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ON DUCCI MATRIX SEQUENCES

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ABSTRACT. For each irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, there is a unique Ducci matrix sequence $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots$ associated with it. We first consider the function j that maps each $\alpha \in (0,1) \setminus \mathbb{Q}$ to the sequence $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ of indexes of its Ducci matrix sequence expansion. While continuity of j and j^{-1} is easily checked, we show that j^{-1} is moreover uniformly continuous. We then study the distribution of Ducci matrices in the Ducci matrix sequence expansion of a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ by considering the following three conditions on the sequence $j(\alpha)$:

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\};$$
$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

We prove that the top implies the middle and the middle implies the bottom. We also give examples witnessing that the converse to these two implications are not true in general. In addition, various equivalent statements to the first condition will be presented. Furthermore, we shall give measure theoretic treatment of the subject: We prove that for almost every α, each Ducci matrix appears in the Ducci matrix sequence expansion of α infinitely often. We then ask if the second (and also the third) condition above holds almost everywhere. Related questions as well as several partial results will be presented.

Key words and phrases. Ducci map; Continued fraction; Ducci matrix sequence; Distribution; Measure; Almost everywhere.