Online
ISSN 1346-0447

ISSN 1346-0862

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

The Forum for Interdisciplinary Mathematics (FIM)
Special Version

Vol. 85, No. 3 Whole Number 302

March 2023

International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES
Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meeetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21st Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM). MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centurial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, begining from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences " as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathmatical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system** (**IVMS**) which can connect up to twenty sites of a reserch group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit http://www.jams.or.jp.

CORDIAL LABELING ON FAMILIES OF BLADE GRAPH

D Devakirubanithi[1], J Jeba Jesintha[2], R Charishma [3]
[1]Research Scholar (Part-Time), PG Department of Mathematics,
Women's Christian College, University of Madras, Chennai.
Department of Mathematics, St Thomas College of Arts and Science,
University of Madras, Chennai.
[2,3] PG Department of Mathematics, Women's Christian College,
University of Madras, Chennai, India.
E-mail: [1] kiruba.1980@yahoo.com,[2] jjesintha_75@yahoo.com,[3] charishma26042000@gmail.com

Abstract

A binary vertex labeling in a graph G is referred to as cordial labeling[4] if $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$ where $v_g(0)$ and $v_g(1)$ indicate the value of number of vertices with 0's and 1's, similarly $e_g(0)$ and $e_g(1)$ indicate the value of number of edges with 0's and 1's, respectively. In this paper, we show that the Blade graph and its one-point union allows cordial labeling.

Keywords: Graph labeling, Cordial labeling, Path, Blade graph, One point union of graphs.

2010 Mathematics Subject Classification: 05C78

1. Introduction

In graph labeling, vertices, edges, or both are given labels that are expressed by integers when certain factors are met. Rosa's 1967 invention is credited with the greater part of graph labeling methods [12]. In 1987, Cahit[4] made the discovery of cordial labeling. A binary vertex labeling in a graph G is referred to as cordial labeling[4] if $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$ where $v_g(0)$ and $v_g(1)$ indicate the value of number of vertices with 0's and 1's, similarly $e_g(0)$ and $e_g(1)$ indicate the value of number of edges with 0's and 1's, respectively. The graphs below have been shown to be cordial by Andar, Boxwala, and Limaye [1], [2], and [3]. Helms, closed helms, generalized helms, sunflower graphs, flowers made from joining the vertices of a helm's degree one to the central vertex, multiple shells, and one-point unions of helms, gears, closed helms, sunflower graphs and flowers. In [5], Cahit demonstrated the following: All trees are cordial, and $K_{m,n}$ is cordial for all m and n. The cordial nature of the one-point union of multiple subdivided shell graphs of the same order has been demonstrated by Devakirubanithi D and Jeba Jesintha J [7]. Ho, Lee, and Shee [9] demonstrated that the generalised Petersen graph $P(n, k)$ is cordial if and only if $n \not\equiv 2\ (mod\ 4)$ and that a unicyclic graph is not cordial unless it is C_{4k+2} (mod 4). According to Lee and Liu [10], the entire n-partite graph is cordial if and only if three of its partite sets have odd cardinality or less. Every fan is cordial, A 3-regular graph of order n is cordial, according to Liu and Zhu [11] but only if $n \not\equiv 4\ (mod\ 8)$. For $C(n)_m$, the one-point union of n copies of C_m. In his Ph. D. thesis from 2001, Selvaraju [13] proves that any number of copies of a complete bipartite graph can be joined at one point amicable. Sethuraman and Selvaraju [14] have shown a few instances of the union of any number of copies of K_4 with one or many edges eliminated and one edge in common to be amicable. One-point union of n copies of various graphs for cordiality has been studied by Shee and Ho [15]. Labeled graph is used in coding theory, astronomy, communication network, circuit design and crystallographic research. Cordial labeling is used in DNA code word design problems as well as in noisy communication channels. For more results, we may refer Gallian's survey [8].

We show in this paper that the Blade graph allows cordial labeling. We also show that cordial labeling is possible with a one-point union of blade graph.

2. Preliminary definitions

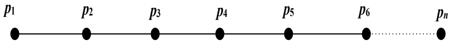
Under this section, we state a definition which is used to prove the theorem.

Definition 2.1

A binary vertex labeling in a graph G is referred to as cordial labeling[4] if $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$ where $v_g(0)$ and $v_g(1)$ indicate the amount of vertices with 0's and 1's, similarly $e_g(0)$ and $e_g(1)$ indicate the amount of edges with 0's and 1's, respectively.

Definition 2.2

A walk of a graph G is an alternating sequence of vertices and edges $v_0e_1v_1e_2v_2e_3,\ldots,v_{n-1}e_nv_n$, with points at the start and end of each line and each line is incident with the two points immediately before and after it. If all the vertices and edges are distinct, it is referred to as $v_0 - v_n$ path [6]. Let P_n is also known as a path of order n points and length $n-1$.



Definition 2.3

The blade graph is defined as follows, Consider the path a_i where $i = 0,1,2,\ldots,m$. Let new edges are formed by a_0 adjacent to a_i ($2 \le i \le m-1$) and a_m is adjacent to a_{m-i} $(2 \le i \le m-1)$. The blade graph is depicted in Figure 1.

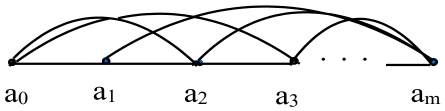


Figure 1. Blade graph

Definition 2.4

One-point union of graphs [7] is defined as the process of joining the apex vertex of each graph to a single common point to create a new graph.

3. Main Results

Under this section, we prove few theorems.

Theorem 1. Blade graph admits cordial labeling.

Proof:

Consider G be the blade graph. The description for the blade graph G is as follows, Consider the path a_i where $i = 0,1,2,\ldots,m$. Let new edges are formed by a_0 adjacent to a_i ($2 \le i \le m-1$) and a_m is adjacent to a_{m-i} $(2 \le i \le m-1)$. The blade graph is depicted in Figure 1.

The collection of vertices is defined as $|V(G)| = m + 1$

The collection of edges is defined as $|E(G)| = 3m - 4$

The vertex labeling is identified as follows:

Case 1: $m \equiv 0(mod 4)$

$$f(a_i) = \begin{cases} 0, & if\ i \equiv 1,2(mod 4) \\ 1, & if\ i \equiv 0,3(mod 4) \end{cases}$$

Case 2: $m \equiv 2(mod 4)$

$$f(a_i) = \begin{cases} 0, & if\ i \equiv 1,0(mod 4) \\ 1, & if\ i \equiv 2,3(mod 4) \end{cases}$$

From the above labeling pattern, we can find the edge labels as follows.

Case 1: $m \equiv 0(mod 4)$

$a_0 a_i = 0 \qquad if\ i \equiv 1,2(mod 4), for\ 2 \leq i \leq m - 1$

$a_{(m-1)} a_{(m-1)-i} = 0 \qquad if\ i \equiv 1,2(mod 4), for\ 2 \leq i \leq m - 1$
$a_i a_{i+1} = 0 \qquad if\ i \equiv 1,2(mod 4), for\ 0 \leq i \leq m - 1$

$a_0 a_i = 1 \qquad if\ i \equiv 0,3(mod 4), for\ 2 \leq i \leq m - 1$

$a_{(m-1)} a_{(m-1)-i} = 1 \qquad if\ i \equiv 0,3(mod 4), for\ 2 \leq i \leq m - 1$
$a_i a_{i+1} = 1 \qquad if\ i \equiv 0,3(mod 4), for\ 0 \leq i \leq m - 1$

Case 2: $m \equiv 2(mod 4)$

$a_0 a_i = 0 \qquad if\ i \equiv 1,0(mod 4), for\ 2 \leq i \leq m - 1$

$a_{(m-1)} a_{(m-1)-i} = 0 \qquad if\ i \equiv 1,0(mod 4), for\ 2 \leq i \leq m - 1$
$a_i a_{i+1} = 0 \qquad if\ i \equiv 1,0(mod 4), for\ 0 \leq i \leq m - 1$

$a_0 a_i = 1 \qquad if\ i \equiv 2,3(mod 4), for\ 2 \leq i \leq m - 1$

$a_{(m-1)} a_{(m-1)-i} = 1 \qquad if\ i \equiv 2,3(mod 4), for\ 2 \leq i \leq m - 1$
$a_i a_{i+1} = 1 \qquad if\ i \equiv 2,3(mod 4), for\ 0 \leq i \leq m - 1$

The number of vertices marked with 0 and 1 is defined as follows.

$V_f(0) = \left\lfloor \frac{m}{2} \right\rfloor + 1$

$V_f(1) = \left\lfloor \frac{m}{2} \right\rfloor$

The number of edges marked with 0 and 1 is defined as follows.

$e_f(0) = \left\lfloor \frac{3m-4}{2} \right\rfloor$

$e_f(1) = \left\lfloor \frac{3m-4}{2} \right\rfloor$

Therefore,

$|V_f(0) - V_f(1)| = \left|\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) - \left(\left\lfloor \frac{m}{2} \right\rfloor\right)\right| = 1$ and

$|e_f(0) - e_f(1)| = \left|\left(\left\lfloor \frac{3m-4}{2} \right\rfloor\right) - \left(\left\lfloor \frac{3m-4}{2} \right\rfloor\right)\right| = 0$

The above labeling technique, satisfies the conditions $|V_f(0) - V_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$

Therefore, cordial labeling is possible for Blade graph.

Illustration for Cordial labeling of blade graph when $m \equiv 0 (mod 4)$ shown in the Figure 2.

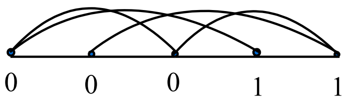


Figure 2. ($m = 4, V_f(0) = 3, V_f(1) = 2, e_f(0) = 4, e_f(1) = 4$)

Illustration for Cordial labeling of blade graph when $m \equiv 2 (mod 4)$ shown in the Figure 3

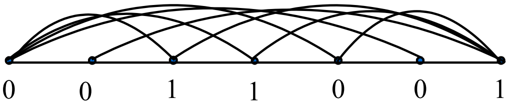


Figure 3 ($m = 6,\ V_f(0) = 4,\ V_f(1) = 3,\ e_f(0) = 7,\ e_f(1) = 7$)

Theorem 2. One-point union of blade graph admits cordial labeling.

Proof:

Consider G to be the one-point union of Blade graph. The description of the graph G is described as follows: The copies of each blade that are connected to the central vertex a_0 represented by the vertices a_i^j $(1 \leq i \leq m, 1 \leq j \leq n)$. Set the central vertex $a_0 = 0$. Let the vertices of the first blade as $a_1^1, a_2^1, a_3^1, \dots, a_m^1$ then the second blade's vertices be $a_1^1, a_2^2, a_3^2, \dots, a_m^2$. Continuing this manner, we obtain $a_1^n, a_2^n, a_3^n, \dots, a_m^n$ as the vertices of n^{th} blade where $m \geq 4$ and $n \geq 3$. The graph is depicted in Figure 4.

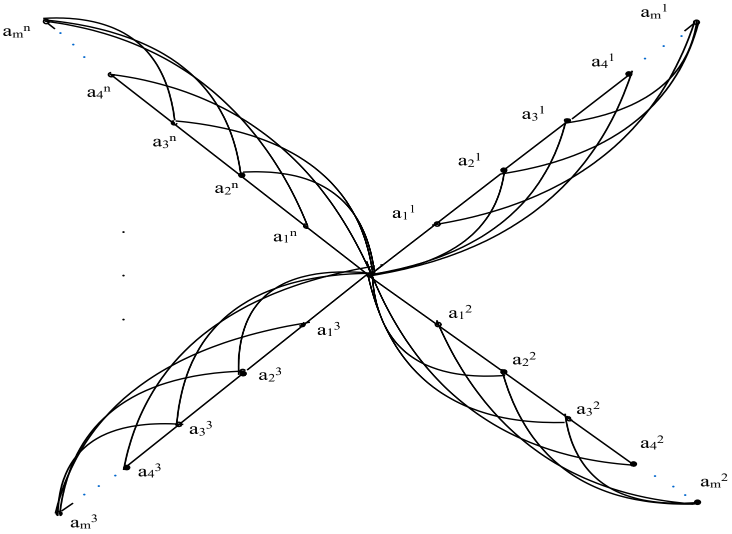


Figure 4. One point union of Blade graph

The collection of vertices is defined as $|V(G)| = nm + 1$

The collection of edges is defined as $|E(G)| = 3nm - 4n$

The vertex labeling is identified as follows:

Case 1: $m \equiv 0(mod4)$

$$f(a_i^j) = \begin{cases} 0, if\ i \equiv 1,2(mod4) \\ 1, if\ i \equiv 0,3(mod4) \end{cases}$$

Case 2: $m \equiv 2(mod4)$

$$f(a_i^j) = \begin{cases} 0, if\ i \equiv 1,0(mod4) \\ 1, if\ i \equiv 2,3(mod4) \end{cases}$$

From the above labeling pattern, we can find the edge labels as follows.

Case 1: $m \equiv 0(mod4), 1 \leq j \leq n$

$a_0 a_i^j = 0 \qquad if\ i \equiv 1, 2(mod4), for\ 2 \leq i \leq m-1$

$a_{(m-1)}^j a_{(m-1)-i}^j = 0 \qquad if\ i \equiv 1, 2(mod4), for\ 2 \leq i \leq m-1$

$a_i^j a_{i+1}^j = 0 \qquad if\ i \equiv 1, 2(mod4), for\ 0 \leq i \leq m-1$

$a_0 a_i^j = 1 \qquad if\ i \equiv 0, 3(mod4), for\ 2 \leq i \leq m-1$

$a_{(m-1)}^j a_{(m-1)-i}^j = 1 \qquad if\ i \equiv 0, 3(mod4), for\ 2 \leq i \leq m-1$

$a_i^j a_{i+1}^j = 1 \qquad if\ i \equiv 0, 3(mod4), for\ 0 \leq i \leq m-1$

Case 2: $\boldsymbol{m} \equiv 2(mod4), 1 \leq j \leq n$

$a_0 a_i^j = 0 \qquad if\ i \equiv 1, 0(mod4), for\ 2 \leq i \leq m-1$

$a_{(m-1)}^j a_{(m-1)-i}^j = 0 \qquad if\ i \equiv 1, 0(mod4), for\ 2 \leq i \leq m-1$

$a_i^j a_{i+1}^j = 0 \qquad if\ i \equiv 1, 0(mod4), for\ 0 \leq i \leq m-1$

$a_0 a_i^j = 1 \qquad if\ i \equiv 2, 3(mod4), for\ 2 \leq i \leq m-1$

$a_{(m-1)}^j a_{(m-1)-i}^j = 1 \qquad if\ i \equiv 2, 3(mod4), for\ 2 \leq i \leq m-1$

$a_i^j a_{i+1}^j = 1 \qquad if\ i \equiv 2, 3(mod4), for\ 0 \leq i \leq m-1$

The number of vertices marked with 0 and 1 is defined as follows.

$$V_f(0) = \left\lfloor \frac{nm}{2} \right\rfloor + 1$$

$$V_f(1) = \left\lfloor \frac{nm}{2} \right\rfloor$$

The number of edges marked with 0 and 1 is defined as follows.

$$e_f(0) = \left\lfloor \frac{3nm - 4n}{2} \right\rfloor$$

$$e_f(1) = \left\lfloor \frac{3nm - 4n}{2} \right\rfloor$$

Therefore

$$\left|V_f(0) - V_f(1)\right| = \left|\left(\left\lfloor \frac{nm}{2} \right\rfloor + 1\right) - \left(\left\lfloor \frac{nm}{2} \right\rfloor\right)\right| = 1 \text{ and}$$

$$\left|e_f(0) - e_f(1)\right| = \left|\left(\left\lfloor \frac{3nm - 4}{2} \right\rfloor\right) - \left(\left\lfloor \frac{3nm - 4}{2} \right\rfloor\right)\right| = 0$$

The above labeling techniques satisfies the conditions that, $|V_f(0) - V_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Therefore, cordial labeling is possible for one-point union of Blade graph.

Illustration for one point union of blade graph admits cordial labeling when $m \equiv 0(mod4)$ is shown in the Figure 5.

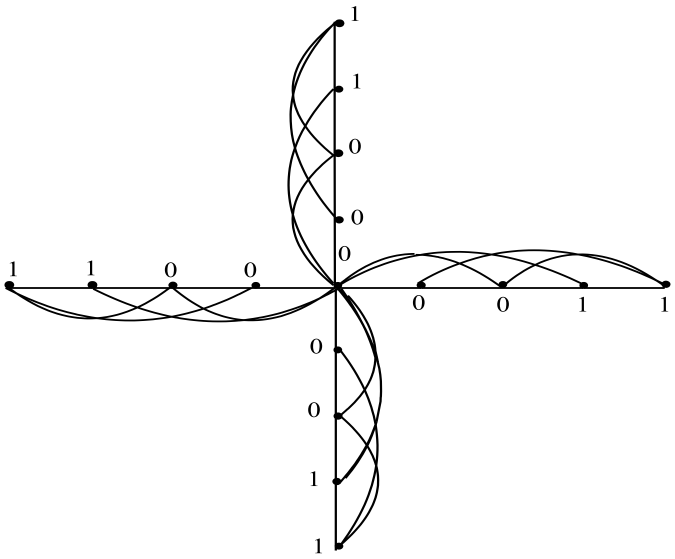


Figure 5. $(m = 4, n = 4, V_f(0) = 9, V_f(1) = 8, e_f(0) = 16, e_f(1) = 16)$

Illustration for one point union of blade graph admits cordial labeling when $m \equiv 2(mod 4)$ is shown in the Figure 6.

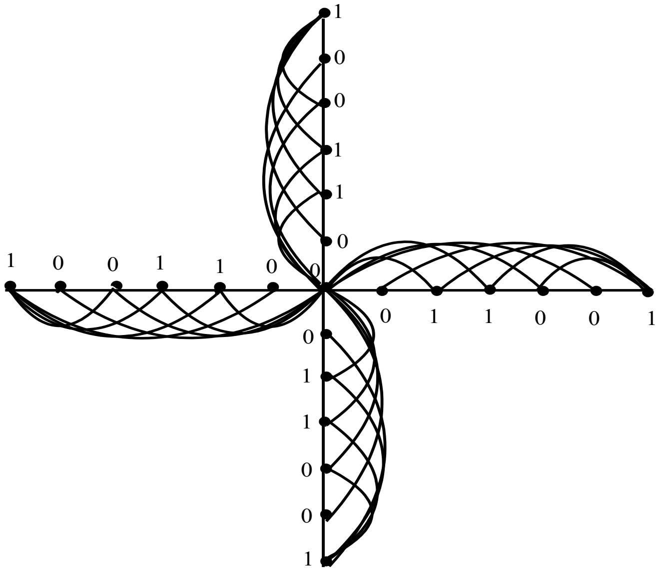


Figure 6. $(m = 6, n = 4, V_f(0) = 13, V_f(1) = 12, e_f(0) = 28, e_f(1) = 28)$

4. Conclusion

The Cordial labeling is used in the problem with the word design in the DNA Code and noisy communication channel. In this paper we have shown that cordial labeling is possible for blade graph and one-point union of blade graph.

References:

[1] M. Andar, S. Boxwala, and N. Limaye, Cordial labelings of some wheel related graphs, J. Combin. Math. Combin. Comput., 41 (2002) 203-208.

[2] M. Andar, S. Boxwala, and N. Limaye, A note on cordial labelings of multiple shells, Trends Math., (2002) 77-80.

[3] M. Andar, S. Boxwala, and N. Limaye, On the cordiality of the t-uniform homeomorphs–I, Ars Combin., 66 (2003) 313-318.

[4] Cahit I, *Cordial graphs: a weaker version of graceful and harmonious graphs*, Ars Combin., 23 (1987), 201-207.

[5] Cahit I, *On cordial and 3-equitable labellings of graphs*, Util. Math., 37 (1990), 189-198.

[6] Bondy J A, Murty, U. S. R. (1976). *Graph Theory with Applications*. North Holland. pp. 12–21. ISBN 0-444-19451-7.

[7] Devakirubanithi D, Jeba Jesintha J, "*Cordial Labeling on Few Graphs of Subdivided shell Graphs*" in "Journal of Algebraic Statistics," Volume 13, No. 2, 2022, p. 3455-3466.

[8] Gallian J. A, *A dynamic survey of Graph labeling*, The Electronic Journal of Combinatorics, (2020), 77-81.

[9] Ho Y S., Lee S M, and Shee S C, *Cordial labellings of unicyclic graphs and generalized Petersen graphs*, Congr. Numer., 68 (1989) 109-122.

[10] Lee S M and Liu A, *A construction of cordial graphs from smaller cordial graphs*, Ars Combin., 32 (1991) 209-214.

[11] Liu Y Z and Zhu B, A *necessary and sufficient condition for a 3-regular graph to be cordial*, Ars Combin., 84 (2007) 225-230.

[12] Rosa A, *On certain valuations of the vertices of a graph, Theory of Graphs* (International Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.

[13] Selvaraju P, *New classes of graphs with α-valuation, harmonious and cordial labelings*, Ph. D. Thesis, Anna University, 2001. Madurai Kamaraj University, 2002.

[14] Sethuraman G and Selvaraju P, *On harmonious and felicitous graphs*: Union of n-copies of edge deleted subgraphs of K_4, preprint.

[15] Shee S C and Ho Y S, The cordiality of one-point union of n-copies of a graph, Discrete Math., 117 (1993) 225-243.

Scientiae Mathematicae Japonicae 85, No.3(2023)(171-182)

FACTORS OF WATSON CRICK PALINDROMES IN DNA COMPUTING

[1]C Annal Deva Priya Dharshini, [2]J Jeba Jesintha, [3]T Jennifer Ruth Nesakirubai

[1] Department of Mathematics, Madras Christian College, University of Madras, Chennai.

[2,3] PG Department of Mathematics, Women's Christian College, University of Madras, Chennai.

E-mail:[1]annaldevapriyadharshini@gmail.com [2]jjesintha_75@yahoo.com, [3]jenniferruth1611@gmail.com

Abstract

DNA is a linear, unbranched polymer with four chemically distinct monomeric subunits: adenine A, thymine T, guanine G and cytosine C, where A binds to T and G binds to C, producing words over the DNA alphabet {A,G,C,T}. The bonding between nucleotides is known as involution mapping [1].

Kari [4, 5] introduced the insertion and deletion operators on DNA strands to study methods of encoding information using DNA in the context of bio - computation. This present study examines the factors of θ - palindromes and factor free words in Watson Crick palindromes.

Keywords : DNA computing, DNA encoding, Watson Crick palindromes, insertion operator.

1 Introduction

Combinatorics on words is a relatively recent school of mathematics that stems from combinatorics, which studies words and formal languages. Combinatorics on words is concerned with non-commutative monoid issues, such as subword complexity of finite or infinite words, infinite word formation and attributes, and inescapable regularities or patterns. Any number can be written as a finite word over an alphabet of digits in various numeration systems[6].

A finite word is a sequence of symbols that span an alphabet. Many fields of research, such as Formal Languages and Automata Theory, Computability and DNA computing, study on words. The nucleotides adenine, guanine, cytosine and thymine, indicated by the symbols $\{A, G, C, T\}$, form a DNA strand, which is the fundamental unit of every living cell. The sequence of nucleotides determine the gene of an organism. These nucleotides bind to each other specifically. DNA computation exploits their structure to encode information as a sequence of nucleotides rather than as a binary sequence listed in conventional computers. In DNA computation, there are three basic phases. The first option is to encode the problem using DNA. The real calculation is done via a sequence of bio-operations. Finally, the output of the computation is sequenced and decoded.

This paper is motivated from [4] and [5], we introduce a variant of the insertion and deletion operator in θ - palindrome. Also, the factor of words and factor free words [2] in θ - palindromes and their properties are studied.

This paper is organized as follows: section 2 reviews the basic concepts, definitions and properties of involutively factor free words. Section 3 deals with the insertion and deletion operators with special reference to factors of θ - palindromes and factor free words in θ - palindromes. Section 4 gives the concluding remarks.

2 Basic Concepts

We first recall the certain basic notions stated in [3].A finite non empty set of symbols denoted by $\sum$ is called **alphabet**. A sequence of symbols being finite or infinite from $\sum$ is **word**. The set of all words over $\sum$ is denoted by $\sum^*$ and the set of all non empty words by $\sum^+$. Let u and v be words in $\sum^+$, if u is a **prefix** of v if there exsits a word $x \in \sum^*$ such that $v = ux$.

A mapping $\theta : \sum^* \to \sum^*$ satisfying the property that $\theta(xy) = \theta(x)\theta(y)$ is a **morphism** on $\sum^*$. If $\theta(xy) = \theta(y)\theta(x)$, then θ is an **antimorphism** on $\sum^*$. θ is an **involution** on $\sum^*$, if $\theta(\theta(x)) = x$, for all $x \in \sum^*$.

In this section, we give some definitions and conditions of palindromic words [4, 5].

Definition 2.1

Let θ be either a morphic or an antimorphic involution on $\sum^*$. Then a word $x \in \sum^*$ is called a **θ - palindrome** if $x = \theta(x)$. For example, let $u = ATAT$. Here, $u = \theta(u)$, where θ is an antimorphic involution with $\theta(A) = T, \theta(T) = A, \theta(C) = G, \theta(G) = C$. The set of all θ - palindromes are denoted as P_θ.

Remark 2.1

1. Let θ be a either morphic or an antimorphic involution and let $u \in \sum^*$,
    (a) $u \in P_\theta$ iff $\sqrt{u} \in P_\theta$.
    (b) $u \in P_\theta$ iff $(u)^n \in P_\theta$, for all $n \geq 1$.
2. Let $\sum$ be such that for all $a \in \sum$, $\theta(a) \neq a$,
    (a) When θ is a morphic involution, then $P_\theta = \{\lambda\}$.
    (b) When θ is an antimorphic involution, then for all $u \in P_\theta$, the length of u is an even number.

Lemma 2.1

Let θ be an antimorphic involution and for all a $\in \sum$, let $a \neq \theta(a)$. Then, x $\in \sum^+$, is a θ -palindrome iff x = ayθ(a), for some a$\in \sum$ and y$\in P_\theta$.

Definition 2.2

Let u, v be words over an alphabet $\sum$, the **insertion** of v into u is defined as: $u \leftarrow v = \{u_1 v u_2 | u = u_1 u_2\}$. For example, let $u = cd, v = a$. The insertion of v into u is $u \leftarrow v = \{acd, cad, cda\}$. Let u, v be words over an alphabet. The **deletion** of v from u is defined as: $u \rightarrow v = \{w \in \sum^* / u = w_1 v w_2, w = w_1 w_2, w_1 w_2\}$.For example, let $u = abababa$ and $v = aba$, the result of the deletion of v from u is $u \rightarrow v = \{baba, abab, abba\}$. The deletion and the insertion operation is neither associative nor commutative [4].

3 Properties of θ - palindromes

The properties of the set of all θ - palindromes across a particular alphabet are discussed in this section. Also, the factor of θ - palindromes and factor free words has been discussed in this paper. The discoveries were made as a result, contributing to the fast growing field of DNA computing. Despite this, these properties aid in the growth of DNA computing.

3.1 Insertion Operator in θ - palindromes

Kari [4] introduced the insertion operation as a generalization of catenation and studied its feasibility for application in DNA computing.

Definition 3.1.1

Let $\sum$ be a finite alphabet. Let $w = w_1w_2\ldots\ldots w_n \in \sum^*$. The word $v \in \sum^*$ is inserted in w after the position k, to get $w' = w_1w_2\ldots w_k\mathbf{v}w_{k+1}w_{k+2}\ldots\ldots w_{2k}$ denoted as $in(w,k,v) = w^{'}$.

Example 3.1.1

Consider $w = AGCTACGTTATTGGCTCCAA$ and let $v = GTGGAAATGGGTACCATG$. Let v be inserted in w after the position $k = 10$. Then, $w^{'} = AGCTACGTTA(GTGGAAATGGGTACCA$ $TG)TTGGCTCCAA = in(w,10,v)$.

Using the insertion operation in θ - palindrome in the following proposition.

Proposition 3.1.1

Let θ be an antimorphic involution. For all, $a \in \sum$, let $a \neq \theta(a)$. Let $u, v \in P_\theta$. Let u be of length $2n$ such that the word u' derived by inserting v in u after the position n will also be a θ - palindrome.

Proof. Let $u \in P_\theta$.Then $u = a_1y_1\theta(a_1)$, where $a_1 \in \sum, y_1 \in P_\theta$, by Lemma 2.1. Since, $y_1 \in P_\theta$. We apply Lemma 2.1 again to the word or set $u = a_1(a_2y_2\theta(a_2))\theta(a_1)$, where $a_2 \in \sum$, $y_2 \in P_\theta$. Apply Lemma 2.1 again, we get, $u = a_1a_2(a_3y_3\theta(a_3))\theta(a_2)\theta(a_1)$, where $a_3 \in \sum$, $y_3 \in P_\theta$. Continuing this way, we reach a stage where $u = a_1a_2a_3...(a_ny_n\theta(a_n))...\theta(a_3)\theta(a_2)\theta(a_1)$, where $y_n = \lambda \in P_\theta$. (i.e), $u = u_1u_2u_3...u_{2n}$ where

$$u_i = \begin{cases} a_i & ;\forall \quad 1 \leq i \leq n \\ a_{n-(i+1)} & ;\forall \quad n+1 \leq i \leq 2n \end{cases} \tag{3.1}$$

Let $v \in P_\theta$ then v is inserted after the position n, we get $u^{'} = u_1u_2\ldots\ldots u_n\mathbf{v}u_{n+1}u_{n+2}\ldots.u_{2n} \in P_\theta$. We illustrate by an example, let $u = ATAATTAT$ and $v_1 = AATT \in P_\theta$. Then, $u^{'} = ATAA(AATT)TTAT \in P_\theta$. We observe that v is not unique. We can also insert $v_2 = ATAT \in P_\theta$ in u, as a result we get, $u^{''} = ATAA(ATAT)TTAT \in P_\theta$. Thus, for a given $u \in P_\theta$ we have a collection of words $v \in P_\theta$ such that v inserted in u will still be a θ - palindrome. The palindrome set derived from u is denoted as $P_i(u,\theta)$ is defined by, for $u \in P_\theta$, $P_i(u,\theta) = \{v \in P_\theta / u^{'} = u_1u_2\ldots\ldots u_n\mathbf{v}u_{n+1}u_{n+2}\ldots.u_{2n} \in P_\theta\}$.

□

Note that the insertion of v into any other position will not result in a θ - palindrome. We'll now look at the various methods for producing a θ - palindrome while powering the words throughout the insertion operation.

Proposition 3.1.2

Let θ be an antimorphic involution. Let $u, v \in P_\theta$ then the word $u^{'} = u_1u_2...u_n\mathbf{v}u_{n+1}u_{n+2}...u_{2n} \in P_\theta$ iff $u_1u_2...u_n = \theta(u_{n+1}u_{n+2}...u_{2n})$.

a. Let $u^k \in P_\theta$ then the word $u^{'} = (u_1u_2...u_n)^kv(u_{n+1}u_{n+2}...u_{2n})^k \in P_\theta$.

b. Similarly, let $v^k \in P_\theta$ then the word $u^{'} = (u_1u_2...u_n)v^k(u_{n+1}u_{n+2}...u_{2n}) \in P_\theta$.

c. Also, when $u^k, v^k \in P_\theta$ then the word $(u^{'})^k = (u_1u_2...u_n)^kv^k(u_{n+1}u_{n+2}...u_{2n})^k \in P_\theta$, which can also be written as, $(u^{'})^k = [(u_1u_2...u_n)v(u_{n+1}u_{n+2}...u_{2n})]_1[(u_1u_2...u_n)v(u_{n+1}u_{n+2}...u_{2n})]_2....[(u_1u_2...u_n)v(u_{n+1}u_{n+2}...u_{2n})]_k \in P_\theta$, where $k \geq 1$.

Example 3.1.2

Drosophila melanogaster, also known as the fruit fly, is one of the most widely used model organisms in biomedical research. The inexpensive cost, quick generation time, and superb genetic tools have made the fly indispensable for basic research for almost a century. Thus, its genome's euchromatic part of 120 megabases has been sequenced and was first published in Science on March 24, 2000.

Consider a strand from fruit fly $u = TATATATATATA$ and $v = GCGC$ Let $k = 3$.

(i)when $(u)^3 \in P_\theta$ then the word $u^{'} = TATATATATATATATATATA(GCGC)TATATATATATATATATATA \in P_\theta$.

(ii)$v^3 \in P_\theta$ then the word, $u^{'} = TATATA(GCGCGCGCGCG)CTATATA \in P_\theta$.

(iii)$u^3, v^3 \in P_\theta \Rightarrow (u')^3 = TATATATATATATATATATA(GCGCGCGCGCGC)TATATATATATATATATATA \in P_\theta$ (or) which can also be written in this form, $(u^{'})^3 = TATATA(GCGC)TATATATATATA(GCGC)TATATA$ $TATATA(GCGC)TATATA \in P_\theta$, and on proceeding further we still obtain a palindrome. Similarly, $u^{'}, (u^{'})^2, (u^{'})^3, ..., (u^{'})^n$.

The insertion operation is observed multiple times in the next definition.

Definition 3.1.2

Let θ be an antimorphic involution. For all, $a \in \sum$, let $a \neq \theta(a)$. For a given $u = u_1u_2 \ldots \ldots u_nu_{n+1}u_{n+2}$ $...u_{2n} \in P_\theta$. We have a collection of words $\{v_k\} \in P_\theta$ such that it is inserted multiple times in u after the position n of that word which will still be a θ - palindrome. $u^{(k)} = u_1^{(k-1)} v^k u_2^{(k-1)}$, where $\{v_k\} \in P_\theta$ is the collection of words inserted in u.

Example 3.1.3

Consider a strand from the genome of Drosophila Melanogaster, which is the biological name of Fruit fly $u = TATATATATATA$ and $\{v_k\} = \{GCGC, ATAT, TA, AAATTT, GCGCGCGC\} \in P_\theta$, where $k = 5$, inserting v_1 in u after the position $n = 6$ then $u^{'} = u_1 v_1 u_2 = TATATA(GCGC)TATATA \in P_\theta$. Again on insertion after the position $n = 8$, then $u^{''} = u_1{'} v_2 u_2^{'} = TATATAGC(ATAT)GCTATATA \in P_\theta$. Similarly, when we proceed further, we obtain, insertion of the fifth word after the position $n = 14$, $u^{(5)} = u_1^{(4)} v_5 u_2^{(4)} = TATATAGCATTAAA(GCGCGCGC)TTTAATGCTATAT \in P_\theta$.

We study the special case of **self insertion** or a word inserted into itself in the following proposition.

Proposition 3.1.3

Let $v \in P_\theta$ and $v = x.y$, then the word $v^{'} = xvy \in P_\theta$ (i.e) $v \in P_\theta$ inserted in v itself will also be a θ - palindrome.

Proof. Given that $v \in P_\theta$ and $v = x.y \Rightarrow xy \in P_\theta$, where y is the image of x. Since, $y = \theta(x)$. Thus, $|x| = |y|$. To prove that, $v^{'} = xvy \in P_\theta$, from the definition of θ - palindrome, $x = \theta(x)$, where θ is an antimorphic involution. We know that, $v = xy \Rightarrow \theta(v) = \theta(y)\theta(x)$. Therefore, $\theta(v^{'}) = \theta(y)\theta(v)\theta(x)$. Thus, it is sufficient to show that, $\theta(v^{'}) = \theta(y)\theta(v)\theta(x) \in P_\theta$. $v^{'} = xvy = \theta(y)\theta(v)\theta(x) \Rightarrow x(xy)y = \theta(y)\theta(xy)\theta(x) \Rightarrow (xx)(yy) = \theta(y)\theta(y)\theta(x)\theta(x) = \theta(yy)\theta(xx) \Rightarrow x^2y^2 = \theta(y)^2\theta(x)^2 \Rightarrow (xy)^2 = (\theta(y)\theta(x))^2 \in P_\theta$ $(Since, u \in P_\theta$ iff $\sqrt{u} \in P_\theta)$[4]. Thus, $\theta(v^{'}) = \theta(y)\theta(v)\theta(x) \in P_\theta$. □

Observation 3.1.1

If $w^{'} = w_1 w_2 \ldots\ldots w_n \mathbf{v} w_{n+1} w_{n+2} \ldots . w_{2n} \in P_\theta$ be the word derived by inserting a word $v \in P_\theta$ in $u = w_1 w_2 w_3 ... w_{2n}$ then $|w^{'}| = |w| + |v|$.

Corollary 3.1.1

Let θ be an antimorphic involution. For all, $a \in \sum$, let $a \neq \theta(a)$. Let $u, v \in P_\theta$ and let $P_i(u, \theta)$ be a derived θ -palindrome set, for

(i) $v \in P_i(u, \theta)$ iff $\theta(v) \in P_i(u, \theta)$.

(ii) $v \in P_i(u,\theta)$ iff $v^n \in P_i(u,\theta)$.

(iii) $v \in P_i(u,\theta)$ iff $\sqrt{v} \in P_i(u,\theta)$.

(iv) if $v \in P_\theta$ is an empty word inserted in $w \in P_\theta$ then w maps into itself.

(v) Every collection of words $\{v_k\}$ inserted in $u = u_1u_2\ldots\ldots u_nu_{n+1}u_{n+2}\ldots.u_{2n} \in P_\theta$ will not always be distinct.

Proof. (i) Given that $v \in P_\theta$ be a word inserted in a derived θ - palindrome set. From the definition of θ - palindrome, $x = \theta(x)$, therefore, if $v \in P_i(u,\theta)$ then $\theta(v) \in P_i(u,\theta)$. And the converse is also true. For example, consider $v = AATT = \theta(AATT) = TTAA = \theta(v)$ such that $P_i(u,\theta) = ATAA\mathbf{AATT}TTAT = TATT\mathbf{TTAA}AATA = P_i(u,\theta)$, where θ be an antimorphic involution.

(ii) Given that $v \in P_\theta$. From Remark 2.2.1, 1(b) $u \in P_\theta$ iff $(u)^n \in P_\theta$, for all $n \geq 1$. Thus, $v \in P_i(u,\theta)$ iff $v^n \in P_i(u,\theta)$.

(iii) Given that $v \in P_\theta$. From Remark 2.2.1, 1(a) $u \in P_\theta$ iff $\sqrt{u} \in P_\theta$. Hence, $v \in P_i(u,\theta)$ iff $\sqrt{v} \in P_i(u,\theta)$.

(iv)Given that, v is an empty word then $v = \lambda$. From the observation 3.1.1, we know that $|w^{'}| = |w| + |v|$. Let us assume that, $|w| = 2n$ and since $v = \lambda \Rightarrow |v| = 0$ then, $|w^{'}| = |w| + |v| = 2n + 0 = 2n = |w|$. Thus, $|w^{'}| = |w|$. In other words, w maps into itself.

(v) When, $k = 3$, then $\{v_k\} = \{v_1, v_2, v_3\}$. Assume that, $v_1 = v_2$, then on inserting v_1 in u, $u^{'} = u_1v_1u_2 \in P_\theta$. Again on inserting v_2 in $u^{'}$, where $v_1 = v_2$ then $u^{''} = u_1^{'}v_2u_2^{'} \in P_\theta$.On insertion of $u^{'''} = u_1^{''}v_3u_2^{''} \in P_\theta$. Therefore, the words inserted in u cannot always be distinct. ☐

The difference between Lila Kari's insertion operator [4] and θ - palindrome's insertion operator is that in the former the insertion can happen at any position, whereas here the insertion can only happen at a specific position by inserting a θ - palindrome sub string into a θ - palindrome string to form a θ - palindrome strand.

3.2 Deletion Operator in θ - palindromes

Lila Kari also made the quotient operation, which is the deletion operator, more generic. There were only two ways to remove symbols or words from an initial word: left quotient or right quotient, where the symbol or word could only be removed from one of the two extremities. Kari devised the deletion operation, which allowed the removal of symbols or words from any location inside a given word[4].

We have modified this deletion opertation by specifing the position at which the letter is deleted. This is specifically useful in the study of θ - palindromes.

Definition 3.2.1

Let $w = w_1 w_2 ... w_n \in \sum^*$. We say that, $v \in \sum^*$ is deleted in w after the position k. Let $v = w_{k+1} ... w_{k+r}$ and we get $w^{'} = w_1 w_2 ... w_k w_{k+r+1} ... w_n$, where v contains a set of words from w. Denoted as $del(w, k, v) = w^{'}$.

Example 3.2.1

We take a sequence from Drosophila Melanogaster $w = CGAAATGTGGGT(GAGAGTTTAAGGCTCCAT)$ $CGGTTTTGACTTTCTGATTGTACAAATATGTATATATATATACTATAGCATACATACCTTCGA$ $CTAATTAATGCAAAACGCTGGCCGGCA$. A word $v = GAGAGTTTAAGCTCCAT$ be deleted in w after the position $k = 12$. Then the derived word be, $w^{'} = CGAAATGTGGGTCGGTTTTGACTTTCTGATTGT$ $ACAAATATGTATATATATATACTATAGCATACATACCTTCGACTAATTAATGCAAAACGCTGG$ $CCGGCA$

Using the deletion operation in θ - palindrome.

Proposition 3.2.1

Let θ be an antimorphic involution. For all $a \in \sum$, let $a \neq \theta(a)$. Let $u, v \in P_\theta$. Let u be of length $2n$ such that the word $u^{'}$ derived by deleting v in u after the position n will also be a θ - palindrome. The palindrome set derived from u is denoted as $P_d(u, \theta)$ is defined by, for $u \in P_\theta$, $P_d(u, \theta) = \{v \in P_\theta / u^{'} = u_1 u_2 \ldots \ldots u_k u_{k+r+1} u_{k+r+2} \ldots .u_{2n} \in P_\theta\}$.

Observartion 3.2.1

If $w^{'} = w_1 w_2 \ldots \ldots w_k w_{k+r+1} w_{k+r+2} \ldots .w_{2n} \in P_\theta$ be the word derived by deleting a word $v \in P_\theta$ in $w = w_1 w_2 w_3 ... w_{2n}$ then $|w^{'}| = |w| - |v|$.

In the following statement, we look at the particular situation of **self deletion** or a term that is deleted into itself.

Proposition 3.2.2

A word $w \in P_\theta$ deleted from itself will also be a θ - palindrome.

Proof. Let $w \in P_\theta$ and $w = x.y \Rightarrow xy \in P_\theta$, where y is the image of $x \Rightarrow y = \theta(x) \in P_\theta$. Thus, $|x| = |y|$. To prove that, w deleted from itself will be a θ - palindrome. (i.e) it is enough to show that $w^{'} = \lambda \in P_\theta$. Let $w = x.y \Rightarrow xy \in P_\theta \Rightarrow w = x\lambda y \in P_\theta$, where λ is an empty word belongs to P_θ [4]. On deleting the word

$w = x.y$ from that word itself, we obtain a derived word in the form of $w = \lambda \in P_\theta$.

The multiple deletion is observed in the next definition. $\square$

Definition 3.2.2

Let θ be an antimorphic involution. For all, $a \in \sum$, let $a \neq \theta(a)$. For a given $u = u_1u_2u_3...u_{2n} \in P_\theta$. We have a collection of words $v_k \in P_\theta$ such that v is deleted multiple times in u after the position n of that word which will still be a θ - palindrome. $u^{(k)} = \lambda \in P_\theta$, where $\{v_k\}$ is the collection of words deleted in u.

Corollary 3.2.1

Let θ be an antimorphic involution. For all, $a \in \sum$, let $a \neq \theta(a)$. Let $u \in P_\theta$ and let $P_d(u, \theta)$ be a derived θ - palindrome set, for

(i) $v \in P_d(u, \theta)$ iff $\theta(v) \in P_d(u, \theta)$.

(ii) $v \in P_d(u, \theta)$ iff $v^n \in P_d(u, \theta)$.

(iii) $v \in P_d(u, \theta)$ iff $\sqrt{v} \in P_d(u, \theta)$.

(iv) If $v \in P_\theta$ is an empty word deleted from $w \in P_\theta$ then w maps into itself.

(v) Let $u = u_1u_2u_3...u_{2n} \in P_\theta$.If a word $|v| = 2n-1$ is deleted from u then $u^{'}$ will not be a θ - palindrome.

(vi) Every collection of words $\{v_k\}$ deleted from $u = u_1u_2.......u_nu_{n+1}u_{n+2}.....u_{2n} \in P_\theta$ will not always be distinct.

Proof. (i) Given that $v \in P_\theta$ be a word deleted from a derived θ - palindrome set. From the definition of θ - palindrome, $x = \theta(x)$, therefore, $v \in P_d(u, \theta)$ iff $\theta(v) \in P_d(u, \theta)$.

(ii) Given that $v \in P_\theta$. From Remark 2.1, $u \in P_\theta$ iff $(u)^n \in P_\theta$, for all $n \geq 1$. Thus, $v \in P_d(u, \theta)$ iff $v^n \in P_d(u, \theta)$.

(iii) Given that $v \in P_\theta$. From Remark 2.1, $u \in P_\theta$ iff $\sqrt{u} \in P_\theta$. Hence, $v \in P_i(u, \theta)$ iff $\sqrt{v} \in P_i(u, \theta)$.

(iv) Given that, v is an empty word then $v = \lambda$. From the Observation 3.2.1, we know that $|w^{'}| = |w| - |v|$. Let us assume that, $|w| = 2n$ and since $v = \lambda \Rightarrow |v| = 0$ then, $|w^{'}| = |w| - |v| = 2n - 0 = 2n = |w|$. Thus, $|w^{'}| = |w|$. In other words, w maps into itself.

(v) Let $u \in P_\theta$ when $n = 1$, then $|v| = 1$ be a word deleted from u. Thus, we get, $u^{'} = u_1u_2u_3...u_{n-1}u_{n+1}...u_{2n} \notin P_\theta$ and $v = u_n \notin P_\theta$. Similarly, when $n = 2$, then $|v| = 3$ be the word deleted from u. Thus, we get, $u^{'} = u_1u_2u_3...u_{n-2}u_{n+}...u_{2n} \notin P_\theta$ where $v = u_{n-1}u_nu_{n+1} \notin P_\theta$. In general, if a word v of odd length is been removed from a word u of even length then the derived word $u^{'}$ of odd length which will not be a θ - Palindrome. Thus, deletion can happen if v is of even length.

(vi) When $k = 3$, then $\{v_k\} = \{v_1, v_2, v_3\}$. Assume that, $v_1 = v_2$ of length 2 and v_3 is of length 4, then on deleting v_1 in u, we get $u^{'} = u_1u_2...u_{n-1}u_{n+2}...u_{2n} \in P_\theta$, where $v_1 = u_nu_{n+1} \in P_\theta$. Again on deleting v_2 from $u^{'}$, where $v_1 = v_2$ then $u^{''} = u^{'}_1u^{'}_2...u^{'}_{n-2}u^{'}_{n+3}...u^{'}_{2n} \in P_\theta$, where $v_2 = u_{n-1}u_{n+2} \in P_\theta$ .On deletion of v_3 from $u^{''}$ we obtain, $u^{'''} = u^{''}_1u^{''}_2...u^{''}_{n-4}u^{''}_{n+5}...u^{''}_{2n} \in P_\theta$, where $v_3 = u_{n-3}u_{n-2}u_{n+3}u_{n+4} \in P_\theta$. Therefore, the words deleted from u need not always be distinct. □

The difference between Lila Kari's deletion operator [4] and θ - palindrome's deletion operator is that in the former deletion can happen at any position, whereas here the deletion can only happen at a specific position by deleting a θ - palindrome sub string from a θ - palindrome strand to form a θ - palindrome string.

The insertion and deletion operators are inverse operators. As we have, $in(w, k, v) = w^{'}$ iff $del(w^{'}, k, v) = w, \forall\ w, w^{'}, v \in \sum^*$.

3.3 Factors of θ - palindromes

For a word $w \in \sum^*$, the word $u \in \sum^+$ is a factor of w if there are words $\alpha, \beta \sum \sum^*$ such that $w = \alpha u \beta$. If $\alpha = \lambda$ then u is called a prefix of w and if $\beta = \lambda$ then u is called a suffix of w. The set of all factors of w is denoted by $F(w)$ [2].

Definition 3.3.1

Let θ be a morphic or an antimorphic involution on $\sum^*$. Let $F(w)$ be the factor of words, $F(w) = w_1w_2...w_n$, where $w_i \in \sum^*$. If w_i and w_j be two words in $F(w)$ which are not a θ - palindrome. On catenating the suffix of w_i to the prefix of w_j in such a way that the derived $u \in P_\theta$.

Proposition 3.3.1

Let θ be a morphic or an antimorphic involution on $\sum^*$. Let $F(w)$ be the factor of words, $F(w) = w_1w_2...w_n$, where $w_i \in \sum^*$. If w_i and w_j be two words in $F(w)$ which are not a θ - Palindrome. On catenating the suffix of w_i to the entire word of w_j in such a way that the derived $u \in P_\theta$.

Observation 3.3.1

1. Let $S(w_i)$ be the suffix of the word w_i and let $P(w_j)$ be the prefix of the word w_j, such that,

   (a) If $|S(w_i)| = |P(w_j)|$ then $S(w_i) = \theta(P(w_j))$.
   (b) If $|S(w_i)| \neq |P(w_j)|$ then $S(w_i) \neq \theta(P(w_j))$, for $i = 1, 2, ..., n$.

2. Let $S(w_i)$ be the suffix of the word w_i and let $E(w_j)$ be the entire word of w_j, such that,
   (a) If $|S(w_i)| = |E(w_j)|$ then $S(w_i) = \theta(E(w_j))$.
   (b) If $|S(w_i)| \neq |E(w_j)|$ then $S(w_i) \neq \theta(E(w_j))$, for $i = 1, 2, ..., n$.

3. If the factor of words $F(w) = w_1w_2...w_n$, where $w_i \in \sum^*$ has atleast one θ - palindrome $w_i \in P_\theta$ then the mirror image of the factor of $F\bar{(w)} = w_nw_{n-1}...w_2w_1$ has the same number of θ - palindromes obtained in $F(w)$ ia an image of $F\bar{(w)}$.

3.4 Factor Free Words

Let θ be a morphic or an antimorphic involution on $\sum^*$. A word $w \in \sum^*$ is a θ - factor free or involutive factor free if none of its factors is a θ - factor of w.

For example, consider the word, $w = acccccaaa$ over $\sum = \{a, t, c, g\}$. Let θ be a morphic involution on $\sum^*$ defined by $\theta(a) = t, \theta(t) = a, \theta(c) = g, \theta(g) = c$. No factor u of w is a θ - factor of w, as $\theta(w)$ is not a factor of w [2].

⇒ Factor free words can be a palindrome like *aaccccaa*.

⇒ On mapping, $aaccccaa = \theta(aaccccaa) = ttggggtt$.

⇒ $aaccccaa \neq ttggggtt \notin P_\theta$.

⇒ Thus, it cannot be a θ - palindrome, because $\theta(u) \neq u$.

4 CONCLUSION

Using the insertion and deletion operators within the DNA, we defined θ - palindrome. It may be used to find the θ - palindromic words in DNA. To build a practical DNA computer, a number of challenges must be addressed. Because it is application specialised, it will not replace present computers, but it has the potential to replace high-end and research-oriented systems in the future. We can additionally use the insertion and deletion operators for languages to expand the technique to locate the θ - palindromes. The insertion operator is only used in a finite word within a DNA molecule, but it may be extended to infinite words in DNA. Using the insertion and deletion operators within the DNA, we defined θ - palindromes. It may be used to find the θ - palindromic words in DNA.

References

[1] Adleman L.M.,(1998), “Computing with DNA”, Scientific American, 34 – 41.

[2] Annal Deva Priya Dharshini C, Rajkumar Dare V, Ibrahim Venkat and Subramaniam K.G.,(2013-2014), “Factors of Words under an Involution”, Journal of Mathematics and Informatics, Vol-1, 52-59.

[3] Berstel J, Dominique Perrin,(2007), ”The origins of combinatorics on words”, European Journal of Combinatorics, Vol 28, 996-1022.

[4] Lila Kari,(1992), “Insertion and Deletion of Words: Determinism and Reversibility”, International Symposium on Mathematical Foundations of Computer Science, 315 - 326.

[5] Lila Kari, Kalpana Mahalingam(2009 -2010),”Watson Crick Palindromes in DNA Computing”, Natural Computing, Vol 9, 297–316.

[6] M.Lothaire, ”Combinatorics on Words”, Addison Wesley Publishing Company, 1983.

INTUITIONISTIC MULTI L – FUZZY NORMAL SUBGROUPS

C.GOWRISHANKAR[1] **and M. AKILESH**[2]

[1] Department of Mathematics, Government Arts and Science College, Gudalur, The Nilgiris–643212, India.

[2] Department of Mathematics, SRMV College of Arts and Science, Coimbatore-641 020, Tamil Nadu, India.

*Email: [1]cgowrishankar88@gmail.com and [2]akileshmaths@gmail.com

ABSTRACT

The multi dimensional fuzziness dealt with the multi-fuzzy set theory and which is an extension of fuzzy set theory. In that paper, we have initially start with basic definitions of an Intuitionistic Multi L- Fuzzy Subgroups using Multi L- Fuzzy Subgroups. Further, we have to introduce the definition about an Intuitionistic Multi L – Fuzzy Normal Subgroups followed by Intuitionistic L- Fuzzy Normal Subgroups and discuss some of its properties.

Keywords: Intuitionistic Multi L-Fuzzy Set (IMLFS), Intuitionistic Multi L–Fuzzy Subgroup (IMLFSG), Intuitionistic Multi Anti L–Fuzzy Subgroup (IMALFSG), Intuitionistic Multi L–Fuzzy Normal Subgroup (IMLFNSG).

INTRODUCTION

In 1965, L.A.Zadeh [19] was introduced the notion of fuzzy subset A of a set X as a function from X into $I = [0,1]$. Azriel Rosenfeld [13] defined on fuzzy subgroup of a given group and derived some of their properties. The concept of an anti-fuzzy subgroup was introduced by Biswas [4]. S.Sabu and T.V.Ramakrishnan [14] proposed the theory of multi fuzzy sets in terms of multi dimensional membership functions. The idea of multi fuzzy subgroups was introduced by Souriar Sebastian and S.Babu Sundar [17]. N.Palaniappan, S.Naganathan and K.Arjunan [11] establised the theory of intuitionistic fuzzy subgroup is an augmentation theories of fuzzy subgroups. The notion of intuitionistic fuzzy set was initiated by Atanassov [2] as a generalization of Zadeh's fuzzy sets. After the introduction of the conception of intuitionistic fuzzy group by Biswas.

Several researcher's tried to generalize the concept of intuitionistic fuzzy group, for example J.Zhan and Z.Tan [20] define the notion of intuitionistic fuzzy M- group, R.Muthuraj and S.Balamurugan [10] beliefs the intuitionistic multi fuzzy subgroups, Fathi and salleh [7] introduced the idea of intuitionistic fuzzy group based on the notion of fuzzy space.

The identity of fuzzy normal subgroup was first initiated by Abdul Razak [7] to continue the theory of fuzzy groups obtained by Dib. In 1998 Dib and Hassan [6] introduced and discussed the fuzzy normal group in a same manner to Razak.

Many authors can be defined using degree of membership function in multi L- fuzzy normal subgroups. But in that paper we can used to define both the degree of membership and degree of non-membership function in multi L- fuzzy normal subgroups.

Also, in that paper we extend the study of intuitionistic multi L- fuzzy subgroups by generalizing the notion of intuitionistic multi L-fuzzy normal subgroups and discuss some of its properties.

PRELIMINARIES

2.1 Definition

Let X be a non-empty set. A fuzzy set μ of X is defined by $\mu: X \to [0,1]$.

2.2 Definition

Let $(G,.)$ be a group. A fuzzy subset μ of G is said to be L-fuzzy subgroup (LFSG) of G, if the following conditions are satisfied:

(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) = \mu(x), \forall\, x, y \in G.$

2.3 Definition

Let X be a fixed non-empty set. A Multi L-fuzzy subset (MLFS) μ in X is defined as a set of ordered sequences. $\mu = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_i(x), \ldots.): x\epsilon X\}$, where $\mu_i: X \to [0,1]$, for all i. Also note that, for all $i, \mu_i(x)$ is a decreasingly ordered sequence of elements. ie.,$\mu_1(x) \geq \mu_2(x) \geq. \ldots.. \geq \mu_i(x) \geq. \ldots..$, for all $x\epsilon X$.

2.4 Definition

A Multi L-fuzzy subset μ of a group G is called a Multi L-fuzzy subgroup of G (MLFSG), if

(i) $\mu(x, y) \geq min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) = \mu(x), \forall\, x, y\epsilon G.$

2.5 Definition

A Multi L-fuzzy subset μ of a group G is called a Multi Anti L-fuzzy subgroup of G (MALFSG), if (i) $\mu(x, y) \leq max\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) = \mu(x), \forall\, x, y\epsilon G.$

2.6 Definition

Let X be a fixed non-empty set. An Intuitionistic L-fuzzy subset (ILFS) μ of X is an object of the form $\mu = \{x, \mu_A(x), \gamma_A(x): x\epsilon X\}$, where $\mu_A: X \to [0,1]$ and $\gamma_A: X \to [0,1]$.

Define the degree of membership and degree of non-membership of the element $x\epsilon X$ respectively with $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$, for all $x\epsilon X$.

Remark

(a) When $\mu_A(x) + \gamma_A(x) = 1$, ie.,when $\gamma_A(x) = 1 - \mu_A(x) = \mu_A^c(x)$. Then μ is called as L-fuzzy set.

(b) We use an inscription $\mu = (\mu_A, \gamma_A)$ to denote the Intuitionistic L-fuzzy subset (ILFS) μ of X.

2.7 Definition

Let X be non-empty set. Let $\mu = \{x, \mu_A(x), \gamma_A(x): x\epsilon X\}$ in X is defined [2] as a set of ordered sequences. ie., $\mu = \{x, (\mu_{A_1}(x), \mu_{A_2}(x), \ldots., \mu_{A_i}(x), \ldots.), (\gamma_{A_1}(x), \gamma_{A_2}(x), \ldots., \gamma_{A_i}(x), \ldots.): x\epsilon X\}$. Where $\mu_{A_i}: X \to [0,1]$, $\gamma_{A_i}: X \to [0,1]$ and $0 \leq \mu_{A_i}(x) + \gamma_{A_i}(x) \leq 1$ for all i.

Here, $\mu_1(x) \geq \mu_2(x) \geq. \ldots. \geq \mu_i(x) \geq. \ldots.$, for all $x\epsilon X$ are decreasingly ordered sequence. Then the set μ is said to be an Intuitionistic Multi L-fuzzy subset (IMLFS) of X.

Remark

Since we arrange the membership sequence in decreasing order, the corresponding non-membership sequence may not be in decreasing or increasing order.

2.8 Definition

The Intuitionistic Multi L-fuzzy subset $\mu = \{x, \mu_A(x), \gamma_A(x): x\epsilon X\}$ of a group G is said to be Intuitionistic Multi L-fuzzy subgroup [5] of G (IMLFSG) if it satisfies the following: For all $x, y\epsilon G$,

(i) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$,

(ii) $\mu_A(x^{-1}) = \mu_A(x)$ and $\gamma_A(x^{-1}) = \gamma_A(x)$.

Or Equivalentlyμ is IMLFSG of G iff

$\mu_A(xy^{-1}) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy^{-1}) \leq max\{\gamma_A(x), \gamma_A(y)\}$.

3. PROPERTIES OF INTUITIONISTIC MULTI L – FUZZY NORMAL SUBGROUPS (IMLFNSG)

In this fragment, we discuss about the intuitionistic multi L-fuzzy normal subgroups [6] and some of its properties.

3.1 Definition

A Multi L-fuzzy subgroup μ_A of a group G is called a Multi L-fuzzy normal subgroup (MLFNSG) of G, if for every $x, y \epsilon G$ then $\mu_A(xy) = \mu_A(yx)$ *or* $\mu_A(xyx^{-1}) \geq \mu_A(y)$.

3.2 Definition

The Intuitionistic L-fuzzy subgroup of a group G is called an Intuitionistic L-fuzzy normal subgroup (ILFNSG) of G [8], if for every $x, y \epsilon G$ then

(i) $\mu(xyx^{-1}) \geq \mu(y)$

(ii) $\gamma(xyx^{-1}) \leq \gamma(y)$

3.3 Definition

The Intuitionistic Multi L-fuzzy subgroup of a group G is called an Intuitionistic Multi L-fuzzy normal subgroup (IMLFNSG) of G, if for every $x, y \epsilon G$ then

(i) $\mu_A(xyx^{-1}) \geq \mu_A(y)$

(ii) $\gamma_A(xyx^{-1}) \leq \gamma_A(y)$

Where $\mu_A = \max(\mu_{A_1}, \mu_{A_2}, \mu_{A_3}, \dots\dots)$ and $\gamma_A = \min(\gamma_{A_1}, \gamma_{A_2}, \gamma_{A_3}, \dots\dots.)$.

3.4 Theorem

Let G be a group and μ be an IMLFSG of G, then the following are equivalent.

(i) μ_A and γ_A be an IMLFNSG of G,

(ii) $\mu_A(xyx^{-1}) = \mu_A(y)$ and $\gamma_A(xyx^{-1}) = \gamma_A(y)$, for all $x, y \epsilon G$,

(iii) $\mu_A(xy) = \mu_A(yx)$ and $\gamma_A(xy) = \gamma_A(yx)$, for all $x, y \epsilon G$.

Proof

Given $\mu = \{x, \mu_A(x), \gamma_A(x): x\epsilon X\}$ be an IMLFSG of G.

To prove: (i) ⇒ (ii)

Let us assume that μ_A be an IMLFNSG of G.

Then, $\mu_A(xy) \geq \mu_A(yx)$ for all $x, y \epsilon G$.

By taking an arbitrary property of x we have,

$$\mu_A(x^{-1}y(x^{-1})^{-1}) \geq \mu_A(y)$$

Now, $\mu_A(y) = \mu_A(x^{-1}(xyx^{-1})(x^{-1})^{-1})$

$= \mu_A(xyx^{-1})$

$\geq \mu_A(y)$ [∵ By definition 3.2]

Hence, $\mu_A(xyx^{-1}) = \mu_A(y)$ ……………….. (1)

Similarly, let us assume that γ_A be an IMLFNSG of G.

Then, $\gamma_A(xy) \leq \gamma_A(yx)$ for all $x, y \in G$.

By taking an arbitrary property of x we have,

$$\gamma_A(x^{-1}y(x^{-1})^{-1}) \leq \gamma_A(y)$$

Now, $\gamma_A(y) = \gamma_A(x^{-1}(xyx^{-1})(x^{-1})^{-1})$

$= \gamma_A(xyx^{-1})$

$\leq \gamma_A(y)$ [∵ By definition 3.2]

Hence, $\gamma_A(xyx^{-1}) = \gamma_A(y)$ ……………….. (2)

∴ From (1) & (2), $\mu_A(xyx^{-1}) = \mu_A(y)$ and $\gamma_A(xyx^{-1}) = \gamma_A(y)$, for all $x, y \in G$.

To prove: (ii) ⇒ (iii)

Assume that $\mu_A(xyx^{-1}) = \mu_A(y), \ \forall\, x, y \in G$.

Take $y = yx$, we get

$$\mu_A(x(yx)x^{-1}) = \mu_A(yx)$$
$$\mu_A(xy(xx^{-1})) = \mu_A(yx)$$

Therefore, $\mu_A(xy) = \mu_A(yx)$

Similarly, assume that $\gamma_A(xyx^{-1}) = \gamma_A(y)$, for all $x, y \in G$.

Take $y = yx$, we get

$$\gamma_A(x(yx)x^{-1}) = \gamma_A(yx)$$
$$\gamma_A(xy(xx^{-1})) = \gamma_A(yx)$$

Therefore, $\gamma_A(xy) = \gamma_A(yx)$

To prove: (iii) ⇒ (i)

Let us assume $\mu_A(xy) = \mu_A(yx)$ for all $x, y \in G$.

Post multiple by x^{-1} on both sides, we get

$$\mu_A(xyx^{-1}) = \mu_A(yxx^{-1})$$
$$= \mu_A(y) \geq \mu_A(y)$$

i.e., $\mu\,_A(xyx^{-1}) \geq \mu_A(y)$ ………………….. (3)

Let us assume $\gamma_A(xy) = \gamma_A(yx)$ for all $x, y \in G$.

Post multiple by x^{-1} on both sides, we get

$$\gamma_A(xyx^{-1}) = \gamma_A(yxx^{-1})$$
$$= \gamma_A(y) \leq \gamma_A(y)$$

i.e., $\gamma_A(xyx^{-1}) \leq \gamma_A(y)$ …………………. (4)

Hence from (3) & (4) we conclude that μ_A and γ_A be an IMLFNSG of G.

3.5 Theorem

Let μ be an Intuitionistic Multi L-Fuzzy Subset of a group $(G, .)$. If $\mu_A(e) = 1$ and $\gamma_A(e) = 0$.

Also (i) $\mu_A(xy^{-1}) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\mu_A(xy) = \mu_A(yx)$ for all $x, y \in G$.

(ii) $\gamma_A(xy^{-1}) \leq max\{\gamma_A(x), \gamma_A(y)\}$ and $\gamma_A(xy) = \gamma_A(yx)$ for all $x, y \in G$.

Then prove that $\mu = (\mu_A, \gamma_A)$ is an IMLFNSG of G, where e be the identity element of G.

Proof

Let e be an identity element of G and for all $x, y \epsilon G$.

(a) Given $\mu_A(e) = 1$ and $\mu_A(xy^{-1}) \geq min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \epsilon G$.

To prove that μ_A be an IMLFNSG of G.

Now, $\mu_A(x^{-1}) = \mu_A(ex^{-1})$

$\geq \min\{\mu_A(e), \mu_A(x)\}$

$= min\{1, \mu_A(x)\}$

$= \mu_A(x)$

i.e., $\mu_A(x^{-1}) \geq \mu_A(x), \quad \forall\, x \in G.$ ………………. (1)

From (1), replace x by x^{-1} we get

$\mu_A((x^{-1})^{-1}) \geq \mu_A(x^{-1})$

i.e., $\mu_A(x) \geq \mu_A(x^{-1})$ ………………. (2)

From (1) & (2), gives that

$\mu_A(x) = \mu_A(x^{-1}), \quad \forall\, x \in G$ ………………. (3)

Now, $\mu_A(xy) = \mu_A(x(y^{-1})^{-1})$

$\geq \min\{\mu_A(x), \mu_A(y^{-1})\}$

$= \min\{\mu_A(x), \mu_A(y)\}$ [∵ by using (3)]

i.e., $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}, \quad \forall\, x, y \in G$ ………………. (4)

Hence, from (3) & (4) gives that μ_A is an IMLFSG of G.

Also, given $\mu_A(xy) = \mu_A(yx), \quad \forall\, x, y \epsilon G$.

Post multiple by x^{-1} on both sides we get,

$\mu_A(xyx^{-1}) = \mu_A(yxx^{-1})$

$\geq \min\{\mu_A(y), \mu_A(xx^{-1})\}$

$= \min\{\mu_A(y), \mu_A(e)\}$

$= \min\{\mu_A(y), 1\}$

i.e., $\mu_A(xyx^{-1}) \geq \mu_A(y)$

Hence, μ_A is an IMLFNSG of $(G, .)$.

(b) Given $\gamma_A(e) = 0$ and $\gamma_A(xy^{-1}) \leq max\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y \epsilon G$.

To prove that γ_A be an IMLFNSG of G.

Now, $\gamma_A(x^{-1}) = \gamma_A(ex^{-1})$

$\leq max\{\gamma_A(e), \gamma_A(x)\}$

$= max\{0, \gamma_A(x)\}$

$= \gamma_A(x)$

i.e., $\gamma_A(x^{-1}) \leq \gamma_A(x), \quad \forall\, x \in G.$ ………………. (5)

From (5), replace x by x^{-1} we get

$\gamma_A((x^{-1})^{-1}) \leq \gamma_A(x^{-1})$

i.e., $\gamma_A(x) \leq \gamma_A(x^{-1})$ ………………. (6)

From (5) & (6), gives that

$\gamma_A(x) = \gamma_A(x^{-1}), \quad \forall\, x \in G$ ………………. (7)

Now, $\gamma_A(xy) = \gamma_A(x(y^{-1})^{-1})$

$\leq \max\{\gamma_A(x), \gamma_A(y^{-1})\}$

$= \max\{\gamma_A(x), \gamma_A(y)\}$ [∵ by using (7)]

i.e., $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}, \quad \forall\, x, y \in G$ .... (8)

Hence, from (7) & (8) gives that γ_A is an IMLFSG of G.

Also, given that $\gamma_A(xy) = \gamma_A(yx), \quad \forall\, x, y \in G$.

Post multiple by x^{-1} on both sides we get,

$$\begin{aligned}\gamma_A(xyx^{-1}) &= \gamma_A(yxx^{-1})\\ &\leq \max\{\gamma_A(y), \gamma_A(xx^{-1})\}\\ &= \max\{\gamma_A(y), \gamma_A(e)\}\\ &= \max\{\gamma_A(y), 0\}\end{aligned}$$

i.e., $\gamma_A(xyx^{-1}) \leq \gamma_A(y)$

Hence, γ_A is an IMLFNSG of $(G, .)$.

Hence, from (a) & (b) we conclude that $\mu = (\mu_A, \gamma_A)$ be the IMLFNSG of G.

3.6 Theorem

If A and B are two IMLFNSG of a group $(G, .)$, then prove that their intersection $(A \cap B)$ is an IMLFNSG of $(G, .)$.

Proof

Given A and B are two IMLFNSG of a group $(G, .)$.

To prove that $\mu = (\mu_{(A\cap B)}, \gamma_{(A\cap B)})$ be an IMLFNSG of G.

Let $x, y \in G$.

(a) To prove that $\mu_{(A\cap B)}$ is an IMLFNSG of G.

(i) $$\begin{aligned}\mu_{(A\cap B)}(xy) &= \mu_A(xy) \cap \mu_B(xy)\\ &\geq \min\{\mu_A(xy), \mu_B(xy)\}\\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \{\min\{\mu_B(x), \mu_B(y)\}\}\\ &= \min\{\min\{\mu_A(x), \mu_B(x)\}, \{\min\{\mu_A(y), \mu_B(y)\}\}\\ &= \min\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}\end{aligned}$$

i.e., $\mu_{(A\cap B)}(xy) \geq \mu_{A\cap B}(x) \cap \mu_{A\cap B}(y)$ .... (1)

(ii) $$\begin{aligned}\mu_{(A\cap B)}(x^{-1}) &= \mu_A(x^{-1}) \cap \mu_B(x^{-1})\\ &= \mu_A(x) \cap \mu_B(x) \qquad [\because \textit{by theorem } 3.4]\\ &= \mu_{(A\cap B)}(x)\end{aligned}$$

i.e., $\mu_{(A\cap B)}(x^{-1}) = \mu_{(A\cap B)}(x)$ .... (2)

From (1) & (2), we conclude that $\mu_{(A\cap B)}$ is an IMLFSG of G.

Also, $$\begin{aligned}\mu_{(A\cap B)}(xy) &= \mu_A(xy) \cap \mu_B(xy)\\ &= \mu_A(yx) \cap \mu_B(yx) \qquad [\because A \,\&\, B \textit{ are IMLFNSG of } G]\end{aligned}$$

i.e., $\mu_{(A\cap B)}(xy) = \mu_{A\cap B}(yx)$

Hence, $\mu_{(A\cap B)}$ is an IMLFNSG of G.

(b) To prove that $\gamma_{(A\cap B)}$ is an IMLFNSG of G.

(i) $$\begin{aligned}\gamma_{(A\cap B)}(xy) &= \gamma_A(xy) \cap \gamma_B(xy)\\ &\leq \max\{\gamma_A(xy), \gamma_B(xy)\}\\ &\leq \max\{\max\{\gamma_A(x), \gamma_A(y)\}, \{\max\{\gamma_B(x), \gamma_B(y)\}\}\end{aligned}$$

$= \max\{\max\{\gamma_A(x), \gamma_B(x)\}, \{\max\{\gamma_A(y), \gamma_B(y)\}\}$

$= \max\{\gamma_{A\cap B}(x), \gamma_{A\cap B}(y)\}$

i.e., $\gamma_{(A\cap B)}(xy) \leq \gamma_{A\cap B}(x) \cap \gamma_{A\cap B}(y)$ .... (3)

(ii) $\gamma_{(A\cap B)}(x^{-1}) = \gamma_A(x^{-1}) \cap \gamma_B(x^{-1})$

$= \gamma_A(x) \cap \gamma_B(x)$ [$\because$ *by theorem* 3.4]

$= \gamma_{(A\cap B)}(x)$

i.e., $\gamma_{(A\cap B)}(x^{-1}) = \gamma_{(A\cap B)}(x)$ .... (4)

From (3) & (4), we conclude that $\gamma_{(A\cap B)}$ is an IMLFSG of G.

Also, $\gamma_{(A\cap B)}(xy) = \gamma_A(xy) \cap \gamma_B(xy)$

$= \gamma_A(yx) \cap \gamma_B(yx)$ [$\because A \,\&\, B$ *are IMLFNSG of* G]

i.e., $\gamma_{(A\cap B)}(xy) = \gamma_{A\cap B}(yx)$

Hence, $\gamma_{(A\cap B)}$ is an IMLFNSG of G.

3.7 Theorem

Let $\mu = (\mu_A, \gamma_A)$ be an IMLFSG of a group $(G,.)$ with $\mu_A(y) < \mu_A(x)$ and $\gamma_A(y) > \gamma_A(x)$, for some x and y in G. Then prove that μ is an IMLFNSG of G.

Proof

Given $\mu = (\mu_A, \gamma_A)$ be an IMLFSG of a group $(G,.)$.

Also $\mu_A(y) < \mu_A(x)$ and $\gamma_A(y) > \gamma_A(x)$, for some x and y in G .... (1)

To prove that μ be an IMLFNSG of G.

(i) Let, $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ [$\because$ μ_A *is an IMLFSG of* G]

$= \mu_A(y)$ [$\because$ *by using* (1)]

i.e., $\mu_A(xy) \geq \mu_A(y)$

And, $\mu_A(y) = \mu_A(x^{-1}xy)$

$\geq \min\{\mu_A(x^{-1}), \mu_A(xy)\}$

$\geq \min\{\mu_A(x), \mu_A(xy)\}$ [$\because$ μ_A *is an IMLFSG of* G]

$= \mu_A(xy)$ [$\because$ *again using* (1)]

i.e., $\mu_A(y) \geq \mu_A(xy)$

Therefore, $\mu_A(xy) = \mu_A(y),\ \forall\, x, y \in G$. .... (2)

Similarly, $\mu_A(yx) \geq \min\{\mu_A(y), \mu_A(x)\}$ [$\because$ μ_A *is an IMLFSG of* G]

$= \mu_A(y)$ [$\because$ *by using* (1)]

i.e., $\mu_A(yx) \geq \mu_A(y)$

And, $\mu_A(y) = \mu_A(yxx^{-1})$

$\geq \min\{\mu_A(yx), \mu_A(x^{-1})\}$

$\geq \min\{\mu_A(yx), \mu_A(x)\}$ [$\because$ μ_A *is an IMLFSG of* G]

$= \mu_A(yx)$ [$\because$ *again using* (1)]

i.e., $\mu_A(y) \geq \mu_A(yx)$

Therefore, $\mu_A(yx) = \mu_A(y),\ \forall\, x, y \in G$. .... (3)

Hence, from (2) & (3) we have $\mu_A(xy) = \mu_A(y) = \mu_A(yx),\ \forall\, x, y \in G$.

Hence, $\mu_A(xy) = \mu_A(yx),\ \forall\, x, y \in G$.

Hence, μ_A is an IMLFNSG of G.

(ii) Let, $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$ $[\because\ \gamma_A\ is\ an\ IMLFSG\ of\ G]$
$= \gamma_A(y)$ $[\because\ by\ using\ (1)]$
i.e., $\gamma_A(xy) \leq \gamma_A(y)$
And, $\gamma_A(y) = \gamma_A(x^{-1}xy)$
$\leq \max\{\gamma_A(x^{-1}), \gamma_A(xy)\}$
$\leq max\{\gamma_A(x), \gamma_A(xy)\}$ $[\because\ \gamma_A\ is\ an\ IMLFSG\ of\ G]$
$= \gamma_A(xy)$ $[\because\ again\ using\ (1)]$
i.e., $\gamma_A(y) \leq \gamma_A(xy)$
Therefore, $\gamma_A(xy) = \gamma_A(y),\ \forall\, x, y \in G.$ ………………… (4)
Similarly, $\gamma_A(yx) \leq max\{\gamma_A(y), \gamma_A(x)\}$ $[\because\ \gamma_A\ is\ an\ IMLFSG\ of\ G]$
$= \gamma_A(y)$ $[\because\ by\ using\ (1)]$
i.e., $\gamma_A(yx) \leq \gamma_A(y)$
And, $\gamma_A(y) = \gamma_A(yxx^{-1})$
$\leq max\{\gamma_A(yx), \gamma_A(x^{-1})\}$
$\leq max\{\gamma_A(yx), \gamma_A(x)\}$ $[\because\ \gamma_A\ is\ an\ IMLFSG\ of\ G]$
$= \gamma_A(yx)$ $[\because\ again\ using\ (1)]$
i.e., $\gamma_A(y) \leq \gamma_A(yx)$
Therefore, $\gamma_A(yx) = \gamma_A(y),\ \forall\, x, y \in G.$ ………………… (5)
Hence, from (4) & (5) we have $\gamma_A(xy) = \gamma_A(y) = \gamma_A(yx),\ \forall\, x, y \in G.$
Hence, $\gamma_A(xy) = \gamma_A(yx),\ \forall\, x, y \in G.$
Hence, γ_A be an IMLFNSG of G.
Hence the theorem.

CONCLUSION

In that paper, we have seen a few basic definitions associated with an Intuitionistic Multi L-Fuzzy Subgroups. Next we have to introduce the concept of Intuitionistic Multi L-Fuzzy Normal Subgroups using in view of Intuitionistic L-Fuzzy Normal Subgroups. Also we have explored some properties and theorems based on IMLFNSG.

Further, in future we extend this study for Intuitionistic Multi L-Fuzzy Normal Subgroups in to Homomorphism of Intuitionistic Multi L-Fuzzy Subgroups.

REFERENCES

[1] M.Akilesh, M.Govindaraju and A.K.Abdul Hakeem, "A Study on Multi Anti L-Fuzzy Normal Subgroup", International Journal of Applied Engineering Research, ISSN:0973-4562 Vol.14, Number 1 (2019) , pp.20-23.

[2] K.T.Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 20, pp. 87-96, 1986.

[3] Azriel Rosenfeld, 1971, "Fuzzy Groups", Journal of mathematical analysis and applications, 35, pp. 512- 517.

[4] R.Biswas 1990, "Fuzzy subgroups and anti fuzzy subgroups", Fuzzy Sets and System, 35, pp. 121-124.

[5] K.R.Balasubramaniyan and R.Revathy, "Some Theorems on Intuitionistic Multi L-Fuzzy Subgroups", International Journal of Scientific and Research Publications, Vol.10, Issue 1 (2020), 653-655.

[6] K.A.Dib, "The fuzzy normal subgroup", Fuzzy Sets and Systems, vol. 98, pp. 393-402,1998.

[7] M.Fathi and Abdul Razak Salleh, "Intuitionistic fuzzy groups," Asian Journal of Algebra,vol. 2(1), pp1-10, 2009.

[8] Kul Hur, Hee-Won Kang and Hyeong-Kee Song, "Intuitionistic Fuzzy Normal Subgroup and Intuitionistic Fuzzy Cosets", Proceedings of KFIS spring conference 2004, volume 14, number 1, pp.367-371.

[9] Mohammed F. Marashdeh and Abdul Razak Salleh, "The Intuitionistic Fuzzy Normal Subgroup", International Journal of Fuzzy Logic and Intelligent Systems, vol. 10 no.1, 2010, pp.82-88.

[10] R.Muthuraj and S.Balamurugan, " A Study on Intuitionistic Multi-Fuzzy Subgroups", International Journal of Applications of Fuzzy sets and Artificial Intelligence, Vol.4, 153-172 (2014).

[11] N.Palaniappan, S.Naganathan and K.Arjunan, "A Study on Intuitionistic L-Fuzzy Subgroups", Applied mathematical Sciences, 3 (53) (2009), 2619-2624.

[12] Prabir Bhattacharya, "Fuzzy subgroups", Some characterizations, J. Math. Anal. Appl. 128 (1981), 241-252.

[13] A.Rosenfeld, "Fuzzy groups", J. Math. Anal. Appl, vol. 35, pp. 512-517, 1971.

[14] S.Sabu and T.V.Ramakrishnan, "Multi-fuzzy sets", International Mathematical Forum, 50, pp. 2471-2476.

[15] S.Sabu, T.V.Ramakrishnan, "Multi Fuzzy Sub Groups", Int.J. Contemp. Math. Sciences, Vol.6,No.8, pp.365- 372 (2011) .

[16] P.K.Sharma, "Homomorphism of Intuitionistic fuzzy groups", International Mathematics Forum , Vol. 6, 2011, no. 64, pp.3169-3178.

[17] Souriar Sebastian, S.Babu Sundar, "Commutative L-Fuzzy Subgroups Fuzzy sets and Systems", 68 (1994), pp.115-121.

[18] Yuying Li, Xuzhu Wang and Liqiong Yang, "A Study of (λ, μ)-Fuzzy Subgroups", Journal of Applied Mathematics Volume 2013.

[19] L.A.Zadeh, "Fuzzy sets", Information and control, 8, (1965), pp. 338-353.

[20] J.Zhan and Z.Tan, "Intuitionistic M-fuzzy groups," Soochow journal of mathematics, vol. 30(1), pp. 85-90, 2004.

Scientiae Mathematicae Japonicae 85, No.3(2023)(193-202)

LAYERING INTERCONNECTION NETWORKS

SWATHI N.R. and V. JUDE ANNIE CYNTHIA

Abstract Data storage can be exigent due to the volume and veracity in which data is accumulating. In this paper we analyze graph theoretic ideas such as decomposition in order to propose a framework for data storage and data sharing on a well-defined interconnection network namely, the ringed Petersen network $RP(k)$, $k \geq 2$.

1 Introduction

Graph databases store structured data anchoring to the relationship between the data elements. In simple terms, a graph database is a database of graphs or networks representing the data elements along with their relationships. Navigating through the graph database is the key to efficiently query the database for information. In this paper we propose to give a well defined structure to the storage layer of the graph databse (i.e), design storage layer in the form of an interconnection network. Designing the storage layer in the form of an interconnection network is advantageous as graph data can be clustered based on customised criterion and the clusters can be stored in the nodes of the interconnection network depending on the relationships between the clusters. Also the definition of an interconnection promises a highly fault-tolerant storage layer. Further, we propose to decompose this interconnection network into edge-disjoint non-isomorphic cycles so as to slice the storage layer to capture the intricate relationships between the clusters and thereby providing efficient navigation through the database.

Swathi N. R.
Stella Maris College, Cathedral Road, Poes Garden, Gopalapuram, Chennai, Tamil Nadu 600086
e-mail: swathibindu201@gmail.com

V. Jude Annie Cynthia
Stella Maris College, Cathedral Road, Poes Garden, Gopalapuram, Chennai, Tamil Nadu 600086
e-mail: judeanniecynthia@stellamariscollege.edu.in

Nash Williams [8] was the pioneer to study the decomposition of graphs into Hamilton cycles. Hamilton decomposition of complete graphs was the earliest result in this direction where Hilton [5] proved that a complete graph K_n of odd order can be decomposed into $\frac{n}{2}$ Hamilton cycles and a complete graph K_n of even order can be decomposed into $\frac{n}{2}$ Hamilton cycles and a perfect matching and the existence of path decompositions in K_n of odd order was established by Alspach [2]. Following the complete graph, enumerating the Hamilton cycles in certain generalized Petersen graphs and the necessary and sufficient condition for a generalized Petersen graph to be Hamilton was established by Allen J. Schwenk [1]. Recently, Cynthia *et al.* [6] investigated Hamilton decomposition of Harary graphs.

Decomposition of graphs into subgraphs isomorphic to cycles has tremendous applications in computational environments. Bermond *et al.* [3] proved that a wrapped Butterfly graph of dimension n can be decomposed into Hamilton cycles. Algorithms for factorizations of the hypercube network were designed by Douglas [4]. Decomposition of cartesian product of cycles into isomorphic cycles under certain conditions was investigated by Tapadia *et al.* [9].

Definition 1 [8] A pair of subgraphs H, K of a graph G are called edge-disjoint subgraphs of G if $E(H) \cap E(K) = \phi$.

Definition 2 [8] A set of edge-disjoint subgraphs $H_1, H_2, \ldots, H_r$ of a graph G is called a decompostion of that graph G if $\bigcup_{i=1}^{r} H_i = G$.

Definition 3 [8] If the edge-disjoint subgraphs are cycles of G, then $\bigcup_{i=1}^{r} H_i$ is a cycle decomposition of G.

Definition 4 [8] If the edge-disjoint subgraphs are Hamilton cycles of G, then $\bigcup_{i=1}^{r} H_i$ is a Hamilton decomposition of G.

The ringed Petersen network is an interconnection networks with high fault tolerance suitable for routing algorithms. In view of vulnerability, an interconnection network that can be represented as the combination of cycles of varied lengths or equal lengths becomes highly edge fault tolerant as the structure provides high scope for re-routing under faulty conditions. We propose that these decompositions can provide efficient data storage platforms by exhibiting non-isomorphic cycle decompositions of the ringed Petersen network.

Definition 5 [7] The ringed Petersen network $RP(k)$ is defined as the cartesian product of the Petersen graph $P(5,2)$ and cycle C_k.

$$RP(k) = P(5,2) \times C_k$$

The r^{th} slice of $RP(k)$ is defined to be the r^{th} Petersen graph $P(5,2)$ where $1 \leq r \leq k$.

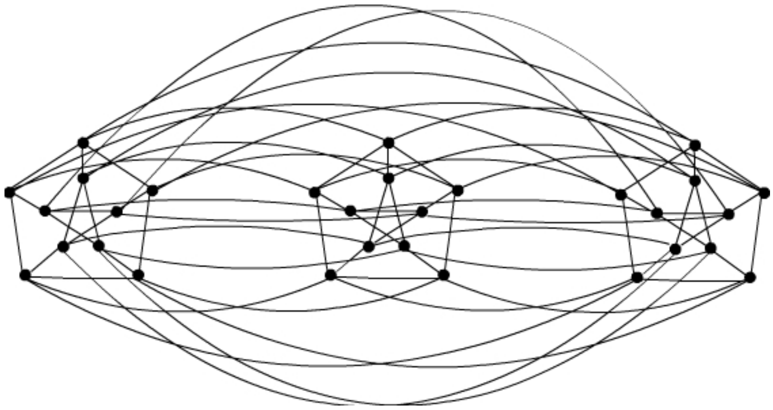


Fig. 1: Ringed Petersen network $RP(3)$

2 Cycle Decomposition of the Ringed Petersen Network $RP(k)$, $k \geq 2$

In this section, we prove that the ringed Petersen network can be decomposed into cycles of varied lengths. Since the regularity is 5, we prove that there exists a perfect matching in the decomposition of the ringed Petersen graph.

Theorem 1 *The ringed Petersen network $RP(k)$, $k \geq 2$ can be decomposed into $2k$ cycles of length 5, ten cycles of length k and a perfect matching.*

Proof Consider the following gadgets: Fig.2 and Fig.3 indicate that gagets 1 and 2 are isomorphic to C_5 whereas Fig.4 indicates that gadget 3 is isomorphic to C_k.

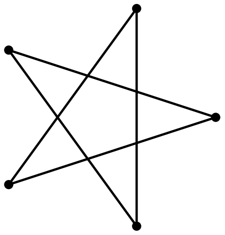


Fig. 2: Gadget 1

We construct the $\{C_5, C_4\}$-decomposition of the ringed Petersen network $RP(k)$, k even, using gadgets 1, 2 and 3 as follows:

Step 1: Define the set of vertices $V(P(r)) \in V(RP(k))$ as follows:

$$V(P(r)) = \{v^r_{i\,j}\} \tag{1}$$

where, $i = 1, 2$, $1 \leq j \leq 5$, $1 \leq r \leq k$.

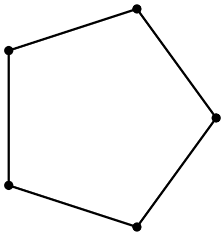


Fig. 3: Gadget 2

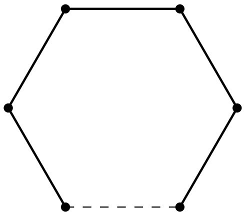


Fig. 4: Gadget 3

Step 2 Obtain the induced subgraph $RP(k)[V(P(r))]$.

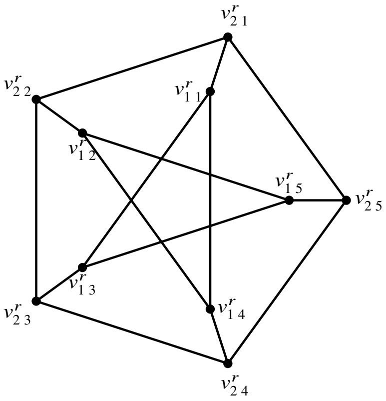


Fig. 5: Induced subgraph $RP(k)[V(P(r))]$

Step 3: The following cycle is isomorphic to gadget 1.

$$C_5^1(r) = v^r_{1\,1} - v^r_{1\,3} - v^r_{1\,5} - v^r_{1\,2} - v^r_{1\,4} - v^r_{1\,1} \tag{2}$$

Therefore, fit gadget 1 to $C_5^1(r)$.

Step 4: The following cycle is isomorphic to gadget 2.

$$C_5^2(r) = v^r_{2\,1} - v^r_{2\,2} - v^r_{2\,3} - v^r_{2\,4} - v^r_{2\,5} - v^r_{2\,1} \tag{3}$$

Therefore, fit gadget 2 to C_5^2.

Step 5: Consider the subgraph $RP(k) \setminus \bigcup_{r=1}^{k} E(C_5^1(r) \cup C_5^2(r))$. We fit gadget 3 to the following C_k cycles:

$$C_k^j = v_{i\,j}^1 - v_{i\,j}^2 - v_{i\,j}^3 - \cdots - v_{i\,j}^k - v_{i\,j}^1 \tag{4}$$

where, $i = 1, 2$, $1 \le j \le 10$.

Since k even, gadget 1 can be fitted to k cycles of length 5 and gadget 2 can be fitted to k cycles of length 5. Each of the gadgets are fitted to independent subgraphs of $RP(k)$. Therefore, the cycles obtained by fitting gadgets 1, 2 and 3 are edge-disjoint. The perfect matching can be obtained by removing all the edges fitted by gadgets 1, 2 and 3 from $RP(k)$. □

Theorem 2 *The ringed Petersen network $RP(k)$, $k \ge 2$, k even is decomposable into*

(i) *k cycles of length 10, $\frac{5k}{2}$ cycles of length 4 and a perfect matching*

(ii) *$2k$ cycles of length 5, $\frac{5k}{2}$ cycles of length 4 and a perfect matching*

Proof Consider the following gadgets:

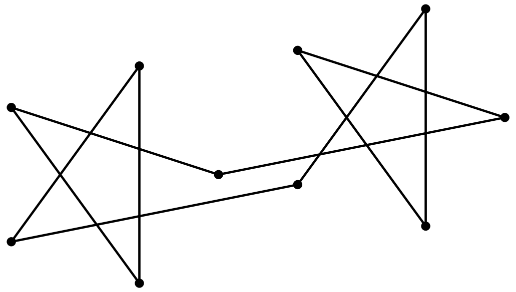


Fig. 6: Gadget 1

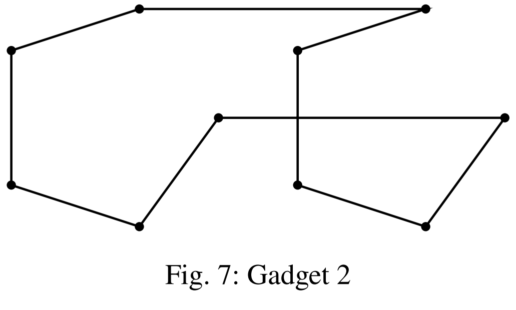


Fig. 7: Gadget 2

Fig. 8: Gadget 3

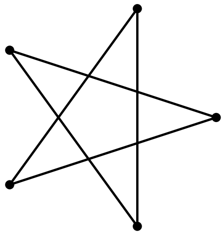


Fig. 9: Gadget 4

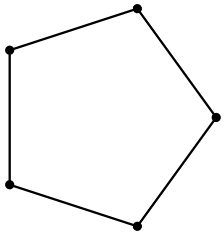


Fig. 10: Gadget 5

Fig.6 and Fig.7 indicate that gagets 1 and 2 are isomorphic to C_{10}, Fig.9 and Fig.10 indicate that gagets 4 and 5 are isomorphic to C_5, whereas Fig.8 indicate that gadget 3 is isomorphic to C_4.
We construct the $\{C_{10}, C_4\}$-decomposition of the ringed Petersen network $RP(k)$, k even, using gadgets 1, 2 and 3 as follows:

Step 1: Define the set of vertices $V(P(r)) \in V(RP(k))$ as follows:

$$V(P(r, r+1)) = \{v_{i\,j}^{r}\} \tag{5}$$

where, $i = 1, 2, 1 \leq j \leq 5, r = 1, 3, 5, \ldots, k-1$.
Step 2: Obtain the induced subgraph $RP(k)[V(P(r)) \cup V(P(r+1))]$.
Step 3: The following cycle is isomorphic to gadget 1.

$$C_{10}^{1}(r) = v_{1\,1}^{r} - v_{1\,4}^{r} - v_{1\,2}^{r} - v_{1\,5}^{r} - v_{1\,5}^{r+1} - v_{1\,2}^{r+1} - v_{1\,4}^{r+1} - v_{1\,1}^{r+1} - v_{1\,3}^{r+1} - v_{1\,3}^{r} - v_{1\,1}^{r} \tag{6}$$

Therefore, fit gadget 1 to C_{10}^{1}.
Step 4: The following cycle is isomorphic to gadget 2.

$$C_{10}^{2}(r) = v_{2\,6}^{r} - v_{2\,10}^{r} - v_{2\,9}^{r} - v_{2\,8}^{r} - v_{2\,7}^{r} - v_{2\,7}^{r+1} - v_{2\,8}^{r+1} - v_{2\,9}^{r+1} - v_{2\,10}^{r+1} - v_{2\,6}^{r+1} - v_{2\,6}^{r} \tag{7}$$

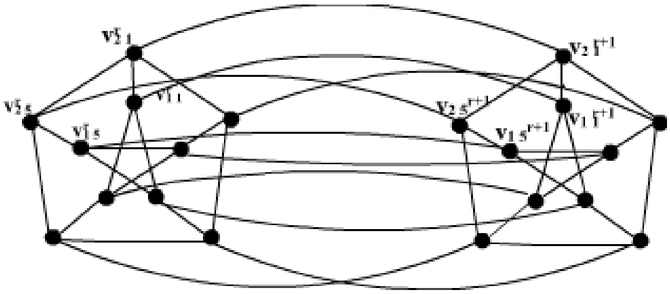


Fig. 11: Induced subgraph $RP(k)[V(P(r)) \cup V(P(r+1))]$

Therefore, fit gadget 2 to C_{10}^2.

Step 5: The following cycle is isomorphic to gadget 4.

$$C_5^1(r) = v_{1\,1}^r - v_{1\,3}^r - v_{1\,5}^r - v_{1\,2}^r - v_{1\,4}^r - v_{1\,1}^r \tag{8}$$

Therefore, fit gadget 4 to C_5^1.

Step 6: The following cycle is isomorphic to gadget 5.

$$C_5^2(r) = v_{2\,1}^r - v_{2\,2}^r - v_{2\,3}^r - v_{2\,4}^r - v_{2\,5}^r - v_{2\,1}^r \tag{9}$$

Therefore, fit gadget 5 to C_5^2.

Step 7: Obtain the subgraph $RP(k)[V(P(r-1)) \cup V(P(r)) \cup V(P(r+1)) \cup V(P(r+2))] \setminus \{C_{10}^1(r) \cup C_{10}^2(r)\}$

Since, the $r-1^{th}$ slice and the r^{th} slice are connected by edges $v_{i\,j}^{r-1} v_{i\,j}^r$, fit gadget 3 to the cycle

$$C_4^j(r) = v_{2\,j}^{r-1} - v_{1\,j}^{r-1} - v_{1\,j}^r - v_{2\,j}^r - v_{2\,j}^{r-1} \tag{10}$$

Similiarly, the $r+1^{th}$ slice and the $r+2^{th}$ slice are connected by edges $v_{i\,j}^{r+1} v_{i\,j}^{r+2}$. Therefore, fit gadget 3 to the cycles

$$C_4^j(r) = v_{2\,j}^{r-1} - v_{1\,j}^{r-1} - v_{1\,j}^r - v_{2\,j}^r - v_{2\,j}^{r-1} \tag{11}$$

where, $1 \leq j \leq 5$.

Step 8: Since, $RP(k)$ is 5-regular, 4 out of 5 edges incident with a vertex contribute to the cycle decomposition. The remaining edges are collected as follows: Consider the induced subgraph $RP(k)[V(P(r)) \cup V(P(r+1))]$ and remove all the edges fitted by gadgets 1, 2 and 3. From Fig.12, it is evident that it is the perfect matching of each of the induced subgraphs $RP(k)[V(P(r)) \cup V(P(r+1))]$, $r = 1, 3, \ldots, k-1$.

Step 9: Since the cycles C_5^1 and C_5^2 defined in Eqn.8 and Eqn.9 respectively are fitted by gadgets 4 and 5, C_4 cycles, we fit gadget 3 to the following C_4 cycles in the induced subgraph $RP(k)[V(P(r)) \cup V(P(r+1))] \setminus E(C_5^1 \cup C_5^2)$:

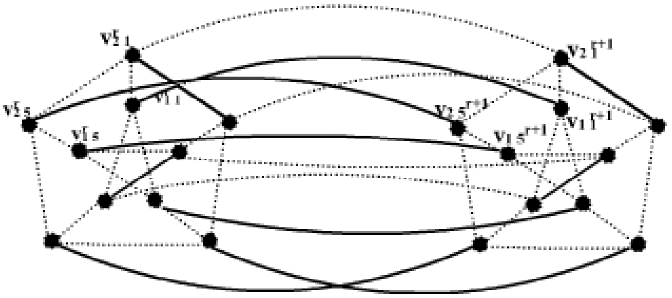


Fig. 12: Edges of the induced subgraph $RP(k)[V(P(r)) \cup V(P(r+1))]$ not fitted by gadgets 1, 2 and 3.

$$C_4^j(r) = v^r_{1\,j} - v^r_{2\,j} - v^{r+1}_{2\,j} - v^{r+1}_{1\,j} - v^r_{1\,j} \tag{12}$$

Step 10: Since $RP(k)$ is regular, 4 out of 5 edges incident with a vertex contribute to the cycle decomposition. The remaining edges are collected as follows: Consider the induced subgraph $RP(k)[V(P(r)) \cup V(P(r-1))], r = 1, 3, 5, \ldots, k-1$ and remove all the edges fitted by gadgets 3, 4 and 5. From Fig.13, it is evident that it is the perfect matching of each of the induced subgraphs $RP(k)[V(P(r-1)) \cup V(P(r+1))], r = 1, 3, \ldots, k-1$. Therefore, each subgraph $RP(k)[V(P(r-1)) \cup V(P(r+1))]$ contributes 10 edges to the perfect matching of $RP(k)$.

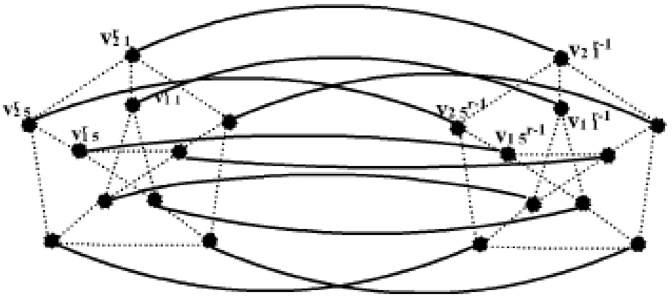


Fig. 13: Edges of the induced subgraph $RP(k)[V(P(r)) \cup V(P(r-1))]$ not fitted by gadgets 1, 2 and 3.

We prove statements (i) and (ii) by collecting subgraphs of $RP(k)$ fitted by gadgets 1, 2, 3 and gadgets 3, 4, 5 separately.

(i) From step 3, since k is even, gadget 1 can be fitted to $\frac{k}{2}$ cycles of length 10. From step 4, since k is even, gadget 2 can be fitted to $\frac{k}{2}$ cycles of length 10. Each pair of the k slices contributes 5 cycles of length 4 and k even. Therefore, using step 7 gadget 3 can be fitted to $\frac{5k}{2}$ cycles of length 4. Each of the gadgets are fitted to independent subgraphs of $RP(k)$. Therefore, the cycles obtained by fitting gadgets 1, 2 and 3 are edge-disjoint. Step 8 implies that each subgraph $RP(k)[V(P(r)) \cup V(P(r+1))]$ contributes 10 edges to the perfect matching of $RP(k)$.

(ii) Similiar to (i), $2k$ edge-disjoint cycles of length 5 can be collected using steps 5 and 6, $\frac{5k}{2}$ edge-disjoint cycles of length 4 can be collected using step 9 and the edges of the perfect matching can be obtained from step 10. □

Theorem 3 *The ringed Petersen network $RP(k)$, $k > 2$, k odd is decomposable into*

(i) $k-1$ *cycles of length* 10*, two cycles of length* 5*,* $\frac{5(k-3)}{2}$ *cycles of length* 4*, five cycles of length* 6 *and a perfect matching*

(ii) $2k$ *cycles of length* 5*,* $\frac{5(k-3)}{2}$ *cycles of length* 4*, five cycles of length* 6 *and a perfect matching.*

Proof (i) $k-1$ even. Therefore, obtain $k-1$ edge-disjoint cycles of length 10, $\frac{5(k-3)}{2}$ edge-disjoint cycles of length 4 and a set of independent edges from $RP(k-1)$ using gadgets 1, 2 and 3 as in Thm.2.

Since k odd, when $r = k$ the induced subgraph $RP(k)[V(P(k))]$ is isomorphic to the Petersen graph on 10 vertices. As the Petersen graph is not hamiltonian, it is impossible to obtain a cycle of length 10 from $RP(k)[V(P(k))]$. Therefore, the following two edge-disjoint cycles of length C_5 are obtained from $RP(k)[V(P(k))]$:

$$C_5^1 = v^k_{1\,1} - v^k_{1\,3} - v^k_{1\,5} - v^k_{1\,2} - v^k_{1\,4} - v^k_{1\,1} \tag{13}$$

$$C_5^2 = v^k_{2\,1} - v^k_{2\,2} - v^k_{2\,3} - v^k_{2\,4} - v^k_{2\,5} - v^k_{2\,1} \tag{14}$$

Gadget 3 is fitted to the following cycles of length 4 from the induced subgraph $RP(k)[V(P(k-1)) \cup V(P(k))]$:

$$C_4^j(r) = v^{k-1}_{2\,j} - v^{k-1}_{1\,j} - v^k_{1\,j} - v^k_{2\,j} - v^{k-1}_{2\,j} \tag{15}$$

where, $1 \le j \le 5$.

From, the induced subgraph $RP(k)[V(P(1)) \cup V(P(k)) \cup V(P(k+1))] \setminus \{\bigcup_r \{E(C^1_{10}(r) \cup C^2_{10}(r))\} \cup C_5^1 \cup C_5^2)\}$, cycles of length 6 can be obtained as follows:

$$C_6^j = v^{k-1}_{2\,j} - v^{k-1}_{1\,j} - v^k_{1\,j} - v^1_{1\,j} - v^1_{2\,j} - v^k_{2\,j} - v^{k-1}_{2\,j} \tag{16}$$

Then, the remaining 5 edges in the k^{th} slice are edge disjoint and therefore can be contributed to the perfect matching of the cycle decomposition of the ringed Petersen graph $RP(k)$.

(ii) $k-1$ even. Therefore, $2(k-1)$ edge-disjoint cycles of length 5, $\frac{5(k-3)}{2}$ edge-disjoint cycles of length 4 and a set of independent edges can be obtainded from $RP(k-1)$ using gadgets 3, 4 and 5 as in Thm.2.

When $r = k$, two cycles of length 5 can be obtained by fitting gadgets 4 and 5.

From, the induced subgraph $RP(k)[V(P(1)) \cup V(P(k)) \cup V(P(k-1))] \setminus \{\bigcup_r E(C^1_{10}(r) \cup C^2_{10}(r) \cup C^1_5(r) \cup C^2_5(r))\}$, cycles of length 6 can be obtained as follows:

$$C_6^j = v_{2\,j}^{k-1} - v_{1\,j}^{k-1} - v_{1\,j}^{k} - v_{1\,j}^{1} - v_{2\,j}^{1} - v_{2\,j}^{k} - v_{2\,j}^{k-1} \tag{17}$$

Then, the remaining 5 edges in the k^{th} slice are edge disjoint and therefore can be contributed to the perfect matching of the cycle decomposition of the ringed Petersen graph $RP(k)$. □

3 Conclusion

In this paper, we propose that the storage layer of a graph database can be modelled as an interconnection network and decomposition of graphs can be used as a tool to enhance data sharing. We have investigated various non-isomorphic cycle decompositon of the ringed Petersen network by defining and fitting gadgets to edge-disjoint subgraphs of the ringed Petersen network.

References

1. Allen J. Schwenk, Enumeration of Hamilton Cycles in Certain generalized Petersen Graphs, Journal of Combinatorial Theory, 47 (1989) 53-59.
2. Brian Alspach, The Wonderful Walecki Construction, Bulletin of the Institute of Combinatorics and its Applications, 52 (2008) 7-20.
3. J-C Bermond, E. Darrot, O. Delmas and S. Perennes, Hamilton Cycle Decomposition of the Butterfly Network, Parallel Processing Letters, 3 (1998) 371-385.
4. Douglas W. Bass and I. Hal Sudborough, Hamilton Decompositions and $(n/2)$-Factorizations of Hypercubes, Journal of Graph Algorithms and Applications, 7 (2003) 79-98.
5. Hilton A. J. W., Hamilton Decompositions of Complete Graphs, Journal of Combinatorial Theory, 36 (1984) 125-134.
6. Jude Annie Cynthia and N. R. Swathi, Hamilton Decomposition of Harary Graphs, International Journal of Mathematics Trends and Technology, 34 (2016) 59-63.
7. LIU Fang'ai, LIU Zhiyong and QIAO Xiangzhen, A Practical Interconnection Network $RP(k)$ and its Routing Algorithms, Science in China (Series F), 44 (2001) 461-473.
8. Nash Williams C. St. J. A., Infinite graphs – A Survey, Journal of Combinatorial Theory, 3 (1967) 286-301.
9. S. A. Tapadia, Y. M. Borse and B. N. Waphare, Cycle Decompositions of the Cartesian Product of Cycles, Australian Journal of Combinatorics, 74 (2019) 443-459.

Scientiae Mathematicae Japonicae 85, No.3(2023)(203-214)

TOTAL LIAR'S DOMINATING SET IN CERTAIN GRAPHS

A.S. SHANTHI and DIANA GRACE THOMAS

Abstract P. J. Slater introduced a new parameter called the liar's dominating set (LDS), L where every vertex should be dominated twice and any two distinct vertices in V should be triple dominated. These conditions were imposed by Slater to identify the intruder in the network and also to correctly identify the location of the intruder allowing one fault to occur. The total liar's dominating set was introduced by Panda and Paul. For a graph G, a vertex set $L \subseteq V(G)$ is a total liar's dominating Set (TLDS) if and only if (i) for every $u \in V(G)$, $|N(u) \cap L| \geq 2$ and (ii) for every pair u, v of distinct vertices we have $|(N(u) \cup N(v)) \cap L| \geq 3$. The total liar's domination number $\gamma_{T\ L}(G)$ is the minimum cardinality of a total liar's dominating set. In this paper, we determine the total liar's domination number for complete sun graphs, line graph of sunlet graphs, triangular snake graphs, Sierpiński gasket graph, Sierpiński triangle graph, Sierpiński complete graph, wheel graph, double wheel graph, fan graph, half gear graph, double fan graph, polar grid graph and Sierpiński cycle graphs.

1 Introduction

In 2009, Slater introduced the liar's dominating set problem. In [15] Slater showed that the LDS problem is NP-hard in the case of general graphs and in the case of trees he has given a lower bound for liar's domination number by proving in general for tree of order n the LDS contains atleast $\frac{3}{4}(n+1)$ vertices and atmost n. Slater has also observed that for a subclass of trees the entire vertex set is the LDS. In [11] Roden and Slater have given that the LDS equals $\frac{3}{4}(n+1)$ for some characterized tree

---

A. S. Shanthi
Stella Maris College, Chennai e-mail: shanthu.a.s@gmail.com

Diana Grace Thomas
Stella Maris College, Chennai e-mail: dianagrace@stellamariscollege.edu.in

classes. In that paper they have shown that for bipartite graphs the LDS problem is NP-hard. Later, in [9], Panda and Paul proved that the LDS problem for split graphs and chordal graphs are NP-hard and for computing the minimum LDS in trees they also proposed a linear time algorithm. Then in [10], Panda et al. gave the algorithm of $O(ln\Delta)$ factor approximation for a LDS problem, where Δ is the degree of the graph. In [7], Panda and Paul studied the LDS problem for proper interval graphs and also computed the minimum LDS by proposing a linear time algorithm for the same. In [10], the problem is studied for p-claw free graphs and bounded degree graphs. In [17], the LDS problem is studied for grid graphs of two dimension and the bounds are given for the same. In [3] Jallu et al. found the LDS problem for unit disk graphs and obtained the NP-completeness of this problem. In [2], LDS problem for bipartite graphs is proved as $W[2]$-hard by Banerjee et al.

In [1] Alimadi et al. gave the LDS number for the characterization of graphs and trees as $|V|$ and $|V| - 1$ respectively. They have observed that for a graph G the LDS is n for graphs having degree $n \geq 3$ if and only if every vertex in G satisfies the conditions given: (i) degree of vertex v is 1 (ii) atleast one component of $G - v$ is of order atmost 2 (iii) v belongs to the block having cut vertex of G of order 3. They have also characterized trees for which the minimum LDS is $n - 1$. Also, the ratio between the minimum LDS and 2-tuple domination is given as a upper bound for connected graphs of girth at least five. The problem of LDS for circulant networks were given by Paul Manuel in [5]. Liar's dominating set is also called 1-distance 2-tuple (3,2)-domination problem. Recently in [4] the authors have worked on this and have proved that 1-distance m-tuple (l, r)-domination problem and d-distance m-tuple $(l, 2)$-domination problem is NP-complete.

In [6], Panda and Paul introduced and studied the connected liar's dominating set problem which is obtained when the induced subgraph L is connected. In [8], Panda and Paul introduced the variant total liar's dominating set L and found the minimum connected liar's dominating set and minimum total liar's dominating set. They also stated that these problems are NP-hard and proposed algorithms of $O(ln\Delta)$ factor approximation. For graphs of maximum degree 4 they proved that the problems are APX-complete. A recent study on total liar's domination problem can also be seen in [12].

In this paper, we discuss the total liar's dominating set and determine the total liar's domination number for certain graphs. A vertex set $L \subseteq V(G)$ is a total liar's dominating set (*TLDS*) if and only if (i) for every $u \in V(G)$, $|N(u) \cap L| \geq 2$ and (ii) for every pair u, v of distinct vertices we have $|(N(u) \cup N(v)) \cap L| \geq 3$. The total liar's domination number $\gamma_{TL}(G)$ is the minimum cardinality of a total liar's dominating set.

Theorem 1 [8] A connected graph G admits a *TLDS* if and only if (i) G has atleast three vertices (ii) deg$(G) \geq 2$ and (iii) every non-adjacent pair of vertices say u,v in V induces a subgraph $G[S]$ which is neither isomorphic to C_4 nor isomorphic to $K_4\backslash\{e\}$ where $S = N[u] \cup N[v]$.

2 Main results in total liar's dominating set

In this section, we study the classes of graphs for which the total liar's dominating set and the liar's dominating set are equal. Also, we study for classes of graphs with total liar's dominating set greater than the liar's dominating set.

2.1 Graphs with equal *LDS* and *TLDS*

In this subsection, we obtain the total liar's dominating set for few graphs that are equal to the liar's dominating set. First we consider k-CS_n complete k-sun graph and line graph of sunlet graph $L(Sl_n)$.

Theorem 2 [13] Let 1-CS_n be a complete sun graph with $n \geq 3$, then γ_L(1-$CS_n) = n$.

Theorem 3 [13] Let 2-CS_n be a complete 2-sun graph with $n \geq 3$, then γ_L(2-$CS_n) = 2n$.

Theorem 4 [13] Let k-CS_n be a complete sun graph with $n \geq 3$. Then $\gamma_L(k$-$CS_n) = (2^{k-1})n$.

Theorem 5 [13] Let $L(Sl_n)$ be a line graph of a sunlet graph with $n \geq 3$. Then $\gamma_L(L(Sl_n)) = n$.

The total liar's dominating set for 1-CS_n, 2-CS_n, k-CS_n and $L(S_n)$ are constructed as in the Theorem 2, 3, 4 and 5 respectively. Since the graph induced by those L forms a cycle, it satisfies both the conditions of total liar's dominating set. Now we discuss the total liar's dominating set for triangular snake graph, Sierpiński gasket, Sierpiński triangle graph, Sierpiński complete graph, wheel graph, double wheel graph, fan graph, half gear graph, double fan graph, polar grid graph and Sierpiński cycle graph.

Theorem 6 [16] Let TS_n be a triangular snake graph with $n \geq 3$. Then $\gamma_L(TS_n) = n + 3$.

Theorem 7 *Let TS_n be a triangular snake graph with $n \geq 3$. Then $\gamma_{TL}(TS_n) = n+3$.*

Proof The L constructed in Theorem 6 constitute total liar's dominating set since v_i's form a path, $N(v_1) \cap L = \{u_1, v_2\}$, $N(v_{n+1}) \cap L = \{u_n, v_{n+1}\}$ and $N(u_i) \cap L = \{v_i, v_{i+1}\}$, $i = 1, n$. □

Theorem 8 [16] Let S_3 be a Sierpiński gasket graph. Then $\gamma_L(S_3) = 9$.

By Theorem 8, the minimum liar's dominating set $L = \{1\{1,2\}, 1\{1,3\}, \{1,3\}, 3\{1,3\}, 3\{2,3\}, \{2,3\}, 2\{2,3\}, 2\{1,2\}, \{1,2\}\}$ is considered. Since L forms a cycle C_9, for any $u, v \in L$ both conditions of *TLDS* are satisfied. Also since the vertices not in L are all independent vertices with $|N(iii) \cap L| = 2, i = 1, 2, 3$ and $|N(1\{2,3\}) \cap L| = 4$. It is clear that for any two vertices not in L, the second condition is also satisfied. Hence we have the following result:

Theorem 9 *Let S_3 be a Sierpiński gasket graph. Then $\gamma_{TL}(S_3) = 9$.*

Let us consider the L obtained for S_3 and determine the liar's domination number for S_4.

Theorem 10 *Let S_4 be a Sierpiński gasket graph. Then $\gamma_{TL}(S_4) = 24$.*

Proof By construction, S_4 consists of 3 copies of S_3 namely S_3^1, S_3^2, S_3^3. Identify each of {(1111, {1, 2}, {1, 3}); ({1, 2}, 2222, {2, 3}); ({1, 3}, {2, 3}, 3333)} of S_4 with (111, 222, 333) of S_3. The vertex {1, 2} is common to both S_3^1 and S_3^2 and $|N(\{1,2\}) \cap L| = 4$. So instead of $21\{1,2\}$ and $21\{1,3\}$ to be in L we take $21\{2,3\}$ thus $|N(\{1,2\}) \cap L| = 2$, $|N(21\{1,2\}) \cap L| = 2$ and $|N(21\{1,3\}) \cap L| = 2$. Also $|(N(\{1,2\}) \cup N(21\{1,2\})) \cap L| = 4$ and $|(N(21\{1,2\}) \cup N(21\{1,3\})) \cap L| = 3$. Similarly instead of $32\{1,2\}, 32\{2,3\}$ we take $32\{1,3\}$, and instead of $13\{1,3\}, 13\{2,3\}$ we take $13\{1,2\}$. Here, we again see that in all the 3 copies we have formed 3 cycles in L. Thus $\gamma_{TL}(S_4) = 3[\gamma_{TL}(S_3)] - 3 = 24$.

Theorem 11 *Let S_n be a Sierpiński gasket graph. Then $\gamma_{TL}(S_n) = 3(\gamma_{TL}(S_{n-1})) - 3 = \frac{1}{2}(5(3^{n-2}) + 3)$ for $n \geq 5$.*

Proof By Theorem 10, the result is true for $n = 4$. Let us assume that the result is true for S_k, $k \leq n$. Let $k = n$. Since S_n consists of 3 copies of S_{n-1} namely S_{n-1}^i, i =1, 2, 3 identify the vertices 111...1, {1, 2}, {1, 3}; 222...2, {1, 2}, {1,3}; 333...3, {1, 2}, {2, 3} respectively by {111...1, 222...2, 333...3} of S_{n-1}. Since {1, 2} is in both S_{n-1}^1 and S_{n-1}^2. Also {1, 3} is in both S_{n-1}^1 and S_{n-1}^3 and {2, 3} is in both S_{n-1}^2 and S_{n-1}^3. Instead of $21...1\{1,3\}$ and $12...2\{2,3\}$; $32...2\{1,2\}$ and $32...2\{2,3\}$; $13...3\{1,3\}$ and $13...3\{2,3\}$ we take $21...1\{2,3\}$; $32...2\{1,3\}$; $13...3\{1,2\}$ respectively. Thus $\gamma_{TL}(S_n) = 3(\gamma_{TL}(S_{n-1})) - 3$. □

Lemma 1 [16] Let $S(2,3)$ be a Sierpiński triangle graph. Then $\gamma_L(S(2,3)) = 6$.

Theorem 12 *Let $S(2,3)$ be a Sierpiński triangle graph. Then $\gamma_{TL}(S(2,3)) = 6$.*

Proof By Lemma 1, we have $L = \{12, 13, 22, 23, 32, 33\}$ which forms a cycle C_6. Hence, all the vertices of $S(2,3)$ satisfy the conditions of a total liar's dominating set. □

Theorem 13 *Let $S(n,3)$ be a Sierpiński triangle graph with $n \geq 2$. Then $\gamma_{TL}(S(n,3)) = 3(\gamma_{TL}(S(n-1,3)) = 2(3^{n-1})$.*

Proof We prove the result by the method of induction. $S(3,3)$ comprises of 3 copies of $S(2,3)$ namely $S_1(2,3)$, $S_2(2,3)$ and $S_3(2,3)$ joined by the linking edges $(122, 211)$, $(233, 322)$ and $(311, 133)$ respectively. In view of Theorem 12 every copy of $S(2,3)$ in $S(3,3)$, L contains 6 vertices which will not include the vertices $\{122, 211, 233, 322, 311, 133\}$. Since each of these vertices is having degree 3 in $S(3,3)$, instead of the vertices 121 or 123 without loss of generality let us say 123 we include 122 in L. Also let 311 belong to L. Now we can make 132 and 213 not to belong to L by including 311 and 233 in L. Also let 322 belong to L. Finally 231 and 321 does not belong to L. By doing so $|L| = 18$ which we can obtain by taking L in every copy of $S(2,3)$ using Theorem 12.

Assume that the result is true for $S(k,3)$, $k \leq n-1$. Let $k = n$. Since $S(k,3)$ comprises of 3 copies of $S(k-1,3)$ namely $S_1(k-1,3)$ whose corner vertices are 111...1, 122....2, 133...3 of $S(k,3)$. Similarly $S_2(k-1,3)$ are identified as 211...1, 222...2, 233...3 and $S_3(k-1,3)$ are identified as 311...1, 322...2, 333...3. By induction hypothesis, corner vertices of $S_1(k-1,3)$, $S_2(k-1,3)$, $S_3(k-1,3)$ does not belong to L and $S(k,3)$ is obtained by joining vertices (211...1, 122...2); (311...1, 133...3); (233...3, 322...2) with an edge. Therefore *TLDS* for $S(n,3)$ is $3(\gamma_L(S(n-1,3)))$. Hence $\gamma_{TL}(S(n,3)) = 3(\gamma_{TL}(S(n-1,3))) = 2(3^{n-1})$. □

Theorem 14 [14] Let $S'(2,K_4)$ be a Sierpiński complete graph. Then $\gamma_L(S'(2,K_4)) = 9$.

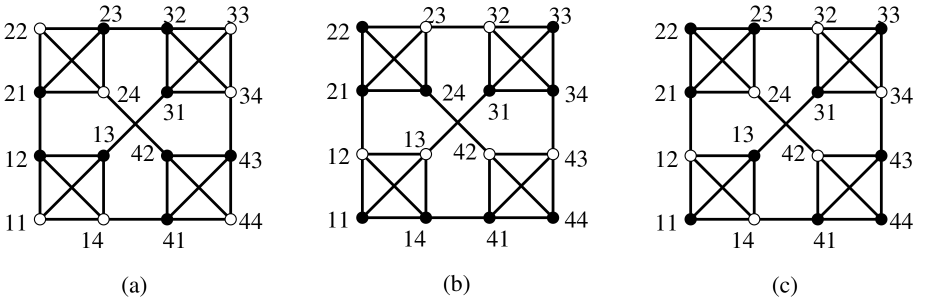


Fig. 1: Bold vertices in each case represent the liar's dominating set

$S(1,K_4) \cong K_4$ hence $\gamma_{TL}(K_n) = 3$. Consider $n = 2$, each $S_i(1,K_4)$ should have 3 vertices in L, let $L = \{12,13,14,21,23,24,31,32,34,41,42,43\}$. For the adjacent vertices in L say 12 and 21 along the linking edges of $S_1(1,K_4)$ and $S_2(1,K_4)$ we have $|(N(12) \cup N(21)) \cap L| = 4$. If any one vertex 13 and 24 from these two copies are unsaturated then $|(N(11) \cup N(13)) \cap L| = 2$ and $|(N(22) \cup N(24)) \cap L| = 2$. Therefore each $S(1,K_4)$ should contain 3 vertices. Thus the minimum *TLDS* for $S(2,K_4)$ is 12. In general, we have the following result:

Theorem 15 *Let* $S(n,K_4)$ *be a Sierpinski complete graph. Then* $\gamma_{TL}S(n,K_4) = 4[\gamma_{TL}S(n-1,K_4)] = 12(4^{n-2})$.

By Theorem 14 and Fig. 1, we observe that the *TLDS* of $S'(n,K_4)$ is selected as in Fig. 1 (a) only but in Fig. 1 (b) and (c) do not admit $TLDS$ because the corner vertices $\{11,44\}$ and $\{11,33\}$ in Fig. 1 (b) and (c) respectively do not satisfy the first condition of *TLDS* i.e., $|N(11) \cap L| = |N(33) \cap L| = |N(44) \cap L| = 1$. Therefore we have:

Theorem 16 *Let* $S'(n,K_4)$ *be a Sierpiński complete graph with* $n \geq 2$. *Then* $\gamma_{TL}S'(n,K_4) = 4[\gamma_{TL}S'(n-1,K_4)] = 9(4^{n-2})$.

In view of the above results we observe that:

Remark 1 Let G be a connected graph and let L be minimum liar's dominating set and if $< L >$ is a component of cycles or component of cycles joined by a path then L has a total liar's dominating set.

In view of the above study on graphs with equal *LDS* and *TLDS* and since $3 \leq \gamma_{TL}(G) \leq |V(G)|$, we present the following results:

Theorem 17 *Let G be a connected graph and L be a liar's dominating set. Then $\gamma_L(G) = \gamma_{TL}(G) = 3$ iff $G = C_3 + H$ where H is any graph such that for any $v \in H$, $deg_H(v) > 2$.*

Proof Let $L = \{v_1, v_2, v_3\}$ be a liar's dominating set. We know that the vertices in L are adjacent to each other and also for any $u \in V$, if $N(u) = \{v_1, v_2\}$ then $|N(u) \cup L| = 2$ but $|(N(u) \cup N(v_3)) \cap L| = 2$ which is a contradiction to $\gamma_L(G) = \gamma_{TL}(G)$. Hence $N(u) = \{v_1, v_2, v_3\}$. So if $S = V \backslash \{v_1, v_2, v_3\}$ then $G[S] = C_3 + H$ for any $v \in H$, $deg_H(v) > 2$. The converse part is clear. □

Theorem 18 *Let G be connected graph with $n \geq 5$. If X is a set of vertices with degree 2 and $N(X) = L$ where L is a total liar's dominating set then $\gamma_L(G) = \gamma_{TL}(G) = n$.*

Proof Since X contains vertices of degree 2, and L is a total liar's dominating set it is obvious that $N(X) \subseteq L = V$. Thus $|L| = |V|$ and it is obvious that L is a liar's dominating set. Since for any $u, v \in X$, $N(u) \cup N(v) = N[u]$ or $N[v]$, the liar's dominating set is minimum and therefore $\gamma_L(G) = \gamma_{TL}(G) = n$. □

2.2 Classes of Graphs with *TLDS* greater than *LDS*

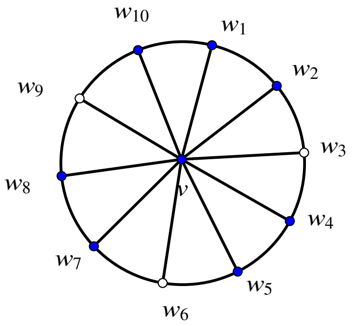


Fig. 2: Wheel Graph (W_{10})

Theorem 19 *Let W_n be a wheel graph with $n \geq 4$, then $\gamma_{TL}(W_n) = [\frac{3n}{4}] + 1$.*

Proof Let L include w_i's, i odd, $1 \leq i \leq n-1$; so that $|N(w_i) \cap L| = 2$, i even and $|N(v) \cap L| \geq 2$. Even if v is included in L, it is not sufficient since each w_i's, i odd are such that $|N(w_i) \cap L| < 2$, i odd. Suppose, L includes two consecutive vertices alternatively and also v. Then for every $w_i \in L$, $|N(w_i) \cap L| = 2$ and each $w_i \notin L$ is triple dominated. Therefore L becomes a total liar's dominating set and $\gamma_{TL}(W_n) = [\frac{3n}{4}] + 1$. □

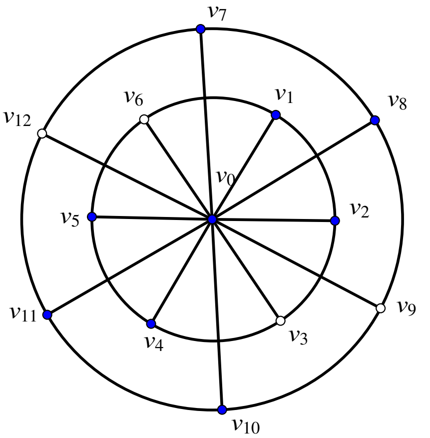


Fig. 3: Double Wheel Graph ($W_{6,6}$)

Theorem 20 *Let $W_{n,n}$ be a double wheel graph. Then $\gamma_{TL}(W_{n,n}) = 2[\frac{3n}{4}] + 1$.*

Proof The graph induced by $\langle v_0, v_1, \ldots, v_n \rangle$ a wheel graph (W_n), by Theorem 19 has $[\frac{3n}{4}] + 1$ vertices in L. Now since $v_0 \in L$, the outer cycle also needs $[\frac{3n}{4}]$ vertices and therefore $\gamma_{TL}(W_{n,n}) = [\frac{3n}{4}] + [\frac{3n}{4}] + 1 = 2[\frac{3n}{4}] + 1$. □

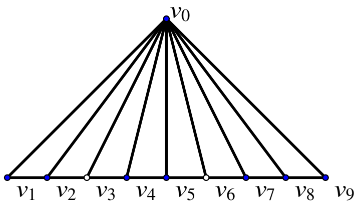


Fig. 4: Fan Graph (F_9)

Theorem 21 *Let F_n be a fan graph. Then*

$$\gamma_{TL}(F_n) = \begin{cases} [\frac{2n}{3}] + 2, & n \equiv 0, 1 \pmod 3 \\ \lceil \frac{2n}{3} \rceil + 1, & n \equiv 2 \pmod 3 \end{cases}$$

Proof Selecting two consecutive vertices alternatively and also v_0 in L we have the following cases:
Case 1: $n \equiv 0 \pmod 3$
In this case, $v_1, v_{n-1} \in L$ and $v_n \notin L$. Now since $N(v_n) \cap N(v_{n-2}) = \{v_0, v_{n-1}\}$, v_n should be in L such that $|(N(v_n) \cup N(v_{n-2})) \cap L| = 3$. Hence $|L| = 1 + [\frac{2n}{3}] + 1$.
Case 2: $n \equiv 1 \pmod 3$
Here we see that $v_n \in L$ and $v_{n-1} \notin L$ that makes $|N(v_n) \cap L| = 1$. Thus v_{n-1} should be in L. Hence, $|L| = 1 + [\frac{2n}{3}] + 1$.
Case 3: $n \equiv 2 \pmod 3$
In this case, $v_1, v_n \in L$ and for any $u, v \in L$ both the conditions of *TLDS* is satisfied and therefore $|L| = 1 + \lceil \frac{2n}{3} \rceil$. □

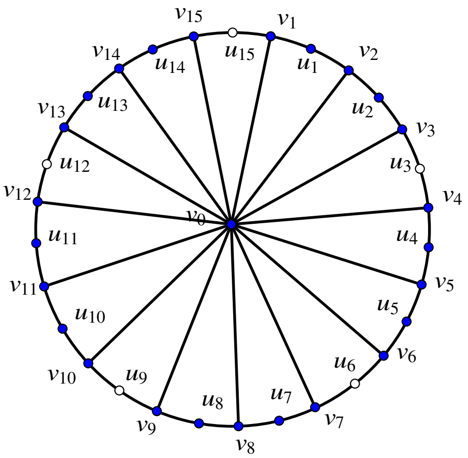


Fig. 5: Gear Graph (G_{15})

Theorem 22 *Let G_n be a gear graph. Then, $\gamma_{TL}(G_n) = n + [\frac{2n}{3}] + 1$.*

Proof Let $L = v_0 \cup \{v_i\}, 1 \leq i \leq n$. Every u_i's lie in between v_i and v_{i+1} and are such that $|N(u_i) \cap L| = 2$. Since each v_i is dominated by itself and v_0, $|N(v_i) \cap L| = 1$. Suppose if u_i, i odd, belongs to L then the graph induced by the vertices $\langle v_i, u_i, v_{i+1}, v_0 \rangle$ is isomorphic to C_4. Thus if consecutive vertices $\{u_i, u_{i+1}, u_{i+3}, u_{i+4}, \ldots\}$ are selected alternatively then $|N(v_i) \cap L| \geq 2$. Thus $|L| = n + [\frac{2n}{3}] + 1$. □

Theorem 23 *Let HG_n be a half gear graph. Then $\gamma_{TL}(HG_n) = n + 3[\frac{n}{4}] + 1$.*

Proof The proof is similar to Theorem 22.

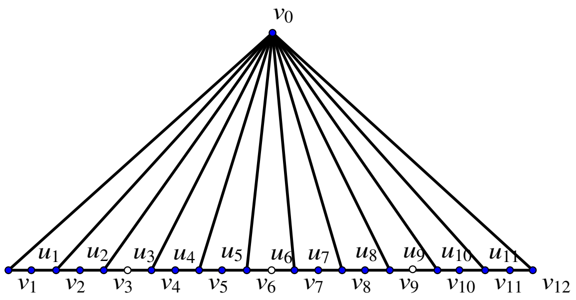


Fig. 6: Half Gear Graph (HG_{12})

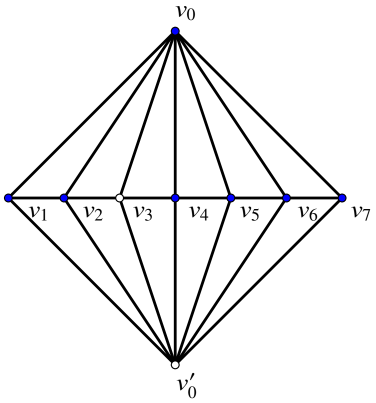


Fig. 7: Double Fan Graph ($F_{2,7}$)

Theorem 24 *Let $F_{2,n}$ be a double fan graph. Then*

$$\gamma_{TL}(F_{2,n}) = \begin{cases} [\frac{2n}{3}] + 2, & n \equiv 0, 1 \pmod 3 \\ [\frac{2n}{3}] + 1, & n \equiv 2 \pmod 3 \end{cases}$$

The proof is similar to Theorem 21.

Theorem 25 *Let $P(m, n)$ be a polar grid graph. Then $\gamma_{TL}(P(m, n)) = m\lceil \frac{n}{2} \rceil + \lceil \frac{n}{4} \rceil + 1$, $m \geq 2$.*

Proof Let us assume that when $m = 1$, it is a normal wheel graph with $n+1$ vertices and thus L consists of $\lceil \frac{n}{2} \rceil$ vertices alternatively and also the central vertex $v_0 \in L$. All the vertices not in L are double dominated and the central vertex is dominated $\lceil \frac{n}{2} \rceil + 1$ times. Now to double dominate the vertices in L let the vertices adjacent to it in the grid $m = 2$ be in L. Since each of the vertices in L dominates each other and for every adjacent $u, v \in L$, $N[u] = N[v] = 2$. Therefore, again $\lceil \frac{n}{2} \rceil$ vertices from grid $m = 3$ belong to L and so on. Since on grid m, for any $u \in L$, $|N(u) \cap L| = 1$,

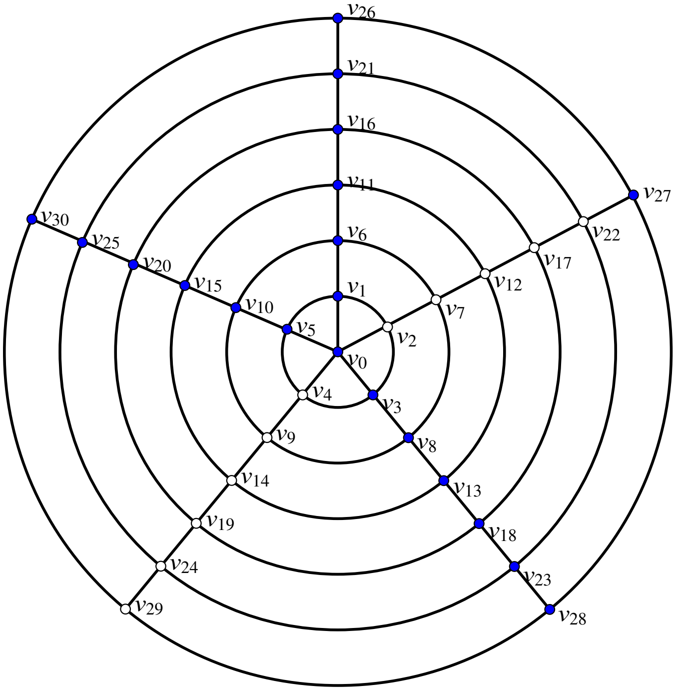


Fig. 8: Polar Grid Graph ($P(6, 5)$)

$[\frac{n}{4}]$ vertices from grid m say $(m-1)n+2, (m-1)n+6, \ldots$ are included in L such that the conditions of *TLDS* are satisfied. Hence $\gamma_{TL}(P(m, n)) = m\lceil\frac{n}{2}\rceil + [\frac{n}{4}] + 1$.□

Fig. 9: Minimum *LDS* of Sierpiński Cycle Graph $S'(2, C_4)$

Theorem 26 *Let $S'(2, C_4)$ be a Sierpiński cycle graph. Then $\gamma_{TL}(S'(2, C_4)) = 12$.*

Proof The minimum liar's dominating sets for $S'(2, C_4)$ are shown in Fig. 9. In which by symmetry, the minimum liar's dominating set that includes either two or one of its extreme vertices is $\{12, 13, 14, 22, 23, 32, 33, 42, 43, 41\}$ or $\{12, 13, 14, 22, 23, 32, 34, 42, 43, 41\}$. But to find the *TLDS* for $S'(2, C_4)$ either $\{21, 34\}$ or $\{21, 31\}$ should be included in L. But in both the cases we have $|(N(11) \cup N(13)) \cap L| = 2$ and also $|(N(44) \cup N(42)) \cap L| = 2$. Hence, instead of the extreme vertices all the inner vertices should be in L i.e., $\{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43\}$ only then $|(N(i\,i+2) \cup N(ii)) \cap L| = 3$ for $i = 1, 2, 3, 4$ ($i \mod 4$). □

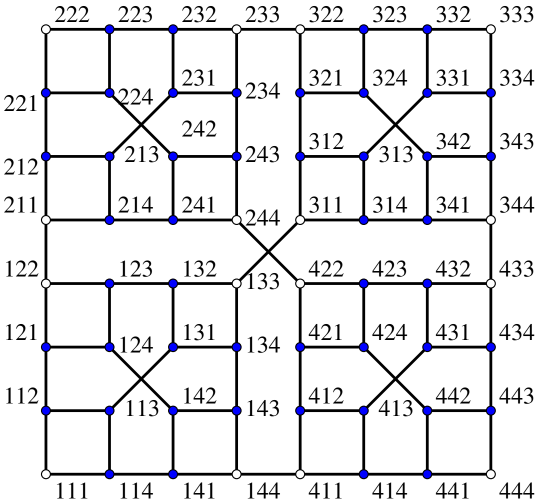


Fig. 10: Bold vertices depict the vertices in the minimum *TLDS* of $S'(3, C_4)$

Theorem 27 *Let* $S'(3, C_4)$ *be a Sierpiński cycle graph. Then* $\gamma_{TL}(S'(3, C_4)) = 4(\gamma_{TL}(S'(2, C_4)) = 48$.

Proof $S'(3, C_4)$ consists of 4 copies of $S'(2, C_4)$ namely $S'_i(2, C_4)$ with the respective corner vertices *iii* for $i = 1, 2, 3, 4$ with the linking edges (122, 211), (233, 322), (344, 433), (411, 144) and cross edges (133,311) and (244,422). Consider the copy $S'_1(2, 3)$. Here, let 112, 114, 113 belong to L along with its adjacent vertices and 123, 143 belong to L. Now instead of 124 if 211 belongs to L then $|N(123) \cap L| = 1$ and $|N(121) \cap L| = 1$. Suppose, instead of 124 and 142 if the vertices 122 and 144 belong to L then 211 and 411 from $S'_2(2, 3)$ and $S'_4(2, 3)$ respectively should be in L. Hence in each $S'_i(2, 3)$, we have 12 vertices. Therefore, $\gamma_{TL}(S'(3, C_4)) = 4(\gamma_{TL}(S'(2, C_4)) = 4(12) = 48$. □

In general we have the following result:

Theorem 28 *Let* $S'(n, C_4)$ *be a Sierpiński cycle graph. Then* $\gamma_{TL}(S'(n, C_4)) = 4(\gamma_{TL}(S'(n-1, C_4))) = 12(4^{n-2})$.

3 Conclusion

In this paper, we have obtained the total liar's domination number for certain classes of graphs and have presented results for general graphs for which the liar's dominating set and total liar's dominating sets are equal and not equal. We extend this study to many more classes of graphs in our future work.

References

1. A. Alimadadi, D. A. Mojdeh, and N. J. Rad. Various bounds for liar's domination number. *Discussiones Mathematicae - Graph Theory*, 36(3):629–641, 2016.
2. S. Banerjee and S. Bhore. Algorithm and hardness results on liar's dominating set and k-tuple dominating set. In *International Workshop on Combinatorial Algorithms*, pages 48–60. Springer, 2019.
3. R. K. Jallu, S. K. Jena, and G. K. Das. Liar's dominating set in unit disk graphs. In *International Computing and Combinatorics Conference*, pages 516–528. Springer, 2018.
4. S. K. Jena, R. K. Jallu, and G. K. Das. On d-distance m-tuple (, r)-domination in graphs. *Information Processing Letters*, page 106178, 2021.
5. P. D. Manuel. Locating and liar domination of circulant networks. *Ars Combinatoria*, 101:309–320, 2011.
6. B. Panda and S. Paul. Connected liar's domination in graphs: complexity and algorithms. *Discrete Mathematics, Algorithms and Applications*, 5(04):1–16, 2013.
7. B. Panda and S. Paul. A linear time algorithm for liars domination problem in proper interval graphs. *Information Processing Letters*, 113(19-21):815–822, 2013.
8. B. Panda and P. S. Hardness results and approximation algorithm for total liar's domination in graphs. *Journal of Combinatorial Optimization*, 27:643–662, 2014.
9. B. S. Panda and S. Paul. Liar's domination in graphs: Complexity and algorithm. *Discrete Applied Mathematics*, 161(7-8):1085–1092, 2013.
10. B. S. Panda, S. Paul, and D. Pradhan. Hardness results, approximation and exact algorithms for liar's domination problem in graphs. *Theoretical Computer Science*, 573:26–42, 2015.
11. M. L. Roden and P. J. Slater. Liar's domination in graphs. *Discrete Mathematics*, 309(19):5884–5890, 2009.
12. N. Seyedi, H. R. Maimani, and A. Tehranian. On the total liar's domination of graphs. *Communications in Combinatorics and Optimization*, 7(2):169–175, 2022.
13. A. S. Shanthi and D. G. Thomas. Liar's domination in sun graphs. *International Journal of Computer Sciences and Engineering*, 7:106–108, 2019.
14. A. S. Shanthi and D. G. Thomas. Liar's domination in sierpi´nski-like graphs. *South East Asian Journal of Mathematics and Mathematical Sciences*, 16:121–130, 2020.
15. P. J. Slater. Liar's domination. *Networks*, 54(2):70–74, 2009.
16. A. S.Shanthi and D. G. Thomas. Liar's domination and connected liar's domination in triangle-like graphs. *Advances and Applications in Mathematical Sciences*, 21.
17. C. Sterling. *Liar's Domination in Grid Graphs*. PhD thesis, East Tennessee State University, 2012.

DEGREE BASED TOPOLOGICAL INDICES OF POLYSILOXANES

MUGE TOGAN,AYSUN YURTTAS GUNES, MUSA DEMIRCI
ISMAIL NACI CANGUL[*]
Bursa Uludag University, Faculty of Arts & Science,
Mathematics Department, Gorukle-Bursa 16059 Turkey

Abstract

Molecules can be modelled by graphs to obtain their required properties by means of only mathematical methods and formulae. In this paper, we consider several degree based topological graph indices of a chemical molecule called polysiloxanes.

AMS 2010 Subject Classification Codes: 05C07, 05C10, 05C30, 68R10
Keywords: Polysiloxane, graph, topological indices

1 Introduction

The term polysiloxane refers to a class of compounds whose molecules consist of a silicon-oxygen backbone $(-Si-O-Si-O-Si-O-Si-)n$ arranged either in a linear or cyclic (ring) pattern. Each silicon in the chain has two additional oxygen atoms attached to it. In many cases, the polysiloxanes also have one or more alkyl groups attached to the main chain replacing one or more of the oxygens. An alkyl group is an alkane, a saturated hydrocarbon, lacking one hydrogen atom. Examples of alkyl groups are the methyl $(-CH3-)$ and ethyl $(-CH2CH3-)$ groups. Its chemical formula is $(R2SiO)n$, where R is usually methylsiloxanes $(-CH3-)$, although it can be H or an alkyl or aryl group. In a common type of siloxane, all the oxygens that are not a part of the backbone of the chain are replaced by methyl groups.

Polysiloxane has shown greater resistance to the effects of UV radiation than organic polymers containing a carbon-carbon backbone. Polysiloxanes can be oils, greases, rubbers or plastics depending on molecular weight. Polysiloxanes exhibit superb abrasion and corrosion resistance, anti-graffiti properties, strong chemical resistance and resilience to dirt pickup. In Figure 1, a single independent unit of the polysiloxane molecule is illustrated. Some useful properties of polysiloxanes include flexibility, resistance to water and oxidation, chemical inertness, permeability to gases, low glass transition temperature and low

---

[*] Corresponding Author

surface energy. These properties make polysiloxanes suitable for a wide range of commercial applications, such as electrical insulation, electronical coatings, household sealants for cooking apparatus, automobile gaskets, airplane seals, textiles or paper coatings and lubricating greases, [12].

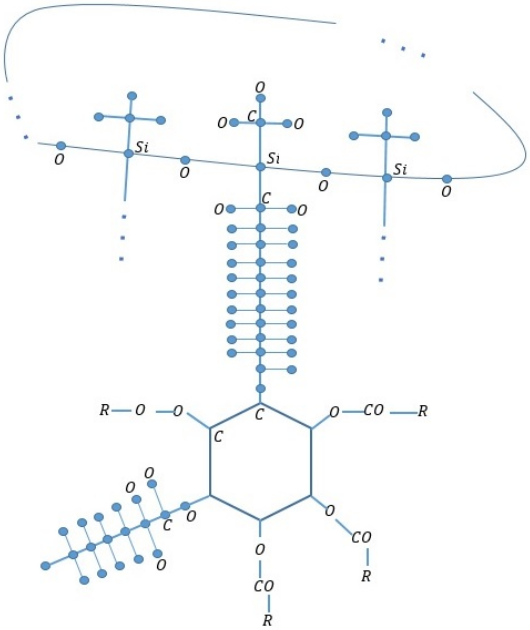


Figure 1: Polysiloxanes

Graph indices are used in different ways to study several properties of chemical substances. They are defined as topological graph invariants measuring several properties of graphs that are modeling real life cases. They can be classified to three groups according to the way they are defined: by vertex degrees, by matrices or by distances, see e.g. [1, 2]. The first and second Zagreb indices $M_1(G)$ and $M_2(G)$ are defined by Gutman and Trinajstic, [9], by $M_1(G) = \sum_{u\in V(G)} d_G^2(u)$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, respectively, and become popular due to their uses in QSAR and QSPR studies. In [4], some results on the first Zagreb index together with some other indices are given and these indices are calculated for some graph operations in [5].

The forgotten index of a graph G denoted by $F(G)$ or $M_3(G)$ was defined by $F(G) = \sum_{u\in V(G)} d_G^3(u)$, see [7, 8]. The hyper-Zagreb index is defined as a variant of the Zagreb indices as $HM(G) = \sum\limits_{uv\in E} (d_u + d_v)^2$, see e.g. [7].

Inspired by the study of heat of formation for some alkanes, Furtula et al. proposed an index called Augmented Zagreb index, which gives a better prediction power in [6]. It is defined by $AZI(G) = \sum_{uv\in E(G)} \frac{d_u d_v}{d_u+d_v-2}$. The

Harmonic index was introduced by Zhong [16], who found that it correlates well with Π-electron energy of benzenoid hydrocarbons and it was defined by $H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$. Ranjini et al., [13], defined the redefined first, second and third Zagreb indices for a graph G as $ReZG_1(G) = \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u \cdot d_v}$, $ReZG_2(G) = \sum_{uv \in E(G)} \frac{d_u \cdot d_v}{d_u + d_v}$ and $ReZG_3(G) = \sum_{uv \in E(G)} (d_u \cdot d_v)(d_u + d_v)$. Milicevic et al., [11], reformulated the Zagreb indices in terms of the edge degrees instead of the vertex-degrees as $RM_1(G) = \sum_{uv \in E(G)} d(e)^2$, $RM_2(G) = \sum_{e,e' \in E(G)} d(e)d(e')$ where e, e' are pairs of incident edges of the graph G. Aram and Dehgardi, [3], introduced the concept of reformulated F-index as $RF(G) = \sum_{uv \in E(G)} d(uv)^3$. Kulli, [10], introduced the first and second K Banhatti indices, intending to take into account the contributions of pairs of incident elements. They are defined as $B_1(G) = \sum_{u,e} [d_G(u) + d(e)]$, $B_2(G) = \sum_{u,e} d_G(u) d(e)$.

2 Main Results

We shall denote the polysiloxane consisting of n independent units by G^*. Now we will determine some well-known topological indices of polysiloxane.

Lemma 1 *The first and second Zagreb indices of G^* are $M_1(G^*) = 812n$ and $M_2(G^*) = 1040n$.*

Proof. We partition the edges of G^* into edges of type $E_{(d_u, d_v)}$ where uv is an edge. In G^*, we get the edges of type $E_{(1,3)}, E_{(1,4)}, E_{(2,3)}, E_{(2,4)}, E_{(3,3)}, E_{(3,4)}$ and $E_{(4,4)}$. The number of edges of these types for one unit are 6, 78, 12, 2, 6, 6 and 31, respectively. First, we calculate $M_1(G)$ for one unit:

$$\begin{aligned} M_1(G^*) &= \mid E_{(1,3)} \mid (1+3) + \mid E_{(1,4)} \mid (1+4) + \mid E_{(2,3)} \mid (2+3) \\ &+ \mid E_{(2,4)} \mid (2+4) + \mid E_{(3,3)} \mid (3+3) \\ &+ \mid E_{(3,4)} \mid (3+4) + \mid E_{(4,4)} \mid (4+4) = 812. \end{aligned}$$

Hence, for n units, we get the final result as $M_1(G^*) = 812n$ due to the independence of the single units and by the additivity of these indices. As $M_2(G) = \sum_{uv \in E(G)} d_u d_v$, we get the result by similar calculations again by the additivity and independence. ■

Lemma 2 *The third Zagreb index (forgotten index) of G^* is $F(G^*) = 2832n$.*

Proof. We know that $F(G) = \sum_{u \in V(G)} {d_u}^3$, i.e.,

$$F(G^*) = \sum_{u \in V} {d_u}^3 = 1^3 \cdot 84 + 2^3 \cdot 7 + 3^3 \cdot 12 + 4^3 \cdot 37 = 2832.$$

For n units, we have $F(G^*) = 2832n$ by the independence of the units and additivity of the forgotten index. ■

Using the same counting method together with additive property due to independence of units of polysiloxane, we can prove the following results in a similar fashion:

Lemma 3 *The hyper Zagreb index of* G^* *is* $HM(G^*) = 4912n$.

Lemma 4 *The Augmented Zagreb index of* G^* *is* $AZI(G^*) = 1056,27 \cdot n$.

Lemma 5 *The harmonic index of* G^* *is* $H(G^*) = 51,13 \cdot n$.

Proof. We know that $H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$, i.e.,

$$\begin{aligned} H(G^*) &= \mid E_{(1,3)} \mid \frac{2}{1+3} + \mid E_{(1,4)} \mid \frac{2}{1+4} + \mid E_{(2,3)} \mid \frac{2}{2+3} \\ &+ \mid E_{(2,4)} \mid \frac{2}{2+4} + \mid E_{(3,3)} \mid \frac{2}{3+3} + \mid E_{(3,4)} \mid \frac{2}{3+4} + \mid E_{(4,4)} \mid \frac{2}{4+4} \\ &= 6 \cdot \frac{1}{2} + 78 \cdot \frac{2}{5} + 12 \cdot \frac{2}{5} + 2 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} + 6 \cdot \frac{2}{7} + 31 \cdot \frac{1}{4} = 51,13. \end{aligned}$$

■ The following can be deduced similarly:

Lemma 6 *The re-defined versions of the Zagreb indices of* G^* *are* $ReZG_1(G) = 140n$, $ReZG_2(G) = 165,25 \cdot n$, $ReZG_3(G) = 6884n$.

Lemma 7 *The reformulated Zagreb indices of* G^* *are* $RM_1(G^*) = 2228n$ *and* $RM_2(G^*) = 3022n$.

Lemma 8 *The reformulated F-index of* G^* *is* $RF(G^*) = 9836n$.

Proof. As $RF(G) = \sum_{uv \in E(G)} d(uv)^3$, the calculations are similar to $RM_1(G^*)$. ■

Lemma 9 *The Banhatti indices of* G^* *are* $B_1(G^*) = 1872n$ *and* $B_2(G^*) = 3288n$.

Proof. We know that $B_1(G) = \sum_{u,e} d_G(u) + d(e)$, i.e.,

$$\begin{aligned} B_1(G^*) &= 6[(1+2)+(2+3)] + 78[(4+3)+(1+3)] + 12[(2+3)+(3+3)] \\ &+ 6[(3+4)+(3+4)] + 2[(4+4)+(2+4)] + 6[(4+5)+(3+5)] \\ &+ 31[(6+4)+(6+4)] = 1872. \end{aligned}$$

For n units, we obtain $B_1(G^*) = 1872n$. For B_2, a similar proof applies. ■

3 Conclusion

Polysiloxane is a very useful chemical compound and although having a relatively complicated structure, it has too many real life applications which are mainly focused on isolation, lubrication and coating. Its modular structure allows us to study with it easily. Doing the calculations for one unit, we can obtain the general results for n units, directly, unlike many other chemical compounds. Using this information, we calculated several topological graph descriptors which are useful in QSAR and QSPR studies in chemistry. These include first and second Zagreb indices, forgotten index, hyper-Zagreb indices, augmented Zagreb indices, redefined Zagreb indices and reformulated Zagreb indices and Banhatti indices,

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

References

[1] Ahmad, A., Elahi, K., Hasni, R., Nadeem, M. F., Computing the degree based topological indices of line graph of benzene ring embedded in P-type-surface in 2D network, Journal of Information and Optimization Sciences, **40 (7)** (2019), 1511-1528.

[2] Ali, F., Salman, M., Hafeez, A., Huang, S., On computation of some distance-based topological indices on circulant networks-II, Journal of Information and Optimization Sciences, **39 (4)** (2018), 759-782.

[3] Aram, H., Dehgardi, N., Reformulated F-index of graph operations, Commun. Comb. Optim., **2** (2017), 87-98.

[4] Das, K. C., Akgunes, N., Togan, M., Yurttas, A., Cangul, I. N., Cevik, A.S., On the first Zagreb index and multiplicative Zagreb coindices of graphs, Analele Stiintifice ale Universitatii Ovidius Constanta, **24 (1)** (2016), 153-176.

[5] Das, K. C., Yurttas, A., Togan, M., Cangul, I. N., Cevik, A. S., The multiplicative Zagreb indices of graph operations, Journal of Inequalities and Applications, **90** (2013).

[6] Furtula B., Graovac, A., Vukicevic, D., Augmented Zagreb Index, Journal of Mathematical Chemistry **48** (2010), 370-380.

[7] Furtula, B., Gutman, I., A Forgotten Topological Index, J. Math. Chem., **53 (4)**, (2015), 1184-1190.

[8] Gutman, I., Trinajstic, N., Graph Theory and Molecular Orbitals. Total pi electron energy of alternant hydrocarbons, Chem. Phys. Lett., **17** (1972), 535-538.

[9] Gutman, I., Trinajstic, N., Graph theory and molecular orbitals III. Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett., **17** (1972), 535-538.

[10] Kulli, V. R., On K Banhatti indices of graphs, J. Comput. Math. Sci., **7**, (2016), 213–218

[11] Milićević, A., Nikolić,S., Trinajstić, N., On reformulated Zagreb indices, Mol. Divers. **8**, (2004), 393–399.

[12] Moretto, H.-H., Schulze, M., Wagner, G., ”Silicones” in Ullmann's Encyclopedia of Industrial Chemistry, Wiley-VCH, Weinheim (2005) doi:10.1002/14356007.a24-057.

[13] Ranjini, P. S., Lokesha, V., Usha, A., Relation between phenylene and hexagonal squeeze using harmonic index. Int J Graph Theory, **1** (2013), 116-21.

[14] Togan, M., Topological Indices of Collagen, Montes Taurus Journal of Pure and Applied Mathematics, 2 (1) (2020), 81-86.

[15] Yurttas Gunes, A., Togan, M., Demirci, M, Cangul, I. N., Harmonic Index and Zagreb Indices of Vertex-Semitotal Graphs, European Journal of Pure and Applied Mathematics, Special Issue Dedicated to Prof. Hari. M. Srivastava on the Occasion of his 80th Birthday, 13 (5) (2020), 1260-1269.

[16] Zhong, L., The harmonic index on graphs, Applied Mathematics Letters, **25**, (2012), 561-566.

Muge TOGAN, Aysun YURTTAS GUNES, Musa DEMIRCI, Ismail Naci CANGUL
capkinm@uludag.edu.tr, ayurttas@uludag.edu.tr, mdemirci@uludag.edu.tr, cangul@uludag.edu.tr

Scientiae Mathematicae Japonicae 85, No.3(2023)(221-235)

FUZZY HYPERBOLIC GENDER INEQUALITY INDEX INVOLVING SOFT FUZZY NUMBER VALUED INFORMATION SYSTEM – A STUDY IN THE INDIAN CONTEXT

FELBIN C. KENNEDY AND ARUL ROSELET MERYLINE.S
STELLA MARIS COLLEGE(AUTONOMOUS)
AFFILATED TO THE UNIVERSITY OF MADRAS, CHENNAI,INDIA
FELBIN@STELLAMARISCOLLEGE.EDU.IN ARULROSELET@STELLAMARISCOLLEGE.EDU.IN

ABSTRACT.

A comprehensive measure of inequality will expose the true gender of economics in a society. Most of the data in Economics encapsulates discrete information which subsumes the reality of discrimination among male and female agricultural workers. This then leads to myopic policies emanating from the database which is cut and dry. Capturing only the ontology is incomplete; an epistemological understanding of the database is imperative. Hence the necessity to annul the lacuna arises. In this paper we consider dynamic multi-dimensional indicators that illustrate the degree of discord between genders. The questionnaire invoked linguistic responses which are then modeled as soft linear octagonal fuzzy number valued information system. A new procedure is proposed to obtain the fuzzy hyperbolic gender inequality index wherein fuzzy importance to the indicators and aggregation of fuzzy hyperbolic inequality index corresponding to the same using Chouqet integral are considered. Primary data is procured in the field of agriculture from two distinct regions rich in paddy cultivation. Using the procedure introduced we yield separate Fuzzy Hyperbolic Gender Inequality Indices for male and female workers in the two populations under consideration. In addition we determine the relative strength of the indicators by performing comparisons between both the samples and the genders.

1 Introduction Women are an integral part of India working population. However, despite their irrefragable contribution to the labour force, a large section of working women still exist at the grassroot level unable to break social constructs. Gender Inequality is thus one of the pressing issues confronting many developing economies. Discussion on gender inequality assumes paramount significance as the informal agriculture sector in India is characteristic of a large female population performing mundane manual labour and receiving paltry wages Discrimination against these women has garnered meager attention by authorities in office until the dawn of the 20th century. This era threw light on the working conditions of women, their temporary guarantee of employment and their eternal burden of balance between farm and family. Extensive research has been undertaken to understand the crack in the glass ceiling and work is in progress to make amends.

1.1 The Case for involving Fuzziness A comprehensive evaluation of gender inequality warrants the researchers to perceive the magnitude of disparity that the subject experiences as close as possible to the truth.This would call for an in-depth understanding of the attitudes and feelings of the sampled individuals. There are by far very few studies that have attempted to deviate from the ontology while capturing the deterrents to women

---

Key words and phrases. Fuzzy importance experts,Fuzzy Hyperbolic Gender Inequality Index, Agriculture, India .

equity. This leads to cut and dry analysis of data describing an imprecise problem and thereby emanating myopic policies.

In this paper importance is placed on socio-cultural and psychological attributes and normative economics. An epistemological standpoint is thus adopted in order to annul the lacuna present. Data surveys corresponding to studies of gender inequality in agriculture have so far recorded responses in binary language. However unlike the first world countries, agricultural labourers in developing economies such as India are often poorly literate and the responses were vague in nature. The vagueness is captured using linguistic concepts. Zadeh L.A [9, 10, 11] handled the quantification of linguistic concepts using fuzzy numbers. The concept of linear octagonal fuzzy number introduced by Malini S.U and Felbin C. Kennedy [8] in 2013 and had found various application for the same.

Thus, when responses are linguistic in nature it necessitates the employment of fuzziness to reflect an accurate picture of ground reality. It is also noteworthy that the responses could be qualitative in nature expressing equality only to a certain degree. There is a need to assess this magnitude of inequality that is characteristic of female wage labourers in agriculture. The linguistic nature of responses call for a fuzzy quantification to assess the degree of discord between genders. Therefore a fuzzy treatment is given to both the data set and the weights while developing the Gender Inequality Index.

In order to capture the linguistic data we employ soft fuzzy number valued information system and fuzzy number valued measure introduced in the earlier paper [1, 2, 3] as tool to measure the Fuzzy Hyperbolic Gender Inequality Index.

This study in the Indian context attempts to develop a Gender Inequality Index for paddy wage labourers. The contribution of this study is in three folds: identifying attributes that assess the degree of discord between genders in paddy farm employment, developing a model that yields a Gender Inequality Index for male and female workers involving soft fuzzy number valued information system, applying the methodology to a random sampling of primary data collected from two different paddy cultivating regions in Tamil Nadu, India.

The paper is organized as follows: The first section provides a brief introduction on Gender Inequality in Agriculture and makes a case for the fuzzy treatment of responses. The next section presents the preliminaries this paper is based on and briefly reviews literature. The third section explains the assessors of inequality and develops the conceptual framework. After this a short description is recorded on the background of the study. The fifth section describes the methodological framework developed. The following section applies the model developed in the previous section using primary data. Results are presented followed by analysis. Finally the seventh section contains concluding remarks and policy recommendations.

2 Preliminaries

Definition 2.1 [8] A fuzzy number $\tilde{A}$ was said to be a *linear octagonal fuzzy number* denoted by $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \in \mathbb{R}$ with membership function $\tilde{A}(x)$ given by

$$\tilde{A}(x) = \begin{cases} k(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ k & a_2 \leq x \leq a_3 \\ k + (1-k)(\dfrac{x-a_3}{a_4-a_3}) & a_3 \leq x \leq a_4 \\ 1 & a_4 \leq x \leq a_5 \\ k + (1-k)(\dfrac{a_6-x}{a_6-a_5}) & a_5 \leq x \leq a_6 \\ k & a_6 \leq x \leq a_7 \\ k(\dfrac{a_8-x}{a_8-a_7}) & a_7 \leq x \leq a_8 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq k \leq 1$

The collection of such octagonal fuzzy numbers is denoted $\mathcal{F}(\mathbb{R})$ and that of non- negative octagonal fuzzy numbers is denoted $\mathcal{F}^*(\mathbb{R})$

Definition 2.2 [8] Let $\tilde{A}$ be an octagonal fuzzy number. The *measure* on $\tilde{A}$ was defined by $M^{Oct}(\tilde{A}) = \frac{1}{4}[(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1-k)]$

Remark 2.1 *[8] Any two linear octagonal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ could be compared using the following:*
1. $\tilde{A} \preceq \tilde{B} \iff M^{Oct}(\tilde{A}) \leq M^{Oct}(\tilde{B})$
2. $\tilde{A} \approx \tilde{B} \iff M^{Oct}(\tilde{A}) = M^{Oct}(\tilde{B})$
3. $\tilde{A} \succeq \tilde{B} \iff M^{Oct}(\tilde{A}) \geq M^{Oct}(\tilde{B})$

Definition 2.3 [5] Let $\tilde{A} \approx (a_1, a_2, \ldots, a_8; k, 1)$ and $\tilde{B} \approx (b_1, b_2, b_3, \ldots, b_8; k, 1)$ be two linear octagonal fuzzy numbers, then addition and scalar multiplication are defined by
1. $\tilde{A} \oplus \tilde{B} \approx (a_1 + b_1, a_2 + b_2, \ldots, a_8 + b_8; k, 1)$
2. $\lambda\tilde{A} \approx (\lambda a_1, \lambda a_2, \ldots, \lambda a_8; k, 1)$ for any non-negative real number λ.

Remark 2.2 $\tilde{A} \oplus \tilde{B}$ *and* $\lambda\tilde{A}$ *yield linear octagonal fuzzy numbers.*

Definition 2.4 *The division of two non-negative octagonal fuzzy numbers* $\tilde{A} \approx (a_1, a_2, \ldots, a_8; k, 1)$ *and* $\tilde{B} \approx (b_1, b_2, b_3, \ldots, b_8; k, 1)$ *with* $a_1 \geq 0$ *and* $b_1 > 0$ *is defined by* $\tilde{A} \oslash \tilde{B} \approx \left(\frac{a_1}{b_8}, \ldots, \frac{a_8}{b_1}; k, 1\right)$.

Whenever the relevance of the group and the importance of the individual criteria are required, the Choquet integral-based aggregation has been used. The Choquet integral is associated to a fuzzy measure that aggregates the interaction between the criteria. One of the author [6] introduced the octagonal fuzzy Choquet integral as explained in the following:

Definition 2.5 [6] Let $\{\tilde{A}_i\}_{i=1}^n$ be a collection of octagonal fuzzy numbers on a finite set X and m be $\lambda-$ measure on X. The octagonal fuzzy Choquet integral of $\tilde{A}_i$ with respect to m is defined by OFCI $\left(\tilde{A}_1, ..., \tilde{A}_n\right) = \oplus\sum_{i=1}^n \left(m(E_{(i)}) - m(E_{(i+1)})\right)\tilde{A}_{(i)}$ where $(\cdot)$ indicates the permutation on X such that $\tilde{A}_{(1)}, \preceq \tilde{A}_{(2)} \preceq \cdots \preceq \tilde{A}_{(n)}$ and $E_{(i)} = \{x_i, \ldots, x_n\}$, $E_{(n+1)} = \emptyset$

Remark 2.3 *The ordering of linear octagonal fuzzy numbers in the above definition are obtained using radius of gyration introduced by one of the author [4] and and $\lambda-$ measure of the powerset of the attribute set and aggregation of the attributes for the objects under consideration of the algorithm presented [5] is considered in the procedure proposed in section 5 as the identified indicators are interactive in nature.*

Definition 2.6 [1] A soft linear octagonal fuzzy number is defined as a mapping $\tilde{\tilde{f}} : E \to \mathcal{F}^*(\mathbb{R})$, E the parameter set, $\mathcal{F}^*(\mathbb{R})$ collection of linear octagonal fuzzy numbers. The collection of soft fuzzy numbers was denoted as $\tilde{\mathcal{F}}^*(\mathbb{R})(E)$.

Definition 2.7 [1] Let $(\tilde{\tilde{f}}, E) \in \tilde{\mathcal{F}}(\mathbb{R})(E)$ with $E = \{e_j\}_{j=1}^{l}$. A fuzzy number valued measure on $\tilde{\tilde{f}}$ was defined by
$\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{f}}, E)\right] = \sum_{j=1}^{l}\left[w_j \tilde{\tilde{f}}(e_j)\right]$ where $w_j \geq 0$ are weights of the parameters in E with $\sum_{j=1}^{l} w_j = 1$.

Definition 2.8 [1] A soft linear octagonal fuzzy number valued information system is a quadruple
$\widetilde{IS} = (U, A, \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E}), \tilde{I})$ where
$U = \{u_i\}_{i=1}^{m}$ is the set of objects under consideration,
$A = \{a_j\}_{j=1}^{n}$ is the attribute set,
$\mathcal{E} = \{E_1, E_2, ..., E_n\}$ and $E_j = \{e_{jk}\}_{k=1}^{l_j}$ is the parameter set associated with attribute a_j, l_j representing the number of parameters in E_j and
if $\tilde{I} : U \times A \to \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E})$ is a mapping such that $\tilde{I}(u_i, a_j) = (\tilde{\tilde{f}}_{ij}, E_j) \in \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E})$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ where $(\tilde{\tilde{f}}_{ij}, E_j)$ is a soft linear octagonal fuzzy number.

3 Developing the Conceptual Framework – Indicators of Inequality In this section we present the rationale for ten main indicators on gender inequality. The details are furnished below:

1. **Education and skill development :** This attribute captures the individual's knowledge about different farming practices, their access to various extension programs and their awareness about the existence of technology used for cultivation. There is asymmetric information in the market between genders which needs to be considered as extension services and innovation programmes are not equitably accessible.

2. **Financial Independence :** It is convenient to consider the partners in a household as players of a co-operative game having a coalition strategy so as to maximize the gains for the household unit. However in reality we see that there is inefficient allocation of gains. A parameter considered here is their allocation of earnings to satisfy personal desires that refers to wants beyond their basic necessities like the money they keep aside to buy themselves jewellery, clothes magazines, movie tickets etc. Despite children's expenditure being a public good within the unit, often women contribute a major portion of their earnings to fulfill their off-springs wants rather then their own needs. Such irrational decision making of the intra-household agents make it important to incorporate these two parameters while studying the inequalities in gender. It is very worthy of investigation in Indian agrarian households where spousal control over earnings arise and men tend to keep away their income in private while the women in most cases deposit their income with the head of the household which is usually a man.

3. **Decision Making Power and Empowerment :** An unitary household assumption is often presumed where the entire household is treated as one agent. However in our context of a male-dominated agrarian backdrop in a developing nation it is more realistic to study the behavior of each individual agent as men tend to have a more dominant strategy when it comes to decision making. In our study we deem it

significant to consider decisions made regarding the workplace - in determining the nature and time of employment; household - with respect to expenditure and purchase of utilities and contribution to household chores.

4. **Mobility and Freedom :** Rural Indian agrarian women clad in traditional sarees often do not commute using even the relatively inexpensive bicycles. In addition to mobility their active participation in social activities and frequency of visiting the parental home is often decided by their husbands or in-laws. All of these in turn impact their on farm performance.

5. **Rights :** Awareness of the fundamental rights of equal wages; information and education and inheritance are our three sub parameters in the attribute-Rights. The researchers feel that knowledge of basic constitutional rights is integral as most rural women are usually unaware about the Minimum Wages Act 1948. The bargaining power is paltry and women succumb to the landlords discretion. There is an absence of Labour unions in the informal set up. Additionally with respect to our sample female population it is often perceived a waste of time to educate them as they are ultimately going to be employed in mundane farm jobs only.

6. **Political Participation :** Active membership in self-help groups and participation in its activities are indicative of a woman's access and the freedom to raise her voice. The researchers included this sub-attribute to investigate if activities of the self-help group contribute to reducing a women's mental and financial burden. Filial control over exercising one's franchise is commonplace where illiteracy prevails. Thus the influence of family friends or the media in casting your ballot is suggestive of disparity.

7. **Health :** Certain jobs like weeding and transplanting are often deemed as 'women's work' in the field. These seemingly routine jobs require the woman to squat in muddy water for hours together. Apart from causing severe physical pain that the women tend to ignore, these unsanitary conditions can be symptomatic of gynecological ailments, skin irritation and water borne diseases. In addition paltry remuneration, demanding toil as well as conduct of male co-workers or the landlord can also impact the emotional well-being. Despite not beholding a direct linkage, we need to understand that the health of a woman affects her performance in the field which in turn affects her wages and working hours.

8. **Safety and Security :** The informal farm employment is unprotected by medical insurances or work safety norms. Wage labourers work in wilderness,not far from snakes and other insects and pests. With no first aid medical-kits in hand or hospitals in the immediate neighborhood, these labourers are often left to tend to themselves. While safety is a concern for both male and female, it is an important indicator for our study as the differential negative impact for the women is greater. Since women do the transplanting which requires the farm to be flooded in soiled water they are more vulnerable to dangerous lurking reptiles.

9. **Gender Pay Gap :** In our context we measure the perception of whether the labourers believe this inequality in wage rate is actually unfair given gendered segregation of jobs and differential working hours.Attitude towards receiving an unfair wage rate is a key determinant of inequality.

10. **Freedom of Choice :** A pre-conclusive judgement of inequality based on socio-cultural assumptions need not be an ideal definition for our study. It is worthy to note

that being a paddy wage labourer could have been an informed or preferred choice. A women might stay a farm worker not as her family compelled her to but as she desired to be one. She derives a greater utility from her statuesque then she would have from any other choice. In this case access to alternative employment or higher education has not been hampered and there is no deprivation in her rights and freedom. Thus while formulating a measure of inequality it is integral to understand the agents capabilities and functionings of the agent. Though several studies of inequality in the past have incorporated capabilities involving exogenous variables such as leisure, education and health, to the best of the researchers' knowledge no previous work formulating a Fuzzy Hyperbolic Gender Inequality Index has accounted for the opportunities of freedom involving functioning and capabilities. Inspired by the seminal work of Nobel Laureate Amartya Sen this study incorporates the invaluable idea as described below. We seek to address three key questions:
a) **Leisure :** Are you able to enjoy leisure of personal value?
b) **Literacy :** Were you able to attain the level of literacy you wished to have so that you would have gone for a better employment?
c) **Health :** Are you able to visit the doctor whenever you experience discomfort or ailment?

Though most indicators selected are based on normative evaluations this attribute assumes prominence as it reflects the freedoms and un-freedoms in our sample.
The importance of the 10 indicators are collected from 6 subject experts $DM_1, \ldots DM_6$ as shown in Table 1.

Table 1: Expert's weights

Experts	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	VHI	VHI	VHI	HI	HI	MI	MI	MI	LI	LI
2	HI	LI	HI	VHI	VHI	MI	VHI	HI	HI	HI
3	MI	VHI	HI	MI	MI	LI	VHI	MI	LI	MI
4	HI	VHI	HI	VHI	LI	MI	MI	MI	LI	VHI
5	VHI	VHI	VHI	LI	LI	HI	VHI	HI	MI	MI
6	VHI	VHI	VHI	HI	HI	MI	HI	LI	LI	MI

where
$LI =$ less important, $MI =$ moderately important, $HI =$ highly important,
$VHI =$ very highly important

4 Background of the Study Area of Study :

After taking into account the various dimensions the subject lends itself to, the researcher found it apt to obtain data from two different districts namely Thanjavur and Thiruvallur. Thiruvallur district was chosen as it was closer to the Chennai city and therefore might have a relatively decent living cost. In contrast the standards of living and costs are significantly lesser in Thanjavur. This district was also chosen due to it's consistently remarkable trends in rice production. Data was collected from 36 male, 45 female and 35 male, 38 female wage labourers engaged in paddy cultivation in Papanasam of Thanjavur District and Ikkadu Kandigai, Selai Villages of Thiruvallur district respectively.

Period of Study :

Data was collected in the time period from July to August 2019.

Objective of the study :

To formulate a comprehensive measure of gender inequality for female agricultural wage labourers.

Methodologies opted for collection of data:

The present study is a largely descriptive one based on primary data. Different tools of data collection are employed to cull out as much information as possible.

Questionnaire: A structured questionnaire was prepared and used to solicit responses. Efforts were taken to keep it as objective as possible. However the nature of subject chosen also lent significant importance to make use of available qualitative data.

Vignettes : A few questions required respondents to imagine a hypothetical scenario and select their preferences accordingly. Vignettes are employed to procure data for such questions.

Observation : Since the study is from a gender perspective, the researcher felt it pertinent to pay details to aspects otherwise branded trivial. The words unspoken conveyed greater depth and meaning.

Interview : A face to face interview was carried out with two doctors, general physicians in specific. This was undertaken to reveal the health hazards the labourers are prone to and their long term implications.

The responses to the questionnaire are obtained as qualitative data and formulated in the following.

5 Mathematical Model of the problem In this section we consider the problem of measuring fuzzy hyperbolic gender inequality index of each group of wage laborers. A mathematical formulation and procedure are considered to obtain the same.

We consider the female and male wage laborers in two districts as collection of 4 groups (say) $\{U_t\}_{t=1}^{4}$ of size $m_1 = 36, m_2 = 45, m_3 = 35$ and $m_4 = 38$ $U_1 = \{u_h^1\}_{h=1}^{36}$, $U_2 = \{u_h^2\}_{h=1}^{45}$, $U_3 = \{u_h^3\}_{h=1}^{35}$ and $U_4 = \{u_h^4\}_{h=1}^{38}$. Considering the indicators as attribute set A and questions associated with it as parameter set , the qualitative responses of the individuals in U_t are captured as soft fuzzy numbers associated with A and modeled as collection of soft fuzzy number valued information systems $\widetilde{IS}_t = (U_t, A, \tilde{\mathbb{R}}(\mathcal{E}_t), \tilde{\tilde{I}}_t)$ for $t = 1, 2, \ldots, 4$, where $A = \{a_1, \ldots, a_{10}\}$ the attribute set consists of the 10 indicators (See section 3),

$\mathcal{E}_t = \{E_{t,1}, \ldots, E_{t,10}\}$ the associated parameter sets consisting of the corresponding questions (See Appendix C) given by

$E_{t,1} = \{e_{t,1,1}\}$; $E_{t,2} = \{e_{t,2,1}, e_{t,2,2}\}$; $E_{t,3} = \{e_{t,3,1}, e_{t,3,2}, e_{33}\}$; $E_{t,4} = \{e_{t,4,1}\}$;
$E_{t,5} = \{e_{t,5,1}, e_{t,5,2}, e_{t,5,3}\}$; $E_{t,6} = \{e_{t,6,1}\}$; $E_{t,7} = \{e_{t,7,1}\}$; $E_{t,8} = \{e_{t,8,1}\}$;
$E_{t,9} = \{e_{t,9,1}\}$; $E_{t,10} = \{e_{t,10,1}, e_{t,10,2}, e_{t,10,3}\}$; where
$e_{t,1,1} =$ i.; $e_{t,2,1} =$ ii.; $e_{t,2,2} =$ iii.; $e_{t,3,1} =$ iv.;
$e_{t,3,2} =$ v.; $e_{33} =$ vi.; $e_{t,4,1} =$ vii.; $e_{t,5,1} =$ viii.; $e_{t,5,2} =$ ix., $e_{t,5,3} =$ x.;
$e_{t,6,1} =$ xia., for $t = 1, 3$; $e_{t,6,1} =$ xib., for $t = 2, 4$;
$e_{t,7,1} =$ xii.; $e_{t,7,2} =$ xiii.; $e_{t,8,1} =$ xiv.; $e_{t,9,1} =$ xv.; $e_{t,10,1} =$ xvi.; $e_{t,10,2} =$ xvii.; $e_{t,10,3} =$ xviii.; The octagonal fuzzy numbers quantifying the linguistic terms (See Appendix) are given in Table 2.

The soft linear octagonal fuzzy numbers expressing the responses of a worker in group 1 are given by $\tilde{\tilde{I}}_1(u_h^1, a_j) = (\tilde{\tilde{f}}_{hj}^1, E_{1,j})$ for $h = 1, \ldots, 36$ and $j = 1, \ldots, 10$.

The soft linear octagonal fuzzy numbers corresponding to attribute 1 for an individual (say u_1^1) is given by

$(\tilde{\tilde{f}}_{11}^1, E_{1,1}) = \left\{\tilde{\tilde{f}}_{11}^1(e_{1,1,1})\right\}$, where $\tilde{\tilde{f}}_{11}^1(e_{1,1,1}) = (0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00; 0.3)$

Similarly the responses of other workers in all the 4 groups corresponding to the attributes are expressed as soft linear octagonal fuzzy numbers.

We propose the following procedure to solve the problem

Table 2: Fuzzy numbers representing linguistic terms

Linguistic terms	**Linear octagonal fuzzy numbers**
SG, SK, FS, HC, HM, AW	(0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00;0.3)
MD, SA, PD, MC , MM	(0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70;0.3)
HL, MK, FD, LC, LM, NA	(0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45;0.3)
VI, HG, VH, VHI	(0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00, 1.00;0.3)
HI,MS	(0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75;0.3)
PC, LW, PI, MI	(0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50;0.3)
NC, HL, NI, LI	(0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45;0.3)

Step1. For each $t = 1, \ldots, 4$ input the soft fuzzy number valued information system $\widetilde{IS}_t$

Step1.1. Determine $\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{f}}^t_{hj}, E_j)\right]$ and call it $\tilde{\boldsymbol{M}}^t_{hj}$ for each $j = 1, 2, \ldots, n$ and $h = 1, 2, \ldots, m_t$

Step1.2. For each j compute $\frac{\sum_{h=1}^{m_t} \tilde{\boldsymbol{M}}^t_{hj}}{m_t}$, denote it $\tilde{\boldsymbol{M}}^t_j$

Step1.3. For each j compute $\sum_{t=1}^{4} \frac{\tilde{M}^t_j}{4}$, call $\tilde{M}^{avg}_j = \sum_{t=1}^{4} \frac{\tilde{M}^t_j}{4}$

Step1.4. By using suitable defuzzification method $\boldsymbol{M}$ compute $\boldsymbol{M}(\tilde{M}^t_j)$ and $\boldsymbol{M}(\tilde{M}^{avg}_j)$ for each j to determine relative strength among the populations by comparison

Step2. For each j compute fuzzy hyperbolic inequality index
$\widetilde{FH}^t_j = \frac{\sum_h^{m_t} (\tilde{\boldsymbol{M}}^t_j) \oslash \tilde{\boldsymbol{M}}^t_{hj}}{m_t} - \tilde{1}$

Step3. Compute $OFCI(\widetilde{FH}^t_1, \tilde{FH}^t_2, \ldots, \tilde{FH}^t_n)$, call $\widetilde{FH}^t \in \mathcal{F}(\mathbb{R})$ the fuzzy hyperbolic gender inequality index for the t^{th} population

Step4. Compute $\boldsymbol{M}(\widetilde{FH}^t)$ to compare the degree of inequality among the populations.

Remark 5.1 *Step3 is computed using Definition 2.5 steps given in Remark 2.3 considering the indicators as criteria and the groups as alternatives.*

Remark 5.2 $\tilde{\boldsymbol{M}}$ *and* $\boldsymbol{M}$ *are computed using Definitions 2.7 and 2.2*

To determine the relative strength of the indicators among the groups we compute the following steps.

Step i: Compute $\widetilde{FH}^t_j$ by computing the above procedure till Step2 and perform the following

Step ii: Using suitable defuzzification method, compute $\boldsymbol{M}(\widetilde{FH}^t_j)$

6 Computations and inferences

Input the soft octagonal fuzzy number valued information system for the 4 groups. For group 1, soft linear octagonal fuzzy numbers modeling the linguistic responses are formatted in matrix form and are given as input for program (See Appendix) The methodology given in Section 5 is computed using the scientific program developed in MATLAB 2016 a and the fuzzy hyperbolic gender inequality index for each group are obtained as shown in table 3.

Table 3: Fuzzy hyperbolic gender inequality index

Groups	$\widetilde{FH}^t$	$\boldsymbol{M}(\widetilde{FH}^t)$
U_1	(-0.5453, -0.3766, -0.1714, 0.0814, 0.4039, 0.8371, .4684, 2.5170;0.3)	0.4312
U_2	(-0.6168, -0.4283,-0.1946, 0.1014, 0.4922, 1.0396, 1.8814, 3.3939;0.3)	0.5690
U_3	(-0.5719, -0.4047, - 0.2022, 0.0466, 0.3629, 0.7855, 1.3966, 2.4033;0.3)	0.3855
U_4	(-0.6117, -0.4275,-0.1995, 0.0875, 0.4630, 0.9823, 1.7668, 3.1265;0.3)	0.5224

The female paddy wage labourers in both the districts suffer high levels of inequality.

The degree of the fuzzy hyperbolic inequality index of the 4 groups corresponding to each indicator are computed and shown in table 4

Table 4: Fuzzy hyperbolic gender inequality index

Indicators	$M(\widetilde{FH}^{1}_{j})$	$M(\widetilde{FH}^{2}_{j})$	$M(\widetilde{FH}^{3}_{j})$	$M(\widetilde{FH}^{4}_{j})$
a_1	0.297926	0.491088	0.295876	0.458897
a_2	0.178841	0.272268	0.141207	0.221967
a_3	0.096864	0.229631	0.117033	0.257802
a_4	0.349715	0.438864	0.168681	0.43222
a_5	0.158284	0.272308	0.144859	0.136093
a_6	0.322515	0.39009	0.346078	0.409099
a_7	0.265902	0.294155	0.272622	0.204532
a_8	0.374837	0.40724	0.311588	0.424788
a_9	0.452144	0.561085	0.448994	0.473047
a_{10}	0.157449	0.209801	0.122203	0.205005

From table 2 the relative strength among the groups are identified and shown in table 5

Table 5: Fuzzy hyperbolic gender inequality index

Indicators	Comparison among the groups
a_1	$M(\widetilde{FH}^{3}_{1}) \leq M(\widetilde{FH}^{1}_{1}) \leq M(\widetilde{FH}^{4}_{1}) \leq M(\widetilde{FH}^{2}_{1})$
a_2	$M(\widetilde{FH}^{3}_{2}) \leq M(\widetilde{FH}^{1}_{2}) \leq M(\widetilde{FH}^{4}_{2}) \leq M(\widetilde{FH}^{2}_{2})$
a_3	$M(\widetilde{FH}^{1}_{3}) \leq M(\widetilde{FH}^{3}_{3}) \leq M(\widetilde{FH}^{2}_{3}) \leq M(\widetilde{FH}^{4}_{3})$
a_4	$M(\widetilde{FH}^{3}_{4}) \leq M(\widetilde{FH}^{1}_{4}) \leq M(\widetilde{FH}^{4}_{4}) \leq M(\widetilde{FH}^{2}_{4})$
a_5	$M(\widetilde{FH}^{4}_{5}) \leq M(\widetilde{FH}^{3}_{5}) \leq M(\widetilde{FH}^{1}_{5}) \leq M(\widetilde{FH}^{2}_{5})$
a_6	$M(\widetilde{FH}^{1}_{6}) \leq M(\widetilde{FH}^{3}_{6}) \leq M(\widetilde{FH}^{2}_{6}) \leq M(\widetilde{FH}^{4}_{6})$
a_7	$M(\widetilde{FH}^{4}_{7}) \leq M(\widetilde{FH}^{1}_{7}) \leq M(\widetilde{FH}^{3}_{7}) \leq M(\widetilde{FH}^{2}_{7})$
a_8	$M(\widetilde{FH}^{3}_{8}) \leq M(\widetilde{FH}^{1}_{8}) \leq M(\widetilde{FH}^{2}_{8}) \leq M(\widetilde{FH}^{4}_{8})$
a_9	$M(\widetilde{FH}^{3}_{9}) \leq M(\widetilde{FH}^{1}_{9}) \leq M(\widetilde{FH}^{4}_{9}) \leq M(\widetilde{FH}^{2}_{9})$
a_{10}	$M(\widetilde{FH}^{3}_{10}) \leq M(\widetilde{FH}^{1}_{10}) \leq M(\widetilde{FH}^{2}_{10}) \leq M(\widetilde{FH}^{4}_{10})$

7 Conclusion The fuzzy nature of this index is of great relevance in capturing the persisting intensity of inequality as it reflects the ground reality. Therefore this technique will be a boon to policy makers who desire to assess the magnitude of disparity and design empowerment schemes.

Acknowledgment

- We thank DST (FIST 2015) MATLAB 2016a which is used for computational purpose.
- I thank Dr. Crystal David, Associate Professor, Department of Economics, Stella Maris College (Autonomous), Chennai and Ms. Abinaya Rengarajan, Post Graduate (2018-2020) , Trinity College Dublin, for their support in analyzing the primary data..

References

[1] Arul Roselet Meryline S., Felbin C. Kennedy:Representation of Knowledge involving Soft Fuzzy Numbers and its Application in Decision Making problems. Recent Trends in Parallel Computing **Vol.6(1)**, 38-47 (2019)

[2] Arul Roselet Meryline S., Felbin C. Kennedy: Solution to a Soft Fuzzy Group Decision-Making Problem Involving a Soft Fuzzy Number Valued Information System. Fuzzy Information and Engineering **11:3**, 320-356 (2019)

[3] Arul Roselet Meryline S., Felbin C. Kennedy: Solution to Challenges of Self- Assessment in Learning using Soft Fuzzy Number Valued Information System. Global and Stochastic Analysis (To Appear)

[4] Dhanalakshmi V., Felbin C. Kennedy: Some Ranking Methods For Octagonal Fuzzy Numbers. International Journal of Mathematical Archive **5(6)**, 177–188 (2014)

[5] Felbin C. Kennedy., V. Dhanalakshmi: Cone properties of Linear Fuzzy numbers. Global and Stochastic Analysis **4(1)**, 95–109 (2017)

[6] Felbin C. Kennedy., Dhanalakshmi V: Octagonal Fuzzy Choquet Integral Operator For Multi-Attribute Decision Making. Scientiae Mathematicae Japonicae **81(3)**, 205-218 (2019)

[7] Felbin C. Kennedy., Masilla Moses Kennedy., Arul Roselet Meryline S., Jayachandiran M: A decision making problem in soft fuzzy environment: Energy efficient Light Emitting Diode bulbs. Forum for Interdisciplinary Mathematics, Scopus Indexed Springer-FIM series (To Appear)

[8] Malini S. U. and Felbin C. Kennedy, An approach for solving fuzzy transportation problem using octagonal fuzzy numbers. International journal of applied mathematical sciences **7(54)**, 2661-2673 (2013)

[9] L. A. Zadeh:Fuzzy Sets, Information and Control **8**, 338–353 (1965)

[10] L.A. Zadeh:The concept of a linguistic variable and its application to approximate reasoning I. Information Sciences **8**, 199–249 (1975)

[11] L.A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning II. Information Sciences **8**, 301–35 (1975)

APPENDIX

Information from Agricultural workers

Here we present the input from one group.
The responses of women in the Thiruvallur District is considered as input in the format given below: For simplicity we have coded the linguistic terms as
SG, SK, FS, HC, HM, AW - a1; MD, SA, PI, MC , MM - a2;
HL, MK, FD, LC, LM, NA - a3; NC, HL, NI - b1; PC, LW, PI - b2; HI, MS - b3;
VI, HG, VH - b4;

```
Enter the number of people 36
enter the number of attributes 10
Enter the weight vector for the attributes of size 10
[0.13 0.14 0.13 0.11 0.09 0.07
0.12 0.074 0.06 0.08]
enter the number of parameters for attribute 11
Enter the matrix for 1 parameter of attribute 1{a1 a3 a2 a2 a3 a1 a1 a3 a3 a1 a2 a3
a3 a2 a3 a3 a2 a2 a1 a3 a3 a3 a1 a2 a3 a2 a3 a3 a3 a1 a2 a1 a3 a3 a3 a1}
Enter the weight vector of size 1 1
enter the number of parameters for attribute 21
Enter the matrix for 1 parameter of attribute 2{a1 a3 a2 a1 a2 a1 a2 a1 a1 a3 a1 a1
```

```
a2 a1 a2 a2 a1 a1 a2 a1 a2 a3 a3 a1 a2 a1 a1 a1 a1 a1 a2 a1 a1 a2 a2 a2 a1 a3 a2 a3
a1 a1 a1 a2 a1;a2 a3 a2 a1 a2 a1 a3 a1 a1 a3 a1 a1 a1 a2 a3 a1 a1 a1 a2 a1 a2 a2 a2
a1 a1 a1 a1 a2 a1 a2 a2 a2 a1 a2 a2 a2 a1 a1 a3 a1 a1 a2 a2 a2 a1}
Enter the weight vector of size 2 [.6 .4]
enter the number of parameters for attribute 31
Enter the matrix for 1 parameter of attribute 3{b1 b3 b1 b1 b3 b2 b2 b2 b3 b4 b1 b1
b3 b1 b2 b3 b1 b1 b1 b1 b3 b1 b1 b2 b2 b1 b3 b1 b2 b2 b4 b3 b2 b3 b1 b2 b2 b1 b3 b2
b2 b3 b1 b1 b2;b2 b2 b3 b1 b4 b1 b2 b1 b2 b3 b1 b1 b3 b2 b1 b4 b2 b2 b3 b2 b2 b1 b1
b2 b2 b1 b2 b3 b1 b1 b3 b2 b3 b3 b2 b2 b3 b1 b2 b3 b1 b2 b1 b1 b2;b1 b2 b2 b1 b1 b1
b1 b3 b4 b1 b3 b1 b1 b3 b2 b1 b1 b2 b2 b1 b1 b2 b2 b1 b3 b1 b1 b4 b1 b2 b1 b2 b1 b1
b2 b1 b1 b3 b1 b1 b1 b3 b1 b1 b2}
Enter the weight vector of size 3 [.2 .4 .4]
enter the number of parameters for attribute 41
Enter the matrix for 1 parameter of attribute 4{a1 a1 a1 a1 a1 a1 a2 a2 a1 a3 a2 a1
a1 a1 a1 a1 a1 a1 a1 a1 a1 a3 a3 a2 a1 a3 a1 a3 a1 a1 a2 a2 a3 a3 a1 a2 a1 a1 a1 a2
a2 a1 a3 a1 a3}
Enter the weight vector of size 1 1
enter the number of parameters for attribute 51
Enter the matrix for 1 parameter of attribute 5{a1 a2 a1 a2 a2 a1 a2 a2 a2 a3 a2 a1
a2 a2 a1 a1 a2 a2 a1 a2 a1 a3 a3 a2 a2 a3 a1 a3 a1 a1 a2 a2 a3 a3 a1 a2 a2 a1 a2 a2
a2 a2 a3 a1 a3;a3 a2 a2 a1 a1 a1 a2 a1 a2 a2 a3 a1 a2 a1 a2 a1 a1 a2 a2 a2 a2 a2 a1
a2 a1 a3 a2 a3 a2 a1 a2 a1 a2 a2 a1 a2 a2 a2 a3 a1 a2 a1 a3 a2 a3;a2 a1 a1 a1 a3 a1
a1 a1 a3 a1 a2 a1 a1 a1 a1 a1 a1 a2 a1 a2 a2 a2 a3 a1 a1 a2 a1 a2 a1 a2 a2 a2 a3 a2
a1 a2 a3 a3 a1 a1 a3 a1 a2 a3 a2}
Enter the weight vector of size 3 [.5 .2 .3]
enter the number of parameters for attribute 71
Enter the matrix for 1 parameter of attribute 7{b3 b3 b1 b3 b3 b3 b4 b3 b2 b3 b3 b2
b2 b2 b2 b4 b3 b3 b1 b1 b2 b1 b1 b2 b2 b1 b1 b2 b3 b3 b3 b4 b4 b4 b1 b4 b4 b4 b4 b3
b3 b3 b3 b3 b4;b2 b3 b3 b1 b1 b1 b3 b1 b3 b3 b4 b1 b2 b4 b3 b2 b3 b4 b1 b1 b4 b3 b3
b4 b2 b2 b1 b3 b4 b3 b2 b1 b3 b2 b3 b1 b2 b1 b4 b3 b3 b4 b3 b2 b3}
Enter the weight vector of size 2 [.6 .4]
enter the number of parameters for attribute 81
Enter the matrix for 1 parameter of attribute 8{b2 b1 b1 b3 b2 b2 b1 b2 b3 b3 b4 b3
b3 b3 b3 b2 b3 b2 b1 b1 b2 b3 b2 b3 b1 b1 b1 b1 b2 b2 b4 b3 b4 b1 b4 b1 b2 b3 b1 b3
b1 b2 b2 b4 b3}
Enter the weight vector of size 1 1
enter the number of parameters for attribute 10 1
Enter the matrix for 1 parameter of attribute 10{a2 a1 a1 a1 a1 a2 a2 a1 a2 a1 a2 a2
a2 a2 a2 a1 a1 a1 a2 a1 a1 a1 a1 a1 a2 a1 a1 a1 a1 a2 a2 a3 a2 a3 a1 a2 a1 a1 a3 a2
a1 a1 a2 a2 a3;a3 a3 a2 a3 a3 a3 a1 a1 a1 a1 a2 a3 a2 a2 a3 a2 a2 a1 a2 a1 a2 a2 a3
a1 a2 a3 a1 a2 a1 a3 a1 a2 a2 a2 a1 a1 a2 a2 a1 a3 a2 a1 a2 a1 a2;a2 a2 a2 a1 a1 a2
a1 a2 a1 a3 a2 a2 a1 a1 a1 a3 a2 a1 a2 a1 a1 a1 a1 a2 a1 a2 a1 a1 a3 a3 a1 a3 a2 a1
a3 a1 a2 a3 a2 a3 a2 a3 a3 a1 a3}
Enter the weight vector of size 3 [.2 .2 .6]
```

GENDER INEQUALITY OF WORKERS IN PADDY CULTIVATION - QUESTIONNAIRE

PERSONAL DETAILS:

1).Age of the respondent:

2).Gender: male: ☐ female:☐

3).Daily wage: Rs.

Gender Inequality index Female

EDUCATION AND SKILL DEVELOPMENT

i. Do you have knowledge about the different farming practices, access to training programs and concessions, awareness about the existence of technology used for cultivation?

Minimal Knowledge:☐ Somewhat aware:☐ Sufficient Knowledge:☐

FINANCIAL INDEPENDENCE

ii. What do you do with your wages?

Financially Dependent: ☐ Partly Independent:☐ Financially Stable:☐

iii. Allocation of your earnings to satisfy personal desires?

Low contribution to personal needs:☐ Medium contribution to personal needs:☐

High contribution to personal needs:☐

DECISION MAKING POWER

iv. Are you involved in taking decisions regarding your nature and time of employment, no. of holidays you avail?

Not Consulted:☐ Partially Involved:☐ Highly Involved:☐ Very Highly Involved:☐

v. Are you involved in taking household related decisions?

Not Consulted:☐ Partially Involved:☐ Highly Involved:☐ Very Highly Involved:☐

vi. Your share in burden of household work?

Hardly ☐ Low:☐ Moderate:☐ High:☐

Who helps?____

MOBILITY

vii. Is there active social participation (public events), any restriction on travel or curb on your independence?
Less Mobility:☐ Moderate Mobility:☐ High Mobility:☐

RIGHTS

viii. Do you have the right to equal wages?
Not aware:☐ Somewhat Aware:☐ Aware:☐

ix. Do you have the right to education & information?
Not aware:☐ Somewhat Aware:☐ Aware:☐

x. Do you have the right to equal inheritance & property?
Not aware:☐ Somewhat Aware:☐ Aware:☐

HEALTH

xi. Does employment during reproductive and menstrual cycles impact your physical or emotional well-being?
Hardly☐: Moderately☐: Significantly☐:

POLITICAL PARTICIPATION

xii. To what extent are you influenced by family friends or media while exercising your franchise
Very Highly Influenced:☐ Highly Influenced:☐ Partially Influenced:☐ Not Influenced:☐

xiii. To what extent are you involved in /receptive of SHG's/Panchayat activities etc
Hardly ☐ Low:☐ Moderate:☐ High☐:

SAFETY AND SECURITY

xiv. How do you rate your safety at the field based on co –male farmers and work environment
Hardly safe ☐ Less safe☐ moderately safe:☐ Highly safe☐:

GENDER PAY GAP

xv. Are you disturbed by the existence of a gender pay gap:
Hardly☐: Moderately☐: Significantly☐:

NOTIONS OF FREEDOM

xvi. LEISURE: Do you experience leisure of personal value:
Hardly☐: Moderately☐: Significantly☐:

xvii. EDUCATION:If you were better educated would you still wish to be a farm labourer?
Hardly☐: Moderately☐: Significantly☐:

xviii. HEALTH: Do you visit a doctor immediately despite having farm work when unwell?
Hardly☐: Moderately☐: Significantly☐:

Note 1:

Question xi is modified accordingly for men workers in the following:

xi b. Does the scarcity of employment and the compulsion to exert strength on field impact your physical or emotional well-being?
Hardly☐: Moderately☐: Significantly☐:

Note 2:

The abbreviations of the linguistic terms:

Significantly (SG), Sufficient Knowledge (SK), Financially Stable (FS), High contribution to personal needs (HC), High Mobility (HM), Aware (AW), Moderately (MD), Somewhat aware (SA), Partly Independent (PD) , Medium contribution to personal needs (MC) , Moderate Mobility (MM), Hardly, Hardly safe (HL), Minimal Knowledge (MK), Financially Dependent (FD), Low contribution to personal needs (LC), Less Mobility (LM), Not Aware (NA), Not Consulted (NC), NI (Not Influenced), Partially Involved (PC), Low, Less safe (LW), Partially Influenced (PI), Highly Involved , Highly Influenced (HI), Moderate, Moderately safe (MS), Very Highly Involved (VI), High, Highly safe (HG) , Very Highly Influenced (VH)

INTERNATIONAL ADVISORS

J.Aczél (Univ. of Waterloo, Canada) **J.Musielak** (A.Mickiewicz Univ., Poland)
A.V.Arkhangelskii (Ohio Univ., U.S.A.) **A.Blass** (Univ.of Michigan, U.S.A.)
W.W.Comfort (Wesleyan Univ., U.S.A.) **D.A. Drake** (Univ.of Florida, Gainsville, U.S.A.)
S.Lajos (Budapest Univ.of Economics, Hungary) **P.K.Sen** (Univ. of North Carolina, U.S.A.)
O.Takenouchi (P.E.Osaka Univ.) **J.D.Monk** (Univ. of Colorado, U.S.A.)
M. Valdivia (Univ.de Valencia, Spain)

BOARD OF EDITORS

****

Honorary Editors
Arto Salomaa (Academy of Finland, Finland)
Saharon Shelah (Hebrew University, Israel and Rutgers University, U.S.A)

****

(a) Name
(b) Postal Address
(b') E-mail address
(c) Reviewable Area
(d) additional information

AUSTRALIA

(a) **Phillip Isaac**
(b) School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072,Australia
(c) psi@maths.uq.edu.au
(d) Representation theory of Lie algebras and Lie superalgebras, quantum integrable systems, Bethe ansatz,Yang-Baxter equation

BRAZIL

(a) **Jair Minoro Abe**
(b) 1) Institute of Exact Sciences and Technology – Paulista University, UNIP, Rua Dr. Bacelar, 1212, 04026-002 – SAO PAULO, S.P. – BRAZIL FAX: 0055 11 55864010
2) Institute For Advanced Studies, University of Sao Paulo, Rua Praca do Relógio, 109, bloco K. 5° andar, Cidade Universitária, 05508-050, SAO PAULO, S.P. – BRAZIL
(b') jairabe@uol.com.br
(c) Mathematical Logic, Algebraic Logic, Foundations of Mathematics

CANADA

(a) **Anthony To-Ming Lau**
(b) Dept. of Mathematical and Statistical Sciences, Univ. of Alberta, Edmonton, Alberta, Canada T6G 2G1
(b') anthonyt@ualberta.ca
(c) harmonic analysis and functional analysis

(a) **János Aczél**
(b) Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
(b') jdaczel@uwaterloo.ca
(c) Functional Equations

(a) **M. S. Srivastava**
(b) Department of Statistics, University of Toronto, 100 St. George Street Toronto, Ontario, M5S 3G3, Canada
(b') srivasta@utstat.toronto.edu
(c) Multivariate Analysis, Sequential Analysis, Quality Control

CZECH REPUBLIC

(a) **Milan Vlach**
(b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
(b’) milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
(c) Game theory, Fair division, Optimization

FINLAND

(a) **Arto Salomaa**
(b) Jaanintie 34 A 26, 20540 Turku, Finland
(b') asalomaa@utu.fi
(c) Formal languages, automata, computability, cryptography

(a) **Jarkko Kari**
(b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
(b') jkari@utu.fi
(c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

(a)**Klaus Denecke**
(b) 14542 Werder (Havel), Germany, Zanderweg 3
(b') klausdenecke@hotmail.com
(c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

(a) **Maria Fragoulopoulou**
(b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
(b') fragoulop@math.uoa.gr
(c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory,
Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

(a) **Gyula Maksa**
(b) Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary
(b') maksa@science.unideb.hu
(c) Functional equations

(a) **Kálmán Györy**
(b) University of Debrecen, Institute of Mathematics, 4010 Debrecen, Hungary
(b') gyory@science.unideb.hu
(c) Number Theory (mainly Diophantine and Algebraic Number Theory)

(a) **Pál Dömösi**
(b) Institute of Mathematics and Informatics, Nyíregyháza University, Nyíregyháza, Sóstói út 31/B, H-4400, Hungary
(b') domosi.pal@nye.hu
(c) Theoretical Computer Science, Algebra

ISRAEL

(a) **Dany Leviatan**
(b) School of Mathematics, Tel Aviv University, 6139001 Tel Aviv, Israel
(b') leviatan@post.tau.ac.il
(c) Approximation Theory, Computer Added Geometric Design, Summability

ITALY

(a) **Angelo Favini**
(b) Dept. of Mathematics, Bologna Univ., Piazza di Porta S. Donato, 5, 40126, Bologna, Italy
(b') angero.favini@unibo.it
(c) Evolution equations and Control theory for abstract differential equations and PDE.

(a) **Antonio Di Crescenzo**
(b) Università di Salerno, Dipartimento di Matematica, Via Giovanni Paolo II, n.132, 84084 Fisciano (SA), Italy
(b') adicrescenzo@unisa.it
(c) Applied Probability, Stochastic Processes and Applications, Reliability Theory, Queueing Systems, Stochastic Models in Biology, Information Measures

(a) **Tonia Ricciardi**
(b) Department of Mathematics and its Applications, Federico II University, Via Cintia, 80126 Naples, Italy.
(b') tonia.ricciardi@unina.it
(c) Nonlinear elliptic partial differential equations

NETHERLANDS

(a) **Grzegorz Rozenberg**
(b) Leiden Institute of Advanced Computer Science (LIACS) Leiden University, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
(b') rozenber@liacs.nl
(c) Natural computing, Formal languages and automata theory

POLAND

(a) **Dariusz Zagrodny**
(b) Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszynski University, Dewajtis 5, 01-815 Warsaw, Poland
(b')dariusz.zagrodny@wmii.uni.lodz.pl
(c) Nonsmooth Analysis (this is my main field of research), Nonlinear Programming (convex, nonconvex), Maximal Monotonicity (with respect to duality)

(a) **Henryk Hudzik**
(b) Faculty of Economics and Information Technology, The State University of Applied Sciences in Płock, Nowe Trzepowo 55, 09-402 Płock, Poland
and
Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska Street 87, 61-614 Poznań, Poland.
(b') hudzik@amu.edu.pl
(c) Function spaces theory and abstract Banach spaces theory, Banach Lattices, Geometry of Banach Spaces, Composition and multiplication operators between Köthe spaces

(a) **Krzysztof Szajowski**
(b) Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, Wybrzeze, Wyspianskiego 27, PL-50-370 Wroclaw, Poland
(b') Krzysztof.Szajowski@pwr.edu.pl
(c) Applied Probability, Game Theory, Operations Research, Mathematical Statistics

(a) **Piotr Multarzynski**
(b) Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland
(b') multarz@mini.pw.edu.pl
(c) Algebraic analysis (calculus of right invertible operators); Algebraic approach to differential geometry; Discrete counterparts of the classical concepts in analysis and differential geometry; q-calculus, Sikorski and Froelicher differential (or smooth) spaces; 12H10, 39A12, 39A70, 47B39, Groupoids - theory and applications in physics.

(a) **Tomasz Kubiak**
(b) Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
(b') tkubiak@amu.edu.pl
(c) Many-valued topology (fuzzy topology), Pointfree topology (frames and locales)

P.R.OF CHINA

(a) **Minghao Chen**
(b) Xidazhi Street 92,Harbin ,150001,China Department of Mathematics,Harbin Institute of Technology
(b') chenmh130264@aliyun.com;chenmh130264@hit.edu.cn
(c) Uncertain dynamical systems;Fuzzy differential equation; Fuzzy optimization;Fuzzy sets theory

ROMANIA

(a) **Adrian Petrusel**
(b) Babes-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Department of Mathematics, Kogalniceanu street no. 1, 400084 Cluj-Napoca, Romania
(b') petrusel@math.ubbcluj.ro
(c) Nonlinear Analysis

(a) **Ioan A. Rus**
(b) Department of Mathematics, Babes-Bolyai University, Str. Kogalniceanu No. 1, 400084 Cluj-Napoca, Romania
(b') iarus@math.ubbcluj.ro
(c) Fixed Point Theory

(a) **Vasile Berinde**
(b) Department of Mathematics and Computer Science, Faculty of Sciences Technical University of Cluj-Napoca North University Center at Baia Mare, Victoriei Nr. 76, 430122 Baia Mare, Romania
(b') vberinde@cubm.utcluj.ro
(c) Fixed Point Theory, iterative approximation of fixed points

RUSSIA

(a) **Andrei Vesnin**
(b) Sobolev Institute of Mathematics, pr. ak. Koptyuga 4, Novosibirsk, 630090, Russia
(b') vesnin@math.nsc.ru
(c) Low-dimensional topology, Knot theory, Hyperbolic manifolds and orbifolds.

(a) **Semen S. Kutateladze**
(b) The Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Academician Koptyug's Avenue 4, Novosibirsk, 630090, RUSSIA
(b') sskut@member.ams.org and sskut@math.nsc.ru
(c) Functional Analysis, Operator Theory, Convex Geometry, Optimization and Programming, Nonstandard Analysis, Boolean Valued Models

(a) **Vladimir V. Mazalov**
(b) Institute of Applied Mathematical Research, Karelia Research Center of Russian Academy of Sciences Pushkinskaya str., 11, Petrozavodsk 185610, Russia
(b') vmazalov@krc.karelia.ru
(c) Optimal Stopping Theory, Game with Optimal Stopping, Stochastic Dynamic Programming, Applications in Behavioral Ecology

(a) **Elena Parilina**
(b) Department of Mathematical Game Theory and Statistical Decisions, Saint Petersburg State University,7/9 Universitetskaya nab., Saint Petersburg 199034, Russia
(b') e.parilina@spbu.ru<mailto-e.parilina@spbu.ru>
(c) Game Theory, Stochastic Games, Applied Mathematical Statistics.

SOUTH AFRICA

(a) **Joachim Schröder**
(b) Department van Wiskunde, Universiteit van die Vrystaat, Posbus 339, Bloemfontein 9300, South Africa
(b') schroderjd@ufs.ac.za
(c) Enumerative combinatorics, Categorical methods in topology, Set theoretic topology (cardinal invariants, elementary submodels)

SPAIN

(a) **Javier Gutierrez Garcia**
(b) Departamento de Matematicas, Universidad del Pais Vasco/Euskal Herriko Unibertsitatea UPV/EHU, Apartado 644, 48080, Bilbao, Spain
(b') javier.gutierrezgarcia@ehu.eus
(c) General topology (in particular, insertion and extension of functions), Pointfree topology,Many-valued topology

(a) **Jorge Galindo**
(b) Instituo de Matemáticas y Aplicaciones de Castellón (IMAC), Departamento de Matemáticas, Universidad Jaume I, 12071-Castellón, Spain.
(b') jgalindo@mat.uji.es
(c) Topological Algebra, Abstract Harmonic Analysis, General Topology.

(a) **Luis M. Sanchez Ruiz**
(b) ETSID-Depto. de Matematica Aplicada & CITG, Universitat Politècnica de València, E-46022 Valencia, Spain
(b') LMSR@mat.upv.es
(c) Functional Analysis, Topological Vector Spaces, Barrelledness Properties, Baire-like Spaces, Continuous Function Spaces, Wavelets

(a) **Salvador Hernandez**
(b) Departamento de Matematicas, Universitat Jaume I, 12071 Castellon, Spain
(b') hernande@uji.es
(c) Topological groups and semigroups, Spaces of continuous functions, Operators defined between spaces of continuous functions, General Topology.

TAIWAN

(a) **Hang-Chin Lai**
(b) Department of Mathematics, National Tsing Hua University, Hsin Chu City, Taiwan
(b’) laihc@mx.nthu.edu.tw
(c) Nonlinear analysis and convex analysis, Optimization theory, Harmonic analysis

UNITED STATES OF AMERICA

(a) **Andreas Blass**
(b) Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1043, USA
(b') ablass@umich.edu
(c) Mathematical logic, set theory, category theory

(a) **John B Conway**
(b) Professor Emeritus, George Washington University,Phillip Hall 801 22nd St. NW
Washington, DC 20052, U.S.A
(b') Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**
(b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**
(b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**
(b) non-public
(b') yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Haruo Maki**
(b) non-public
(b') makih@pop12.odn.ne.jp
(c) (Topological) digital n-spaces (n>0), Generalized closed sets (after Levine),
Operation theory in topology (in the sense of Kasahara and Ogata)

(a) **Kohzo Yamada**
(b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b') kohzo.yamada@shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**
(b) Shimane Univ., Matsue, Shimane 690-8504, Japan
(b') hattori@riko.shimane-u.ac.jp
(c) General Topology

(a) **Yoshikazu Yasui**
(b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho,
Kitakaturagi-gun, Nara, 635-0832, Japan
(b') y.yasui@kio.ac.jp
(c) General Topology

(a) **Eiichi Nakai**
(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
(b') eiichi.nakai.math@vc.ibaraki.ac.jp
(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**
(b) Division of Mathematics and Physics, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan
(b') jkawabe@shinshu-u.ac.jp
(c) Measure and integration, Vector measure, Nonadditive measure

(a) **Jun Ichi Fujii**
(b) Department of Educational Collaboration(Science, Mathematics and Information),
Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
(b') fujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Masaru Nagisa**
(b) Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Yayoi-cho,
Chiba, 263-8522, Japan
(b’) nagisa@math.s.chiba-u.ac.jp
(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**
(b) Graduate School of Science and Engineering, Ritsumeikan University, 1-1-1 Noji-higashi, Kusatsu,
Shiga 525-8577 Japan
(b') osaka@se.ritsumei.ac.jp
(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**
(b) non-public
(b’) mfujii@cc.osaka-kyoiku.ac.jp
(c) Operator Theory

(a) **Shigeo Akashi**
(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science,
2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan
(b’) akashi@is.noda.tus.ac.jp
(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**
(b) Department of Mathematics, Osaka Metropolitan University, 1-1, Gakuencho, Naka-ku,
Sakai, Osaka 599-8531, Japan
(b’) yo.kabeya@omu.ac.jp
(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**
(b) non-public
(b’) yagi-atsushi-ch@alumni.osaka-u.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yasumasa Fujisaki**
(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology,
Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
(b') fujisaki@ist.osaka-u.ac.jp
(c) Control Systems Theory

(a) **Naruhiko Aizawa**
(b) Department of Physics, Graduate School of Science, Osaka Metropolitan University, Sakai, Osaka 599-8531,Japan
(b')aizawa@omu.ac.jp
(c) Representation theory (Lie groups and algebras and their extensions)

(a) **Hisao Nagao**
(b) non-public
(b') nagao.hisao@aqua.plala.or.jp
(c) Multivariate Analysis, Sequential Analysis, Jackknife Statistics and Bootstrap Method

(a) **Masanobu Taniguchi**
(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University,
3-4-1, Okubo, Shinjuku-ku, Tokyo,169-8555, Japan, Tel : 070-7775-9662
(b') taniguchi@waseda.jp
(c) Statistical Inference for Stochastic Processes

(a) **Masao Kondo**
(b) non-public
(b') kondo@sci.kagoshima-u.ac.jp
(c) Time Series Analysis

(a) **Ryusuke Hohzaki**
(b) Tsukuba Gakuin University 3-1 Azuma Tsukuba City, Ibaragi Prefecture,Japan
(b') hozaki@tsukuba-g.ac.jp
(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Junzo Watada, PhD**
(b) Dr. Junzo Watada, WCICME, 2-10-8-407 Kobai, Yahatanishi, Kitakyushu 806-0011 Japan
(c) +81-90-3464-4929
(d) Affiliation
invited Research, IPS Research Center, Waseda University
Professor Emeritus, Waseda University, Japan
http://www.wcicme.com/watada/
Mobile: +81-90-3464-4929
Research Professor, Zhejiang Gongshang University, China
Research Professor, Indian Instutute of Finance, Delhi, India

(a) **Kensaku Kikuta**
(b) School of Business Administration, University of Hyogo,
8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN
(b') kikuta@biz.u-hyogo.ac.jp
(c) Game Theory, Operations Research,

(a) **Wuyi Yue**
(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto,
Higashinada-ku , Kobe 658-8501, JAPAN
(b') yue@konan-u.ac.jp
(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov Processes, Probabilistic Methods, Systems Engineering

(a) **Hiroaki Sandoh**
(b) Faculty of Policy Studies Kwansei Gakuin University 1, Gakuen-Uegahara, Sanda-shi, Hyogo 669-1330 Japan
(b’) sandoh@kwansei.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

(a) Katsunori Ano
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku
Saitama-city, 337-8570, Japan
(b’) k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

(a) **Koyu Uematsu**
(b) Graduate School of Management and Information Sciense Faculty of Global Business ,Osaka International University
6-21-57 Tohdacho, Moriguchi-Shi, Osaka,570-8555,Japan
(b’) uematsu@oiu.jp
(c) Stochastic Process and its Applications,Reliability Analysis,and Game Theory

(a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences , Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa
259-1293, Japan
(b’) yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

(a)**Tadashi Takahashi**
(b)Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto,
Higashinada, Kobe, Hyogo 658-8501, Japan
(b’) takahasi@konan-u.ac.jp
(c)Mathematics Education

(a) **Yoko Watamori**
(b) Department of Mathematics, Graduate School of Science, Osaka Metropolitan University, Sakai, Osaka 599-8531, Japan
(b') w318y@omu.ac.jp
(c) Directional statistics, Multivariate Analysis

(a) **Koichi Osaki**
(b) Department of Mathematical Sciences, School of Science, Kwansei Gakuin University,
1 Gakuen Uegahara, Sanda, 669-1330, Japan.
(b') osaki@kwansei.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

****

Managing Editor
Koyu Uematsu (Professor of Osaka International University)
International Society for Mathematical Sciences
1-5-12-202 Kaorigaoka-cho, Sakai-ku, Sakai-city, 590-0011,Japan
uematsu@jams.jp

Submission to the SCMJ

In September 2012, the way of submission to Scientiae Mathematicae Japonicae (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (scmjlt2e.sty) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp .

b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:

1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html)and would like to take in charge of the paper for refereeing.
2. Title of the paper.
3. Authors' names.
4. Corresponding author's name, e-mail address and postal address (affiliation).
5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and sent it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.

b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.

c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.

(4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.

b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.

c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names
3'. 3. in Japanese for Japanese authors
4. Corresponding author's name and postal address (affiliation)
4'. 4. in Japanese for Japanese authors
5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal Scientiae Mathematicae Japonicae at the yearly price of US$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

Subscription of SCMJ Check one of the two.	☐Print (US$225) ☐Print ＋ Online (US$225)
University (Institution)	
Department	
Postal Address where SCMJ should be sent	
E-mail address	
Person in charge	Name: Signature:
Payment Check one of the two.	☐Bank transfer ☐Credit Card (Visa, Master)
Name of Associate Membership	1.
	2.

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in "**Join ISMS !**" on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4's address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in "**Join ISMS !**")

Individual Membership Application Form

1. **Name**	
2. **Birth date**	
3. **Academic background**	
4. **Affiliation**	
5. **4's address**	
6. **Doctorate**	
7. **Contact address**	
8. **E-mail address**	
9. **Special fields**	
10. **Membership category**	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.

The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

(1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan

(2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

****

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.

(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:

(1) Post Office Transfer Account - 00930-3-73982 or

(2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.

All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the "21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae" and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
3) **ISMS Annual Meeting**: Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

(1) No publication charges
(2) Free access (**including printing out**) to the online version of SCMJ
(3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2019

Categories	**Domestic**	**Overseas**	**Developing countries**
1-year Regular member	¥8,000	US$80 , Euro75	US$50, Euro47
1-year Students member	¥4,000	US$50 , Euro47	US$30 , Euro28
Life member*	Calculated as below*	US$750 , Euro710	US$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers,we apply above presented new fee after April 15 in 2015 on registoration date.) * Regular member between 63 - 73 years old can apply the category.

(73－age) × ¥3,000

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

Notices from the ISMS

発行：一般社団法人国際数理科学協会
大阪府堺市堺区香ケ丘町1丁5-12浅香山マンション202号
TEL　072-222-1850
FAX　072-222-1850
email　trsr@jams.jp