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A BOUNDARY ELEMENT GALERKIN METHOD FOR THE DIRICHLET PROBLEM OF THE HEAT EQUATION IN NON-SMOOTH DOMAIN

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Abstract. We consider a numerical method for the Volterra-Fredholm integral equation of the first kind corresponding to the Dirichlet problem of heat conduction in a solid with piecewise Lyapunov surface with corners and edges. To approximate the ill-posed boundary integral equation we adopt the Galerkin method using boundary finite element and one-dimensional finite element in the time variable. We show the convergence property and the stability of the semi-discretized approximate solution using boundary finite elements. We estimate the error bound for the full-discretized approximate solution.

1. INTRODUCTION. Recently, numerical solutions of initial-boundary value problems of the heat equation are often obtained by boundary element methods based on boundary integral equations, because the approach enables us to treat heat conduction problems with domains extending to infinity, with polygonal domains and non-smooth data with much ease. For the Dirichlet problem, direct methods lead to the approximation of a Volterra integral equation of the first kind. The kernel function involved in the boundary integral equation corresponds to the single-layer heat potential, which is weakly singular.

The approximation of boundary integral equations in transient heat conduction problems has been considered by several authors; see Brebbia[2] et al. for example in engineering applications. They used the collocation method with boundary finite elements as trial functions on the boundary. As regard to the mathematical analysis, Costabel[4] et al. and Onishi[17] discussed the Neumann problem and they showed the existence of the solution of a corresponding Volterra integral equation of the second kind on a non-smooth boundary. They showed the convergence and the stability of the projection method in the space of continuous functions. Yang[23], Arnold and Noon[1], and Noon[14] presented some attempts at boundary element methods using the single-layer heat potential to the solution of Dirichlet problem on a smooth surface. Okamoto[15] showed an application of Fourier transform to the Dirichlet problem and proved unconditional stability as well as conditional convergence of the boundary element approximation for the heat operator in L^2-sense. Pointwise convergence in time was obtained by Lubich and Schneider[11] on a smooth boundary. The uniform convergence of boundary element solutions and conditional stability of the boundary element collocation method are proved by Iso[7] for the boundary integral equation corresponding to an initial-boundary value problem of the heat equation with the Robin boundary condition on a boundary of class C^3.

In this paper, we will show the convergence property and the stability of Galerkin's method applied to the solution of the boundary integral equation of the Dirichlet problem

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of heat conduction in a solid with piecewise Lyapunov surface with corners and edges in a more general class. The discussion is carried out for three-dimensional problems, but the validity of the results remains also for problems in two dimensions.

2. DIRICHLET PROBLEM IN A NON-SMOOTH DOMAIN. We shall confine the geometry of the domain in question. Let Ω be a simply connected bounded open domain in the three-dimensional Euclidean space $\boldsymbol{R}^3$ and assume that the closed bounded surface $\Gamma = \partial\Omega$ consists of a finite number of open smooth subsurfaces Γ_i $(i = 1, \dots, N)$ so that $\Gamma = \bigcup_{1\le i\le N} \overline{\Gamma}_i$, where $\overline{\Gamma}_i = \Gamma_i \cup \partial\Gamma_i$. Then the surface has a tangent plane at every point $x \in \overline{\Gamma}_i$ if the tangent plane at the edge point of Γ_i is understood to be the corresponding half plain. Moreover, the angle ν between the exterior normal vector $n(x)$ to Γ_i at $x \in \Gamma_i$ and the vector $(x-y)$ for an arbitrary point $y \in \Gamma_i$ $(x \neq y)$ satisfies the Lyapunov condition, see Michlin[12, p. 285] for example:

$$|\cos\nu| \le L(\Gamma)|y-x|^{\kappa} \quad (0 < \kappa < 1), \tag{1}$$

where L is a global constant depending only on Γ. The set of points on Γ where the surface is not smooth forms corners and edges. This is denoted by $\delta\Gamma = \bigcup_{1\le i\le N} \partial\Gamma_i$, which has zero Lebesgue volume measure.

Let $d\Theta_x(y)$ denote an infinitesimal solid angle at $x \in \boldsymbol{R}^3$ subtending the infinitesimal surface area $d\Gamma(y)$ at $y \in \Gamma - \delta\Gamma$; see Michlin[12, p. 287]. Then

$$\begin{aligned} d\Theta_x(y) &= -\frac{\partial}{\partial n(y)}\left(\frac{1}{|y-x|}\right) d\Gamma(y) \\ &= \frac{(y-x)\cdot n(y)}{|y-x|^3} d\Gamma(y). \end{aligned} \tag{2}$$

Remark 2.1. Let $I(x)$ be the index set attributed to the point x, for which $x \in \overline{\Gamma}_i$ with $i \in I(x)$. If $x \notin \Gamma$, then $I(x)$ is the null set. Put $I^C(x) = \{1, 2, \dots, N\} - I(x)$. For $i \in I(x)$ it follows from (1) that $|(y-x)\cdot n(y)|/|y-x|^3 = |\cos\nu|/|y-x|^2 \le L/|y-x|^{2-\kappa}$ for any $y \in \overline{\Gamma}_i$. Therefore, the integral $\int_{\Gamma_i} d\Theta_x(y)$ is absolutely convergent. For $j \in I^C(x)$, $\int_{\Gamma_j} d\Theta_x(y)$ is also convergent since $|y-x| \ge C(x) > 0$ for $y \in \overline{\Gamma}_j$. Hence, $\Theta(x) = \int_\Gamma d\Theta_x(y)$ is well defined for every $x \in \boldsymbol{R}^3$.

For $x \in \Gamma$, $\Theta(x)$ is equal to the interior solid angle at the vertex x of the cone, whose side surface is constructed by all the half ray tangential lines to the surface Γ radiating from x. For a piecewise Lyapunov surface Γ it follows that

$$\sup_{x\in\boldsymbol{R}^3}\int_\Gamma |d\Theta_x(y)| = \sup_{x\in\boldsymbol{R}^3}\int_\Gamma \frac{|(y-x)\cdot n(y)|}{|y-x|^3} d\Gamma(y) = A < +\infty \tag{3}$$

with some constant A. In addition we require Γ to satisfy

$$\lim_{\delta\to 0}\sup_{x\in\Gamma} W_\delta(x) = \omega < 1, \tag{4}$$

where $W_\delta(x)$ is defined by the expression:

$$W_\delta(x) := \frac{1}{2\pi}\left\{\int_{0<|y-x|\le\delta} |d\Theta_x(y)| + |2\pi - \Theta(x)|\right\}. \tag{5}$$

Remark 2.2. The piecewise Lyapunov surface satisfying (4) is called Wendland surface. The condition (4) (Wendland[21]) allows the splitting of the integral operator of the double-layer potential into the sum of a contraction operator and a completely continuous one in $C(,\ )$, which is basic for the validity of the Fredholm-Radon method in potential theory.

We consider the heat equation for unknown temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = \Delta u, \qquad (x,t) \in (\Omega \cup \Omega^e) \times (0,T] \tag{6}$$

for some finite value T, in which Δ is the Laplacian in $\boldsymbol{R}^3$ with respect to the variable x and Ω^e denotes the exterior of the domain Ω.

On the boundary we consider the Dirichlet condition:

$$u(x,t) = \hat{u}(x,t), \quad (x,t) \in ,\ \times [0,T]. \tag{7}$$

In addition, we consider the initial condition:

$$u(x,0) = u_0(x), \qquad x \in \Omega \cup \Omega^e \tag{8}$$

for the bounded Cauchy datum u_0 in $C(\Omega \cup \Omega^e)$. In Ω^e, the corresponding Cauchy datum u_0 may be assumed to grow at most exponentially:

$$|u_0(x)| \le \alpha_1 \exp[\beta_1 |x|^\sigma] \tag{9}$$

with some constants $\alpha_1 > 0$, $\beta_1 > 0$ and $0 < \sigma < 2$; see Krzyzanski[9, p. 455] for example. We can assume without loss of generality by considering the Weierstrass integral that $u_0 = 0$ in $\Omega \cup \Omega^e$.

3. BOUNDARY INTEGRAL EQUATION OF THE FIRST KIND. We shall derive a boundary integral equation corresponding to the Dirichlet problem (6)–(8) and investigate some properties of the integral operator. We start the discussion with definitions of single-layer heat potential:

$$Gq(x,t) := \int_0^t \int_\Gamma q(y,\tau) v(y,\tau : x,t) d,\ (y) d\tau, \tag{10}$$

with the density q and double-layer heat potential:

$$Hu(x,t) := \int_0^t \int_\Gamma u(y,\tau) \frac{\partial v(y,\tau : x,t)}{\partial n(y)} d,\ (y) d\tau, \tag{11}$$

with the density u, where $n(y)$ is the external normal at y to the boundary , . Here, v is the fundamental solution of the heat operator $\partial/\partial t - \Delta$:

$$v(y,\tau : x,t) = \begin{cases} \left(\dfrac{1}{2\sqrt{\pi(t-\tau)}}\right)^3 \exp\left[-\dfrac{r^2}{4(t-\tau)}\right] & (t > \tau) \\ 0 \quad (t < \tau) \end{cases} \tag{12}$$

with $r = |y - x|$. Put

$$g(x,t) := \frac{1}{2}\hat{u}(x,t) + H\hat{u}(x,t), \qquad x \in , \tag{13}$$

with the expression:

$$H\hat{u}(x,t) = \left(1 - \frac{\Theta(x)}{2\pi}\right)\hat{u}(x,t) + \int_0^t \int_\Gamma \hat{u}(y,\tau) \frac{r^3}{(t-\tau)} v(y,\tau : x,t) d\Theta_x(y) d\tau. \tag{14}$$

According to Costabel[4] et al., unknown boundary flux $q = \partial u/\partial n$ in the normal direction is given as a solution of the linear Volterra-Fredholm boundary integral equation of the first kind:

$$Gq(x,t) = g(x,t), \qquad (x,t) \in \Sigma = \,,\; \times [0,T] \tag{15}$$

Next lemma shows that G can be understood as a linear bounded operator from $C(L^p(\,,\,) : [0,T])$ into $C(\Sigma)$.

Lemma 3.1. *The operator $G : C(L^p(\,,\,) : [0,T]) \to C(\Sigma)$ defined by (10) is bounded for $p > 2$.*

Proof. Using the idea in Pogorzelski[18, p. 353], we have for any μ $(0 < \mu < 3/2)$ the inequality:

$$\begin{aligned} v(y,\tau : x,t) &= \frac{1}{2^{2\mu}\pi^{3/2}} \frac{1}{(t-\tau)^\mu} \frac{1}{r^{3-2\mu}} \left[\frac{r^2}{4(t-\tau)}\right]^{(3/2)-\mu} \exp\left[-\frac{r^2}{4(t-\tau)}\right] \\ &\le \frac{1}{(t-\tau)^\mu} \frac{G_1}{r^{3-2\mu}} \end{aligned} \tag{16}$$

with $G_1 = s^s e^{-s}/(2^{2\mu}\pi^{3/2})$ and $s = 3/2 - \mu$. We apply the Hölder's inequality to $Gq(x,t)$ and obtain from (16) that

$$\begin{aligned} |Gq(x,t)| &\le \int_0^t \left\{\int_\Gamma |q(y,\tau)|^p d,\right\}^{1/p} \left\{\int_\Gamma |v|^{p'} d,\right\}^{1/p'} d\tau \\ &\le G_1 \int_0^t \|q(\cdot,\tau)\|_p \left\{\int_\Gamma \frac{1}{(t-\tau)^{\mu p'}} \frac{1}{r^{(3-2\mu)p'}}\, d,\right\}^{\frac{1}{p'}} d\tau \\ &= G_1 \int_0^t \frac{\|q(\cdot,\tau)\|_p}{(t-\tau)^\mu} \left(\int_\Gamma \frac{d,}{r^{(3-2\mu)p'}}\right)^{\frac{1}{p'}} d\tau \\ &\le G_1 \left(\int_0^t \frac{d\tau}{(t-\tau)^\mu}\right) \left(\int_\Gamma \frac{d,}{r^{(3-2\mu)p'}}\right)^{\frac{1}{p'}} |||q|||_{C(L^p(\Gamma):[0,T])} \end{aligned}$$

with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. In view of the inequality (16) we can see that inequalities above are valid only for such values of μ satisfying $(3-2\mu)p' < 2$, i.e., $\dfrac{1}{2}(3 - \dfrac{2}{p'}) < \mu$. If we take μ with $0 < \mu < 1$, the integral $\displaystyle\int_0^t \frac{d\tau}{(t-\tau)^\mu}$ is convergent. Consequently, there exists a constant C depending only on , and T such that $\|Gq\| \le C|||q|||_{C(L^p(\Gamma):[0,T])}$ for the value $\dfrac{1}{2}(3 - \dfrac{2}{p'}) < 1$. The assumption $p > 2$ is equivalent to $\dfrac{1}{2}(3 - \dfrac{2}{p'}) < 1$. □

Remark 3.1. In order to apply the Hilbert space approach in the approximation method in the next section, we shall regard G as an operator:

$$G : H^{-1/2,-1/4}(\Sigma) \to H^{1/2,1/4}(\Sigma).$$

Lemma 3.2. *Under the assumption (3) and for $u \in C(\Sigma)$, the continuous function $g(x,t)$ of (13) satisfies the inequality:*

$$\|g\| \le \left(\frac{3}{2} + \frac{A}{2\pi}\right) \|u\|.$$

Proof. The continuity of $g(x,t)$ is shown in Costabel[3] et al.. We shall prove the inequality of the lemma: By the variable transformation; $\tau \mapsto \sigma = r/2\sqrt{t-\tau}$, $H\hat{u}(x,t)$ in (14) can be expressed as

$$\begin{aligned} H\hat{u}(x,t) &= \left(1 - \frac{\Theta(x)}{2\pi}\right)\hat{u}(x,t) \\ &\quad + \frac{1}{2\pi}\int_\Gamma \left\{ \frac{4}{\sqrt{\pi}} \int_{r/2\sqrt{t}}^\infty \sigma^2 e^{-\sigma^2} \hat{u}(y, t - \frac{r^2}{4\sigma^2}) d\sigma \right\} d\Theta_x(y). \end{aligned}$$

Consequently, we have

$$\begin{aligned} |H\hat{u}(x,t)| &\leq \left\{ \left|1 - \frac{\Theta(x)}{2\pi}\right| + \frac{1}{2\pi}\int_\Gamma \left(\frac{4}{\sqrt{\pi}} \int_0^\infty \sigma^2 e^{-\sigma^2} d\sigma \right) |d\Theta_x(y)| \right\} \|\hat{u}\| \\ &\leq \left(1 + \frac{A}{2\pi}\right)\|\hat{u}\|. \end{aligned}$$

The last inequality follows from (3) and from $0 < \Theta(x) < 4\pi$, $\int_0^\infty \sigma^2 e^{-\sigma^2} d\sigma = \sqrt{\pi}/4$. □

Properties of the integral operator G are now discussed in the space $H^{1/2,1/4}(\Sigma)$ and its dual space $H^{-1/2,-1/4}(\Sigma)$, introduced by Lions and Magenes[10, p. 10 and p. 44]: Let $H^{1/2,1/4}(\Sigma)$ be a Sobolev space defined by

$$H^{1/2,1/4}(\Sigma) = L^2(H^{1/2}(\,\,) : [0,T]) \cap H^{1/4}(L^2(\,\,) : [0,T])$$

equipped with the norm:

$$|||w|||^2_{H^{1/2,1/4}(\Sigma)} = \int_0^T \|w(\cdot,t)\|^2_{H^{1/2}(\Gamma)} dt + \int_0^T \int_0^T \frac{\|w(\cdot,t) - w(\cdot,s)\|^2_{L^2(\Gamma)}}{|t-s|^{3/2}} ds dt.$$

We denote by $((\cdot,\cdot))_0$ the scalar product:

$$((w_1, w_2))_0 := \int_0^T (w_1(\cdot,t), w_2(\cdot,t))_{L^2(\Gamma)} dt.$$

Next two important lemmas are much due to Costabel[3].

Lemma 3.3. *There exists a constant $\alpha > 0$ depending only on Σ such that*

$$\alpha^{-1}|||q|||_{H^{-1/2,-1/4}(\Sigma)} \leq |||Gq|||_{H^{1/2,1/4}(\Sigma)} \leq \alpha|||q|||_{H^{-1/2,-1/4}(\Sigma)}.$$

The next lemma shows strong coerciveness of the operator G.

Lemma 3.4. *There exists a constant $\beta > 0$ depending only on Σ such that*

$$((Gq, q))_0 \geq \beta |||q|||^2_{H^{-1/2,-1/4}(\Sigma)}$$

for all q in $H^{-1/2,-1/4}(\Sigma)$.

4. APPROXIMATION ON THE BOUNDARY. In this section, we shall consider the semi-discretization of the solution by Galerkin method using boundary finite elements. We shall show convergence and accuracy of the semi-discretized approximate solution. The way of arguments is much due to Nedelec and Planchard[13] as well as Hsiao and Wendland[5].

Let V_h be finite-dimensional subspaces of the Hilbert space $H^{-1/2}(,\ )$, approximating the solution $q(x,t)$ of the Volterra-Fredholm integral equation (10) and (15), such that $\cup_{h>0}V_h$ is dense in $L^2(,\ )$ and $V_h \subset V_{h'}$ for $h > h'$. Put $\dim(V_h) = n$ by assuming that $n = 1/h$ for $n = 1, 2, \ldots$. Let $\{\varphi_j(x)\}_{j=1,2,\ldots,n}$ denote the basis of V_h. We consider the approximation of $q(x,t)$ in the form:

$$q_h(x,t) = \sum_{j=1}^{n} \hat{q}_j(t)\varphi_j(x) \tag{17}$$

with coefficient functions $\hat{q}_j(t)$ $(0 \le t \le T)$ to be determined later.

We shall consider the semi-discrete Galerkin approximation: Find unknown q_h in $H^{-1/2,-1/4}(\Sigma)$ satisfying that

$$((Gq_h, q'_h))_0 = ((g_h, q'_h))_0 \quad \text{for all } q'_h \in V_h, \tag{18}$$

where g_h is an L^2-orthogonal projection of $g \in H^{1/2,1/4}(\Sigma)$ into $L^2(V_h : (0,T))$: That is, with the projector

$$P_h : g \in L^2(\Sigma) \to g_h \in L^2(V_h : (0,T)).$$

We assume that $\|P_h g\|_{H^{1/2,1/4}(\Sigma)} \le \|g\|_{H^{1/2,1/4}(\Sigma)}$. This is equivalent to the proposition:

$$((Gq_h, \varphi_i))_0 = ((g, \varphi_i))_0 \quad \text{for all } \varphi_i \in V_h, \quad i = 1, 2, \ldots, n. \tag{19}$$

Theorem 4.1. *Let q be the solution of (15) in $H^{-1/2,-1/4}(\Sigma)$ and q_h be a solution of (18). Then, there exists a constant $\rho(\Sigma) > 0$ such that*

$$|||q - q_h|||_{H^{-1/2,-1/4}(\Sigma)} \le \rho\{\inf_{q'_h \in V_h} |||q - q'_h|||_{H^{-1/2,-1/4}(\Sigma)} + |||g - g_h|||_{H^{1/2,1/4}(\Sigma)}\}. \tag{20}$$

Proof. From (15) we have

$$((Gq, q'))_0 = ((g, q'))_0 \quad \text{for all } q' \in H^{-1/2,-1/4}(\Sigma). \tag{21}$$

For an arbitrary q'_h in V_h, it follows from Lemma 3.4 that

$$((G(q_h - q'_h), q_h - q'_h))_0 \ge \beta |||q_h - q'_h|||^2_{H^{-1/2,-1/4}(\Sigma)}.$$

On the other hand, we can see that

$$\begin{aligned}
((G(q_h - q'_h), q_h - q'_h))_0 &= ((G((q - q'_h) - (q - q_h)), q_h - q'_h))_0 \\
&= ((G(q - q'_h), q_h - q'_h))_0 - ((G(q - q_h), q_h - q'_h))_0 \\
&= ((G(q - q'_h), q_h - q'_h))_0 - ((g - g_h, q_h - q'_h))_0 \\
&\le |||G(q - q'_h)|||_{H^{1/2,1/4}(\Sigma)} |||q_h - q'_h|||_{H^{-1/2,-1/4}(\Sigma)} \\
&\quad + |||g - g_h|||_{H^{1/2,1/4}(\Sigma)} |||q_h - q'_h|||_{H^{-1/2,-1/4}(\Sigma)} \\
&\le \{\alpha |||q - q'_h|||_{H^{-1/2,-1/4}(\Sigma)} + |||g - g_h|||_{H^{1/2,1/4}(\Sigma)}\} \\
&\quad \times |||q_h - q'_h|||_{H^{-1/2,-1/4}(\Sigma)}.
\end{aligned}$$

The third equality follows from (18) and (21). The last inequality follows from Lemma 3.3. Combining the above inequalities, we have

$$\beta|||q_h - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} \le \alpha|||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} + |||g - g_h|||_{H^{1/2,1/4}(\Sigma)},$$

from which it follows that

$$\begin{aligned} |||q - q_h|||_{H^{-1/2,-1/4}(\Sigma)} &\le |||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} + |||q_h - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\le (1 + \frac{\alpha}{\beta})|||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} + \frac{1}{\beta}|||g - g_h|||_{H^{1/2,1/4}(\Sigma)}. \end{aligned}$$

This leads to the desired inequality (20) with $\rho = \max\{(1 + \alpha/\beta), 1/\beta\}$. □

We can obtain a stronger result in the next theorem, which shows the optimal rate of convergence of the Galerkin approximation in $H^{-1/2,-1/4}(\Sigma)$.

Theorem 4.2 (Cea's lemma). *The semi-discrete Galerkin approximation (*18*) is inverse stable: For the Galerkin solution q_h it holds that*

$$|||q - q_h|||_{H^{-1/2,-1/4}(\Sigma)} \le (1 + \frac{\alpha}{\beta}) \inf_{q_h' \in V_h} |||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)}.$$

Proof. The Galerkin approximation (18) is equivalent to the problem of finding the unknown q_h of the form (17) in $H^{-1/2,-1/4}(\Sigma)$, satisfying the operator equation:

$$P_h G P_h q_h = P_h g. \tag{22}$$

By the same way of arguments as in Wendland[22, p. 21], we can see from Lemma 3.4 that

$$\begin{aligned} \beta|||q_h|||^2_{H^{-1/2,-1/4}(\Sigma)} &\le ((Gq_h, q_h))_0 = ((P_h G P_h q_h, q_h))_0 \\ &\le |||P_h G P_h q_h|||_{H^{1/2,1/4}(\Sigma)} |||q_h|||_{H^{-1/2,-1/4}(\Sigma)}. \end{aligned}$$

The first equality followed from the relation:

$$((GP_h q_h, q_h))_0 - ((P_h G P_h q_h, q_h))_0 = (((I - P_h) G P_h q_h, q_h))_0 = 0,$$

since $q_h \in V_h$ and $(I - P_h) G P_h q_h \in V_h^\perp$, the orthogonal complement of V_h, for every $t \in [0, T]$. The we have

$$\beta|||q_h|||_{H^{-1/2,-1/4}(\Sigma)} \le |||P_h G P_h q_h|||_{H^{1/2,1/4}(\Sigma)}.$$

Since this inequality holds for all q_h, we know that $P_h G P_h : H^{-1/2,-1/4}(\Sigma) \to H^{1/2,1/4}(\Sigma)$ is invertible. The inverse is bounded as follows:

$$|||(P_h G P_h)^{-1}|||_{H^{1/2,1/4}(\Sigma), H^{-1/2,-1/4}(\Sigma)} := \sup_{g_h \neq 0} \frac{|||(P_h G P_h)^{-1} g_h|||_{H^{-1/2,-1/4}(\Sigma)}}{|||g_h|||_{H^{1/2,1/4}(\Sigma)}} \le \frac{1}{\beta} \tag{23}$$

see Kantorowitsch and Akilow[8, Satz 2, 2.V] for example. From (15) and (22) it follows that

$$q_h = (P_h G P_h)^{-1} P_h G q.$$

This defines the Galerkin projector $G_h = (P_h G P_h)^{-1} P_h G : H^{-1/2,-1/4}(\Sigma) \to H^{-1/2,-1/4}(\Sigma)$. We shall show that G_h is bounded. For this purpose, we put $g_h = P_h G q$. From (23) and Lemma 3.3 it follows that

$$\begin{aligned} &|||(P_h G P_h)^{-1} P_h G q|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\le |||(P_h G P_h)^{-1}|||_{H^{1/2,1/4}(\Sigma), H^{-1/2,-1/4}(\Sigma)} |||P_h G q|||_{H^{1/2,1/4}(\Sigma)} \\ &\le \frac{\alpha}{\beta}|||q|||_{H^{-1/2,-1/4}(\Sigma)}. \end{aligned}$$

This implies that G_h is bounded as desired: Namely

$$|||G_h|||_{H^{-1/2,-1/4}(\Sigma),H^{-1/2,-1/4}(\Sigma)} \leq \frac{\alpha}{\beta}. \tag{24}$$

Note that $G_h q_h' = q_h'$ for all $q_h' \in V_h$ because

$$P_h G P_h q_h' = P_h G q_h'. \tag{25}$$

Consequently we have

$$\begin{aligned} |||q - q_h|||_{H^{-1/2,-1/4}(\Sigma)} &\leq |||q - q_h{}'|||_{H^{-1/2,-1/4}(\Sigma)} + |||q_h - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} \\ &= |||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} + |||G_h q - G_h q_h'|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\leq (1 + |||G_h|||_{H^{-1/2,-1/4}(\Sigma),H^{-1/2,-1/4}(\Sigma)})|||q - q_h'|||_{H^{-1/2,-1/4}(\Sigma)} \end{aligned}$$

which leads to the assertion of the theorem from (24). □

For the concreteness of the discussion, as V_h we shall consider the regular finite element spaces S_h with the following two conditions for some positive integer m; see Hsiao and Wendland[6, p. 4] for example:

Convergence property: Let $t \leq s$ be such that $-(m+1) \leq t \leq s \leq m+1$, $-m \leq s$ and $t \leq m$ for some non-negative integer m. Then for any $q \in H^s(\Gamma)$ there exists a $q_h' \in S_h$ such that

$$\|q - q_h'\|_{H^t(\Gamma)} \leq C_1 h^{s-t} \|q\|_{H^s(\Gamma)} \tag{26}$$

with some constant C_1 which is independent on q_h and h.

Inverse assumption: Let $t \leq s$ be such that $|t|,\ |s| \leq m$. Then there exists a constant C_2 independent on h such that

$$\|q_h\|_{H^s(\Gamma)} \leq C_2 h^{t-s} \|q_h\|_{H^t(\Gamma)} \quad \text{for all } q_h \in S_h. \tag{27}$$

Remark 4.1. Nedelec and Planchard[13, Lemma 3.1 and Lemma 3.2] showed that, if Γ is a polyhedron, linear triangular finite element spaces satisfy (26) and (27) with $m = 1$, provided that all the angles θ in the triangulation satisfy $\theta \geq \theta_0 > 0$ with a constant θ_0, which is independent of the maximum diameter h among all triangles. For constant triangular finite element spaces, the convergence property (26) is satisfied with $m = 0$; see Nedelec and Planchard[13, Lemme 3.4]. However, (27) holds only for $-1 \leq t \leq s \leq 0$, see Nedelec and Planchard[13, Lemme 3.3].

As an immediate consequence of Theorem 4.2 and (26) we have

Theorem 4.3. *For the semi-discrete Galerkin solution q_h with constant boundary finite elements on the triangulation of the polyhedron Γ, it holds that*

$$|||q - q_h|||_{H^{-1/2,-1/4}(\Sigma)} \leq (1 + \frac{\alpha}{\beta}) C_1 h^{s+1/2} |||q|||_{H^{s,-1/4}(\Sigma)}$$

with $0 \leq s \leq 1$.

It happens often that Dirichlet data $\hat{u}(x,t)$ of (7) or Cauchy data $u_0(x)$ of (8) are given imprecisely due to measurements. The right hand side $g(x,t)$ or (15) can not be obtained exactly because of the approximate evaluation of the term $H\hat{u}(x,t)$ in (13) and (14). Due to the limitation of a finite number of digits available in the numerical computation, round-off errors are not avoidable. These cause the additional impreciseness involved in the right hand side $g(x,t)$. We assume that the polluted g, denoted here by $\tilde{g}$, belong to $H^{0,1/4}(\Sigma)$. Instead of (15), we have to solve the equation:

$$G\tilde{q}(x,t) = \tilde{q}(x,t), \quad (x,t) \in \Sigma. \tag{28}$$

In this situation we have the ill-posedness of the Galerkin approximation as next theorem shows.

Theorem 4.4. *For the semi-discrete Galerkin solution $\tilde{q}_h$ of (28) with $\varphi_i \in S_h$, it holds that*

$$|||q - \tilde{q}_h|||_{H^{-1/2,-1/4}(\Sigma)} \leq C_3\{h^{s+1/2}|||q|||_{H^{s,-1/4}(\Sigma)} + h^{-1/2}|||g - \tilde{g}|||_{H^{0,1/4}(\Sigma)}\}$$

with some constant $C_3 > 0$ and $0 \leq s \leq 1$.

Proof. Our proof is due to Hsiao and Wendland[6, p. 9]. From (22) it follows that

$$P_hGP_h(q - \tilde{q}_h) = P_hGP_hq - P_h\tilde{g} = (P_hGP_h - P_hG)q + P_h(g - \tilde{g}).$$

Using (25) we can see that

$$P_hGP_h(q - \tilde{q}_h) = (P_hGP_h - P_hG)(q - {q_h}') + P_h(g - \tilde{g})$$

for all $q'_h \in S_h$. Application of $(P_hGP_h)^{-1}$ to both side of the equality yields that

$$q - \tilde{q}_h = (I - G_h)(q - q'_h) - (P_hGP_h)^{-1}P_h(g - \tilde{g}).$$

Consequently, from (23) and (24) it follows that

$$\begin{aligned} & |||q - \tilde{q}_h|||_{H^{-1/2,-1/4}(\Sigma)} \\ \leq \;& (1 + \frac{\alpha}{\beta})|||q - q'_h|||_{H^{-1/2,-1/4}(\Sigma)} + \frac{1}{\beta}|||P_h(g - \tilde{g})|||_{H^{1/2,1/4}(\Sigma)} \\ \leq \;& (1 + \frac{\alpha}{\beta})C_1h^{s+1/2}|||q|||_{H^{s,-1/4}(\Sigma)} + \frac{1}{\beta}C_2h^{-1/2}|||g - \tilde{g}|||_{H^{0,1/4}(\Sigma)}. \end{aligned} \tag{29}$$

The last inequality follows from (26) and (27). □

Remark 4.2. For constant elements we can obtain only the first inequality of (29). Hence it is suggested that, when constant elements are used, numerical computations must proceed so that $|||P_h(g - \tilde{g})|||_{H^{1/2,1/4}(\Sigma)}$ is evaluated as small as possible. In other words, the right hand side is required to be smooth and it should be calculated with high accuracy.

Remark 4.3. A rough estimate of the optimal choice of h may be given from Theorem 4.4 by minimization of the expression in $\{\cdots\}$ with respect to h: From the relation

$$h^{s+1} = \frac{1}{2}\frac{|||g - \tilde{g}|||_{H^{0,1/4}(\Sigma)}}{(s + \frac{1}{2})|||q|||_{H^{s,1/4}(\Sigma)}},$$

we have the guideline:

$$h_{opt} = O(|||g - \tilde{g}|||^{\frac{1}{s+1}}_{H^{0,1/4}(\Sigma)}).$$

5. APPROXIMATION IN TIME. In this section, we shall consider a constructive theory in the full-discretization of the solution by Galerkin method using one-dimensional finite elements in the time variable. We shall estimate the condition number of the coefficient matrix in the linear system of equations for a time-stepping procedure. We shall obtain convergence and accuracy of the approximate solution.

Let us subdivide the interval $[0,T]$ into N small segments of equal length with nodes $t_k = t_{k-1} + \Delta t$; $k = 1, 2, \dots, N$ $(= T/\Delta t)$. Let $T_{\Delta t}$ be corresponding finite element subspaces of $C([0,T])$, approximating coefficient functions $\hat{q}_j(t)$ in the expression (17). Let $\{\Psi_k(t)\}_{k=0,1,\dots,N}$ denote the basis of $T_{\Delta t}$. From (15) it follows that

$$\sum_{j=1}^{n} \int_0^t \hat{q}_j(\tau) V_{ij}(\tau, t) d\tau = G_i(t), \quad i = 1, 2, \dots, n, \tag{30}$$

where

$$V_{ij}(\tau, t) = \int_\Gamma \int_\Gamma \varphi_i(x) \varphi_j(y) v(y, \tau : x, t) d,\ (x) d,\ (y) \tag{31}$$

$$G_i(t) = \int_\Gamma g(x, t) \varphi_i(x) d,\ (x). \tag{32}$$

This is the linear system of Volterra integral equations of the first kind for unknowns $\hat{q}_j(t)$ with kernels $V_{ij}(\tau, t)$. Let $q_j(t)$ be the orthogonal projection of $\hat{q}_j(t)$ into $T_{\Delta t}$:

$$q_j(t) = \sum_{k=0}^{m} q_j^k \Psi_k(t), \quad 0 \le t \le t_m \tag{33}$$

with coefficients q_j^k, which stand for approximate values of $\hat{q}_j(t_k)$. As an approximation, we consider the Galerkin method: Namely, we will find unknown $q_j(t)$ satisfying that

$$\int_0^{t_m} \Psi_m(t) \left\{ \sum_{j=1}^{n} \int_0^t q_j(\tau) V_{ij}(\tau, t) d\tau \right\} dt = \int_0^{t_m} \Psi_m(t) G_i(t) dt \tag{34}$$

for $m = 1, 2, \dots, N$. Substitution of (33) into these equations yields the linear system of algebraic equations for unknowns q_j^k:

$$\sum_{j=1}^{n} \sum_{k=0}^{m} q_j^k a_{ij}^k = b_i^m, \tag{35}$$

where

$$a_{ij}^k = \int_0^{t_m} \Psi_m(t) \int_0^t \Psi_k(\tau) V_{ij}(\tau, t) d\tau dt, \tag{36}$$

$$b_i^m = \int_0^{t_m} \Psi_m(t) G_i(t) dt. \tag{37}$$

Note that a_{ij}^k depends on the number m of the time step, in general. Inductively suppose that all q_j^k $(k \le m-1)$ are known. Then, the system of equations (35) can be written in the form:

$$\sum_{j=1}^{n} q_j^m a_{ij}^m = b_i^m - \sum_{j=1}^{n} \sum_{k=0}^{m-1} q_j^k a_{ij}^k. \tag{38}$$

We shall express this form using matrices and column vectors as follows:

$$[A^{(m)}]\{q^m\} = \{b^m\} - \sum_{k=0}^{m-1}[A^{(k)}]\{q^k\}. \tag{39}$$

Note that all square matrices $[A^{(k)}]$ $(k = 0, 1, \dots, m)$ are symmetric since $V_{ij} = V_{ji}$ in view of the reciprocity $v(t, \tau : x, t) = v(x, \tau : y, t)$ in (31).

Lemma 5.1. *The matrix $[A^{(m)}]$ is symmetric, positive definite, and all eigenvalues $\lambda(A^{(m)})$ satisfy*

$$\frac{\beta}{C_3} h\Delta t^2 \lambda_{min}(B) \le \lambda(A^{(m)}) \le \alpha C_4 \Delta t \lambda_{max}(B) \tag{40}$$

with some constant $C_3 > 0$ and $C_4 > 0$, where $[B]$ is the Gram matrix of the basis $\{\varphi_i(x)\}_{i=1,2,\dots,n}$ in L^2-sense: $b_{ij} = (\varphi_i, \varphi_j)_{L^2(\Gamma)}$, $\lambda_{mix}(B)$ and $\lambda_{max}(B)$ are smallest and largest eigenvalues of $[B]$, respectively.

Proof. The basic idea of the proof is due to Richter[19]. With real numbers ξ_i $(i = 1, 2, \dots, n)$, consider the quadratic form:

$$\begin{aligned} Q(A) &= \sum_{i,j=1}^{n} a_{ij}^m \xi_i \xi_j \\ &= \int_0^{t_m} \Psi_m(t) \int_0^t \Psi_m(\tau) \int_\Gamma \int_\Gamma \eta_h(x)\eta_h(y) v(y, \tau : x, t) d,\ (x) d,\ (y) d\tau dt. \end{aligned}$$

Here we put: $\eta_h(x) = \sum_{i=1}^{n} \xi_i \varphi_i(x) \in V_h$. Set $q_h'(x,t) = \Psi_m(t)\eta_h(x)$. Then we have

$$Q(A) = \int_0^{t_m} (Gq_h'(\cdot, t), q_h'(\cdot, t))_{L^2(\Gamma)} dt. \tag{41}$$

From Lemma 3.4 with $T = t_m$ it follows for any $q \in H^{1/2,1/4}(\Sigma)$ that

$$\begin{aligned} Q(A) &= ((Gq_h', q_h'))_0 \ge \beta |||q_h'|||^2_{H^{-1/2,-1/4}(\Sigma)} \\ &\ge \beta |((q_h', g))_0|^2 / |||g|||^2_{H^{1/2,1/4}(\Sigma)}. \end{aligned}$$

The last inequality followed from the definition:

$$|||q_h'|||_{H^{-1/2,-1/4}(\Sigma)} = \sup_{g \ne 0} |((q_h', g))_0| / |||g|||_{H^{1/2,1/4}(\Sigma)}.$$

Take $g(x) = \eta_h(x)$. Then it becomes

$$\begin{aligned} Q(A) &\ge \beta \left| \int_0^T (\eta_h(\cdot), \Psi_m(t)\eta_h(\cdot))_{L^2(\Gamma)} dt \right|^2 / |||\eta_h|||^2_{H^{1/2,1/4}(\Sigma)} \\ &= \beta \left| \int_0^T \Psi_m(t) dt \right|^2 \|\eta_h\|^4_{L^2(\Gamma)} / \{T \|\eta_h\|^2_{H^{1/2}(\Gamma)}\} \\ &\ge \frac{\beta}{C_2^2} h \left| \int_0^T \Psi_m(t) dt \right|^2 \|\eta_h\|^2_{L^2(\Gamma)} / T. \end{aligned}$$

The last inequality followed from the inverse assumption (27) with $s = 1/2,\ t = 0$. This implies the positive definiteness of $[A^{(m)}]$. For the finite element base $\Psi_k(t)$, there exists an integer ρ, which is independent on k, such that $supp\ (\Psi_k) \subset [t_{k-\rho}, t_{k+\rho}]$. We have

$$\int_0^{t_m} \Psi_m(t)dt = O(\Delta t), \quad \int_0^{t_m} |\Psi_m(t)|^2 dt = O(\Delta t)$$

independently of m. We have also that

$$\|\eta_h\|^2_{L^2(\Gamma)} = \sum_{i,j=1}^{n} b_{ij}\xi_i\xi_j.$$

Consequently, there exists a constant $C_3(\Sigma)$ such that

$$Q(A) \geq \frac{\beta}{C_3} h\Delta t^2 \lambda_{min}(B)|\xi|^2. \tag{42}$$

On the other hand, from (41) it follows that

$$\begin{aligned} Q(A) &\leq |||Gq'_h|||_{H^{1/2,1/4}(\Sigma)}|||q'_h|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\leq \alpha|||q'_h|||^2_{H^{-1/2,-1/4}(\Sigma)} \leq \alpha\gamma(\Sigma)^2|||q'_h|||^2_{L^2(\Sigma)} \\ &= \alpha\gamma^2 \int_0^{t_m} |\Psi_m(t)|^2 dt \|\eta_h\|^2_{L^2(\Gamma)} \\ &\leq \alpha\gamma^2\Delta t C_5\|\eta_h\|^2_{L^2(\Gamma)} \end{aligned}$$

with some constant $C_5 > 0$. The second inequality followed from Lemma 3.3. The third inequality followed from the continuous imbedding: $H^{-1/2,-1/4}(\Sigma) \supset L^2(\Sigma) ihtecntnam >$ Consequently there exists a constant $C_4(\Sigma)$ such that

$$Q(A) \leq \alpha C_4 \Delta t \lambda_{max}(B)|\xi|^2. \tag{43}$$

By combining (42) with (43), we can obtain (40). □

Corollary 5.1. *The condition number $\kappa(A^{(m)})$ of the coefficient matrix in the linear system of equations (39) satisfies*

$$\kappa(A^{(m)}) \leq \frac{\alpha}{\beta} C_3 C_4 \frac{1}{h\Delta t} \kappa(B). \tag{44}$$

Proof. From (40) the assertion follows immediately, since

$$\kappa(A^{(m)}) := \frac{\lambda_{max}(A^{(m)})}{\lambda_{min}(A^{(m)})} \leq \frac{\alpha C_3 C_4}{\beta h \Delta t} \frac{\lambda_{max}(B)}{\lambda_{min}(B)}.$$

□

In order to obtain error estimates of fully discretized approximate solution $q_j(t)$ of (33), let us introduce the interpolates $q_j^I(t)$ defined by

$$q_j^I(t) = \sum_{k=0}^{m} \hat{q}_j(t_k)\Psi_k(t), \quad 0 \leq t \leq t_m \tag{45}$$

and we assume the *convergence property*:

$$|q_j^I(t) - \hat{q}_j(t)| \leq C_6 \Delta t^\sigma, \quad 0 < t < T \tag{46}$$

with some constants $C_6 > 0$ and $\sigma > 0$.

Remark 5.1. If linear finite element shape functions (roof functions) are used for $T_{\Delta t}$ and $\hat{q}_j(t)$ has bounded second derivative, then (46) holds with $\sigma = 2$; see Strang and Fix[20, Theorem 3.1] for example.

From (30) we have the next semi-exact equations:

$$\int_0^{t_m} \Psi_m(t) \left\{ \sum_{j=1}^{n} \int_0^t \hat{q}_j(\tau) V_{ij}(\tau,t) d\tau \right\} dt = \int_0^{t_m} \Psi_m(t) G_i(t) dt.$$

Substituting (34) from this equation we have

$$\int_0^{t_m} \Psi_m(t) \left\{ \sum_{j=1}^{n} \int_0^t [q_j(\tau) - q_j(\tau)] V_{ij}(\tau,t) d\tau \right\} dt = 0.$$

Hence

$$\begin{aligned} &\int_0^{t_m} \Psi_m(t) \left\{ \sum_{j=1}^{n} \int_0^t [q_j^I(\tau) - q_j(\tau)] V_{ij}(\tau,t) d\tau \right\} dt \\ &= \int_0^{t_m} \Psi_m(t) \left\{ \sum_{j=1}^{n} \int_0^t [q_j^I(\tau) - q_j(\tau)] V_{ij}(\tau,t) d\tau \right\} dt. \end{aligned}$$

Put $e_j^k = \hat{q}_j(t_k) - q_j^k$. This indicates the error committed in the time-discretization at time-space lattice point (P_j, t_k) with $P_j \in$, . From

$$q_j^I(t) - q_j(t) = \sum_{k=0}^{m} e_j^k \Psi_k(t),$$

we have

$$\sum_{j=1}^{n} \sum_{k=1}^{m} e_j^k a_{ij}^k = v_i^m \tag{47}$$

where

$$v_i^m = \sum_{j=1}^{n} \int_0^{t_m} \Psi_m(t) \int_0^t \{q_j^I(\tau) - q_j(\tau)\} V_{ij}(\tau,t) d\tau dt.$$

Theorem 5.1. *Let $\xi(h, \Delta t) = h/\{\Delta t \lambda_{min}(B)\}$. Then the maximum norm $\|\{e^m\}\|_\infty$ of the error column vector $\{e^m\} = (e_1^m, \ldots, e_n^m)'$ satisfies*

$$\|\{e^m\}\|_\infty \le G_2 \Delta t^\sigma \xi Exp[G_2 T \xi] \tag{48}$$

with some constant $G_2 > 0$.

Proof. By using the inequalities (16) and (46), v_i^m can be estimated as follows:

$$|v_i^m| \le C_6 \Delta t^\sigma \int_0^{t_m} |\Psi_m(t)| \int_0^t \frac{d\tau}{(t-\tau)^\mu} dt \int_\Gamma |\varphi_i(x)| \sum_{j=1}^{n} \int_\Gamma |\varphi_j(y)| \frac{G_1 d, (y)}{r^{3-2\mu}} d, (x).$$

Since finite element bases have the properties; $-1 \le \varphi_i(x) \le 1$ and $-1 \le \Psi_k(t) \le 1$, all integrals involved in the right-hand side are convergent for $1/2 < \mu < 1$. We can see that

$$\int_0^{t_m} |\Psi_m(t)| \int_0^t \frac{d\tau}{(t-\tau)^\mu} dt \le \frac{1}{1-\mu} \int_{t_{m-\rho}}^{t_m} t^{1-\mu} dt \le \rho \Delta t \frac{t_m^{1-\mu}}{1-\mu}$$

since *supp* $(\Psi_m) \subset [t_{m-\rho}, t_m]$. Since φ_i has a locally compact support, that is,

$$\int_\Gamma |\varphi_i(x)| d, \ (x) \leq C_7 h^2$$

with some constant $C_7 > 0$, independent on i, we can see that

$$\int_\Gamma |\varphi_i(x)| \sum_{j=1}^n \int_\Gamma |\varphi_j(y)| \frac{G_1 d, \ (y)}{r^{3-2\mu}} d, \ (x) \leq \int_\Gamma |\varphi_i(x)| d, \ (x) \max_{x \in \Gamma} \sum_{j=1}^n \int_\Gamma |\varphi_j(y)| \frac{G_1 d, \ (y)}{r^{3-2\mu}}.$$

From (3) and the fact that φ_j has a locally compact support, there exists a constant G_3, depending only on , , such that

$$\max_{x \in \Gamma} \sum_{j=1}^n \int_\Gamma |\varphi_j(y)| \frac{G_1 d, \ (y)}{r^{3-2\mu}} \leq G_3.$$

Consequently we have the estimate:

$$\|\{v^m\}\|_\infty := \max_i |v_i^m| \leq C_6 C_7 G_3 \rho h^2 \Delta t^{\sigma+1} \frac{t_m^{1-\mu}}{1-\mu} \leq G_4 h^2 \Delta t^{\sigma+1} \tag{49}$$

with the constant $G_4 = C_6 C_7 G_3 \rho T^{1-\mu}/(1-\mu)$. Similarly, we can see from (36) that

$$\begin{aligned} \sum_{j=1}^n |a_{ij}^k| &\leq \int_{t_{m-\rho}}^{t_m} |\Psi_m(t)| \int_{t_{k-\rho}}^{t_{k+\rho}} |\Psi_k(\tau)| \frac{d\tau}{(t-\tau)^\mu} dt \\ &\quad \times \int_\Gamma |\varphi_i(x)| \sum_{j=1}^n \int_\Gamma |\varphi_j(y)| \frac{G_1 d, \ (y)}{r^{3-2\mu}} d, \ (x) \\ &\leq 2\rho^2 \Delta t^2 C_8 C_7 h^2 G_3. \end{aligned}$$

Here we have used the inequality:

$$\int_{t_{k-\rho}}^{t_{k+\rho}} |\Psi_k(\tau)| \frac{d\tau}{(t-\tau)^\mu} \leq 2\rho \Delta t C_8$$

with some constant $C_8 > 0$. Therefore, there exists a constant G_5 such that

$$\|[A^{(k)}]\|_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}^{(k)}| \leq G_5 h^2 \Delta t^2. \tag{50}$$

Now we shall show (48). For this purpose, we express (47) in the matrix form:

$$[A^{(m)}]\{e^m\} = \{v^m\} - \sum_{k=0}^{m-1} [A^{(k)}]\{e^k\}.$$

Using (49) and (50), we can see that

$$\begin{aligned} \frac{\|\{e^m\}\|_\infty}{\|[A^{(m)}]^{-1}\|_\infty} &\leq \|[A^{(m)}]\{e^{(m)}\}\|_\infty \leq \|\{v^m\}\|_\infty + \sum_{k=0}^{m-1} \|[A^{(k)}]\|_\infty \|\{e^k\}\|_\infty \\ &\leq h^2 \Delta t^2 (G_4 \Delta t^{\sigma-1} + G_5 \sum_{k=0}^{m-1} \|\{e^k\}\|_\infty). \end{aligned}$$

From (40) we know that

$$\|[A^{(m)}]^{-1}\|_\infty \le \frac{C_3}{\beta h \Delta t^2 \lambda_{min}(B)}.$$

Consequently we have the recursive relation for $\|\{e^m\}\|_\infty$:

$$\|\{e^m\}\|_\infty \le G_6 \frac{h}{\lambda_{min}(B)}(\Delta t^{\sigma-1} + \sum_{k=0}^{m-1} \|\{e^k\}\|_\infty)$$

with some constant $G_6 > 0$. Using the result in Onishi[16, Eq. 4.45], we can see that

$$\|\{e^m\}\|_\infty \le (\hat{\alpha}\|\{e^0\}\|_\infty + \hat{\beta})(1+\alpha)^{m-1}$$

where

$$\hat{\alpha} = G_6 \frac{h}{\lambda_{min}(B)}, \quad \hat{\beta} = G_6 \frac{h\Delta t^{\sigma-1}}{\lambda_{min}(B)}.$$

From the inequality $1 + x \le e^x$ $(x \ge 0)$, we have

$$(1+\hat{\alpha})^{m-1} \le Exp[G_6 hm/\lambda_{min}(B)].$$

If we can assume that $\|\{e^0\}\|_\infty = 0$, we arrive at (48) by noting that $m \le T/\Delta t$ and $G_2 = G_6$. □

So far we have considered several solutions; the exact solution $q(x,t)$ of the equation (15) in $H^{-1/2,-1/4}(\Sigma)$, semi-discretized solution $q_h(x,t)$ of the form (17), fully discretized solution

$$q_{h,\Delta t}(x,t) = \sum_{j=1}^{n} q_j(t)\varphi_j(x) \quad \text{in } T_{\Delta t} \times S_h \tag{51}$$

with $q_j(t)$ defined by (33), and interpolated solution

$$q_h^I(x,t) = \sum_{j=1}^{n} q_j^I(t)\varphi_j(x) \quad \text{in } T_{\Delta t} \times S_h \tag{52}$$

with $q_j^I(t)$ defined by (45). Put $e(x,t) = q(x,t) - q_{h,\Delta t}(x,t)$. This indicates the total error of the boundary finite element solution $q_{h,\Delta t}$. We shall estimate $e(x,t)$ in $H^{-1/2,-1/4}(\Sigma)$.

Theorem 5.2. *Under the assumptions (26), (27) and (46), the total error $e(x,t)$ is bounded by*

$$\begin{aligned} |||e|||_{H^{-1/2,-1/4}(\Sigma)} &\le G_7 h^{s+1/2} |||q|||_{H^{s,-1/4}(\Sigma)} \\ &+ \frac{\Delta t^{\sigma-1}}{h}\left\{ G_8 \Delta t \sqrt{\lambda_{max}(B)} + G_9 h \sqrt{\frac{\kappa(B)}{\lambda_{min}(B)}} Exp[G_2 T \xi] \right\} \end{aligned} \tag{53}$$

for $0 \le s \le 1$, with some constants $G_7 > 0$ and $G_8 > 0$.

Proof. From (17), (51) and (52) it follows that

$$\begin{aligned} e(x,t) &= \{q(x,t)-q_h(x,t)\}+\{q_h(x,t)-q_h^I(x,t)\} \\ &\quad +\{q_h^I(x,t)-q_{h,\Delta t}(x,t)\} \\ &= \{q(x,t)-q_h(x,t)\}+\sum_{j=1}^{n}\{\hat{q}_j(t)-q_j^I(t)\}\varphi_j(x) \\ &\quad +\sum_{j=1}^{n}\{q_j^I(t)-q_j(t)\}\varphi_j(x) \end{aligned} \tag{54}$$

The second term on the most right hand side can be bounded as follows:

$$\begin{aligned} &|||\sum_{j=1}^{n}(\hat{q}_j-q_j^I)\varphi_j|||^2_{H^{-1/2,-1/4}(\Sigma)} \\ &\quad \le \gamma^2|||\sum_{j=1}^{n}\{\hat{q}_j-q_j^I\}\varphi_j|||^2_{L^2(\Sigma)} \\ &\quad = \gamma^2\sum_{i,j=1}^{n}\int_0^T\{\hat{q}_i(t)-q_i^I(t)\}\{\hat{q}_j(t)-q_j^I(t)\}dt\,(\varphi_i,\varphi_j)_{L^2(\Gamma)} \\ &\quad \le \gamma^2TC_6^2\Delta t^{2\sigma}n\lambda_{max}(B). \end{aligned}$$

The last inequality followed from (46). The third term on the most right hand side can be bounded similarly as follows:

$$\begin{aligned} |||\sum_{j=1}^{n}(q_j^I-q_j)\varphi_j|||^2_{H^{-1/2,-1/4}(\Sigma)} &\le \gamma^2|||\sum_{j=1}^{n}\{q_j^I-q_j\}\varphi_j|||^2_{L^2(\Sigma)} \\ &= \gamma^2|||\sum_{j=1}^{n}\left\{\sum_{k=0}^{m}e_j^k\Psi_k\right\}\varphi_j|||^2_{L^2(\Sigma)}. \end{aligned}$$

Notice that the inequality:

$$\left|\sum_{k=0}^{m}e_j^k\Psi_k(t)\right| \le C_9(\max_{0\le k\le m}\|\{e^k\}\|_\infty)$$

holds with some constant $C_9>0$. It follows from Theorem 5.1 that

$$\begin{aligned} |||\sum_{j=1}^{n}(q_j^I-q_j)\varphi_j|||^2_{H^{-1/2,-1/4}(\Sigma)} &\le \gamma^2C_9^2(\max_{0\le k\le m}\|\{e^k\}\|_\infty)^2|||\sum_{j=1}^{n}\varphi_j|||^2_{L^2(\Sigma)} \\ &\le \gamma^2C_9^2G_2^2\Delta t^{2\sigma}\xi^2Exp[2G_2T\xi]Tn\lambda_{max}(B). \end{aligned}$$

From Theorem 4.3 and (54), the total error satisfies that

$$\begin{aligned} &|||e|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\le (1+\frac{\alpha}{\beta})C_1h^{s+1/2}|||q|||_{H^{s,-1/4}(\Gamma)} \\ &\quad +\gamma\sqrt{T}\Delta t^{\sigma-1}\sqrt{n}\{C_6\Delta t\sqrt{\lambda_{max}(B)}+C_9G_2h\sqrt{\frac{\kappa(B)}{\lambda_{min}(B)}}Exp\left[\frac{G_2Th}{\Delta t\lambda_{min}(B)}\right]\} \end{aligned}$$

with $0\le s\le 1$. This completes the proof. □

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