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MAXIMAL ARITHMETICAL ALGEBRAS

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Abstract. In this paper we define the notions of semilocal algebra and of maximal algebra. We prove that any maximal algebra of an arithmetical variety is semilocal. The main result shows that an algebra of an arithmatical variety is semisimple and maximal iff it is isomorphic to a finite product of simple algebras.

All algebras are assumed to be of the same fixed type. Because no confusion will result we use the same letter, e.g., A, to denote both an algebra and its base set. For notions not defined here, we refer the reader to [4],[8].

For any algebra A, Con$A$ denotes the congruence lattice on A, 1_A is the universal congruence and 0_A is the trivial congruence on A. We say that the **Chinese Remainder Theorem** (CHRT, for short) holds in the algebra A if for any $\theta_1, \dots, \theta_n \in \mathrm{Con}A$, $a_1, \dots, a_n \in A$, the condition $(a_i, a_j) \in \theta_i \vee \theta_j$ for $1 \leq i, j \leq n$ implies the existence of an $a \in A$ with $(a, a_i) \in \theta_i$, for $i = 1, \dots, n$. An algebra A verifies the Chinese Remainder Theorem if and only if A is **arithmetical** (i.e., Con$A$ is a distributive lattice and the congruence relations permute) [8]. A variety composed of algebras with this property is called **arithmetical variety**.

Let Max$A$ denotes the set of maximal congruences on A and $\mathrm{Rad}A = \cap\{\theta | \theta \in \mathrm{Max}A\}$. We say that A is **semisimple** if $\mathrm{Rad}A = 0_A$ (i.e., A is a subdirect product of simple algebras).

Definition 1. An algebra A is called **semilocal** if Max$A$ is finite.

Lemma 2. Let A be an algebra that verifies CHRT. If A is semilocal, then $A/\mathrm{Rad}A$ is isomorphic to a finite direct product of simple algebras.

Proof. Let $\mathrm{Max}A = \{\theta_1, \dots, \theta_n\}$ and let be $\varphi : A/\mathrm{Rad}A \to \prod_{i=1}^{n} A/\theta_i$ the morphism given by $\varphi(x/\mathrm{Rad}A) = (x/\theta_1, \dots, x/\theta_n)$. This map is injective. We shall prove that φ is also surjective; if $y = (x/\theta_1, \dots, x/\theta_n) \in \prod_{i=1}^{n} A/\theta_i$, then because $\theta_i \vee \theta_j = 1_A$ for $1 \leq i, j \leq n$, $i \neq j$ and A verifies CHRT there is $x \in A$ such that $x/\theta_i = x_i/\theta_i$, for $i = 1, \dots, n$ so $\varphi(x/\mathrm{Rad}A) = y$.

Corollary 3. Let A be an algebra that verifies CHRT. If A is semisimple and semilocal, then A is isomorphic to finite direct product of simple algebras.

Definition 4. An algebra A is called **artinian** if every descending chain of congruences in A eventually stops.

Proposition 5. Let A be an algebra that verifies CHRT. Then A is semilocal iff $A/\mathrm{Rad}A$ is artinian.

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Proof. If A is semilocal, by Lemma 2 it follows that $A/\mathrm{Rad}A$ is isomorphic to $\prod_{i=1}^{n} A_i$, where A_i are simple algebras for $i = 1, \ldots, n$. Let $p_i : \prod_{i=1}^{n} A_i \to A_i$ be the canonical projections, for $i = 1, \ldots, n$; because A_i for $i = 1, \ldots, n$ are simple, it is easy to see that $\ker p_i \in \mathrm{Max}(\prod_{i=1}^{n} A_i)$. For $\theta \in \mathrm{Max}(\prod_{i=1}^{n} A_i)$ we have $0 = \ker p_1 \wedge \ldots \wedge \ker p_n \leq \theta$, then there exists i such that $\theta = \ker p_i$; it follows that $\mathrm{Max}(\prod_{i=1}^{n} A_i) = \{\ker p_1, \ldots, \ker p_n\}$. Now let be $\theta \in \mathrm{Con}(\prod_{i=1}^{n} A_i)$ and suppose $\theta \subseteq \ker p_i$ for $i \in I \subseteq \{1, \ldots, n\}$ and for $i \in \{1, \ldots, n\} \setminus I$ the previous inclusion is not valid. Then $\theta = \theta \vee 0 = \theta \vee (\ker p_1 \wedge \ldots \wedge \ker p_n) = (\theta \vee \ker p_1) \wedge \ldots \wedge (\theta \vee \ker p_n) = \wedge\{\ker p_i | i \in I\}$, because $\theta \vee \ker p_i = 1$ for $i \in \{1, \ldots, n\} \setminus I$. In this way we proved that $\mathrm{Con}(\prod_{i=1}^{n} A_i)$ is finite, in particular $\prod_{i=1}^{n} A_i$ is artinian, i.e. $A/\mathrm{Rad}A$ is artinian.

Conversely, suppose that $A/\mathrm{Rad}A$ is artinian and A is not semilocal. Then there exists a sequence $(\theta_n)_{n \in N}$ of distinct elements in $\mathrm{Max}A$, and we obtain a descending sequence $\sigma_1 \supseteq \sigma_2 \supseteq \sigma_3 \supseteq \ldots$, $\sigma_n \neq \sigma_{n+1}$, for all $n \in N$, where $\sigma_n = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n$. Then the sequence $(\sigma_n/\mathrm{Rad}A)_{n \in N}$ is an infinite descending chain in $A/\mathrm{Rad}A$, a contradiction.

Remarks.

1) The proof of the previous proposition shows that the following assertions are true for an algebra A which verifies CHRT:
   a) A is semilocal iff $\mathrm{Con}(A/\mathrm{Rad}A)$ is finite;
   b) If A is semisimple then A is semilocal iff $\mathrm{Con}A$ is finite.
2) Theorem 2.6 of [9] is a particular case of the previous proposition.

Definition 6.

a) Let A be an algebra. We shall say that the family $\{(x_i, \theta_i) | i \in I\}$, $x_i \in A$, $\theta_i \in \mathrm{Con}A$ has the **property** $(\bullet)$ if for any finite subset $\triangle$ of I there exists $x_\triangle \in A$ such that $(x_\triangle, x_i) \in \theta_i$ for any $i \in \triangle$.
b) The algebra A will be called **maximal** if for any family $\{(x_i, \theta_i) | i \in I\}$ having property $(\bullet)$ there exists $x \in A$ such that $(x, x_i) \in \theta_i$ for any $i \in I$.

Lemma 7. If A is an algebra such that $\mathrm{Con}A$ is finite then A is maximal.

Proof. Let $\{(x_i, \theta_i) | i \in I\}$ be a family having property $(\bullet)$. If (x_i, θ_i) and (x_j, θ_j) are two elements such that $\theta_i = \theta_j$ it follows by $(\bullet)$ that $(x_i, x_j) \in \theta_i$. Because $\mathrm{Con}A$ is finite and the family $\{(x_i, \theta_i) | i \in I\}$ has the property $(\bullet)$ it follows that there exists $x \in A$ such that $(x, x_i) \in \theta_i$ for any $i \in I$.

Proposition 8. Let A be an algebra that verifies CHRT. If A is maximal then A is semilocal.

Proof. We start the proof with the remark that any family $\{(x_\theta, \theta) | \theta \in \mathrm{Max}A\}$ has the property $(\bullet)$, because A verifies CHRT and for $\theta, \theta' \in \mathrm{Max}A$, $\theta \neq \theta'$ we have $\theta \vee \theta' = 1_A$. A being maximal there exists $x^* \in A$ such that $(x^*, x_\theta) \in \theta$ for any $\theta \in \mathrm{Max}A$.

Consider the binary relation on A given by:

$\theta^* = \{(x, y) \in A \times A | \{\theta \in \mathrm{Max}A | (x, y) \notin \theta\}$ is finite $\}$.

Then θ^* is a congruence on A, because:

-θ^* is obviously an equivalence relation on A;

- if f is an p-ary operation on A and $(a_i, b_i) \in \theta^*$ for $i = 1, \ldots, p$, denoting by $I_i = \{\theta \in \mathrm{Max}A | (a_i, b_i) \in \theta\}$ and using the definition of θ^* we obtain $\mathrm{Max}A \setminus I_i$ is finite for $i = 1, \ldots, p$.

For $\theta \in \cap\{I_i | i = 1, \ldots, p\}$ it follows that $(a_i, b_i) \in \theta$, hence $(f(a_1, \ldots, a_p), f(b_1, \ldots, b_p)) \in \theta$ so $\{\theta \in \mathrm{Max}A | (f(a_1, \ldots, a_p), f(b_1, \ldots, b_p)) \notin \theta\} \subseteq \mathrm{Max}A \backslash (\cap\{I_i | i = 1, \ldots, p\}) = \cup\{\mathrm{Max}A \backslash I_i | i = 1, \ldots, p\}$. The last set being finite we obtain $(f(a_1, \ldots, a_p), f(b_1, \ldots, b_p)) \in \theta^*$.

Let be $a \in A$ fixed; for each $\theta \in \mathrm{Max}A$ there exists $b_\theta \in A$ such that $(a, b_\theta) \notin \theta$, because $\theta \neq 1_A$. The family $\{(a, \theta^*)\} \cup \{(b_\theta, \theta) | \theta \in \mathrm{Max}A\}$ has the property $(\bullet)$. In fact if we consider a finite subfamily $\{(a, \theta^*)\} \cup \{(b_1, \theta_1), \ldots, (b_n, \theta_n)\}$ of the given family, using the definition of θ^* we obtain $\cap\{\theta | \theta \in \mathrm{Max}A \backslash \{\theta_1, \ldots, \theta_n\}\} \subseteq \theta^*$. Let be the family $\{(b_1, \theta_1), \ldots, (b_n, \theta_n)\} \cup \{(a, \theta) | \theta \in \mathrm{Max}A \backslash \{\theta_1, \ldots, \theta_n\}\}$. Using the start of the proof there exists $x^* \in A$ such that $(x^*, b_i) \in \theta_i$ for $i = 1, \ldots, n$ and $(x^*, a) \in \theta$ for any $\theta \in \mathrm{Max}A \backslash \{\theta_1, \ldots, \theta_n\}$, hence $(x^*, a) \in \cap\{\theta | \theta \in \mathrm{Max}A \backslash \{\theta_1, \ldots, \theta_n\}\}$ and by the definition of θ^* we obtain $(x^*, a) \in \theta^*$, so the given family has property $(\bullet)$.

The algebra A being maximal there is $a^* \in A$ such that $(a^*, a) \in \theta^*$ and $(a^*, b_\theta) \in \theta$ for any $\theta \in \mathrm{Max}A$. But $(a, b_\theta) \notin \theta$, for any $\theta \in \mathrm{Max}A$, hence $(a, a^*) \notin \theta$ for any $\theta \in \mathrm{Max}A$. So $\mathrm{Max}A = \{\theta \in \mathrm{Max}A | (a, a^*) \notin \theta\}$ and using the definition of θ^* and the fact that $(a^*, a) \in \theta^*$ we obtain that $\mathrm{Max}A$ is finite, i.e. the algebra A is semilocal.

Lemma 9. Let V be a congruence distributive variety. A finite direct product of maximal algebras of V is maximal.

Proof. Let $A_1, A_2 \in V$ be two maximal algebras and $A = A_1 \times A_2$. Consider $\{((a_{i1}, a_{i2}), \theta_i) | i \in I\}$ $a_{i1} \in A_1$, $a_{i2} \in A_2$, $\theta_i \in \mathrm{Con}A$ a family in A having property $(\bullet)$. For $\theta \in \mathrm{Con}A$ we denote by θ_1 the congruence on A_1 given by the following stipulation: $(a, b) \in \theta_1$ if there exist $c, d \in A_2$ such that $((a, c), (b, d)) \in \theta \vee \ker p_1$, where $p_1 : A_1 \times A_2 \to A_1$ is the canonical projection; similarly we associate to θ the congruence θ_2 on A_2.

Now consider the family $\{(a_{i1}, \theta_{i1}) | i \in I\}$ in A_1 which obviously has property $(\bullet)$. Because A_1 is maximal there is $x \in A_1$ such that $(x, a_{i1}) \in \theta_{i1}$ for any $i \in I$. Similarly there is $y \in A_2$ such that $(y, a_{i2}) \in \theta_{i2}$ for any $i \in I$. By construction of θ_{i1} and θ_{i2} there exist $c_i, d_i \in A_2$ and $s_i, t_i \in A_1$ such that:

$((x, c_i), (a_{i1}, d_i)) \in \theta_i \vee \ker p_1$ and $((s_i, y), (t_i, a_{i2})) \in \theta_i \vee \ker p_2$ for any $i \in I$. Then $((x, y), (a_{i1}, a_{i2})) \in \theta_i \vee \ker p_1$ and $((x, y), (a_{i1}, a_{i2})) \in \theta_i \vee \ker p_2$ for any $i \in I$. Hence $((x, y), (a_{i1}, a_{i2})) \in (\theta_i \vee \ker p_1) \wedge (\theta_i \vee \ker p_2)$ for any $i \in I$; but $(\theta_i \vee \ker p_1) \wedge (\theta_i \vee \ker p_2) = \theta_i \vee (\ker p_1 \wedge \ker p_2) = \theta_i \vee 0_A = \theta_i$. Finally $A = A_1 \times A_2$ is maximal.

Theorem 10. Let V be an aritmethical variety and $A \in V$. The following assertions are equivalent:

1) A is semisimple and maximal;
2) A is isomorphic to finite direct product of simple algebras.

Proof. 1) $\Rightarrow$ 2) By proposition 8 A is semilocal and then we apply Corollary 3.

2) $\Rightarrow$ 1) Each simple algebra is maximal by Lemma 8, hence by Lemma 9 it follows that A is maximal.

Now we give some applications of the previous theorem in concrete varieties of algebras. The varieties **B** of Boolean algebras, **LM-n** of n-valued Lukasiewicz-Moisil algebras (see [2]), **MV** of MV-algebras (see [6]), **1-Ab** of abelian 1-groups (see [1]), are aritmethical varieties. Any algebra of **B** or **LM-n** is semisimple. In these particular varieties, using Theorem 10 we obtain the following results:

Corollary 11.

1) Let $A \in \mathbf{B}$. Then A is maximal iff there is a natural number n such that $A \cong \mathbf{2}^n$ where $\mathbf{2} = \{0, 1\}$.

2) Let be $A \in \mathbf{LM\text{-}n}$. Then A is maximal iff A is isomorphic to a finite direct product of subalgebras of $L_n = \{0, 1/n-1, \ldots, n-2/n-1, 1\}$.
3) Let be $G \in \mathbf{1\text{-}Ab}$ an archimedean group. Then G is maximal iff G is isomorphic to a finite direct product of real groups.
4) Let be $A \in \mathbf{MV}$. Then A is semisimple and maximal iff A is isomorphic to a finite direct product of locally-finite MV-algebras.

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