SOME FIXED POINT THEOREMS IN COMPLETE SEMI–METRIC SPACES

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Abstract. In this paper, we give some fixed point theorems in complete semi-metric spaces by using the concept of w-distance. Our results generalize some well-Known fixed point theorems.

I. Introduction. Recently, Kada-Suzuki-Takahashi [3] introduced the concept of w-distance and, by using this concept, proved a nonconvex minimization theorems and some fixed point theorems in complete metric spaces.

In this paper, we also prove some fixed point theorems in complete semi-metric spaces by using the concept of w-distance. Our results generalize some fixed point theorems of authors ([1], [2], [4]).

II. Fixed Point Theorems in Semi-Metric Spaces. First, we introduce some definitions, examples and lemmas for our main theorems. Throughout this paper, let X be a semi-metric space with a semi-metric d, which is denoted by (X, d), i.e., $d(x, y) = d(y, x) \geq 0$, $d(x, x) = 0$ and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.1. [13] Let (X, d) be a semi-metric space. A function $p : X \times X \to [0, \infty)$ is called a *w-semi-distance* on X if the following conditions are satisfied:

(wsd-1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$,
(wsd-2) For all $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,
(wsd-3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply that $d(x, y) \leq \epsilon$.

Some examples and properties of w-distances are given in [3].

Definition 2.2. A sequence $\{x_n\}$ in a semi-metric space (X, d) is said to be *convergent* if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. We write $x_n \to x$ as $n \to \infty$.

Definition 2.3. A sequence $\{x_n\}$ in a semi-metric space (X, d) is called a *Cauchy sequence* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

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Definition 2.4. A semi-metric space (X,d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Let (X,d) be a semi-metric space, p be a w-semi-continuous on X and T be a mapping of X into itself. For $A \subset X$, denote

$$\delta_d(A) = \sup\{d(x,y) : x, y \in A\},$$
$$\delta_p(A) = \sup\{p(x,y) : x, y \in A\}$$

and, for each $x \in X$ and $n = 1, 2, \cdots$,

$$O(x,n) = \{x, Tx, T^2x, \cdots, T^nx\},$$

$$O(x,\infty) = \{x, Tx, T^2x, \cdots\}.$$

We need some lemmas for our main Theorems:

Lemma 2.1. [3] *Let (X,d) be a semi-metric space and p be a w-semi-distance on X. Let $\{x_n\}$ be a sequence in X and $\{\alpha_n\}$ be a sequence in $[0,\infty)$ converging to 0. If $p(x_n, x_m) \le \alpha_n$ for any $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. Let $\epsilon > 0$ be given. From the definition of a w-semi-distance, there exists $\delta > 0$ such that $p(x,y) \le \delta$ and $p(x,z) \le \delta$ for $x, y, z \in X$ imply $d(y,z) \le \epsilon$. Since $\lim_{n\to\infty} \alpha_n = 0$ and $p(x_n, x_m) \le \alpha_n$ for any $m, n \in N$ with $n < m$, there exists $n_0 \in N$ such that

$$p(x_n, x_m) \le \alpha_n < \delta$$

for $m > n \ge n_0$. Thus, for $m, n \ge 1 + n_0$,

$$p(x_{n_0}, x_n) \le \alpha_{n_0} < \delta, \quad p(x_{n_0}, x_m) \le \alpha_{n_0} < \delta$$

and so, by (wsd-3),

$$d(x_n, x_m) \le \epsilon,$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. This completes the proof.

Lemma 2.2. *Let (X,d) be a semi-metric space and p be a w-semi-continuous on X. Let E be a nonempty closed subset of X and $T : E \to E$ be a mapping such that*

$$p(Tx,Ty) \le a_1p(x,y) + a_2p(x,Tx) + a_3p(y,Ty) + a_4p(x,Ty) + a_5p(y,Tx) \tag{2.1}$$

for all $x, y \in E$, where $a_i > 0$ for $i = 1, 2, \cdots, 5$ and $r = \sum_{i=1}^{5} a_i < 1$. Then we have the following:

(1) *For each $x \in X$, $n \in N$ and $i, j \in N$ with $i, j \le n$,*

$$p(T^ix, T^jx) \le r\delta_p(O(x,n)),$$

(2) *For each $x \in E$ and $n \in N$, there exist $k,\ l \in N$ with $k,\ l \le n$ such that*

$$\delta_p(O(x,n)) = \max\{p(x,x), p(x,T^kx), p(T^lx,x)\},$$

(3) *For each* $x \in E$,

$$\delta_p(O(x,\infty)) \le \frac{1}{1-r}\{p(x,x)+p(x,Tx)+p(Tx,x)\},$$

(4) *For each* $x \in E$, $\{T^n x\}$ *is a Cauchy sequence in* E.

Proof. (1) Let $x \in E$, $n \in N$ and $i,j \in N$ with $i,j \le n$. Then $T^{i-1}x, T^i x$, $T^{j-1}x$, $T^j x \in O(x,n)$, where $T^0 x = x$, From (2.1), we have

$$\begin{aligned} p(T^i x, T^j x) &= p(TT^{i-1}x, TT^{j-1}x) \\ &\le a_1 p(T^{i-1}x, T^{j-1}x) + a_2 p(T^{i-1}x, T^i x) + a_3 p(T^{j-1}x, T^j x) \\ &\quad + a_4 p(T^{i-1}x, T^j x) + a_5 p(T^{j-1}x, T^i x) \\ &\le r\delta_p(O(x,n)) \end{aligned}$$

and so (1) is proved.

(2) From (1), it follows that, for each $x \in E$ and $n \in N$, there exist $k,l \in N$ with $k,l \le n$ such that

$$\delta_p(O(x,n)) = \max\{p(x,x), p(x,T^k x), p(T^l x, x)\},$$

which proves (2).

(3) Appling the condition (wsd-1), we have

$$p(x,T^k x) \le p(x,Tx) + p(Tx,T^k x) \le p(x,Tx) + r\delta_p(O(x,n))$$

and

$$p(T^l x, Tx) \le p(T^l x, Tx) + p(Tx,x) \le r\delta_p(O(x,n)) + p(Tx,x).$$

Thus we have

$$\delta_p(O(x,n)) \le \frac{1}{1-r}\{p(x,x)+p(x,Tx)+p(Tx,x)\}.$$

Since n is arbitrary, the proof of (3) is completed.

(4) Let $x \in E$ and define a sequence $\{x_n\}$ in E by $x_{n+1} = T^{n+1}x$ for $n = 1,2,\cdots$. Then, by (2,1), we have

$$\begin{aligned} p(x_n,x_{n+1}) &= p(Tx_{n-1},Tx_n) \\ &\le a_1 p(x_{n-1},x_n) + a_2 p(x_{n-1},x_n) + a_3 p(x_n,x_{n+1}) \\ &\quad + a_4 p(x_{n-1},x_{n+1}) + a_5 p(x_n,x_n) \\ &\le r\max\{p(x_{n-1},x_n), p(x_n,x_{n+1}), p(x_{n-1},x_{n+1}), p(x_n,x_n)\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} p(x_{n-1},x_n) &= p(Tx_{n-2},Tx_{n-1}) \\ &\le a_1 p(x_{n-2},x_{n-1}) + a_2 p(x_{n-2},x_{n-1}) + a_3 p(x_{n-1},x_n) \\ &\quad + a_4 p(x_{n-2},x_n) + a_5 p(x_{n-1},x_{n-1}) \\ &\le r\max\{p(x_{n-2},x_{n-1}), p(x_{n-1},x_n), p(x_{n-2},x_n), p(x_{n-1},x_{n-1})\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} p(x_{n-1},x_{n+1}) &= p(Tx_{n-2},Tx_n) \\ &\le a_1 p(x_{n-2},x_n) + a_2 p(x_{n-2},x_{n-1}) + a_3 p(x_n,x_{n+1}) \\ &\quad + a_4 p(x_{n-2},x_{n+1}) + a_5 p(x_n,x_{n-1}) \\ &\le r\max\{p(x_{n-2},x_n), p(x_{n-2},x_{n-1}), \\ &\qquad p(x_n,x_{n+1}), p(x_{n-2},x_{n+1}), p(x_n,x_{n-1})\}, \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
p(x_n, x_n) &= p(Tx_{n-1}, Tx_{n-1}) \\
&\leq a_1 p(x_{n-1}, x_{n-1}) + a_2 p(x_{n-1}, x_n) + a_3 p(x_{n-1}, x_n) \\
&\quad + a_4 p(x_{n-1}, x_n) + a_5 p(x_{n-1}, x_n) \\
&\leq r \max\{p(x_{n-1}, x_{n-1}), p(x_{n-1}, x_n)\}.
\end{aligned} \tag{2.5}
$$

Thus, from (2.2)∼(2.5), it follows that

$$
\begin{aligned}
p(x_n, x_{n+1}) &\leq r^2 \max\{p(x_i, x_j) : n-2 \leq i \leq n, n-1 \leq j \leq n+1\} \\
&\leq r^3 \max\{p(x_i, x_j) : n-3 \leq i \leq n, n-2 \leq j \leq n+1\} \\
&\quad \cdots \\
&\leq r^{n-1} \max\{p(x_i, x_j) : 1 \leq i \leq n, 2 \leq j \leq n+1\} \\
&\leq \frac{r^n}{1-r} a(x),
\end{aligned} \tag{2.6}
$$

where $a(x) = p(x, x) + p(x, Tx) + p(Tx, x)$. If $n < m$, then, by (2.6), we have

$$
\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\
&= \sum_{k=0}^{m-n-1} p(x_{n+k}, x_{n+k+1}) \\
&\leq \sum_{k=0}^{m-n-1} \frac{r^{n+k}}{1-r} a(x) \\
&= \frac{r^n}{1-r} a(x) \sum_{k=0}^{m-n-1} r^k \\
&\leq \frac{r^n}{(1-r)^2} a(x).
\end{aligned} \tag{2.7}
$$

Therefore, by Lemma 2.1, $\{T^n x\}$ is a Cauchy sequence in E. This completes the proof.

Theorem 2.3. *Let (X, d) be a complete semi-metric space and p be a w-semi-continuous on X. Let E be a nonempty closed subset of X and $T : E \to E$ be a mapping satisfying the conditions (2.1) and (2.8): For all $y \in E$ with $y \neq Ty$,*

$$
\inf\{p(x, y) + p(x, Tx) : x \in E\} > 0. \tag{2.8}
$$

Then we have the following:

(1) $\lim_{n\to\infty} T^n x = z$,

(2) $p(T^n x, y) \leq \frac{r^n}{(1-r)^2} a(x)$, where $a(x) = p(x, x) + p(x, Tx) + p(Tx, x)$,

(3) T has a unique fixed point z in E,

Proof. (1) Let $x \in E$ and define a sequence $\{x_n\}$ in E by $x_{n+1} = T^{n+1} x$ for $n = 1, 2, \cdots$. Then, by Lemma 2.2, $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, the Cauchy sequence $\{x_n\}$ converges to a point $z \in X$, i.e., $\lim_{n\to\infty} T^n x = z$.

(2) From (2.6), we have

$$
p(x_n, z) \leq \lim_{m\to\infty} \inf p(x_n, x_m) \leq \frac{r^n}{(1-r)^2} a(x) \tag{2.9}
$$

and so (2) is proved.

(3) Assume that $z \neq Tz$ for all $z \in E$. By the hypothesis, since

$$\inf\{p(x,z) + p(x,Tx) : x \in E\} > 0,$$

from (2.6) and (2.9), it follows that

$$\begin{aligned} 0 &< \inf\{p(x,z) + p(x,Tx) : x \in E\} \\ &\leq \inf\{p(x_n,z) + p(x_n,Tx_n) : n \in N\} \\ &\leq \inf\left\{\frac{r^n}{1-r}a(x) + \frac{r^n}{(1-r)^2}a(x) : n \in N\right\} \\ &\leq \frac{2-r}{(1-r)^2}\inf\{r^n : n \in N\} \\ &= 0, \end{aligned}$$

which is a contradiction. Therefore, T has a fixed point z in E. The uniqueness of the fixed point z follows from the condition (2.1). This completes the proof.

If we put $p = d$ and $E = X$ in Theorem 2.3, we have the following:

Corollary 2.4. [2] *Let (X,d) be a complete metric space and T be a mapping of X into itself. If there exist nonnegative real numbers $a_1, a_2, \cdots, a_5$ with $r = \sum_{i=1}^{5} a_i < 1$ such that*

$$\begin{aligned} d(Tx,Ty) \leq\ & a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) \\ & + a_4 d(x,Ty) + a_5 d(y,Tx) \end{aligned}$$

for all $x, y \in X$, then T has a unique fixed point in X.

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