

Scientiae Mathematicae Japonicae

(Scientiae Mathematicae / Mathematica Japonica New Series)

Vol. 88, No. 1

Whole Number 309

February 2026



International Society for Mathematical Sciences

INTERNATIONAL SOCIETY FOR MATHEMATICAL SCIENCES Scientiae Mathematicae Japonicae, Notices from the ISMS

The International Society for Mathematical Sciences (ISMS) is an international society consisting of mathematical scientists throughout the world.

The main activities of the ISMS are to publish (1) the (**print and online**) journal **Scientiae Mathematicae Japonicae (SCMJ)** and (2) Notices from the ISMS and to hold **assembly meetings** in Japan and **international internet meetings** (distance symposium) of mathematical sciences (IVMS) accessible from all over the world.

SCMJ is the 21st Century New Unified Series of **Mathematica Japonica (MJ)** and **Scientiae Mathematicae (SCM)**. **MJ** was first published in 1948 and was one of the oldest mathematical journals in Japan. **SCM** was an online and print journal started in 1998 in celebration of the semi-centennial anniversary and received 26000 visits per month from 50 countries in the world. **SCMJ** contains original papers in mathematical sciences submitted from all over the world and receives **38000** visits per month now. Not only papers in pure and applied mathematics but those devoted to mathematical statistics, operations research, informatics, computer science, biomathematics, mathematical economics and other mathematical sciences are also welcome. The journal is published in January, March, May, July, September, and November in each calendar year.

The ISMS has enhanced the journal, beginning from July 1995, by including excellent Research-Expository papers in the section "**International Plaza** for Mathematical Sciences" as well as original research papers. The section provides papers dealing with broad overviews of contemporary mathematical sciences, written by experts mainly at our invitation. Papers shedding lights on open problems or new directions or new breakthroughs for future research are especially welcome.

As is shown in the **Editorial Board of SCMJ**, we have invited **many distinguished professors** of 20 countries as editors, who will receive and referee the papers of their special fields with their **high standard**.

Beginning from 2007, we make the online version of SCMJ more readable and convenient to the readers by adding the specialized contents. By this, the readers can access to the online version, in which the papers appear in the order of acceptance, from (i) the contents of the printed version, and (ii) the specialized contents of a volume. From 2007, the subscription fee of the printed version plus the online version of SCMJ becomes lower and the same of the printed version only. Therefore, the subscribers of the printed version can read the online version without no additional cost.

For benefit of the ISMS members, we publish "Notices from the ISMS" 6 times a year. We are enhancing it by adding interesting articles, including book reviewing, written by eminent professors.

The ISMS has set up a **videoconferencing system (IVMS)** which can connect up to twenty sites of a research group in the same or **different countries in the world**. Using this system, speakers of the session can write on a white board or an OHP sheet or use PowerPoint. On the other hand participants can ask questions or make comments from any connected site in the world. All these are performed similarly to the traditional meetings.

To connect with our system, you can use your own videoconferencing system only if it satisfies the International Telecommunication Union-Technical Committee Standards (ITU-T Standard).

Copyright Transfer Agreement

A copyright transfer agreement is required before a paper is published in this journal. By submitting a paper to this journal, authors are regarded to certify that the manuscript has not been submitted to nor is it under consideration for publication by another journal, conference proceedings or similar publication.

For more information, please visit <http://www.jams.or.jp>.

Copyright Copyright ©2014 by International Society for Mathematical Sciences. All rights reserved.

REAL HYPERSURFACES AND CURVE THEORY IN A NONFLAT COMPLEX SPACE FORM

SADAHIRO MAEDA

Received November 27,2023; revised March 25,2024

ABSTRACT. In the class of all real hypersurfaces M^{2n-1} of a nonflat complex space form $\widetilde{M}_n(c)$, both of type (A) hypersurfaces in this space and the homogeneous ruled real hypersurface in a complex hyperbolic space are fundamental examples. We characterize these two real hypersurfaces from the viewpoint of curve theory in those ambient spaces.

1. INTRODUCTION

We denote by $\widetilde{M}_n(c)$ ($n \geq 2$) a complex n -dimensional nonflat complex space form of constant holomorphic sectional curvature $c (\neq 0)$ furnished with Kähler structure J and Riemannian metric g , that is, $\widetilde{M}_n(c)$ is either a complex projective space $\mathbb{C}P^n(c)$ ($c > 0$) or a complex hyperbolic space $\mathbb{C}H^n(c)$ ($c < 0$). Real hypersurfaces isometrically immersed into $\widetilde{M}_n(c)$ provide rich sources and significant notions in submanifold geometry such as Hopf hypersurfaces and ruled real hypersurfaces (for the notion of Hopf hypersurfaces and ruled real hypersurfaces, see Sections 2 and 3). In this paper we treat two fundamental examples of such real hypersurfaces.

The former is a type (A) hypersurface which is a significant Hopf hypersurface in $\widetilde{M}_n(c)$. There are many theorems which characterize type (A) hypersurfaces (for example, see [21]). Every type (A) hypersurface in $\widetilde{M}_n(c)$ is either a tube of radius r ($0 < r < \pi/\sqrt{c}$) around a totally geodesic $\mathbb{C}P^\ell(c)$ ($0 \leq \ell \leq n-2$) in $\mathbb{C}P^n(c)$, a tube of radius r ($0 < r < \infty$) around a totally geodesic $\mathbb{C}H^\ell(c)$ ($0 \leq \ell \leq n-1$) or a horosphere HS in $\mathbb{C}H^n(c)$.

In Theorem 1 which is a characterization of type (A) hypersurfaces M , observing all geodesics on M , we give a geometric meaning of a well-known algebraic condition $\phi A = A\phi$ on M with structure tensor ϕ defined by $g(\phi X, Y) = g(JX, Y)$ for all $X, Y \in TM$ and shape operator A of M in the ambient space $\widetilde{M}_n(c)$.

The latter is the *homogeneous* ruled real hypersurface, say HR which is an important non-Hopf hypersurface in $\mathbb{C}H^n(c)$. In order to construct HR, we first take a circle $\gamma = \gamma(s)$ of curvature $\sqrt{|c|}/2$ on a totally real totally geodesic real hyperbolic plane $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$, $n \geq 2$. Here the curve γ is so-called a *horocycle* on $\mathbb{R}H^2(c/4)$. Next,

2020 *Mathematics Subject Classification.* Primary 53B25, Secondary 53C40.

Key words and phrases. type (A) hypersurfaces, nonflat complex space forms, complex hyperbolic spaces, homogeneous ruled real hypersurfaces, geodesics, circles, integral curves of the characteristic vector field ξ .

attaching a totally geodesic holomorphic hyperplane $\mathbb{C}H_s^{n-1}(c)(:= \mathbb{C}H^{n-1}(c))$ orthogonally to a holomorphic line generated by $\dot{\gamma}(s)$ at every s along the curve $\gamma = \gamma(s)$, we see that $\text{HR} = \bigcup_{-\infty < s < \infty} \mathbb{C}H_s^{n-1}(c)$. We remark that the full isometry group $I(\text{HR})$ is a direct product of the full isometry group $I(\mathbb{C}H^{n-1}(c))$ of a totally geodesic $\mathbb{C}H^{n-1}(c)$ and a one-parameter subgroup $\{\varphi_s\}_{-\infty < s < \infty}$ (of the full isometry group $I(\mathbb{C}H^2(c))$ of a totally geodesic $\mathbb{C}H^2(c)$) whose orbit is the above horocycle $\gamma = \gamma(s)$, which acts transitively on HR (see p. 320 in [15]). Hence our ruled real hypersurface HR is diffeomorphic to a direct product of a complex hyperbolic space $\mathbb{C}H^{n-1}(c)$ and a real line \mathbb{R} . This homogeneous non-Hopf hypersurface HR is the unique example which is *minimal* in the class of all homogeneous real hypersurfaces, namely they are expressed as orbits under subgroups of the full isometry group $I(\mathbb{C}H^n(c))(= U(1, n))$ of the ambient space $\mathbb{C}H^n(c)$ (see [12, 7, 19]).

In Theorem 2 we characterize HR by investigating some geodesics and all integral curves of the characteristic vector field ξ on HR , which is defined by $\xi = -J\mathcal{N}$, where \mathcal{N} is a unit normal local vector field on HR .

2. BASIC FORMULAS AND TERMINOLOGY

We first start by recalling some basic formulas and terminology on real hypersurfaces M isomerically immersed into a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$. The ambient space $\widetilde{M}_n(c)$ is furnished with the standard Riemannian metric g and canonical Kähler structure J . We use the same notation g for the induced Riemannian metric on M for simplicity. Let \mathcal{N} denote a unit normal local vector field on M . Then the Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$(2.1) \quad \widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N},$$

$$(2.2) \quad \widetilde{\nabla}_X \mathcal{N} = -AX$$

for vector fields X and Y tangent to M , where A is the shape operator of M in $\widetilde{M}_n(c)$.

The hypersurface M can be equipped with an *almost contact metric structure* (ϕ, ξ, η, g) which consists of a tensor field ϕ of type $(1, 1)$, a vector field ξ , a 1-form η and the induced Riemannian metric g . That is, we define ϕ , ξ and η on M by

$$\xi := -J\mathcal{N}, \quad \eta(X) := g(X, \xi) = g(JX, \mathcal{N}), \quad \phi X := JX - \eta(X)\mathcal{N}$$

for each tangent vector $X \in TM$. The structure satisfies

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, & g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(\xi) &= 1, & \phi\xi &= 0 \quad \text{and} \quad \eta(\phi X) = 0 \end{aligned}$$

for all vectors $X, Y \in TM$. We call ϕ , ξ and η the *structure tensor*, the *characteristic vector field* and the *contact form* on M , respectively. By use of the contact form η we define the *holomorphic distribution* T^0M on M as

$$T^0M := \{X \in TM \mid \eta(X) = 0\}.$$

Moreover, we have the following equations:

$$(2.3) \quad \nabla_X \xi = \phi AX,$$

$$(2.4) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

$$(2.5) \quad (\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

The last one is known as the equation of Codazzi. An eigenvalue of the shape operator A is called a *principal curvature* of M , and an eigenvector of A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$.

A real hypersurface M is said to be a *Hopf hypersurface* in $\widetilde{M}_n(c)$ if the characteristic vector ξ is a principal curvature vector at each point \underline{p} of M . Every tube of sufficiently small radius around a Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface (cf. [9]). So the set of all Hopf hypersurfaces is a huge subset in the class of real hypersurfaces of $\widetilde{M}_n(c)$. We can see easily that a real hypersurface M in $\widetilde{M}_n(c)$ is a Hopf hypersurface with shape operator A satisfying $A\xi = \delta\xi$ if and only if every integral curve γ of ξ on M is lying locally on a totally geodesic holomorphic line $M_1(c) (= \mathbb{C}P^1(c)$ or $\mathbb{C}H^1(c))$. In this case, the function δ is locally constant on M and every integral curve γ of ξ is a circle of the same curvature $|\delta|$ (see [21]). These facts give us a fundamental fact that all integral curves of the characteristic vector field ξ on every Hopf hypersurface in $\widetilde{M}_n(c)$, considered as curves in the ambient space $\widetilde{M}_n(c)$, are congruent one another by isometries $\varphi \in I(\widetilde{M}_n(c))$ (cf. [4, 2]). Here, $I(\widetilde{M}_n(c)) = U(n+1)$ ($c > 0$) or $I(\widetilde{M}_n(c)) = U(1, n)$ ($c < 0$).

We here recall the following fact on Hopf hypersurfaces:

Proposition 1 ([8]). *There does not exist a Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ satisfying the condition that the holomorphic distribution T^0M is integrable.*

Type (A) hypersurfaces are the simplest Hopf hypersurfaces in $\widetilde{M}_n(c)$. The following is a well-known characterization of all type (A) hypersurfaces in this space.

Lemma 1 ([22]). *For a real hypersurface M of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$, M is of type (A) if and only if $\phi A = A\phi$ holds on M with structure tensor ϕ and shape operator A of M in the space $\widetilde{M}_n(c)$.*

We remark that every homogeneous real hypersurface of $\mathbb{C}P^n(c)$, i.e., it is an orbit of some subgroup of the full isometry group $U(n+1)$ of the ambient space $\mathbb{C}P^n(c)$, must be a Hopf hypersurface but there *do* exist many homogeneous *non-Hopf* hypersurfaces of $\mathbb{C}H^n(c)$ (for details, see [23, 7]).

3. RULED REAL HYPERSURFACES

A real hypersurface M in $\widetilde{M}_n(c)$, $n \geq 2$ is said to be *ruled* if the holomorphic distribution T^0M is integrable and each of its leaves is locally congruent to a totally geodesic holomorphic hyperplane $M_{n-1}(c)$ of $\widetilde{M}_n(c)$. By the definition

of ruled real hypersurfaces we see easily that a real hypersurface M in $\widetilde{M}_n(c)$ is ruled in this space if and only if $\widetilde{\nabla}_X Y \in T^0M$ holds for all $X, Y \in T^0M$.

Every ruled real hypersurface in $\widetilde{M}_n(c)$ can be constructed in the following way. We take an arbitrary smooth real curve $\gamma : I \rightarrow \widetilde{M}_n(c)$ parametrized by its arclength s defined on some open interval $I(\subset \mathbb{R})$. At each point $\gamma(s)$ ($s \in I$) we attach a unique totally geodesic holomorphic hyperplane $M_s \cong M_{n-1}(c)$ which is orthogonal to the real plane spanned by $\dot{\gamma}(s)$ and $J\dot{\gamma}(s)$. Then, the union $M = \bigcup_{s \in I} M_s$ is a ruled real hypersurface in $\widetilde{M}_n(c)$. This construction implies that a ruled real hypersurface M may have singularities in general. Hence we must omit such points. If a ruled real hypersurface M has no singularities, then M is diffeomorphic to a direct product of a leaf M_s and the above interval I .

The smooth curve $\gamma = \gamma(s)$ in the definition of a ruled real hypersurface M is called a *generating curve associated to the ruled real hypersurface* M^{2n-1} . Note that a generating curve γ is nothing but one of integral curves of the characteristic vector field ξ of the ruled real hypersurface M . However, given a ruled real hypersurface M , a generating curve γ associated to M is not unique in general, that is, there exist many generating curves associated to M which are *not* congruent one another by isometries of the ambient space $\widetilde{M}_n(c)$.

For an arbitrary ruled real hypersurface M in $\widetilde{M}_n(c)$, $n \geq 2$ we define two functions $\mu, \nu : M \rightarrow \mathbb{R}$ by

$$(3.1) \quad \mu := g(A\xi, \xi) \quad \text{and} \quad \nu := \|A\xi - \mu\xi\|.$$

We also define an open dense subset M_* of M by

$$M_* := \{x \in M \mid \nu(x) \neq 0\}.$$

On the subset M_* there exists a unit vector field U orthogonal to ξ satisfying the following:

$$(3.2) \quad A\xi = \mu\xi + \nu U, \quad AU = \nu\xi \quad \text{and} \quad AX = 0 \quad \text{for each vector } X \perp \xi, U.$$

Evidently, a ruled real hypersurface M is minimal if and only if the function μ vanishes. Every ruled real hypersurface M in $\widetilde{M}_n(c)$ is decomposed as:

$$(3.3) \quad M = M_* \cup M_0 \cup M_\infty,$$

where $M_0 = \{x \in M \mid \nu(x) = 0\}$ and $M_\infty = \{x \in M \mid \nu(x) = \infty\}$. In the decomposition (3.3) of a ruled real hypersurface M , M_∞ is the set of singularities of M . So, needless to say when M_∞ is not empty, our ruled real hypersurface $M(= M_* \cup M_0)$ is *not* complete.

In order to prove Theorem 2 in section 5, we prepare the following:

Lemma 2 ([10, 15]). *For every ruled real hypersurface M in $\widetilde{M}_n(c)$ ($n \geq 2$), the function ν in (3.1) satisfies the following differential equation along each integral curve $\rho = \rho(s)$ of the vector field ϕU on M_* :*

$$(3.4) \quad \phi U \nu = \nu^2 + \frac{c}{4}.$$

We note that every integral curve $\rho = \rho(s)$ of the vector field ϕU on each ruled real hypersurface M_* is mapped to a geodesic in the ambient space $\widetilde{M}_n(c)$, so that it is also a geodesic on M_* .

4. REAL CURVES IN A NONFLAT COMPLEX SPACE FORM

For two smooth real curves $\gamma_1 = \gamma_1(s)$, $\gamma_2 = \gamma_2(s)$ parametrized by their arclengths s on a Riemannian manifold M , we say that these two curves γ_1, γ_2 are *congruent* if there exist an isometry φ on M and some constant s_0 with $\gamma_2(s) = (\varphi \circ \gamma_1)(s + s_0)$ for each s . Here, we regard a smooth real curve $\gamma = \gamma(s)$ on M as a smooth mapping $\gamma : I(\subset \mathbb{R}) \rightarrow M$. So curves $\gamma_1 = \gamma_1(s)$ and $\gamma_2 = \gamma_2(s) := \gamma_1(s + s_0)$ for every s are different in our sense.

In this section, we review the real curve theory in a nonflat complex space form. Let $\gamma : I \rightarrow \widetilde{M}_n(c)$ be a smooth real curve parametrized by its arclength s defined on an open interval $I(\subset \mathbb{R})$. The curve γ is said to be a *Frenet curve of proper order d* ($2 \leq d \leq 2n$) if there exist an orthonormal system $\{V_1 := \dot{\gamma}, V_2, \dots, V_d\}$ of vector fields along γ and positive smooth functions $\kappa_1(s), \dots, \kappa_{d-1}(s)$ satisfying the following:

$$(4.1) \quad \widetilde{\nabla}_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad 1 \leq j \leq d,$$

where $\kappa_0 V_0$ and $\kappa_d V_{d+1}$ are null vector fields along γ and $\widetilde{\nabla}_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the standard Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$. The functions $\kappa_1, \dots, \kappa_{d-1}$ and the orthonormal frame $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of the curve γ , respectively. Roughly speaking, Frenet curves have no inflection points.

A Frenet curve γ is said to be a *helix* if all of its curvatures $\kappa_1, \dots, \kappa_{d-1}$ are constant functions. In particular, a helix of proper order 2, namely, a curve which satisfies

$$\widetilde{\nabla}_{\dot{\gamma}} \dot{\gamma}(s) = kV_2(s) \quad \text{and} \quad \widetilde{\nabla}_{\dot{\gamma}} V_2(s) = -k\dot{\gamma}(s),$$

is called a *circle* of curvature $k(> 0)$. A geodesic is regarded as a circle of null curvature.

For a Frenet curve γ of proper order d with Frenet frame $\{V_1, \dots, V_d\}$ in $\widetilde{M}_n(c)$, we define its *holomorphic torsions* by

$$\tau_{ij}(s) := g(V_i(s), JV_j(s)), \quad 1 \leq i < j \leq d.$$

The notion of holomorphic torsions plays an important role in the study of Frenet curves in $\widetilde{M}_n(c)$. A real curve γ in $\widetilde{M}_n(c)$ is said to be *homogeneous* if it is an orbit of one-parameter subgroup of the full isometry group $I(\widetilde{M}_n(c))$ of the space $\widetilde{M}_n(c)$. We can give the necessary and sufficient condition for a Frenet curve to be homogeneous in $\widetilde{M}_n(c)$ by using the notion of holomorphic torsions ([16]): A Frenet curve γ is homogeneous in $\widetilde{M}_n(c)$ if and only if all of its curvatures and all of its holomorphic torsions are constant functions.

For a circle γ of positive curvature k in $\widetilde{M}_n(c)$, we have just one holomorphic torsion $\tau_{12}(s) = g(V_1(s), JV_2(s))$. It is easy to see that the holomorphic torsion τ_{12} of the circle γ is automatically constant along γ . In the following, for simplicity

we denote by τ the holomorphic torsion of a circle of positive curvature in $\widetilde{M}_n(c)$. Thus, circles are necessarily homogeneous (cf. [16]), but in the class of helices of proper order $d(\geq 3)$ we have many helices which are not homogeneous (see [3]).

It is well-known that all geodesics on a Riemannian symmetric space of rank one are congruent one another. So, we next pay particular attention to the congruence theorem for circles of positive curvature in a nonflat complex space form $\widetilde{M}_n(c)$. The congruence theorem can be described as follows:

Lemma 3 ([4, 2]). *We take two circles γ_1 and γ_2 on a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$ with positive curvatures k_1 and k_2 , and with holomorphic torsions τ_1 and τ_2 , respectively. Then γ_1 and γ_2 are congruent if and only if the equalities hold either $k_1 = k_2$, $\tau_1 = \tau_2$ or $k_1 = k_2$, $\tau_1 = -\tau_2$. The former holds when γ_1 , γ_2 are congruent by some holomorphic isometry, and the latter holds when γ_1 , γ_2 are congruent by some anti-holomorphic isometry.*

The congruence theorem for homogeneous curves in $\widetilde{M}_n(c)$ is described in [16]. A circle of positive curvature in $\widetilde{M}_n(c)$ is called *Kähler* (resp. *totally real*) if its holomorphic torsion τ satisfies either $\tau = 1$ or $\tau = -1$ (resp. $\tau = 0$). It is easy to see that a circle of positive curvature is a Kähler circle (resp. a totally real circle) if and only if it lies on a totally geodesic holomorphic line $M_1(c)(= \mathbb{C}P^1(c)$ or $\mathbb{C}H^1(c)$) (resp. a totally real totally geodesic surface $\mathbb{R}M^2(c/4)(= \mathbb{R}P^2(c/4)$ or $\mathbb{R}H^2(c/4)$) of constant sectional curvature $c/4$) in the ambient space $\widetilde{M}_n(c)$.

5. STATEMENT OF RESULTS

The following gives a geometric meaning of an algebraic condition that $\phi A = A\phi$ (see Lemma 1).

Theorem 1. *Let M be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$ through an isometric immersion. Then the structure tensor ϕ and the shape operator A of M satisfy an equality $\phi A = A\phi$ on M , i.e., M is locally congruent to a type (A) hypersurface if and only if the structure torsion $\rho_\gamma(s) := g(\dot{\gamma}(s), \xi_{\gamma(s)})$ of each geodesic $\gamma = \gamma(s)$ on M is constant along the curve γ .*

Proof. (\implies): Suppose that $\phi A = A\phi$ holds on M . Then, from (2.3) we see that

$$\begin{aligned} \dot{\gamma}\rho_\gamma &= \dot{\gamma}(g(\dot{\gamma}, \xi)) = g(\dot{\gamma}, \phi A\dot{\gamma}) = g(\dot{\gamma}, A\phi\dot{\gamma}) \\ &= g(A\dot{\gamma}, \phi\dot{\gamma}) = -g(\phi A\dot{\gamma}, \dot{\gamma}) = 0 \end{aligned}$$

for every geodesic $\gamma = \gamma(s)$ on our type (A) hypersurface M .

(\impliedby): By assumption we have

$$0 = \dot{\gamma}\rho_\gamma = \dot{\gamma}(g(\dot{\gamma}, \xi)) = g(\dot{\gamma}, \phi A\dot{\gamma}) = (1/2)g((\phi A - A\phi)\dot{\gamma}, \dot{\gamma})$$

holds for each geodesic γ on M , which shows that $g((\phi A - A\phi)X, X) = 0$ for every vector X of M . This, together with a fact that $\phi A - A\phi$ is symmetric, yields that $g((\phi A - A\phi)X, Y) = 0$ for every $X, Y \in TM$. Hence we get the desirable equality $\phi A = A\phi$. \square

Remark 1. (1) For a geodesic $\gamma = \gamma(s)$ stated in Theorem 1, the constant ρ_γ depends on the choice of the curve γ .

(2) There are many conditions which characterize type (A) hypersurfaces M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$ (cf. [21]). For example, the following is an analytic condition characterizing type (A) hypersurfaces (see [20]).

A real hypersurface M isometrically immersed into $\widetilde{M}_n(c)$ is of type (A) if and only if the derivative of shape operator A of M satisfies $(\nabla_X A)Y = -(c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X)$ for all $X, Y \in TM$.

The following gives a geometric meaning of this analytic condition (see [13]). A real hypersurface M isometrically immersed into $\widetilde{M}_n(c)$ satisfies the above differential equation of the shape operator A of M if and only if for every geodesic γ on M the normal curvature $\kappa_\gamma := g(A\dot{\gamma}, \dot{\gamma})$ is constant along the curve γ .

Theorem 2. *Let M be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$ through an isometric immersion. Then $c < 0$ and M is locally congruent to the homogeneous ruled real hypersurface HR in a complex hyperbolic space $\mathbb{C}H^n(c)$ if and only if M satisfies the following two conditions:*

- (1) *At an arbitrary point $x \in M$, there exist such orthonormal vectors $v_1, v_2, \dots, v_{2n-2} \in T_x M$ orthogonal to the characteristic vector ξ_x that every geodesic $\gamma_{ij,x}$ on M through the point $\gamma_{ij,x}(0) = x$ in the direction of $v_i + v_j$ ($1 \leq i \leq j \leq 2n-2$) is also mapped to a geodesic in the ambient space $\widetilde{M}_n(c)$;*
- (2) *All integral curves of the characteristic vector field ξ on M , considered as curves in the ambient space $\widetilde{M}_n(c)$, are congruent one another by isometries $\varphi \in I(\widetilde{M}_n(c))$.*

Proof. We first verify that a real hypersurface M is ruled in the ambient space $\widetilde{M}_n(c)$ if and only M satisfies Condition (1) in Theorem 2.

Suppose that M is ruled in this space. We take each geodesic $\rho = \rho(s)$ on M whose initial vector $\dot{\rho}(0)$ is orthogonal to the characteristic vector $\xi_{\rho(0)}$. Let M_0 be the leaf through the point $\rho(0)$ for the holomorphic distribution T^0M . We here take a geodesic $\rho_1 = \rho_1(s)$ on M_0 with the same initial condition as $\rho_1(0) = \rho(0)$ and $\dot{\rho}_1(0) = \dot{\rho}(0)$. Since M_0 is locally congruent to a totally geodesic holomorphic hyperplane $M_{n-1}(c)$ of $\widetilde{M}_n(c)$, M_0 is also totally geodesic in our ruled real hypersurface M . This implies that the curve ρ_1 is also a geodesic in M . Hence the uniqueness theorem on geodesics shows that these two curves ρ and ρ_1 are coincidental, so that every geodesic $\rho = \rho(s)$ on M with initial vector $\dot{\rho}(0)$ orthogonal to the characteristic vector $\xi_{\rho(0)}$ is mapped to a geodesic in the ambient space $\widetilde{M}_n(c)$. Thus we see that M satisfies Condition (1) in a trivial sense.

Conversely, we assume that M satisfies Condition (1). We take a geodesic $\gamma_{ii,x} = \gamma_{ii,x}(s)$ ($1 \leq i \leq 2n-2$) on M with initial condition that $\gamma_{ii,x}(0) = x$ and $\dot{\gamma}_{ii,x}(0) = v_i$. Then it follows from the hypothesis and (2.1) that $g(A\dot{\gamma}_{ii,x}(s), \dot{\gamma}_{ii,x}(s)) = 0$ for every s . So, in particular at the point $x(= \gamma_{ii,x}(0)) \in M$ we find

$$(5.1) \quad g(Av_i, v_i) = 0 \quad \text{for } 1 \leq i \leq 2n-2.$$

Next, we take a geodesic $\gamma_{ij,x} = \gamma_{ij,x}(s)$ ($1 \leq i < j \leq 2n - 2$) on M with initial condition that $\dot{\gamma}_{ij,x}(0) = x$ and $\dot{\gamma}_{ij,x}(0) = (v_i + v_j)/\sqrt{2}$. Then, applying the same discussion as above to the curve $\dot{\gamma}_{ij,x}$, we have

$$(5.2) \quad g(A(v_i + v_j)/\sqrt{2}, (v_i + v_j)/\sqrt{2}) = 0 \quad \text{for } 1 \leq i < j \leq 2n - 2.$$

Thus, from (5.1) and (5.2) we see that

$$g(Av_i, v_j) = 0 \quad \text{for } i, j = 1, 2, \dots, 2n - 2,$$

which yields

$$(5.3) \quad g(AX, Y) = 0 \quad \text{for all vectors } X, Y \in T^0M.$$

Therefore, in consideration of (2.1), (2.3) and (5.3) we can see that $\tilde{\nabla}_X Y \in T^0M$ for each $X, Y \in T^0M$, that is, our real hypersurface M is ruled in $\tilde{M}_n(c)$.

In the following, we study on the open dense subset $M_* = \{x \in M \mid \nu(x) \neq 0\}$ of a ruled real hypersurface M satisfying Condition (2) in Theorem 2. We take an arbitrary integral curve $\gamma = \gamma(s)$ of the vector field ξ on M_* , and take two different points p, q on the curve γ . Set $\gamma(0) = p$ and consider another integral curve $\gamma_1 = \gamma_1(s)$ of the characteristic vector field ξ with $\gamma_1(0) = q$. Then by assumption there exist an isometry $\varphi \in \text{I}(\tilde{M}_n(c))$ and some $s_0 (\neq 0)$ with $\gamma_1(s) = (\varphi \circ \gamma)(s + s_0)$ for each s , so that $\gamma(s_0) = q$. Hence the first curvature $\kappa_1(s) := \|\tilde{\nabla}_{\dot{\gamma}(s)} \dot{\gamma}(s)\|$ on each integral curve γ of ξ satisfies $\kappa_1(0) = \kappa_1(s_0)$, which implies that the function $\kappa_1(s)$ is constant along the curve γ , where $\tilde{\nabla}$ is the Riemannian connection of the ambient space $\tilde{M}_n(c)$. This, together with the same discussion as above for an arbitrary integral curve of ξ which does *not* pass the point p , shows that the first curvature $\kappa_1 := \|\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}\|$ is a constant function which is independent of the choice of γ . We here rewrite our argument in tensor calculus. Since $\dot{\gamma} = \xi$, thanks to (2.1), (2.3) and (3.2) we have

$$(5.4) \quad \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = \phi A\xi + g(A\xi, \xi)\mathcal{N} = \nu\phi U + \mu\mathcal{N},$$

so that the first curvature κ_1 of each integral curve γ of ξ is given by

$$\kappa_1 = \sqrt{\mu^2 + \nu^2} (\neq 0),$$

which is a constant on M_* .

We next consider the first holomorphic torsion $\tau_{12} := g(\dot{\gamma}, JV_2)$ of the above integral curve γ of ξ on M_* satisfying the hypothesis in Theorem 2. It follows from (5.4) that the unit curvature vector V_2 in the Frenet formula (4.1) for the curve γ is expressed as:

$$V_2(s) = (\nu\phi U + \mu\mathcal{N})/\sqrt{\mu^2 + \nu^2},$$

which yields immediately that the first holomorphic torsion τ_{12} of the curve γ can be given by

$$\tau_{12}(s) = -\mu/\sqrt{\mu^2 + \nu^2}$$

on M_* .

Since the isometry $\varphi \in I(\widetilde{M}_n(c))$ satisfies $\varphi(p) = q$ with $\gamma(0) = p$ and $\gamma(s_0) = q$, the derivative $d\varphi$ of the mapping φ gives $\xi_q = d\varphi_p(\xi_p)$ and $V_2(s_0) = d\varphi_p(V_2(0))$. Hence, when φ is holomorphic, we find that

$$\begin{aligned}\tau_{12}(s_0) &= g(\xi_q, J_q V_2(s_0)) = g(d\varphi_p(\xi_p), (J_q \circ d\varphi_p)(V_2(0))) \\ &= g(d\varphi_p(\xi_p), (d\varphi_p \circ J_p)(V_2(0))) = g(\xi_p, J_p V_2(0)) = \tau_{12}(0),\end{aligned}$$

and when φ is anti-holomorphic, we have

$$\begin{aligned}\tau_{12}(s_0) &= g(\xi_q, J_q V_2(s_0)) = g(d\varphi_p(\xi_p), (J_q \circ d\varphi_p)(V_2(0))) \\ &= -g(d\varphi_p(\xi_p), (d\varphi_p \circ J_p)(V_2(0))) = -g(\xi_p, J_p V_2(0)) = -\tau_{12}(0).\end{aligned}$$

This shows that the function $|\tau_{12}(s)|$ is constant along the curve γ . This, combined with the same discussion as above for an arbitrary integral curve of ξ which does *not* pass the point p , shows that the absolute value of the first holomorphic torsion $|\tau_{12}|$ is a constant function which is independent of the choice of γ . Thus we see that both of $|\mu|$ and ν are constant on M_* , which together with the continuity of μ and ν , yields that $M_* = M$ and both of the functions μ and ν are constant on the whole set M .

We first study the case of $c > 0$. Since the function ν is a nonzero constant, we find from (3.4) that

$$0 = \phi U \nu = \nu^2 + \frac{c}{4} > \frac{c}{4},$$

which is a contradiction.

We next investigate the case of $c < 0$. The expression of the shape operator A of M in (3.2) shows that M has three distinct constant principal curvatures λ_1 , λ_2 and zero at each point of M , where λ_1 and λ_2 are solutions to a quadratic equation $\lambda^2 - \mu\lambda - \nu^2 = 0$. On the other hand, by virtue of (3.4) the constant ν satisfies

$$0 = \phi U \nu = \nu^2 + \frac{c}{4},$$

so that $\nu \equiv \sqrt{|c|}/2$. Hence, the principal curvatures of our ruled real hypersurface M are given by either zero or constants λ_1 , $\lambda_2 = (\mu \pm \sqrt{\mu^2 - c})/2$. Therefore by virtue of the classification theorem of real hypersurfaces with three constant principal curvatures in $\mathbb{C}H^n(c)$ (see [5, 6]) we can see that the function μ vanishes, i.e., M is minimal in the ambient space, and that our real hypersurface M is locally congruent to a ruled real hypersurface which is determined by a horocycle on a totally real totally geodesic real hyperbolic plane $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$, that is, it is congruent to HR in local. Thus, the ‘‘if’’ part is proved.

Conversely, let M be the homogeneous ruled real hypersurface HR which is explained in Introduction. Then, from the construction of this hypersurface we know that every integral curve of the characteristic vector field ξ on HR is mapped to a totally real circle of the same curvature $\sqrt{|c|}/2$ in the ambient space $\mathbb{C}H^n(c)$, $n \geq 2$. So, these curves, considered as in $\mathbb{C}H^n(c)$, are congruent one another (see Lemma 3). Therefore we obtain the desired conclusion. \square

Characterization theorems of the homogeneous ruled real hypersurface in $\mathbb{C}H^n(c)$, $n \geq 2$ are given in [15, 11, 17].

6. REMARKS ON THEOREM 2

We here comment on minimal ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$.

- (1) We describe the solution $\nu = \nu(s)$ to a differential equation (3.4) in Lemma 2. On each integral curve of ϕU , when $c > 0$, the function $\nu = \nu(s)$ is written as:

$$\nu(s) = \frac{\sqrt{c}}{2} \tan\left(\frac{\sqrt{c}}{2}s + s_0\right) \quad \text{with some } s_0,$$

and, when $c < 0$, the function $\nu = \nu(s)$ is expressed as:

$$\nu(s) = \begin{cases} -\frac{\sqrt{|c|}}{2} \tanh\left(\frac{\sqrt{|c|}}{2}s + s_0\right) & \text{with some } s_0 \text{ if } \nu(0) < \sqrt{|c|}/2, \\ -\frac{\sqrt{|c|}}{2} \coth\left(\frac{\sqrt{|c|}}{2}s + s_0\right) & \text{with some } s_0 \text{ if } \nu(0) > \sqrt{|c|}/2, \\ \frac{\sqrt{|c|}}{2} & \text{if } \nu(0) = \sqrt{|c|}/2. \end{cases}$$

- (2) It is known that a ruled real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$ is minimal in this space if and only if all integral curves of the characteristic vector field ξ (i.e., generating curves associated to M) are mapped to totally real circles of curvature (k , say) in the ambient space $\widetilde{M}_n(c)$ (cf. [12, 14, 1]), and that there exists just one minimal ruled real hypersurface when $c > 0$ and there exist three minimal ruled real hypersurfaces when $c < 0$ up to isometries of the space $\widetilde{M}_n(c)$. Here, if $c > 0$, then the curvature k satisfies $0 \leq k < \infty$, and if $c < 0$, then the curvature k satisfies either $0 \leq k < \sqrt{|c|}/2$, $k > \sqrt{|c|}/2$ or $k = \sqrt{|c|}/2$. These three minimal ruled real hypersurfaces in $\mathbb{C}H^n(c)$ are said to be of *axial type*, *elliptic type* and *parabolic type* (i.e., *homogeneous type*), respectively. Note that minimal ruled real hypersurfaces of axial type and elliptic type are *not* homogeneous in the space $\mathbb{C}H^n(c)$ ([1]). Furthermore, a minimal non-homogeneous ruled real hypersurface $M = M_* \cup M_0$ (see (3.3)) of axial type is diffeomorphic to a direct product of a totally geodesic $\mathbb{C}H^{n-1}(c)$ and a real line \mathbb{R} , so that it is complete. In this case, M_∞ is empty and M_0 is not empty. But a minimal non-homogeneous ruled real hypersurface M of elliptic type is *not* complete because it has singularities.
- (3) It is known that the sectional curvature K of *every* ruled real hypersurface M in $\mathbb{C}H^n(c)$ satisfies sharp inequalities either $-\infty < K \leq c/4$ on M or $c \leq K \leq c/4$ on M . Note that a minimal ruled real hypersurface M , which is one of parabolic type and axial type, satisfies the latter inequalities and a minimal ruled real hypersurface M of elliptic type satisfies the former inequalities (for details, see [18]).

- (4) Motivated by Condition (1) in Theorem 2, we pose the following open problem:

Problem. In a complex hyperbolic space $\mathbb{C}H^n(c)$, $n \geq 2$, does there exist a non-ruled real hypersurface M satisfying that at an arbitrary point $x \in M$, there exist such orthonormal vectors $v_1, v_2, \dots, v_{2n-2} (\in T_x M)$ orthogonal to the characteristic vector ξ_x that every geodesic $\gamma_{i,x}$ on M through the point $\gamma_{i,x}(0) = x$ with initial vector $\dot{\gamma}_{i,x}(0) = v_i$ ($1 \leq i \leq 2n-2$) is also mapped to a geodesic in the ambient space $\mathbb{C}H^n(c)$?

We remark that this problem is solved affirmatively in the case that the ambient space is a complex projective space $\mathbb{C}P^n(c)$. In fact, we denote by M a tube of radius $\pi/(2\sqrt{c})$ around a totally geodesic $\mathbb{C}P^{(n-1)/2}(c)$ in $\mathbb{C}P^n(c)$, where $n(\geq 3)$ is an odd number. That is, M is a member of homogeneous real hypersurfaces of type (A_2) in the ambient space $\mathbb{C}P^n(c)$. Then the tangent bundle TM of M is decomposed as: $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\sqrt{c}/2} \oplus V_{-\sqrt{c}/2}$ where $A\xi = 0$ and $\dim_{\mathbb{R}} V_{\sqrt{c}/2} = \dim_{\mathbb{R}} V_{-\sqrt{c}/2} = n-1$. We take orthonormal bases $\{e_1, e_2, \dots, e_{n-1}\}$ and $\{f_1, f_2, \dots, f_{n-1}\}$ of principal distributions $V_{\sqrt{c}/2}$ and $V_{-\sqrt{c}/2}$, respectively. Using these vectors, we take an orthonormal basis $\{v_1, \dots, v_{2n-2}\}$ of the holomorphic distribution T^0M on M in such a way that $v_i := (e_i + f_i)/\sqrt{2}$ and $v_{n-1+i} := (e_i - f_i)/\sqrt{2}$, where $i \in \{1, 2, \dots, n-1\}$. We also take and fix an arbitrary point x of our real hypersurface M and choose the above orthonormal vectors $v_1, \dots, v_{2n-2} (\in T_x(M))$ perpendicular to ξ_x . Then we find easily that $g(Av_i, v_i) = 0$ for $i \in \{1, 2, \dots, 2n-2\}$. We here recall a fact that the shape operator A of the manifold M satisfies (see [20]) $(\nabla_X A)Y = -(c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X)$ for all $X, Y \in TM$, which shows that $g((\nabla_X A)X, X) = 0$ for each vector X of M . This, combined with Gauss formula (2.1) and initial condition $g(Av_i, v_i) = 0$, implies that the geodesic $\gamma_i = \gamma_i(s)$ ($1 \leq i \leq 2n-2$) on M through the point $x = \gamma_i(0)$ with initial vector $\dot{\gamma}_i(0) = v_i$ perpendicular to the characteristic vector ξ_x is also mapped to a geodesic in the ambient space $\mathbb{C}P^n(c)$.

At the end of this paper we give a final comment on Theorem 2. Note that all integral curves γ_ξ of the characteristic vector fields ξ on HR are homogeneous and moreover, they are congruent one another. Motivated by this algebraic condition, we establish Theorem 2.

On the other hand, we pay particular attention to another property of these curves γ_ξ . We remark that all curves γ_ξ are mapped to circles of the same curvature on a totally real totally geodesic surface $\mathbb{R}H^2(c/4)$ of constant sectional curvature $c/4$ in the ambient space $\mathbb{C}H^n(c)$. Inspired by this geometric condition, we obtain another characterization of the homogeneous ruled real hypersurface HR in $\mathbb{C}H^n(c)$ (for details, see p. 321 in [15]).

REFERENCES

- [1] T. Adachi, T. Bao and S. Maeda, *Congruence classes of minimal ruled real hypersurfaces in a nonflat complex space form*, Hokkaido Math. J. **43** (2014), 1–14.

- [2] T. Adachi and S. Maeda, *Global behaviours of circles in a complex hyperbolic space*, Tsukuba J. Math. **21** (1997), 29–42.
- [3] T. Adachi and S. Maeda, *A construction of closed helices with self-intersections in a complex projective space by using submanifold theory*, Hokkaido Math. J. **28** (1999), 133–145.
- [4] T. Adachi, S. Maeda and S. Udagawa, *Circles in a complex projective space*, Osaka J. Math. **32** (1995), No.3, 709–719.
- [5] J. Berndt and J.C. Diaz-Ramos, *Real hypersurfaces with constant principal curvatures in the complex hyperbolic spaces*, J. London Math. Soc. **74** (2006), 778–798.
- [6] J. Berndt and J.C. Diaz-Ramos, *Real hypersurfaces with constant principal curvatures in complex hyperbolic plane*, Proc. Amer. Math. Soc. **135** (2007), 3349–3357.
- [7] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces of rank one*, Trans. Amer. Math. Soc. **359** (2007), 3425–3438.
- [8] B.Y. Chen and S. Maeda, *Hopf hypersurfaces with constant principal curvatures in complex projective or complex hyperbolic spaces*, Tokyo J. Math. **24** (2001), 133–152.
- [9] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), No. 2, 481–499.
- [10] M. Kimura, *Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(C)$* , Math. Ann. **276** (1987), no. 3, 487–497.
- [11] M. Kimura, S. Maeda and H. Tanabe, *Integral curves of the characteristic vector field of minimal ruled real hypersurfaces in non-flat complex space forms*, Hokkaido Math. J. **48** (2019), 589–609.
- [12] M. Lohnherr and H. Reckziegel, *On ruled real hypersurfaces in complex space forms*, Geom. Dedicata **79** (1999), 267–286.
- [13] S. Maeda, *Real hypersurfaces in complex projective spaces*, Math. Ann. **263** (1983), 473–478.
- [14] S. Maeda and T. Adachi, *Integral curves of characteristic vector fields of real hypersurfaces in nonflat complex space forms*, Geom. Dedicata **123** (2006), 65–72.
- [15] S. Maeda, T. Adachi and Y.H. Kim, *A characterization of the homogeneous minimal ruled real hypersurface in a complex hyperbolic space*, J. Math. Soc. Japan **61** (2009), No. 1, 315–325.
- [16] S. Maeda and Y. Ohnita, *Helical geodesic immersions into complex space forms*, Geom. Dedicata **30** (1989), 93–114.
- [17] S. Maeda and H. Tanabe, *A characterization of the homogeneous ruled real hypersurface in a complex hyperbolic space in terms of the first curvature of some integral curves*, Arch. Math. (Basel) **105** (2015), 593–599.
- [18] S. Maeda and H. Tanabe, *Sectional curvatures of ruled real hypersurfaces in a complex hyperbolic space*, Differential Geometry and its Applications **51** (2017), 1–8.
- [19] S. Maeda, H. Tamaru and H. Tanabe, *Curvature properties of homogeneous real hypersurfaces in nonflat complex space forms*, Kodai Math. J. **41** (2018), 315–331.
- [20] Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976), 529–540.
- [21] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, Tight and taut submanifolds (T.E. Cecil and S.S. Chern, eds.), Cambridge Univ. Press, 1997, 233–305.
- [22] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.
- [23] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

Communicated by *Yasunao Hattori*

PROFESSOR EMERITUS, DEPARTMENT OF MATHEMATICS, SAGA UNIVERSITY AND SHIMANE UNIVERSITY, JAPAN

E-mail address: sadahiromaeda0801@gmail.com

A STRUCTURE OF THE EXTENDED INTEGRAL

TOSHIHARU KAWASAKI

Received February 25, 2024; revised April 27, 2024

ABSTRACT. The biggest obstacle to expansion for integrals is that the indefinite integral must be continuous. Let us consider an integral, named the extended integral, where the indefinite integral is not necessarily continuous. Furthermore, in this paper, we define a seminorm-like functional on the set of all extended integrable functions and consider a topology of that set.

1. INTRODUCTION

Integral is an extended concept of length, area, and volume, and also has the aspect of being an inverse operation of differential. When considering integrals in the former sense, nowadays it is common to use the Lebesgue integral, which have sufficient properties for practical applications such as the convergence theorem and Fubini's theorem. However, there are cases where the Lebesgue integral alone is not sufficient, and in such cases the improper Lebesgue integral is used. On the other hand, when considering integrals in the latter sense, we use the Newton integral, that is, the primitive. A function f is said to be Newton integrable if there exists a function F such that $F' = f$ holds everywhere. The condition that it is differentiable everywhere is too strict, therefore, we loosen the condition a little more and consider the case where F is continuous and $F' = f$ holds except for a countable number of points, which called the improper Newton integral.

Unfortunately, there are functions that are improper Lebesgue integrable but not Newton integrable, and functions that are improper Newton integrable but not Lebesgue integrable [5].

If we want to consider about differential equations straightforwardly, the Newton integral is more appropriate, but there is no convergence theorem like the Lebesgue integral. Therefore, an integral that includes both of these and has a convergence theorem is required. This solution is the Denjoy integral, the Perron integral, and the Henstock-Kurzweil integral. Although these are defined independently, subsequent research has shown that they are equivalent, that is, the sets of integrable functions are the same and the integral values are also the same; for instance, see [2, 7]. In addition, convergence theorems that are sufficient for practical applications have been obtained. For details on the Denjoy, Perron, and Henstock-Kurzweil integrals, refer to the following references [2, 3, 7–13].

Therefore, the Denjoy-Perron-Henstock-Kurzweil integral seems to be sufficient, but considering the framework of integrals, it seems difficult to extend it any further. The biggest obstacle to expansion for integrals is that the indefinite integral must be continuous. Let us consider an integral where the indefinite integral is not necessarily continuous.

Let

$$\mathcal{I}_0 = \{[a, b] \mid a, b \in \mathbb{R}\} \cup \{(-\infty, b] \mid b \in \mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, \infty)\}.$$

2010 *Mathematics Subject Classification*. Primary: 26A36; Secondary: 26A39.

Key words and phrases. Extended integral, Indefinite integral, Primitive function, Denjoy integral, Perron integral, Henstock-Kurzweil integral.

In this paper, the elements of \mathcal{I}_0 are called intervals. A set $\{I_k\}$ of intervals is said to be non-overlapping if, for any k_1 and for any k_2 , $I_{k_1}^\circ \cap I_{k_2}^\circ = \emptyset$ holds if $k_1 \neq k_2$. Furthermore, it is said to be adjacent if it is non-overlapping, and for any k_1 , there exists $k_2 \neq k_1$ such that $I_{k_1} \cap I_{k_2} \neq \emptyset$ holds.

Generally, the integral T on $D \in \mathcal{I}_0$ is one that satisfies the following; see [7]:

- (I1) Let $\mathcal{I} \stackrel{\text{def}}{=} \{I \mid I \in \mathcal{I}_0, I \subset D\}$. There exists an appropriate function space \mathcal{F} consisting of functions with domain D . Then T is a mapping from $\mathcal{F} \times \mathcal{I}$ into \mathbb{R} . A function belonging to \mathcal{F} is said to be T -integrable.
- (I2) (Linearity) Suppose that \mathcal{F} is linear. Then T is linear with respect to the first variable, that is,

$$T(\alpha f + \beta g, I) = \alpha T(f, I) + \beta T(g, I)$$

holds for any $\alpha, \beta \in \mathbb{R}$, for any $f, g \in \mathcal{F}$, and for any $I \in \mathcal{I}$.

- (I3) (Interval additivity) T has interval additivity with respect to the second variable, that is,

$$T(f, I_1 \cup I_2) = T(f, I_1) + T(f, I_2)$$

holds for any $f \in \mathcal{F}$ and for any adjacent $I_1, I_2 \in \mathcal{I}$.

- (I4) (Continuity) For any $f \in \mathcal{F}$, $T(f, [\inf D, \cdot])$, where $[\inf D, \cdot]$ is interpreted as $(\inf D, \cdot]$ if $\inf D = -\infty$ (similarly, $[\cdot, \sup D]$ is interpreted as $[\cdot, \sup D)$ if $\sup D = \infty$), is a continuous function when we consider it as a function from D into \mathbb{R} . This function is called indefinite T -integral.

We would like to consider something that relaxes the fourth condition above. In particular, we would like to consider a case where the indefinite integral takes an infinite value.

2. THE EXTENDED INTEGRAL

The extended integral was discussed in [6]. However, since there are some improvements, we redefine in this paper.

Let T be an integral from $\mathcal{F} \times \mathcal{I}$ into \mathbb{R} . For any mapping f from D into \mathbb{R} , define

$$\mathcal{N}_f \stackrel{\text{def}}{=} \left\{ z \mid \begin{array}{l} \text{for any } a \in D \text{ with } a < z, f \text{ is not } T\text{-integrable on } [a, z] \\ \text{or} \\ \text{for any } b \in D \text{ with } b > z, f \text{ is not } T\text{-integrable on } [z, b] \end{array} \right\}.$$

Lemma 2.1. \mathcal{N}_f is closed.

Proof. We show that, if $\{x_n\} \rightarrow x_0$ holds for $\{x_n\} \subset \mathcal{N}_f$, then $x_0 \in \mathcal{N}_f$ holds.

Since $\{x_n\} \rightarrow x_0$, there exists a subsequence $\{x_{n(k)}\}$ such that $\{x_{n(k)}\} \uparrow x_0$ or $\{x_{n(k)}\} \downarrow x_0$.

In the case of $\{x_{n(k)}\} \uparrow x_0$, we only need to consider the case of $x_{n(k)} < x_0$ for any k . Assume that there exists $a < x_0$ such that f is T -integrable on $[a, x_0]$. Then there exists k such that $a < x_{n(k)} < x_0$. Since $[a, x_{n(k)}], [x_{n(k)}, x_0] \subset [a, x_0]$, f is T -integrable on both $[a, x_{n(k)}]$ and $[x_{n(k)}, x_0]$. However, it contradicts $x_{n(k)} \in \mathcal{N}_f$. Therefore, f is not T -integrable on $[a, x_0]$ for any $a < x_0$, that is, $x_0 \in \mathcal{N}_f$ holds.

In the case of $\{x_{n(k)}\} \downarrow x_0$, it is possible to be similarly shown. \square

Since \mathcal{N}_f is closed, putting $D \setminus \mathcal{N}_f = \bigcup_{p=1}^{\infty} I_p$, each I_p is an open interval with respect to the relative topology in D or the empty set.

Define

$$\mathcal{F}^* \stackrel{\text{def}}{=} \{f \mid \mathcal{N}_f \text{ is a null set}\}.$$

Furthermore, for any $f \in \mathcal{F}^*$, define

$$\mathcal{I}_f \stackrel{\text{def}}{=} \{I \mid f \text{ is } T\text{-integrable on } I \in \mathcal{I}\}.$$

Then, for any $f \in \mathcal{F}^*$ and for any $I \in \mathcal{I}_f$, a mapping

$$T^* : \bigcup_{f \in \mathcal{F}^*} (\{f\} \times \mathcal{I}_f) \longrightarrow \mathbb{R}$$

is defined by

$$T^*(f, I) = T(f, I).$$

T^* is called the extended integral of T .

By the definition, the following hold.

Theorem 2.1. *Let T^* be the extended integral of T on D . If f is T -integrable on D , then f is T^* -integrable on D .*

Theorem 2.2. *Let T^* be the extended integral of T on D , let $\alpha, \beta \in \mathbb{R}$, let $f, g \in \mathcal{F}^*$, and let $I \in \mathcal{I}_f \cap \mathcal{I}_g$. Then the following holds:*

$$T^*(\alpha f + \beta g, I) = \alpha T^*(f, I) + \beta T^*(g, I).$$

Theorem 2.3. *Let T^* be the extended integral of T on D , let $f \in \mathcal{F}^*$, and let $I_1, I_2 \in \mathcal{I}_f$ be adjacent intervals. Then the following holds:*

$$T^*(f, I_1 \cup I_2) = T^*(f, I_1) + T^*(f, I_2).$$

Usually, we consider the condition $a \leq b$ for any interval $[a, b]$. However, when we want to consider b as a variable, we should also consider the case of $a > b$. As it often used, define $T^*(f, [a, b]) = -T^*(f, [b, a])$ if $a > b$. Furthermore, when $a > b$, if $[b, a] \in \mathcal{I}_f$, then it interpret as $[a, b] \in \mathcal{I}_f$.

Theorem 2.4. *Let T^* be the extended integral of T on D , let $f \in \mathcal{F}^*$, let $c \in D$, and let $D_c = \{x \mid [c, x] \in \mathcal{I}_f\}$. Then $T^*(f, [c, x])$ is continuous on D_c with respect to the variable x .*

Let

$$\mathcal{I}_{\text{compact}} \stackrel{\text{def}}{=} \{[a, b] \mid a, b \in \mathbb{R}, [a, b] \subset D\}$$

and let

$$L_{\text{loc}}^1 \stackrel{\text{def}}{=} \{f \mid f \in \mathcal{F}^*, \mathcal{I}_{\text{compact}} \subset \mathcal{I}_f\}.$$

A function belonging to L_{loc}^1 is said to be locally T -integrable. By the definition, $L_{\text{loc}}^1 \subset \mathcal{F}^*$ holds.

3. THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we consider the Denjoy-Perron-Henstock-Kurzweil integral for T .

The set of all extended primitives of f corresponding to the extended Denjoy-Perron-Henstock-Kurzweil integral is defined by

$$\mathcal{P}_f^{\text{DPHK}^*} \stackrel{\text{def}}{=} \{F \mid F \in \mathbf{ACG}^*(I) \cap \mathbf{C}(I) \text{ for any } I \in \mathcal{I}_f \text{ and } F' = f \text{ a.e.}\},$$

where $\mathbf{C}(I)$ is the set of all continuous functions on I , and $\mathbf{ACG}^*(I)$ is the set of all restricted generalized absolutely continuous functions on I , that is, there exists a sequence

$\{E_p \mid p = 1, 2, \dots\}$ of measurable sets such that $\bigcup_{p=1}^{\infty} E_p = I$, and for any p and for any $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that

$$\sum_{k=1}^{k_0} (\beta_k - \alpha_k) < \delta \text{ implies } \sum_{k=1}^{k_0} \omega(F, [\alpha_k, \beta_k]) \stackrel{\text{def}}{=} \sum_{k=1}^{k_0} \sup_{x, y \in [\alpha_k, \beta_k]} |F(y) - F(x)| < \varepsilon$$

for any $\{[\alpha_k, \beta_k] \mid k = 1, \dots, k_0\}$ such that it is non-overlapping and $\alpha_k, \beta_k \in E_p$.

Then the following are clear.

Theorem 3.1. $F - G \in \mathcal{P}_0^{\text{DPHK}^*}$ holds for any $F, G \in \mathcal{P}_f^{\text{DPHK}^*}$.

Lemma 3.1. For any $I \in \mathcal{I}_f$, let $\mathcal{P}_0^{\text{DPHK}^*} \big|_I \stackrel{\text{def}}{=} \{F|_I \mid F \in \mathcal{P}_0^{\text{DPHK}^*}\}$. Then $\mathcal{P}_0^{\text{DPHK}^*} \big|_I$ is equivalent to \mathbb{R} .

Let $\{J_p\}$ be a set of open, closed, and half-open intervals with $J_p \subset D$. Suppose that $J_{p_1} \cap J_{p_2} = \emptyset$ if $p_1 \neq p_2$. Let \mathcal{G} be a set of functions from $\bigcup_p J_p$ into \mathbb{R} . Then $F \in \mathcal{G}$ is said to be piecewise unique except for constant differences if, for any $G \in \mathcal{G}$, there exists $c \in \mathbb{R}$ such that $F(x) - G(x) = c$ for any $x \in J_p$.

For any $f \in \mathcal{F}^*$ and for any $I_p \neq \emptyset$,

- let $J_p = [\inf I_p, \sup I_p]$ if there exist $z_1 > \inf I_p$ and $z_2 < \sup I_p$ such that $[\inf I_p, z_1], [z_2, \sup I_p] \in \mathcal{I}_f$;
- let $J_p = [\inf I_p, \sup I_p)$ if there exists $z_1 > \inf I_p$ such that $[\inf I_p, z_1] \in \mathcal{I}_f$ and $[z_2, \sup I_p] \notin \mathcal{I}_f$ for any $z_2 < \sup I_p$;
- let $J_p = (\inf I_p, \sup I_p]$ if there exists $z_2 < \sup I_p$ such that $[z_2, \sup I_p] \in \mathcal{I}_f$ and $[\inf I_p, z_1] \notin \mathcal{I}_f$ for any $z_1 > \inf I_p$;
- let $J_p = I_p$ if $[\inf I_p, z_1], [z_2, \sup I_p] \notin \mathcal{I}_f$ for any $z_1 > \inf I_p$ and for any $z_2 < \sup I_p$.

Note that the domain of any $F \in \mathcal{P}_f^{\text{DPHK}^*}$ is $\bigcup_p J_p$, $\{J_p\}$ satisfies $J_{p_1} \cap J_{p_2} = \emptyset$ if $p_1 \neq p_2$,

and for any $I \in \mathcal{I}_f$, there exists a unique p such that $I \subset J_p$. By Theorem 3.1 and Lemma 3.1, if the extended primitive F of f exists, then it is piecewise unique except for constant differences.

Theorem 3.2. A function f from D into \mathbb{R} is extended Denjoy-Perron-Henstock-Kurzweil integrable if and only if $T^*(f, [c, \cdot]) \in \mathcal{P}_f^{\text{DPHK}^*}$ for any $[c, \cdot] \in \mathcal{I}_f$.

Theorem 3.2 means that the extended primitive is equivalent to the extended Denjoy-Perron-Henstock-Kurzweil indefinite integral. Therefore, if the extended Denjoy-Perron-Henstock-Kurzweil indefinite integral $T^*(f, [c, \cdot])$ of f exists, then it is piecewise unique except for constant differences. Furthermore, the following holds.

Theorem 3.3. If f from D into \mathbb{R} is extended Denjoy-Perron-Henstock-Kurzweil integrable, then $T^*(f, I) = F(\sup I) - F(\inf I)$ holds for any $F \in \mathcal{P}_f^{\text{DPHK}^*}$ and for any $I \in \mathcal{I}_f$.

4. A TOPOLOGY OF \mathcal{F}^*

For any $f \in \mathcal{F}^*$ and for any $I \in \mathcal{I}_f$, define a mapping

$$p : \bigcup_{f \in \mathcal{F}^*} (\{f\} \times \mathcal{I}_f) \longrightarrow \mathbb{R}$$

by

$$p(f; I) \stackrel{\text{def}}{=} \sup_{J \subset I} |T^*(f, J)|.$$

Theorem 4.1. *The following hold:*

- (1) $p(f; I) \geq 0$; $p(0; I) = 0$;
- (2) For any $\lambda \in \mathbb{R}$, for any $f \in \mathcal{F}^*$, and for any $I \in \mathcal{I}_f$, $p(\lambda f; I) = |\lambda|p(f; I)$ holds;
- (3) For any $f \in \mathcal{F}^*$, for any $g \in \mathcal{F}_f^*$, and for any $I \in \mathcal{I}_{f+g}$, $p(f+g; I) \leq p(f; I) + p(g; I)$ holds, where

$$\mathcal{F}_f^* \stackrel{\text{def}}{=} \{g \mid g \in \mathcal{F}^*, \mathcal{I}_{f+g} = \mathcal{I}_f \cap \mathcal{I}_g\};$$

- (4) For any $f \in \mathcal{F}^*$ and for any adjacent $I_1, I_2 \in \mathcal{I}_f$, $p(f; I_1 \cup I_2) \leq p(f; I_1) + p(f; I_2)$ holds;
- (5) For any $f \in \mathcal{F}^*$ and for any $I_1, I_2 \in \mathcal{I}_f$, $I_1 \subset I_2$ implies $p(f; I_1) \leq p(f; I_2)$.

Furthermore, if T is the Denjoy-Perron-Henstock-Kurzweil integral, then the following hold:

- (1)' $p(f; I) = 0$ for any $I \in \mathcal{I}_f$ if and only if $f = 0$ a.e.;
- (3)' For any $\varepsilon \in (0, \infty)$, for any $f, g \in \mathcal{F}^*$, and for any $I \in \mathcal{I}_{f+g} \cap \mathcal{I}_{\text{compact}}$, there exists non-overlapping $\{I_k \mid I_k \subset I, k = 1, \dots, k_0\} \subset \mathcal{I}_f \cap \mathcal{I}_g$ such that

$$p(f+g; I) \leq \sum_{k=1}^{k_0} (p(f; I_k) + p(g; I_k)) + \varepsilon.$$

Proof. (1), (1)', (2), (3), (4), and (5) are clear. We show (3)'. We need to show only in the case of $I \notin \mathcal{I}_f \cap \mathcal{I}_g$.

Firstly, we show $I \cap \mathcal{N}_f = I \cap \mathcal{N}_g$.

In the case of $I \cap \mathcal{N}_f = \emptyset$, assume that $I \cap \mathcal{N}_g \neq \emptyset$. Let $z \in I \cap \mathcal{N}_g$. Then $[z_1, z] \notin \mathcal{I}_g$ holds for any $z_1 \in I$ with $z_1 < z$ or $[z, z_2] \notin \mathcal{I}_g$ holds for any $z_2 \in I$ with $z_2 > z$. Since $[z_1, z] \subset I$ holds if $[z_1, z] \neq \emptyset$, $[z, z_2] \subset I$ holds if $[z, z_2] \neq \emptyset$, and at least either $[z_1, z] \neq \emptyset$ or $[z, z_2] \neq \emptyset$ holds, $[z_1, z] \in \mathcal{I}_f$ or $[z, z_2] \in \mathcal{I}_f$ holds. Therefore, $[z_1, z] \notin \mathcal{I}_{f+g}$ or $[z, z_2] \notin \mathcal{I}_{f+g}$ holds. However, since $I \in \mathcal{I}_{f+g}$, $[z_1, z] \in \mathcal{I}_{f+g}$ or $[z, z_2] \in \mathcal{I}_{f+g}$ holds and it is a contradiction. Therefore, $I \cap \mathcal{N}_g = \emptyset$ holds and hence $I \cap \mathcal{N}_f = I \cap \mathcal{N}_g$.

In the case of $I \cap \mathcal{N}_g = \emptyset$, we show similarly.

In the case of $I \cap \mathcal{N}_f \neq \emptyset$ and $I \cap \mathcal{N}_g \neq \emptyset$, let $z \in I \cap \mathcal{N}_f$. Then $[z_1, z] \notin \mathcal{I}_f$ holds for any $z_1 \in I$ with $z_1 < z$ or $[z, z_2] \notin \mathcal{I}_f$ holds for any $z_2 \in I$ with $z_2 > z$. Assume that $z \notin I \cap \mathcal{I}_g$. Then there exists $z_1 \in I$ such that $z_1 < z$ and $[z_1, z] \in \mathcal{I}_g$, or there exists $z_2 \in I$ such that $z_2 > z$ and $[z, z_2] \in \mathcal{I}_g$. Therefore, $[z_1, z] \notin \mathcal{I}_{f+g}$ or $[z, z_2] \notin \mathcal{I}_{f+g}$ holds. However, since $I \in \mathcal{I}_{f+g}$, $[z_1, z] \in \mathcal{I}_{f+g}$ or $[z, z_2] \in \mathcal{I}_{f+g}$ holds and it is a contradiction. Therefore, $z \in I \cap \mathcal{I}_g$ holds and hence $I \cap \mathcal{N}_f \subset I \cap \mathcal{N}_g$. Similarly, we show $I \cap \mathcal{N}_g \subset I \cap \mathcal{N}_f$. Therefore, $I \cap \mathcal{N}_f = I \cap \mathcal{N}_g$ holds.

Next, since $T^*(f+g, [\min I, \cdot]) \in \mathbf{ACG}^*(I)$, there exists a sequence $\{E_p \mid p \in \mathbb{N}\}$ of measurable sets such that $\bigcup_{p=1}^{\infty} E_p = I$, and for any $p \in \mathbb{N}$ and for any $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that for any $\alpha_\ell, \beta_\ell \in E_p$ with $\alpha_\ell < \beta_\ell$,

$$\sum_{\ell} (\beta_\ell - \alpha_\ell) < \delta \text{ implies } \sum_{\ell} \sup_{J \subset [\alpha_\ell, \beta_\ell]} |T^*(f+g, J)| < \frac{\varepsilon}{2^p}.$$

Since $I \cap \mathcal{N}_f (= I \cap \mathcal{N}_g)$ is a compact null set, there exists a sequence $\{(\alpha_\ell, \beta_\ell) \mid \ell = 1, \dots, \ell_0\}$ of open intervals covering $I \cap \mathcal{N}_f (= I \cap \mathcal{N}_g)$ such that

$$\sum_{\ell=1}^{\ell_0} (\beta_\ell - \alpha_\ell) < \delta.$$

Since α_ℓ, β_ℓ are finite, we are able to assume $\alpha_\ell \in E_p$ if and only if $\beta_\ell \in E_p$. Let $I \setminus \bigcup_{\ell=1}^{\ell_0} (\alpha_\ell, \beta_\ell) = \bigcup_{k=1}^{k_0} I_k$. Then $\{I_k\}$ is a set of non-overlapping intervals. Let $J \subset I$. Since

$$\begin{aligned}
|T^*(f+g; J)| &\leq \sum_{k=1}^{k_0} |T^*(f+g, I_k \cap J)| \\
&\quad + \sum_{p=1}^{\infty} \sum_{\alpha_\ell, \beta_\ell \in E_p} |T^*(f+g; [\alpha_\ell, \beta_\ell] \cap J)| \\
&\leq \sum_{k=1}^{k_0} \left(\sup_{J \subset I_k} |T^*(f, J)| + \sup_{J \subset I_k} |T^*(g, J)| \right) \\
&\quad + \sum_{p=1}^{\infty} \sum_{\alpha_\ell, \beta_\ell \in E_p} \sup_{J \subset [\alpha_\ell, \beta_\ell]} |T^*(f+g; J)| \\
&\leq \sum_{k=1}^{k_0} (p(f; I_k) + p(g; I_k)) + \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} \\
&= \sum_{k=1}^{k_0} (p(f; I_k) + p(g; I_k)) + \varepsilon,
\end{aligned}$$

the following holds:

$$p(f+g; I) \leq \sum_{k=1}^{k_0} (p(f; I_k) + p(g; I_k)) + \varepsilon.$$

□

Theorem 4.2. For any $\varepsilon \in (0, \infty)$, define

$$U(\varepsilon) \stackrel{\text{def}}{=} \{g \mid g \in \mathcal{F}, p(g; D) < \varepsilon\}$$

and define

$$\mathcal{U}(0) \stackrel{\text{def}}{=} \{U(\varepsilon) \mid \varepsilon \in (0, \infty)\}.$$

Then $\mathcal{U}(0)$ is a fundamental open neighborhood system of 0. Therefore, defining

$$\mathcal{U}(f) \stackrel{\text{def}}{=} \{f + U(\varepsilon) \mid \varepsilon \in (0, \infty)\}$$

for each $f \in \mathcal{F}^*$, $\mathcal{U}(f)$ is a fundamental open neighborhood system of f .

Proof. We need to show the following:

- (1) $0 \in U(\varepsilon)$ for any $U(\varepsilon) \in \mathcal{U}(0)$;
- (2) For any $U(\varepsilon_1), U(\varepsilon_2) \in \mathcal{U}(0)$, there exists $U(\varepsilon_3) \in \mathcal{U}(0)$ such that $U(\varepsilon_3) \subset U(\varepsilon_1) \cap U(\varepsilon_2)$;
- (3) For any $U(\varepsilon_1) \in \mathcal{U}(0)$ and for any $g \in U(\varepsilon_1)$, there exists $U(\varepsilon_2) \in \mathcal{U}(0)$ such that $g + U(\varepsilon_2) \subset U(\varepsilon_1)$ holds.

(1) is clear.

To show (2), it is sufficient to let $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$.

We show (3). Since $g \in U(\varepsilon_1)$,

$$g \in \mathcal{F}, p(g; D) < \varepsilon_1$$

holds. Let $\varepsilon_2 \leq \varepsilon_1 - p(g; D)$. Let $h \in g + U(\varepsilon_2)$. Then

$$h \in \mathcal{F}, p(h; D) < \varepsilon_2.$$

Therefore, $g + h \in \mathcal{F}$ and

$$\begin{aligned} p(g + h; D) &\leq p(g; D) + p(h; D) \\ &< p(g; D) + \varepsilon_2 \\ &\leq \varepsilon_1. \end{aligned}$$

Therefore, $g + h \in U(\varepsilon_1)$ holds. \square

Theorem 4.3. \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ is Hausdorff.

Proof. We need to show the following: for any $f, g \in \mathcal{F}^*$, there exists $U(\varepsilon_1), U(\varepsilon_2) \in \mathcal{U}(0)$ such that $(f + U(\varepsilon_1)) \cap (g + U(\varepsilon_2)) = \emptyset$ holds.

In the case of $f - g \in \mathcal{F}$, let $\varepsilon_1 + \varepsilon_2 \leq p(f - g; D)$. Assume that there exists $h \in \mathcal{F}^*$ such that $h \in (f + U(\varepsilon_1)) \cap (g + U(\varepsilon_2))$. Then

$$\begin{aligned} p(f - g; D) &\leq p(f - h; D) + p(h - g; D) \\ &< \varepsilon_1 + \varepsilon_2 \\ &\leq p(f - g; D), \end{aligned}$$

and hence it contradicts. Therefore,

$$(f + U(\varepsilon_1)) \cap (g + U(\varepsilon_2)) = \emptyset$$

holds.

In the case of $f - g \in \mathcal{F}^* \setminus \mathcal{F}$, assume that there exists $h \in \mathcal{F}^*$ such that $h \in (f + U(\varepsilon_1)) \cap (g + U(\varepsilon_2))$. Then $h - f, h - g \in \mathcal{F}$. Since $u \in \mathcal{F}$ implies $-u \in \mathcal{F}$,

$$f - g = (f - h) + (h - g) \in \mathcal{F}$$

and hence it contradicts. Therefore,

$$(f + U(\varepsilon_1)) \cap (g + U(\varepsilon_2)) = \emptyset$$

holds. \square

Theorem 4.4. \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ satisfies the first axiom of countability.

Proof. As ε , it is sufficient to take a value that is $\frac{1}{n}$ for any $n \in \mathbb{N}$. \square

Theorem 4.5. $U(\varepsilon)$ is convex.

Proof. Let $g_1, g_2 \in U(\varepsilon)$, $\lambda \in [0, 1]$. Since

$$\begin{aligned} p((1 - \lambda)g_1 + \lambda g_2; D) &\leq (1 - \lambda)p(g_1; D) + \lambda p(g_2; D) \\ &< (1 - \lambda)\varepsilon + \lambda\varepsilon \\ &= \varepsilon, \end{aligned}$$

$(1 - \lambda)g_1 + \lambda g_2 \in U(\varepsilon)$ holds. \square

Theorem 4.6. \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ becomes a topological additive group.

Proof. We need to show the following:

- (1) For any $U(\varepsilon_1) \in \mathcal{U}(0)$, there exists $U(\varepsilon_2), U(\varepsilon_3) \in \mathcal{U}(0)$ such that $U(\varepsilon_2) + U(\varepsilon_3) \subset U(\varepsilon_1)$ holds;
- (2) For any $U(\varepsilon_1) \in \mathcal{U}(0)$, there exists $U(\varepsilon_2) \in \mathcal{U}(0)$ such that $-U(\varepsilon_2) \subset U(\varepsilon_1)$.

We show (1). Let $\varepsilon_2 + \varepsilon_3 \leq \varepsilon_1$. Let $g \in U(\varepsilon_2)$ and let $h \in U(\varepsilon_3)$. Then

$$\begin{aligned} g &\in \mathcal{F}, p(g; D) < \varepsilon_2, \\ h &\in \mathcal{F}, p(h; D) < \varepsilon_3. \end{aligned}$$

Therefore, $g + h \in \mathcal{F}$ and

$$\begin{aligned} p(g + h; D) &\leq p(g; D) + p(h; D) \\ &< \varepsilon_2 + \varepsilon_3 \\ &\leq \varepsilon_1, \end{aligned}$$

and hence $g + h \in U(\varepsilon_1)$ holds.

We show (2). Since

$$-U(\varepsilon) = U(\varepsilon)$$

for any $U(\varepsilon)$, (2) holds. \square

Theorem 4.7. \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ is metrizable.

Proof. By the Birkoff-Kakutani theorem [1, 4], a topological group is metrizable if and only if it is Hausdorff and satisfies the first axiom of countability. Therefore, by Theorems 4.3, 4.4, and 4.6, \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ is metrizable. \square

Example 4.1. Actually, let

$$d(f, g) \stackrel{\text{def}}{=} \begin{cases} \frac{p(f - g; D)}{1 + p(f - g; D)}, & \text{if } f - g \in \mathcal{F}, \\ 1, & \text{if } f - g \in \mathcal{F}^* \setminus \mathcal{F}. \end{cases}$$

If T is the Denjoy-Perron-Henstock-Kurzweil integral, then d is a metric on \mathcal{F}^* . $d(f, g) \geq 0$, $d(f, g) = 0$ if and only if $f = g$ a.e., and $d(f, g) = d(g, f)$ are clear. We show the triangle inequality. Let $f, g, h \in \mathcal{F}^*$. If $f - g \in \mathcal{F}$, then $f - h, h - g \in \mathcal{F}$ or $f - h, h - g \in \mathcal{F}^* \setminus \mathcal{F}$. In the case of $f - h, h - g \in \mathcal{F}$,

$$\begin{aligned} d(f, g) &= \frac{p(f - g; D)}{1 + p(f - g; D)} \\ &\leq \frac{p(f - h; D) + p(h - g; D)}{1 + p(f - h; D) + p(h - g; D)} \\ &\leq \frac{p(f - h; D)}{1 + p(f - h; D)} + \frac{p(h - g; D)}{1 + p(h - g; D)} \\ &= d(f, h) + d(h, g). \end{aligned}$$

In the case of $f - h, h - g \in \mathcal{F}^* \setminus \mathcal{F}$,

$$d(f, g) = \frac{p(f - g; D)}{1 + p(f - g; D)} < 1 < 1 + 1 = d(f, h) + d(h, g).$$

In the case of $f - g \in \mathcal{F}^* \setminus \mathcal{F}$, then $f - h \in \mathcal{F}^* \setminus \mathcal{F}$ or $h - g \in \mathcal{F}^* \setminus \mathcal{F}$. In the case of $f - h \in \mathcal{F}^* \setminus \mathcal{F}$ and $h - g \in \mathcal{F}$,

$$d(f, g) = 1 \leq 1 + \frac{p(h - g; D)}{1 + p(h - g; D)} = d(f, h) + d(h, g).$$

In the case of $f - h \in \mathcal{F}$ and $h - g \in \mathcal{F}^* \setminus \mathcal{F}$,

$$d(f, g) = 1 \leq \frac{p(f - h; D)}{1 + p(f - h; D)} + 1 = d(f, h) + d(h, g).$$

In the case of $f - h \in \mathcal{F}^* \setminus \mathcal{F}$ and $h - g \in \mathcal{F}^* \setminus \mathcal{F}$,

$$d(f, g) = 1 < 1 + 1 = d(f, h) + d(h, g).$$

Lastly, we consider the completeness of \mathcal{F}^* .

Theorem 4.8. *Let T be the Denjoy-Perron-Henstock-Kurzweil integral. Then \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ is incomplete.*

Proof. Let $D = [0, 1]$ and let $\mathbb{Q} \cap (0, 1] = \{r_m \mid m \in \mathbb{N}\}$. For any $m, n \in \mathbb{N}$, let

$$\varphi_{m,n}(x) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2^{m-n-1}r_m}, & \text{if } x \in \left[\left(\sum_{k=1}^{n-1} \frac{1}{2^k} \right) r_m, \left(\sum_{k=1}^{n-1} \frac{1}{2^k} + \frac{1}{2^{n+1}} \right) r_m \right), \\ -\frac{1}{2^{m-n-1}r_m}, & \text{if } x \in \left[\left(\sum_{k=1}^{n-1} \frac{1}{2^k} + \frac{1}{2^{n+1}} \right) r_m, \left(\sum_{k=1}^n \frac{1}{2^k} \right) r_m \right), \\ 0, & \text{otherwise,} \end{cases}$$

and for any $\ell \in \mathbb{N}$, let

$$f_\ell(x) \stackrel{\text{def}}{=} \sum_{m=1}^{\ell} \sum_{n=1}^{\ell} \varphi_{m,n}(x).$$

Then $f_\ell \in \mathcal{F} \subset \mathcal{F}^*$ for any $\ell \in \mathbb{N}$ and

$$p(f_{\ell_1} - f_{\ell_2}; D) \leq \sum_{\ell=\ell_1+1}^{\ell_2} \frac{1}{2^\ell}$$

for any $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_1 < \ell_2$, therefore, $\{f_\ell\}$ is a Cauchy sequence. Let

$$f_\infty(x) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{m,n}(x).$$

f_∞ is not Denjoy-Perron-Henstock-Kurzweil integrable on $[a, r_m]$ for any $m \in \mathbb{N}$ and for any $a < r_m$, that is, $\{r_m \mid m \in \mathbb{N}\} \subset \mathcal{N}_{f_\infty}$. Since, by Lemma 2.1, \mathcal{N}_{f_∞} is closed, $\mathcal{N}_{f_\infty} = [0, 1]$ holds. Therefore, $f_\infty \notin \mathcal{F}^*$ and hence \mathcal{F}^* is incomplete. \square

Theorem 4.9. *Let T be the Denjoy-Perron-Henstock-Kurzweil integral. Then the completion of \mathcal{F}^* whose topology is made from $\mathcal{U}(f)$ is $\mathcal{F}^* + \mathcal{A}_C$, where*

$$\mathcal{A}_C \stackrel{\text{def}}{=} \left\{ f \mid \begin{array}{l} f \in \mathcal{D}' \text{ and there exists } F \in C(D) \text{ such that} \\ F(\inf D) = 0, |F(\sup D)| < \infty \text{ and } \mathcal{D}F = f \end{array} \right\}$$

when \mathcal{D}' is the space of all Schwartz distributions, that is, the dual of space

$$\mathcal{D} \stackrel{\text{def}}{=} \left\{ \phi \mid \begin{array}{l} \phi \text{ is an infinitely differentiable mapping from } D \text{ into } \mathbb{R} \\ \text{and has a compact support} \end{array} \right\}$$

of all test functions, and \mathcal{D} is the distributional differentiation.

Proof. Let $\{f_n\}$ be a Cauchy sequence in \mathcal{F}^* . Then there exists $f \in \mathcal{F}^*$ such that $\{f_n - f\}$ is a Cauchy sequence in \mathcal{F} . By [14, Theorem 3], the completion of the space \mathcal{F} of all Denjoy-Perron-Henstock-Kurzweil integrable functions is \mathcal{A}_C . Therefore, the completion of \mathcal{F}^* is $\mathcal{F}^* + \mathcal{A}_C$. \square

REFERENCES

- [1] G. Birkoff, *A note on topological groups*, *Compositio Mathematica* **3** (1936), 427–430.
- [2] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, Providence, 1994.
- [3] R. Henstock, *The general theory of integration*, Clarendon Press, Oxford, 1991.
- [4] S. Kakutani, *Über die Metrisation der topologischen Gruppen.*, Proceedings of the Imperial Academy **12** (1936), 82–84.
- [5] T. Kawasaki, *Some integrals between the Lebesgue integral and the Denjoy integral*, *Scientiae Mathematicae Japonicae* **e-2016** (2016), 25 pages.
- [6] ———, *An extension of integrals*, Differential Geometry, Algebra, and Analysis (M. H. Shahid, M. Ashraf, F. Al-Solamy, Y. Kimura, and G. E. Vilcu, eds.), Springer Proceedings in Mathematics & Statistics, vol. 327, Springer, Singapore, 2020, pp. 147–171.
- [7] Y. Kubota, *Integration Theory (in Japanese)*, Maki, Tokyo, 1977.
- [8] S. Nakanishi (formerly S. Enomoto), *Sur une totalisation dans les espaces de plusieurs dimensions, I*, *Osaka Mathematical Journal* **7** (1955), 59–102.
- [9] ———, *Sur une totalisation dans les espaces de plusieurs dimensions, II*, *Osaka Mathematical Journal* **7** (1955), 157–178.
- [10] S. Nakanishi, *The Denjoy integrals defined as the completion of simple functions*, *Mathematica Japonica* **37** (1992), 89–101.
- [11] ———, *A new definition of the Denjoy's special integral by the method of successive approximation*, *Mathematica Japonica* **41** (1995), 217–230.
- [12] W. F. Pfeffer, *The Riemann approach to integration*, Cambridge University Press, Oxford, 1993.
- [13] S. Saks, *Theory of the integral*, Warsaw, 1937.
- [14] E. Talvila, *The distributional Denjoy integral*, *Real Analysis Exchange* **33** (2007/2008), 51–82.

Communicated by *Jun Kawabe*

FACULTY OF ENGINEERING, TAMAGAWA UNIVERSITY, TOKYO 194–8610, JAPAN; FACULTY OF SCIENCE,
CHIBA UNIVERSITY, CHIBA 263–8522, JAPAN

Email address: toshiharu.kawasaki@nifty.ne.jp

Continuity and computability: A framework for transition and extension

Mariko Yasugi, Yoshiki Tsujii

Received August 5, 2024 ; revised November 10, 2024

Subject Classification 03-08, 26-08, 03A99

Abstract

LW-computability, the effectivization of the general definition of continuity of real functions (called LW-continuity), demonstrates a feature different from that of the traditional computability, which is the effectivization of the classically equivalent local uniform continuity. LW-continuity is extensible to irrational based IB-continuity. LW-computability can be naturally extended to IB-computability, which is the theme of [3], and which is the effectivization of IB-continuity. The mathematical circumstances of those notions are first clarified. Then a common mechanism which explains appropriateness of the extension from LW to IB as well as that of the transition from continuity to computability is presented.

Keywords Computable sequences of numbers, continuity of real functions, computability of real functions, conjunctive schema, conjunctive transition, conjunctive extension

1 Introduction

The universe of discourse will be confined to real numbers, real number sequences and real functions throughout, and, by continuity of a function, we mean the continuity with respect to the Euclidean topology.

The conventional computability of a function is the effectivization of ‘locally uniform continuity’, whereas one could effectivize the general form of ‘continuity’ (without any modifier), although those two kinds of continuity are classically equivalent.

The continuity is here called ‘LW(location-wise)-continuity’, and its effectivization, called ‘LW-computability’, is first introduced. Crudely delineated, a function is LW-computable if it is computable on the domain of computable real sequences. The continuity relative to irrational numbers (IB-continuity) will then be defined. Its effectivization turns out to be IB-computability, whose properties have been broadly investigated in [3].

The main objective of this treatise is to present a common mechanism, which explains appropriateness of the extension from LW to IB as well as that of the

transition from continuity to computability, with the goal of justifying the theory of IB-computability.

The knowledge about continuous computable functions will be assumed. It has long been defined and generally agreed (cf. [1] for example). There are, however, many discontinuous functions of interest and importance in mathematics as well as in science and technology, which appeal to the human mind as computable ‘in some sense’. Extending the notion of computability to such functions had been a necessity. At the same time, computability in a wider sense must be a natural continuation of that of continuous functions.

We have thus studied computability of some real functions which are not necessarily continuous, following the functional space approach initiated by Pour-El and Richards ([1]). We have worked with several mathematical theories such as Fine spaces, uniform topological spaces and Fréchet spaces, endowed with ‘computability structures’. In its essence, some discontinuous functions are made computable by changing the topology of the real line.

While those background mathematical theories had been selected for respective discontinuous functions of interest, we also sought for a theory which encompasses all the functions we had so far treated. The type two Turing machine approach by Weihrauch ([2]) is one successful method to systematically deal with the computability of certain functions, discontinuous ones inclusive, but our stance is to stay within classical mathematics as much as possible, without resorting to matters outside. We then came up with the idea of IB-computability, and the foundational speculation on it is our present major concern.

The effect of basing the computability of functions on irrational numbers is exhibited in [3] and [4]. In passing, since a comprehensive list of relevant works is provided in [3], we do not detail them here.

This article is composed of six sections. The classical definitions of continuity are compared and the traditional computability of continuous functions is reviewed in Section 2. In Section 3, ‘almost everywhere continuity’ and its effectivization are surveyed for later reference.

LW-computability of a function is defined and some of its basic properties are contained in Section 4.1. In the same section, IB-continuity is introduced and the definition of IB-computability is quoted from [3]. While it is not intended to develop the mathematics of LW-computability, a sort of closure property of LW-computable functions is treated in Section 4.2.

Section 5 is the central part of this work. It is devoted to a search for a mechanism of the transfer from continuity to computability (Section 5.1) as well as from LW to IB (Section 5.2). It is attained systematically through a framework, called ‘conjunctive schema’, proposed in Definition 5.2. Continuity and computability of partial functions are sketched in Section 6. ¹

¹The exploratory study of this article was presented at CCA2023, Dubrovnik, Croatia, September 9, 2023. cf.

<http://cca-net.de/cca2023/abstracts/Yasugi-Tsujii.pdf>

2 Preliminaries: continuity and computability

Let us repeat here that the universe of our interest is the family of (totally defined) real functions.

There are standard definitions of continuity of a function with regards to Euclidean topology, which are listed here for use in due course.

Uniform continuity

$$(1) \quad \forall \varepsilon > 0 \exists \delta > 0. \forall x, y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(δ does not depend on the location.)

Locally uniform continuity

$$(2) \quad \forall N \forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in [-N, N]. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(δ depends on each positive integer N , which specifies an interval location, but is constant on $[-N, N]$.) For convenience, we abbreviate ‘locally uniform continuity’ to ‘LU-continuity’.

Continuity

$$(3) \quad \forall x \forall \varepsilon > 0 \exists \delta > 0 \forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(δ depends on the individual location.)

It is a common knowledge that LU-continuity and continuity are classically equivalent. However, when computability is involved, they manifest different phases.

Remark 1 1) In effectivizing a definition of continuity, it is a common practice to confine $\varepsilon (> 0)$ and $\delta (> 0)$ to rational numbers.

2) ε and δ can be further replaced respectively with $\frac{1}{2^p}$ and $\frac{1}{2^d}$, where $p, d = 1, 2, 3, \dots$. Consider (2) as an instance. It is classically equivalent to:

$$(4) \quad \forall N \forall p \exists d \forall x, y \in [-N, N]. |x - y| < \frac{1}{2^d} \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

The traditional computability is recapitulated below. It is the effectivization of LU-continuity as formulated in (4).

Definition 2.1 ((LU-)Computability: cf. [1]) A real function f is computable (LU-computable) if it satisfies (i) and (ii) below.

(i) (Sequential computability) For any computable sequence of real numbers, say $\{x_m\}$, the real number sequence $\{f(x_m)\}$ is computable.

(ii) (Effective continuity) There is a recursive function δ satisfying,

$$(5) \quad \forall N \forall p \forall x, y \in [-N, N]. |x - y| < \frac{1}{2^{\delta(N,p)}} \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

The definition can be easily extended to a sequence of functions $\{f_n\}$.

Remark 2 1) It is this definition of computability that has commonly been adopted, and, based on which, the theory of computable real functions has been developed (cf. [1]).

2) Effective continuity (ii) of Definition 2.1 trivially implies LU-continuity of f .

Definition 2.2 (LW-continuity) The condition (3) of continuity signifies that a modulus of continuity depends on the locations of individual points. This case will be here in this treatment called *location-wise continuity*, and abbreviated to LW-continuity.

Whereas IB-computability was brought up in [3] for its own sake, not as an effectivization of some kind of continuity, IB-continuity can be defined, whose effectivization turns out to be IB-computability.

Definition 2.3 (IB-continuity) Let \mathbf{I} denote the set of irrational numbers. A real function $f(x)$ is IB-continuous if it satisfies the following condition.

$$(6) \quad \forall x \in \mathbf{I} \forall \varepsilon > 0 \exists \delta > 0 \forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Proposition 2.1 LW-continuity implies IB-continuity.

Remark 3 The opposite direction does not hold. Thomae's function is a good counter-example (cf. [3]).

The effectivization of LW-/IB-continuity is discussed in Section 4.

3 Almost everywhere continuity and effectivization

There are many real functions which are not necessarily continuous but are 'continuous in an extended sense'. One class of such functions is identified as 'almost everywhere continuous', where LW-continuity and IB-continuity fall into this category. By developing the idea of almost everywhere continuity and its effectivization, done in this section, LW-computability and IB-computability can be systematically treated. It is also useful in exploring the transfer mechanism of transition from continuity to computability as well as of extending the scope of continuity and that of computability.

Let \mathbf{J} be a set of real numbers satisfying

$$(7) \quad \forall N. \mu(\mathbf{J} \cap [N, N + 1]) = 1,$$

where N ranges over all the integers and μ denotes the Lebesgue measure.

Let f be a real total function.

Definition 3.1 (\mathbf{J} -continuity) f is said to be \mathbf{J} -continuous if:

$$(8) \quad \forall x \in \mathbf{J} \forall \varepsilon > 0 \exists \delta > 0 \forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Proposition 3.1 1) **J**-continuity for the case $\mathbf{J} = \mathbf{R}$, \mathbf{R} being the set of all real numbers, is the standard continuity, or LW-continuity (cf. (3)). .

2) The condition on \mathbf{J} admits the case $\mathbf{J} = \mathbf{I}$. **J**-continuity is then IB-continuity (cf. (6)).

Example 1 1) Let $\{t_l\}$ be any number sequence, where $t_l < t_{l+1}$ for all integers l , and $\lim_{l \rightarrow -\infty} t_l$ and $\lim_{l \rightarrow \infty} t_l$ are both divergent. Put $\mathbf{J} = \mathbf{R} \setminus \{t_l\}$. Then any function which is continuous on the open intervals (t_l, t_{l+1}) is **J**-continuous.

2) (Counter-example) The function g in Example 8 of [3], which distinguishes rational numbers and irrational numbers, is not **I**-continuous.

An appropriate effectivization of **J**-continuity would be the following. \mathbf{C} will denote the set of all computable real numbers.

Definition 3.2 (**J**-computability: conceptual formulation) (i) (**J**-effective evaluation) For every $x \in \mathbf{J} \cap \mathbf{C}$, the value $f(x)$ is ‘computed’, and is hence computable.

(ii) (**J**-effective continuity) Given any $x \in \mathbf{J} \cap \mathbf{C}$ and every positive rational number r , a positive number δ is computed, so that

$$(9) \quad \forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < r.$$

Although this definition effectivizes **J**-continuity, it is in its form inconvenient for developing mathematics. As is the case of (LU-)computability in Definition 2.1, it has been a general practice to remold such a conceptual formulation into a mathematical form with the help of computable sequences of numbers and recursive functions (cf. [1]).

Definition 3.3 (**J**-computability: mathematical expression) f is said to be **J**-computable if, for every computable sequence from \mathbf{J} , say $\{x_m\}$, hold the following.

(i) (**J**-sequential computability) $\{f(x_m)\}$ is computable.

(ii) (**J**-effective continuity: **JE**-continuity) There is a recursive function $\delta(m, p)$, for which holds

$$(10) \quad \forall m, p \forall y. |x_m - y| < \frac{1}{2^{\delta(m, p)}} \Rightarrow |f(x_m) - f(y)| < \frac{1}{2^p}.$$

$\delta(m, p)$ is a **JE**-modulus of continuity for $(f \text{ and } \{x_m\})$.

The definition can be easily extended to a sequence of functions $\{f_n\}$: computability of the double sequence $\{f_n(x_m)\}$ is required in (i), and $\delta(m, p)$ is a recursive function of n, m, p in (ii).

Note The y in (8), (9) and (10) needs be neither in \mathbf{J} nor in \mathbf{C} .

Remark 4 The equivalence of Definitions 3.2 and 3.3 can be proved by returning to the definition of computable real numbers.

4 LW-computability and IB-computability

4.1 Basic properties

Definition 4.1 (LW-computability and IB-computability) The effectivization of LW-/IB-continuity can be defined by letting \mathbf{J} be respectively \mathbf{R} and \mathbf{I} in Definitions 3.2 and 3.3.

Remark 5 The mathematical expression of thus defined IB-computability coincides with the definition in [3].

Since LW-computability is the basis of the present treatise, its definition in two forms are explicitly stated below.

Definition 4.2 (LW-computability: conceptual formulation) A real function f is *location-wise computable* (LW-computable) if it satisfies the two conditions (i) and (ii).

(i) (Effective evaluation) Given any computable real number x , the value $f(x)$ is ‘computed’.

(ii) (Effective continuity) Given a computable real number x and a positive rational number r , a positive rational number δ is ‘computed’, so that

$$\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < r.$$

Definition 4.3 (LW-computability: mathematical expression) A real function f is *LW-computable* if the following conditions hold.

(i) (Sequential computability) For every computable sequence of real numbers $\{x_m\}$, $\{f(x_m)\}$ is a computable sequence of real numbers.

(ii) (LW-effective continuity) f is *location-wise effectively* (LWE-) continuous. Namely, for each computable sequence of real numbers $\{x_m\}$, there is a recursive function, say $\delta(m, p)$, satisfying

$$(11) \quad \forall m \forall p \forall y. |x_m - y| < \frac{1}{2^{\delta(m, p)}} \Rightarrow |f(x_m) - f(y)| < \frac{1}{2^p}.$$

Some basic facts about LW-computability are summarised in a proposition. (The properties and examples of IB-computable functions have been worked out in [3].)

Proposition 4.1

- 1) A computable (hence continuous) function is LW-computable.
- 2) An LW-computable function is IB-computable.
- 3) An LW-computable function is continuous at all computable numbers, hence, in particular, at rational numbers.
- 4) The functions f in Example 7 and θ in Example 13 of [3] are LW-computable.

f is an example of a continuous, not computable and LW-computable function. θ is an example of a discontinuous and LW-computable function.

5) Many of the examples of IB-computable functions in [3] are *not* LW-computable.

6) LW-computability is closed under composition, unlike the case of IB-computability (cf. (2) of Remark 3 in [3]).

7) The patching theorem (Theorem 2 in Chapter 0 of [1]) holds for LW-computable functions.

Proof

1) can be proved in a manner similar to the proof of Theorem 1 in [3].

2), 3) and 6) are immediate from the definitions.

4) LW-computability of f and that of θ can be shown by imitating the proofs in [3], considering any computable sequence $\{\gamma_k\}$ instead of just computable irrational sequences.

5) The floor function (Example 2), the functions in Example 4, Thomae's function (Example 5), probability distribution function G (Example 6) are not LW-computable, for they are discontinuous at some computable numbers. Fine computable functions (Example 3) are not necessarily LW-computable.

7) The proof in [1] goes through.

Remark 6 From the way it is defined, which respects the attribute of 'location-wise continuity', LW-computability can be regarded as a natural effectivization of continuity, and yet it covers some discontinuous functions. On the other hand, it does not contain most of the functions which have been made computable in our preceding works and which are IB-computable (cf. 5) of Proposition 4.1). Those are the functions that are discontinuous not only at some non-computable numbers, but also at some rational numbers, and still maintain some features of computability.

4.2 LW-computability: Closure property

LW-computability is not a big leap from (LU-)computability, since an LW-computable function may be discontinuous only at some noncomputable points. In fact, the next proposition holds.

Proposition 4.2 For an LW-computable function f , the set of discontinuous points of f is nowhere dense.

Proof Let r be any rational number. Since r is computable, f is continuous at r (cf. 3) of Proposition 4.1), so that there is a small interval around r on which f is continuous, and hence there is no discontinuity point of f in that interval.

Remark 7 Recall that LW-continuity is the standard continuity, hence it should be reasonable to deem LW-computability as the basis of various kinds of computability. The traditional computability is then its special case (cf. 1) of Proposition 4.1).

The following definition is in need when dealing with the effective limit of a function sequence. It is similar to IBAQ-computability in Definition 7 of [3].

Definition 4.4 (LWAQ-computability of function sequence) A function sequence $\{f_n\}$ is called *location-wise asymptotically equi-computable*, abbreviated to LWAQ-computable, if it satisfies 1° and 2° below.

Let $\{x_m\}$ be a computable real sequence.

1°. (Sequential computability) $\{f_n(x_m)\}$ is computable as a double sequence of real numbers.

2°. (LW-effective asymptotical equi-continuity: LWEAQ-continuity) There are recursive functions $\beta_1(m, p)$ and $\beta_2(m, p)$ such that

$$(12) \quad \forall n \geq \beta_1(m, p) \forall y. |x_m - y| < \frac{1}{2^{\beta_2(m, p)}} \Rightarrow |f_n(x_m) - f_n(y)| < \frac{1}{2^p}.$$

$\beta_2(m, p)$ may be called a recursive asymptotic modulus of continuity (for $\{f_n\}$ and) $\{x_m\}$.

Definition 4.5 (LWE-convergence) A function sequence $\{f_n\}$ will be said to *location-wise effectively converge* (LWE-converge) to f if the following condition holds.

For any computable $\{x_m\}$, there are recursive functions $\alpha_1(m, p)$ and $\alpha_2(m, p)$ such that

$$(13) \quad \forall n \geq \alpha_1(m, p) \forall y. |y - x_m| < \frac{1}{2^{\alpha_2(m, p)}} \Rightarrow |f_n(y) - f(y)| < \frac{1}{2^p}.$$

Theorem 1 (Closure property) Suppose $\{f_n\}$ is an LWAQ-computable sequence and LWE-converges to f . Then f is LW-computable.

Proof $\{x_m\}$ will denote any computable real sequence.

We have the following (a), (b) and (c) at our disposal.

(a) Existence of α_1 and α_2 (Definition 4.5)

(b) Computability of the double sequence $\{f_n(x_m)\}$ (Condition 1° in Definition 4.4)

(c) Existence of β_1 and β_2 (Condition 2° in Definition 4.4)

(i) (Sequential computability) From (a) with $y = x_m$, (b), and the closure property under the effective limit of computable real number sequences (Proposition 1 in Chapter 0 of [1]), follows the computability of $\{f(x_m)\}$.

(ii) (LWE-continuity) What is to be shown is: for every $\{x_m\}$, there is a recursive function $\gamma(m, p)$ satisfying,

$$|x_m - y| < \frac{1}{2^{\gamma(m, p)}} \Rightarrow |f(x_m) - f(y)| < \frac{1}{2^p}.$$

By (c), there are recursive β_1 and β_2 for which (12) holds, and, by (a), there are recursive α_1 and α_2 so that (13) holds. Put

$$\delta_1(m, p) = \max(\alpha_1(m, p), \beta_1(m, p)); \quad \delta_2(m, p) = \max(\alpha_2(m, p), \beta_2(m, p)),$$

and let $n \geq \delta_1(m, p)$. If $|x_m - y| < \frac{1}{2^{\delta_2(m, p)}}$, then the premises of (13) and (12) both hold. So, for $n = \delta_1(m, p)$, in particular,

$$\begin{aligned} |f(x_m) - f(y)| &\leq |f(x_m) - f_n(x_m)| + |f_n(x_m) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{2^p} = \frac{3}{2^p} < \frac{1}{2^{p-2}}, \end{aligned}$$

where the three evaluations are due to respectively (13) with $y = x_m$, (12) and (13). Now, $\gamma(m, p) = \delta_2(m, p + 2)$ will do.

Remark 8 1) In 2° of Definition 4.4, LWE-continuity is not assumed for $\{f_n\}$. It could be added as a premise, but since the condition 2° secures LWE-continuity of $\{f_n\}$ for all $n \geq \beta_1(m, p)$, we may view that Theorem 1 asserts the closure property of LW-computable sequences of functions.

2) Although it is not intended to develop mathematics of LW-computability, the closure property has been taken up as a distinguished topic, since the limit of a function sequence is an important subject in the study of mathematics on the continuum.

5 Conjunctive transfer relations

In order to indicate that the effectivizations of continuities and the extensions from LW to IB are not makeshift ideas, we present a contrivance for such transferring processes.

The general concept can be described as follows.

Definition 5.1 (Conjunctive transfer) A transfer from a theory to another is called ‘conjunctive’ if the two theories are characterized with a same formal law.

In particular, it will be called ‘conjunctive transition’ in the case of the transfer from continuity to computability, and ‘conjunctive extension’ if the transfer is the extension of the universe of a theory.

Remark 9 1) The term ‘conjunctive’ has been chosen since, in such a case, the two theories are connected with a common kernel, secured by a formal scheme, entailing that the transfer is regarded as natural or admissible.

2) The word ‘formal’ does not necessarily require symbolic expressions. What it signifies is being devoid of content.

Instead of making the above statement more exact, we work with our specific cases: the transition from continuity to computability and the extension from location-wise properties to irrational based ones.

We first set a common schema that will yield a law when symbols are specified.

Definition 5.2 (Conjunctive schema) Let $\Sigma(f, A)$ stand for the schema:

$$f : A[\Gamma] \mapsto^{\Xi} B[\Delta]; \Theta[A, B]$$

where the letters symbolize the following.

- f : a map from A to B ;
- A/B : one or more objects with the properties respectively Γ and Δ ;
- Ξ : a specified method to obtain B from A ;
- $\Theta[A, B]$: a formula concerning A and B .

$\Sigma(f, A)$ now reads “ f assigns B , which satisfies the requirement Δ , to A , which satisfies Γ , with the method Ξ so that $\Theta[A, B]$ holds,” and is called the conjunctive schema.

Remark 10 1) The schema is called ‘conjunctive’, since it connects one theory to another.

2) Ξ being void means that the method is not specified, or that any method is allowed.

5.1 Transition from continuity to computability

Computability of a function can be envisaged in its essence, aside from the preservation of computability of individual values, as an effectivization of continuity. Compare respectively the formula (2) and Definition 2.1, the formula (3) and Definition 4.2 (or Definition 3.2 with $\mathbf{J} = \mathbf{R}$), and Definitions 2.3 and 3.2 with $\mathbf{J} = \mathbf{I}$.

The mechanism of transition from continuity to computability is to be articulated in terms of \mathbf{J} -continuity and \mathbf{J} -computability (cf. Definitions 3.1 and 3.2) as a general principle.

For later use, let $S[x, r; \delta]$ stand for the formula

$$\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < r.$$

(cf. (8): ε can be restricted to an arbitrary positive rational number, here denoted by r .)

Let us analyze the state of a matter with regard to the \mathbf{J} -continuity of f . f being a totally defined and \mathbf{J} -continuous function necessitates that f assigns (a) a single value, denoted by $f(x)$, to every number x , and (b) a number $\delta = \delta_{x,r} > 0$ to every pair of $x \in \mathbf{J}$ and $r > 0$ so that the continuity requirement $S[x, r; \delta]$ holds at x . This scene may be depicted as a schema.

- (a) $f : x \mapsto f(x)$.
- (b) $f : x \in \mathbf{J}, r > 0 \mapsto \delta = \delta_{x,r} > 0 ; S[x, r; \delta]$.

The image of transition from this to \mathbf{J} -computability of f may be illustrated with (c) and (d) below, which indicate effectivizations of the correspondences \mapsto in (a) and (b). f being a (totally defined) function is presupposed.

Let \mathbf{C} denote the set of computable real numbers.

- (c) ‘ $f : x \in \mathbf{C} \cap \mathbf{J} \mapsto f(x)$ ’ is effective.
- (d) ‘ $f : x \in \mathbf{C} \cap \mathbf{J}, r > 0 \mapsto \delta = \delta_{x,r} > 0$ ’ is effective.

(c) expresses the fact that, from the information of a computable x , the value $f(x)$ can be ‘computed’, and hence $f(x)$ is a computable number. (d) can be interpreted likewise. The formula $S[x, r; \delta]$ in (b) remains unaltered in (d).

Proposition 5.1 (Transition mechanism: conceptual expression) The conjunctive schema of Definition 5.2 provides the formal law common to \mathbf{J} -continuity and \mathbf{J} -computability. In other words, the transition from \mathbf{J} -continuity to \mathbf{J} -computability conserves the essential structure.

Proof Compare the specifications of parameters in the conjunctive schema for (a) and for (c).

(a) $A \equiv x; B \equiv f(x); \Gamma, \Xi, \Delta, \Theta : \text{void}$

(c) $A \equiv x; B \equiv f(x); \Gamma : x \in \mathbf{C} \cap \mathbf{J}; \Xi : \text{effective}; \Delta, \Theta : \text{void}$

Next compare (b) and (d).

(b) $A \equiv x, r; \Gamma : x \in \mathbf{J}, r > 0 \text{ rational}; \Xi : \text{void}; B \equiv \delta_{x,r}; \Delta : \delta_{x,r} > 0; \Theta[A; B] \equiv S[x, r; \delta]$.

(d) $A \equiv x, r; \Gamma : x \in \mathbf{J} \cap \mathbf{C}, r > 0 \text{ rational}; \Xi : \text{effective}; B \equiv \delta_{x,r}; \Delta : \delta_{x,r} > 0; \Theta[A; B] \equiv S[x, r; \delta]$.

Theorem 2 (From continuity to computability: conjunctive transition) The transitions from LW-continuity to LW-computability and the transition from IB-continuity to IB-computability are conjunctive, hence admissible (cf. 1) of Remark 9).

Proof In Proposition 5.1, the transition is from LW-continuity to LW-computability if $\mathbf{J} = \mathbf{R}$, and is from IB-continuity to IB-computability if $\mathbf{J} = \mathbf{I}$.

In order to give a mathematical form to the conjunctive schema in Definition 5.2, we replace the conceptual word ‘effective’ with the device of sequential expressions as is the case of Definition 3.3.

Definition 5.3 (Transition mechanism: mathematical expression) The transformation from (a) and (b) to the following (i) and (ii) is regarded as the transition mechanism from continuity to computability, where \mathcal{R} denotes the family of all computable sequences from $\mathcal{C} \cap \mathbf{J}$ and \mathcal{B} denotes the family of all binary recursive functions.

(i) $f : \{x_m\} \in \mathcal{R} \mapsto \{f(x_m)\} \in \mathcal{R}$.

(ii) $f : \{x_m\} \in \mathcal{R} \mapsto \delta \in \mathcal{B}; (10)$,

where (10) is the formula in Definition 3.3.

Remark 11 1) In Definition 5.3, the metamathematical word ‘effective’ does not occur in reference to the correspondence sign \mapsto . The effectivities claimed in (c) and (d) are reflected in the conditions on the resulting objects; namely, $\in \mathcal{R}$ and $\in \mathcal{B}$, where \mathcal{R} and \mathcal{B} are mathematically defined objects, and \mapsto itself is a classical relation.

2) For the case of LU-continuity and its effectivization, i.e. the usual computability, compare the formulas (2) and (5). Now, the requisites corresponding to (b) and (d) stand as follows. Let S denote the formula

$$\forall x, y \in [-N, N]. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(b') $f : N, \varepsilon > 0 \mapsto \delta = \delta_{N, \varepsilon} > 0; S$.

(d') ' $f : N, \varepsilon > 0 \mapsto \delta = \delta_{N, \varepsilon} > 0$ ' is effective; S .

The effectivity in (d') entails the existence of a recursive function $\delta(N, p)$ (cf. 2) of Remark 1).

5.2 Conjunctive theory extension

We now turn to the extensions from LW-continuity to IB-continuity and from LW-computability to IB-computability.

Recall that LW-continuous functions are embedded in the family of IB-continuous functions (cf. Proposition 2.1). So, the latter is an extension of the former in this sense.

Theorem 3 (Conjunctive extension: continuity) IB-continuity is a conjunctive extension of LW-continuity.

Proof LW-continuity and IB-continuity both satisfy (a) and (b) as specified in the proof of Proposition 5.1 with $\mathbf{J} = \mathbf{R}$ and $\mathbf{J} = \mathbf{I}$ respectively. So, they have the identical conjunctive schema. That means the extension is conjunctive (Definition 5.1).

Next we come down to computability. Notice that the theory of IB-computable functions is an extension of that of LW-computable functions due to 2) of Proposition 4.1.

Theorem 4 (Conjunctive extension: computability) IB-computability is a conjunctive extension of LW-computability.

Proof LW-computability and IB-computability both satisfy (c) and (d) as specified in the proof of Proposition 5.1 with $\mathbf{J} = \mathbf{R}$ and $\mathbf{J} = \mathbf{I}$ respectively, and hence they obey the same formal law (conjunctive schema), certifying the conjunctive extension.

Remark 12 As for the mathematical expression of computability, the conjunctive property of the extension from LW-computability to IB-computability immediately follows from Definition 3.3, which produces LW-computability if $\mathbf{J} = \mathbf{R}$ and IB-computability if $\mathbf{J} = \mathbf{I}$. Namely, they satisfy the same laws (i) and (ii).

6 Note on partial functions

So far, all the functions have been assumed total. A function such as $\tan x$ is then artificially assigned some values at the points $\frac{2n+1}{2}\pi$, where the function is primarily undefined. Such a maneuver has been adopted in order to make the story simple. However, it is perhaps more revealing to deal with the original partial function as it is. For a \mathbf{J} having the property (7) in Section 3, \mathbf{J} -continuity and \mathbf{J} -computability of this kind of functions are succinctly looked at.

Definition 6.1 (**J**-partial continuity) Let f be a real function which is defined on and only on \mathbf{J} . f is said to be **J**-partially continuous if

(*) (**J**-continuity)

$$\forall x \in \mathbf{J} \forall \varepsilon > 0 \exists \delta > 0 \forall y \in \mathbf{J}. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Example 2 1) The definition above amounts to LW-continuity if $\mathbf{J} = \mathbf{R}$.

2) With $\mathbf{J} = \mathbf{R} \setminus \{\frac{2n+1}{2}\pi : n = 0, \pm 1, \pm 2 \dots\}$, $f(x) = \tan x$ is **J**-partially continuous.

3) If $\mathbf{J} = \mathbf{R} \setminus \{0\}$, then $f(x) = \log |x|$ and $f(x) = \frac{1}{x}$ are **J**-partially continuous.

4) Let g be a partial function defined only on the domain $\mathbf{J} = \mathbf{I}$, whose value $g(x)$ is constant 0 on \mathbf{I} . Then g is **J**-partially continuous. (Compare this fact with 2) of Example 1.)

An appropriate effectivization of Definition 6.1 should be the following.

Definition 6.2 (**J**-partial computability: mathematical expression) Let f be a partial function defined invariably on \mathbf{J} . f is said to be **J**-partially computable if, for every computable sequence $\{x_m\}$ from \mathbf{J} , (i) and (ii) hold.

(i) (Sequential computability) $\{f(x_m)\}$ is computable.

(ii) (**J**-partial effective continuity) There is a recursive function $\delta(m, p)$, for which holds:

$$\forall m, p \forall y \in \mathbf{J}. |x_m - y| < \frac{1}{2^{\delta(m, p)}} \Rightarrow |f(x_m) - f(y)| < \frac{1}{2^p}.$$

Example 3 1) Definition 6.2 with $\mathbf{J} = \mathbf{R}$ claims LW-computability.

2) The functions in 2), 3) and 4) of Example 2 are **J**-partially computable.

References

- [1] Marian B. Pour-El and Jonathan I. Richards, “Computability in Analysis and Physics”, Springer-Verlag, 1989: Available at <http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&page=toc&handle=euclid.pl/1235422916>
- [2] Klaus Weihrauch, “Computable Analysis”, Springer-Verlag, 2000.
- [3] Mariko Yasugi, Yoshiki Tsujii, Takakazu Mori, *Irrational-based computability of functions*, Advances in Mathematical Logic: Dedicated to the Memory of Professor Gaisi Takeuti, SAML 2018, Kobe, Japan, September 2018, Selected, Revised Contributions, Springer Proceedings in Mathematics & Statistics (2021), 181-204. Available also at <https://independent.academia.edu/MarikoYasugi>
- [4] Mariko Yasugi, Yoshiki Tsujii, Takakazu Mori, *Applications of recursive continued fractions and extended transfer principle*, SMJ(Scientiae Mathematicae Japonica), e-2024; Whole Number 36 (2023). available at <https://www.jams.jp/notice/scmjol/2024.html>

Communicated by *Yasunao Hattori*

On the maximum of Cramér's V

ETSUO HAMADA

Received January 18, 2025

ABSTRACT. The Cramér's V is popular as an association coefficient in goodness-of-fit tests for contingency tables and its maximum value is known to be 1, but it is not true. We propose a modified Cramér's V.

1 Cramér's V The Cramér's V is popular as an association coefficient in goodness-of-fit tests for contingency tables. For a contingency table

Table 1: Contingency table

	B_1	\cdots	B_j	\cdots	B_c	sum
A_1	x_{11}	\cdots	x_{1j}	\cdots	x_{1c}	$x_{1\cdot}$
\vdots	\vdots		\vdots		\vdots	\vdots
A_i	x_{i1}	\cdots	x_{ij}	\cdots	x_{ic}	$x_{i\cdot}$
\vdots	\vdots		\vdots		\vdots	\vdots
A_r	x_{r1}	\cdots	x_{rj}	\cdots	x_{rc}	$x_{r\cdot}$
sum	$x_{\cdot 1}$	\cdots	$x_{\cdot j}$	\cdots	$x_{\cdot c}$	n

the definition of the Cramér's V is

$$(1.1) \quad V = \sqrt{\frac{\chi^2}{n \min(c-1, r-1)}}$$

where c is the number of columns, r is the number of rows, and χ^2 is the chi-square statistic of the contingency table as follows:

$$(1.2) \quad \chi^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(x_{ij} - e_{ij})^2}{e_{ij}},$$

where e_{ij} is the expectation with respect to the observation x_{ij} . As the probability version for (1.2), [2] wrote, in page 282, that

On the other hand, by means of the inequalities $p_{ij} \leq p_{i\cdot}$ and $p_{ij} \leq p_{\cdot j}$ it follows from the last expression that $\varphi^2 \leq q - 1$, where $q = \min(r, c)$ denotes the smaller of the numbers r and c , or their common value if both are equal.

Note that symbols, etc. above are adapted to the format of this paper and the mean square contingency is

$$(1.3) \quad \varphi^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(p_{ij} - p_{i\cdot} p_{\cdot j})^2}{p_{i\cdot} p_{\cdot j}},$$

2010 Mathematics Subject Classification. subject classifications .
 Key words and phrases. Cramér's V.

PROPOSITION 1.1 ([2])

$$0 \leq \frac{\varphi^2}{q-1} = \frac{\varphi^2}{\min(r,c)-1} \leq 1.$$

The Cramér's V has since been defined as (1.1) whose maximum value has been used as 1. For example, see [1], [4], and [5].

2 The maximum value of V Since Cramér introduced the contingency coefficient V , its maximum value has been recognized as $n(\min(r,c)-1)$, as in Proposition 1.1. However, we recognize that this is a mistake and that the correct value is $n(rc-1)$.

THEOREM 2.1 *For the chi² statistic (1.2) of the contingency table, its maximum value is as follows:*

$$\max \chi^2 = n(rc-1).$$

LEMMA 2.1 *For the mean square contingency (1.3) of the contingency table, its maximum value is as follows:*

$$\max \varphi^2 = rc-1.$$

We first prove the Lemma. The proof of the theorem can be derived naturally from the result.

(Proof of Lemma 2.1) By using the relations $p_{ij} \leq p_i$ and $p_{ij} \leq p_j$ that [2] showed, it holds that

$$\begin{aligned} \varphi^2 &= \sum_{i=1}^r \sum_{j=1}^c \frac{(p_{ij} - p_i \cdot p_j)^2}{p_i \cdot p_j} \\ &= \sum_{i=1}^r \sum_{j=1}^c \frac{p_{ij}^2}{p_i \cdot p_j} - 2 \sum_{i=1}^r \sum_{j=1}^c p_{ij} + \sum_{i=1}^r \sum_{j=1}^c p_i \cdot p_j \\ &= \sum_{i=1}^r \sum_{j=1}^c \frac{p_{ij}^2}{p_i \cdot p_j} - 1 \\ &\leq \sum_{i=1}^r \sum_{j=1}^c \frac{p_i \cdot p_j}{p_i \cdot p_j} - 1 = \left(\sum_{i=1}^r \sum_{j=1}^c 1 \right) - 1 = rc - 1. \end{aligned}$$

□

Thus we propose a modified Cramér's V as follows:

$$(2.4) \quad \text{modified } V = \sqrt{\frac{\chi^2}{n(cr-1)}}$$

3 Simulation For two contingency tables, 2×2 and 3×3 , for the total number $n = 200$, we randomly generate the number of cells. We assume that the probabilities of the cells are all equal and that the number of simulations is 1000.

Table 2: simulation results for Cramér's V and a modified Cramér's V (the number of data is 200 and the number of simulations is 1000.)

	2×2 contingency table		3×3 contingency table	
	V	modified V	V	modified V
Min.	0.1356	0.0783	0.4462	0.2231
1st Qu.	0.6719	0.3879	0.9977	0.4989
Median	0.9178	0.5299	1.2328	0.6164
Mean	0.9607	0.5547	1.272	0.636
3rd Qu.	1.2409	0.7164	1.5263	0.7632
Max.	1.7321	1	2	1

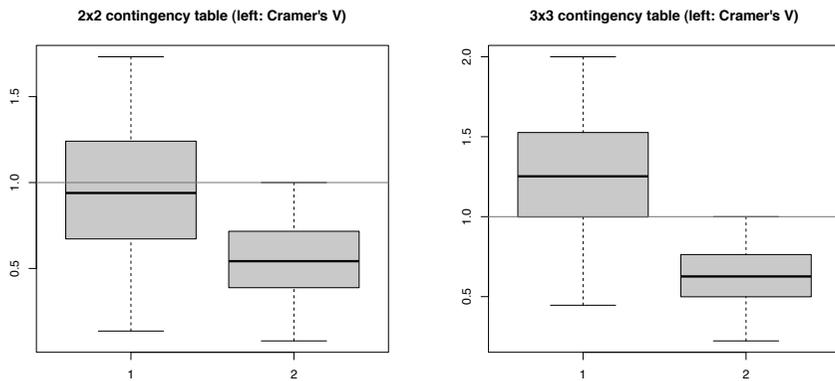


Figure 1: simulation results for Cramér's V and a modified Cramér's V (the number of data is 200 and the number of simulations is 1000.)

4 Conclusion The proof and simulation results regarding the maximum clearly show that we need to modify the Cramér's V. The previous contingency coefficients must be modified by using a modified Cramér's V with the maximum value 1.

Acknowledgment. This paper was partially supported by Grant-in-Aid for Scientific Research (C) (general) 22K11946 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

REFERENCES

- [1] Haldun Akoglu, User's guide to correlation coefficients, *Turkish Journal of Emergency Medicine*, **18**, 91–93, 2018.
- [2] Harald Cramér, *Mathematical methods of statistics*, Princeton University Press, 1946.
- [3] Etsuo Hamada(2025), On the maximum of Cramér's V, *arXiv:2501.12411*, 18 Jan 2025.
- [4] Charles C. Okeke, Alternative methods of solving biasedness in Chi-square contingency table, *Academic Journal of Applied Mathematical Sciences*, **5**(1), 1–6, 2019.
- [5] Wataru Urasaki, Tomoyuki Nakagawa, Tomotaka Momozaki, and Sadao Tomizawa, Generalized Cramér's coefficient via f -divergence for contingency tables, *Advances in Data Analysis and Classification*, **18**, 893–910, 2024.

A SPECIAL CLASS OF BCK-ALGEBRAS WITH CONDITION (S)

Dedicated to the late professor Kiyoshi Iséki (1919-2011)

Received July 7, 2025 ; revised August 6, 2025

Masaaki Kumazawa

ABSTRACT. In this article, we will study two problems. First, to characterize the existence of the least upper bound for two elements in a BCK-algebra. Secondly, to find a new class of upper semilattices in BCK-algebras. Two characterizations for this problem have already been given by K.Iséki [6] and W.H.Cornish [2], and we will show the relationship between these two classes and the new class that we have obtained.

1 Introduction

A lattice is a representative algebra with a partially ordered structure. In a lattice, two binary operations are defined. One is the "meet" \cap , which is an operation that gives the greatest lower bound of any two elements in the lattice. And the other is the "join" \cup , which is an operation that gives the least upper bound of any two elements in the lattice.

On the other hand, algebras derived from various logics as known. Boolean algebra is a Lindenbaum-Tarski algebra for classical propositional logic. Heyting algebra is a Lindenbaum-Tarski algebra for intuitionistic propositional logic. Furthermore, Kleene algebra is considered to be an algebraization of fuzzy logic. All three of these algebras are special lattices. Many algebras that are so-called as logical algebra have structures of lattice.

In 1966 [3], Kiyoshi Iséki introduced BCK-algebras with intention of algebraizing the BCK logic (Meredith's System B-C-K).

Definition 1.1 (BCK-algebra). An algebra $X = \langle X; *, 0 \rangle$ of type $\langle 2, 0 \rangle$ satisfying the following five conditions is called a *BCK-algebra* :

For any x, y, z in X ,

BCK 1. $\{(x * y) * (x * z)\} * (z * y) = 0,$

BCK 2. $\{x * (x * y)\} * y = 0,$

BCK 3. $x * x = 0,$

BCK 4. $0 * x = 0,$

BCK 5. $x * y = 0$ and $y * x = 0$ imply $x = y.$

2000 Mathematics Subject Classification. 03G25, 06A12, 06F35.

Keywords and phrases. BCK-algebra with condition (S), upper semilattice, Cornish's identity, BCK-algebra with a supremum, the condition (Sp) $_{x,y}$, BCK-algebra with Condition (Sp).

At this time, a relation \leq is defined in a BCK-algebra X using the operation $*$ as follows.

We denote $x \leq y$ if and only if $x * y = 0$ for any x, y in X .

It is known that the relation \leq is an partially order in a BCK-algebra X . However, unlike algebras based on various logics, any BCK-algebra does not have a structure of lattice. In other words, a greatest lower or least upper bound does not necessarily exist for any two elements in a BCK-algebra. So, what conditions must be satisfies for a greatest lower or least upper bound to exist in a BCK-algebra ? We would like to find these conditions.

The existence of a greatest lower bound in a BCK-algebra

The first problem concerns the existence of a greatest lower bound in a BCK-algebra. In 1975 [11], Shōtarō Tanaka, a collaborator of K. Isēki, introduced the condition of "commutativity" to BCK-algebras. This condition was imposed in order to give BCK-algebras the properties of natural correct inference in propositional logic.

Definition 1.2 (commutative BCK-algebra) In a BCK-algebra X , we denote $y * (y * x)$ as $x \wedge y$ for two elements x, y in X . If $x \wedge y = y \wedge x$ holds for any two elements x, y in X , then X is called to be *commutative*.

Theorem 1.1 (S. Tanaka [11]) *Any commutative BCK-algebra $X = \langle X; *, 0 \rangle$ is a lower semilattice with respect to the operation \wedge in X . That is, $x \wedge y$ is the greatest lower bound of x and y in X .*

Inspired by this result, we wanted to investigate the relationship between the following two points : " two elements are algebraically commutative" and "there exists a greatest lower bound for two elements as an ordered structure" in a BCK-algebra. In this regard, we obtained the following result by defining "the condition $(I)_{x,y}$ ", which is an expansion of the existence of a greatest lower bound for two elements in a BCK-algebra (see, M. Kumazawa [8], [9]).

Definition 1.3 (the condition $(I)_{x,y}$ in a BCK-algebra) For two elements x, y in a BCK-algebra X , z exists in X , we say that z satisfies the condition $(I)_{x,y}$ when z satisfies the following condition (i)~(iii).

- (i) $z \leq x, z \leq y,$
- (ii) $x * z \leq x * y,$
- (iii) $y * z \leq y * x.$

Based on this concept, we defined a new class in BCK-algebras.

Definition 1.4 (BCK-algebra with Condition (I)) In a BCK-algebra X , let the set $I(x, y) = \{z \in X \mid z \text{ satisfies the condition } (I)_{x,y}\}$. In this case, if the set $I(x, y)$ satisfies the following two conditions (1), (2) for any x, y in X ;

- (1) $I(x, y) \neq \phi,$
- (2) There exists a maximum element in $I(x, y)$ with the respect to \leq .

Then, X is called to be *BCK-algebra with Condition (I)*.

This definition gave us a lower semilattice in the BCK-algebras.

Theorem 1.2 (M. Kumazawa [9])

Let X be a BCK-algebra with Condition (I). For any two elements x, y in X , let m be a maximum element in $I(x, y)$, and denote $m = x \times y$. Then, an algebra $\langle X; \times, 0 \rangle$ is a lower semilattice with respect to the binary operation \times .

Furthermore, the commutative BCK-algebras are a special class of BCK-algebras with Condition (I), i.e., in a commutative BCK-algebra, $x \wedge y = x \times y$ always holds. The class of BCK-algebras with Condition (I) is the largest class of BCK-algebras that can be regarded as a lower semilattice currently known.

2 Upper semilattices in BCK-algebras

Next, we consider the problem that the existence of a least upper bound for any two elements in a BCK-algebra. Regarding this problem, K. Iséki and William Hugh Cornish each presented conditions under which a BCK-algebra can be thought about an upper semilattice.

First, we will give two type definitions of upper semilattices.

Definition 2.1 (Upper Semilattice) A set L is a partially ordered set $\langle L; \leq \rangle$, and we assume that there exists a least common upper bound z of any two elements x, y in L . At this time, a binary operation \cup is defined by $z = x \cup y$. In this case, a set L with the binary operation \cup is called an upper semilattice $L = \langle L; \cup \rangle$.

An upper semilattice can also be defined algebraically as follows.

Definition 2.2 (Upper Semilattice : G. Birkhoff [1]) An algebra $L = \langle L; \cup \rangle$ with a binary operation \cup satisfying the following three conditions is called an upper semilattice :

For any x, y, z in X ,

- UL 1. (Idempotent)** $x \cup x = x,$
- UL 2. (Commutative)** $x \cup y = y \cup x,$
- UL 3. (Associative)** $x \cup (y \cup z) = (x \cup y) \cup z.$

In this algebra $\langle L; \cup \rangle$, a partially order \leq is defined by using the operation \cup as follows.

We denote $x \leq y$ if and only if $x \cup y = y$.

Next, we would like to introduce a special class of BCK-algebras that can be viewed as upper semilattice. The binary operation $*$ defined in a BCK-algebra X has the following fundamental property that is not found in other algebras.

For any two elements x, y in X then $x * y \leq x$

In other words, in a BCK-algebra, when you apply the operation $*$ to any element x and any element y , the product $x * y$ becomes smaller than the original element x . However, if it is a common upper bound on two elements x, y in X , it must at least be an upper bound on x . Taking the above characteristics into consideration, K.Iséki introduced the following "condition (S)" in 1977 [5].

Definition 2.3 (BCK-algebra with condition (S)) A BCK-algebra X is said to be *with condition (S)*, if the set

$$S(x, y) = \{z \in X \mid z * x \leq y\}$$

has a greatest element for any x, y in X . And, its element denoted by $x \circ y$.

Here, we define an important condition in logical algebra, which is constructed by algebraizing "implication" in propositional logic.

Definition 2.4 (positive implicative : H. Rasiowa [10], K. Iséki [4])

Let X be a BCK-algebra. For any x, y, z in X , if the equality

$$(x * z) * (y * z) = (x * y) * z$$

holds, then X is said to be *positive implicative*.

Under the above conditions, K. Iséki obtained the following result.

Theorem A (K. Iséki [6]) *Let X be a positive implicative BCK-algebra with condition (S). Then an algebra $\langle X; \circ, 0 \rangle$ is an upper semilattice with the respect to the binary operation \circ , and the partially order \leq .*

This result gives us a class of BCK-algebras that can be regarded as upper semilattice. We call "Class A" for this class. Inspired by this result, in 1982 [2], W.H.Cornish gave the following the Definition 2.6 as a special class of BCK-algebras with condition (S).

Definition 2.5 (Cornish's identity) Let X be a BCK-algebra with condition (S). If two elements x and y in X satisfy the following identity :

$$x \circ (y * x) = y \circ (x * y),$$

then that is said to *the Cornish's identity* holds for x and y .

Definition 2.6 (BCK-algebra with a supremum) Let X be a BCK-algebra with condition (S). If the Cornish's identity holds for any two elements x and y in X , then X is said to be *with a supremum*.

In this algebra X , for any x, y in X , let z be $x \circ (y * x)$. In this case, z is uniquely determined for any x, y in X . So, we define a binary operation \vee as $x \vee y = z$. Under this assumption, W. H. Cornish proved the following theorem.

Theorem B (W. H. Cornish [2]) *Let X be a BCK-algebra with a supremum. Then an algebra $\langle X; \vee, 0 \rangle$ is an upper semilattice with the respect to the binary operation \vee and the partially order \leq .*

As this result, W. H. Cornish looked out a class that can be regarded as upper semilattice in BCK-algebras, and be different from the class found by K. Iséki. We call "Class B" for this class.

Problem In the BCK-algebras, K.Iséki and W.H.Cornish each found classes that can be viewed as upper semilattices. In this paper, we call these special classes "Class A" and "Class B", respectively. So what is the relationship between these two classes ? We want to examine the inclusion relationship between these classes. Further, are there any classes of BCK-algebras that can be viewed as upper semilattice larger than these classes ?

3 Basic properties of the condition $(S_p)_{x,y}$ in a BCK-algebra

In this section, we define a special class of BCK-algebras with condition (S). And, we will show that this class can also be viewed as upper semilattice.

First, for x, y in a BCK-algebra X , we define the "condition $(Sp)_{x,y}$ ".

Condition $(Sp)_{x,y}$ For x, y in a BCK-algebra X , z exists in X , we say that z satisfies the condition $(Sp)_{x,y}$ when z satisfies the following conditions (i), (ii), and (iii) ;

- (i) $x \leq z, y \leq z,$
- (ii) $z * x \leq y * x,$
- (iii) $z * y \leq x * y.$

Below, we would like to state the basic properties of the condition $(Sp)_{x,y}$. But as a preliminary step, we will first introduce the Yutani's identity (see [12]), which plays an important role in proving these properties.

Theorem 3.1(Yutani's identity : [12]) *In a BCK-algebra $X = \langle X; *, 0 \rangle$, the condition given next is necessary and sufficient for X that to be with condition (S). The condition : The newly defined a binary operation \circ in X satisfies the equation $(x * y) * z = x * (y \circ z)$ for any three elements x, y, z in X .*

The following Proposition 3.1 shows that the condition $(Sp)_{x,y}$ is an extension of the Cornish's identity.

Proposition 3.1 *Let X be a BCK-algebra with condition (S). For x, y in X , assume that the Cornish's identity hold. Then, the product $x \circ (y * x)$ satisfies the condition $(Sp)_{x,y}$.*

Proof We will show in order that the condition (i), (ii), and (iii) hold. We denote that $z = x \circ (y * x) = y \circ (x * y)$. By the Yutani's identity,

$$x * z = x * \{x \circ (y * x)\} = (x * x) * (y * x) = 0.$$

Then, $x \leq z$. Similarly, by $z = y \circ (x * y)$ implies $y \leq z$. The condition (i) holds.

Next,

$$(z * x) * (y * x) = z * \{x \circ (y * x)\} = z * z = 0.$$

Then, the condition (ii) holds. Similarly, we will show the condition (iii).

□

The next Proposition 3.2 plays an important role for the proof of the Proposition 3.3.

Proposition 3.2 *Let X be a BCK-algebra with condition (S). The following inequalities (1) and (2) are equivalent.*

- (1) $z * x \leq y * x,$
- (2) $x \circ (y * x) \geq z.$

Proof First, we will derive inequality (2) from inequality (1). Even if we apply the operation \circ to the both side of an inequality, the partially order \leq is preserved in X . From the assumption $z * x \leq y * x$ implies,

$$(z * x) \circ x \leq (y * x) \circ x = x \circ (y * x). \quad (3.1)$$

On the other hand, by BCK. 2, the inequality $z * (z * x) \leq x$ holds. From this and the definition of the operation \circ , implies,

$$z \leq (z * x) \circ x. \quad (3.2)$$

Then, by (3.1) and (3.2), we have $x \circ (y * x) \geq z$.

Secondly, we will derive inequality (1) from inequality (2). We denote that $z' = x \circ (y * x)$. By the definition of the operation \circ ,

$$z' * x \leq y * x \quad (3.3)$$

On the other hand, by the assumption $x \circ (y * x) = z' \geq z$ implies

$$z * x \leq z' * x \quad (3.4)$$

From (3.3) and (3.4), then we have that $z * x \leq y * x$.

□

Proposition 3.3 *Let X be a BCK-algebra with condition (S). If z is the supremum of two elements x and y in X , then z satisfies the condition $(\text{Sp})_{x,y}$ in X .*

Proof First, since z is the supremum of x and y in X , the condition (i) is clear. Secondly, we will show that the product $x \circ (y * x)$ is a common upper bound of x and y in X . By the Yutani's identity, imply the next equations ;

$$\begin{aligned} x * \{x \circ (y * x)\} &= (x * x) * (y * x) = 0, \\ y * \{x \circ (y * x)\} &= (y * x) * (y * x) = 0. \end{aligned}$$

Then, two inequalities $x \leq x \circ (y * x)$, $y \leq x \circ (y * x)$ hold. Hence, $x \circ (y * x)$ is a common upper bound of x and y in X .

Similarly, two inequalities $x \leq y \circ (x * y)$, $y \leq y \circ (x * y)$ hold. That is, $y \circ (x * y)$ is also a common upper bound of x and y in X .

Further, since the supremum z is the smallest upper bound of x and y in X , it follows the next inequality from Proposition 3.2.

$$z \leq x \circ (y * x) \implies z * x \leq y * x.$$

Hence, the condition (ii) holds. Similarly, the next also follows.

$$z \leq y \circ (x * y) \implies z * y \leq x * y.$$

Therefore, the condition (iii) holds.

□

The Proposition 3.3 shows that satisfying the condition $(\text{Sp})_{x,y}$ is an extended notion of the existence of a common least upper bound for two elements x and y in a BCK-algebra X .

Next, we want to give a characterization of the existence of a common least upper bound for two elements x and y in X by using the condition $(\text{Sp})_{x,y}$. Here, we define a set $\text{Sp}(x, y)$. In a BCK-algebra X , let the set

$$\text{Sp}(x, y) = \{z \in X \mid z \text{ satisfies the condition } (\text{Sp})_{x,y}\}.$$

First, we show that the next lemma holds.

Lemma 3.4 *Let X be a BCK-algebra. For two elements x and y in X , suppose that there exists a smallest element in the set $\text{Sp}(x, y)$. Then, the common least upper bound of x and y exists in X .*

Proof Let m be the smallest element in $\text{Sp}(x, y)$. For two elements x, y in X , suppose that there is no common least upper bound for x and y in X . Then, there must be two distinct minimal elements u, u' in X which are common upper bounds for x and y . This implies the next inequalities :

$$x \leq u, y \leq u, x \leq u', y \leq u', u \neq u'. \quad (3.5)$$

Here, u and u' are minimal elements in the common upper bound of x and y , and m is also one of the common upper bound of x and y , then, the next holds;

$$u \leq m, u' \leq m. \quad (3.6)$$

From (3.6) and $m \in \text{Sp}(x, y)$, imply

$$u * x \leq m * x \leq y * x, u * y \leq m * y \leq x * y. \quad (3.7)$$

$$u' * x \leq m * x \leq y * x, u' * y \leq m * y \leq x * y. \quad (3.8)$$

By (3.5), (3.7), imply $u \in \text{Sp}(x, y)$. And, by (3.5), (3.8), imply $u' \in \text{Sp}(x, y)$.

Then, these lead to the next following ;

$$u, u' \in \text{Sp}(x, y), u \leq m, u' \leq m, u \neq u'.$$

However, this result contradicts the assumption that m is the smallest element in $\text{Sp}(x, y)$. This contradiction arose out because we assumed that there is no common least upper bound of x and y in X . Therefore, the common least upper bound of x and y exists in X .

□

Proposition 3.5 *Let X be a BCK-algebra. For two elements x and y in X , suppose that there exists a smallest element in $\text{Sp}(x, y)$. Then, this smallest element is the common least upper bound of x and y in X .*

Proof Let m be a smallest element in $\text{Sp}(x, y)$. Furthermore, by the results of Lemma 3.4, let z be the common least upper bound of x and y in X . Here, assume that m and z are not equal. In this case, since z is the common least upper bound of x and y in X , it satisfies the following inequations ;

$$x \leq z, y \leq z, z < m. \quad (3.9)$$

By (3.9) and $m \in \text{Sp}(x, y)$, we have

$$z * x < m * x \leq y * x, z * y < m * y \leq x * y. \quad (3.10)$$

From (3.9), (3.10), we have $z \in \text{Sp}(x, y)$. Further, we already know that $z < m$. These results contradict m being the smallest element in $\text{Sp}(x, y)$. Hence, the assumption that m is not equal z was false. Therefore, we can conclude that $z = m$.

□

Here, we use "the condition $(\text{Sp})_{x,y}$ " to define a special class of BCK-algebras with condition (S).

Definition 3.2 (BCK-akgebra with Condition (Sp))

Let X be a BCK-algebra. For any two elements x, y in X , suppose the set $\text{Sp}(x, y)$ satisfies the following two conditions (1), (2) ;

- (1) $\text{Sp}(x, y) \neq \phi$,
- (2) There exists a smallest element in $\text{Sp}(x, y)$ with the respect to \leq .

Then, X is called to be *BCK-algebra with Condition (Sp)*.

From Proposition 3.5, it is easy to see that the class of BCK-algebras defined in Definition 3.2 satisfies the definition of an upper semilattice (Definition 2.2), and therefore the following Theorem C holds.

Theorem C

Let X be a BCK-algebra with Condition (Sp). For any two elements x, y in X , let m be a smallest element in $\text{Sp}(x, y)$, and denote $m = x + y$. Then, an algebra $\langle X; +, 0 \rangle$ is an upper semilattice with respect to the binary operation $+$.

This shows that BCK-algebra with Condition (Sp) is also one class of BCK-algebras as upper semilattice. We will call this class "Class C".

A characterization for 3 classes of upper semilattices in BCK-algebras

So far, we have seen that these are three classes of BCK-algebras that can be considered as upper semilattice. We would like to investigate the relationship between these three classes, Class A, Class B, and Class C.

Theorem 3.2 *Let $X = \langle X; *, \circ, 0 \rangle$ be a positive implicative BCK-algebra with condition (S). Then, for any x, y in X , the equality $x \circ (y * x) = y \circ (x * y)$ holds. That is, the binary operation \vee can be defined on X , and $\langle X; \vee, 0 \rangle$ is a BCK-algebra with a supremum.*

Proof For any x, y in X , suppose that $z = x \circ (y * x)$, $z' = y \circ (x * y)$. By the assumption ; positive implicative and Yutani's identity, we obtain

$$\begin{aligned} z * z' &= z * \{y \circ (x * y)\} \\ &= (z * y) * (x * y) \\ &= (z * x) * y \\ &= z * (x \circ y) \\ &= \{x \circ (y * x)\} * (x \circ y) \\ &= 0 \end{aligned}$$

Then, $z \leq z'$. Similarly, $z' \leq z$. Hence $x \circ (y * x) = y \circ (x * y)$.

□

From Theorem 3.2, we see that all of Class A is included in Class B.

Theorem 3.3 *Let $X = \langle X; \vee, 0 \rangle$ be a BCK-algebra with a supremum. Then, $X = \langle X; +, 0 \rangle$ is a BCK-algebra with Condition (Sp). At this time, for any x, y in X , the identity $x + y = x \vee y$ holds.*

Proof From Proposition 3.1, it is clear that this theorem is true.

□

By Theorem 3.3, the Class B is contained in Class C.

From Theorem 3.2 and 3.3 above, there is the following inclusion relationship among the three classes ; $\{\text{Class A}\} \subseteq \{\text{Class B}\} \subseteq \{\text{Class C}\}$.

However, we cannot know that the converse of Theorem 3.2, and 3.3 is true, so we cannot say whether there are any equivalent classes among these three classes. Then, we will investigate the relationship between the three classes using an example.

4 Examples of BCK-algebras with condition (S)

Example 4.1 Let $X = \{0, a, b, 1\}$ be a set with four elements. Furthermore, this set satisfies the following totally order $0 < a < b < 1$, as ordered structure in X .

At this time, there are six on algebraic structure of non-isomorphic BCK-algebras in totally ordered set X . These six BCK-algebras have the next Cayley Table A, B, C, D, E, and F.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	1	0

Cayley Table A.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	b	0

Cayley Table B.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	1	1	0

Cayley Table C.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	b	a	0

Cayley Table D.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Cayley Table E.

*	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

Cayley Table F.

First, since the set X has a totally ordered structure, it is obvious that all BCK-algebras in X satisfies condition (S). Below are Cayley Table A(\circ) \sim F(\circ) for the binary operation \circ , which can be derived from the Cayley Table A. \sim F. for the operation $*$.

\circ	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table A(\circ)

\circ	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	1	1
1	1	1	1	1

Cayley Table B(\circ)

◦	0	a	b	1
0	0	a	b	1
a	a	b	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table C(◦)

◦	0	a	b	1
0	0	a	b	1
a	a	b	1	1
b	b	1	1	1
1	1	1	1	1

Cayley Table D(◦)

◦	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	1	1
1	1	1	1	1

Cayley Table E(◦)

◦	0	a	b	1
0	0	a	b	1
a	a	1	1	1
b	b	1	1	1
1	1	1	1	1

Cayley Table F(◦)

Secondly, all six BCK-algebras in Example 4.1 are BCK-algebras with Condition (Sp). Below, we list the Cayley Tables for these operations +, in order to $A(+) \sim F(+)$.

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table A(+)

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table B(+)

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table C(+)

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table D(+)

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table E(+)

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table F(+)

Thirdly, we check whether the six BCK-algebras in Example 4.1 satisfy the Cornish's identity. In this case, the four BCK-algebras of the Cayley Table A, B, C, and D satisfy the Cornish's identity and become upper semilattices with respect to the operation \vee . These Cayley Tables are shown below.

\vee	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table A(\vee)

\vee	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table B(\vee)

\vee	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table C(\vee)

\vee	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

Cayley Table D(\vee)

However, BCK-algebras of the Cayley Tables E, and F do not satisfy the Cornish's identity. Examples are shown below.

In the Cayley Table E, $a \circ (b * a) = a \circ b = 1 \neq b = b \circ 0 = b \circ (a * b)$,

In the Cayley Table F, $a \circ (b * a) = a \circ a = 1 \neq b = b \circ 0 = b \circ (a * b)$.

Then, BCK-algebras of the Cayley Tables E, and F are not BCK-algebras with a supremum.

Fourthly, we want to examine whether these six BCK-algebras be positive implicative. To do so, we prepare the following a proposition.

Proposition 4.1 In a BCK-algebra, the following equivalence hold:

For any x, y, z in X , $(x * z) * (y * z) = (x * y) * z \iff (x * y) * y = x * y$.

Proof See, K. Iséki and S. Tanaka [7].

Proposition 4.1 shows that in order to check be positive implicative, it is sufficient to verify equality between two variables, rather than three variables. So, when we actually check, we find that the BCK-algebras consisting of the Cayley Tables B, C, D, E, and F is not positive implicative. The cases whose this does not hold are shown below.

In the Cayley Table B, $(1 * b) * b = b * b = 0 \neq b = 1 * b$,

In the Cayley Table C, $(b * a) * a = a * a = 0 \neq a = b * a$,

In the Cayley Table D, $(b * a) * a = a * a = 0 \neq a = b * a$, $(1 * a) * a = b * a = a \neq b = 1 * a$, $(1 * b) * b = a * b = 0 \neq a = 1 * b$,

In the Cayley Table E, $(1 * b) * b = a * b = 0 \neq a = 1 * b$,

In the Cayley Table F, $(b * a) * a = a * a = 0 \neq a = b * a$, $(1 * a) * a = a * a = 0 \neq a = 1 * a$, $(1 * b) * b = a * b = 0 \neq a = 1 * b$,

At the end, the BCK-algebra with the Cayley Table A is positive implicative, which we will show below. Since the set X consists of four elements, we need to check whether the following 12 equalities hold.

$$(0 * a) * a = 0 * a = 0 = 0 * a, (a * 0) * 0 = a * 0 = a = a * 0,$$

$$(0 * b) * b = 0 * b = 0 = 0 * b, (b * 0) * 0 = b * 0 = b = b * 0,$$

$$(0 * 1) * 1 = 0 * 1 = 0 = 0 * 1, (1 * 0) * 0 = 1 * 0 = 1 = 1 * 0,$$

$$(a * b) * b = 0 * b = 0 = a * b, (b * a) * a = b * a = b = b * a,$$

$$(a * 1) * 1 = 0 * 1 = 0 = a * 1, (1 * a) * a = 1 * a = 1 = 1 * a,$$

$$(b * 1) * 1 = 0 * 1 = 0 = b * 1, (1 * b) * b = 1 * b = 1 = 1 * b.$$

As all 12 equations are satisfied, the BCK-algebra with the Cayley Table A is positive implicative.

From our observations of Example 4.1 so far, we have the following facts.

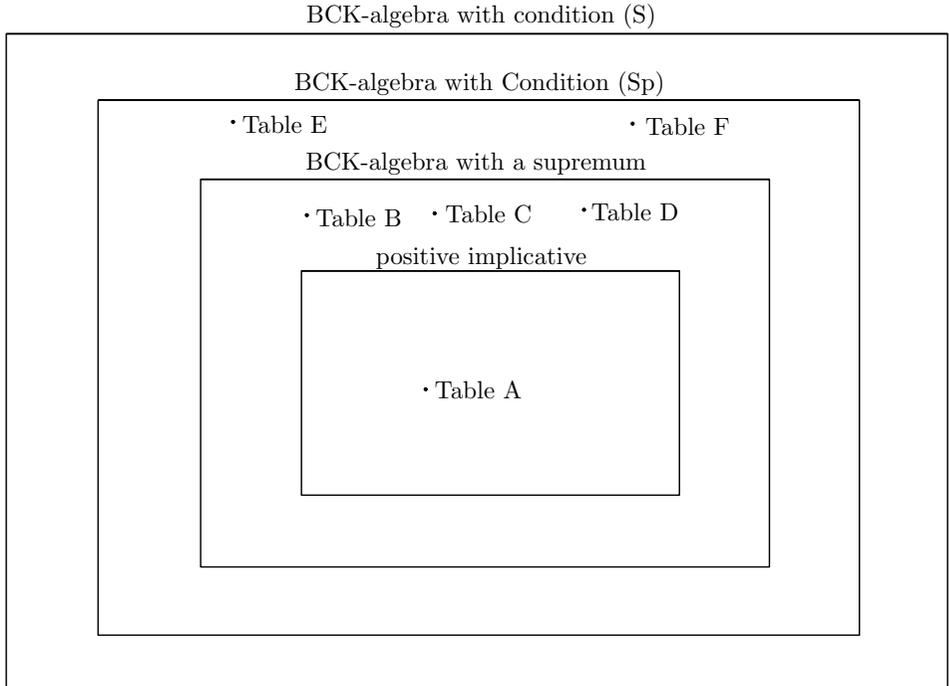
Fact 1. The six BCK-algebras consisting of the Cayley Tables A \sim F all are with condition (S). On the other hand, a totally ordered set X has an upper semilattice that can be defined by the operation \cup . Now, when comparing this upper semilattice with the algebra having the operation \circ that derived from the Cayley Tables A \sim F, the only algebra having the operation \circ that derived from the Cayley Table A has the same algebraic structure as upper semilattice.

Fact 2. All BCK-algebras that defined by the Cayley Tables A \sim F are BCK-algebras with Condition (Sp). And, these six algebras determined by the operation $+$ are identical to the upper semilattice having with the operation \cup .

Fact 3. Among the six BCK-algebras defined by the Cayley Tables A \sim F, the four BCK-algebras defined by the Cayley Tables A, B, C, and D are BCK-algebras with a supremum.

Fact 4. In the six BCK-algebras that defined by the Cayley Tables A \sim F, the only BCK-algebra defined by the Cayley Table A is positive implicative.

The following is a Venn diagram showing the relationships between the six BCK-algebras with condition (S) of Example 4.1; such as with Condition (Sp), with a supremum, and be positive implicative.



A most important observations from Example 4.1 is that the converse of Theorem 3.2 and Theorem 3.3 do not hold.

5 Main Result: Relations between three classes of upper semilattices in BCK-algebras

Finally, we present the following theorem, which expresses the relationships between three classes of upper semilattices in BCK-algebras that appear in this article.

Theorem 5.1 *Let Class A is positive implicative BCK-algebras with condition (S), Class B is BCK-algebra with a supremum, Class C is BCK-algebras with Condition (Sp). These three classes have the following inclusion relation :*

$$\{\mathbf{Class\ A}\} \subsetneq \{\mathbf{Class\ B}\} \subsetneq \{\mathbf{Class\ C}\}.$$

Proof The correctness of this theorem can be seen from all the results in Section 3 and the existence of the six BCK-algebras examined in Section 4.

□

Open Problem 1. Is the Class C the largest class that be viewed as upper semilattice in the BCK-algebras ?

The role of the three operations \circ, \vee , and $+$ that we introduced in this article can be as follows. A set X is an upper semilattice $\langle X; \leq \rangle$ as ordered structure. Here, the operation $*$ of BCK-algebra is defined on this ordered set X , but that algebraic structure is not necessarily an upper semilattice. So, K.Iseki [6], W.H.Cornish [2], and this article redefine the new operations \circ, \vee , and $+$ using the operation $*$ already defined in the BCK-algebra X , and the BCK-algebra can be understood as an algebraic structure of the upper semilattice under these newly defined operations.

The six BCK-algebras in Example 4.1 have the same Hasse diagram as ordered structure. But, the algebraic structure defined from the operation $*$ are different in each algebra. We have seen that these differences can be divided into the three classes according to the degree to which they can be correctly corrected by redefining that using the operation $*$. That classification are Class A, Class B, and Class C ; and redefined by the operations \circ, \vee , and $+$.

In the theory of universal algebra, we think a question concerning the interrelationships between the "operations" and the "relations" defined in one algebra is a topic of considerable interest. The class of BCK-algebras with condition (S) is a variety (see, H. Yutani [12]). Within its variety class in BCK-algebras, there are three classes : Class A, Class B, and Class C, and each with slightly different operations and order relations. For this fact, could we use it to study the interrelationship of the "operations" and the "relations" in the universal algebra ?

6 Future Work

Here, we would like to shore our thought on above results. We introduced "the condition $(Sp)_{x,y}$ " as an analogy of "the condition $(I)_{x,y}$ " into the theory of BCK-algebra. The condition $(I)_{x,y}$ is a notion that describes the structure of a greatest lower bound of x and y in a BCK-algebra X . On the other hand, the condition $(Sp)_{x,y}$ is a notion that describes the structure of a least upper bound of x and y in X . And, after examining these two structures, we want to wonder if there is some kind of dual between the condition $(I)_{x,y}$ and the condition $(Sp)_{x,y}$.

Finally, let us pose the following question :

Open Problem 2. Let $\langle X; *, \times, +, 0 \rangle$ be a BCK-algebra with both Condition (I) and Condition (Sp). Can we find any duality between the operations \times and $+$ in this algebra ?

Acknowledgement The author would like to express his hearty thanks to the editor, Professor Jair Minoro Abe for his kind and helpful advice.

References

- [1] G. Birkhoff, Lattice Theory (Third Edition), Colloquium Publication, **25**, American Mathematical Society, (1993).
- [2] W. H. Cornish, BCK-Algebras with a Supremum, *Math. Japonicae*, **27**, No. 1 (1982), 63-73.
- [3] K. Iséki, An Algebra Related with a Propositional Calculus, *Proc. Japan Acad.*, **42**, No.1(1966), 26-29.
- [4] K. Iséki, Some Properties of BCK-algebras, *Mathematics Seminar Notes Kobe University*, **2**(1974), 193-201.
- [5] K. Iséki, A Special Class of BCK-Algebras, *Mathematics Seminar Notes Kobe University*, **5**(1977), 191-198.
- [6] K. Iséki, BCK-Algebra with Condition (S), *Math. Japonicae*, **24**, No.1(1979), 107-119.
- [7] K. Iséki and S. Tanaka, Introduction to the Theory of BCK-Algebras, *Math. Japonicae*, **23**, No.1 (1978), 1-26.
- [8] M. Kumazawa, On the Commutativity in a BCK-algebra, *Report: 9th Lattice Theory Discussion Conference (Partially Ordered Sets and Its Related Topics)*, Tokai University (edit H.Narushima and M.Tsuchiya), (1990), 28-38. (in Japanese)
- [9] M. Kumazawa, A New Class in BCK-Algebras, *Scientiae Math. Japonicae*, **84**, No.1 (2021), 61-75.
- [10] H. Rasiowa, An Algebraic Approach Non-classical Logics, *Studies in Logic and The Foundations of Mathematics*, **78**, North-Holland, (1974).
- [11] S. Tanaka, A New Class of Algebra, *Mathematics Seminar Notes Kobe University*, **3** (1975), 37-43.
- [12] H. Yutani, An Axiom System for a BCK-Algebra with the Condition (S), *Mathematics Seminar Notes Kobe University*, **7** (1979), 427-432.

Communicated by *Jair Minoro Abe*

E-mail: kumazawamasaaki0717@gmail.com

THE PARTIAL CLONE OF ESSENTIALLY k -ARY TERMS

Klaus Denecke

Received March 24,2022

ABSTRACT. The concept of a term belongs to the mathematical basic concepts. Terms are defined using individual variables and operation symbols. The complexity of a term can be measured by counting the occurrences of individual variables or of operation symbols in the term. The combination of terms to new terms, i.e. calculation with terms, can be described by certain composition operations defined on sets of terms. This paper shows once more that the calculation with terms and the properties of the algebraic structures which can be defined on the basis of these composition operations depend on the complexity of the considered terms.

1 Basic concepts

For a natural number $n \geq 1$ let $X_n := \{x_1, \dots, x_n\}$ be an n -element set of variables. We think of X_n as an n -element alphabet of a formal language. To define terms of this language we need a set $\{f_i \mid i \in I\}$ of operation symbols indexed by set I . To each f_i we assign a natural number n_i and call it the arity of f_i . The sequence $\tau = (n_i)_{i \in I}$ is said to be the type of the language. N -ary terms of type τ are formal expressions defined in the following way.

Definition 1.1. Let $n \geq 1$.

- (i) Every variable $x_i \in X_n$ is an n -ary term of type τ .
- (ii) If t_1, \dots, t_{n_i} are n -ary terms of type τ and if f_i is an n_i -ary operation symbol, then $f_i(t_1, \dots, t_{n_i})$ is an n -ary term of type τ .
- (iii) The set $W_\tau(X_n) = W_\tau(x_1, \dots, x_n)$ of all n -ary terms is the smallest set which contains x_1, \dots, x_n and is closed under finite application of (ii).

Note that by our definition any term of arity n is also a term of arity l with $l > n$. This means, for instance, that for $X_3 = \{x_1, x_2, x_3\}$ and $\tau = (2)$ with the binary operation symbol f , the terms $f(x_1, x_2)$ and $f(x_1, f(x_1, x_1))$ or $f(x_2, x_2)$ are ternary, although, $f(x_1, x_2)$ uses only x_1 and x_2 , $f(x_1, f(x_1, x_1))$ uses only x_1 and $f(x_2, x_2)$ uses only x_2 . For $X_2 = \{x_1, x_2\}$ the terms $f(x_1, f(x_1, x_1))$ and $f(x_2, x_2)$ are binary and for $X_1 = \{x_1\}$ the term $f(x_1, f(x_1, x_1))$ is unary. Let X be the union of all sets X_n , i.e. the countably infinite alphabet $X := \{x_1, \dots, x_n, \dots\}$.

Let $W_\tau(X)$ be the set of all finitary terms of type τ over X : $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$.

Using operations defined on sets of terms from given terms one obtains new ones.

On the set $W_\tau(X_n)$ an $(n+1)$ -ary operation $S_n^n : W_\tau(X_n)^{n+1} \rightarrow W_\tau(X_n)$ can inductively be defined by the following steps:

Definition 1.2. (i) If $x_j \in X_n$ is a variable and $t_1, \dots, t_n \in W_\tau(X_n)$, then $S_n^n(x_j, t_1, \dots, t_n) := t_j$, for $1 \leq j \leq n$;

(ii) If $f_i(s_1, \dots, s_{n_i})$ is a composite term, then

$$S_n^n(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) := f_i(S_n^n(s_1, t_1, \dots, t_{n_i}), \dots, S_n^n(s_{n_i}, t_1, \dots, t_{n_i})).$$

With these operations we may consider the algebras $(W_\tau(X_n); S_n^n)$ and $(W_\tau(X_n); S_n^n, x_1, \dots, x_n)$ of type $(n+1)$ and $(n+1, 0, \dots, 0)$, respectively.

It is not difficult to check that the first algebra satisfies the identity:

$$(C1) \quad \tilde{S}_n^n(\tilde{U}, \tilde{S}_n^n(\tilde{Y}_1, \tilde{U}_1, \dots, \tilde{U}_n), \dots, \tilde{S}_m^n(\tilde{Y}_p, \tilde{U}_1, \dots, \tilde{U}_n)) \\ \approx \tilde{S}_n^n(\tilde{S}_n^n(\tilde{U}, \tilde{Y}_1, \dots, \tilde{Y}_p), \tilde{U}_1, \dots, \tilde{U}_n), \quad (n = 1, 2, \dots),$$

and the second one satisfies (C1) and

$$(C2) \quad \tilde{S}_n^n(\tilde{e}_i^n, \tilde{U}_1, \dots, \tilde{U}_n) \approx \tilde{U}_i, \quad (n = 1, 2, \dots, 1 \leq i \leq n)$$

$$(C3) \quad \tilde{S}_n^n(\tilde{Y}, \tilde{e}_1^n, \dots, \tilde{e}_n^n) \approx \tilde{Y}, \quad (n = 1, 2, \dots).$$

Here $\tilde{U}, \tilde{Y}_1, \dots, \tilde{Y}_p, \tilde{U}_1, \dots, \tilde{U}_n$, are variables for terms, \tilde{S}_n^n are operation symbols and \tilde{e}_i^n are symbols for variables.

(C1) is called the *superassociative law*. It generalizes the associative law which we obtain from (C1) if we set $n = 1$.

Definition 1.3. An algebra of type $\tau = (n+1)$ is said to be a *Menger algebra of rank n* if it satisfies (C1). An algebra of type $(n+1, 0, \dots, 0)$ is said to be a *unitary Menger algebra* if it satisfies (C1), (C2), (C3).

These algebras were introduced by the Austrian mathematician Karl Menger in 1961 ([11]), (for Menger algebras see also [2] or [7]). Another way to consider algebras whose universes are sets of terms starts with the sequence $(W_\tau(X_n))_{n \geq 1}$ as universes of many-sorted algebras. The operations

$$S_n^m : W_\tau(X_n) \times W_\tau(X_m)^n \rightarrow W_\tau(X_m)$$

of these algebras are inductively defined similar as in Definition 1.2. Now the first input of S_n^m is an n -ary term and the other inputs are m -ary terms. The resulting term is m -ary. Then we get a many-sorted algebra

$$\text{clone}_\tau := ((W_\tau(X_n))_{n \geq 1}; (S_n^m)_{m, n \geq 1}, (x_i)_{1 \leq i \leq n})$$

which is called the term clone of type τ . clone_τ satisfies the clone axioms (C1), (C2), (C3), where the superassociative law (C1) now has the form

$$(C1) \quad \tilde{S}_m^p(\tilde{U}, \tilde{S}_n^m(\tilde{Y}_1, \tilde{U}_1, \dots, \tilde{U}_n), \dots, \tilde{S}_n^m(\tilde{Y}_p, \tilde{U}_1, \dots, \tilde{U}_n)) \\ \approx \tilde{S}_n^m(\tilde{S}_n^p(\tilde{U}, \tilde{Y}_1, \dots, \tilde{Y}_p), \tilde{U}_1, \dots, \tilde{U}_n), \quad (m, n, p = 1, 2, \dots).$$

A clone can also be regarded as a category. The duals of those categories are the so-called Lawvere theories (see [9]).

The complexity of an n -ary term t can be measured by the variables or by the operation symbols occurring in it. Let $vb_l(t)$ be the number of occurrences of the variable x_l in t , for $1 \leq l \leq n$ and let $vb(t) = \sum_{x_l \in t} vb_l(t)$ be the number of all occurrences of variables in t . If $var(t)$ is the set of variables occurring in t , then $|var(t)|$ is the number of different variables occurring in term t . For more details on the complexity of terms see [6].

A term t is said to be linear if each variable occurring in t occurs only once, i.e. if $vb_l(t) = 1$ for each variable x_l in t . The set $W_\tau^{lin}(X_n)$ of all n -ary linear terms of type τ is in general not closed under the operation S_n^m . This holds also for the many-sorted operations S_n^m , $m, n \geq 1$.

As an example we consider $\tau = (2)$ with a binary operation symbol f . The terms $f(x_1, f(x_2, x_3))$, $f(x_1, x_3)$, $f(x_1, x_2)$ are linear, but $S_2^3(f(x_1, x_2), f(x_1, f(x_2, x_3)), f(x_1, x_3)) = f(f(x_1, f(x_2, x_3)), f(x_1, x_3))$ is not linear. That means, calculation in the set of linear terms is different from calculation with arbitrary terms. Let $W_\tau^{lin}(X_n)$ be the set of all n -ary linear terms of type τ and let $W_\tau^{lin}(X) = \bigcup_{n=1}^{\infty} W_\tau^{lin}(X_n)$ be the set of all linear terms of this type. In [3] the following many-sorted partial operations

$$\bar{S}_n^m : W_\tau^{lin}(X_n) \times (W_\tau^{lin}(X_m))^n \dashrightarrow W_\tau^{lin}(X_m)$$

defined by

$$\bar{S}_n^m(t, s_1, \dots, s_n) := \begin{cases} S_n^m(t, s_1, \dots, s_n) & \text{if } var(s_j) \cap var(s_k) = \emptyset \text{ for all} \\ 1 \leq j < k \leq n \\ \text{not defined,} & \text{otherwise} \end{cases}$$

for $m, n \geq 1$ and a many-sorted partial algebra

$$Partclone_{in\tau} := ((W_\tau^{lin}(X_n))_{n \geq 1}, (\bar{S}_n^m)_{m, n \geq 1}, (x_i)_{i \leq n, n \geq 1})$$

were introduced.

It turns out that (C1),(C2),(C3) are satisfied in the following weaker sense. If the right hand side of these equations is defined, then the left hand side is defined and if the left hand side is defined, then the right hand side is defined and then both sides are equal. Many-sorted partial algebras of the same type as $Partclone_{in\tau}$ which have this property are called partial clones (see [4]). Lekkoksung and Lekkoksung generalized in [10] linear terms to k -terms in the following way. A term $t \in W_\tau(X_n)$ is said to be an n -ary k -term of type τ if $vb_l(t) \leq k$ for all $1 \leq l \leq n$. Linear terms are particular k -terms for $k = 1$. Let $W_\tau^{(k)}(X_n)$ be the set of all n -ary k -terms of type τ . In [10] for $k \geq 1$ partial operations

$$\bar{S}_n^m := W_\tau^{(k)}(X_n) \times (W_\tau^{(k)}(X_m))^n \dashrightarrow W_\tau^{(k)}(X_m)$$

were defined by

$$\bar{S}_n^m(s, t_1, \dots, t_n) := \begin{cases} S_n^m(s, t_1, \dots, t_n) & \text{if } \sum_{j=1}^n vb_j(s)vb_k(t_j) \leq r, \text{ for all } 1 \leq k \leq n \\ \text{not defined,} & \text{otherwise} \end{cases}$$

Since these operations generalize the superposition operations for linear terms we used the same notation. With these operations the authors defined partial clones

$$Partclone_{k\tau} := ((W_\tau^{(k)}(X_n))_{n \geq 1}, (\bar{S}_n^m)_{m, n \geq 1}, (x_i)_{i \leq n, n \geq 1})$$

Endomorphisms of the many-sorted total algebra $clone\tau$ are mappings from terms to terms which preserve arities and which are compatible with the operations of $clone\tau$, especially with the operations S_n^m , $m, n \geq 1$. Such mappings can be obtained as extensions of hypersubstitutions. Any mapping

$$\{f_i \mid i \in I\} \rightarrow W_\tau(X)$$

which assigns to every operation symbol f_i of type τ an n_i -ary term $\sigma(f_i)$ is called a hypersubstitution of type τ . Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ . Any such σ induces a mapping

$$\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$$

which is defined as follows. For any $t \in W_\tau(X)$ the resulting term $\hat{\sigma}[t]$ is defined inductively by

- (i) $\hat{\sigma}[x] := x$ for variables,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] = S_n^m(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$

It is not difficult to prove (see [5]) that $\hat{\sigma}$ and S_n^m are permutable, i.e. that

$$\hat{\sigma}[S_n^m(t, s_1, \dots, s_n)] = S_n^m(\hat{\sigma}[t], \hat{\sigma}[s_1], \dots, \hat{\sigma}[s_n])$$

and therefore the sequence $(\hat{\sigma}_n)_{n \geq 1}$ with $\hat{\sigma}_n : W_\tau(X_n) \rightarrow W_\tau(X_n)$ is an endomorphism of the many-sorted algebra $clone\tau$.

If $\sigma_1, \sigma_2 \in Hyp(\tau)$, then one defines a product $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where \circ is the usual composition of mappings and together with the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, \dots, x_{n_i})$ one obtains a monoid $(Hyp(\tau); \circ_h, \sigma_{id})$. Hypersubstitutions can be used to get new insight into the equational theory of Universal Algebra. Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra with n_i -ary fundamental operations f_i^A corresponding to the operation symbols f_i . The pair $s \approx t$ of terms of type τ is satisfied as a hyperidentity in \mathcal{A} if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in \mathcal{A} for all $\sigma \in Hyp(\tau)$. If \mathcal{M} is a submonoid of $(Hyp(\tau); \circ_h, \sigma_{id})$, then $s \approx t$ is said to be an M -hyperidentity in \mathcal{A} if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in \mathcal{A} for all $\sigma \in M$. For more details on hyperidentities and M -hyperidentities see [5].

For partial algebras there are several possibilities to define homomorphisms and identities. If \mathcal{A}, \mathcal{B} are partial algebras of the same type with indexed sets $\{f_i^A \mid i \in I\}, \{f_i^B \mid i \in I\}$ of partial fundamental operations on A and on B , respectively, then a mapping $h : A \rightarrow B$ is said to be a weak homomorphism if for all fundamental operations the following is satisfied:

if $(a_1, \dots, a_{n_i}) \in dom f_i^A$, then $(h(a_1), \dots, h(a_{n_i})) \in dom f_i^B$ and then

$$h(f_i^A(a_1, \dots, a_{n_i})) = f_i^B(h(a_1), \dots, h(a_{n_i})), i \in I,$$

where $dom f_i^A$ is the domain of the operation f_i^A . Weak endomorphisms are weak homomorphisms with $h : A \rightarrow A$.

The equation $s \approx t$ is said to be a weak identity of the partial algebra \mathcal{A} if after evaluation there holds: if the right side is defined then the left side is defined and both sides are equal or if the left side is defined then the right side is defined and both sides are equal. For more information on partial algebras see [1].

Considering sets of terms of restricted complexities, it turns out that as in the case of linear terms or of r -terms the operations which compose terms are not longer total, and the corresponding many-sorted partial

algebras of terms satisfy (C1), (C2), (C3) as weak identities. For linear terms the set of all hypersubstitutions which map operation symbols to linear terms forms a monoid. This is not the case for r -terms with $r > 1$ as was shown in [10]. In this paper we consider sets of terms t with $|\text{var}(t)| \leq k$ and show that the composition operations are partial, and the corresponding partial many-sorted algebras satisfy (C1), (C2), (C3) as weak identities. Hypersubstitutions mapping n_i -ary operation symbols to n_i -ary terms t with $|\text{var}(t)| \leq k$ for some $k < n_i$ form monoids if and only if $k = 1$, for $k > 1$ one obtains semigroups, which are not monoids. The extensions of such hypersubstitutions are weak endomorphisms of the corresponding many-sorted partial algebras.

2 Essentially unary terms

The set $W_\tau(X_n)$ of all n -ary terms of type τ contains the set of all terms which use only one variable x_1 , the set of all terms which use only x_2 , etc. i.e. $W_\tau(\{x_1\}) \subseteq W_\tau(X_n)$, $W_\tau(\{x_2\}) \subseteq W_\tau(X_n)$, \dots , $W_\tau(\{x_n\}) \subseteq W_\tau(X_n)$.

Definition 2.1. Any n -ary term which uses only one variable is said to be essentially unary.

$W_\tau(\{x_1\}) \cup W_\tau(\{x_2\}) \dots \cup W_\tau(\{x_n\}) \subseteq W_\tau(X_n)$ is the set of all essentially unary terms of type τ . If $t \in W_\tau(\{x_i\})$, $x_i \in \{1, \dots, n\}$, we will write $t = t(x_{l_1}, \dots, x_{l_n})$. The set $\bigcup_{i=1}^n W_\tau(\{x_i\})$ is closed under the operation S_n^n . Indeed, if

$$t = t(x_j, \dots, x_j), s_1 = s_1(x_{l_1}, \dots, x_{l_1}), \dots, s_n = s_n(x_{l_n}, \dots, x_{l_n}), l_1, \dots, l_n \in \{1, \dots, n\},$$

then

$$S_n^n(t(x_j, \dots, x_j), s_1, \dots, s_n) = t(s_j(x_{l_j}, \dots, x_{l_j}), \dots, s_j(x_{l_j}, \dots, x_{l_j})) \in W_\tau(\{x_j\}) \subseteq \bigcup_{i=1}^n W_\tau(\{x_i\}).$$

Therefore, $(\bigcup_{i=1}^n W_\tau(\{x_i\}); S_n^n)$ is a subalgebra of $(W_\tau(X_n); S_n^n)$ and $(\bigcup_{i=1}^n W_\tau(\{x_i\}); S_n^n, x_1, \dots, x_n)$ is a subalgebra of $(W_\tau(X_n); S_n^n, x_1, \dots, x_n)$.

As a consequence we get

Lemma 2.2. $(\bigcup_{i=1}^n W_\tau(\{x_i\}); S_n^n)$ is a Menger algebra of rank n and $(\bigcup_{i=1}^n W_\tau(\{x_i\}); S_n^n, x_1, \dots, x_n)$ is a unitary Menger algebra of rank n .

This can be formulated also for clones, since it is easy to see that the many-sorted set $(\bigcup_{i=1}^n W_\tau(\{x_i\}))_{n \geq 1}$ is closed under the many-sorted operations

$$S_n^m : \bigcup_{i=1}^n W_\tau(\{x_i\}) \times \left(\bigcup_{i=1}^m W_\tau(\{x_i\}) \right)^n \rightarrow \bigcup_{i=1}^m W_\tau(\{x_i\})$$

for $m, n \geq 1$ and contains $(x_i)_{1 \leq i \leq n, n \geq 1}$.

Lemma 2.3. The many-sorted algebra $((\bigcup_{i=1}^n W_\tau(\{x_i\}))_{n \geq 1}; (S_n^m)_{m, n \geq 1}, (x_i)_{1 \leq i \leq n, n \geq 1})$ is a subclone of clone_τ , i.e. satisfies (C1), (C2), (C3) as identities.

In [12] the authors introduced “terms of a fixed variable” in the following inductive way.

Definition 2.4. ([12]) *N-ary terms of type τ of a fixed variable are defined inductively by*

- (i) $x_j \in X_n$ are *n-ary terms of a fixed variable.*
- (ii) *If t_1, \dots, t_{n_i} are n-ary terms of a fixed variable and if $\text{var}(t_j) = \text{var}(t_k)$ for all $1 \leq j < k \leq n_i$, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of a fixed variable. Let $W_\tau^{fv}(X_n)$ be the set of all n-ary terms of a fixed variable.*

Lemma 2.5. $W_\tau^{fv}(X_n) = \bigcup_{i=1}^n W_\tau(\{x_i\})$, i.e., *t is an n-ary term of a fixed variable iff t is an essentially unary term of type τ .*

Proof. Let $t \in \bigcup_{i=1}^n W_\tau(\{x_i\})$. Then there is an element $j \in \{1, \dots, n\}$ such that $t \in W_\tau(\{x_j\})$. If $t = x_j$, then $t \in W_\tau^{fv}(X_n)$. Otherwise, $t = f_i(t_1, \dots, t_{n_i})$. Since $\text{var}(t) = \{x_j\}$, we have $\text{var}(t_l) = \text{var}(t_k) = \{x_j\}$ for all $1 \leq l < k \leq n_i$ and therefore by Definition 2.4, (ii), $t \in W_\tau^{fv}(X_n)$ and $\bigcup_{i=1}^n W_\tau(\{x_i\}) \subseteq W_\tau^{fv}(X_n)$.

If $t \in W_\tau^{fv}(X_n)$ and $t = x_j$, then $t \in \bigcup_{i=1}^n W_\tau(\{x_i\})$. If $t = f_i(t_1, \dots, t_{n_i})$ then $\text{var}(t_l) = \text{var}(t_k)$ for $1 \leq l < k \leq n_i$. Inductively we suppose that $t_1, \dots, t_{n_i} \in \bigcup_{i=1}^n W_\tau(\{x_i\})$. Therefore, there is a variable $x_j \in X_n$ such that $t_1, \dots, t_{n_i} \in W_\tau(\{x_j\})$ and then also $f_i(t_1, \dots, t_{n_i}) \in W_\tau(\{x_j\}) \subseteq \bigcup_{i=1}^n W_\tau(\{x_i\})$. This shows $W_\tau^{fv}(X_n) \subseteq \bigcup_{i=1}^n W_\tau(\{x_i\})$. \square

In [12] was proved that the many-sorted algebra $((W_\tau^{fv}(X_n))_{n \geq 1}; (S_m^m)_{m, n \geq 1}, (x_i)_{1 \leq i \leq n, n \geq 1})$ is a clone. With Lemma 2.5 this corresponds to Lemma 2.3.

3 Essentially k -ary terms

We want to generalize essentially unary terms to essentially k -ary terms of $W_\tau(X_n)$, $n \geq 2, k < n$. Those terms use at most $k < n$ variables from X_n and belong to $W_\tau(\{x_1, \dots, x_k\})$ or to $W_\tau(Y)$ for another k -element subset Y of X_n . Thus, all these terms belong to $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$. Clearly,

$$t \in \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y) \Leftrightarrow |\text{var}(t)| \leq k.$$

Example 3.1. Let $n = 3$, $\tau = (2)$ with the binary operation symbol f . Then $t \in W_{(2)}(X_3)$ is essentially binary if $|\text{var}(t)| \leq 2$, i.e., if $t \in W_{(2)}(\{x_1, x_2\}) \cup W_{(2)}(\{x_1, x_3\}) \cup W_{(2)}(\{x_2, x_3\})$. The set $\bigcup_{Y \subseteq X_n, |Y| \leq k} W_\tau(Y)$ is in general not closed under S_n^n . As an example we consider $t = f(f(x_1, x_3), x_1) \in W_{(2)}(\{x_1, x_3\})$, $t_1 = f(x_1, x_2) \in W_{(2)}(\{x_1, x_2\})$, $t_2 = f(f(x_1, f(x_3, x_3)) \in W_{(2)}(\{x_1, x_3\})$ and $t_3 = f(x_2, x_3) \in W_{(2)}(\{x_2, x_3\})$.

$$\begin{aligned} \text{Then } S_3^3(t, t_1, t_2, t_3) &= S_3^3(f(f(x_1, x_3), x_1), f(x_1, x_2), f(f(x_1, f(x_3, x_3))), f(x_2, x_3)) \\ &= f(f(f(x_1, x_2)f(x_2, x_3)), f(x_1, x_2)) \notin \bigcup_{Y \subseteq X_n, |Y| \leq 2} W_{(2)}(Y). \end{aligned}$$

Therefore, S_3^3 is not an operation on $\bigcup_{Y \subseteq X_3, |Y| \leq 2} W_{(2)}(Y)$. In the example the reason is that t contains x_1 and x_3 , but the second input of S_3^3 is in $W_{(2)}(\{x_1, x_2\})$ and the fourth input of S_3^3 is in $W_{(2)}(\{x_2, x_3\})$ and therefore $S_3^3(t, t_1, t_2, t_3) \notin \bigcup_{Y \subseteq X_3, |Y| \leq 2} W_{(2)}(Y)$.

More general, if $t \in W_\tau(Y)$ for a subset $Y \subseteq X_n$ with $|Y| \leq k$ and $|\bigcup_{x_j \in \text{var}(t)} \text{var}(t_j)| \leq k$, then $S_n^n(t, t_1, \dots, t_n) \in \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$.

Therefore, we may define the partial $(n + 1)$ -ary operation

$$\overset{\bullet}{S}_n^n: W_\tau(Y) \times (W_\tau(Y_1) \times \cdots \times W_\tau(Y_n)) \dashrightarrow W_\tau(Z)$$

with $Y, Y_1, \dots, Y_n \subseteq X_n$ and $|Y| = |Y_1| = \cdots = |Y_n| = k$ by

$$\overset{\bullet}{S}_n^n(t, t_1, \dots, t_n) := \begin{cases} S_n^n(t, t_1, \dots, t_n) & \text{if } |\bigcup_{x_j \in \text{var}(t)} \text{var}(t_j)| \leq k, \\ \text{not defined,} & \text{otherwise} \end{cases}$$

One may ask for which $k < n$ the set $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ is closed under S_n^n .

Lemma 3.2. *The set $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ is closed under S_n^n if and only if $k = 1$.*

Proof. If $k = 1$, then by Lemma 2.3 the set $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ is closed under S_n^n .

Now let $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$, $n > 2$, $k < n$ be closed under S_n^n and let $k > 1$. Let $t \in W_\tau(X_n)$ be a term with at least two different variables, say x_1 and x_2 . Let t_1, t_2 be terms with $\text{var}(t_1) \neq \text{var}(t_2)$, but $|\text{var}(t_1)| = |\text{var}(t_2)| = k$. Then $|\text{var}(S_n^n(t, t_1, \dots, t_n))| > k$ is a contradiction since $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ is assumed to be closed under S_n^n . Therefore, $k = 1$. \square

4 Partial Menger algebras of essentially k -ary terms from $W_\tau(X_n)$

Let $t, t_1, \dots, t_n \in W_\tau(X_n)$ with $|\text{var}(t)| = |\text{var}(t_i)| \leq k$, $i = 1, \dots, n$, $k < n$, i.e., $t, t_1, \dots, t_n \in \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ and let

$$\overset{\circ}{S}_n^n: \left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y) \right)^{n+1} \dashrightarrow \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$$

be partial operations defined by

$$\overset{\circ}{S}_n^n(t, t_1, \dots, t_n) := \begin{cases} S_n^n(t, t_1, \dots, t_n) & \text{if } |\bigcup_{x_j \in \text{var}(t)} \text{var}(t_j)| \leq k, \\ \text{not defined,} & \text{otherwise} \end{cases}$$

Since the $\overset{\circ}{S}_n^n$ are well-defined with $\overset{\circ}{S}_n^n(t, t_1, \dots, t_n) \in \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$, we obtain partial algebras

$$\left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \overset{\circ}{S}_n^n \right) \text{ and } \left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \overset{\circ}{S}_n^n, x_1, \dots, x_n \right).$$

Lemma 4.1. *The partial algebras*

$$\left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \overset{\circ}{S}_n^n \right) \text{ and } \left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \overset{\circ}{S}_n^n, x_1, \dots, x_n \right)$$

are in general not partial Menger algebras of rank n and unitary partial Menger algebras of rank n , respectively.

Proof. To show that (C1) is in general not a weak identity in these algebras we give counterexamples for the following cases:

1. If the left hand side of (C1) is defined, then the right hand side is not defined.
2. If the right hand side of (C1) is defined, then the left hand side is not defined.

1. Let $t, t_1, t_2, t_3, s_1, s_2, s_3 \in W_{(2)}(X_3)$ be given by $t = t_1 = f(x_1, x_3)$, $t_2 = f(x_3, x_1)$, $t_3 = f(x_3, x_2)$, $s_1 = f(x_1, x_2)$, $s_2 = f(x_2, x_3)$, $s_3 = f(x_2, x_1)$. Then $|\text{var}(t)|, |\text{var}(t_i)|, |\text{var}(s_i)| \leq 2, i = 1, 2, 3$, i.e. $k = 2$ and the left hand side of (C1), i.e.,

$$\begin{aligned} & \overset{\circ}{S}_3^3(\overset{\circ}{S}_3^3(t, s_1, s_2, s_3), t_1, t_2, t_3) \\ &= \overset{\circ}{S}_3^3(\overset{\circ}{S}_3^3(f(x_1, x_3), f(x_1, x_2), f(x_2, x_3), f(x_2, x_1)), f(x_1, x_3), f(x_3, x_1), f(x_3, x_2))) \end{aligned}$$

is defined because of the definition of the partial operation $\overset{\circ}{S}_3^3$ and $|\text{var}(f(x_1, x_2)) \cup \text{var}(f(x_2, x_1))| \leq 2$ and $|\text{var}(f(x_1, x_3)) \cup \text{var}(f(x_3, x_1))| \leq 2$.

The right hand side of (C1) has the form

$$\begin{aligned} & \overset{\circ}{S}_3^3(t, \overset{\circ}{S}_3^3(s_1, t_1, t_2, t_3), \overset{\circ}{S}_3^3(s_2, t_1, t_2, t_3), \overset{\circ}{S}_3^3(s_3, t_1, t_2, t_3)) \\ &= \overset{\circ}{S}_3^3(f(x_1, x_3), \overset{\circ}{S}_3^3(f(x_1, x_2), f(x_1, x_3), f(x_3, x_1), f(x_3, x_2)), \overset{\circ}{S}_3^3(f(x_2, x_3), f(x_1, x_3), f(x_3, x_1), \\ & \quad f(x_3, x_2)), \overset{\circ}{S}_3^3(f(x_2, x_1), f(x_1, x_3), f(x_3, x_1), f(x_3, x_2))) \end{aligned}$$

Since $|\text{var}(f(x_3, x_1)) \cup \text{var}(f(x_3, x_2))| = |\{x_1, x_2, x_3\}| = 3$, the right hand side is not defined.

2. Let $t = t_1 = f(x_1, x_3)$, $t_2 = t_3 = f(x_3, x_1)$, $s_1 = f(x_1, x_2)$, $s_2 = f(x_2, x_3)$, $s_3 = f(x_3, x_3)$. Then $|\text{var}(t)|, |\text{var}(t_i)|, |\text{var}(s_i)| \leq 2, i = 1, 2, 3$, i.e. $k = 2$. Now the right hand side of (C1), i.e.,

$$\begin{aligned} & \overset{\circ}{S}_3^3(t, (\overset{\circ}{S}_3^3(s_1, t_1, t_2, t_3), \overset{\circ}{S}_3^3(s_2, t_1, t_2, t_3), \overset{\circ}{S}_3^3(s_3, t_1, t_2, t_3))) \\ &= \overset{\circ}{S}_3^3(f(x_1, x_3), \overset{\circ}{S}_3^3(f(x_1, x_2), f(x_1, x_3), f(x_3, x_1), f(x_3, x_1)), \\ & \quad \overset{\circ}{S}_3^3(f(x_2, x_3), f(x_1, x_3), f(x_3, x_1), f(x_3, x_1)), \overset{\circ}{S}_3^3(f(x_3, x_1), f(x_1, x_3), f(x_3, x_1), f(x_3, x_1)))) \\ &= f(f(f(x_1, x_3), f(x_3, x_1)), f(f(x_3, x_1), f(x_3, x_1))) \end{aligned}$$

is defined.

The left hand side of (C1) has the form

$$\overset{\circ}{S}_3^3(\overset{\circ}{S}_3^3(f(x_1, x_3), f(x_1, x_2), f(x_2, x_3), f(x_3, x_3)), f(x_1, x_3), f(x_3, x_1), f(x_3, x_1))).$$

Because of

$$\begin{aligned} & |\text{var}(\overset{\circ}{S}_3^3(f(x_1, x_3), f(x_1, x_2), f(x_2, x_3), f(x_3, x_3))))| \\ &= \text{var}(f(f(x_1, x_2), f(x_3, x_3))) = \text{var}(\{x_1, x_2, x_3\}) = 3, \end{aligned}$$

the left hand side is not defined. □

Now we define another partial $(n + 1)$ -ary operation

$$\tilde{S}_n^n: \left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y) \right)^{n+1} \dashrightarrow \bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$$

on $\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y)$ by

$$\tilde{S}_n^n(t, t_1, \dots, t_n) := \begin{cases} S_n^n(t, t_1, \dots, t_n) & \text{if } |\bigcup_{i=1}^n \text{var}(t_i)| \leq k, \\ \text{not defined,} & \text{otherwise.} \end{cases}$$

Theorem 4.2. *The partial algebra*

$$\left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \tilde{S}_n^n \right)$$

is a partial Menger algebra of rank n and

$$\left(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \tilde{S}_n^n, x_1, \dots, x_n \right)$$

is a unitary partial Menger algebra of rank n, respectively.

Proof. We have to show that the first algebra satisfies (C1) and the second one satisfies (C1),(C2),(C3). Assume that $t, t_1, \dots, t_n, s_1, \dots, s_n \in W_\tau(X_n)$ and that $|\text{var}(t)|, |\text{var}(t_i)|, |\text{var}(s_i)| \leq k < n, i = 1, \dots, n$. Assume that $\tilde{S}_n^n (S_n^n (t, s_1, \dots, s_n), t_1, \dots, t_n)$ is defined. Then

$$\tilde{S}_n^n (S_n^n (t, s_1, \dots, s_n), t_1, \dots, t_n) = S_n^n (S_n^n (t, s_1, \dots, s_n), t_1, \dots, t_n)$$

and by definition, $|\bigcup_{j=1}^n \text{var}(t_j)| \leq k$ and $|\bigcup_{j=1}^n \text{var}(s_j)| \leq k$. Therefore, $\tilde{S}_n^n (s_1, t_1, \dots, t_n) \dots, \tilde{S}_n^n (s_n, t_1, \dots, t_n)$ are defined and equal to $S_n^n (s_1, t_1, \dots, t_n), \dots$ and to $S_n^n (s_n, t_1, \dots, t_n)$, respectively. If we can show that $|\bigcup_{j=1}^n S_n^n (s_j, t_1, \dots, t_n)| \leq k$, then the right hand side of (C1) is defined. Indeed,

$$\text{var}(\tilde{S}_n^n (s_j, t_1, \dots, t_n)) = \bigcup_{x_i \in \text{var}(s_j)} \text{var}(t_i) \subseteq \bigcup_{i=1}^n \text{var}(t_i)$$

and then

$$\bigcup_{j=1}^n \text{var}(\tilde{S}_n^n (s_j, t_1, \dots, t_n)) = \bigcup_{j=1}^n \bigcup_{x_i \in \text{var}(s_j)} \text{var}(t_i) \subseteq \bigcup_{j=1}^n \bigcup_{i=1}^n \text{var}(t_i) = \bigcup_{i=1}^n \text{var}(t_i)$$

and

$$\left| \bigcup_{j=1}^n \text{var}(\tilde{S}_n^n (s_j, t_1, \dots, t_n)) \right| = \left| \bigcup_{j=1}^n \bigcup_{x_i \in \text{var}(s_j)} \text{var}(t_i) \right| \leq \left| \bigcup_{i=1}^n \text{var}(t_i) \right| \leq k.$$

Therefore, $\tilde{S}_n^n (t, \tilde{S}_n^n (s_1, t_1, \dots, t_n), \dots, \tilde{S}_n^n (s_n, t_1, \dots, t_n))$ is defined and is equal to $S_n^n (t, S_n^n s_1, t_1, \dots, t_n), \dots, S_n^n (s_n, t_1, \dots, t_n)$. Since (C 1) is satisfied in clone_τ as an identity, it is satisfied as a weak identity in $(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \tilde{S}_n^n)$ and in $(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \tilde{S}_n^n, x_1, \dots, x_n)$.

Now we prove that (C2) is satisfied in $(\bigcup_{Y \subseteq X_n, |Y|=k} W_\tau(Y); \tilde{S}_n^n, x_1, \dots, x_n)$ as a weak identity. The right hand side of (C2) is always defined. Then the left hand side is defined and

$$\tilde{S}_n^n (x_i, t_1, \dots, t_n) = S_n^n (x_i, t_1, \dots, t_n) = t_i, 1 \leq i \leq n.$$

(C3) can be proved in a similar way. □

5 Partial clones of essentially k -ary terms

Now we consider the sequence $(W_\tau(X_n))_{n \geq 1}$ and for each $n \geq 1$ the set $\bigcup_{Y \subseteq X_n, |Y|=k_n} W_\tau(Y)$ and get a sequence $(\bigcup_{Y \subseteq X_n, |Y|=k_n} W_\tau(Y))_{n \geq 1}$. This sequence is in general not closed under the many-sorted superposition operations S_n^m , $m, n \geq 1$.

Example 5.1. Let $\tau = (2)$ with a binary operation symbol f , $X_3 = \{x_1, x_2, x_3\}$, $X_2 = \{x_1, x_2\}$, $k_2 = 1$, $k_3 = 2$. Then

$$\bigcup_{Y \subseteq X_3, |Y|=2} W_{(2)}(Y) = W_{(2)}(\{x_1, x_2\}) \cup W_{(2)}(\{x_1, x_3\}) \cup W_{(2)}(\{x_2, x_3\}),$$

$$\bigcup_{Y \subseteq X_2, |Y|=1} W_{(2)}(Y) = W_{(2)}(\{x_1\}) \cup W_{(2)}(\{x_2\}).$$

We consider the restriction of S_2^3 onto the direct product $\bigcup_{Y \subseteq X_2, |Y|=1} W_{(2)}(Y) \times (\bigcup_{Y \subseteq X_3, |Y|=2} W_{(2)}(Y))^2$ (using the same notation for the restriction). Then

$$S_2^3(f(x_1, x_1), f(x_1, x_2), f(x_3, x_3)) = f(f(x_1, x_2), f(x_1, x_2)) \in \bigcup_{Y \subseteq X_3, |Y|=2} W_{(2)}(Y),$$

but for the restriction onto

$$\bigcup_{Y \subseteq X_2, |Y|=2} W_{(2)}(Y) \times (\bigcup_{Y \subseteq X_3, |Y|=1} W_{(2)}(Y))^2$$

we have

$$S_2^3(f(x_1, x_2), f(x_1, x_1), f(x_3, x_3)) = f(f(x_1, x_1), f(x_3, x_3)) \notin \bigcup_{Y \subseteq X_3, |Y|=1} W_{(2)}(Y),$$

i.e. S_2^3 is not defined.

Lemma 5.2.

$$\tilde{S}_m^n: \bigcup_{Y \subseteq X_n, |Y|=k_n} W_{(\tau)}(Y) \times (\bigcup_{Y \subseteq X_m, |Y|=k_m} W_{(\tau)}(Y))^n \dashrightarrow \bigcup_{Y \subseteq X_m, |Y|=k_m} W_{(\tau)}(Y)$$

is a total operation if and only if $k_n = 1$.

Proof. Let $t \in \bigcup_{Y \subseteq X_n, |Y|=1} W_{(\tau)}(Y)$ and $t_1, \dots, t_n \in \bigcup_{Y \subseteq X_m, |Y|=k_m} W_{(\tau)}(Y)$, i.e. $|var(t)| = 1$, $|var(t_i)| \leq k_m$, and assume that $var(t) = \{x_j\}$, for some $j \in \{1, \dots, n\}$. Then $|var(S_n^m(t, t_1, \dots, t_n))| = |var(t_j)| \leq k_m$ and therefore $S_n^m(t, t_1, \dots, t_n) \in \bigcup_{Y \subseteq X_m, |Y| \leq k_m} W_{(\tau)}(Y) \leq k_m$ is defined.

Suppose that the restriction of S_n^m onto

$$\bigcup_{Y \subseteq X_n, |Y|=k_n} W_{(\tau)}(Y) \times (\bigcup_{Y \subseteq X_m, |Y|=k_m} W_{(\tau)}(Y))^n$$

is defined and there are at least two different variables $x_i \neq x_j$ in $var(t)$. Suppose that $|var(t_i)| = |var(t_j)| = k_m$ for some $1 \leq i < j \leq n$ and that $var(t_i) \neq var(t_j)$. Then

$$|var(S_n^m(t, t_1, \dots, t_n))| = |\bigcup_{x_i \in var(t)} var(t_i)| > k_m,$$

a contradiction. □

Now on the many-sorted set $(\bigcup_{Y \subseteq X_n, |Y|=k_n} W_\tau(Y))_{n \geq 1}$ we define partial operations

$$\tilde{S}_n^m: \bigcup_{Y \subseteq X_n, |Y|=k_n} W_\tau(Y) \times \left(\bigcup_{Y \subseteq X_m, |Y|=k_m} W_\tau(Y) \right)^n \dashrightarrow \bigcup_{Y \subseteq X_m, |Y|=k_m} W_\tau(Y)$$

by

$$\tilde{S}_n^m(t, t_1, \dots, t_n) := \begin{cases} S_n^m(t, t_1, \dots, t_n) & \text{if } |\bigcup_{i=1}^n \text{var}(t_i)| \leq k_m, \\ \text{not defined,} & \text{otherwise.} \end{cases}$$

and obtain a many-sorted partial algebra

$$\text{Partclone}_{\text{ess}\tau} := \left(\left(\bigcup_{Y \subseteq X_n, |Y|=k_n} W_\tau(Y) \right)_{n \geq 1}; \left(\tilde{S}_n^m \right)_{m, n \geq 1}, (x_i)_{i \leq n, n \geq 1} \right).$$

Theorem 5.3. *The many-sorted partial algebra $\text{Partclone}_{\text{ess}\tau}$ is a partial clone, i.e. it satisfies (C1), (C2), (C3) as weak identities.*

Proof. Assume that $t \in W_\tau(X)$, $|\text{var}(t)| \leq k_p$, $t_1, \dots, t_n \in W_\tau(X_m)$, $|\text{var}(t_i)| \leq k_m$, $i = 1, \dots, n$, $s_1, \dots, s_n \in W_\tau(X_n)$, $|\text{var}(s_j)| \leq k_n$, $j = 1, \dots, n$, and that $\tilde{S}_n^m(S_p^m(t, s_1, \dots, s_p), t_1, \dots, t_n)$ is defined and

$$\tilde{S}_n^m(\tilde{S}_p^m(t, s_1, \dots, s_p), t_1, \dots, t_n) = S_n^m(S_p^m(t, s_1, \dots, s_p), t_1, \dots, t_n).$$

This is the case if and only if $|\bigcup_{j=1}^p \text{var}(s_j)| \leq k_n$ and $|\bigcup_{i=1}^n \text{var}(t_i)| \leq k_m$. Then $\tilde{S}_n^m(s_l, t_1, \dots, t_n)$ are defined for all $l = 1, \dots, p$ and

$$\tilde{S}_n^m(s_l, t_1, \dots, t_n) = S_n^m(s_l, t_1, \dots, t_n)$$

for $l = 1, \dots, p$. Moreover, we have

$$\bigcup_{j=1}^p \text{var}(S_n^m(s_j, t_1, \dots, t_n)) = \bigcup_{j=1}^p \bigcup_{x_i \in s_j} \text{var}(t_i) \subseteq \bigcup_{i=1}^n \text{var}(t_i)$$

and then

$$\left| \bigcup_{j=1}^p \text{var}(S_n^m(s_j, t_1, \dots, t_n)) \right| = \left| \bigcup_{i=1}^n \text{var}(t_i) \right| \leq k_m.$$

Therefore,

$$\tilde{S}_n^m(t, \tilde{S}_n^m(s_1, t_1, \dots, t_n), \dots, \tilde{S}_n^m(s_p, t_1, \dots, t_n)) = S_p^m(t, S_n^m(s_1, t_1, \dots, t_n), \dots, S_n^m(s_p, t_1, \dots, t_n)).$$

Since (C3) is satisfied as identity in $\text{clone}\tau$, it is satisfied as weak identity in $\text{Partclone}_{\text{ess}\tau}$. (C2) and (C3) can be proved similar as in the proof of Theorem 4.2. \square

We notice that recently Kumduang and Leeratanavalee, see ([8]) defined n -ary terms with fixed variable count as another generalization of n -ary terms of a fixed variable in the following inductive way

Definition 5.4. ([8])

1. Every variable $x_i \in X_n$ is an n -ary term with fixed variable count.
2. If t_1, \dots, t_{n_i} are n -ary terms with fixed variable count and if the number of variables occurring in t_k and in t_l is equal for all $k, l \in \{1, \dots, n_i\}$, then $f_i(t_1, \dots, t_{n_i})$ is an n -ary term with fixed variable count.

Defining many-sorted partial operations on the many-sorted set of all terms with fixed variable count in an appropriate way one obtains one more partial clone. (see [8])

6 Hyperidentities and weak endomorphisms

Definition 6.1. We assume that $\tau = (n_i)_{i \in I}$ with $n_i > 1$ for all $i \in I$. Let $k := (k_{n_i})_{i \in I}$ be an indexed set of integers with $0 < k_{n_i} < n_i$ for all $i \in I$. Any mapping $\sigma_{ess_k} : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ with

$$\sigma_{ess_k}(f_i) \in \bigcup_{Y \subseteq X_{n_i}, |Y|=k_{n_i}} W_\tau(Y) \subseteq W_\tau(X_{n_i}) \text{ for all } i \in I$$

is said to be an essentially k -ary hypersubstitution of type τ , i.e., σ_{ess_k} is essentially k -ary iff $|\text{var}(\sigma_{ess_k}(f_i))| \leq k_{n_i}$ for all $i \in I$. Let $\text{Hyp}_{ess_k}(\tau)$ be the set of all essentially k -ary hypersubstitutions of type τ . Let $\hat{\sigma}_{ess_k}$ be the extension of σ_{ess_k} as defined in section 1.

Example 6.2. Let $\tau = (3)$ with a ternary operation symbol f . Then σ_{ess} with $\sigma_{ess_2}(f) = f(x_1, x_2, x_2) \in W_{(3)}(\{x_1, x_2\}) \subseteq W_{(3)}(X_3)$ is an essentially binary hypersubstitution of type (3) .

Lemma 6.3. For all $\sigma_{ess_k} \in \text{Hyp}_{ess}(\tau)$, for any $k \geq 1$ and all $t \in W_\tau(X)$ there holds:

$$\text{if } |\text{var}(t)| \leq k, \text{ then } |\text{var}(\hat{\sigma}_{ess_k}[t])| \leq k \text{ for all } i \in I.$$

Proof. We prove by induction that for any hypersubstitution σ and any term t there holds $\text{var}(\hat{\sigma}[t]) \subseteq \hat{\sigma}[t]$. By definition of the extension $\hat{\sigma}$ for any hypersubstitution σ of type τ and for any variable x , we have $\text{var}(\hat{\sigma}[x]) = \{x\} = \text{var}(x)$. Now let $t = f_i(t_1, \dots, t_{n_i})$ and inductively we suppose that $\text{var}(\hat{\sigma}[t_j]) \subseteq \text{var}(t_j)$ for $j = 1, \dots, n_i$. Then

$$\begin{aligned} \text{var}(\hat{\sigma}[t]) &= \text{var}(\hat{\sigma}[f_i(t_1, \dots, t_{n_i})]) = \text{var}(S_{n_i}^n(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])) \\ &= \bigcup_{x_i \in \text{var}(\sigma(f_i))} \text{var}(\hat{\sigma}[t_i]) \subseteq \bigcup_{j=1}^{n_i} \text{var}(\hat{\sigma}[t_j]) \subseteq \bigcup_{j=1}^{n_i} \text{var}(t_j) = \text{var}(t). \end{aligned}$$

Then also for any essentially k -ary hypersubstitution σ_{ess_k} and any term t of type τ there holds

$$\text{var}(\hat{\sigma}_{ess_k}[t]) \subseteq \text{var}(t).$$

If $|\text{var}(t)| \leq k$, then

$$|\text{var}(\hat{\sigma}_{ess_k}[t])| \leq |\text{var}(t)| \leq k.$$

□

We notice that in the proof the properties of essentially k -ary hypersubstitutions are not used. Therefore, Lemma 6.1 holds for any hypersubstitution.

Lemma 6.4. $(Hyp_{ess_k}; \circ_h)$ is a semigroup.

Proof. Let \circ_h denote the restriction of the product introduced in section 1 onto $Hyp_{ess_k}(\tau)$. We have to prove that this restriction is an operation on $Hyp_{ess_k}(\tau)$. Let $\sigma_1, \sigma_2 \in Hyp_{ess_k}(\tau)$ and let $f_i, i \in I$, be an n_i -ary operation symbol. Therefore, we have $|var(\sigma_2(f_i))| \leq k_{n_i}$. Then by Lemma 6.3,

$$|var((\sigma_1 \circ_h \sigma_2)(f_i))| = |var(\hat{\sigma}_1[\sigma_2(f_i)])| \leq k_{n_i}.$$

and thus $\sigma_1 \circ_h \sigma_2 \in Hyp_{ess_k}(\tau)$. As a subalgebra of $(Hyp(\tau); \circ_h)$ the groupoid $(Hyp_{ess_k}(\tau); \circ_h)$ is a semigroup. \square

The identity hypersubstitution σ_{id} with $\sigma_{id}(f_i) = f_i(x_1, \dots, x_{n_i})$ does not belong to $(Hyp_{ess_k}(\tau); \circ_h)$, since $|var(f_i(x_1, \dots, x_{n_i}))| = n_i$ and k_{n_i} must be smaller than n_i . This has big influences on the properties of hyperidentities which can be defined using essentially k_{n_i} -ary hypersubstitutions.

Essentially k -ary hypersubstitutions may be considered as many-sorted mappings $(\sigma_n)_{n \geq 1}$ defined as follows. Let $I_n \subseteq I$ be the set of all indices j such that $f_j, j \in I_n$ is n -ary and let $F_\tau^n := \{f_j \mid j \in I_n\}$. Let σ_n for $k_n < n$ be an essentially k_n -ary hypersubstitution, i.e., $\sigma_n(F_\tau) \subseteq \bigcup_{Y \subseteq X_n, |Y| \leq k_n} W_\tau(Y)$. Let $Hyp_{ess_{k_n}}^n(\tau)$ be the set of all σ_n and let $(Hyp_{ess_{k_n}}^n(\tau))_{n \geq 1}$ be the many-sorted set of all essentially k_n -ary hypersubstitutions of type τ . As in the one-sorted case we may define compositions $\sigma_{1_n} \circ_h^n \sigma_{2_n} = \hat{\sigma}_{1_n} \circ \sigma_{2_n}$ for the usual composition \circ of mappings and obtain a many-sorted semigroup

$$((Hyp_{ess_{k_n}}(\tau))_{n \geq 1}; (\circ_h^n)_{n \geq 1}).$$

Let $\sigma := (\sigma_n)_{n \geq 1} \in (Hyp_{ess_{k_n}}(\tau))_{n \geq 1}$ and let $\hat{\sigma} := (\hat{\sigma}_n)_{n \geq 1}$ be its many-sorted extension. We call σ an essentially k -ary hypersubstitution with $k := (k_n)_{n \geq 1}$.

Theorem 6.5. *The extension $\hat{\sigma}$ of an essentially k -ary hypersubstitution is a weak endomorphism of the partial clone $Partclone_{ess_k}(\tau)$, i.e., there holds: if $(t, t_1, \dots, t_n) \in dom \tilde{S}_n^m$, i.e., if $|\bigcup_{j=1}^n var(t_j)| \leq k_m$, then $(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]) \in dom \tilde{S}_n^m$ and*

$$\hat{\sigma}_m[\tilde{S}_n^m(t, t_1, \dots, t_n)] = \tilde{S}_n^m(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]).$$

Proof. By assumption we have $|var(t)| \leq k_n, |var(t_j)| \leq k_m, j = 1, \dots, n$, and $|\bigcup_{j=1}^n var(t_j)| \leq k_m$. Then by Lemma 6.3, $|var(\hat{\sigma}_n[t])| \leq k_n$ and $|var(\hat{\sigma}_m[t_j])| \leq k_m, j = 1, \dots, n$. By the fact (used in the proof of Lemma 3) that $var(\hat{\sigma}[t]) \subseteq var(t)$ for any term t , we get

$$\bigcup_{j=1}^n var(\hat{\sigma}_m[t_j]) \subseteq \bigcup_{j=1}^n var(t_j)$$

and then

$$|\bigcup_{j=1}^n var(\hat{\sigma}_m[t_j])| \leq |\bigcup_{j=1}^n var(t_j)|,$$

i.e.

$$(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]) \in dom \tilde{S}_n^m.$$

$$(t, t_1, \dots, t_n) \in \text{dom } \tilde{S}_n^m \text{ and } (\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]) \in \text{dom } \tilde{S}_n^m$$

implies

$$\tilde{S}_n^m(t, t_1, \dots, t_n) = S_n^m(t, t_1, \dots, t_n)$$

and

$$\tilde{S}_n^m(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]) = S_n^m(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]).$$

Since $(\hat{\sigma}_n)_{n \geq 1}$ is an endomorphism of $\text{clone}\tau$, i.e. satisfies

$$\hat{\sigma}_m[S_n^m(t, t_1, \dots, t_n)] = S_n^m(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]),$$

we get

$$\hat{\sigma}_m[\tilde{S}_n^m(t, t_1, \dots, t_n)] = \tilde{S}_n^m(\hat{\sigma}_n[t], \hat{\sigma}_m[t_1], \dots, \hat{\sigma}_m[t_n]).$$

□

References

- [1] P. Burmeister, *A Model Theoretic Oriented Approach to Partial Algebras, Introduction to Theory and Application of Partial Algebras*, Vol. **32** of Mathematical Research, Akad. Verlag, Berlin 1986.
- [2] K. Denecke, Menger Algebras and Clones of Terms, *East-West J. of Math.* Vol. **5** No. 2 (2003), 179-193.
- [3] K. Denecke, The Partial Clone of Linear Terms, *Sib. Math. J.* Vol. **57** No. 4 (2016), 589-598.
- [4] K. Denecke, Partial Clones, *Asian-European J. of Math.* Vol. **13**, No. 88, 2050161 (2020).
- [5] K. Denecke and S.L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Science Publishers 2000.
- [6] K. Denecke and S.L. Wismath, Complexity of Terms, Composition and Hypersubstitution, *Int. J. of Math. and Math. Sciences* **15** (2003), 959-969.
- [7] W. A. Dudek and V. S. Trokhimenko, *Menger Algebras of Multiplaced Functions*, 2006.
- [8] T. Kumduang and S. Leeratanavalee, The partial many-sorted algebras of terms and formulas with fixed variables count, *Discussiones Mathematicae: General Algebra and Applications* **43** (2) (2023).
- [9] F. W. Lawvere, *Functorial Semantics of Algebraic Theories*, Diss. Columbia University, New York, 1963.
- [10] N. Lekkoksung and S. Lekkoksung, On Partial Clones of k -Terms, *Discussiones Mathematicae* **41**(2) (2021), 361-379.
- [11] K. Menger, The Algebra of Functions, Past, Present, Future, *Rend. Math.* **20** (1961), 409-430.
- [12] K. Wattanatripop and Th. Changphas, Clones of Terms of a fixed Variable, *Mathematics* **8**(2) (2020), <https://doi.org/math/8020260/10.3390>.

Communicated by *Koyu Uematsu*

Klaus Denecke
University of Potsdam
Institute of Mathematics
14476 Potsdam
Karl-Liebknecht-Str. 24
e-mail:klausdenecke@hotmail.com

INTERNATIONAL ADVISORS

J.Aczel (Univ. of Waterloo, Canada) **J.Musielak** (A.Mickiewicz Univ., Poland)
A.V.Arkhangel'skii (Ohio Univ., U.S.A.) **A.Blass** (Univ.of Michigan, U.S.A.)
W.W.Comfort (Wesleyan Univ., U.S.A.) **D.A. Drake** (Univ.of Florida, Gainesville, U.S.A.)
S.Lajos (Budapest Univ.of Economics, Hungary) **P.K.Sen** (Univ. of North Carolina, U.S.A.)
O.Takenouchi (P.E.Osaka Univ.) **J.D.Monk** (Univ. of Colorado,U.S.A.)
M. Valdivia (Univ.de Valencia, Spain)

BOARD OF EDITORS

Honorary Editors

Arto Salomaa (Academy of Finland, Finland)

Saharon Shelah (Hebrew University, Israel and Rutgers University, U.S.A)

- (a) Name
- (b) Postal Address
- (b') E-mail address
- (c) Reviewable Area
- (d) additional information

AUSTRALIA

- (a) **Phillip Isaac**
- (b) School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia
- (c) psi@maths.uq.edu.au
- (d) Representation theory of Lie algebras and Lie superalgebras, quantum integrable systems, Bethe ansatz, Yang-Baxter equation

BRAZIL

- (a) **Jair Minoro Abe**
- (b) 1) Institute of Exact Sciences and Technology – Paulista University, UNIP, Rua Dr. Bacelar, 1212, 04026-002 – SAO PAULO, S.P. – BRAZIL FAX: 0055 11 55864010
2) Institute For Advanced Studies, University of Sao Paulo, Rua Praca do Relógio, 109, bloco K. 5° andar, Cidade Universitária, 05508-050, SAO PAULO, S.P. – BRAZIL
- (b') jairabe@uol.com.br
- (c) Mathematical Logic, Algebraic Logic, Foundations of Mathematics

CANADA

- (a) **Anthony To-Ming Lau**
- (b) Dept. of Mathematical and Statistical Sciences, Univ. of Alberta, Edmonton, Alberta, Canada T6G 2G1
- (b') anthonyt@ualberta.ca
- (c) harmonic analysis and functional analysis

- (a) **János Aczél**
- (b) Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
- (b') jdaczcel@uwaterloo.ca
- (c) Functional Equations

- (a) **M. S. Srivastava**
- (b) Department of Statistics, University of Toronto, 100 St. George Street Toronto, Ontario, M5S 3G3, Canada
- (b') srivasta@utstat.toronto.edu
- (c) Multivariate Analysis, Sequential Analysis, Quality Control

CZECH REPUBLIC

- (a) **Milan Vlach**
- (b) Charles University, Faculty of Mathematics and Physics
Malostranske namesti 25, 118 00 Prague 1, Czech Republic
- (b') milan.vlach@mff.cuni.cz and mvlach@ksi.ms.mff.cuni.cz
- (c) Game theory, Fair division, Optimization

FINLAND

- (a) **Arto Salomaa**
 - (b) Jaanintie 34 A 26, 20540 Turku, Finland
 - (b') asalomaa@utu.fi
 - (c) Formal languages, automata, computability, cryptography
- (a) **Jarkko Kari**
 - (b) Department of Mathematics and Statistics, FI-20014 University of Turku, Finland
 - (b') jkari@utu.fi
 - (c) automata theory, cellular automata, tilings, symbolic dynamics

GERMANY

- (a) **Klaus Denecke**
- (b) 14542 Werder (Havel), Germany, Zanderweg 3
- (b') klausdenecke@hotmail.com
- (c) General Algebra, Discrete Mathematics, Multiple-valued Logic, Ordered Sets and Lattices, Theory of Semigroups

GREECE

- (a) **Maria Fragoulopoulou**
- (b) Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, Greece
- (b') fragoulop@math.uoa.gr
- (c) Non-normed Topological Algebras, Topological Algebras with an Involution, Unbounded Operator Theory, Tensor products of Topological Algebras and Topological *-Algebras.

HUNGARY

- (a) **Gyula Maksa**
 - (b) Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary
 - (b') maksa@science.unideb.hu
 - (c) Functional equations
- (a) **Kálmán Györy**
 - (b) University of Debrecen, Institute of Mathematics, 4010 Debrecen, Hungary (b') gyory@science.unideb.hu
 - (c) Number Theory (mainly Diophantine and Algebraic Number Theory)
- (a) **Pál Dömösi**
 - (b) Institute of Mathematics and Informatics, Nyíregyháza University, Nyíregyháza, Sóstói út 31/B, H-4400, Hungary
 - (b') domosi.pal@nye.hu
 - (c) Theoretical Computer Science, Algebra

ISRAEL

- (a) **Dany Leviatan**
- (b) School of Mathematics, Tel Aviv University, 6139001 Tel Aviv, Israel
- (b') leviatan@post.tau.ac.il
- (c) Approximation Theory, Computer Added Geometric Design, Summability

ITALY

- (a) **Angelo Favini**
 - (b) Dept. of Mathematics, Bologna Univ., Piazza di Porta S. Donato, 5, 40126, Bologna, Italy
 - (b') angero.favini@unibo.it
 - (c) Evolution equations and Control theory for abstract differential equations and PDE.
- (a) **Antonio Di Crescenzo**
 - (b) Università di Salerno, Dipartimento di Matematica, Via Giovanni Paolo II, n.132, 84084 Fisciano (SA), Italy
 - (b') adicrescenzo@unisa.it
 - (c) Applied Probability, Stochastic Processes and Applications, Reliability Theory, Queueing Systems, Stochastic Models in Biology, Information Measures
- (a) **Tonia Ricciardi**
 - (b) Department of Mathematics and its Applications, Federico II University, Via Cintia, 80126 Naples, Italy.
 - (b') tonia.ricciardi@unina.it
 - (c) Nonlinear elliptic partial differential equations

NETHERLANDS

- (a) **Grzegorz Rozenberg**
- (b) Leiden Institute of Advanced Computer Science (LIACS) Leiden University, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
- (b') rozenber@liacs.nl
- (c) Natural computing, Formal languages and automata theory

POLAND

- (a) **Dariusz Zagrodny**
 - (b) Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszyński University, Dewajtis 5, 01-815 Warsaw, Poland
 - (b') dariusz.zagrodny@wmii.uni.lodz.pl
 - (c) Nonsmooth Analysis (this is my main field of research), Nonlinear Programming (convex, nonconvex), Maximal Monotonicity (with respect to duality)
- (a) **Henryk Hudzik**
 - (b) Faculty of Economics and Information Technology, The State University of Applied Sciences in Płock, Nowe Trzepowo 55, 09-402 Płock, Poland
and
Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska Street 87, 61-614 Poznań, Poland.
 - (b') hudzik@amu.edu.pl
 - (c) Function spaces theory and abstract Banach spaces theory, Banach Lattices, Geometry of Banach Spaces, Composition and multiplication operators between Köthe spaces
- (a) **Krzysztof Szajowski**
 - (b) Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wybrzeże, Wyspińskiego 27, PL-50-370 Wrocław, Poland
 - (b') Krzysztof.Szajowski@pwr.edu.pl
 - (c) Applied Probability, Game Theory, Operations Research, Mathematical Statistics

(a) **Piotr Multarzyński**

(b) Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland

(b') multarz@mini.pw.edu.pl

(c) Algebraic analysis (calculus of right invertible operators); Algebraic approach to differential geometry; Discrete counterparts of the classical concepts in analysis and differential geometry; q-calculus, Sikorski and Froelicher differential (or smooth) spaces; 12H10, 39A12, 39A70, 47B39, Groupoids - theory and applications in physics.

(a) **Tomasz Kubiak**

(b) Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

(b') tkubiak@amu.edu.pl

(c) Many-valued topology (fuzzy topology), Pointfree topology (frames and locales)

P.R.OF CHINA

(a) **Minghao Chen**

(b) Xidazhi Street 92, Harbin, 150001, China Department of Mathematics, Harbin Institute of Technology

(b') chenmh130264@aliyun.com; chenmh130264@hit.edu.cn

(c) Uncertain dynamical systems; Fuzzy differential equation; Fuzzy optimization; Fuzzy sets theory

ROMANIA

(a) **Adrian Petrusel**

(b) Babes-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Department of Mathematics, Kogalniceanu street no. 1, 400084 Cluj-Napoca, Romania

(b') petrusel@math.ubbcluj.ro

(c) Nonlinear Analysis

(a) **Ioan A. Rus**

(b) Department of Mathematics, Babes-Bolyai University, Str. Kogalniceanu No. 1, 400084 Cluj-Napoca, Romania

(b') iarus@math.ubbcluj.ro

(c) Fixed Point Theory

(a) **Vasile Berinde**

(b) Department of Mathematics and Computer Science, Faculty of Sciences Technical University of Cluj-Napoca North University Center at Baia Mare, Victoriei Nr. 76, 430122 Baia Mare, Romania

(b') vberinde@cubm.utcluj.ro

(c) Fixed Point Theory, iterative approximation of fixed points

RUSSIA

(a) **Andrei Vesnin**

(b) Sobolev Institute of Mathematics, pr. ak. Koptyuga 4, Novosibirsk, 630090, Russia

(b') vesnin@math.nsc.ru

(c) Low-dimensional topology, Knot theory, Hyperbolic manifolds and orbifolds.

(a) **Semen S. Kutateladze**

(b) The Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Academician Koptyug's Avenue 4, Novosibirsk, 630090, RUSSIA

(b') sskut@member.ams.org and sskut@math.nsc.ru

(c) Functional Analysis, Operator Theory, Convex Geometry, Optimization and Programming, Nonstandard Analysis, Boolean Valued Models

(a) **Vladimir V. Mazalov**

(b) Institute of Applied Mathematical Research, Karelia Research Center of Russian Academy of Sciences Pushkinskaya str., 11, Petrozavodsk 185610, Russia

(b') vmazalov@krc.karelia.ru

(c) Optimal Stopping Theory, Game with Optimal Stopping, Stochastic Dynamic Programming, Applications in Behavioral Ecology

(a) **Elena Parilina**

(b) Department of Mathematical Game Theory and Statistical Decisions, Saint Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg 199034, Russia

(b') e.parilina@spbu.ru<mailto-e.parilina@spbu.ru>

(c) Game Theory, Stochastic Games, Applied Mathematical Statistics.

SOUTH AFRICA

(a) **Joachim Schröder**

(b) Department van Wiskunde, Universiteit van die Vrystaat, Posbus 339, Bloemfontein 9300, South Africa

(b') schroderjd@ufs.ac.za

(c) Enumerative combinatorics, Categorical methods in topology, Set theoretic topology (cardinal invariants, elementary submodels)

SPAIN

(a) **Javier Gutierrez Garcia**

(b) Departamento de Matematicas, Universidad del Pais Vasco/Euskal Herriko Unibertsitatea UPV/EHU, Apartado 644, 48080, Bilbao, Spain

(b') javier.gutierrezgarcia@ehu.eus

(c) General topology (in particular, insertion and extension of functions), Pointfree topology, Many-valued topology

(a) **Jorge Galindo**

(b) Instituto de Matemáticas y Aplicaciones de Castellón (IMAC), Departamento de Matemáticas, Universidad Jaume I, 12071-Castellón, Spain.

(b') jgalindo@mat.uji.es

(c) Topological Algebra, Abstract Harmonic Analysis, General Topology.

(a) **Luis M. Sanchez Ruiz**

(b) ETSID-Depto. de Matematica Aplicada & CITG, Universitat Politècnica de València, E-46022 Valencia, Spain

(b') LMSR@mat.upv.es

(c) Functional Analysis, Topological Vector Spaces, Barrelledness Properties, Baire-like Spaces, Continuous Function Spaces, Wavelets

(a) **Salvador Hernandez**

(b) Departamento de Matematicas, Universitat Jaume I, 12071 Castellon, Spain

(b') hernande@uji.es

(c) Topological groups and semigroups, Spaces of continuous functions, Operators defined between spaces of continuous functions, General Topology.

TAIWAN

(a) **Hang-Chin Lai**

(b) Department of Mathematics, National Tsing Hua University, Hsin Chu City, Taiwan

(b') laihc@mx.nthu.edu.tw

(c) Nonlinear analysis and convex analysis, Optimization theory, Harmonic analysis

UNITED STATES OF AMERICA

(a) **Andreas Blass**

- (b) Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1043, USA
(b') ablass@umich.edu
(c) Mathematical logic, set theory, category theory

(a) **John B Conway**

- (b) Professor Emeritus, George Washington University, Phillip Hall 801 22nd St. NW
Washington, DC 20052, U.S.A
(b') Conway@gwu.edu
(c) Functional Analysis and Operator Theory

(a) **Paul Cull**

- (b) Computer Science, Kelley Engineering Center, Oregon State University, Corvallis, OR 97331, USA
(b') pc@cs.orst.edu
(c) Difference Equations and Dynamical Systems, Computer Science (Theory, Algorithms, Networks), Mathematical Biology (Population Models, Neural Nets)

(a) **W. Wistar Comfort**

- (b) Department of Mathematics, Wesleyan University, Wesleyan Station, Middletown, CT USA 06459
(b') wcomfort@wesleyan.edu
(c) Topological theory of topological groups, General (set-theoretic) topology

JAPAN

(a) **Mariko Yasugi**

- (b) non-public
(b') yasugi@cc.kyoto-su.ac.jp
(c) Logic Oriented Mathematics

(a) **Kohzo Yamada**

- (b) Faculty of Education, Shizuoka Univ., 836 Ohya, Shizuoka 422-8529, Japan
(b') kohzo.yamada@shizuoka.ac.jp
(c) General Topology

(a) **Yasunao Hattori**

- (b) National Institution for Academic Degrees and
Quality Enhancement of Higher Education
(b') hyasu2014@gmail.com
(c) General Topology

(a) **Yoshikazu Yasui**

- (b) Department of Modern Education, Faculty of Education, Kio University, 4-2-2, Umami-naka, Koryo-cho,
Kitakaturagi-gun, Nara, 635-0832, Japan
(b') y.yasui@kio.ac.jp
(c) General Topology

(a) **Eiichi Nakai**

(b) Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan

(b') eiichi.nakai.math@vc.ibaraki.ac.jp

(c) Real analysis, harmonic analysis, Fourier analysis, function spaces, singular and fractional integrals

(a) **Jun Kawabe**

(b) Division of Mathematics and Physics, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan

(b') jkawabe@shinshu-u.ac.jp

(c) Measure and integration, Vector measure, Nonadditive measure

(a) **Jun Ichi Fujii**

(b) Department of Educational Collaboration (Science, Mathematics and Information),
Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan

(b') fujii@cc.osaka-kyoiku.ac.jp

(c) Operator Theory

(a) **Masaru Nagisa**

(b) Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Yayoi-cho,
Chiba, 263-8522, Japan

(b') nagisa@math.s.chiba-u.ac.jp

(c) operator algebra, operator theory

(a) **Hiroyuki Osaka**

(b) Graduate School of Science and Engineering, Ritsumeikan University, 1-1-1 Noji-higashi, Kusatsu,
Shiga 525-8577 Japan

(b') osaka@se.ritsumei.ac.jp

(c) Operator Theory and Operator Algebras

(a) **Masatoshi Fujii**

(b) non-public

(b') mfujii@cc.osaka-kyoiku.ac.jp

(c) Operator Theory

(a) **Shigeo Akashi**

(b) Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science,
2641, Yamazaki, Noda-City, Chiba-Prefecture, 278-8510, Japan

(b') akashi@is.noda.tus.ac.jp

(c) Information Theory, Entropy Analysis, Applied Mathematics, Functional Analysis

(a) **Yoshitsugu Kabeya**

(b) Department of Mathematics, Osaka Metropolitan University, 1-1, Gakuencho, Naka-ku,
Sakai, Osaka 599-8531, Japan

(b') yo.kabeya@omu.ac.jp

(c) Partial Differential Equations, Ordinary Differential Equations

(a) **Atsushi Yagi**

(b) non-public

(b') yagi-atsushi-ch@alumni.osaka-u.ac.jp

(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

(a) **Yasumasa Fujisaki**

(b) Department of Information and Physical Sciences, Graduate School of Information Science and Technology,
Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan

(b') fujisaki@ist.osaka-u.ac.jp

(c) Control Systems Theory

(a) **Naruhiko Aizawa**

(b) Department of Physics, Graduate School of Science, Osaka Metropolitan University, Sakai, Osaka 599-8531, Japan

(b') aizawa@omu.ac.jp

(c) Representation theory (Lie groups and algebras and their extensions)

(a) **Masanobu Taniguchi**

(b) Dept. of Applied Mathematics, School of Fundamental Science & Engineering, Waseda University,
3-4-1, Okubo, Shinjuku-ku, Tokyo, 169-8555, Japan, Tel : 070-7775-9662

(b') taniguchi@waseda.jp

(c) Statistical Inference for Stochastic Processes

(a) **Ryusuke Hohzaki**

(b) Tsukuba Gakuin University 3-1 Azuma Tsukuba City, Ibaragi Prefecture, Japan

(b') hohzaki@tsukuba-g.ac.jp

(c) Reviewable area: Operations Research, Search theory, Game theory

(a) **Junzo Watada, PhD**

(b) Dr. Junzo Watada, WCICME, 2-10-8-407 Kobai, Yahatanishi, Kitakyushu 806-0011 Japan

(c) Meta-Heuristics, Deep Learning Neural Networks, Engineering Mathematics, Fuzzy Systems, Economics,
Management, FinTech.

#Specially Appointed Professor

Faculty of Data Science.

Shimonoseki City University

invited Research, IPS Research Center, Waseda University

Professor Emeritus, Waseda University, Japan

<http://www.wcicme.com/watada/>

Mobile: [+81-90-3464-4929](tel:+81-90-3464-4929)

Research Professor, Zhejiang Gongshang University, China

Research Professor, Indian Institute of Finance, Delhi, India

(a) **Kensaku Kikuta**

(b) School of Business Administration, University of Hyogo,

8-2-1 Gakuen-nishi-machi, Nishi-ku, Kobe City 651-2197 JAPAN

(b') kikuta@biz.u-hyogo.ac.jp

(c) Game Theory, Operations Research,

(a) **Wuyi Yue**

(b) Dept. of Intelligence and Informatics, Faculty of Intelligence and Informatics, Konan University, 8-9-1 Okamoto,
Higashinada-ku, Kobe 658-8501, JAPAN

(b') yue@konan-u.ac.jp

(c) Queueing Networks, Performance Analysis and Modeling, Communications Networks, Operations Research, Markov
Processes, Probabilistic Methods, Systems Engineering

- (a) **Hiroshi Sakai**
(b) Department of Data Science, Shimonoseki City University
〒751-8510 2-1-1, Daigakuchi
(b') sakai-hi@shimonoseki-cu.ac.jp
(d) Rough set theory, Data mining, Mathematical logic, Logic programming

- (a) **Hiroaki Sandoh**
(b) Faculty of Policy Studies Kwansai Gakuin University 1, Gakuen-Uegahara, Sanda-shi, Hyogo 669-1330 Japan
(b') sandoh@kwansai.ac.jp
(c) Operations Research and Management Science, Stochastic modeling

- (a) **Katsunori Ano**
(b) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku
Saitama-city, 337-8570, Japan
(b') k-ano@shibaura-it.ac.jp
(c) Optimal Stopping, Mathematical Finance, Applied Probability

- (a) **Koyu Uematsu**
(b) Graduate School of Management and Information Science Faculty of Global Business, Osaka International University
6-21-57 Tohdacho, Moriguchi-Shi, Osaka, 570-8555, Japan
(b') uematsu@oiu.jp
(c) Stochastic Process and its Applications, Reliability Analysis, and Game Theory

- (a) **Yoshiki Kinoshita**
(b) Dept. of Information Sciences, Faculty of Science, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa
259-1293, Japan
(b') yoshiki@kanagawa-u.ac.jp
(c) Software Science, Programming language semantics

- (a) **Tadashi Takahashi**
(b) Department of Intelligence and Informatics, Konan University, 8-9-1 Okamoto,
Higashinada, Kobe, Hyogo 658-8501, Japan
(b') takahashi@konan-u.ac.jp
(c) Mathematics Education

- (a) **Yoko Watamori**
(b) Department of Mathematics, Graduate School of Science, Osaka Metropolitan University, Sakai, Osaka 599-8531,
Japan
(b') w318y@omu.ac.jp
(c) Directional statistics, Multivariate Analysis

- (a) **Koichi Osaki**
(b) Department of Mathematical Sciences, School of Science, Kwansai Gakuin University,
1 Gakuen Uegahara, Sanda, 669-1330, Japan.
(b') osaki@kwansai.ac.jp
(c) Nonlinear partial differential equations, Infinite-dimensional dynamical systems

Managing Editor
Koyu Uematsu (Professor of Osaka International University)
International Society for Mathematical Sciences
1-5-12-202 Kaorigaoka-cho, Sakai-ku, Sakai-city, 590-0011, Japan
uematsu@jams.jp

Submission to the SCMJ

In September 2012, the way of submission to *Scientiae Mathematicae Japonicae* (SCMJ) was changed. Submissions should be sent electronically (in PDF file) to the editorial office of International Society for Mathematical Sciences (ISMS).

(1) Preparation of files and Submission

- a. Authors who would like to submit their papers to the SCMJ should make source files of their papers in LaTeX2e using the ISMS style file (`scmjlt2e.sty`) Submissions should be in PDF file compiled from the source files. Send the PDF file to s1bmt@jams.jp.
- b. Prepare a Submission Form and send it to the ISMS. The required items to be contained in the form are:
 1. Editor's name whom the author chooses from the Editorial Board (http://www.jams.or.jp/hp/submission_f.html) and would like to take in charge of the paper for refereeing.
 2. Title of the paper.
 3. Authors' names.
 4. Corresponding author's name, e-mail address and postal address (affiliation).
 5. Membership number in case the author is an ISMS member.

Japanese authors should write 3 and 4 both in English and in Japanese.

At http://www.jams.or.jp/hp/submission_f.html, the author can find the Submission Form. Fulfill the Form and send it to the editorial office by pushing the button "transmission". Or, without using the Form, the author may send an e-mail containing the items 1-5 to s1bmt@jams.jp

(2) Registration of Papers

When the editorial office receives both a PDF file of a submitted paper and a Submission Form, we register the paper. We inform the author of the registration number and the received date. At the same time, we send the PDF file to the editor whom the author chooses in the Submission Form and request him/her to begin the process of refereeing. (Authors need not send their papers to the editor they choose.)

(3) Reviewing Process

- a. The editor who receives, from the editorial office, the PDF file and the request of starting the reviewing process, he/she will find an appropriate referee for the paper.
 - b. The referee sends a report to the editor. When revision of the paper is necessary, the editor informs the author of the referee's opinion.
 - c. Based on the referee report, the editor sends his/her decision (acceptance of rejection) to the editorial office.
- (4) a. Managing Editor of the SCMJ makes the final decision to the paper valuing the editor's decision, and informs it to the author.
- b. When the paper is accepted, we ask the author to send us a source file and a PDF file of the final manuscript.
 - c. The publication charges for the ISMS members are free if the membership dues have been paid without delay. If the authors of the accepted papers are not the ISMS members, they should become ISMS members and pay ¥6,000 (US\$75, Euro55) as the membership dues for a year, or should just pay the same amount without becoming the members.

Items required in Submission Form

1. Editor's name who the authors wish will take in charge of the paper
2. Title of the paper
3. Authors' names
- 3'. 3. in Japanese for Japanese authors
4. Corresponding author's name and postal address (affiliation)
- 4'. 4. in Japanese for Japanese authors
5. ISMS membership number
6. E-mail address

Call for ISMS Members

Call for Academic and Institutional Members

Discounted subscription price: When organizations become the Academic and Institutional Members of the ISMS, they can subscribe our journal *Scientiae Mathematicae Japonicae* at the yearly price of US\$225. At this price, they can add the subscription of the online version upon their request.

Invitation of two associate members: We would like to invite two persons from the organizations to the associate members with no membership fees. The two persons will enjoy almost the same privileges as the individual members. Although the associate members cannot have their own ID Name and Password to read the online version of SCMJ, they can read the online version of SCMJ at their organization.

To apply for the Academic and Institutional Member of the ISMS, please use the following application form.

Application for Academic and Institutional Member of ISMS

<p>Subscription of SCMJ Check one of the two.</p>	<p><input type="checkbox"/> Print (US\$225)</p> <p><input type="checkbox"/> Print + Online (US\$225)</p>
<p>University (Institution)</p>	
<p>Department</p>	
<p>Postal Address where SCMJ should be sent</p>	
<p>E-mail address</p>	
<p>Person in charge</p>	<p>Name: Signature:</p>
<p>Payment Check one of the two.</p>	<p><input type="checkbox"/> Bank transfer</p> <p><input type="checkbox"/> Credit Card (Visa, Master)</p>
<p>Name of Associate Membership</p>	<p>1.</p>
	<p>2.</p>

Call for Individual Members

We call for individual members. The privileges to them and the membership dues are shown in “**Join ISMS !**” on the inside of the back cover.

Items required in Membership Application Form

1. Name
2. Birth date
3. Academic background
4. Affiliation
5. 4’s address
6. Doctorate
7. Contact address
8. E-mail address
9. Special fields
10. Membership category (See **Table 1** in “**Join ISMS !**”)

Individual Membership Application Form

1. Name	
2. Birth date	
3. Academic background	
4. Affiliation	
5. 4’s address	
6. Doctorate	
7. Contact address	
8. E-mail address	
9. Special fields	
10. Membership category	

Contributions (Gift to the ISMS)

We deeply appreciate your generous contributions to support the activities of our society.

The donation are used (1) to make medals for the new prizes (Kitagawa Prize, Kunugi Prize, and ISMS Prize), (2) to support the IVMS at Osaka University Nakanoshima Center, and (3) for a special fund designated by the contributors.

Your remittance to the following accounts of ours will be very much appreciated.

- (1) Through a post office, remit to our giro account (in Yen only):
No. 00930-1-11872, Japanese Association of Mathematical Sciences (JAMS)
or send International Postal Money Order (in US Dollar or in Yen) to our
address:
International Society for Mathematical Sciences
2-1-18 Minami Hanadaguchi, Sakai-ku, Sakai, Osaka 590-0075, Japan
- (2) A/C 94103518, ISMS
CITIBANK, Japan Ltd., Shinsaibashi Branch
Midosuji Diamond Building
2-1-2 Nishi Shinsaibashi, Chuo-ku, Osaka 542-0086, Japan

Payment Instructions:

Payment can be made through a post office or a bank, or by credit card. Members may choose the most convenient way of remittance. Please note that we do not accept payment by bank drafts (checks). For more information, please refer to an invoice.

Methods of Overseas Payment:

Payment can be made through (1) a post office, (2) a bank, (3) by credit card, or (4) UNESCO Coupons.

Authors or members may choose the most convenient way of remittance as are shown below. Please note that **we do not accept payment by bank drafts (checks)**.

(1) Remittance through a post office to our giro account No. 00930-1-11872 or send International Postal Money Order to our postal address (2) Remittance through a bank to our account No. 94103518 at Shinsaibashi Branch of CITIBANK (3) **Payment by credit cards** (AMEX, VISA, MASTER or NICOS), or (4) Payment by UNESCO Coupons.

Methods of Domestic Payment:

Make remittance to:

- (1) Post Office Transfer Account - 00930-3-73982 or
- (2) Account No.7726251 at Sakai Branch, SUMITOMO MITSUI BANKING CORPORATION, Sakai, Osaka, Japan.

All of the correspondences concerning subscriptions, back numbers, individual and institutional memberships, should be addressed to the Publications Department, International Society for Mathematical Sciences.

Join ISMS !

ISMS Publications: We published **Mathematica Japonica (M.J.)** in print, which was first published in 1948 and has gained an international reputation in about sixty years, and its offshoot **Scientiae Mathematicae (SCM)** both online and in print. In January 2001, the two publications were unified and changed to **Scientiae Mathematicae Japonicae (SCMJ)**, which is the “21st Century New Unified Series of Mathematica Japonica and Scientiae Mathematicae” and published both online and in print. Ahead of this, the online version of SCMJ was first published in September 2000. The whole number of SCMJ exceeds 270, which is the largest amount in the publications of mathematical sciences in Japan. The features of SCMJ are:

- 1) About 80 eminent professors and researchers of not only Japan but also 20 foreign countries join the Editorial Board. The accepted papers are published both online and in print. SCMJ is reviewed by Mathematical Review and Zentralblatt from cover to cover.
- 2) SCMJ is distributed to many libraries of the world. The papers in SCMJ are introduced to the relevant research groups for the positive exchanges between researchers.
- 3) **ISMS Annual Meeting:** Many researchers of ISMS members and non-members gather and take time to make presentations and discussions in their research groups every year.

The privileges to the individual ISMS Members:

- (1) No publication charges
- (2) Free access (**including printing out**) to the online version of SCMJ
- (3) Free copy of each printed issue

The privileges to the Institutional Members:

Two associate members can be registered, free of charge, from an institution.

Table 1: Membership Dues for 2019

Categories	Domestic	Overseas	Developing countries
1-year Regular member	¥8,000	US\$80 , Euro75	US\$50, Euro47
1-year Students member	¥4,000	US\$50 , Euro47	US\$30 , Euro28
Life member*	Calculated as below*	US\$750 , Euro710	US\$440, Euro416
Honorary member	Free	Free	Free

(Regarding submitted papers, we apply above presented new fee after April 15 in 2015 on registration date.) * Regular member between 63 - 73 years old can apply the category.

$$(73 - \text{age}) \times \text{¥}3,000$$

Regular member over 73 years old can maintain the qualification and the privileges of the ISMS members, if they wish.

Categories of 3-year members were abolished.

CONTENTS

SADAHIRO MAEDA : REAL HYPERSURFACES AND CURVE THEORY IN A NONFLAT COMPLEX SPACE FORM	1
TOSHIHARU KAWASAKI : A STRUCTURE OF THE EXTENDED INTEGRAL	13
MARIKO YASUGI , YOSHIKI TSUJII : CONTINUITY AND COMPUTABILITY: A FRAMEWORK FOR TRANSITION AND EXTENSION	23
ETSUO HAMADA : ON THE MAXIMUM OF CRAMÉR'S V.....	37
MASAAKI KUMAZAWA : A SPECIAL CLASS OF BCK-ALGEBRAS WITH CONDITION (S) :	41
KLAUS DENECKE : THE PARTIAL CLONE OF ESSENTIALLY k-ARY TERMS	57

Notices from the ISMS

Call for Papers for SCMJ.....	1
Call for ISMS Members.....	12

発行：一般社団法人国際数理科学協会
大阪府堺市堺区香ヶ丘町1丁5-12浅香山マンション202号
TEL 072-222-1850
FAX 072-222-1850
email trsr@jams.jp