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A NEW PROOF OF A FIXED POINT THEOREM OF EDELSTEIN

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Abstract. In this paper, we give a new proof of a well known theorem of Edelstein [3] concerning the fixed point of contractive mappings. The method of the new proof is also used to prove the related results of Smithson [8] and Park [7].

1. INTRODUCTION.

Let (X, d) be a metric space and $f: X \to X$. Suppose that there exists a constant $h, 0 \leq h < 1$, such that

$$d(f(x), f(y)) < h\, d(x, y) \qquad \text{for all } x, y \in X, x \neq y. \tag{1}$$

A classical theorem of Banach states that a mapping f defined on a complete metric space (X, d) that satisfies condition (1) has a unique fixed point in X. Edelstein [3] made significant contribution in the area of fixed point theory by considering the contractive condition in which h is allowed to be 1, -i.e.

$$d(f(x), f(y)) < d(x, y) \qquad \text{for all } x, y \in X, x \neq y. \tag{2}$$

A mapping satisfying condition (2) is called **contractive**. It is the purpose of this note to provide a new proof of this well know result of Edelstein. Two additional related results of Smithson [8] and Park [7] are quoted and given new proofs to demonstrate that the technique of new proof provided here can be modified to prove other results. Edelstein subsequently gave conditions under which a contractive mapping has a fixed point. For an element $x \in X$, the sequence of iterates by f will be called an **orbit** of x and denoted by $\mathcal{O}(x)$, -i.e. $\mathcal{O}(x) = \{x_n : x_n = f(x_{n-1}),\, x_0 = x\}$. Edelstein proved the following result;

Theorem 1.1. *If f is a contractive mapping of X and if an orbit $\mathcal{O}(x)$ has a cluster point ξ, then ξ is a unique fixed point of f.*

The theorem of Edelstein was extended to hold by Smithson [8] in the setting of multivalued contractive mappings. We denote the class of all nonempty closed bounded subsets of X by $CB(X)$. For $A,B \in CB(X)$, $H(A, B)$ denotes the Hausdorff distance between A and B. $T: X \to CB(X)$ will be called **a multivalued contractive mapping** if

$$H(T(x), T(y)) < d(x, y) \qquad \text{for all } x,\, y \in X,\, x \neq y. \tag{3}$$

An orbit $\mathcal{O}(x)$ of a multivalued mapping T at x is a sequence $\{x_n | x_n \in Tx_{n-1},\, x_0 = x\}$. Moreover, an orbit $\mathcal{O}(x)$ is called a **regular orbit** if

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \text{ and } d(x_{n+1}, x_{n+2}) \leq H(Tx_n, Tx_{n+1}). \tag{4}$$

The following is the result of Smithson from [8].

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Theorem 1.2. *Let (X, d) be a metric space and $T: X \to CB(X)$ a multivalued contractive mapping. If there is a regular orbit $\mathcal{O}(x)$ for T which contains a convergent subsequence $x_{n_i} \to y_0$ such that $x_{n_i+1} \to y_1$, then $y_1 = y_0$ (-i.e. T has a fixed point in X).*

Another extension of Edelstein's theorem was made by Park [7]. Park considered Edelstein's result in the setting of f-contractive mappings. Let f be a continuous mapping of X into X. A mapping $g: X \to X$ is said to be f-**contractive** if

$$d(g(x), g(y)) < d(f(x), f(y)) \qquad \text{for all } x,\ y \in X,\ g(x) \neq g(y). \tag{5}$$

The idea of this extension originates with the paper of Jungck [4] who considered fixed point theorem for the class of f-contractions that generalizes the Banach contraction mapping principle. An extension made by Park (Theorem 1.3 below) to the result of Edelstein is one of numerous fixed point results that were also obtained by many other authors who studied the classes of f-contractive and f-contraction mappigs. For example, Kaneko [6] proved a fixed point theorem for f-contraction multi-valued mappings and his result and that of Smithson mentioned above were recently generalized to multi-valued f-contractive mappings by Daffer and Kaneko [2]. Let C_f denote the family of all mappings $g: X \to X$ such that $g(X) \subseteq f(X)$ and $g \circ f = f \circ g$. Given $x \in X$, and a mapping $g \in C_f$, an f**-orbit** $\mathcal{O}_f(x)$ **of** x **under** g is defined by

$$\mathcal{O}_f(x) = \{x_n | f(x_n) = g(x_{n-1}),\ x_0 = x\}.$$

The result of Park [7] is the following;

Theorem 1.3. *A continuous self-mapping f of X has a fixed point if and only if there exists an f contractive mapping $g \in C_f$ such that for some $x_0 \in X$, there is an f-orbit $\mathcal{O}_f(x_0)$ which has a cluster point $\xi \in X$. Indeed, f and g have the unique common fixed point $f(\xi)$.*

In the recent paper [1], the present authors observed that the commutativity condition imposed upon f and g restricts an application of theorem 1.3 to solving certain classes of operator equations. It was demonstrated that without the commutativity assumption, it is still possible to guarantee the existence of a coincidence point of f and g,-i.e., a point x such that $f(x) = g(x)$. It was also shown in that paper that the existence of such a coincidence point is all that is required to establish the solvability of certain classes of nonlinear integral equations.

2. FIXED POINT THEOREMS.

In this section, we present new proofs of theorems 1.1, 1.2 and 1.3 which are described in Introduction. The current method of proofs is, in our opinion, more succinct and straightforward than the original proofs of these theorems.

Proof of Theorem 1.1. Let $\{x_{n_k}\}$ be a convergent subsequence of $\mathcal{O}(x) = \{x_n\}$ such that $x_{n_k} \to \xi$. If anywhere along the orbit $\mathcal{O}(x)$, we have $d(x_{n-1}, x_n) = 0$, this means that $f(x_{n-1}) = x_{n-1}$ and x_{n-1} is a fixed point of f. Thus without loss of generality we assume that $d(x_{n-1}, x_n) > 0$, for all n. Since f is contractive, it is continuous on X and the sequence $\{c_n\}$ defined by $c_n = d(x_n, x_{n+1})$ is monotonically strictly decreasing. Hence

$$\begin{aligned} \lim_{k\to\infty} c_{n_k} &= \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) \\ &= \lim_{k\to\infty} d(x_{n_k}, f(x_{n_k})) \\ &= d(\xi, f(\xi)) = c. \end{aligned}$$

If $c > 0$, since f is contractive, we get

$$c' = d(f(\xi), f(f(\xi))) < d(\xi, f(\xi)) = c.$$

By continuity, $\lim_{k\to\infty} d(f(x_{n_k}), f(f(x_{n_k}))) = c'$. Thus

$$d(f(x_{n_k}), f(f(x_{n_k}))) = d(x_{n_k+1}, x_{n_k+2}) < c,$$

for almost all k. This is a contradiction, since

$$c < d(x_{n_{k+2}}, x_{n_{k+2}+1}) = d(x_{n_{k+2}}, f(x_{n_{k+2}})) < d(x_{n_k+1}, x_{n_k+2}) < c.$$

Hence $c = 0$ and $f(\xi) = \xi$. The uniqueness of ξ follows since if $f(\xi) = \xi, f(\eta) = \eta$ and $\xi \neq \eta$, then we obtain a contradiction by $d(\xi, \eta) = d(f(\xi), f(\eta)) < d(\xi, \eta)$. Q.E.D.

The proof of theorem 1.2 given by Smithson [8], for most of its part, is based upon the proof of Edelstein in [3]. A new and shorter proof is now presented.

Proof of Theorem 1.2. As in the proof of theorem 1.1 given above, if, for some n, $d(x_{n-1}, x_n) = 0$, then $d(x_{n-1}, T(x_{n-1})) \leq d(x_{n-1}, x_n) = 0$. Since $T(x_{n-1})$ is closed, $x_{n-1} \in T(x_{n-1})$ and x_{n-1} is a fixed point of T. Hence assume that $d(x_{n-1}, x_n) > 0$ for all n. Since T is contractive and $\mathcal{O}(x)$ is regular, $c_n = d(x_n, x_{n+1})$ is monotonically strictly decreasing. Let $c = \lim_{n\to\infty} c_n$. By assumption,

$$\lim_{k\to\infty} c_{n_k} = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(y_0, y_1) = c.$$

If $c > 0$, then since T is contractive,

$$c' = H(T(y_0), T(y_1)) < d(y_0, y_1) = c.$$

By the continuity of T, given $\epsilon > 0, c' + \epsilon < c$, there exists N such that for all $k \geq N$,

$$H(T(x_{n_k}), T(x_{n_k+1})) \leq c' + \epsilon < c.$$

Thus we obtain a contradiction that

$$c < d(x_{n_{k+1}}, x_{n_{k+1}+1}) \leq d(x_{n_k+1}, x_{n_k+2}) \leq H(T(x_{n_k}), T(x_{n_k+1})) < c.$$

Thus $y_0 = y_1$ and $y_0 \in T(y_0)$. Q.E.D.

We are now ready to prove theorem 1.3. To put a special emphasis on a coincidence point whose importance was mentioned in the comments following Theorem 1.3, we restate the theorem in a slightly different way. Without the commutativity condition, one can guarantee the existence of a coincidence point. Our proof, however, requires stronger conditions on f. We recall from [5] the definition of compatible mappings. Self mappings f and g of a metric space (X, d) are **compatible** if and only if whenever $\{x_n\}$ is a sequence in X such that $f(x_n)$, $g(x_n) \to t \in X$, then $d(f(g(x_n)), g(f(x_n))) \to 0$.

Theorem 2.1. *Let X be a metric space and g an f-contractive self-mapping of X with $g(X) \subseteq f(X)$. Let f be continuous and one-to-one with f^{-1} also continuous on $f(X)$. If, for some $x \in X$, an f-orbit of x under g has a cluster point, then f and g have a coincidence point x^*,-i.e., $g(x^*) = f(x^*)$. Moreover, if f and g are compatible and if $f^n(x^*) \to y$, then y is the unique common fixed point of f and g.*

Proof. Let $c_n = d(f(x_{n+1}), f(x_n))$ for each $n = 0, 1, \cdots, x_0 = x$. If $c_n = 0$, then $c_m = 0$ for all $m \geq n$. To see this, let $c_n = 0$ but $c_{n+1} > 0$. Then since g is f-contractive,

$$0 < d(f(x_{n+2}), f(x_{n+1})) = d(g(x_{n+1}), g(x_n)) < d(f(x_{n+1}), f(x_n)) = c_n.$$

This is a contradiction. Now if $c_n = 0$ for some n, then $0 = c_n = d(f(x_{n+1}), f(x_n))$ $= d(g(x_n), f(x_n))$, and x_n is a coincidence point of f and g, (hence also x_m for all $m \geq n$). We therefore assume that $c_n > 0$, for all n. First observe that $\{c_n\}$ is monotonically strictly decreasing, since

$$c_{n+1} = d(f(x_{n+2}), f(x_{n+1})) = d(g(x_{n+1}), g(x_n)) < d(f(x_{n+1}), f(x_n)) = c_n.$$

Thus $\lim_{n\to\infty} c_n = c \geq 0$. we need to show that $c = 0$. By hypothesis, there is a subsequence $\{x_{n_k}\}$ of the f-orbit that converges, say $\lim_{k\to\infty} x_{n_k} = x^*$. It follows from the hypotheses that g is continuous. We thus have

$$\begin{aligned} c = \lim_{k\to\infty} c_{n_k} &= \lim_{k\to\infty} d(f(x_{n_k+1}), f(x_{n_k})) \\ &= \lim_{k\to\infty} d(g(x_{n_k}), f(x_{n_k})) \\ &= d(g(x^*), f(x^*)). \end{aligned}$$

Now if $c > 0$, then since $g(X) \subseteq f(X)$ we obtain the following contradiction.

$$\begin{aligned} c = d(g(x^*), f(x^*)) &= d(f(f^{-1}(g(x^*))), f(x^*)) \\ &> d(g(f^{-1}(g(x^*))), g(x^*)) \\ &= \lim_{k\to\infty} d(g(f^{-1}(g(x_{n_k}))), g(x_{n_k})) \\ &= \lim_{k\to\infty} d(g(f^{-1}(f(x_{n_k+1}))), g(x_{n_k})) \\ &= \lim_{k\to\infty} d(g(x_{n_k+1}), g(x_{n_k})) \\ &= \lim_{k\to\infty} d(f(x_{n_k+2}), f(x_{n_k+1})) \\ &= \lim_{k\to\infty} c_{n_k+1} = c. \end{aligned}$$

Thus, $c = 0$ and we have $g(x^*) = f(x^*)$.

If f and g are compatible, then by Proposition 2.2 of [5], they commute at their coincidence point. Hence $f^n(g(x^*)) = g(f^n(x^*)) = f^{n+1}(x^*)$ for all n, and if $\lim_{n\to\infty} f^n(x^*) = y$, then $g(y) = y = f(y)$. If f and g have a common fixed point $z \neq y$, then $d(f(y), f(z)) > 0$, because f is one-to-one. Hence by contractivity, $d(y, z) = d(g(y), g(z)) < d(f(y), f(z)) =$ $d(y, z)$. This contradiction leads to conclude that y is a unique common fixed point. Q.E.D.

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