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ON A CLASS OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. We obtain inclusion relations and convolution characterization for functions that are analytic in the open unit disk and are bounded above by $1 + (1-\alpha)(\pi^2 - 6)/3, \alpha < 1$. We also show that the class of such functions is invariant under convolution with convex functions.

1. Introduction. Let $\mathcal{A}$ denote the family of functions f that are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

For $\alpha < 1$ and for n a whole number we define

$$M_n(\alpha) := \{f \in \mathcal{A} : Re(D^n f)' > \alpha,\ |z| < 1\} \tag{1.2}$$

where $D^n f$ is the Ruscheweyh derivative [5] of f defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} = f(z) * \frac{z}{(1-z)^{n+1}}.$$

The operator $*$ stands for the Hadamard product or convolution of two power series $f(z) = \Sigma_{k=o}^{\infty} a_k z^k$ and $g(z) = \Sigma_{k=o}^{\infty} b_k z^k$, that is, $(f * g)(z) = f(z) * g(z) = \Sigma_{k=o}^{\infty} a_k b_k z^k$. From (1.2) it is easy to see that $f \in M_n(\alpha)$ if and only if $D^n f \in M_o(\alpha)$, and $M_n(\beta) \subset M_n(\alpha)$ whenever $\alpha < \beta$. We also know [4] that $M_{n+1}(\alpha) \subset M_n(\alpha)$. In [1] the authors showed that if $f \in M_n(\alpha)$ then

$$|f(z)| \leq 1 + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{n!(k-1)!}{(k+n-1)!k}. \tag{1.3}$$

From (1.3) when $n = 1$ it follows that if $f \in M_n(\alpha) \subset M_1(\alpha)$ then

$$|f(z)| \leq 1 + 2(1-\alpha)(\frac{\pi^2}{6} - 1). \tag{1.4}$$

The inequality (1.4) for $M_1(\alpha)$ was also obtained in [1] and [8]. The above inequality (1.4) shows that if $n \geq 1$ then the family $M_n(\alpha)$ is bounded in Δ for all real $\alpha,\ \alpha < 1$. Note that, by (1.3), the functions in $M_o(\alpha)$ need not be bounded. Alexander [3] showed that $M_o(0)$ is

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a subfamily of analytic univalent functions. We conclude that if $0 \leq \alpha < 1$ then $M_n(\alpha)$ is a subfamily of analytic univalent function. Note that the functions in $M_n(\alpha)$ when $\alpha < 0$ need not be univalent. Singh and Singh [9] proved that the functions in $M_1(0)$ are starlike in Δ and in [10] they showed that a function in $M_1(0)$ need not be convex in Δ. For $0 \leq \beta < 1$ and for suitable $\alpha = \alpha(\beta)$ and $n = n(\alpha, \beta)$ we will show that $M_n(\alpha) \subset K(\beta)$ where

$$K(\beta) = \{f \in \mathcal{A} : \ Re\frac{(zf')'}{f'} > \beta, \quad |z| < 1\}$$

is the well-known class of convex functions of order β. Note that $K(\beta) \subset K(0)$ for $0 < \beta < 1$. We also show that the functions in $M_n(\alpha)$ are invariant under convolution with convex functions. Finally, a convolution characterization for functions in $M_n(\alpha)$ is introduced.

2. Main Results. The first theorem is on the convexity of the functions in $M_n(\alpha)$.

2.1. Theorem. Let $0 \leq \beta < 1$. If $\alpha \leq \alpha_o = \frac{41+23\beta}{64}$ and if $n \geq n_o = \frac{15+\beta-16\alpha}{1-\beta}$ then

$$M_n(\alpha) \subset K(\beta).$$

To prove the above theorem we shall need the following two lemmas, the first of which is given in [1] and the second one can be deduced from a result of Silverman [7].

2.2. Lemma. If f is of the form (1.1) and belongs to $M_n(\alpha)$ then

$$|a_k| \leq \frac{2(1-\alpha)(n!)(k-1)!}{(k+n-1)!k}.$$

2.3. Lemma. Let f be of the form (1.1). Then f belongs to $K(\beta)$ if

$$\sum_{k=2}^{\infty} k^2 |a_k| \leq 1 - \beta, \quad z \in \Delta.$$

Proof of Theorem 2.1. Let $f \in M_n(\alpha)$. To show that $f \in K(\beta)$, by Lemmas 2.2 and 2.3 it suffices to show that if $\alpha \leq \alpha_o$ and $n \geq n_o$ then

$$\sum_{k=2}^{\infty} k^2 |a_k| \leq \sum_{k=2}^{\infty} \frac{2(1-\alpha)(n!)(k!)}{(k+n-1)!} \leq 1 - \beta. \tag{2.1}$$

Here we will use an argument similar to that used by the first author and Silverman ([2] Theorem 1). Since $\Sigma_{k=2}^{\infty} 1/k^2 < 1$, (2.1) is true if we can show that

$$\sum_{k=2}^{\infty} \frac{2(n!)(k!)}{(k+n-1)!} \leq \frac{1-\beta}{1-\alpha} \sum_{k=2}^{\infty} \frac{1}{k^2} \ . \tag{2.2}$$

Note that (2.2) holds if

$$d_k = \frac{2k^3(n!)(k-1)!}{(k+n-1)!} \leq \frac{1-\beta}{1-\alpha} \ , \quad k \geq 2.$$

Since $d_2 \leq \frac{1-\beta}{1-\alpha}$ when $n \geq n_o$ and since $n!(k-1)!/(k+n-1)!$ is a decreasing function of n, the proof is complete if we can show that d_k is a decreasing function of k. To show that

$d_{k+1} \leq d_k$ we are required to have $(n-3)k^2 - 3k - 1 \geq 0$ when $n \geq n_o$. This is true since for $\alpha \leq \alpha_o$ and $k \geq 2$ we have

$$(n-3)k^2 - 3k - 1 \geq (n_o - 3)k^2 - 3k - 1 \geq \frac{7}{4}k^2 - 3k - 1 \geq 13k^2 - 3k - 1 > 0.$$

The following lemma which is due to Ruscheweyh and Sheil-Small [6] will be used to prove our next theorem.

2.4. Lemma. If $\phi \in K(0)$ and if $g \in \mathcal{A}$ is starlike in Δ, then the function $(\phi * gF)/(\phi * g)$ takes values in the convex hull of $F(\Delta)$ for every function F in $\mathcal{A}$.

2.5. Theorem. $M_n(\alpha)$ is closed under convolution with convex functions.

Proof. Let $g(z) = z$ and $F(z) = (D^n f)'$. Then for $\phi \in K(0)$ we have

$$\frac{\phi * zF}{\phi * z} = \frac{\phi * z(D^n f)'}{z} = (\phi * D^n f)' = (D^n(\phi * f))'.$$

By Lemma 2.4 we conclude that $(D^n(\phi * f))' \in M_o(\alpha)$. This means that $\phi * f \in M_n(\alpha)$. So the proof is complet.

Next we introduce a convolution characterization for the functions in $M_n(\alpha)$.

2.6. Theorem. A function $f \in \mathcal{A}$ belongs to $M_n(\alpha)$ if and only if

$$\frac{f(z)}{z} * \frac{1 + \frac{n(x+\alpha)+x+2\alpha-1}{1-\alpha}z - \frac{x+2\alpha-1}{2(1-\alpha)}\Sigma_{k=2}^{n+2}(-1)^k\binom{n+2}{k}z^k}{(1-z)^{n+2}} \neq 0, \quad |x| = 1,\ z \in \Delta.$$

Proof. Let $f \in M_n(\alpha)$. Since $(D^n f)' = 1$ at the origin, we can write $f \in M_n(\alpha)$ if and only if

$$\frac{(D^n f)' - \alpha}{1 - \alpha} \neq \frac{x-1}{x+1}, \quad |x| = 1,\ z \in \Delta.$$

This is equivalent to

$$(1+x)(D^n f)' + (1 - 2\alpha - x) \neq 0. \tag{2.3}$$

Writing $g(z) = z/(1-z)^{n+1}$ we observe that

$$z(D^n f)' = z(g * f)' = zf' * g = f * (zg)'.$$

From this and (2.3) we conclude that $f \in M_n(\alpha)$ if and only if

$$\frac{1}{z}\left[f * \{(1+x)zg' + (1-2\alpha-x)z\}\right] \neq 0$$

or if and only if

$$\frac{1}{z}\left[f * \frac{(1+x)(z+nz^2) + (1-2\alpha-x)z(1-z)^{n+2}}{(1-z)^{n+2}}\right] \neq 0$$

which implies the theorem.

2.7. Corollaries. Let $|x| = 1$ and $z \in \Delta - \{0\}$. Then

2.7.1. $f \in M_o(0)$ if and only if $f * \frac{z+((x-1)/2)(2z^2-z^3)}{(1-z)^2} \neq 0$.

2.7.2. $f \in M_1(0)$ if and only if $f * \frac{z+(2x-1)z^2-((x-1)/2)(3z^3-z^4)}{(1-z)^3} \neq 0$.

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