GROUPOID FACTORIZATIONS IN THE SEMIGROUP OF BINARY SYSTEMS

HIBA F. FAYOUMI

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ABSTRACT. Let $(X, \bullet)$ be a groupoid (binary algebra) and $Bin(\dot{X})$ denote the collection of all groupoids defined on X. We introduce two methods of factorization for this binary system under the binary groupoid product "$\diamond$" in the semigroup $(Bin(X), \diamond)$. We conclude that a strong non-idempotent groupoid can be represented as a product of its *similar*- and *signature*- derived factors. Moreover, we show that a groupoid with the orientation property is a product of its *orient*- and *skew*- factors. These unique factorizations can be useful for various applications in other areas of study. Application to algebras such as $B/BCH/BCI/BCK/BH/BI/d$-algebra are widely given throughout this paper.

1. Introduction

Algebraic structures play a vital role in mathematical applications such as information science, network engineering, computer science, cell biology, etc. This encourages sufficient motivation to study abstract algebraic concepts and review previously obtained results. One such concept of interest to many mathematicians over the past two decades or so is that of a simple yet very interesting notion of a single set with one binary operation, historically known as magma and more recently referred to as groupoid. Bruck [8] published the book,"A Survey of Binary Systems" in which the theory of groupoids, loops, quasigroups, and several algebraic structures were discussed. Borůvka in [7] explained the foundations for the theory of groupoids, set decompositions and their application to binary systems.

Given a binary operation "$\bullet$" on a non-empty set X, the groupoid $(X, \bullet)$ is a generalization of the very well-known structure of a group. H. S. Kim and J. Neggers in [33] investigated the structure $(Bin(X), \diamond)$ where $Bin(X)$ is the collection of all binary systems (groupoids or algebras) defined on a non-empty set X along with an associative binary product $(X, *) \diamond (X, \circ) = (X, \bullet)$ such that $x \bullet y = (x * y) \circ (y * x)$ for all x, $y \in X$. They recognized that the left-zero-semigroup serves as the identity of this semigroup. The present author in [11] introduced the notion of the center $ZBin(X)$ in the semigroup $(Bin(X), \diamond)$, and proved that $(X, \bullet) \in ZBin(X)$, if and only if $(X, \bullet)$ is locally-zero. Han and Kim in [13] introduced the notion of hypergroupoids $HBin(X)$, and showed that $(HBin(X), \diamond)$ is a supersemigroup of the semigroup $(Bin(X), \diamond)$ via the identification $x \longleftrightarrow \{x\}$. They proved that $(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra.

In this paper, we investigate the following problems:

Main Problem: Consider the semigroup $(Bin(X), \diamond)$. Let the left-zero-semigroup be denoted as $id_{Bin(X)}$. Given a groupoid (binary system) $(X, \bullet) \in Bin(X)$, is it possible to find two groupoid factors $(X, *)$ and $(X, \circ)$ such that

$$(X, \bullet) = (X, *) \diamond (X, \circ)?$$

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