PARAMETRIC OPERATOR FUNCTION VIA FURUTA INEQUALITY

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Dedicated to the memory of the late Professor Hiroyuki Kuroda with deep sorrow

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Abstract. We give a result related to parametric operator function on two parameters via Furuta inequality, which is an extension of recent Kamei's result [11].

1. Introduction. In what follows, a capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible. α-mean is defined by

$$A \natural_\alpha B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$$

for any $\alpha \in [0,1]$ for positive operators A and B by [12]. Very recently, Professor E.Kamei [11] has obtained the following excellent results.

Theorem A [11]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq \delta_2 \geq \delta_1 \geq 1$,*

$$(A^t \natural_{\frac{\delta_2 - t}{p-t}} B^p)^{\frac{1}{\delta_2}} \geq (A^t \natural_{\frac{\delta_1 - t}{p-t}} B^p)^{\frac{1}{\delta_1}},$$

that is, for each $t \leq 0$, $f(\delta) = (A^t \natural_{\frac{\delta - t}{p-t}} B^p)^{\frac{1}{\delta}}$ is increasing for δ such that $p \geq \delta \geq 1$.

Theorem B [11]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq \delta \geq 1$,*

$$A \geq B \geq (A^t \natural_{\frac{\delta - t}{p-t}} B^p)^{\frac{1}{\delta}} \geq A^t \natural_{\frac{1-t}{p-t}} B^p.$$

2. Parametric operator function. Theorem A is related to an operator function on one parameter δ, here we show Theorem 1 related to parametric operator function on two parameters r and s as an extension of Theorem A.

Theorem 1. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}} \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is increasing for s such that $1 \geq s \geq \frac{1-t}{p-t}$ and decreasing for r such that $0 \geq r \geq t$.

Corollary 2 can be considered as a precise estimation of Theorem B.

Corollary 2. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq 1$,*

$$\begin{aligned} A \geq B &\geq (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \\ &\geq A^{r-t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \geq A^t \natural_{\frac{1-t}{p-t}} B^p \end{aligned}$$

holds for $0 \geq r \geq t$ and $1 \geq s \geq \frac{1-t}{p-t}$.

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3. Proofs of the results. We cite the following results to give a proof of Theorem 1.

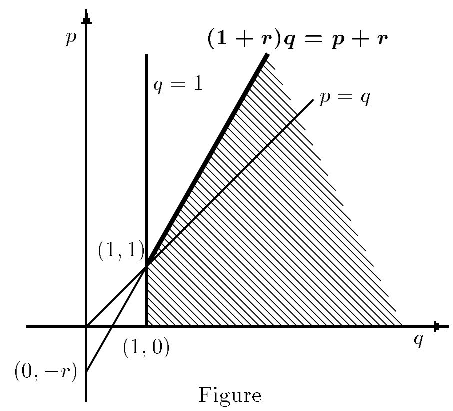
Theorem F (Furuta inequality) [5].
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

Theorem F ensures the famous Löwner-Heinz inequality when we put $r = 0$ in (i) or (ii) of Theorem F; $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. Alternative proofs of Theorem F are given [2][10] and one page proof is in [6]. It is shown in [13] that the domain drawn for p q and r in Fugure is the best possible one for (i) and (ii) of Theorem F.

Lemma 1.[9] *Let A be invertible operator and let B be positive invertible operator. For any real number λ,*

$$(ABA^*)^\lambda = AB^{\frac{1}{2}}(B^{\frac{1}{2}}A^*AB^{\frac{1}{2}})^{\lambda-1}B^{\frac{1}{2}}A^*.$$

Lemma 2. [3][7][8] *If $A \geq B \geq 0$, then for a fixed $q \geq 0$ and $t \leq 0$,*

$$F_q(p) = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}$$

is decreasing for $p \geq q$.

Proof of Theorem 1.

(a) *Proof of the result that $F_{p,t}(A,B,r,s)$ is increasing for s.*

$A \geq B \geq 0$ ensures the following (1) for $p \geq q \geq 1$ and $t \leq 0$

(1) $$A^{\frac{-t}{2}}B^qA^{\frac{-t}{2}} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}} \text{ by Lemma 2}$$

Multiplying $A^{\frac{r}{2}}$ on both sides of (1), we have

(2) $$A^{\frac{r-t}{2}}B^qA^{\frac{r-t}{2}} \geq A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}} \text{ for } 0 \geq r \geq t.$$

Then we have

(3) $$\begin{aligned} A^{1-t+r} &\geq (A^{\frac{r-t}{2}}B^qA^{\frac{r-t}{2}})^{\frac{1-t+r}{q-t+r}} \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q-t+r}} \quad \text{for } 0 \geq r \geq t, \end{aligned}$$

and the first inequality follows by Furuta inequality and the second one follows by applying Löwner-Heinz inequality to (2). In (3) put $A_1 = A^{1-t+r}$ and

$B_1 = \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q-t+r}}$. Then $A_1 \geq B_1 \geq 0$ with $A_1 > 0$, so that repeating (3) again for $p_1 \geq q_1 \geq 1$, we have

(4) $$\begin{aligned} &A_1^{1-t_1+r_1} \\ &\geq (A_1^{\frac{r_1-t_1}{2}}B_1^{q_1}A_1^{\frac{r_1-t_1}{2}})^{\frac{1-t_1+r_1}{q_1-t_1+r_1}} \end{aligned}$$

$$\geq \{A_1^{\frac{r_1}{2}}(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{\frac{q_1-t_1}{p_1-t_1}}A_1^{\frac{r_1}{2}}\}^{\frac{1-t_1+r_1}{q_1-t_1+r_1}}$$

holds for any $0 \geq r_1 \geq t_1$. In (4), put

$$p_1 = \frac{q-t+r}{1-t+r}, \qquad q_1 = \frac{q'-t+r}{1-t+r}$$

for $p \geq q \geq q' \geq 1$. Then $p_1 \geq q_1 \geq 1$. Also put $r_1 = t_1 = \dfrac{r}{1-t+r} \leq 0$. Then

$$A_1^{\frac{r_1}{2}} = A_1^{\frac{t_1}{2}} = A^{\frac{r}{2}}, \qquad \frac{q_1-t_1}{p_1-t_1} = \frac{q'-t}{q-t}$$

and

$$B_1^{p_1} = A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}.$$

Therefore (4) implies

$$\begin{aligned} A_1 &\geq B_1 \\ &\geq \{A^{\frac{r}{2}}[A^{\frac{-r}{2}}A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}A^{\frac{-r}{2}}]^{\frac{q'-t}{q-t}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q'-t+r}}, \end{aligned}$$

that is,

$$\begin{aligned} &A^{1-t+r} \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q-t+r}}, \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q'-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q'-t+r}}, \end{aligned} \tag{5}$$

for $p \geq q \geq q' \geq 1$ and $0 \geq r \geq t$. Replacing $s = \frac{q-t}{p-t}$ and $s' = \frac{q'-t}{p-t}$ in (5), then $1 \geq s \geq s' \geq \frac{1-t}{p-t}$ since $p \geq q \geq q' \geq 1$, so the proof of (a) is complete by (5).

(b) *Proof of the result that* $F_{p,t}(A, B, r, s)$ *is decreasing for* r.

We recall the following (6) by (3) and Löwner-Heinz theorem

$$A^u \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{u}{(p-t)s+r}} \qquad \text{for } 1-t+r \geq u \geq 0 \tag{6}$$

Put $D = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{s}{2}}$. Then

$$\begin{aligned} F_{p,t}(A,B,r,s) &= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \\ &= D(DA^rD)^{\frac{1-t-(p-t)s}{(p-t)s+r}}D \qquad \text{by Lemma 1} \\ &= D\{(DA^rD)^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{1-t-(p-t)s}{(p-t)s+r+u}}D \\ &= D\{DA^{\frac{r}{2}}(A^{\frac{r}{2}}D^2A^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}A^{\frac{r}{2}}D\}^{\frac{1-t-(p-t)s}{(p-t)s+r+u}}D \qquad \text{by Lemma 1} \\ &\geq D(DA^{\frac{r}{2}}A^uA^{\frac{r}{2}}D)^{\frac{1-t-(p-t)s}{(p-t)s+r+u}}D \\ &= D(DA^{r+u}D)^{\frac{1-t-(p-t)s}{(p-t)s+r+u}}D \\ &= F_{p,t}(A,B,r+u,s), \end{aligned}$$

and the last inequality follows by (6) and Löwner-Heinz theorem since $\frac{1-t-(p-t)s}{(p-t)s+r+u} \in [-1, 0]$ and finally taking inverses on both sides, so the proof of (b) is complete.

Whence the proof of theorem 1 is complete.

Proof of Corollary 2. Theorem 1 asserts that the following interpolation result.

If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq 1$,

$$F_{p,t}(A,B,t,1) \geq F_{p,t}(A,B,t,s) \geq F_{p,t}(A,B,r,s) \geq F_{p,t}(A,B,r,\tfrac{1-t}{p-t})$$

holds for $0 \geq r \geq t$ and $1 \geq \frac{1-t}{p-t}$, that is,

$$\begin{aligned}
&A^{\frac{-t}{2}}BA^{\frac{-t}{2}}\\
&\geq A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}}A^{\frac{-t}{2}}\\
&\geq A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}\\
&\geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{1-t}{p-t}}
\end{aligned}$$

Multiplying $A^{\frac{t}{2}}$ on both sides of the inequalities stated above, we have Corollary 2.

Proof of Theorem A. In Theorem 1, put $s = \frac{\delta-t}{p-t}$ for $p \geq \delta \geq 1$ and $r = t$. Then we have Theorem A.

Proof of Theorem B. We have only to put $s = \frac{\delta-t}{p-t}$ for $p \geq \delta \geq 1$ and $r = t$ in Corollary 2.

4. Concluding remark. We established the following Theorem G [9] which interpolates Theorem F and the inequality equivalent to the main result of log majorization by Ando-Hiai [1] and an alternative mean theoretic proof of Theorem G is given in [4].

Theorem G. [4][9] *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$G_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for both r and s such that $r \geq t$ and $s \geq 1$ ..

Remark 1. It is interesting to point out that our Theorem 1 is parallel result to Theorem G, that is, $F_{p,t}(A,B,r,s)$ in Theorem 1 is the same form as $G_{p,t}(A,B,r,s)$ in Theorem G and the differences between these two operator functions are nothing but the differences of the ranges of the parameters t, r and s, that is, the range of the former is

(f) $\quad t \leq 0,\ p \geq 1,\ 1 \geq s \geq \frac{1-t}{p-t}$ and $0 \geq r \geq t$

one of the latter is

(g) $\quad t \in [0,1],\ p \geq 1,\ s \geq 1$ and $r \geq t$.

We would like to emphasize that the two operator functions $F_{p,t}(A,B,r,s)$ in Theorem 1 and $G_{p,t}(A,B,r,s)$ in Theorem G are very important forms in order to research several problems associated with operator functions.

We would like to express our cordial thanks to Professor E.Kamei for sending his excellent Theorem A to us.

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