EQUIVALENCES IN THE $\star$-SHAPE CATEGORY

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Abstract. We shall present in this paper the results on recognition of equivalences in the shape category of arbitrary $C^\star$-algebras described recently by the author. These results resemble analogous G. Kozlowski's characterization of shape equivalences in the usual (commutative) shape theory. As an application, we give also some characterizations of $C^\star$-algebras of trivial shape.

1. Introduction

In the manuscript entitled "Shape theory for arbitrary $C^\star$-algebras" the author has recently described the non-commutative shape category. This category is called the $\star$-shape category. It has $C^\star$-algebras as objects and $\star$-homotopy classes of fundamental $\star$-sequences as morphisms. Our description of the $\star$-shape category followed the original Borsuk's method based on the notion of a fundamental sequence (see [3] and [4]). An analogous approach to the strong shape theory of separable $C^\star$-algebras based on the notion of an asymptotic homomorphism was earlier considered by A. Connes and N. Higson [9].

The present article is a continuation of our studies of the $\star$-shape category. The problem which we address here is the recognition of isomorphisms in the $\star$-shape category. The motivation for our results comes from G. Kozlowski's theorem in his famous unpublished paper "Images of ANR's" which claims that a map $f : X \to Y$ between topological spaces is a shape equivalence if and only if the induced functions $f_P^\sharp : [Y; P] \to [X; P]$ of homotopy classes of maps into any polyhedron P are all bijections.

For $C^\star$-algebras the role of polyhedra is taken by classes of internally movable and internally calm $C^\star$-algebras that we defined in [8, §7]. Our results for $C^\star$-algebras are in certain details more satisfying than Kozlowski's theorem for topological spaces. They show complete duality between $\star$-homotopy classes into and from the test $C^\star$-algebras.

In the final section we shall apply these characterizations in order to get better understanding of the simplest of all $C^\star$-algebras from the $\star$-shape theory point of view. These are the $C^\star$-algebras which are isomorphic in the $\star$-shape category with the trivial $C^\star$-algebra $\mathbb{O}$.

The organization of this paper is briefly as follows. The §2 explains our notation and recalls some standard conventions in our exposition. In §3 we recall briefly the definition of the $\star$-shape category from [8]. This requires to define first $\star^\varepsilon$-homomorphisms and the relation of $\star^\varepsilon$-homotopy for them. The $\star^\varepsilon$-homomorphisms are nonexpansive functions which satisfy conditions for $\star$-homomorphisms only approximately. The $\star^\varepsilon$-homotopy is obtained from the $\star$-homotopy by a similar approximation or asymptotic approach. Next

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we introduce fundamental $\star$-sequences and the notion of the $\star$-homotopy for them. The most demanding is the description of composition of $\star$-homotopy classes of fundamental $\star$-sequences. We close this preliminary section with definitions of internally movable and internally calm $C^\star$-algebras.

The long §4 contains eight theorems that deal with the Whitehead type results which search for conditions under which will a $\star$-homomorphism $f : A \to B$ between $C^\star$-algebras be a $\star$-shape equivalence. The conditions are expressed in terms of the induced functions $f_\sharp^C$ and $f_C^\sharp$ between $\star$-homotopy classes of $\star$-homomorphisms from A and B into a $C^\star$-algebra C belonging to some of the classes from §3 and from C into A and B, respectively. These results are similar to G. Kozlowski's characterization of shape equivalences [12]. However, in contrast with [12] where only functions $f_\sharp^C$ occur, here for $C^\star$-algebras both functions appear and statements for them are dual to each other.

The final §5 treats $C^\star$-algebras of trivial $\star$-shape. We show that they agree with approximately $\star$-contractible $C^\star$-algebras and as an illustration of use of results from the previous section we prove three more theorems that are analogous to corresponding results about spaces of trivial shape. However, in the realm of $C^\star$-algebras in general much richer theory develops. The arguments are usually simpler and more direct due to abundance of structure and the fact that everything can be done with sequences.

2. Preliminaries and notation

In this paper by a *$C^\star$-algebra* we mean a complete normed algebra A over the field $\mathbb{C}$ of complex numbers with an involution $\star$ such that

(1) $x^{\star\star} = x$,
(2) $(\lambda x + \mu y)^\star = \overline{\lambda} x^\star + \overline{\mu} y^\star$,
(3) $(xy)^\star = y^\star x^\star$,
(4) $(\|x\|_A)^2 = \|x^\star x\|_A$,

for all x, $y \in A$ and all λ, $\mu \in \mathbb{C}$, where $\overline{\lambda}$ is complex conjugate of λ and $\| \ \|_A$ denotes the norm on A. Any algebraic $\star$-homomorphism (i. e., respecting the involution) between two $C^\star$-algebras is norm-decreasing thus uniformly continuous and every $\star$-isomorphism between two $C^\star$-algebras is isometric. When speaking of homomorphisms between $C^\star$-algebras we shall always assume that they are $\star$-homomorphisms. We recommend the books [5], [10], [13], and [14], as general references for the theory of $C^\star$-algebras.

The symbol 0_A denotes the zero element of the $C^\star$-algebra A.

For a $C^\star$-algebra B and a compact topological space X, let $C(X; B)$ denote the $C^\star$-algebra of all continuous functions from X into B. The norm $\| \ \|_{C(X;\,B)}$ on $C(X; B)$ is given by

$$\|f\|_{C(X;\,B)} = sup\{\|f(t)\|_B \,|\, t \in X\}.$$

Let I denote the unit closed segment $[0, 1]$ of real numbers. For every t in I, there is a natural evaluation $\star$-homomorphism $e_t^B : C(I; B) \to B$ defined by $e_t^B(f) = f(t)$ for every f in $C(I; B)$.

Our shape theory is an improvement of the $\star$-homotopy theory for $C^\star$-algebras which studies the equivalence relation of $\star$-homotopy on $\star$-homomorphisms. Recall that $\star$-homomorphisms f and g between $C^\star$-algebras A and B are *$\star$-homotopic* and we write $f \simeq_\star g$ provided there is a $\star$-homomorphism $h : A \to C(I; B)$ such that $h_0 = e_0^B \circ h = f$ and $h_1 = e_1^B \circ h = g$. The $\star$-homomorphism h is said to be a *$\star$-homotopy* that joins f and g or which realizes the relation $f \simeq_\star g$. For an efficient introduction to some aspects of $\star$-homotopy theory the reader should consult P. Kohn's thesis [11], J. Rosenberg's excellent expository article [15], and the author's paper [7].

3. Description of the $\star$-shape category

This section includes an efficient description of the $\star$-shape category from [8]. We recall the basic definitions and constructions necessary for our main results in §§4 and 5.

We begin with the definition of a $\star^{\varepsilon}$-homomorphism that resembles asymptotic homomorphisms from [9].

Let A and B be $C^\star$-algebras. Let ε be a positive real number. A function $f : A \to B$ is a *$\star^{\varepsilon}$-homomorphism* provided

(1) f takes the zero element 0_A of A into the zero element 0_B of B,
(2) f is nonexpansive, i. e., the relation $\|f(x) - f(y)\|_B \leqslant \|x - y\|_A$ holds for all $x, y \in A$, and
(3) $\|f(x + y) - f(x) - f(y)\|_B < \varepsilon(\|x\|_A + \|y\|_A)$ for all $x, y \in A$.
(4) $\|f(bx) - bf(x)\|_B < \varepsilon\|x\|_A$ for each $x \in A$ and each $b \in \mathbb{C}$.
(5) $\|f(xy) - f(x)f(y)\|_B < \varepsilon\|x\|_A\,\|y\|_A$ for all $x, y \in A$.
(6) $\|f(x^\star) - f(x)^\star\|_B < \varepsilon\|x\|_A$ for each $x \in A$.

Observe that a $\star^{\varepsilon}$-homomorphism is a uniformly continuous function. Moreover, for every real number ε between 0 and 1 and every $C^\star$-algebra A the function f from A into itself which takes an $x \in A$ into the product of ε and x is an example of a $\star^{\varepsilon}$-homomorphism which is not a $\star$-homomorphism.

Another basic notion is that of the *$\star^{\varepsilon}$-homotopy* for nonexpansive functions of $C^\star$-algebras.

Let $\varepsilon > 0$. Nonexpansive functions f and g between $C^\star$-algebras A and B are *$\star^{\varepsilon}$-homotopic* and we write $f \overset{\varepsilon}{\simeq}_\star g$ provided there is an $\star^{\varepsilon}$-homomorphism $h : A \to C(I; B)$ with $h_0 = e_0^B \circ h = f$ and $h_1 = e_1^B \circ h = g$. We shall also say that h is a $\star^{\varepsilon}$-homotopy which *joins* f and g or that it *realizes* the relation or $\star^{\varepsilon}$-homotopy $f \overset{\varepsilon}{\simeq}_\star g$.

We can now introduce fundamental $\star$-sequences and define the relation of $\star$-homotopy for them. These definitions correspond to Borsuk's definitions in [3] and [4] of a fundamental sequence and a homotopy for fundamental sequences.

Let A and B be $C^\star$-algebras. A family $\varphi = \{f_i\}_{i=1}^{\infty}$ of nonexpansive functions $f_i : A \to B$ is a *fundamental $\star$-sequence* from A into B provided for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_\star f_i$ for every $j \geqslant i$.

We use functional notation $\varphi : A \to B$ to indicate that φ is a fundamental $\star$-sequence from A into B. Let $F_\star(A, B)$ denote all fundamental $\star$-sequences $\varphi : X \to Y$.

Two families $\varphi = \{f_i\}_{i=1}^{\infty}$ and $\psi = \{g_i\}_{i=1}^{\infty}$ of nonexpansive functions $f_i, g_i : A \to B$ are *$\star$-homotopic* and we write $\varphi \simeq_\star \psi$ provided for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_\star g_j$ for every $j \geqslant i$.

The relation of $\star$-homotopy is an equivalence relation on the set $F_\star(A, B)$. The $\star$-homotopy class of a fundamental $\star$-sequence φ is denoted by $[\varphi]_\star$ and the set of all $\star$-homotopy classes by $\mathcal{S}h_\star(A, B)$.

In order to organize $C^\star$-algebras and $\star$-homotopy classes of fundamental $\star$-sequences into a $\star$-shape category $\mathcal{S}h_\star$, we must define a composition for $\star$-homotopy classes of fundamental $\star$-sequences and establish it's associativity.

The definition of the composition is the only tricky part in setting up the category $\mathcal{S}h_\star$. Our idea is to associate to every fundamental $\star$-sequence $\varphi : A \to B$ two increasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. The first function associates to an index $i \in \mathbb{N}$ of the sequence $\varphi = \{f_i\}$ a much larger index $\varphi(i)$ in $\mathbb{N}$ such that f_j and f_k are joined by a $\star^{1/i}$-homotopy whenever $j, k \geqslant \varphi(i)$. The second function associates to an $i \in \mathbb{N}$ an element $f(i)$ of $\mathbb{N}$ such that the reciprocal value $1/f(i)$ of $f(i)$ is sufficiently small. This is a rough description of these functions and now we proceed with the details.

Let us agree that an *increasing* function $f : P \to P$ of a partially ordered set $(P, <)$ into itself is a function which satisfies $x < f(x)$ for every $x \in P$ and $x < y$ in P implies $f(x) < f(y)$. In the case when the domain and the codomain of a function f are different, the first requirement is dropped.

When defining increasing functions that connect indexing sets we shall repeatedly use the following simple lemma.

(3.1) Lemma. *Let $\{f_1, \dots, f_n\}$ be functions from a cofinite directed set $(M, <)$ into a directed set $(L, <)$. Then there is an increasing function $g : M \to L$ such that $g(x) \geq f_1(x), \dots, f_n(x)$ for every $x \in M$.*

For a positive real number ε and a natural number n, let

$$\varepsilon^{\langle n \rangle} = \{i \in \mathbb{N} \mid i > n/\varepsilon\}.$$

The sets $\varepsilon^{\langle 1 \rangle}$ and $\varepsilon^{\langle 2 \rangle}$ are denoted by ε^* and ε^{**}, respectively.

Let $\varphi = \{f_i\} : A \to B$ be a fundamental $\star$-sequence between $C^\star$-algebras. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that for every $i \in \mathbb{N}$ the relation $j, k \geqslant \varphi(i)$ implies the relation $f_j \overset{1/i}{\simeq}_\star f_k$. The multiple use of notation here can not lead to confusion provided one keeps in mind that fundamental $\star$-sequences act only between $C^\star$-algebras and that they can not be evaluated in an index (which is a natural number).

Let $\mathcal{L}_\varphi = \{(i, j, k) \mid i, j, k \in \mathbb{N}, \ j, k \geqslant \varphi(i)\}$. Then $\mathcal{L}_\varphi$ is a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ that becomes a cofinite directed set when we define that $(i, j, k) \geqslant (m, n, p)$ if and only if $i \geqslant m$, $j \geqslant n$, and $k \geqslant p$.

We shall use the same notation φ for an increasing function $\varphi : \mathcal{L}_\varphi \to \mathbb{N}$ such that $\varphi(i, j, k) \geqslant \varphi(i)$ whenever $(i, j, k) \in \mathcal{L}_\varphi$.

It was observed [8, Claim (5.1)] that there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

(1) $f(i) \geqslant \varphi(i, \varphi(i), \varphi(i))$ for every $i \in \mathbb{N}$, and
(2) f is cofinal in φ, i. e., for every $(i, j, k) \in \mathcal{L}_\varphi$ there is an $m \in \mathbb{N}$ with $f(m) \geqslant \varphi(i, j, k)$.

The above discussion shows that every fundamental $\star$-sequence $\varphi : A \to B$ determines two increasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. With the help of these functions we shall define the composition of $\star$-homotopy classes of fundamental $\star$-sequences as follows.

Let $\varphi = \{f_i\}_{i \in \mathbb{N}} : A \to B$ and $\psi = \{g_i\}_{i \in \mathbb{N}} : B \to C$ be fundamental $\star$-sequences. Let $\psi \circ \varphi$ denote the collection $\chi = \{h_i\}_{i \in \mathbb{N}}$, where we define $h_i = g_{\psi(i)} \circ f_{\varphi(g(i))}$ for every $i \in \mathbb{N}$. Observe that each h_i is a nonexpansive function because it is the composition of two nonexpansive functions.

Since the collection $\psi \circ \varphi$ is a fundamental $\star$-sequence from A into C, we can now define an associative composition of $\star$-homotopy classes of fundamental $\star$-sequences by the rule $[\psi]_\star \circ [\varphi]_\star = [\psi \circ \varphi]_\star$ (see [8, Claims (5.2), (5.3) and (5.4)]).

Finally, it remains to observe that for every fundamental $\star$-sequence $\varphi : A \to B$, the following relations hold.

$$[\varphi]_\star \circ [\iota^A]_\star = [\varphi]_\star = [\iota^B]_\star \circ [\varphi]_\star,$$

where for a $C^\star$-algebra A, we let $\iota^A = \{I_i\} : A \to A$ be the identity fundamental $\star$-sequence defined by $I_i = id_A$ for every $i \in \mathbb{N}$.

We can summarize the above with the following main theorem from [8].

(3.2) Theorem. *The $C^\star$-algebras as objects together with the $\star$-homotopy classes of fundamental $\star$-sequences as morphisms, the $\star$-homotopy classes $[\iota^A]_\star$ as identity morphisms, and the above composition of $\star$-homotopy classes form the $\star$-shape category $\mathcal{S}h_\star$.*

We close this review section with definitions of two classes of $C^\star$-algebras that play crucial role in the following two sections.

A $C^\star$-algebra B is *internally movable* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that every $\star^\delta$-homomorphism $f : A \to B$ from any $C^\star$-algebra A into B is $\star^\varepsilon$-homotopic to a $\star$-homomorphism.

A $C^\star$-algebra B is *internally calm* provided there is a $\gamma > 0$ such that $\star$-homomorphisms $f,\, g : A \to B$ from any $C^\star$-algebra A into B which are $\star^\gamma$-homotopic are $\star$-homotopic.

Of course, the above classes of $C^\star$-algebras are analogues of the corresponding classes of internally movable spaces(see [2] and [16]) and calm spaces (see [6]). Also, one can prove that every semiprojective [1] separable $C^\star$-algebra has both of these properties.

4. Characterization of $\star$-shape equivalences

In this section we shall use classes of internally movable and internally calm $C^\star$-algebras to get the non-commutative analogue of G. Kozlowski's characterization of shape equivalences in terms of induced functions of homotopy classes [12]. The results for $C^\star$-algebras are more satisfying. They show complete duality between $\star$-homotopy classes into and from the test $C^\star$-algebras.

For $C^\star$-algebras A and B, let $[A;\, B]_\star$ denote the set of all $\star$-homotopy classes of $\star$-homomorphisms from A into B. Every $\star$-homomorphism $f : A \to B$ induces for every $C^\star$-algebra C two set functions

$$f^\sharp_C : [C;\, A]_\star \to [C;\, B]_\star \qquad \text{and} \qquad f^C_\sharp : [B;\, C]_\star \to [A;\, C]_\star$$

which are defined by the rules $f^\sharp_C([a]_\star) = [f \circ a]_\star$ for every $\star$-homomorphism $a : C \to A$ and $f^C_\sharp([b]_\star) = [b \circ f]_\star$ for every $\star$-homomorphism $b : B \to C$.

Recall that a morphism φ of a category $\mathcal{C}$ is called *left $\mathcal{C}$-invertible* provided there is another morphism ψ of $\mathcal{C}$ with the composition $\psi \circ \varphi$ equal to the identity morphism. The *right $\mathcal{C}$-invertible* morphisms (or shortly *$\mathcal{C}$-dominations*) are defined similarly. A morphism of $\mathcal{C}$ is *$\mathcal{C}$-invertible* or a *$\mathcal{C}$-equivalence* if and only if it is both left $\mathcal{C}$-invertible and right $\mathcal{C}$-invertible. When applied to $\star$-homomorphisms and the $\star$-shape category $\mathcal{S}h_\star$ this notions mean that the $\star$-shape morphism determined by this $\star$-homomorphism has them.

The following eight theorems are similar to the characterization of shape equivalences from [12]. We get information on conditions which imply that the functions $f^C_\sharp$ and $f^\sharp_C$ induced by a $\star$-homomorphism $f : A \to B$ and a $C^\star$-algebra C are surjections and injections and we get an information on what kind of $C^\star$-algebras C one can use.

(4.1) Theorem. *If a $\star$-homomorphism $f : A \to B$ between $C^\star$-algebras is left $\mathcal{S}h_\star$-invertible, then for every $C^\star$-algebra C which is at the same time internally movable and internally calm the induced function $f^C_\sharp$ is a surjection.*

Proof. Let $h : A \to C$ be a $\star$-homomorphism. The assumptions about the $C^\star$-algebra C imply the existence of positive real numbers ε and δ such that $\delta \leqslant \varepsilon$, $\star^\varepsilon$-homotopic $\star$-homomorphisms into C are $\star$-homotopic, and that every $\star^\delta$-homomorphism into C is $\star^\varepsilon$-homotopic to a $\star$-homomorphism. Observe that h (as any other $\star$-homomorphism) has the property that the relation $\|x - y\|_A < \delta$ implies the relation $\|h(x) - h(y)\|_C < \delta$.

Since f is left $\mathcal{S}h_\star$-invertible, there is a fundamental $\star$-sequence $\psi : B \to A$ with $\psi \circ f$ $\star$-homotopic to the identity fundamental $\star$-sequence ι^A on A. In particular, there is an index $i \in \mathbb{N}$ such that $\delta \geqslant 1/i$ and $g_p \circ f \overset{\delta}{\simeq}_\star id_A$, where $p = \psi(i)$.

The composition $h \circ g_p$ is a $\star^\delta$-homomorphism from B into C so that there is a $\star$-homomorphism $k : B \to C$ which is $\star^\varepsilon$-homotopic to it.

It follows that $k \circ f \overset{\varepsilon}{\simeq}_\star h \circ g_p \circ f \overset{\delta}{\simeq}_\star h$, i. e., that $\star$-homomorphisms $k \circ f$ and h are $\star^\varepsilon$-homotopic and therefore also $\star$-homotopic. In other words, $f_\sharp^C([k]) = [h]$. $\square$

(4.2) Theorem. *If $f : A \to B$ is a right $\mathcal{S}h_\star$-invertible $\star$-homomorphism from an internally movable $C^\star$-algebra A into an internally calm $C^\star$-algebra B, then for every $C^\star$-algebra C the induced function $f_C^\sharp$ is a surjection.*

Proof. The assumptions about the $C^\star$-algebras A and B imply the existence of the positive real numbers γ and δ such that $\star^\gamma$-homotopic $\star$-homomorphisms into B are $\star$-homotopic and every $\star^\delta$-homomorphism into A is $\star^\gamma$-homotopic to a $\star$-homomorphism. Let $i \in \gamma^*$.

Since f is right $\mathcal{S}h_\star$-invertible, there is a fundamental $\star$-sequence $\psi = \{g_m\}_{m=1}^\infty$ from B into A with $f \circ \psi$ $\star$-homotopic to the identity fundamental $\star$-sequence ι^B on B. In particular, there is an index $m \geqslant i$ such that $f \circ g_{\psi(m)} \overset{\gamma}{\simeq}_\star 1_B$. Pick an index $j \geqslant \psi(m)$ such that g_j is a $\star^\delta$-homomorphism. We can assume that g_j and $g_{\psi(m)}$ are joined by a $\star^{1/m}$-homotopy $E : B \to C(I;\, A)$.

Let C be a $C^\star$-algebra. In order to prove that the induced function $f_C^\sharp$ is a surjection, let $a : C \to B$ be a $\star$-homomorphism. The composition $g_j \circ a$ is then a $\star^\delta$-homomorphism of C into A so that it is $\star^\gamma$-homotopic to some $\star$-homomorphism $b : C \to A$ via a $\star^\gamma$-homotopy $D : C \to C(I;\, A)$. Then $f \circ b$ and $f \circ g_j \circ a$ are $\star^\gamma$-homotopic via the composition $\hat f \circ D$, where $\hat f : C(I;\, A) \to C(I;\, B)$ is a $\star$-homomorphism induced by the $\star$-homomorphism f. On the other hand, the composition $\hat f \circ E \circ a$ is a $\star^{1/m}$-homotopy joining $f \circ g_j \circ a$ and $f \circ g_{\psi(m)} \circ a$. Finally, let $F : B \to C(I;\, B)$ be a $\star^\gamma$-homotopy between $f \circ g_{\psi(m)}$ and 1_B. The composition $F \circ a$ than realizes the relation $f \circ g_{\psi(m)} \circ a \overset{\gamma}{\simeq}_\star a$. Thus, we have

$$f \circ b \overset{\gamma}{\simeq}_\star f \circ g_j \circ a \overset{1/m}{\simeq}_\star f \circ g_{\psi(m)} \circ a \overset{\gamma}{\simeq}_\star a\,.$$

Since $\gamma \geqslant 1/m$, it follows that $f \circ b \overset{\gamma}{\simeq}_\star a$ and therefore also $f \circ b \simeq_\star a$. Hence, the function $f_C^\sharp$ is onto. $\square$

(4.3) Theorem. *If a $\star$-homomorphism $f : A \to B$ between $C^\star$-algebras is right $\mathcal{S}h_\star$-invertible, then for every internally calm $C^\star$-algebra C the induced function $f_\sharp^C$ is an injection.*

Proof. Let $h,\, k : B \to C$ be $\star$-homomorphisms and assume that $f_\sharp^C([h]) = f_\sharp^C([k])$, i. e., that $h \circ f$ and $k \circ f$ are joined by a $\star$-homotopy $m : A \to C(I;\, C)$.

Since C is internally calm, there is an $\varepsilon > 0$ such that $\star$-homomorphisms into C which are $\star^\varepsilon$-homotopic are also $\star$-homotopic. Since f is a $\mathcal{S}h_\star$-domination, there is an $i \in \mathbb{N}$ with $\varepsilon > 1/i$ and a $\star^\varepsilon$-homotopy $L : B \to C(I;\, B)$ joining $f \circ g_{\psi(i)}$ and 1_B. Observe $g_{\psi(i)}$ is a $\star^{1/i}$-homomorphism. It follows that the composition $m \circ g_{\psi(i)}$ is a $\star^\varepsilon$-homotopy joining $h \circ f \circ g_{\psi(i)}$ and $k \circ f \circ g_{\psi(i)}$. Similarly, the compositions $h \circ L$ and $k \circ L$ provide $\star^\varepsilon$-homotopies joining h with $h \circ f \circ g_{\psi(i)}$ and k with $k \circ f \circ g_{\psi(i)}$, respectively. Hence, $h \overset{\varepsilon}{\simeq}_\star k$ and therefore h and k are $\star$-homotopic. $\square$

(4.4) Theorem. *If $f : A \to B$ is a left $\mathcal{S}h_\star$-invertible $\star$-homomorphism from an internally calm $C^\star$-algebra A into a $C^\star$-algebra B, then for every $C^\star$-algebra C the induced function $f_C^\sharp$ is an injection.*

Proof. Since A is internally calm, there is an $\varepsilon > 0$ such that $\star$-homomorphisms into A which are $\star^\varepsilon$-homotopic are also $\star$-homotopic. By assumption, there is a fundamental $\star$-sequence $\psi : B \to A$ such that $\psi \circ f \simeq_\star \iota^X$. Let $i \in \varepsilon^*$.

Let C be a $C^\star$-algebra and assume that $a, b : C \to A$ are $\star$-homomorphisms with $f \circ a \simeq_\star f \circ b$. Pick an index $j \geqslant i$ in $\mathbb{N}$ such that $g_{\psi(j)} \circ f \overset{\varepsilon}{\simeq}_\star 1_X$. Observe that $g_{\psi(j)}$ is a $\star^{1/j}$-homomorphism, so we get

$$a \overset{\varepsilon}{\simeq}_\star g_{\psi(j)} \circ f \circ a \overset{1/j}{\simeq}_\star g_{\psi(j)} \circ f \circ b \overset{\varepsilon}{\simeq}_\star b.$$

It follows that $a \overset{\sigma}{\simeq}_\star b$ so that $a \simeq_\star b$ and $f_C^\sharp$ is injective. ☐

In order to formulate a partial converse to Theorem (4.1) and for some other results below we shall need the following notion.

Let $\mathcal{P}$ be a class of $C^\star$-algebras. By an *inverse $\mathcal{P}$-exposition* of a $C^\star$-algebra A we mean a sequence $\{A_i\}_{i\in\mathbb{N}}$ of members of $\mathcal{P}$, a family $\{p_j^i \,|\, i, j \in \mathbb{N},\, i \leqslant j\}$ of $\star$-homomorphisms $p_j^i : A_j \to A_i$, a sequence $\{p^i \,|\, i \in \mathbb{N}\}$ of $\star$-homomorphisms $p^i : A \to A_i$, and a sequence $\{J_i \,|\, i \in \mathbb{N}\}$ of $\star^{1/i}$-homomorphisms $J_i : A_i \to A$ such that

(1) the relation $i \leqslant j$ in $\mathbb{N}$ implies $p^i \simeq_\star p_j^i \circ p^j$,

(2) for every $i \in \mathbb{N}$, we have $J_i \circ p^i \overset{1/i}{\simeq}_\star 1_A$, and

(3) the relation $i \leqslant j$ in $\mathbb{N}$ implies $J_j \overset{1/i}{\simeq}_\star J_i \circ p_j^i$.

(4.5) Theorem. *Let $\mathcal{P}$ be a class of $C^\star$-algebras. If a $C^\star$-algebra A has an inverse $\mathcal{P}$-exposition and a $\star$-homomorphism $f : A \to B$ is such that the induced function $f_\sharp^C$ is a bijection for every $C \in \mathcal{P}$, then f is left $\mathcal{S}h_\star$-invertible.*

Proof. By assumption the $C^\star$-algebra A has an inverse $\mathcal{P}$-exposition formed by $C^\star$-algebras A_i from $\mathcal{P}$, $\star$-homomorphisms p_j^i and p^i, and $\star^{1/i}$-homomorphisms J_i. We shall define a fundamental $\star$-sequence $\psi : B \to A$ such that $\psi \circ f$ is $\star$-homotopic to the identity fundamental $\star$-sequence ι^A on A.

Let $i \in \mathbb{N}$. Since the induced set function $f_\sharp^{A_i}$ is a surjection, there is a $\star$-homomorphism $r^i : B \to A_i$ with $p^i \simeq_\star r^i \circ f$. Put $g_i = J_i \circ r^i$ and $\psi = \{g_i\}_{i\in\mathbb{N}}$.

In order to verify that ψ is a fundamental $\star$-sequence, let an $\varepsilon > 0$ be given. Let $i \in \varepsilon^*$ (i. e., $i \in \mathbb{N}$ satisfies $1/i < \varepsilon$). For every $j \geqslant i$, we have

$$r^i \circ f \simeq_\star p^i \simeq_\star p_j^i \circ p^j \simeq_\star p_j^i \circ r^j \circ f.$$

Since $f_\sharp^{A_i}$ is also an injection, we obtain $r^i \simeq_\star p_j^i \circ r^j$. It follows that

$$g_i = J_i \circ r^i \overset{1/i}{\simeq}_\star J_i \circ p_j^i \circ r^j \overset{1/i}{\simeq}_\star J_j \circ r^j = g_j$$

and therefore $g_i \overset{\varepsilon}{\simeq}_\star g_j$.

We shall now check that $\psi \circ f$ is $\star$-homotopic to ι^A, i. e., that for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $g_{\psi(j)} \circ f \overset{\varepsilon}{\simeq}_\star 1_A$ for every $j \geq i$.

Let $\varepsilon > 0$ and let $i \in \varepsilon^{**}$. For every $j \geqslant i$, we get $\psi(j) \geqslant \psi(i) \geqslant i$ so that $g_{\psi(j)} \overset{1/i}{\simeq}_\star g_{\psi(i)}$. Hence,

$$g_{\psi(j)} \circ f \overset{1/i}{\simeq}_\star g_{\psi(i)} \circ f = J_{\psi(i)} \circ r^{\psi(i)} \circ f \overset{2/\psi(i)}{\simeq}_\star J_{\psi(i)} \circ p^{\psi(i)} \overset{1/\psi(i)}{\simeq}_\star 1_A.$$

Thus, $g_{\psi(j)} \circ f \overset{\varepsilon}{\simeq}_\star 1_A$. ☐

(4.6) Theorem. *Let $\mathcal{P}$ and $\mathcal{R}$ be classes of $C^\star$-algebras such that each member of $\mathcal{R}$ is both internally movable and internally calm. If a $C^\star$-algebra A has an inverse $\mathcal{P}$-exposition, a $C^\star$-algebra B has an inverse $\mathcal{R}$-exposition, and a $\star$-homomorphism $f : A \to B$ is such that the induced function $f_\sharp^C$ is a bijection for every $C \in \mathcal{P}$ and an injection for every $C \in \mathcal{R}$, then f is a $\star$-shape equivalence.*

Proof. First we shall show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that the relation $u \circ f \overset{\delta}{\simeq}_\star v \circ f$ for $\star^\delta$-homomorphisms $u,\, v : B \to B$ implies the relation $u \overset{\varepsilon}{\simeq}_\star v$.

Let an $\varepsilon > 0$ be given. Let $i \in \varepsilon^{**}$. By assumption the $C^\star$-algebra B has an inverse $\mathcal{R}$-exposition formed by $C^\star$-algebras B_m from $\mathcal{R}$, $\star$-homomorphisms q_n^m and q^m, and $\star^{1/m}$-homomorphisms K_m.

Since the $C^\star$-algebra B_i is internally calm, there is a positive real number α such that $\alpha < 1/i$ and $\star^\alpha$-homotopic $\star$-homomorphisms into B_i are $\star$-homotopic.

Since Y_i is also internally movable, there is a $\delta > 0$ with the property that $\delta < \alpha/2$ and every $\star^\delta$-homomorphism into B_i is $\star^{\alpha/2}$-homotopic to a $\star$-homomorphism.

Let $u,\, v : B \to B$ be $\star^\delta$-homomorphisms and assume that $u \circ f \overset{\delta}{\simeq}_\star v \circ f$. Then $q^i \circ u$ and $q^i \circ v$ are $\star^\delta$-homomorphisms so that there are $\star$-homomorphisms a and b from B into B_i with $a \overset{\alpha/2}{\simeq}_\star q^i \circ u$ and $b \overset{\alpha/2}{\simeq}_\star q^i \circ v$. Our choices imply the following chain of relations:

$$a \circ f \overset{\alpha}{\simeq}_\star q^i \circ u \circ f \overset{\delta}{\simeq}_\star q^i \circ v \circ f \overset{\alpha}{\simeq}_\star b \circ f.$$

It follows that $a \circ f \simeq_\star b \circ f$ and therefore $a \simeq_\star b$ because $f_\sharp^{B_i}$ is an injection. Now, we have

$$u \overset{1/i}{\simeq}_\star K_i \circ q^i \circ u \overset{2/i}{\simeq}_\star K_i \circ a \overset{1/i}{\simeq}_\star K_i \circ b \overset{2/i}{\simeq}_\star K_i \circ q^i \circ v \overset{1/i}{\simeq}_\star v.$$

Hence, $u \overset{\varepsilon}{\simeq}_\star v$.

Now we are ready to prove the theorem. Let ψ be a fundamental $\star$-sequence constructed in the proof of Theorem (4.5). We shall show that $f \circ \psi$ is $\star$-homotopic to ι^B, i. e., that for every $\varepsilon > 0$ there is an index $i \in \mathbb{N}$ such that $f \circ g_{\psi(j)} \overset{\varepsilon}{\simeq}_\star 1_B$ for every $j \geq i$.

Let an $\varepsilon > 0$ be given. Choose a $\delta > 0$ with the property described above. Let $i \in \delta^*$. For every $j \geqslant i$, $f \circ g_{\psi(j)}$ and 1_B are $\star^\delta$-homomorphisms of B into itself and

$$f \circ g_{\psi(j)} \circ f = f \circ J_{\psi(j)} \circ r^{\psi(j)} \circ f \overset{1/\psi(j)}{\simeq}_\star f \circ J_{\psi(j)} \circ p^{\psi(j)} \overset{1/\psi(j)}{\simeq}_\star f \;=\; 1_B \circ f.$$

It follows that $f \circ g_{\psi(j)} \overset{\varepsilon}{\simeq}_\star 1_B$. $\square$

In order to formulate a partial converse to Theorem (4.2) and for some other results below we shall need the following notion which is dual to the notion of an inverse $\mathcal{P}$-exposition.

Let $\mathcal{P}$ be a class of $C^\star$-algebras. By a *direct $\mathcal{P}$-exposition* of a $C^\star$-algebra A we mean a sequence $\{A_i\}_{i \in \mathbb{N}}$ of members of $\mathcal{P}$, a family $\{p_i^j \,|\, i,\, j \in \mathbb{N},\, i \leqslant j\}$ of $\star$-homomorphisms $p_i^j : A_i \to A_j$, a sequence $\{p_i \,|\, i \in \mathbb{N}\}$ of $\star$-homomorphisms $p_i : A_i \to A$, and a sequence $\{J^i \,|\, i \in \mathbb{N}\}$ of $\star^{1/i}$-homomorphisms $J^i : A \to A_i$ such that

(1) the relation $i \leqslant j$ in $\mathbb{N}$ implies $p^i \simeq_\star p_j \circ p_i^j$,

(2) for every $i \in \mathbb{N}$, we have $p_i \circ J^i \overset{1/i}{\simeq}_\star 1_A$, and

(3) the relation $i \leqslant j$ in $\mathbb{N}$ implies $J^j \overset{1/i}{\simeq}_\star p_i^j \circ J^i$.

(4.7) Theorem. *Let $\mathcal{R}$ be a class of $C^\star$-algebras. If a $C^\star$-algebra B has a direct $\mathcal{R}$-exposition and a $\star$-homomorphism $f : A \to B$ from a $C^\star$-algebra A into B is such that the induced function $f_C^\sharp$ is a bijection for every $C \in \mathcal{R}$, then f is right $\mathcal{S}h_\star$-invertible.*

Proof. By assumption the $C^\star$-algebra B has a direct $\mathcal{R}$-exposition formed by $C^\star$-algebras B_i from $\mathcal{R}$, $\star$-homomorphisms q_i^j and q_i, and $\star^{1/i}$-homomorphisms K^i. We shall define a fundamental $\star$-sequence $\psi : B \to A$ such that $f \circ \psi$ is $\star$-homotopic to the identity fundamental $\star$-sequence ι^B on B.

Let $i \in \mathbb{N}$. Since the induced set function $f_{B_i}^\sharp$ is a surjection, there is a $\star$-homomorphism $r_i : B_i \to A$ with $q_i \simeq_\star f \circ r_i$. Put $g_i = r_i \circ K^i$ and $\psi = \{g_i\}_{i \in \mathbb{N}}$.

In order to verify that ψ is a fundamental $\star$-sequence, let an $\varepsilon > 0$ be given. Let $i \in \varepsilon^*$ (i. e., $i \in \mathbb{N}$ satisfies $1/i < \varepsilon$). For every $j \geqslant i$, we have

$$f \circ r_i \simeq_\star q_i \simeq_\star q_j \circ q_i^j \simeq_\star f \circ r_j \circ q_i^j .$$

Since $f_{B_i}^\sharp$ is also an injection, we obtain $r_i \simeq_\star r_j \circ q_i^j$. It follows that

$$g_i = r_i \circ K^i \overset{1/i}{\simeq}_\star r_j \circ q_i^j \circ K^i \overset{1/i}{\simeq}_\star r_j \circ K^j = g_j$$

and therefore $g_i \overset{\varepsilon}{\simeq}_\star g_j$.

We shall now check that $f \circ \psi$ is $\star$-homotopic to ι^B, i. e., that for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f \circ g_{\psi(j)} \overset{\varepsilon}{\simeq}_\star 1_B$ for every $j \geq i$.

Let $\varepsilon > 0$ and let $i \in \varepsilon^{**}$. For every $j \geqslant i$, we get $\psi(j) \geqslant \psi(i) \geqslant i$ so that $g_{\psi(j)} \overset{1/i}{\simeq}_\star g_{\psi(i)}$. Hence,

$$f \circ g_{\psi(j)} \overset{1/i}{\simeq}_\star f \circ g_{\psi(i)} = f \circ r_{\psi(i)} \circ K^{\psi(i)} \overset{1/\psi(i)}{\simeq}_\star q_{\psi(i)} \circ K^{\psi(i)} \overset{1/\psi(i)}{\simeq}_\star 1_B.$$

Thus, $f \circ g_{\psi(j)} \overset{\varepsilon}{\simeq}_\star 1_B$. □

(4.8) Theorem. *Let A be an internally movable $C^\star$-algebra and let B be an internally calm $C^\star$-algebra. Let $\mathcal{P}$ and $\mathcal{R}$ be classes of $C^\star$-algebras. Suppose that A has a direct $\mathcal{P}$-exposition while B has a direct $\mathcal{R}$-exposition. If a $\star$-homomorphism $f : A \to B$ is such that the induced function $f_C^\sharp$ is a bijection for every $C \in \mathcal{R}$ and an injection for every $C \in \mathcal{P}$, then f is a $\star$-shape equivalence.*

Proof. First we shall show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that the relation $f \circ u \overset{\delta}{\simeq}_\star f \circ v$ for $\star^\delta$-homomorphisms $u,\, v : A \to A$ implies the relation $u \overset{\varepsilon}{\simeq}_\star v$.

Let an $\varepsilon > 0$ be given. Since B is internally calm, there is a positive real number σ such that $\star^\sigma$-homotopic $\star$-homomorphisms into B are $\star$-homotopic. Let $\xi < min\{\sigma,\, \varepsilon/4\}$.

Since A is internally movable, there is a $\delta > 0$ with the property that $\delta < \xi$ and every $\star^{2\delta}$-homomorphism into A is $\star^\xi$-homotopic to a $\star$-homomorphism. Then δ is a positive real number that we need.

Indeed, consider $\star^\delta$-homomorphisms $u,\, v : A \to A$ and assume that $f \circ u \overset{\delta}{\simeq}_\star f \circ v$. By assumption the $C^\star$-algebra A has a direct $\mathcal{P}$-exposition formed by $C^\star$-algebras A_m from $\mathcal{P}$, $\star$-homomorphisms p_n^m and p_m, and $\star^{1/m}$-homomorphisms J^m. Let $i \in \delta^*$.

The compositions $u \circ p_i$ and $v \circ p_i$ are $\star^{2\delta}$-homomorphisms so that there are $\star$-homomorphisms a and b from A_i into A with $a \overset{\xi}{\simeq}_\star u \circ p_i$ and $b \overset{\xi}{\simeq}_\star v \circ p_i$. Our choices imply the following chain of relations:

$$f \circ a \overset{\xi}{\simeq}_\star f \circ u \circ p_i \overset{\delta}{\simeq}_\star f \circ v \circ p_i \overset{\xi}{\simeq}_\star f \circ b.$$

Since $\delta < \xi < \sigma$, we conclude that $f \circ a \overset{\sigma}{\simeq}_\star f \circ b$ and therefore $f \circ a \simeq_\star f \circ b$ because $f \circ a$ and $f \circ b$ are $\star$-homomorphisms into B. Since the function $f^\sharp_{A_i}$ is an injection, it follows that $a \simeq_\star b$. Now, we have

$$u \circ p_i \overset{\xi}{\simeq}_\star a \simeq_\star b \overset{\xi}{\simeq}_\star v \circ p_i$$

and therefore $u \circ p_i \overset{\xi}{\simeq}_\star v \circ p_i$. Since J^i is a $\star^{1/i}$-homomorphism (and thus also a $\star^\xi$-homomorphism because $i \in \delta^*$ and $\delta < \xi$), we obtain $u \circ p_i \circ J^i \overset{2\xi}{\simeq}_\star v \circ p_i \circ J^i$. But, since u is a $\star^\delta$-homomorphism, the lifting $\hat{u} : C(I; A) \to C(I; A)$ is a $\star^{2\delta}$-homomorphism. So the composition $\hat{u} \circ K$ of a $\star^{1/i}$-homotopy $K : A \to C(I; A)$ which joins $p_i \circ J^i$ and 1_A with $\hat{u}$ is a $\star^{4\delta}$-homotopy joining $u \circ p_i \circ J^i$ and u. Similarly, $v \circ p_i \circ J^i \overset{4\delta}{\simeq}_\star v$. Hence,

$$u \overset{4\delta}{\simeq}_\star u \circ p_i \circ J^i \overset{2\xi}{\simeq}_\star v \circ p_i \circ J^i \overset{4\delta}{\simeq}_\star v,$$

and finally, $u \overset{\varepsilon}{\simeq}_\star v$.

Now we are ready to prove the theorem. Let ψ be a fundamental $\star$-sequence constructed in the proof of Theorem (4.7). We shall show that $\psi \circ f$ is $\star$-homotopic to ι^A, i. e., that for every $\varepsilon > 0$ there is an index $i \in \mathbb{N}$ such that $g_{\psi(j)} \circ f \overset{\varepsilon}{\simeq}_\star 1_A$ for every $j \geq i$.

Let an $\varepsilon > 0$ be given. Choose a $\delta > 0$ with the property described above. Let $i \in \delta^*$. For every $j \geqslant i$, $g_{\psi(j)} \circ f$ and 1_A are $\star^\delta$-homomorphisms of A into itself and

$$f \circ g_{\psi(j)} \circ f = f \circ r_{\psi(j)} \circ K^{\psi(j)} \circ f \overset{1/\psi(j)}{\simeq_\star} q_{\psi(j)} \circ K^{\psi(j)} \circ f \overset{1/\psi(j)}{\simeq_\star} f = f \circ 1_A.$$

It follows that $g_{\psi(j)} \circ f \overset{\varepsilon}{\simeq}_\star 1_A$. □

5. Approximate $\star$-contractibility and trivial $\star$-shape

As an illustration of the new insight offered in $\star$-shape theory by our approach via fundamental $\star$-sequences we will characterize $C^\star$-algebras of trivial $\star$-shape as approximately $\star$-contractible $C^\star$-algebras.

Let us use $\mathbb{O}$ to denote the trivial $C^\star$-algebra which has a single element. A $C^\star$-algebra A has *trivial $\star$-shape* or is *$\star$-shape trivial* provided it is equivalent in the $\star$-shape category $\mathcal{S}h_\star$ to the trivial $C^\star$-algebra $\mathbb{O}$. A $\star$-homomorphism $f : A \to B$ between $C^\star$-algebras is *trivial* provided it sends all elements of A into the zero element of B.

As we shall see shortly, $C^\star$-algebras of trivial $\star$-shape coincide with approximately $\star$-contractible $C^\star$-algebras which we define next. A $C^\star$-algebra A is said to be *approximately $\star$-contractible* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that every two $\star^\delta$-homomorphisms into A are $\star^\varepsilon$-homotopic.

(5.1) Theorem. *A $C^\star$-algebra A is approximately $\star$-contractible if and only if A has trivial $\star$-shape.*

Proof. ($\Longrightarrow$). Let $a : A \to \mathbb{O}$ and $b : \mathbb{O} \to A$ be trivial $\star$-homomorphisms. Let α and β be simple fundamental $\star$-sequences determined by $\star$-homomorphisms a and b, respectively. Then obviously $\alpha \circ \beta \simeq_\star \iota^{\mathbb{O}}$. On the other hand, $\beta \circ \alpha \simeq_\star \iota^A$ because $\beta \circ \alpha$ is determined by a trivial $\star$-homomorphism $d : A \to A$ and for every $\varepsilon > 0$, $d \overset{\varepsilon}{\simeq}_\star 1_A$ because the $C^\star$-algebra A is approximately $\star$-contractible (d and 1_A are $\star^\delta$-homomorphisms into A for every $\delta > 0$).

($\Longleftarrow$). Since the trivial $C^\star$-algebra $\mathbb{O}$ is $\star$-contractible and therefore also approximately $\star$-contractible and since one can easily prove that approximate $\star$-contractibility is preserved under $\star$-shape domination it follows that A is approximately $\star$-contractible. □

(5.2) Theorem. *If a $C^\star$-algebra A has trivial $\star$-shape, then every two $\star$-homomorphisms a and b from A into a $C^\star$-algebra C which is both internally movable and internally calm are $\star$-homotopic.*

Proof. Let C be a $C^\star$-algebra which is both internally movable and internally calm. Let $a, b : A \to C$ be $\star$-homomorphisms. The trivial $\star$-homomorphism $c : A \to \mathbb{O}$ is a $\star$-shape equivalence since A has trivial $\star$-shape. It follows from Theorem (4.1) that there are $\star$-homomorphisms $f, g : \mathbb{O} \to C$ with $a \simeq_\star f \circ c$ and $b \simeq_\star g \circ c$. But, f and g are both the trivial $\star$-homomorphism so that $f \circ c \simeq_\star g \circ c$. Therefore, $a \simeq_\star b$. $\square$

(5.3) Theorem. *Let $\mathcal{P}$ be a class of $C^\star$-algebras. If a $C^\star$-algebra A has an inverse $\mathcal{P}$-exposition and every two $\star$-homomorphisms from A into any member of $\mathcal{P}$ are $\star$-homotopic, then A has trivial $\star$-shape.*

Proof. Let $c : A \to \mathbb{O}$ be the trivial $\star$-homomorphism from A into the trivial $C^\star$-algebra $\mathbb{O}$. We shall show that c is left $\mathcal{S}h_\star$-invertible. Then the $C^\star$-algebra A will be $\star$-shape dominated by the approximately $\star$-contractible $C^\star$-algebra $\mathbb{O}$ and will therefore have trivial $\star$-shape. According to Theorem (4.5), we must in fact check that the induced function $c_\sharp^C$ is a bijection for every $C \in \mathcal{P}$.

The induced function $c_\sharp^C$ is trivially an injection. In order to see that $c_\sharp^C$ is a surjection for every $C \in \mathcal{P}$, let C be from $\mathcal{P}$ and let $f : A \to C$ be a $\star$-homomorphism. Let $g : \mathbb{O} \to C$ be the trivial $\star$-homomorphism. Then f and $g \circ c$ are two $\star$-homomorphisms from A into a member C of $\mathcal{P}$. By assumption, they are $\star$-homotopic so that $[f]_\star = c_\sharp^C([g]_\star)$ and $c_\sharp^C$ is onto. $\square$

(5.4) Theorem. *Let $\mathcal{R}$ be a class of $C^\star$-algebras. If a $C^\star$-algebra B has a direct $\mathcal{R}$-exposition and every two $\star$-homomorphisms from any member of $\mathcal{R}$ into B are $\star$-homotopic, then B has trivial $\star$-shape.*

Proof. Let $c : \mathbb{O} \to B$ be the trivial $\star$-homomorphism from the trivial $C^\star$-algebra $\mathbb{O}$ into B. We shall show that c is right $\mathcal{S}h_\star$-invertible. Then B will be $\star$-shape dominated by the approximately $\star$-contractible $C^\star$-algebra $\mathbb{O}$ and will therefore have trivial $\star$-shape. According to Theorem (4.7), we must in fact check that the induced function $c^\sharp_C$ is a bijection for every $C \in \mathcal{R}$.

The induced function $c_\sharp^C$ is again trivially an injection. In order to see that $c^\sharp_C$ is a surjection for every $C \in \mathcal{R}$, let C be from $\mathcal{R}$ and let $f : C \to B$ be a $\star$-homomorphism. Let $g : C \to \mathbb{O}$ be the trivial $\star$-homomorphism. Then f and $c \circ g$ are two $\star$-homomorphisms from a member C of $\mathcal{R}$ into B. By assumption, they are $\star$-homotopic so that $[f]_\star = c^\sharp_C([g]_\star)$ and $c^\sharp_C$ is onto. $\square$

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