SOME OPERATOR DIVERGENCES BASED ON PETZ-BREGMAN DIVERGENCE

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Received October 9, 2015 ; revised November 16, 2015

Abstract. Let A and B be strictly positive operators on a Hilbert space. For relative operator entropies $S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, $T_\alpha(A|B) \equiv \frac{1}{\alpha}(A \sharp_\alpha B - A)$ and $S_\alpha(A|B) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, we showed

$$(*) \qquad S_1(A|B) \geq -T_{1-\alpha}(B|A) \geq S_\alpha(A|B) \geq T_\alpha(A|B) \geq S(A|B) \quad \text{for } \alpha \in (0, 1).$$

Petz gave an operator divergence $D_0(A|B) = B - A - S(A|B)$ which we call Petz-Bregman divergence. Petz also defined Bregman divergence $D_\Psi(X, Y)$ for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space.

In this paper, firstly, we define new operator divergences as the differences between two terms in $(*)$ and represent them by using $D_0(A|B)$. Secondly, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x, y) = D_0(A \natural_y B|A \natural_x B)$ for $\Psi(t) = A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1, 0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Bregman divergences $D_\Psi(1, 0)$ for several functions Ψ which relate to the operator divergences defined above.

Key words and phrases. operator divergence, operator valued α-divergence, relative operator entropy, Tsallis relative operator entropy, Petz-Bregman divergence.