NON-LIPSCHITZ FUNCTIONS WHICH OPERATE ON FUNCTION SPACES

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Abstract. Sufficient conditions for non-Lipschitz functions to operate only in the space of all continuous functions among weakly normal real Banach function spaces. If the operating function h does not satisfy the conditions, then the both cases can occur:h operates only in the space of all continuous functions; there exsits a non-trivial normal real Banach function space on which h operates.

Introduction. In this paper we consider the question "Une fonction non Lipschitzienne peutelle opérer sur un espace de Banach de fonctions non trivial?" posed by A. Bernard [2]. There are non-Lipschitz functions which operate in non-trivial real Banach function spaces. We give a sufficient condition for those functions which cannot operate in them. By a real Banach function space on a compact Hausdorff space X we mean a linear subspace E of $C_R(X)$, the space of all real-valued continuous functions on X, which contains constant functions, separates the different points of X and is a Banach space in a norm $\|\cdot\|_E$ which dominates the uniform norm $\|\cdot\|_{\infty(X)}$ on X and is normalized so that $\|1\|_E = 1$. The space E is said to be non-trivial if $E \neq C_R(X)$. We say that a real Banach function space E is *weakly normal* if for every pair of disjoint compact subsets K_0 and K_1 of X, there exists a function $f \in E$ such that $f = 0$ on K_0 and $f = 1$ on K_1. We say that E is normal if for every pair of disjoint compact subsets K_0 and K_1 of X and $g \in E$, there exists a function $f \in E$ such that $f = 0$ on K_0 and $f = g$ on K_1. We say that E satisfies the condition $(*)$ if for every point x_0 in X, there exist a compact neighborhood G_0 of x_0, an infinite number of points $\{x_\alpha\}$ in X, compact neighborhood G_α of each x_α with $G_0 \cap G_\alpha = \emptyset$ and a homeomorphism π_α from G_0 onto G_α such that $E|G_0 = E|G_\alpha \circ \pi_\alpha$. If $A_{\mathbb{R}}(\mathbb{T})$ is the space of all real-valued functions in the Wiener algebra $A(\mathbb{T})$, then $A_{\mathbb{R}}(\mathbb{T})$ is a non-trivial real Banach function space on $\mathbb{T}$ and satisfies the condition $(*)$. The real part of the disk algebra on the unit disk also satisfies the condition $(*)$.

Let φ be a real valued function defined on an interval I. We say that φ operates in E if $\varphi \circ f$ is in E for every $f \in E$ with $f(X) \subset I$. de Leeuw and Katznelson [3] showed that if a non-trivial real Banach function space E on X is uniformly closed, then only affine functions operate on E, which is a generalization of the Stone-Weierstrass theorem. It is not the case for non-uniformly closed spaces: by a theorem of Wiener and Levy [8, p. 138] every real-valued real-analytic function operates on $A_{\mathbb{R}}(\mathbb{T})$. On the other hand by a theorem of Katznelson [6] we see that if φ is a real-valued continuous function which is not real-analytic, then φ never operates in $A_{\mathbb{R}}(\mathbb{T})$. In general, if E is a non-trivial real Banach function space, then there exists a fucntion which does not operate in E. Although the study of these functions is still far from beeing satisfactory, Katznelson's square root theorem is

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well-known: the function $\sqrt{t}$ on $[0,1)$ never operates on a non-trivial real Banach function algebra (cf. [2, 4, 5]). One might conjecture that non-Lipschitz functions never operate on non-trivial E. We showed that it is the case for certain real Banach function spaces. We can prove the following theorem in a way similar to the proof of [5, Proposition 25].

Theorem 0.1. *Suppose that E is a non-trivial normal real Banach function space which satisfies the condition $(*)$. Suppose also that φ is a real-valued function defined on the open interval $(-1,1)$. If φ operates in E, then φ satisfies the Lipschitz condition on every compact subset of $(-1,1)$.*

Let h be a real-valued function defined on the open interval $(-1,1)$. Suppose that h does not satisfy the Lipschitz condition on a comapct subset K of $(-1,1)$, i.e.,

$$\sup\{|h(t)-h(s)|/|t-s| : t,s \in K,\ t \neq s\} = \infty.$$

We consider two cases: i)for every $t_0 \in (-1,1)$,

$$\overline{\lim}_{s\to t_0}|h(t_0)-h(s)|/|t_0-s| < \infty;$$

ii) there exists a $t_0 \in (-1,1)$ such that

$$\overline{\lim}_{s\to t_0}|h(t_0)-h(s)|/|t_0-s| = \infty.$$

Put

$$E = \{f \in C_R(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} |f(n)-f(\infty)| < \infty\},$$

where $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ is the one point compactification of the space of all positive integers $\mathbb{N}$. Then E is a non-trivial normal real Banach function space on $\mathbb{N}_\infty$ with the norm $\|f\|_E = \sum_{n=1}^{\infty} |f(n)-f(\infty)| + |f(\infty)|$. It is easy to see that if h satisfies the condition i) above, then h operates in E. Thus our problem is to consider whether a real valued function on $(-1,1)$ which satisfies the condition ii) above can operate on a non-trivial real Banach function space or not.

Systematic study of operating non-Lipschitz functions by using an ultraseparation argument, which is originated by Bernard [1], has just begun recently and we believe it is a powerful tool to attack the problem involving operating functions. We proved that the Cantor function and t^p on $[0,1)$ for a p with $0 < p < 1$ never operate in a non-trivial E [4, 5]. Similar results were obtained independently by Bernard [2]. We also proved the following [4, 5].

Theorem 0.2. *If φ is a real-valued function on $(-1,1)$ such that $(\varphi(t)-\varphi(0))/t \to \infty$ as $t \to +0$, then φ never operates in a non-trivial weakly normal real Banach function space.*

In the same way as in the proof of Proposition 24 in [5] we see that there is a non-Lipschitz function which does operate on a non-trivial real Banach function space.

Theorem 0.3. *Let $X = \mathbb{N}_\infty$ and*

$$E = \{f \in C_R(X) : \sum_{n=1}^{\infty} |f(n)-f(\infty)|M_n < \infty\},$$

where $M_n = 2^{n^2}$. Then E is a non-trivial normal real Banach function space on X. Let φ be a continuous function defined on the interval $(-1, 1)$ such that

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad t \in (-1, 0] \cup [\frac{1}{2}, 1) \cup \left(\bigcup_{n=1}^{\infty} (\frac{1}{M_{n+1}-1}, \frac{1}{M_n})\right) \\ c_n(t - \frac{1}{M_{n+1}}) & \text{if} \quad \frac{1}{M_{n+1}} \le t \le \frac{1}{2M_{n+1}} + \frac{1}{2(M_{n+1}-1)} \\ -c_n(t - \frac{1}{M_{n+1}-1}) & \text{if} \quad \frac{1}{2M_{n+1}} + \frac{1}{2(M_{n+1}-1)} \le t \le \frac{1}{M_{n+1}-1}, \end{cases}$$

where we denote $c_n = \frac{2^{-(n^2+n-1)}}{-(M_{n+1})^{-1} + (M_{n+1}-1)^{-1}}$. Then $t_n = (1/M_{n+1} + 1/(M_{n+1}-1))/2 \to 0$ and $\varphi(t_n)/t_n \to \infty$ and φ operates in E.

We may say that t_n in Theorem 0.3 *rapidly* converges to 0 in the sense that $t_{n+1}/t_n \to 0$ as $n \to \infty$. In this paper we consider the intermediate case of the above two theorems, that is, we consider the case where there exists a *slowly* decreasing sequence $\{t_n\}$ with $(\varphi(t_n) - \varphi(0))/t_n \to \infty$. The proofs in this paper implicitly and heavily depend on an ultraseparation argument.

1. Sufficient conditions for $E = C_R(X)$. We say that a subset S of X is a *uniqueness set* for a real Banach function space E if $f = 0$ on S implies that $f = 0$ on X for $f \in E$.

Lemma 1.1. *Let B be a real Banach function space on a compact Hausdorff space K. Let $\{\mu_n\}$ be a sequence of positive real numbers and $\{f_n\}$ a sequence of functions in B. Let S be a subset of K which is a uniqueness set for B. Suppose that for every sequence $\{a_n\}$ of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n\mu_n < \infty$, there exists a function $f \in B$ such that $\sum_{n=1}^{\infty} a_n f_n$ converges pointwisely on S to f. Then there exists a positive real number Msuch that the inequality*

$$\|f_n\|_B \le M\mu_n$$

holds for every positive integer n.

Proof. First we consider the case where $\mu_n = 1$ for every $n \in \mathbb{N}$. The corresponding function $f \in B$ for each sequence $\{a_n\}$ of non-negative real numbers with $\sum a_n < \infty$ is unique since S is a uniqueness set and $\sum_{n=1}^{\infty} a_n f_n(y) = f(y)$ for every $y \in S$. Put $T(\{a_n\}) = f$. Then T can be extended in a way natural as a linear operator on the usual Banaach space ℓ^1 of all sequences of complex numbers $\{c_n\}$ such that $\sum |c_n| < \infty$ to B. It is easy to see that

$$T(\{c_n\})(y) = \sum_{n=1}^{\infty} c_n f_n(y)$$

holds for every $\{c_n\} \in \ell^1$ and every $y \in S$. We show that T is bounded. If we prove it, it will follow that

$$\|f_n\|_E \le \|T\|$$

holds for every $n \in \mathbb{N}$ since $T(\{\delta_{mn}\}_{m=1}^{\infty}) = f_n$. First we show that $\{f_n(y)\}$ is a bounded sequence for each $y \in S$. Suppose not. Then, for every $m \in \mathbb{N}$, there exists $n(m)$ such that $|f_{n(m)}(y)| \ge m^2$. Put

$$a_n = \begin{cases} \dfrac{1}{m^2}, & n = n(m) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{a_n\} \in \ell^1$ and $\sum_{n=1}^{\infty} a_n f_n(y)$ diverges since $|a_{n(m)} f_{n(m)}| \ge 1$ for every m, a contradiction. Suppose that $\{c_n^{(k)}\}_{n=1}^{\infty} \in \ell^1$ converges to $\{c_n\} \in \ell^1$ and $T(\{c_n^{(k)}\})$ converges in

B to a function $F \in B$. If we show that $F = T(\{c_n\})$, it will follow by the closed graph theorem that T is bounded. Let $y \in S$. Since $\|\cdot\|_{\infty(S)} \le \|\cdot\|_B$, we have

$$|\sum_{n=1}^{\infty} c_n^{(k)} f_n(y) - F(y)| \le \|T(\{c_n^{(k)}\}) - F\|_B \to 0$$

as $k \to \infty$. We also have

$$|\sum_{n=1}^{\infty} c_n^{(k)} f_n(y) - \sum_{n=1}^{\infty} c_n f_n(y)| \to 0$$

as $k \to \infty$ since $\{f_n(y)\}$ is bounded and $\{c_n^{(k)}\} \to \{c_n\}$ in ℓ^1. Henceforce we see that the equality

$$F(y) = \sum_{n=1}^{\infty} c_n f_n(y) = T(\{c_n\})(y)$$

holds for every $y \in S$, thus we have

$$F = T(\{c_n\})$$

since S is a uniquness set for B. We have proven that T is bounded.

Next we consider the general case. Put

$$g_n = f_n/\mu_n.$$

Then for every $\{c_n\} \in \ell^1$ with $c_n \ge 0$, we have

$$\sum_{n=1}^{\infty} d_n \mu_n < \infty,$$

where $d_n = c_n/\mu_n$. Then by the condition there exists a function $f \in E$ such that $\sum_{n=1}^{\infty} d_n f_n$ converges pointwisely on S to f, hence $\sum_{n=1}^{\infty} c_n g_n$ converges pointwisely to f. It follows by the first part of the proof that there exists a positive real number M such that the inequality

$$\|g_n\| \le M$$

holds for every $n \in \mathbb{N}$, henceforce

$$\|f_n\|_E \le M\mu_n$$

holds for every $n \in \mathbb{N}$. □

The function φ in Theorem 0.3 satisfies $t_{n+1}/t_n \to 0$ and $\varphi(t_{n+1})/t_n \to 0$ as $n \to \infty$. We consider operating functions which does not satisfy these properties.

Theorem 1.2. *Let E be a weakly normal real Banach function space on a compact Hausdorff space X. Suppose that h is a real-valued function defined on the open interval $(-1, 1)$ such that $h(0) = 0$. Suppose also that there exists a strictly decreasing sequence $\{t_n\}$ of positive real numbers such that $t_n \to 0$ as $n \to \infty$ which satisfies $\lim_{n\to\infty} h(t_{n+1})/t_n = \infty$. If h operates in E, then $E = C_R(X)$.*

Proof. Suppose that X is a finite set. Then we have $E = C_R(X)$ since E is weakly normal. So we consider the case where X is infinite. In the same way as in the first part of the proof of Theorem 1 in [4] we see that h is continuous on $(-1, 1)$.

Suppose that $E \ne C_R(X)$. Then by Theorem 9 in [5] or Théorèm 3 in [2], there exists $x \in X$ such that $E|G \ne C_R(G)$ holds for every compact neighborhood G of x since h is non-affine and continuous on $(-1, 1)$. Let $E_x = \{u \in E : u(x) = 0\}$. Then by Lemma 27 in

[5] there are two sequences $\{G_0^{(n)}\}_{n=1}^{\infty}$ and $\{G_1^{(n)}\}_{n=1}^{\infty}$ of compact subsets of $X \setminus \{x\}$ which satisfy that

$$G_\alpha^{(n)} \cap (\overline{\bigcup_{(\beta,m)\neq(\alpha,n)} G_\beta^{(m)}}) = \emptyset$$

for every $(\alpha, n) \in \{0,1\} \times \mathbb{N}$ and that for evey $n \in \mathbb{N}$ a function $u \in E_x$ with $u \geq 1$ on $G_1^{(n)}$ and $u \leq 0$ on $G_0^{(n)}$ implies that $\|u\|_E > n$, where $\bar{\cdot}$ denotes the closure in X. For $n \in \mathbb{N}$ put

$$M_n = \inf\{\|u\|_E : u \in E_x,\ u = 1 \text{ on } G_1^{(n)},\\ u = 0 \text{ on } G_\beta^{(m)} \text{ for every } (\beta, m) \in \{0,1\} \times \mathbb{N} \text{ with } (\beta, m) \neq (1, n)\}.$$

Since E is weakly normal, we see that $n \leq M_n < \infty$. Put $\alpha_n = h(t_n)/t_{n-1}$ for $n \geq 2$. Since $\alpha_n \to \infty$ as $n \to \infty$, for every $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that

$$\alpha_k > n2^{n+1}$$

holds for every $k \geq k(n)$. Then there exists $l(n) \in \mathbb{N}$ such that

$$M_{l(n)} t_{k(n)} 2^{n+1} > 1$$

since $M_n \to \infty$ as $n \to \infty$. We may suppose that $l(n) < l(n+1)$. Then there exists $m(n) \in \mathbb{N}$ such that

$$t_{m(n)} < (2^{n+1} M_{l(n)})^{-1} \leq t_{m(n)-1}.$$

We have $m(n) > k(n)$ for every $n \in \mathbb{N}$ since $(2^{n+1} M_{l(n)})^{-1} < t_{k(n)}$. So we see that

$$\alpha_{m(n)} > n2^{n+1}.$$

Let

$$K = \{x\} \cup (\overline{\bigcup G_\alpha^{(m)}}),\ S = \{x\} \cup (\bigcup G_\alpha^{(m)}),$$

where (α, m) varies through $\{0,1\} \times \mathbb{N}$. Then S is a uniqueness set for a real Banach function space $B = E|K$ on K. Note that $E|K = \{u|K : u \in E\}$ is a real Banach function space with the quotient norm $\|\cdot\|_{E|K}$, where $\|u|K\|_{E|K} = \inf\{\|v\|_E : v|K = u|K,\ v \in E\}$. By the definition of $M_{l(n)}$, there exists $u_{l(n)} \in E_x$, for every n, such that

$$u_{l(n)} = \begin{cases} 1, & \text{on } G_1^{(l(n))} \\ 0, & \text{on } \displaystyle\bigcup_{(\beta,m)\neq(1,l(n))} G_\beta^{(m)}, \end{cases}$$

and $\|u_{l(n)}\|_E < 2M_{l(n)}$. We see that

$$M_{l(n)} \leq \|u_{l(n)}|K\|_{E|K}.$$

by the definition of the quotient norm. For every $n \in \mathbb{N}$, put a positive real number

$$\mu_n = (2^{n+1} M_{l(n)})/\alpha_{m(n)}.$$

Let

$$f_n = u_{l(n)}|K$$

for every $n \in \mathbb{N}$. For every sequence $\{a_n\}$ of non-negative real numbers such that

$$\sum a_n \mu_n < \infty,$$

we show that there exists a function in $f \in B$ and $\sum_{n=1}^{\infty} a_n f_n$ converges pointwisely on S to the function f. Choose a sufficiently large D such that

$$\sum_{n=1}^{\infty} a_n \mu_n / D < 1.$$

Since every $a_n \mu_n / D$ is non-negative, we have $a_n/D < 1/\mu_n$, hence

$$0 \leq a_n/D < \alpha_{m(n)} (M_{l(n)} 2^{n+1})^{-1} \leq \alpha_{m(n)} t_{m(n)-1} = h(t_{m(n)}).$$

By the intermediate value theorem for continuous functions, there exists $0 \leq s_n < t_{m(n)}$ such that $h(s_n) = a_n/D$. Since $\|s_n u_{l(n)}\|_E \leq 2M_{l(n)} t_{m(n)} < 1/2^n$, the series $\sum_{n=1}^{\infty} s_n u_{l(n)}$ converges in E, say g. Since $\|\cdot\|_E$ dominates $\|\cdot\|_{\infty(X)}$, $\sum_{n=1}^{\infty} s_n u_{l(n)}$ also converges uniformly on X to g and $g(X) \subset (-1, 1)$, so $h \circ g$ is a function in E. Then $f = D \cdot (h \circ g)|K$ is the desired function. Let $y \in S$. Then $y = x$ or $y \in G_\alpha^{(m)}$ for some $(\alpha, m) \in \{0, 1\} \times \mathbb{N}$. If $y = x$, then $\sum_{n=1}^{\infty} a_n f_n(x) = 0 = f(x)$ since $f_n(x) = 0$ and $h(0) = 0$. If $y \in G_\alpha^{(m)}$ for $m \in \mathbb{N} \setminus \{l(n)\}_{n=1}^{\infty}$, then

$$\sum_{n=1}^{\infty} a_n f_n(y) = 0 = f(y).$$

If $y \in G_\alpha^{(l(n))}$ for some $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n f_n(y) = a_n u_{l(n)}(y)$$

and

$$f(y) = D \cdot h(\sum_{n=1}^{\infty} s_n f_n(y)) = a_n u_{l(n)}(y).$$

Thus we have

$$f(y) = \sum_{n=1}^{\infty} a_n f_n(y)$$

for every $y \in S$. We have proved that $\sum_{n=1}^{\infty} a_n f_n$ converges pointwisely on S to f. It follows by Lemma 1.1 that there exists a poisitive real number M such that the inequality

$$\|u_{l(n)}|K\|_{E|K} \leq M \mu_n$$

holds for every $n \in \mathbb{N}$. Thus we see that

$$M_{l(n)} \leq M \cdot 2^{n+1} M_{l(n)} / \alpha_{m(n)},$$

so

$$\alpha_{m(n)} 2^{-(n+1)} \leq M$$

holds for every $n \in \mathbb{N}$, which is a contradiction since

$$\alpha_{m(n)} > 2^{n+1} n$$

holds for every $n \in \mathbb{N}$. □

Corollary 1.3. *Let E be a weakly normal real Banach function space on a compact Hausdorff space X. Suppose that h is a real-valued function defined on the open interval $(-1, 1)$ such that $h(0) = 0$. Suppose also that there exists a strictly decreasing sequence $\{t_n\}$ of positive real numbers such that $t_n \to 0$ as $n \to \infty$ which satisfies that $\lim_{n\to\infty} h(t_n)/t_n = \infty$ and $\inf\{t_{n+1}/t_n\} > 0$. If h operates in E, then $E = C_R(X)$.*

Proof. Put $\delta = \inf\{t_{n+1}/t_n\}$. Then we have

$$h(t_{n+1})/t_n \geq \delta h(t_{n+1})/t_{n+1} \to \infty$$

as $n \to \infty$, henceforce we see that h satisfies the condition of Theorem 1.2. It follows that $E = C_R(X)$. □

2. Operating functions with mild conditions. Let h be a real valued-continuous function defined on $(-1, 1)$ with $h(0) = 0$. For $0 \leq t < 1$, put

$$H(t) = \max\{h(s) : 0 \leq s \leq t\}.$$

Then the function h satisfies the condition in Theorem 1.2 if and only if

$$\lim_{t\to+0} \frac{H(t)}{t} = \infty.$$

Suppose that there exists a decreasing sequence $\{t_n\}$ of positive real numbers with $t_n \to 0$ such that $\lim_{n\to\infty} h(t_{n+1})/t_n = \infty$. For every $t > 0$, there exist a positive integer n such that $t_{n+1} < t \leq t_n$. Then $h(t_{n+1})/t_n \leq H(t)/t$, so $H(t)/t \to \infty$ as $t \to +0$. Suppose conversely that $H(t)/t \to \infty$ as $t \to +0$. Let $t_1 = 1/2$. Suppose that $t_1, \ldots t_n$ are choosen. Then put

$$t_{n+1} = \inf\{t : H(t) = H(t_n/2)\}.$$

By induction we define a sequence $\{t_n\}$. Since H is continuous, we have $H(t_{n+1}) = H(t_n/2)$. Then by the definition of t_{n+1}, we see that $h(t_{n+1}) = H(t_{n+1})$. Thus

$$h(t_{n+1})/t_n = H(t_n/2)/t_n \to \infty$$

as $n \to \infty$. Thus by Theorem 1.2 we see that, for short, $E = C_R(X)$ if $\lim_{t\to+0} H(t)/t = \infty$.

Next we consider the case that $\lim_{t\to+0} H(t)/t \neq \infty$. The following examples show that if $\underline{\lim}_{t\to+0} H(t)/t > 0$, then both two cases are posible.

Example 2.1. *Let φ be the function defined in Theorem 0.3. Put*

$$h(t) = \begin{cases} \varphi(t) + 2t, & 0 \leq t < 1 \\ 0, & -1 < t < 0. \end{cases}$$

Then $\overline{\lim}_{t\to+0} \frac{H(t)}{t} = \infty$ and $\underline{\lim}_{t\to 0} \frac{H(t)}{t} = 2$. We also see that h operates in the space E defined in Theorem 0.3, that is, h operates in a non-trivial real Banach function space.

Example 2.2. *Put a decreasing sequence $\{t_n\}$ defined inductively by $t_0 = 1/2$, $t_{n+1} = t_n/(n + 2)$ and put*

$$h(t) = \begin{cases} 1/2, & 1/4 \leq t < 1 \\ t_n, & t_{n+1} \leq t < t_n/2 \\ \frac{2(t_{n-1} - t_n)}{t_n} t - t_{n-1} + 2t_n, & t_n/2 \leq t < t_n \\ 0, & -1 < t < 0. \end{cases}$$

Then $\overline{\lim}_{t\to+0} \frac{H(t)}{t} = \infty$ and $\underline{\lim}_{t\to 0} \frac{H(t)}{t} = 2$. Suppose that h operates in a weakly normal real Banach function space E on a compact Hausdorff space X. Then $h \circ h$ also opertates in E. It follows by Theorem 1.2 that $E = C_R(X)$ since

$$h \circ h(t_{n+1})/t_n = t_{n-1}/t_n \to \infty$$

as $n \to \infty$.

We consider the case where $\underline{\lim}_{t\to+0}\frac{H(t)}{t}=0$. For $0\le t<1$, put

$$\widetilde{H}(t)=\max\{|h(s)|:0\le s\le t\}$$

for a real-valued continuous function h on $(-1,1)$.

Theorem 2.3. *Let h be a real-valued continuous function defined on the open interval $(-1,1)$ with $h(0)=0$. Suppose that $\overline{\lim}_{t\to t_0}|h(t)-h(t_0)|/|t-t_0|<\infty$ for every $t_0\in(-1,1)\setminus\{0\}$ and $\overline{\lim}_{t\to-0}|h(t)/t|<\infty$. Suppose also that $\underline{\lim}_{t\to+0}\widetilde{H}(t)/t=0$. Then there exists a non-trivial normal real Banach function space E in which h operates.*

Proof. Since $\underline{\lim}_{t\to+0}\widetilde{H}(t)/t=0$, there exists a decreasing sequence $\{t_n\}$ of positive real numbers such that $\widetilde{H}(t_n)/t_n<2^{-n}$. Let $\mathbb{N}_\infty=\mathbb{N}\cup\{\infty\}$ be the one point compactification of the space of all positive integers. Put

$$E=\{f\in C_R(\mathbb{N}_\infty):\sum_{n=1}^{\infty}|f(n)-f(0)|/t_n<\infty\}.$$

Then E is a non-trivial normal real Banach function space on $\mathbb{N}_\infty$ with the norm $\|f\|_E=\sum_{n=1}^{\infty}|f(n)-f(\infty)|/t_n+|f(\infty)|$. We show that h operates in E. Suppose that $f\in E$ with $f(\mathbb{N}_\infty)\subset(-1,1)$. If $f(\infty)\neq0$, then by the condition there exists $c>0$ such that $|h(t)-h(f(\infty))|\le c|t-f(\infty)|$ for t near $f(\infty)$, so $\sum_{n=1}^{\infty}|h\circ f(n)-h\circ f(\infty)|/t_n<\infty$, that is, $h\circ f\in E$. Suppose that $f(\infty)=0$. Since $\overline{\lim}_{t\to-0}|h(t)/t|<\infty$, there exists $c'>0$ such that $|h(t)|\le c'|t|$ holds if $-t_1\le t\le0$. For a sufficiently large n, we have $|f(n)|<t_n$ since $f\in E$. If $f(n)<0$, then $|h\circ f(n)|\le c'|f(n)|$. If $f(n)>0$, then $|h\circ f(n)|\le\widetilde{H}(f(n))\le2^{-n}t_n$. It follows that $\sum_{n=1}^{\infty}|h\circ f(n)|/t_n<\infty$, that is, $h\circ f\in E$. □

Note that the function φ in Theorem 0.3 satisfies the condition of h in Theorem 2.3.

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