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FIXED POINTS OF COMPATIBLE MAPPINGS IN COMPLETE MENGER SPACES

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ABSTRACT. A common fixed point theorem for four selfmaps on a complete Menger space is established. This result is even new in a metric space setting.

1. Introduction. Seghal and Bharucha–Reid [9] initiated the study of fixed points in Menger spaces, a subclass of probabilistic metric spaces (PM–spaces). In PM–spaces the concept of distance is considered to be probabilistic, rather than deterministic, that is to say, given any two points x and y of a set, a distribution function $F_{x,y}(\varepsilon)$ is introduced which gives the probabilistic interpretation as the distance between x and y is less than ε ($\varepsilon > 0$). There has been an extensive investigation on fixed point theory in PM–spaces in the last twenty years, cf. [1], [2], [3], [6], [9], [11]. For topological preliminaries on PM–spaces we refer the reader to [7] and [8].

In this paper, we shall prove mainly a common fixed point theorem for four selfmaps on a complete Menger space. This result is new even in a metric space setting. In addition to the above result a generalization of Hadžić fixed point theorem in [3] is also established.

We now recall some basic definitions and results. A mapping $F : \mathbb{R} \to [0, 1]$ is called a distribution function if it is nondecreasing, left continuous and $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to\infty} F(t) = 1$. The set of all distribution functions is denoted by $\mathcal{D}$. A probabilistic metric space (PM–space) is an ordered pair $(X, \mathcal{F})$ consisting of a nonempty set X and a mapping $\mathcal{F} : X \times X \to \mathcal{D}$, whose value $\mathcal{F}(x, y)$ at (x, y) is denoted by $F_{x,y}$, such that the following conditions are satisfied.

(i) $F_{x,y}(a) = 1$ for all $a > 0$ if and only if $x = y$;
(ii) $F_{x,y}(0) = 0$ for all x, y in X;
(iii) $F_{x,y} = F_{y,x}$ for all x, y in X;
(iv) if $F_{x,y}(a) = 1$ and $F_{y,z}(b) = 1$ then $F_{x,z}(a + b) = 1$ for all x, y, z in X and $a, b > 0$.

A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a t–norm if it is commutative, associative, nondecreasing in each coordinate, and $t(a, 1) = a$ for all $a \in [0, 1]$. An important t–norm is the min which is defined by min (a, b) = minimum of a and b.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM–space and t is a t–norm such that the generalized triangle inequality

$$F_{x,z}(a + b) \geq t\,(F_{x,y}(a), F_{y,z}(b))$$

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holds for all x, y, z in X and $a, b > 0$. The concept of neighborhoods in a PM–space $(X, \mathcal{F})$ was introduced by Schweizer and Sklar [7]. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then an (ε, λ)–neighborhood of x, denoted by $U_x(\varepsilon; \lambda)$, is defined to be

$$U_x(\varepsilon; \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

It is well–known that if $(X, \mathcal{F}, t)$ is a Menger space with the continuous t–norm t, then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family $\{U_x(\varepsilon; \lambda) : x \in X, \varepsilon > 0, \lambda \in (0,1)\}$ of neighborhoods. A sequence $\{x_n\}$ in a PM–space is said to be Cauchy if for any $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $N = N(\varepsilon, \lambda) \in \mathbb{N}$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$. The sequence $\{x_n\}$ is said to be convergent to a point x in X if for any $\varepsilon > 0$ $\lim\limits_{n\to\infty} F_{x_n,x}(\varepsilon) = 1$. If every Cauchy sequence in X is convergent, then $(X, \mathcal{F})$ is called a complete PM–space.

The following result is a special case of Schweizer and Sklar [7, Theorems 8.1 and 8.2].

Lemma 1.1. Suppose $(X, \mathcal{F}, \min)$ is a Menger space then for any $\varepsilon > 0$ $\underline{\lim}_{n\to\infty} F_{x_n,y_n}(\varepsilon) \geq F_{x,y}(\varepsilon)$ provided that $x_n \to x$ and $y_n \to y$. Moreover, if $F_{x,y}$ is continuous at ε, then $\lim\limits_{n\to\infty} F_{x_n,y_n}(\varepsilon) = F_{x,y}(\varepsilon)$.

2. A New Fixed Point Theorem in Menger Spaces. To start with, suppose $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Then T.H. Chang [2] showed there exists a strictly increasing continuous function $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(0) = 0$ and $\varphi(t) \leq \alpha(t) < t$ for all $t > 0$. The function α is invertible and for any $t > 0$ $\lim\limits_{n\to\infty} \alpha^{-n}(t) = \infty$, where α^{-n} denotes the n–th iterates of α^{-1} (α^{-1} composed with itself n times) and α^{-1} denotes the inverse of α.

In order to prove our main result we need some lemmas.

Lemma 2.1. Suppose $(X, \mathcal{F}, \min)$ is a complete Menger space and $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. If $\{y_n\}$ is a sequence in X such that for any $\varepsilon > 0$ and any $n \in \mathbb{N}$

$$F_{y_n,y_{n+1}}(\varphi(\varepsilon)) \geq F_{y_{n-1},y_n}(\varepsilon)$$

then $\{y_n\}$ is a Cauchy sequence in X.

Proof. Choose a strictly increasing continuous function $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(0) = 0$ and $\varphi(t) \leq \alpha(t) < t$ for all $t > 0$. Then for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ one has $F_{y_n,y_{n+1}}(\alpha(\varepsilon)) \geq F_{y_n,y_{n+1}}(\varphi(\varepsilon)) \geq F_{y_{n-1},y_n}(\varepsilon)$, so

$$\begin{aligned} F_{y_n,y_{n-1}}\Big(\alpha^{-1}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big) &\geq F_{y_{n-1},y_{n-2}}\Big(\alpha^{-2}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big) \\ &\ \ \vdots \\ &\geq F_{y_1,y_0}\Big(\alpha^{-n}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big). \qquad (1) \end{aligned}$$

Since $\lim\limits_{n\to\infty} \alpha^{-n}(\frac{\varepsilon-\alpha(\varepsilon)}{2}) = \infty$, we have $\lim\limits_{n\to\infty} F_{y_1,y_0}\Big(\alpha^{-n}(\frac{\varepsilon-\alpha(\varepsilon)}{2})\Big) = 1$, and hence (1) shows that $\lim\limits_{n\to\infty} F_{y_n,y_{n-1}}\Big(\alpha^{-1}(\frac{\varepsilon-\alpha(\varepsilon)}{2})\Big) = 1$, which says that for any $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $N \in \mathbb{N}$ such that

$$F_{y_n,y_{n-1}}\Big(\alpha^{-1}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big) > 1 - \lambda \quad \text{whenever } n \geq N \qquad (2)$$

We now prove by induction that for $n \geq N$ and $m \in \mathbb{N}$,

$$F_{y_n, y_{n+m}}(\varepsilon) > 1 - \lambda \tag{3}$$

When $m = 1$, it follows from (2) that

$$\begin{aligned} F_{y_n, y_{n+1}}(\varepsilon) \geq F_{y_{n-1}, y_n}(\alpha^{-1}\varepsilon) &\geq F_{y_{n-1}, y_n}\Big(\alpha^{-1}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big) \\ &> 1 - \lambda, \quad \forall\, n \geq N. \end{aligned}$$

Suppose (3) holds for any $n \geq N$ and for any $m = 1, 2, \ldots, r$. Then when $m = r + 1$, we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} &F_{y_n, y_{n+r+1}}(\varepsilon) \\ \geq &\min\Big\{F_{y_n, y_{n+1}}(\frac{\varepsilon - \alpha(\varepsilon)}{2}), F_{y_{n+1}, y_{n+r+1}}(\frac{\varepsilon + \alpha(\varepsilon)}{2})\Big\} \\ \geq &\min\Big\{F_{y_{n-1}, y_n}\Big(\alpha^{-1}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big), F_{y_{n+1}, y_{n+r+1}}(\frac{\varepsilon + \alpha(\varepsilon)}{2})\Big\} \\ \geq &\min\Big\{1 - \lambda, \min\Big\{F_{y_{n+1}, y_{n+2}}(\frac{\varepsilon - \alpha(\varepsilon)}{2}), F_{y_{n+2}, y_{n+r+1}}\big(\alpha(\varepsilon)\big)\Big\}\Big\} \\ \geq &\min\Big\{1 - \lambda, \min\Big\{F_{y_n, y_{n+1}}\Big(\alpha^{-1}(\frac{\varepsilon - \alpha(\varepsilon)}{2})\Big), F_{y_{n+1}, y_{n+r}}(\varepsilon)\Big\}\Big\} \\ \geq &\min\Big\{1 - \lambda, \min\Big\{1 - \lambda, F_{y_{n+1}, y_{n+r}}(\varepsilon)\Big\}\Big\} \\ = &\min\Big\{1 - \lambda, F_{y_{n+1}, y_{(n+1)+(r-1)}}(\varepsilon)\Big\} \\ \geq &\min\big\{1 - \lambda, 1 - \lambda\big\} \quad \text{by induction hypothesis} \\ = &1 - \lambda. \end{aligned}$$

Therefore $\{y_n\}$ is a Cauchy sequence in X. ///

The following Lemma 2.2 is well–known, cf. [2].

Lemma 2.2. Suppose $(X, \mathcal{F})$ is a PM–space and $\alpha : [0, \infty) \to [0, \infty)$ is strictly increasing and satisfies $\alpha(0) = 0$ and $\alpha(t) < t$ for all $t > 0$. If x, y are two members in X such that

$$F_{x,y}(\alpha(\varepsilon)) \geq F_{x,y}(\varepsilon)$$

for all $\varepsilon > 0$, then $x = y$.

The commutative notion was first generalized by Sessa [10] in the following way:
Two selfmaps f, g on a metric space (X, d) are said to be weakly commutative if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Later Jungck [4] made a further generalization:
Two selfmaps f, g on a metric space (X, d) are said to be compatible if whenever $\{x_n\}$ is a sequence in X such that both $\{fx_n\}$ and $\{gx_n\}$ are convergent to a same point x in X then $d(fgx_n, gfx_n) \to 0$.
The counterpart of the compatibility in a PM–space is the following

Definition 2.3. Two selfmaps S, A on a PM–space $(X, \mathcal{F})$ are compatible if $\lim_{n\to\infty} F_{SAx_n, ASx_n}(\varepsilon) = 1$ for all $\varepsilon > 0$ whenever $\{x_n\}$ is a sequence in X such that $\{Ax_n\}$ and $\{Sx_n\}$ are convergent to some point x in X.

By taking $x_n = x$ for all n it follows from the compatibility of A and S that $ASx = SAx$ if $Ax = Sx$.

We are now in a position to prove our main result.

Theorem 2.4. Suppose $(X, \mathcal{F}, \min)$ is a complete Menger space and $S, T, A, B : X \to X$ are four selfmaps on X satisfying the following conditions:

(i) $SX \subseteq BX$ and $TX \subseteq AX$;

(ii) (S, A) and (T, B) are compatible pairs;

(iii) one of S, T, A, B is continuous;

(iv) there exists an upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$ such that

$$\Big(F_{Sx,Ty}\big(\varphi(\varepsilon)\big)\Big)^2 \geq \min\Big\{F_{Ax,Sx}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Ax,Ty}(2\varepsilon)F_{By,Sx}(2\varepsilon), F_{Ax,Sx}(\varepsilon)F_{Ax,Ty}(2\varepsilon), F_{By,Sx}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}$$

for all x, y in X and $\varepsilon > 0$.

Then S, T, A and B have a unique common fixed point.

Proof. In view of condition (iv) and the remark at the begining of this section, we may assume that φ is a strictly increasing continuous function with $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$. Fix an $x_0 \in X$ and define a sequence $\{y_n\}$ recursively by

$$\begin{cases} y_{2n} = Sx_{2n} = Bx_{2n+1} \\ y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}, \end{cases} \qquad n \in \mathbb{N} \cup \{0\}.$$

We shall prove that for any $n \in \mathbb{N}$ and $\varepsilon > 0$

$$F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon)) \geq F_{y_{2n},y_{2n+1}}(\varepsilon). \tag{1}$$

Suppose (1) is not true. Then there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon)) < F_{y_{2n},y_{2n+1}}(\varepsilon). \tag{2}$$

It follows from (iv) and (2) that

$$\begin{aligned}
&\Big(F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))\Big)^2\\
=&\Big(F_{Sx_{2n+2},Tx_{2n+1}}(\varphi(\varepsilon))\Big)^2\\
\geq&\min\Big\{F_{Ax_{2n+2},Sx_{2n+2}}(\varepsilon)F_{Bx_{2n+1},Tx_{2n+1}}(\varepsilon),\\
&\quad F_{Ax_{2n+2},Tx_{2n+1}}(2\varepsilon)F_{Bx_{2n+1},Sx_{2n+2}}(2\varepsilon),\\
&\quad F_{Ax_{2n+2},Sx_{2n+2}}(\varepsilon)F_{Ax_{2n+2},Tx_{2n+1}}(2\varepsilon),\\
&\quad F_{Bx_{2n+1},Sx_{2n+2}}(2\varepsilon)F_{Bx_{2n+1},Tx_{2n+1}}(\varepsilon)\Big\}\\
=&\min\Big\{F_{y_{2n+1},y_{2n+2}}(\varepsilon)F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n+1},y_{2n+1}}(2\varepsilon)F_{y_{2n},y_{2n+2}}(2\varepsilon),\\
&\quad F_{y_{2n+1},y_{2n+2}}(\varepsilon)F_{y_{2n+1},y_{2n+1}}(2\varepsilon),F_{y_{2n},y_{2n+2}}(2\varepsilon)F_{y_{2n},y_{2n+1}}(\varepsilon)\Big\}\\
\geq&\min\Big\{F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n},y_{2n+2}}(2\varepsilon),\\
&\quad F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon)),F_{y_{2n},y_{2n+2}}(2\varepsilon)F_{y_{2n},y_{2n+1}}(\varepsilon)\Big\},\quad\text{since }\varphi(\varepsilon)<\varepsilon\\
&\quad\text{and } F \text{ is nondecreasing,}\\
\geq&\min\Big\{F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))F_{y_{2n},y_{2n+1}}(\varepsilon),\min\{F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n+1},y_{2n+2}}(\varepsilon)\},\\
&\quad F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon)),\min\{F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n+1},y_{2n+2}}(\varepsilon)\}F_{y_{2n},y_{2n+1}}(\varepsilon)\Big\}. \qquad (3)
\end{aligned}$$

Now, note that

$$\begin{cases}
(a)\ F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))F_{y_{2n},y_{2n+1}}(\varepsilon)>\Big(F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))\Big)^2\\
(b)\ \min\Big\{F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n+1},y_{2n+2}}(\varepsilon)\Big\}\geq F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon)),\\
(c)\ \min\Big\{F_{y_{2n},y_{2n+1}}(\varepsilon),F_{y_{2n+1},y_{2n+2}}(\varepsilon)\Big\}F_{y_{2n},y_{2n+1}}(\varepsilon)
\end{cases}$$
$$\geq F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))F_{y_{2n},y_{2n+1}}(\varepsilon)$$
$$>\Big(F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))\Big)^2.$$

So we get from (3) that

$$\Big(F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))\Big)^2>\Big(F_{y_{2n+1},y_{2n+2}}(\varphi(\varepsilon))\Big)^2,\quad\text{a contradiction.}$$

Therefore, (1) holds, for any $n\in\mathbb{N}$ and $\varepsilon>0$. Using a similar argument we obtain that for any $n\in\mathbb{N}$ and $\varepsilon>0$

$$F_{y_{2n},y_{2n+1}}(\varphi(\varepsilon))\geq F_{y_{2n-1},y_{2n}}(\varepsilon). \qquad (4)$$

Thus putting (1) and (4) together, we see that $F_{y_n,y_{n+1}}(\varphi(\varepsilon))\geq F_{y_{n-1},y_n}(\varepsilon)$ for any $n\in\mathbb{N}$ and $\varepsilon>0$, and hence by Lemma 2.1 $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists z in X such that

$$\begin{cases}
Sx_{2n}\longrightarrow z\\
Bx_{2n+1}\longrightarrow z\\
Tx_{2n+1}\longrightarrow z\\
Ax_{2n+2}\longrightarrow z,
\end{cases}\quad\text{as } n\to\infty.$$

Now, suppose A is continuous. Then

$$A^2x_{2n} \to Az \text{ and } ASx_{2n} \to Az \text{ as } n \to \infty. \tag{5}$$

Since both of $\{Ax_{2n}\}$ and $\{Sx_{2n}\}$ are convergent to z, the compatibility of A and S implies that $\lim_{n\to\infty} F_{ASx_{2n},SAx_{2n}}(\varepsilon) = 1$. This in conjunction with (5) and the inequality

$$F_{SAx_{2n},Az}(\varepsilon) \geq \min\{F_{SAx_{2n},ASx_{2n}}\big(\frac{\varepsilon}{2}\big), F_{ASx_{2n},Az}\big(\frac{\varepsilon}{2}\big)\}$$

shows that $SAx_{2n} \to Az$ as $n \to \infty$. Let $E = \{\varepsilon > 0 : F_{Az,z}$ is continuous at $\varepsilon\}$. Since $F_{Az,z}$ is nondecreasing, it can be discontinuous at only denumerably many points. We now show that $F_{Az,z}(\varepsilon) \geq F_{Az,z}(\varphi^{-1}(\varepsilon))$ for any $\varepsilon \in E$. By (iv).

$$\begin{aligned}\Big(F_{SAx_{2n},Tx_{2n+1}}(\varepsilon)\Big)^2 \geq \min\Big\{&F_{A^2x_{2n},SAx_{2n}}(\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon)),\\ &F_{A^2x_{2n},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},SAx_{2n}}(2\varphi^{-1}(\varepsilon)),\\ &F_{A^2x_{2n},SAx_{2n}}(\varphi^{-1}(\varepsilon))F_{A^2x_{2n},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon)),\\ &F_{Bx_{2n+1},SAx_{2n}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon))\Big\}.\end{aligned} \tag{6}$$

It is easy to see that we can choose a subsequence $\{n_j\}$ of natural numbers such that all the limits in (6) exist as $j \to \infty$ and satify

$$\begin{aligned}&\lim_{j\to\infty}\Big(F_{SAx_{2n_j},Tx_{2n_j+1}}(\varepsilon)\Big)^2\\ &\geq \min\Big\{\lim_{j\to\infty}\Big(F_{A^2x_{2n_j},SAx_{2n_j}}(\varphi^{-1}(\varepsilon))F_{Bx_{2n_j+1},Tx_{2n_j+1}}(\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \lim_{j\to\infty}\Big(F_{A^2x_{2n_j},Tx_{2n_j+1}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n_j+1},SAx_{2n_j}}(2\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \lim_{j\to\infty}\Big(F_{A^2x_{2n_j},SAx_{2n_j}}(\varphi^{-1}(\varepsilon))F_{A^2x_{2n_j},Tx_{2n_j+1}}(2\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \lim_{j\to\infty}\Big(F_{Bx_{2n_j+1},SAx_{2n_j}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n_j+1},Tx_{2n_j+1}}(\varphi^{-1}(\varepsilon))\Big)\Big\}\\ &\geq \min\Big\{\varliminf_{n\to\infty}\Big(F_{A^2x_{2n},SAx_{2n}}(\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \varliminf_{n\to\infty}\Big(F_{A^2x_{2n},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},SAx_{2n}}(2\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \varliminf_{n\to\infty}\Big(F_{A^2x_{2n},SAx_{2n}}(\varphi^{-1}(\varepsilon))F_{A^2x_{2n},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon))\Big),\\ &\qquad \varliminf_{n\to\infty}\Big(F_{Bx_{2n+1},SAx_{2n}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon))\Big)\Big\}\\ &\geq \min\Big\{F_{Az,Az}(\varphi^{-1}(\varepsilon))F_{z,z}(\varphi^{-1}(\varepsilon)), F_{Az,z}(2\varphi^{-1}(\varepsilon))F_{z,Az}(2\varphi^{-1}(\varepsilon)),\\ &\qquad F_{Az,Az}(\varphi^{-1}(\varepsilon))F_{Az,z}(2\varphi^{-1}(\varepsilon)), F_{z,Az}(2\varphi^{-1}(\varepsilon))F_{z,z}(\varphi^{-1}(\varepsilon))\Big\}\\ &\geq\Big(F_{Az,z}(\varphi^{-1}(\varepsilon))\Big)^2,\end{aligned} \tag{7}$$

where the penultimate inequality follows from Lemma 1.1.

Also, since $\varepsilon \in E$, it follows from Lemma 1.1 that $\lim_{n\to\infty} F_{SAx_{2n},Tx_{2n+1}}(\varepsilon) = F_{Az,z}(\varepsilon)$, which in conjunction with (7) shows that

$$F_{Az,z}(\varepsilon) \geq F_{Az,z}(\varphi^{-1}(\varepsilon)) \quad \text{for } \varepsilon \in E. \tag{8}$$

To conclude that $Az = z$ we must show that $F_{Az,z}(\varepsilon) = 1$ for any $\varepsilon > 0$. For this, let ε be any member in E and put $\varepsilon_1 = \varepsilon$. Then we have

$$\varepsilon_1 < \varphi^{-1}(\varepsilon_1) < \varphi^{-2}(\varepsilon_1) < \cdots < \varphi^{-n}(\varepsilon_1) < \cdots, \text{ and } \lim_{n\to\infty} \varphi^{-n}(\varepsilon_1) = \infty. \tag{9}$$

Let $\eta > 0$ be any given positive number. Since $F_{Az,z}$ is left continuous at $\varphi^{-2}(\varepsilon_1)$, there is $\delta > 0$ such that

$$F_{Az,z}(\varphi^{-2}(\varepsilon_1)) \leq F_{Az,z}(\omega) + \frac{\eta}{2} \tag{10}$$

for all $\omega \in (\varphi^{-2}(\varepsilon_1) - \delta, \varphi^{-2}(\varepsilon_1))$.
By the continuity of φ^{-1} at $\varphi^{-1}(\varepsilon_1)$, we can choose $\varepsilon_2 \in (\varepsilon_1, \varphi^{-1}(\varepsilon_1)) \cap E$ so that $\varphi^{-1}(\varepsilon_2) \in (\varphi^{-2}(\varepsilon_1) - \delta, \varphi^{-2}(\varepsilon_1))$, and hence with the aid of (10)

$$F_{Az,z}(\varphi^{-1}(\varepsilon_2)) \geq F_{Az,z}(\varphi^{-2}(\varepsilon_1)) - \frac{\eta}{2}. \tag{11}$$

By induction, for any $n \in \mathbb{N}$ we can choose $\varepsilon_{n+1} \in E$ so that

$$\begin{aligned} &\varphi^{-n+1}(\varepsilon_1) < \varepsilon_{n+1} < \varphi^{-n}(\varepsilon_1), \quad \text{and} \\ &F_{Az,z}(\varphi^{-1}(\varepsilon_{n+1})) \geq F_{Az,z}(\varphi^{-(n+1)}(\varepsilon_1)) - \frac{\eta}{2^n} \end{aligned} \tag{12}$$

So we have

$$\begin{aligned} F_{Az,z}(\varepsilon) =& F_{Az,z}(\varepsilon_1) \\ \geq& F_{Az,z}(\varphi^{-1}(\varepsilon_1)) \\ \geq& F_{Az,z}(\varepsilon_2) \\ \geq& F_{Az,z}(\varphi^{-1}(\varepsilon_2)), \quad \text{since } \varepsilon_2 \in E \\ \geq& F_{Az,z}(\varphi^{-2}(\varepsilon_1)) - \frac{\eta}{2} \qquad \text{by (11)} \\ \geq& F_{Az,z}(\varepsilon_3) - \frac{\eta}{2} \\ \geq& F_{Az,z}(\varphi^{-1}(\varepsilon_3)) - \frac{\eta}{2} \\ \geq& F_{Az,z}(\varphi^{-3}(\varepsilon_1)) - \frac{\eta}{2^2} - \frac{\eta}{2} \\ &\vdots \\ \geq& F_{Az,z}(\varphi^{-n}(\varepsilon_1)) - \big(\frac{\eta}{2^{n-1}} + \frac{\eta}{2^{n-2}} + \cdots \frac{\eta}{2}\big) \\ =& F_{Az,z}(\varphi^{-n}(\varepsilon_1)) - \eta(1 - \frac{1}{2^{n-1}}), \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13) and noting $\lim_{n\to\infty} \varphi^{-n}(\varepsilon_1) = \infty$, we obtain that

$$F_{Az,z}(\varepsilon) \geq 1 - \eta \quad \text{for any } \eta > 0.$$

Since $\eta > 0$ is arbitrary, we conclude that $F_{Az,z}(\varepsilon) = 1$ for any $\varepsilon \in E$. Since E is dense in $(0, \infty)$ and $F_{Az,z}$ is left continuous on $(0, \infty)$, we see that $F_{Az,z}(\varepsilon) = 1$ for all $\varepsilon > 0$, and so $Az = z$. As for $Sz = z$, using $(F_{Sz,z}(\varepsilon))^2 = \lim_{n\to\infty}(F_{Sz,Tx_{2n+1}}(\varepsilon))^2$ and (iv), we can just follow as before to obtain $F_{Sz,z}(\varepsilon) \geq F_{Sz,z}(\varphi^{-1}(\varepsilon))$ for any $\varepsilon > 0$ where $F_{Sz,z}$ is continuous. Then in a similar argument as before, we conclude $F_{Sz,z}(\varepsilon) = 1\ \forall\, \varepsilon > 0$, and so $Sz = z$. Since $SX \subseteq BX$, there exists y in X such that $By = Sz = z$. So for any $\varepsilon > 0$

$$\begin{aligned}
\Big(F_{z,Ty}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{Sz,Ty}\big(\varphi(\varepsilon)\big)\Big)^2 \\
&\geq \min\Big\{F_{Az,Sz}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Az,Ty}(2\varepsilon)F_{By,Sz}(2\varepsilon), \\
&\qquad F_{Az,Sz}(\varepsilon)F_{Az,Ty}(2\varepsilon), F_{By,Sz}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\} \\
&= \min\Big\{F_{z,Ty}(\varepsilon), F_{z,Ty}(2\varepsilon)\Big\} \\
&\geq \Big(F_{z,Ty}(\varepsilon)\Big)^2.
\end{aligned}$$

Thus $F_{z,Ty}\big(\varphi(\varepsilon)\big) \geq F_{z,Ty}(\varepsilon)$, and hence $Ty = z$. Up to now we have shown that $Sz = Az = z = By = Ty$. We are now going to show that z is a common fixed point of S, T, A and B. Since T and B are compatible, we have $BTy = TBy$, that is , $Bz = Tz$. Therefore, for $\varepsilon > 0$, we have the following inequalities:

$$\begin{aligned}
\Big(F_{z,Tz}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{Sz,Tz}\big(\varphi(\varepsilon)\big)\Big)^2 \\
&\geq \min\Big\{F_{Az,Sz}(\varepsilon)F_{Bz,Tz}(\varepsilon), F_{Az,Tz}(2\varepsilon)F_{Bz,Sz}(2\varepsilon), \\
&\qquad F_{Az,Sz}(\varepsilon)F_{Az,Tz}(2\varepsilon), F_{Bz,Sz}(2\varepsilon)F_{Bz,Tz}(\varepsilon)\Big\} \\
&= \min\Big\{\big(F_{z,Tz}(2\varepsilon)\big)^2, F_{z,Tz}(2\varepsilon)\Big\} \\
&\geq \Big(F_{z,Tz}(\varepsilon)\Big)^2.
\end{aligned}$$

So $Tz = z$ by Lemma 2.2. This completes the proof for z being the common fixed point of S, T, A and B provided that A is continuous. By symmetry, if B is continuous, we can prove that S, T, A and B have a common fixed point in a similar way.

Next, assume that S is continuous. Then $SAx_{2n} \to Sz$ and $SBx_{2n+1} \to Sz$ as $n \to \infty$, and, since S and A are compatible and both of $\{Ax_{2n}\}$ and $\{Sx_{2n}\}$ are convergent to z, $\lim_{n\to\infty} F_{ASx_{2n},SAx_{2n}}(\varepsilon) = 1$ for $\varepsilon > 0$. Noting that for $\varepsilon > 0$ $F_{ASx_{2n},Sz}(\varepsilon) \geq \min\{F_{ASx_{2n},SAx_{2n}}(\frac{\varepsilon}{2}), F_{SAx_{2n},Sz}(\frac{\varepsilon}{2})\}$ and both of $\{F_{ASx_{2n},SAx_{2n}}(\frac{\varepsilon}{2})\}$ and $\{F_{SAx_{2n},Sz}(\frac{\varepsilon}{2})\}$ are convergent to 1, we see that $\lim_{n\to\infty} F_{ASx_{2n},Sz}(\varepsilon) = 1$ for all $\varepsilon > 0$, and so $\lim_{n\to\infty} ASx_{2n} = Sz$. In the inequality

$$\begin{aligned}
\Big(F_{SBx_{2n+1},Tx_{2n+1}}(\varepsilon)\Big)^2 \geq \min\Big\{&F_{ABx_{2n+1},SBx_{2n+1}}(\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon)), \\
&F_{ABx_{2n+1},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},SBx_{2n+1}}(2\varphi^{-1}(\varepsilon)), \\
&F_{ABx_{2n+1},SBx_{2n+1}}(\varphi^{-1}(\varepsilon))F_{ABx_{2n+1},Tx_{2n+1}}(2\varphi^{-1}(\varepsilon)), \\
&F_{Bx_{2n+1},SBx_{2n+1}}(2\varphi^{-1}(\varepsilon))F_{Bx_{2n+1},Tx_{2n+1}}(\varphi^{-1}(\varepsilon))\Big\},
\end{aligned}$$

we can imitate the procedure for the case that A is continuous to show that $F_{Sz,z}(\varepsilon) \geq F_{Sz,z}(\varphi^{-1}(\varepsilon))$ for any $\varepsilon > 0$ where $F_{Sz,z}$ is continuous, and then show that $F_{Sz,z}(\varepsilon) = 1$ for any $\varepsilon > 0$. So $Sz = z$. Since $SX \subseteq BX$, we can choose $y \in X$ such that $By = Sz = z$. Then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\Big(F_{SBx_{2n+1},Ty}(\varepsilon)\Big)^2 \geq \min\Big\{&F_{ABx_{2n+1},SBx_{2n+1}}(\varphi^{-1}(\varepsilon))F_{By,Ty}(\varphi^{-1}(\varepsilon)),\\ &F_{ABx_{2n+1},Ty}(2\varphi^{-1}(\varepsilon))F_{By,SBx_{2n+1}}(2\varphi^{-1}(\varepsilon)),\\ &F_{ABx_{2n+1},SBx_{2n+1}}(\varphi^{-1}\varepsilon)F_{ABx_{2n+1},Ty}(2\varphi^{-1}(\varepsilon)),\\ &F_{By,SBx_{2n+1}}(2\varphi^{-1}(\varepsilon))F_{By,Ty}(\varphi^{-1}(\varepsilon))\Big\}\\ = \min\Big\{&F_{ASx_{2n},S^2x_{2n}}(\varphi^{-1}(\varepsilon))F_{z,Ty}(\varphi^{-1}(\varepsilon)),\\ &F_{ASx_{2n},Ty}(2\varphi^{-1}(\varepsilon))F_{z,S^2x_{2n}}(2\varphi^{-1}(\varepsilon)),\\ &F_{ASx_{2n},S^2x_{2n}}(\varphi^{-1}(\varepsilon))F_{ASx_{2n},Ty}(2\varphi^{-1}(\varepsilon)),\\ &F_{z,S^2x_{2n}}(2\varphi^{-1}(\varepsilon))F_{z,Ty}(\varphi^{-1}(\varepsilon))\Big\}.\end{aligned}$$

As the case that A is continuous, we can take limit via a suitable subsequence $\{n_j\}$ of natural numbers to get

$$\begin{aligned}\Big(F_{z,Ty}\varphi(\varepsilon)\Big)^2 &\geq \min\Big\{F_{z,Ty}(\varphi^{-1}(\varepsilon)), F_{z,Ty}(2\varphi^{-1}(\varepsilon))\Big\}\\ &\geq \big(F_{z,Ty}(\varphi^{-1}(\varepsilon))\big)^2, \quad \text{for any } \varepsilon > 0 \text{ where } F_{z,Ty} \text{ is continuous.}\end{aligned}$$

Thus $Ty = z$. In summary we have shown that $By = Ty = Sz = z$. Now, since $TX \subseteq AX$, there exists $x \in X$ such that $z = Sz = By = Ty = Ax$. Then we get $Ax = Sx$ from the following inequalities:

$$\begin{aligned}\Big(F_{Sx,Ax}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{Sx,Ty}\big(\varphi(\varepsilon)\big)\Big)^2\\ &\geq \min\Big\{F_{Ax,Sx}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Ax,Ty}(2\varepsilon)F_{By,Sx}(2\varepsilon),\\ &\qquad F_{Ax,Sx}(\varepsilon)F_{Ax,Ty}(2\varepsilon), F_{By,Sx}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}\\ &= \min\Big\{F_{Ax,Sx}(\varepsilon), F_{Ax,Sx}(2\varepsilon)\Big\}\\ &\geq \Big(F_{Ax,Sx}(\varepsilon)\Big)^2 \quad \text{for any } \varepsilon > 0.\end{aligned}$$

Let $\xi = Ax = Sx = Ty = By$. Since S and A are compatible and since $Ax = Sx$, we get $ASx = SAx$, that is, $A\xi = S\xi$. Then for any $\varepsilon > 0$,

$$\begin{aligned}\Big(F_{S\xi,\xi}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{S\xi,Ty}\big(\varphi(\varepsilon)\big)\Big)^2\\ &\geq \min\Big\{F_{A\xi,S\xi}(\varepsilon)F_{By,Ty}(\varepsilon), F_{A\xi,Ty}(2\varepsilon)F_{By,S\xi}(2\varepsilon),\\ &\qquad F_{A\xi,S\xi}(\varepsilon)F_{A\xi,Ty}(2\varepsilon), F_{By,S\xi}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}\\ &= \min\Big\{\big(F_{S\xi,\xi}(2\varepsilon)\big)^2, F_{S\xi,\xi}(2\varepsilon)\Big\}\\ &\geq \big(F_{S\xi,\xi}(\varepsilon)\big)^2,\end{aligned}$$

which implies that $S\xi = \xi = A\xi$. Next, choose $v \in X$ such that $Bv = S\xi = \xi$. Then

$$\begin{aligned}\Big(F_{\xi,Tv}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{S\xi,Tv}\big(\varphi(\varepsilon)\big)\Big)^2 \\ &\geq \min\Big\{F_{A\xi,S\xi}(\varepsilon)F_{Bv,Tv}(\varepsilon), F_{A\xi,Tv}(2\varepsilon)F_{Bv,S\xi}(2\varepsilon), \\ &\qquad F_{A\xi,S\xi}(\varepsilon)F_{A\xi,Tv}(2\varepsilon), F_{Bv,S\xi}(2\varepsilon)F_{Bv,Tv}(\varepsilon)\Big\} \\ &= \min\Big\{F_{\xi,Tv}(\varepsilon), F_{\xi,Tv}(2\varepsilon)\Big\} \\ &\geq \big(F_{\xi,Tv}(\varepsilon)\big)^2 \quad \text{for any } \varepsilon > 0.\end{aligned}$$

Hence $Tv = \xi$. Since T and B are compatible and $Tv = Bv$, we have $TBv = BTv$, that is, $T\xi = B\xi$. Then we conclude that $T\xi = \xi$ from the following inequalities:

$$\begin{aligned}\Big(F_{\xi,T\xi}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{S\xi,T\xi}\big(\varphi(\varepsilon)\big)\Big)^2 \\ &\geq \min\Big\{F_{A\xi,S\xi}(\varepsilon)F_{B\xi,T\xi}(\varepsilon), F_{A\xi,T\xi}(2\varepsilon)F_{B\xi,S\xi}(2\varepsilon), \\ &\qquad F_{A\xi,S\xi}(\varepsilon)F_{A\xi,T\xi}(2\varepsilon), F_{B\xi,S\xi}(2\varepsilon)F_{B\xi,T\xi}(\varepsilon)\Big\} \\ &= \min\Big\{\big(F_{\xi,T\xi}(2\varepsilon)\big)^2, F_{\xi,T\xi}(2\varepsilon)\Big\} \\ &\geq \big(F_{\xi,T\xi}(\varepsilon)\big)^2 \quad \forall\, \varepsilon > 0.\end{aligned}$$

Thus ξ is a common fixed point of S, T, A and B provided that S is continuous. By symmetry, if T is continuous we can prove that S, T, A and B have a common fixed point in a similar way. This completes the proof for the existence of common fixed points of S, T, A and B. It remains to show the uniqueness of the common fixed point. Assume y and z are two common fixed points of S, T, A and B. Since

$$\begin{aligned}\Big(F_{y,z}\big(\varphi(\varepsilon)\big)\Big)^2 &= \Big(F_{Sy,Tz}\big(\varphi(\varepsilon)\big)\Big)^2 \\ &\geq \min\Big\{F_{Ay,Sy}(\varepsilon)F_{Bz,Tz}(\varepsilon), F_{Ay,Tz}(2\varepsilon)F_{Bz,Sy}(2\varepsilon), \\ &\qquad F_{Ay,Sy}(\varepsilon)F_{Ay,Tz}(2\varepsilon), F_{Bz,Sy}(2\varepsilon)F_{Bz,Tz}(\varepsilon)\Big\} \\ &= \min\Big\{\big(F_{y,z}(2\varepsilon)\big)^2, F_{y,z}(2\varepsilon)\Big\} \\ &\geq \big(F_{y,z}(\varepsilon)\big)^2 \quad \text{for all } \varepsilon > 0,\end{aligned}$$

we conclude that $y = z$ by virtue of Lemma 2.2. ///

3. Connection with Metric Spaces. Every metric space (M, d) is a Menger space $(M, \mathcal{F}, \min)$, where the mapping $\mathcal{F}(x, y) = F_{x,y}$ is defined by $F_{x,y}(\varepsilon) = H(\varepsilon - d(x, y))$, and H is the distribution function defined by

$$H(\varepsilon) = \begin{cases} 0, & \text{if } \varepsilon \leq 0, \\ 1, & \text{if } \varepsilon > 0. \end{cases}$$

The space $(M, \mathcal{F}, \min)$ is called the induced Menger space.

Lemma 3.1. Suppose $(M, \mathcal{F}, \min)$ is the induced complete Menger space associated with the complete metric space (M, d) and $\varphi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function satisfying that $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$. If $S, T, A, B : X \to X$ are four selfmaps on M then the following two statements are equivalent:

(i) For x, y in M and $\varepsilon > 0$, if $\varphi(\varepsilon) \le d(Sx, Ty)$ then either $\varepsilon \le \max\{d(Ax, Sx), d(By, Ty)\}$ or $2\varepsilon \le \max\{d(Ax, Ty), d(By, Sx)\}$.

(ii)
$$\big(F_{Sx,Ty}\big(\varphi(\varepsilon)\big)\big)^2 \ge \min\Big\{F_{Ax,Sx}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Ax,Ty}(2\varepsilon)F_{By,Sx}(2\varepsilon), F_{Ax,Sx}(\varepsilon)F_{Ax,Ty}(2\varepsilon), F_{By,Sx}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}$$
for all x, y in M and $\varepsilon > 0$.

Proof. For simplicity put
$$\begin{cases} \alpha = F_{Ax,Sx}(\varepsilon) \\ \beta = F_{By,Ty}(\varepsilon) \\ \gamma = F_{Ax,Ty}(2\varepsilon) \\ \delta = F_{By,Sx}(2\varepsilon) \\ \omega = F_{Sx,Ty}\big(\varphi(\varepsilon)\big). \end{cases}$$
Then (ii) says that
$$\omega^2 \ge \min\{\alpha\beta, \gamma\delta, \alpha\gamma, \beta\delta\}$$
for all x, y in M and $\varepsilon > 0$.
Now, assume (i) holds and suppose x, y are any two points in M and ε is any positive number. Then either $\varphi(\varepsilon) > d(Sx, Ty)$ or $\varphi(\varepsilon) \le d(Sx, Ty)$. For the case that $\varphi(\varepsilon) > d(Sx, Ty)$ we have $\omega^2 = \big(H\big(\varphi(\varepsilon) - d(Sx, Ty)\big)\big)^2 = 1 \ge \min\{\alpha\beta, \gamma\delta, \alpha\gamma, \beta\delta\}$. On the other hand, if $\varphi(\varepsilon) \le d(Sx, Ty)$ then by (i) we have either $\varepsilon \le \max\{d(Ax, Sx),\ d(By, Ty)\}$ or $2\varepsilon \le \max\{d(Ax, Ty), d(By, Sx)\}$, and so we see that at least one of the following four inequalities
$$\begin{cases} \text{(a)}\ \ \varepsilon \le d(Ax, Sx) \\ \text{(b)}\ \ \varepsilon \le d(By, Ty) \\ \text{(c)}\ \ 2\varepsilon \le d(Ax, Ty) \\ \text{(d)}\ \ 2\varepsilon \le d(By, Sx) \end{cases}$$
occurs. Hence at least one of $\alpha, \beta, \gamma, \delta$ is zero. Consequently, $\omega^2 = 0 = \min\{\alpha\beta, \gamma\delta, \alpha\gamma, \beta\delta\}$. Thus (i) implies (ii).

Next, suppose (ii) holds. Let $\varepsilon > 0$ and x, y be any two points in M satisfying that $\varphi(\varepsilon) \le d(Sx, Ty)$. Then $\omega = H\big(\varphi(\varepsilon) - d(Sx, Ty)\big) = 0$, and so (ii) implies that $\min\{\alpha\beta, \gamma\delta, \alpha\gamma, \beta\delta\} = 0$. Thus at least one of $\alpha, \beta, \gamma, \delta$ is zero, that is, at least one of (a), (b), (c), (d) in the previous paragraph holds. Therefore we have either $\varepsilon \le \max\{d(Ax, Sx), d(By, Ty)\}$ or $2\varepsilon \le \max\{d(Ax, Ty), d(By, Sx)\}$. So (ii) implies (i). ///

In view of Theorem 2.4 and Lemma 3.1 the following theorem follows immediately.

Theorem 3.2. Suppose (M, d) is a complete metric space and S, T, A, B are four self-maps on M satisfying the following conditions:

(i) $SM \subseteq BM$ and $TM \subseteq AM$;
(ii) (S, A) and (T, B) are compatible pairs;
(iii) one of S, T, A, B is continuous;
(iv) there exists an upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$ such that for $\varepsilon > 0$ and x, y in M if $\varphi(\varepsilon) \le d(Sx, Ty)$ then either $\varepsilon \le \max\{d(Ax, Sx), d(By, Ty)\}$ or $2\varepsilon \le \max\{d(Ax, Ty), d(By, Sx)\}$.

Then S, T, A and B have a unique common fixed point.

All notations are just as in Lemma 3.1. Assume that for any $\varepsilon > 0$ and for any x, y in M if $\varepsilon > \max\{d(Ax, Sx), d(By, Ty)\}$ then $\varphi(\varepsilon) > d(Sx, Ty)$. We now check that condition (ii) of Lemma 3.1 holds for any x, y in M and any $\varepsilon > 0$. Indeed, let x, y be any two points in M and ε be any positive number. In case $\varepsilon > \max\{d(Ax, Sx), d(By, Ty)\}$. Then we have $H\big(\varepsilon - d(Ax, Sx)\big) = 1 = H\big(\varepsilon - d(By, Ty)\big)$ and $\varphi(\varepsilon) > d(Sx, Ty)$. So the following inequalities hold:

$$\begin{aligned} 2\varepsilon - d(Ax, Ty) &\ge 2\varepsilon - d(Ax, Sx) - d(Sx, Ty) \\ &\ge 2\varepsilon - \varepsilon - \varphi(\varepsilon) \\ &= \varepsilon - \varphi(\varepsilon) > 0. \end{aligned}$$

Thus $H\big(2\varepsilon - d(Ax, Ty)\big) = 1$. Similarly, $H\big(2\varepsilon - d(By, Sx)\big) = 1$. Hence,

$$\begin{aligned} \Big(F_{Sx,Ty}\big(\varphi(\varepsilon)\big)\Big)^2 = 1 = \min\Big\{&F_{Ax,Sx}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Ax,Ty}(2\varepsilon)F_{By,Sx}(2\varepsilon), \\ &F_{Ax,Sx}(\varepsilon)F_{Ax,Ty}(2\varepsilon), F_{By,Sx}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}. \end{aligned}$$

On the other hand, if $\varepsilon \le \max\big\{d(Ax, Sx), d(By, Ty)\big\}$, then we

$$\begin{aligned} \Big(F_{Sx,Ty}\big(\varphi(\varepsilon)\big)\Big)^2 \ge 0 = \min\Big\{&F_{Ax,Sx}(\varepsilon)F_{By,Ty}(\varepsilon), F_{Ax,Ty}(2\varepsilon)F_{By,Sx}(2\varepsilon), \\ &F_{Ax,Sx}(\varepsilon)F_{Ax,Ty}(2\varepsilon), F_{By,Sx}(2\varepsilon)F_{By,Ty}(\varepsilon)\Big\}. \end{aligned}$$

Therefore, the following corollary follows from Theorem 2.4.

Corollary 3.3. Except condition (iv) of Theorem 3.2 is replaced by

(iv)′ For x, y in M and $\varepsilon > 0$ if $\varepsilon > \max\{d(Ax, Sx), d(By, Ty)\}$ then $\varphi(\varepsilon) > d(Sx, Ty)$,

assume all assumptions are just as in Theorem 3.2. Then S, T, A, B have a unique common fixed point.

In the remainder of this section we give a concrete example for Corollary 3.3.

Example 3.4. Let $M = [0, 1]$ with the usual Euclidean distance $d(x, y) = |x - y|$ and let $A, B, S, T : [0,1] \to [0,1]$ be four functions defined by

$$Ax = \begin{cases} \dfrac{3}{4}x + \dfrac{1}{8}, & \text{if } 0 \le x \le \dfrac{1}{2} \\ \dfrac{1}{2}, & \text{if } \dfrac{1}{2} < x \le 1, \end{cases}$$

$$Bx = \begin{cases} \dfrac{1}{2}x + \dfrac{1}{4}, & \text{if } 0 \le x \le \dfrac{1}{2} \\ \dfrac{5}{8}, & \text{if } \dfrac{1}{2} < x \le 1, \end{cases}$$

$$Sx = \begin{cases} \dfrac{1}{4}x + \dfrac{3}{8}, & \text{if } 0 \le x \le \dfrac{1}{2} \\ \dfrac{1}{2}, & \text{if } \dfrac{1}{2} < x \le 1, \end{cases}$$

and

$$Tx = \begin{cases} \dfrac{1}{2}, & \text{if } 0 \le x \le \dfrac{1}{2} \\ \dfrac{3}{8}, & \text{if } \dfrac{1}{2} < x \le 1. \end{cases}$$

Then

(i) $TM \subseteq AM$ and $SM \subseteq BM$,

(ii) $d(ASx, SAx) = 0 \le d(Sx, Ax)$ for $0 \le x \le 1$; $d(TBx, BTx) = 0 \le d(Bx, Tx)$ for $0 \le x \le \frac{1}{2}$ and $d(TBx, BTx) = \frac{1}{16} < \frac{1}{4} = d(Tx, Bx)$ for $\frac{1}{2} < x \le 1$. So (A, S) and (B, T) are compatible pairs.

(iii) A is continuous,

(iv)

$$d(Ax, Sx) = \begin{cases} \dfrac{1}{4} - \dfrac{1}{2}x, & \text{if } 0 \le x \le \dfrac{1}{2} \\ 0, & \text{if } \dfrac{1}{2} < x \le 1, \end{cases}$$

$$d(By, Ty) = \begin{cases} \dfrac{1}{4} - \dfrac{1}{2}y, & \text{if } 0 \le y \le \dfrac{1}{2} \\ \dfrac{1}{4}, & \text{if } \dfrac{1}{2} < y \le 1, \end{cases}$$

$$d(Ty, Sx) = \begin{cases} \dfrac{1}{8} - \dfrac{1}{4}x, & \text{if } 0 \le x \le \dfrac{1}{2},\ 0 \le y \le \dfrac{1}{2} \\ \dfrac{1}{4}x, & \text{if } 0 \le x \le \dfrac{1}{2},\ \dfrac{1}{2} < y \le 1 \\ 0, & \text{if } \dfrac{1}{2} < x \le 1,\ 0 \le y \le \dfrac{1}{2} \\ \dfrac{1}{8}, & \text{if } \dfrac{1}{2} < x \le 1,\ \dfrac{1}{2} < y \le 1 \end{cases}$$

So if we put $\varphi : [0, \infty) \to [0, \infty) : \varphi(x) = \frac{3}{4}x$, then it is easy to check that for $\varepsilon > 0$ and x, y in M if $\varepsilon > \max\{d(Ax, Sx), d(By, Ty)\}$ then $\varphi(\epsilon) > d(Sx, Ty)$.

Thus the conditions in Corollary 3.3 are satisfied and in this case $\frac{1}{2}$ is the unique common fixed point of A, B, S and T.

4. A Generalization of Hadžić Fixed Point Theorem. In [3] the following fixed point theorem is proved.

Hadžić Fixed Point Theorem: Suppose $c \in [0,1)$ is a constant and $(X, \mathcal{F}, t)$ is a complete Menger space with continuous t–norm t and f is a selfmap on X such that for each x in X there is $n(x) \in \mathbb{N}$ so that for all $y \in X$

$$F_{f^{n(x)}(x), f^{n(x)}(y)}(c\varepsilon) \geq F_{x,y}(\varepsilon)$$

for all $\varepsilon > 0$. If there is x_0 in X such that $\sup_{\varepsilon>0} G_{x_0}(\varepsilon) = 1$, where $G_{x_0}(\varepsilon) = \inf\{F_{f^k x_0, x_0}(\varepsilon) : k \in \mathbb{N}\}$, then f has a unique fixed point ξ and for any $x \in X$ $\lim_{n\to\infty} f^n(x) = \xi$.

In what follows we shall show that this theorem holds true if the constant $c \in [0,1)$ is replaced by an upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Actually we have the following theorem.

Theorem 4.1. Suppose $(X, \mathcal{F}, t)$ is a complete Menger space with continuous t–norm t and f is a selfmap on X. If there is an upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$ such that for each x in X there is $n(x) \in \mathbb{N}$ so that for all y in X $F_{f^{n(x)}(x), f^{n(x)}(y)}\big(\varphi(\varepsilon)\big) \geq F_{x,y}(\varepsilon)$ for all $\varepsilon > 0$ and if there exists a point x_0 in X such that $\sup_{\varepsilon>0} G_{x_0}(\varepsilon) = 1$, where $G_{x_0}(\varepsilon) = \inf\{F_{f^k(x_0), x_0}(\varepsilon) : k \in \mathbb{N}\}$, then f has a unique fixed point ξ in X and for any x in X $\lim_{n\to\infty} f^n(x) = \xi$.

Proof. Choose a continuous function $\alpha : [0,\infty) \to [0,\infty)$ such that
(i) α is strictly increasing,
(ii) $\alpha(0) = 0$, and
(iii) $\varphi(t) \leq \alpha(t) < t$ for all $t > 0$.
Then for any $\varepsilon > 0$ we have

$$F_{f^{n(x)}(x), f^{n(x)}(y)}(\alpha(\varepsilon)) \geq F_{f^{n(x)}(x), f^{n(x)}(y)}(\varphi(\varepsilon)) \geq F_{x,y}(\varepsilon) \tag{1}$$

for all $y \in X$.

Define the sequence $\{x_n\}$ recursively in the following way:

$$x_n = f^{n(x_{n-1})}(x_{n-1}), \quad n \in \mathbb{N}.$$

Then for any $n, p \in \mathbb{N}$,

$$\begin{aligned} F_{x_{n+p}, x_n}(\varepsilon) =& F_{f^{n(x_{n+p-1})} f^{n(x_{n+p-2})} \cdots f^{n(x_{n-1})}(x_{n-1}), f^{n(x_{n-1})}(x_{n-1})}(\varepsilon) \\ \geq& F_{f^{n(x_{n+p-1})} \cdots f^{n(x_n)}(x_{n-1}), x_{n-1}}(\alpha^{-1}\varepsilon) \\ &\vdots \\ \geq& F_{f^{n(x_{n+p-1})} \cdots f^{n(x_n)} x_0, x_0}(\alpha^{-n}\varepsilon) \\ \geq& G_{x_0}(\alpha^{-n}\varepsilon). \end{aligned} \tag{2}$$

Since $\sup_{\varepsilon>0} G_{x_0}(\varepsilon) = 1$, it follows from (2) that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there is ξ in X such that $\lim_{n\to\infty} x_n = \xi$. Then in view of $F_{f^{n(\xi)}(x_n), f^{n(\xi)}(\xi)}(\varepsilon) \geq$

$F_{x_n,\xi}(\alpha^{-1}\varepsilon)$ we see that $\lim\limits_{n\to\infty} f^{n(\xi)}(x_n) = f^{n(\xi)}(\xi)$. Now

$$\begin{aligned}
&F_{f^{n(\xi)}(\xi),\xi}(\varepsilon)\\
\geq& t\Big\{F_{f^{n(\xi)}(\xi),f^{n(\xi)}(x_n)}\big(\varepsilon-\alpha(\varepsilon)\big), F_{f^{n(\xi)}(x_n),\xi}\big(\alpha(\varepsilon)\big)\Big\}\\
\geq& t\Big\{F_{\xi,x_n}\big(\alpha^{-1}(\varepsilon-\alpha(\varepsilon))\big), t\big\{F_{f^{n(\xi)}(x_n),x_n}\big(\frac{\alpha(\varepsilon)}{2}\big), F_{x_n,\xi}\big(\frac{\alpha(\varepsilon)}{2}\big)\big\}\Big\}\\
=& t\Big\{F_{\xi,x_n}\big(\alpha^{-1}(\varepsilon-\alpha(\varepsilon))\big), t\big\{F_{f^{n(x_{n-1})}(x_{n-1}),f^{n(x_{n-1})}f^{n(\xi)}(x_{n-1})}\big(\frac{\alpha(\varepsilon)}{2}\big), F_{x_n,\xi}\big(\frac{\alpha(\varepsilon)}{2}\big)\big\}\Big\}\\
\geq& t\Big\{F_{\xi,x_n}\big(\alpha^{-1}(\varepsilon-\alpha(\varepsilon))\big), t\big\{F_{x_{n-1},f^{n(\xi)}(x_{n-1})}\big(\alpha^{-1}\big(\frac{\alpha(\varepsilon)}{2}\big)\big), F_{x_n,\xi}\big(\frac{\alpha(\varepsilon)}{2}\big)\big\}\Big\}\\
&\vdots\\
\geq& t\Big\{F_{\xi,x_n}\big(\alpha^{-1}(\varepsilon-\alpha(\varepsilon))\big), t\big\{F_{x_0,f^{n(\xi)}(x_0)}\big(\alpha^{-n}\big(\frac{\alpha(\varepsilon)}{2}\big)\big), F_{x_n,\xi}\big(\frac{\alpha(\varepsilon)}{2}\big)\big\}\Big\}. \qquad (3)
\end{aligned}$$

Noting that $\lim\limits_{n\to\infty} F_{\xi,x_n}\big(\alpha^{-1}(\varepsilon-\alpha(\varepsilon))\big) = 1$ and $\lim\limits_{n\to\infty} F_{x_0,f^{n(\xi)}(x_0)}\big(\alpha^{-n}\big(\frac{\alpha(\varepsilon)}{2}\big)\big) = 1$ and $\lim\limits_{n\to\infty} F_{x_n,\xi}\big(\frac{\alpha(\varepsilon)}{2}\big) = 1$, it follows from (3) that $f^{n(\xi)}(\xi) = \xi$. We claim that ξ is the unique fixed point of $f^{n(\xi)}$. Suppose y is another fixed point of $f^{n(\xi)}$. Then, for any $\varepsilon > 0$, $F_{\xi,y}(\varepsilon) = F_{f^{n(\xi)}\xi,f^{n(\xi)}y}(\varepsilon) \geq F_{\xi,y}(\alpha^{-1}\varepsilon)$, which by Lemma 2.2 implies that $\xi = y$. Now, since $f(\xi) = f\big(f^{n(\xi)}(\xi)\big) = f^{n(\xi)}(f(\xi))$, we see that $f(\xi)$ is a fixed point of $f^{n(\xi)}$. By the uniqueness of the fixed point of $f^{n(\xi)}$, we get that $f(\xi) = \xi$. For the uniqueness of the fixed point of f, assume y is another fixed point of f. Then for any $\varepsilon > 0$

$$\begin{aligned}
F_{\xi,y}(\varepsilon) =& F_{f^{n(\xi)}(\xi),f^{n(\xi)}y}(\varepsilon)\\
\geq& F_{\xi,y}(\alpha^{-1}\varepsilon),
\end{aligned}$$

which implies that $\xi = y$.

Finally, we show that for any x in X, $\lim\limits_{n\to\infty} f^n(x) = \xi$. For any $m \in \mathbb{N}$ choose $k \in \mathbb{N}$ so that

$$kn(\xi) < m \leq (k+1)n(\xi).$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned}
F_{f^m(x),\xi}(\varepsilon) =& F_{f^m(x),f^{n(\xi)}\xi}(\varepsilon)\\
\geq& F_{f^{m-n(\xi)}(x),\xi}(\alpha^{-1}\varepsilon)\\
&\vdots\\
\geq& F_{f^{m-kn(\xi)}(x),\xi}(\alpha^{-k}\varepsilon) \qquad (4)
\end{aligned}$$

Since $0 < m - kn(\xi) \leq n(\xi)$ and each of $F_{f(x),\xi}(\alpha^{-k}\varepsilon)$, $F_{f^2(x),\xi}(\alpha^{-k}\varepsilon), \ldots$ and $F_{f^{n(\xi)}x,\xi}(\alpha^{-k}\varepsilon)$ converges to 1 as $n \to \infty$, we obtain that $\lim\limits_{m\to\infty} F_{f^{m-kn(\xi)}(x),\xi}(\alpha^{-k}\varepsilon) = 1$, and hence (4) gives us that $\lim\limits_{m\to\infty} F_{f^m(x),\xi}(\varepsilon) = 1$ for any $\varepsilon > 0$. This means $\lim\limits_{m\to\infty} f^m(x) = \xi$. ///

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