \times p_i)(j(\prod_{i\in I}\mathcal{F}_i)) \subset (p_i \times p_i)[K] \subset X_i \times X_i$ can be obtained for all but finitely many $i \in I$, i.e., $X_i \times X_i = (p_i \times p_i)[K]$ is (weakly) precompact for all but finitely many $i \in I$ □

Proposition 4.5. **LPC-PUConv** is closed under the formation of subspaces.

Proof. Let $(X,\mathcal{J}_X) \in$ be a locally precompact preuniform convergence space and $(A,\mathcal{J}_A)$ a subspace of X. Let $\mathcal{F} \in \mathcal{J}_A$. It follows $(i \times i)(\mathcal{F}) \in \mathcal{J}_X$ ($\mathcal{J}_A$ initial), where $i : A \to X$ denotes in inclusion map. By assumption, there is a precompact subset K of $(X,\mathcal{J}_X) \times (X,\mathcal{J}_X)$ such that $K \in (i \times i)(\mathcal{F})$. Therefore, the subspace $K \cap (A \times A)$ of $(K,\mathcal{J}_K)$ is precompact (Proposition 3.4) and it is $K \cap (A \times A) \in \mathcal{F}$. □

Theorem 4.2. **LPC-PUConv** is a topological universe.

Proof. 1. **LPC-PUConv** is cartesian closed: According to Proposition 4.4, **LPC-PUConv** is a bicoreflective subconstruct of **PUConv**. Furthermore, **LPC-PUConv** is closed under the formation of finite products (Theorem 4.1). Now the assertion follows from the fact that **PUConv** is cartesian closed and topological [6, Proposition 1.2]).

2. **LPC-PUConv** is extensional: This follows from the fact that **PUConv** is extensional (see [6]), that **LPC-PUConv** is closed under the formation of subspaces (Proposition 4.5) and that it is a bicoreflective subconstruct of **PUConv** (Proposition 4.4). □

Theorem 4.2 can be considered as a generalization of the respective theorem by Preuß with respect to semiuniform convergence spaces [4, Theorem 3.12]).

Theorem 4.3. The construct **LWPC-PUConv** is cartesian closed and topological.

Proof. This can be verified similarly to 1. in the proof of Theorem 4.2. □

5. (Weak) Compact and (Weak) Precompact Generation

Definition 5.1. 1. A preuniform convergence space $(X,\mathcal{J}_X)$ is called *(weakly) compactly generated* provided that $\mathcal{J}_X$ is the final **PUConv**-structure with respect to the family $(j_i : (K_i,\mathcal{J}_{K_i}) \to (X,\mathcal{J}_X))_{i\in I}$ of inclusion maps of all (weakly) compact subsets of $(X,\mathcal{J}_X)$.

2. A preuniform convergence space $(X,\mathcal{J}_X)$ is called *(weakly) precompactly generated* provided that $\mathcal{J}_X$ is the final **PUConv**-structure with respect to the family $(j_i : (K_i,\mathcal{J}_{K_i}) \to (X,\mathcal{J}_X))_{i\in I}$ of inclusion maps of all (weakly) precompact subsets of $(X,\mathcal{J}_X)$.

Compact generation and weak compact generation resp. precompact generation and weak precompact generation are equivalent concepts in uniform limit spaces and in **Fil**-determined preuniform convergence spaces.

Theorem 5.1. A preuniform convergence space is (weakly) compactly generated iff it is locally (weakly) compact.

Proof. "$\Rightarrow$". $(X,\mathcal{J}_X)$ is (weakly) compactly generated, i.e., $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : (\exists i \in I$ and $\mathcal{F}_i \in \mathcal{J}_{K_i}$ such that $\mathcal{F} \supset (j_i \times j_i)(\mathcal{F}_i))\}$ (K_i from Definition 5.1). Let $\mathcal{F} \in \mathcal{J}_X$. By assumption, there is $i \in I$ and $\mathcal{F}_i \in \mathcal{J}_{K_i}$ such that $\mathcal{F} \supset (j_i \times j_i)(\mathcal{F}_i)$. Consequently, the (weakly) compact set $(j_i \times j_i)(K_i \times K_i) = K_i \times K_i$ belongs to $\mathcal{F}$.

"$\Leftarrow$". Let $(X,\mathcal{J}_X)$ be a locally (weakly) compact preuniform convergence space and let $\mathcal{J}_X'$ be the final **PUConv**-structure with respect to the family $(j_i : (K_i,\mathcal{J}_{K_i}) \to (X,\mathcal{J}_X))_{i\in I}$ of all inclusion maps of all (weakly) compact subsets of $(X,\mathcal{J}_X)$. We have to show that $\mathcal{J}_X \subset \mathcal{J}_X'$. Let $\mathcal{F} \in \mathcal{J}_X$. By assumption, there is a (weakly) compact set $K' \in \mathcal{F}$. W.l.o.g. let K' be symmetric (K'^{-1} and $(K' \cup K'^{-1}) = (K' \cup K'^{-1})^{-1} \in \mathcal{F}$ are (weakly) compact when K' is (weakly) compact). Denote $K = p_1[K']$, where $p_1 : X \times X \to X$ is the projection in the first component. Then K is (weakly) compact (see [7, Corollary 2.15]). Let $j_K : K \to X$ be the inclusion map. Because $K' \subset K \times K$ and, hence, $(K \times K) \cap F \in \mathcal{F}$ for all $F \in \mathcal{F}$, it exists $\mathcal{F}_K = (j_K \times j_K)^{-1}(\mathcal{F})$ and it holds $(j_K \times j_K)(\mathcal{F}_K) = \mathcal{F}$. Consequently, $\mathcal{F}_K \in \mathcal{J}_K$ and $\mathcal{F} \in \mathcal{J}'_X$. □

Theorem 5.2. A preuniform convergence space is (weakly) precompactly generated iff it is locally (weakly) precompact.

Proof. This can be verified similarly to Theorem 5.1. □

Theorems 5.1 and 5.2 can be considered as generalizations of the respective theorems by Preuß with respect to semiuniform convergence spaces [5, Theorems 6.2.1.14 and 6.2.2.18] (previously published in [3] and [4]). It follows that (weakly) compactly generated preuniform convergence space can be described by axioms – in contrast to the situation of compactly generated topological spaces [5, Remark 6.2.1.16].

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9. Special fields
10. Membership category (See **Table 1** in "**Join ISMS !**")

Individual Membership Application Form

1. Name	
2. Birth date	
3. Academic background	
4. Affiliation	
5. 4's address	
6. Doctorate	
7. Contact address	
8. E-mail address	
9. Special fields	
10. Membership category	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.
The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

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Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.
(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:
(1) Post Office Transfer Account - 00930-3-73982 or
(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.
All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	￥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	￥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73－age) × ￥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

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Notices from the ISMS

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