REMARKS ON CONTRACTIVE-TYPE MAPPINGS

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Abstract. In a series of recent papers ([7],[8],[9],[10]), Husain and Latif et al defined the notions of contractive-type and of nonexpansive-type multivalued mappings and presented theorems that are designed to generalize the well known theorems concerning the fixed points of nonexpansive and contractive multivalued mappings. In this note, we present some results that generalize theorem 2.3 of [9]. We also point out in this note that most of the theorems in the aforementioned papers, in light of conditions under which they were presented, do not generalize the existing theorems of nonexpansive mappings. Some remarks on Definition 2.1 made by Husain and Tarafdar in [7] are also included.

1. Introduction. In a series of recent papers ([7],[8],[9],[10]), Husain and Latif et al defined the notions of contractive-type and of nonexpansive-type multivalued mappings and presented a number of theorems that are designed to generalize the well known theorems concerning the fixed points of nonexpansive and contractive multivalued mappings. A merit of the generalization seems to lie in the fact that nonexpansive-type and contractive-type mappings can have unbounded images and yet be guaranteed fixed points. In this regard, theorem 2.3 [9] generalizes the well known theorem of Nadler [15] that guarantees the fixed point of multivalued contraction in a complete metric space. In Section 4, we present some results that extend theorem 2.3 of [9]. Sections 2 and 3 are somewhat more critical in nature toward the results presented in [7],[8],[9],[10]. For example, in Section 3, we prove that contractive-type and nonexpansive-type multivalued mapping having closed and bounded images are, respectively, contractive and nonexpansive multivalued mappings. Therefore, most of the theorems in [7],[8],[9],[10], under the conditions that are presented, do not generalize the corresponding fixed point theorems for contractive and nonexpansive multivalued mappings. A remark is made in Section 3 to the effect that, if a nonexpansive-type multivalued mapping has an unbounded image at a point, then the mapping must have an unbounded image at every other point. The remark affirms the claim which we made above that the notions of nonexpansive-type and contractive-type mappings are useful in the setting of multivalued mappings with unbounded images. Section 2 is used to make comments regarding an earlier definition of nonexpansive-type mappings that appeared in [7].

2. Nonexpansive-type Mappings; Earlier Version. In [7], Husain and Tarafdar defined the following concept of nonexpansive-type mappings and proved theorem 1.1 below.
Definition A: Let (E,τ) be a locally convex linear Hausdorff topological space where the topology τ is generated by the family $[p_\alpha\colon \alpha \in I]$ of seminorms on E. Let K be a subset of E. A multivalued (or a single valued) mapping $f\colon K \to 2^K \setminus \emptyset$ is said to be nonexpansive-type on K if f satisfies either of the following conditions:

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(a) for each $\alpha \in I$, there are nonnegative real numbers $a_1(\alpha)$,$a_2(\alpha)$,$a_3(\alpha)$ with $a_1(\alpha) + a_2(\alpha) + a_3(\alpha) \leq 1$ such that for all $x, y \in K$,

$$p_\alpha(u - v) \leq a_1(\alpha)p_\alpha(x - y) + a_2(\alpha)p_\alpha(x - v) + a_3(\alpha)p_\alpha(y - u),$$

whenever $u \in f(x)$ and $v \in f(y)$;

(b) given $x \in K$ and $u \in F(x)$, for each $v \in f(y)$, $y \in K$ and each $\alpha \in I$, there exists $v'(\alpha) \in f(y)$ such that $p_\alpha(u - v) \leq p_\alpha(x - v'(\alpha))$;

(c) Given $x \in K$ and a real number $\epsilon > 0$, there exists for each $\alpha \in I$ a real number $\delta(\alpha) \geq \epsilon$ such that $p_\alpha(u - v) \leq \epsilon$ whenever $u \in f(x)$, $v \in f(y)$, $y \in K$ and $p_\alpha(x - y) \leq \delta(\alpha)$.
Using Definition A, the following theorem was presented in [7].

Theorem 2.1. *Let K be a nonempty weakly compact convex subset of E. Assume that K has a normal structure. Then for each multivalued mappings f of nonexpansive-type on K, there is a point $x \in K$ such that $f(x) = \{x\}$ where $\{x\}$ denotes the set consisting of the single point x.*

Unfortunately, Theorem 2.1 fails for the following simple multivalued contraction mapping; let $E = R$, and let $f: [0, 1] \to 2^{[0,1]} \setminus \emptyset$ be defined by $f(x) = [0, 1]$ for every $x \in [0, 1]$. Therefore, the authors' claim (page 4, [7]) that Theorem 2.1 generalizes the results of Browder[2], Göhde[6], Kirk[13] and Wong[20] is false. The difficulty of Theorem 2.1 lies in Definition A. Therefore we now focus our attention to this definition.

Three parts of Definition A above are clearly not equivalent. To see this, in part (b) of Definition A, define $K = R$ and let $f: R \to R$ be a single valued contraction $f(x) = \frac{1}{2}x$. For $x \in R$, $u = \frac{1}{2}x$, if $y \in R$, then $v = \frac{1}{2}y$ and the only choice for v' is also $\frac{1}{2}y$. We thus obtain $|\frac{1}{2}x - \frac{1}{2}y| \leq |x - \frac{y}{2}|$. This does not hold for $y = 2x$ with $x \neq 0$. The reader should note that part (c) is satisfied by this contraction mapping f. Now let $g: [0, 1] \to [0, 1]$ be defined by $g(x) = 0$ for $x \in [0, 1)$ and $g(1) = 1$. Then g satisfies part (a) but neither part (b) nor part (c) of Definition A. Thus we have shown that three parts of Definition are not equivalent and that Definition A does not encompass the classical contraction mappings. Hence it neither does encompass the class of nonexpansive mappings. Now consider the following mapping. With $0 < \epsilon < 1$, let $f: [0, 4] \to [0, 4]$ be defined by

$$f(x) = \begin{cases} x, & \text{on } [0,1] \\ 1, & \text{on } [1,3] \\ -\frac{x}{\epsilon} + \frac{3}{\epsilon} + 1, & \text{on } [3, 3+\epsilon] \\ 0, & \text{on } [3+\epsilon, 4] \end{cases}$$

f can be shown to satisfy all three parts of Definition A. Taking $3 \leq x < y \leq 3 + \epsilon$, we get $\frac{|f(x)-f(y)|}{|x-y|} = \frac{1}{\epsilon}$. Hence f is not a nonexpansive mapping. Thus, we have shown that Definition A neither implies nor is implied by the classical notion of nonexpansiveness.

3. Nonexpansive-type Mappings; Later Version. In the papers [9],[10], the notion of nonexpansive-type multivalued mappings is defined to be that described in Definition B below. We note that in [8], the same idea is used under the different terminology of weak nonexpansive mappings.

Definition B: Let C be a subset of a metric space (X, d). A multivalued mapping $f: C \to 2^C \setminus \emptyset$ is said to be nonexpansive-type if given x and $u \in f(x)$, there is $v \in f(y)$ for each $y \in C$ such that $d(u, v) \leq d(x, y)$.

The idea of contractive-type mappings is defined similarly by replacing $d(u, v) \leq d(x, y)$ in the above definition by $d(u, v) \leq hd(x, y)$ for $0 \leq h < 1$. As was stated in Introduction, every contractive-type multivalued mapping with closed and bounded images turns out to be a contractive multivalued mapping. This will be demonstrated in the following proposition.

Proposition 3.1. *Let (X, d) be a metric space. Denote by $CB(X)$ the set of all nonempty closed bounded subsets of X. The every contractive-type multivalued mapping $f: X \to CB(X)$ is a contractive mapping.*

Proof: If f is not a contractive mapping, then there is a sequence (x_n, y_n) such that, for every $n \in N$, $H(f(x_n), f(y_n)) > (1 - \frac{1}{n})d(x_n, y_n)$. Hence $x_n \neq y_n$ for every n. Let C be any subset of X containing $\{x_n\} \cup \{y_n\}$. Now suppose that T is of contractive-type with $0 \leq h < 1$. Choose $n \in N$ such that $n > \frac{1}{1-h}$. Also choose ϵ such that $0 < \epsilon < 1 - \frac{1}{n} - h$. Without of loss of generality, we suppose that $H(f(x_n), f(y_n)) = \sup_{a \in f(x_n)} d(a, f(y_n))$ where H denotes the Hausdorff metric on $CB(X)$ induced by the metric d. Now find $x'_n \in f(x_n)$ such that

$$\begin{aligned} d(x'_n, f(y_n)) &> \sup_{a \in f(x_n)} d(a, f(y_n)) - \epsilon d(x_n, y_n) \\ &= H(f(x_n), f(y_n)) - \epsilon d(x_n, y_n). \end{aligned}$$

For any $y'_n \in f(y_n)$, we have

$$\begin{aligned} d(x'_n, y'_n) &\geq d(x'_n, f(y_n)) \\ &> H(f(x_n), f(y_n)) - \epsilon d(x_n, y_n) \\ &> (1 - \tfrac{1}{n} - \epsilon)d(x_n, y_n) \\ &> hd(x_n, y_n). \end{aligned}$$

Since y'_n is arbitrary in $f(y_n)$ for the element $y_n \in C$, for x_n and $x'_n \in f(x_n)$, there is no element y of $f(y_n)$ which satisfies $d(x'_n, y) \leq hd(x_n, y)$, and hence f is not of contractive-type. This contradiction proves our proposition. □

Similarly, we obtain;

Proposition 3.2. *Let (X, d) be a metric space. The every nonexpansive-type multivalued mapping $f: X \to CB(X)$ is a nonexpansive mapping.*

Proof: If f is not nonexpansive, then there exist x and y with $H(f(x), f(y)) > d(x, y)$. As in the proof of Proposition 2.1, without loss of generality that $H(f(x), f(y)) = \sup_{a \in f(x)} d(a, f(y))$ and we choose ϵ such that $0 < \epsilon < H(f(x), f(y)) - d(x, y)$. If f is of nonexpansive-type, then to x and any $x' \in f(x)$, and y, there exists $y' \in f(y)$ such that $d(x', y') < d(x, y)$. We now choose $x' \in f(x)$ to satisfy $d(x', f(y)) > \sup_{a \in f(x)} d(a, f(y)) - \epsilon = H(f(x), f(y)) - \epsilon$. Then for any $y' \in f(y)$, $d(x', y') \geq d(x', f(y)) = H(f(x), f(y)) - \epsilon > d(x, y)$ and f is not of nonexpansive type. □

In [10], the notion of K-multivalued mapping is defined. Namely, let C be a nonempty subset of a normed linear space X. $f: C \to 2^C$ is K-multivalued if for each $x \in C$, $u_x \in f(x)$, there is $u_y \in f(y)$ for all $y \in C$ such that

$$\|u_x - u_y\| \leq \frac{1}{2}\{\|x - u_x\| + \|y - u_y\|\}.$$

Of course, the purpose of defining the idea of K-multivalued mappings is to generalize the idea of Kannan mappins of [11],[12]. We recall that if (X, d) is a metric space, $f: X \to CB(X)$ is called a Kannan mapping if for each x,$y \in X$,

$$H(f(x), f(y)) \leq \frac{1}{2}\{d(x, f(x)) + d(y, f(y))\}.$$

Following the argument used in the proof of Proposition 3.1, it is easy to see the following;

Proposition 3.3. *Let (X,d) be a metric space. Then every K-multivalued mapping $f\colon X\to CB(X)$ is a Kannan mapping.*

We note that Proposition 3.2 is recognized by Husain and Latif in [9] (p. 428). Then what is the advantage of studying the new classes of contractive-type and nonexpansive-type mappings? It seems to the present authors that the advantage of studying such mappings ought to lie on the point that such mappings need not have bounded images to have fixed points. Among the theorems listed in [8],[9] and [10], theorem 2.3 of [9] appears to be the only theorem that takes advantage of this point. Theorem 2.3 of [9] states the following;

Theorem 3.4. *Let M be a nonempty closed subset of a complete metric space (X,d) and $J\colon M\to 2^M$ a multivalued contractive-type mappings with closed subsets of M as values. Then there is a point $x\in M$ such that $x\in J(x)$.*

Most of the other theorems in the papers [8],[9] and [10] require multivalued mappings to have closed bounded images since the mappings are assumed to be compact-valued. Therefore, because of Propositions 3.1 and 3.2, they do not generalize the existing fixed point theorems for contractive and nonexpansive mappings. The following proposition is interesting in that, if a nonexpansive-type multivalued mapping is unbounded at a point, then it has unbounded images everywhere. This confirms our previous claim that a useful application of the notions of nonexpansive-type and contractive-type mappings ought to lie in the area of guaranteeing the existence of fixed points for multivalued mappings having unbounded images.

Proposition 3.5. *Let (X,d) be a metric space and $f\colon X\to 2^X\setminus\emptyset$ is of nonexpansive-type. If there exists $x\in X$ such that $f(x)$ is unbounded, then $f(y)$ is unbounded for every $y\in X$.*

Proof: Suppose that, for $x\in X$, $f(x)$ is unbounded. Then there is a sequence $\{u_n\}$, $u_n\in f(x)$, such that $d(x,u_n)\to\infty$. Since f is of nonexpansive-type, given any $y\in X$, there is, to each u_n, a $v_n\in f(y)$ such that $d(u_n,v_n)\le d(x,y)$. But $d(x,u_n)\le d(x,v_n)+d(v_n,u_n)\le d(x,v_n)+d(x,y)$, hence as $n\to\infty$, we get $d(x,v_n)\to\infty$. This shows that $f(y)$ is an unbounded set. Since y is arbitrary in X, we have that $f(y)$ is unbounded for every $y\in X$. □

4. Generalizations of Contractive-type Mappings. In this section, we obtain several results that guarantee the existence of fixed points of generalized contractive-type multivalued mappings that extend theorem 2.3 of [9]. Throughout this section, (X,d) denotes a complete metric space. $K(X)$ denotes the space of all nonempty compact subsets of X. In [16, p. 40], Reich proved that a mapping $T:X\to K(X)$ has a fixed point in X if it satisfies $H(Tx,Ty)\le k(d(x,y))d(x,y)$ for all $x,y\in X$ with $x\ne y$, where $k:(0,\infty)\to[0,1)$ satisfies $\limsup_{r\to t^+}k(r)<1$ for every $t\in(0,\infty)$. This result generalizes the fixed point theorem for single-valued mappings that was proved by Boyd and Wong [1]. One of the conjectures made by Reich in [17, 18] asks whether or not the range of T can be relaxed. Specifically the question is whether or not the range of T, $K(X)$, can be replaced by $CB(X)$. In response to Reich's conjecture, the following theorem was recently proved by Mizoguchi and Takahashi [14] , and other proofs have been given by Daffer and Kaneko [4] and Tong-Huei Chang [3].

Theorem 4.1. *Let (X,d) be a complete metric space and $T:X\to CB(X)$. Assume that T satisfies*

$$H(Tx,Ty)\le k(d(x,y))d(x,y) \tag{1}$$

for all $x,y\in X$ with $x\ne y$, where $k:(0,\infty)\to[0,1)$ satisfies $\limsup_{r\to t^+}k(r)<1$ for every $t\in[0,\infty)$. Then T has a fixed point in X.

This theorem is now modified to accomodate multivalued mappings with unbounded images.

Theorem 4.2. *Let (X,d) be a complete metric space and T is a mapping of X into the family of all closed subsets of X. If $k\colon (0,\infty) \to [0,1)$ satisfies $\limsup_{r\to t^+} k(r) < 1$ for every $t \in [0,\infty)$, and if T satisfies the following condition;*

(*) to every $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u,v) \leq k(d(x,y))d(x,y),$$

then T has a fixed point in X.

Proof: Let $x_0 \in X$ and $x_1 \in T(x_0)$. By hypothesis, there is $x_2 \in T(x_1)$ such that $d(x_2,x_1) \leq k(d(x_1,x_0))d(x_1,x_0)$. In general, if $x_n \in T(x_{n-1})$, we can find $x_{n+1} \in T(x_n)$ such that

$$d(x_{n+1},x_n) \leq k(d(x_n,x_{n-1}))d(x_n,x_{n-1}).$$

We write $d_n \equiv d(x_n,x_{n-1})$. Then

$$d_{n+1} \leq k(d_n)d_n \leq \prod_{i=1}^{n} k(d_i)d_i.$$

Now $d_{n+1} < d_n$ and so d_n converges to some c as $n \to \infty$. Since $d_{n+1} \leq k(d_n)d_n$, we get $c \leq \limsup_{n\to\infty} k(d_n)c$. This shows that $c = 0$. Since $\limsup_{t\to 0^+} k(t) < 1$, there is $h < 1$ and n_0 such that $k(d_n) < h$ for all $n > n_0$. We then have

$$d_{n+1} \leq \prod_{i=1}^{n} k(d_i)d_1 < \prod_{i=1}^{n_0} k(d_i)d_1 \prod_{i=n_0+1}^{n} k(d_i) < h^{n-n_0}C$$

where C denotes a generic constant throughout the remainder of proof. For $n > n_0$,

$$\begin{aligned} d(x_n,x_{n+m}) &\leq \textstyle\sum_{i+n+1}^{n+m} d_i \leq C\sum_{i=n+1}^{n+m} h^{i-1-n_0} \\ &= C\textstyle\sum_{i=0}^{m-1} h^{i+n-n_0} = Ch^{n-n_0}\frac{1-h^m}{1-h} \\ &\leq Ch^n. \end{aligned}$$

Since $h^n \to 0$ as $n \to \infty$, this shows that the sequence $\{x_n\}$ is Cauchy. Hence, $x_n \to x \in X$, and to x_n and x, and $x_{n+1} \in T(x_n)$, we have $y_n \in T(x)$ such that

$$d(x_{n+1},T(x)) \leq d(x_{n+1},y_n) \leq k(d(x_n,x))d(x_n,x),$$

so that $\lim_{n\to\infty} d(x,T(x)) = 0$. We thus have $d(x,T(x)) = 0$, and since $T(x)$ is closed, $x \in T(x))$. □

Now we focus our attention to the following class of functions that were recently studied by several authors, -e.g. [3, 19].

Definition C Let $\phi : R_+ \to R_+$. The function ϕ is said to satisfy the condition (Φ) (denoted $\phi \in (\Phi)$) if (i) $\phi(t) < t$ for all $t \in (0,\infty)$; (ii) ϕ is upper semicontinuous from the right on $(0,\infty)$; and (iii) there exists a positive real number s such that ϕ is nondecreasing on $(0,s]$ and $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t \in (0,s]$.

Chang [3] observed that if $k : (0,\infty) \to [0,1)$ satisfies $\limsup_{r\to t^+} k(r) < 1$ for every $t \in [0,\infty)$, then there exists a function $\phi \in (\Phi)$ such that $k(t)t \leq \phi(t)$ for all $t \in (0,\infty)$. Subsequently, Chang proved the following theorem that generalizes Theorem 4.1 above.

Theorem 4.3. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ and suppose that there exists a function $\phi \in (\Phi)$ such that*

$$H(Tx, Ty) \leq \phi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right)$$

for all $x, y \in X$. Then T has a fixed point in X.

Theorem 4.2 is now extended using a function from the class (Φ). Specifically, we obtain,

Theorem 4.4. *Let (X, d) be a complete metric space and T is a mapping of X into the family of all closed subsets of X. Suppose that there exists a function $\phi \in (\Phi)$ such that*

(**) to every $x, y \in X$,$u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u, v) \leq \phi(d(x, y)),$$

Then T has a fixed point in X.

Proof: Making use of the argument employed in lemma 2 [3], one may show that $\inf_{x \in X} d(x, T(x)) = 0$. Let $x_1 \in X$ be such that $d(x_1, T(x_1)) < s$ where s is as specified in Definition C. Now to x_1 and $x_2 \in T(x_1)$, choose $x_3 \in T(x_2)$ such that

$$d(x_2, x_3) \leq \phi(d(x_1, x_2)) < d(x_1, x_2).$$

Likewise, since ϕ is nondecreasing on $(0, s]$,

$$d(x_3, x_4) \leq \phi(d(x_2, x_3)) \leq \phi^2(d(x_1, x_2)).$$

In general,

$$d(x_n, x_{n+1}) \leq \phi^{n-1}(d(x_1, x_2)).$$

For positive integers m and n $(n > m)$, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \phi^{n-1}(d(x_1, x_2)) + \cdots + \phi^m(d(x_1, x_2)) \\ &\leq \textstyle\sum_{k=m}^{\infty} \phi^k(d(x_1, x_2)) \end{aligned}$$

By virtue of condition $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for each $t \in (0, s]$ (rf. (iii) Definition C), the above inequalities show that $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$ as $n \to \infty$. Now using (**), choose $y_n \in T(x)$ so that

$$d(x_{n+1}, T(x)) \leq d(x_{n+1}, y_n) \leq \phi(d(x_n, x)) < d(x_n, x).$$

Hence $\lim_{n \to \infty} d(x_n, T(x)) = 0$ and $x \in T(x)$ since $T(x)$ is closed. □

Of course, Theorem 4.4 can be reformulated using the contractive condition described in Theorem 4.3. Finally, we point out the fact that the conjecture of Reich described above is still open. Namely, it is not yet known whether or not Theorem 4.1 remains valid upon replacing the condition $\limsup_{r \to t^+} k(r) < 1$ for every $t \in [0, \infty)$ by the condition $\limsup_{r \to t^+} k(r) < 1$ for every $t \in (0, \infty)$. In a recent paper, the present authors in collaboration with Wu Li constructed a class of functions in (Φ) for which the condition $\limsup_{r \to t^+} k(r) < 1$ is satisfied only in the region $(0, \infty)$. Then utilizing Theorem 4.3 of Chang [3], the following theorem was proved in [5].

Theorem 4.5. *Let (X,d) be a complete metric space and $T : X \to CB(X)$. If there exists an upper right semi-continuous function $\varphi : R_+ \to R_+$ such that (i) $\varphi(t) < t$ for all $t > 0$, (ii) $\varphi(t) \le t - at^b$, $a > 0$, for some $1 < b < 2$ on some interval $[0,s]$, $s > 0$, and (iii)*

$$H(Tx, Ty) \le \varphi(d(x,y))$$

for all $x, y \in X$, then T has a fixed point in X.

Following the development made in Theorem 4.4, we obtain an extension of Theorem 4.5 that accomodates mappings with unbounded images. A proof is left to the reader.

Theorem 4.6. *Let (X,d) be a complete metric space and T is a mapping of X into the family of all closed subsets of X. If there exists an upper right semi-continuous function $\varphi : R_+ \to R_+$ such that (i) $\varphi(t) < t$ for all $t > 0$, (ii) $\varphi(t) \le t - at^b$, $a > 0$, for some $1 < b < 2$ on some interval $[0,s]$, $s > 0$, and (iii) to every $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \le \varphi(d(x,y)).$$

Then T has a fixed point in X.

References

1. D. W. Boyd and J. S. W. Wong, *On Nonlinear Contractions,* Proc. A.M.S., 89(1968), 458-464.
2. F. E. Browder, *Nonexpansive nonlinear operators in Banach space,* Proc. Nat. Acad. Sci. U. S. A. 54 (1965) 1041-1044.
3. T. H. Chang, *Common Fixed Point Theorems for Multi-Valued Mappings,* Math. Japonica (in press).
4. P. Z. Daffer and H. Kaneko, *Fixed Points of Generalized Contractive Multi-Valued Mappings,* Jl. Math. Anal. Appl. 192 (1995), 655-666.
5. P. Z. Daffer, H. Kaneko and W. Li, *On a conjecture of S. Reich,* Proceedings A. M. S. Vol.124 (1996), 3159-3162.
6. D. Göhde, *Zum Prinzip der kontraktiven Abbildung,* Math. Nachr. 30 (1965), 251-258.
7. T. Husain and E. Tarafdar, *Fixed point theorems for multivalued mappings of nonexpansive type,* Yokohama Jl. Math. 28 (1980), 1-6.
8. T. Husain and A. Latif, *Fixed points of multivalued nonexpansive maps I,* Math. Japonica 33 (1988), 385-391.
9. T. Husain and A. Latif, *Fixed points of multivalued nonexpansive maps II,* Int. Jl. Math. and Math. Sci. 14 (1991), 421-430.
10. A. Latif, T. Husain and I. Beg, *Fixed point of nonexpansive type and K-multivalued maps,* Int. Jl. Math. and Math. Sci. 17 (1994), 429-436.
11. R. Kannan, *Some results on fixed points,* Bull. Cal. Math. Soc. 60 (1968), 71-76.
12. R. Kannan, *Some results on fixed points II,* Amer. Math. Monthly 76 (1969), 405-408.
13. W. A. Kirk, *A fixed point theorem for mappings which do not increase distance,* Amer. Math. Monthly 72 (1965), 1004-1006.
14. N. Mizoguchi and W. Takahashi, *Fixed Points Theorems for Multi-Valued Mappings on Complete Metric Spaces,* Jl. Math. Anal. Appl. 141(1989) 177-188.
15. S. B. Nadler, *Multi-valued contraction mappings,* Pacific Jl. Math. 30 (1969), 459-462.
16. S. Reich, *Fixed Points of Contractive Functions,* Boll. Un. Mat. Ital. (4) 5 (1972), 26-42.
17. S. Reich, *Some Fixed Point Problems,* Atti. Acad. Naz. Lincei 57(1974), 194-198.
18. S. Reich, *Some Problems and Results in Fixed Point Theory,* Contemporary Math. A.M.S. 21(1983), 179-187.
19. K. Sastry, S.Naidu and J. Prasad, *Common Fixed Points for Multimaps in a Metric Space,* Nonlinear Anal. Thy. Methods & Appl. 13(3) (1989) 221-229.
20. C. S. Wong, *Fixed point theorems for generalized nonexpansive mappings,* Jl. Australian Math. Soc. 18 (1974), 265-276.

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