FEKETE–SZEGÖ PROBLEM AND LITTLEWOOD–PALEY CONJECTURE FOR POWERS OF CLOSE–TO–CONVEX FUNCTIONS

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(In the memory of Professor M. J. Shah of Kent State University.)

Received March 21, 1997; revised July 3, 1997

Abstract. We obtain sharp Fekete-Szegö inequalities for powers of a class of close-to-convex functions. We also show that the Littlewood-Paley conjecture fails for these functions. A previous result by the second author is also improved in this paper.

1. Introduction. Let $\mathcal{A}$ be the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of functions univalent in Δ. Let $\gamma > 0$. For f of the form (1) and for the Koebe function $k(z) = z/(1-z)^2$, we write

$$\left\{\frac{f(z)}{z}\right\}^{\frac{1}{\gamma}} = 1 + \sum_{n=1}^{\infty} a_n(\gamma) z^n \tag{2}$$

and

$$\left\{\frac{k(z)}{z}\right\}^{\frac{1}{\gamma}} = 1 + \sum_{n=1}^{\infty} b_n(\gamma) z^n. \tag{3}$$

By equating the coefficients of the like terms in (2) and (3) we obtain

$$a_1(\gamma) = \frac{1}{\gamma} a_2 \ , \quad a_2(\gamma) = \frac{1}{\gamma}\left(a_3 - \frac{\gamma-1}{2\gamma} a_2^2\right) \tag{4}$$

and

$$b_n(\gamma) = \frac{2(2+\gamma)(2+2\gamma)...(2+(n-1)\gamma)}{(n!)\gamma^n}. \tag{5}$$

We consider the inequality

$$|a_n(\gamma)| \le b_n(\gamma) \tag{6}$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C45; 30C50.

Key words and phrases. Fekete-Szegö Inequalities, Littlewood-Paley Conjecture, Starlike and Close-to-convex Functions.

The work of the second author was supported by the Kent State University Research Council .

and for $-\infty < \mu < \infty$ we write

$$M(\gamma) = |a_2(\gamma) - \mu a_1^2(\gamma)|. \tag{7}$$

The inequality (6) is true [16] if $\gamma \leq 1$ and is false for $\gamma > 1$. For $\gamma = 1$, (6) was conjectured by Bieberbach [3] in 1900+16 and was proved by de Branges [8] in 2000-16. Since then many authors studied alternative approaches to the Bieberbach conjecture. The most recent and shortest is given by Ekhad and Zielberger [12]. For $\gamma = 2$, (6) is the Littlewood-Paley [26] conjecture which was disproved by Fekete-Szegö [13]. In fact, Fekete and Szegö [13] obtained sharp bounds for $M(1)$ when $0 \leq \mu \leq 1$. The expression $M(\gamma)$ in (7) has many applications and analogous Fekete-Szegö problems for subclasses of $\mathcal{A}$ and $\mathcal{S}$ proved to be of interest. For example, see Kim and Minda [23, Theorems 1 and 2] and Chua [7, Lemma 2]. It is known that (6) holds for functions that are starlike in Δ and does not hold for close-to-convex functions (see [18]) when $\gamma > 3$. It is of interest to see if there exists a subfamily of close-to-convex functions, larger than the class of starlike functions, for which (6) holds. The answer to this question is still open. We note that $M(\gamma)$ of (7) when $\mu = 0$ is an effective tool to check the validity of the inequality (6). The second author in [18 & 20] used $M(\gamma)$ to show that the inequality (6) is false for some subclasses of Bazilevič and close-to-convex functions. The upper bound for $M(\gamma)$ when f belongs to various subclasses of $\mathcal{A}$ and $\mathcal{S}$ has been studied by many different authors including [1,2,4,7,9,10,13-31]. Recently, Darus and Thomas [9] considered the class $K(\alpha, \beta)$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$ consisting of functions $f \in \mathcal{A}$ so that

$$Re\frac{zf'(z)}{g(z)} > \alpha, z \in \Delta \tag{8}$$

for some $g \in \mathcal{A}$ satisfying the condition

$$Re\frac{zg'(z)}{g(z)} > \beta,\ z \in \Delta. \tag{9}$$

Draus and Thomas [9] obtained sharp upper bounds for $M(1)$ when $f \in K(\alpha, \beta)$. In this paper we generalize their results to the case $\gamma \geq 1$ for $M(\gamma)$ given by (7). Furthermore, we disprove the inequality (6) for certain γ when $f \in K(\alpha, \beta)$. This improves an earlier result obtained by the second author [18].

2. Fekete-Szegö Problem.

To prove our theorem in this section we shall need the following well-known lemmas.

2.1. Lemma. Let $p(z) = 1 + \Sigma_{n=1}^{\infty} p_n z^n$ be analytic in Δ so that $Re\{p(z)\} > 0$ in Δ. Then

$$|p_n| \leq 2 \tag{10}$$

and

$$|p_2 + \lambda p_1^2| \leq 2 + \lambda |p_1|^2 \quad if \quad \lambda \geq -\frac{1}{2}. \tag{11}$$

2.2. Lemma. For $0 \leq \beta < 1$ let $g(z) = z + b_2 z^2 + b_3 z^3 + ...$ be in $\mathcal{A}$ and satisfy the condition (9). Then for μ real,

$$|b_3 - \mu b_2^2| \leq (1-\beta) max\{1, |3 - 2\beta - 4(1-\beta)\mu|\}. \tag{12}$$

The inequality (10) was first proved by Carathéodory [5] (also see Duren [11] page 41) and the inequality (11) can be found in [18]. The inequality (12) was given by Keogh and Merkes [22]. We now state and prove our theorem.

2.3. Theorem. For f given by (1) let $f \in K(\alpha, \beta)$ where $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. Then for $\gamma \geq 1$ and for $-\infty < \mu < \infty$ we have the following sharp bounds.

2.3.1. If $\mu \leq \frac{(1-\beta)(\gamma+3)+3(1-\alpha)(1-\gamma)}{6(2-\alpha-\beta)}$ then

$$|a_2(\gamma) - \mu a_1^2(\gamma)| \leq \frac{(3-2\alpha-\beta)[(1-\beta)(\gamma+3-6\mu)+2\gamma]+3(1-\alpha)^2(1-\gamma-2\mu)}{6\gamma^2}.$$

2.3.2. If $\frac{(1-\beta)(\gamma+3)+3(1-\alpha)(1-\gamma)}{6(2-\alpha-\beta)} \leq \mu \leq \frac{3+\gamma}{6}$ then

$$|a_2(\gamma) - \mu a_1^2(\gamma)| \leq \frac{3-2\alpha-\beta}{3\gamma} + \frac{2(1-\beta)^2(\gamma+3-6\mu)}{9\gamma(\gamma-1+2\mu)}.$$

2.3.3. If $\frac{3+\gamma}{6} \leq \mu \leq \frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}$ then

$$|a_2(\gamma) - \mu a_1^2(\gamma)| \leq \frac{3-2\alpha-\beta}{3\gamma}.$$

2.3.4. If $\mu \geq \frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}$ then

$$|a_2(\gamma) - \mu a_1^2(\gamma)| \leq \frac{3-2\alpha-\beta}{3\gamma} + \frac{\mu(2-\alpha-\beta)^2}{\gamma^2} + \frac{4\alpha\gamma(\alpha-1)+(2-\alpha-\beta)[(\alpha+\beta)(3+\gamma)-6(1+\gamma)]}{6\gamma^2}.$$

Proof. For some $g(z) = z + b_2 z^2 + b_3 z^3 + ...$ in $\mathcal{A}$ and satisfying the condition (9) we let $f(z)$ of the form (1) to be in $K(\alpha, \beta)$. Then we can write

$$\frac{zg'(z)}{g(z)} = \beta + (1-\beta)p(z), \tag{13}$$

and

$$\frac{zf'(z)}{g(z)} = \alpha + (1-\alpha)q(z), \tag{14}$$

where both $p(z) = 1 + p_1 z + p_2 z^2 + ...$ and $q(z) = 1 + q_1 z + q_2 z^2 + ...$ are analytic in Δ and $Re\{p(z)\} > 0$ and $Re\{q(z)\} > 0$ in Δ. Equating the coefficients of the like terms in (13) and (14) we obtain

$$b_2 = (1-\beta)p_1, \tag{15}$$

$$2b_3 = (1-\beta)(p_2 + b_2 p_1), \tag{16}$$

$$2a_2 = (1-\alpha)q_1 + (1-\beta)p_1, \tag{17}$$

and

$$6a_3 = 2(1-\alpha)[q_2 + (1-\beta)p_1 q_1] + (1-\beta)[p_2 + (1-\beta)p_1^2]. \tag{18}$$

Substituting for a_2 and a_3 in (4) yields

$$a_1(\gamma) = \frac{a_2}{\gamma} = \frac{(1-\alpha)q_1 + (1-\beta)p_1}{2\gamma}, \tag{19}$$

and

$$a_2(\gamma) = \frac{1}{\gamma}\left(a_3 + \frac{1-\gamma}{2\gamma}a_2^2\right) \tag{20}$$

$= \frac{1}{\gamma}\left\{\frac{1-\alpha}{3}\left[q_2+\frac{3(1-\gamma)(1-\alpha)}{8\gamma}q_1^2\right]+\frac{1-\beta}{6}\left[p_2+\frac{(1-\beta)(3+\gamma)}{4\gamma}p_1^2\right]+\frac{(1-\alpha)(1-\beta)(3+\gamma)}{12\gamma}p_1q_1\right\}.$

Consequently $M(\gamma)$ of (7) can be written as follows:

$\gamma M(\gamma) = \gamma|a_2(\gamma)-\mu a_1^2(\gamma)|$

$= \left|\frac{1-\alpha}{3}\left[q_2+\frac{3(1-\alpha)(1-\gamma-2\mu)}{8\gamma}q_1^2\right]+\frac{1-\beta}{6}\left[p_2+\frac{(1-\beta)(\gamma+3-6\gamma)}{4\gamma}p_1^2\right]+\frac{(1-\alpha)(1-\beta)(\gamma+3-6\gamma)}{12\gamma}p_1q_1\right|$

$= \left|\frac{1-\alpha}{3}\left[q_2+Aq_1^2\right]+\frac{1-\beta}{6}\left[p_2+Bp_1^2\right]+Cp_1q_1\right|.$

Note that if $\mu \le [1/2+\gamma(1+3\alpha)/6(1-\alpha)] = \mu_1$ and $\mu \le [1/2+\gamma(3-\beta)/6(1-\beta)] = \mu_2$ then $A \ge -1/2$ and $B \ge -1/2$, respectively. Also if $\mu \le (3+\gamma)/6 = \mu_3$ then $C \ge 0$. We observe that $\mu_3 \le \mu_1$ and $\mu_3 \le \mu_2$. So we can use Lemma 2.1 if we let $\mu \le \mu_3$.

First we let $\mu \le \frac{3+\gamma}{6}$. Then

$\gamma M(\gamma) = \gamma|a_2(\gamma)-\mu a_1^2(\gamma)|$

$\le \frac{(1-\alpha)^2(1-\gamma-2\mu)}{8\gamma}|p_1|^2+\frac{(1-\alpha)(1-\beta)(\gamma+3-6\mu)}{6\gamma}|q_1|+\frac{4(1-\alpha)\gamma+2(1-\beta)\gamma+(1-\beta)^2(\gamma+3-6\mu)}{6\gamma}$
$= R(|q_1|).$

Calculating $\frac{dR(|q_1|)}{d|q_1|} = R'(|q_1|) = 0$ we obtain

$$|q_1^o| = \frac{2(1-\beta)(\gamma+3-6\mu)}{3(1-\alpha)(\gamma-1+2\mu)}. \tag{21}$$

If $\mu \le \frac{(1-\beta)(\gamma+3)+3(1-\alpha)(1-\gamma)}{6(2-\alpha-\beta)}$ we observe that $|q_1^o| \notin (0,2)$. In this case the maximum of $R(|q_1|)$ occurs at the end points, i.e., when $|q_1|=0$ or when $|q_1|=2$. Calculating $R(0)$ and $R(2)$ we observe that $R(0)<R(2)$. Therefore we obtain Theorem 2.3.1 that

$$\left|a_2(\gamma)-\mu a_1^2(\gamma)\right| \le \frac{(3-2\alpha-\beta)[(1-\beta)(\gamma+3-6\mu)+2\gamma]+3(1-\alpha)^2(1-\gamma-2\mu)}{6\gamma^2}.$$

Equality is attained on choosing $p_1=p_2=q_1=q_2=2$.

If $\frac{(1-\beta)(\gamma+3)+3(1-\alpha)(1-\gamma)}{6(2-\alpha-\beta)} \le \mu \le \frac{3+\gamma}{6}$ then $0 \le |q_1^o| \le 2$ and so $R(|q_1^o|)$ is a maximum since $R''(|q_1^o|) \le 0$. Therefore we obtain Theorem 2.3.2 that

$$\left|a_2(\gamma)-\mu a_1^2(\gamma)\right| \le \frac{3-2\alpha-\beta}{3\gamma}+\frac{2(1-\beta)^2(\gamma+3-6\mu)}{9\gamma(\gamma-1+2\mu)}.$$

Choosing $p_1=p_2=q_2=2$ and $q_1=|q_1^o|$ as given by (21) shows that the result is sharp.

Next we let $\mu \ge (3+\gamma)/6$. We deal first with the case

$$\mu = \frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}.$$

It follows from (11), (12), (15)-(20) and a simple calculation that

$\left|a_2(\gamma)-\frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}a_1^2(\gamma)\right|$

$= |\frac{1-\alpha}{3\gamma}\left[q_2-\frac{(1-\alpha)(2-\beta)(3-2\alpha-\beta)}{2(2-\alpha-\beta)^2}q_1^2\right]$

$$+\tfrac{1-\beta}{6\gamma}\left[p_2-\tfrac{(1-\beta)(2-\alpha^2-\beta)}{(2-\alpha-\beta)^2}p_1^2\right]-\tfrac{(1-\alpha)(1-\beta)(2-\alpha^2-\beta)}{3\gamma(2-\alpha-\beta)^2}p_1q_1|$$

$$\leq \tfrac{3-2\alpha-\beta}{3\gamma}-\tfrac{(1-\alpha)(1-\beta)(2-\alpha^2-\beta)}{6\gamma(2-\alpha-\beta)^2}(|q_1|-|p_1|)^2-\tfrac{\alpha(1-\alpha)(1-\beta)}{6\gamma(2-\alpha-\beta)}\left(|q_1|^2-|p_1|^2\right)\leq \tfrac{3-2\alpha-\beta}{3\gamma}.$$

In this case, we need to consider the following two subcases.

For $\frac{3+\gamma}{6}\leq\mu\leq\frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}$ we write

$$a_2(\gamma)-\mu a_1^2(\gamma)=\tfrac{(2-\alpha-\beta)^2(6\mu-\gamma-3)}{4\gamma(2-\alpha^2-\beta)}\left[a_2(\gamma)-\tfrac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}a_1^2(\gamma)\right]$$
$$+\tfrac{4\gamma(2-\alpha^2-\beta)-(2-\alpha-\beta)(6\mu-\gamma-3)}{4\gamma(2-\alpha^2-\beta)}\left[a_2(\gamma)-\tfrac{3+\gamma}{6}a_1^2(\gamma)\right].$$

Using the bounds obtained for $M(\mu)$ when $\mu=\frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}$ and $\mu=\frac{3+\gamma}{6}$ we obtain Theorem 2.3.3 that

$$\left|a_2(\gamma)-\mu a_1^2(\gamma)\right|\leq\left(\tfrac{3-2\alpha-\beta}{3\gamma}\right)\left(\tfrac{(2-\alpha-\beta)^2(6\mu-\gamma-3)}{4\gamma(2-\alpha^2-\beta)}+\tfrac{4\gamma(2-\gamma^2-\beta)-(2-\alpha-\beta)^2(6\mu-\gamma-3)}{4\gamma(2-\gamma^2-\beta)}\right)$$
$$=\tfrac{3-2\alpha-\beta}{3\gamma}.$$

Equality is attained on choosing $p_1=q_1=0$ and $p_2=q_2=2$.

Finally, we let $\mu\geq\frac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}$. In this case we write

$$a_2(\gamma)-\mu a_1^2(\gamma)=a_2(\gamma)-\tfrac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}a_1^2(\gamma)$$
$$+\left[\tfrac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}-\mu\right]a_1^2(\gamma).$$

Taking the absolute values we obtain Theorem 2.3.4 that

$$\left|a_2(\gamma)-\mu a_1^2(\gamma)\right|\leq\left|a_2(\gamma)-\tfrac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}a_1^2(\gamma)\right|$$
$$+\left[\mu-\tfrac{4\alpha\gamma(1-\alpha)+(2-\alpha-\beta)[6(1+\gamma)-(\alpha+\beta)(3+\gamma)]}{6(2-\alpha-\beta)^2}\right]|a_1(\gamma)|^2$$
$$\leq\frac{3-2\alpha-\beta}{3\gamma}+\frac{4\alpha\gamma(\alpha-1)+(2-\alpha-\beta)[(\alpha+\beta)(3+\gamma)-6(1+\gamma)]}{6\gamma^2}+\frac{\mu(2-\alpha-\beta)^2}{\gamma^2},$$

where we have used the fact that $|a_1(\gamma)|=|[(1-\alpha)q_1+(1-\beta)p_1]/2\gamma|\leq(2-\alpha-\beta)/\gamma$. Choosing $p_1=q_1=2i$ and $p_2=q_2=-2$ concludes the sharpness.

3. Littlewood-Paley Conjecture.

As mentioned earlier , letting $\mu=0$ in $M(\gamma)$ given by (7) we may obtain bounds for $|a_2(\gamma)|$ which is a good criterion to check the validity of $|a_2(\gamma)|\leq b_2(\gamma)$ given by (6). Now for $\mu=0$ and for $|q_1^o|=\frac{2(1-\beta)(\gamma+3)}{3(1-\alpha)(\gamma-1)}$ we obtain

$$|a_2(\gamma)|=\frac{R(|q_1^o|)}{\gamma}=\frac{3-2\alpha-\beta}{3\gamma}+\frac{2(1-\beta)^2(\gamma+3)}{9\gamma(\gamma-1)}. \tag{22}$$

It is easy to see that there are α, β and γ in (22) so that $|a_2(\gamma)|>b_2(\gamma)$. For example, for $\gamma=4$ and for $28\beta^2-74\beta-36\alpha+1>0$ we have

$$|a_2(4)|=\frac{82-74\beta+28\beta^2-36\alpha}{216}>\frac{81}{216}. \tag{23}$$

We see that the inequality (6) is false, by the condition (23). Furthermore, this result is an improvement of an earlier result obtained by the second author [18] for $f\in K(0,0)$.

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