REMARKS ON STRANG'S INEQUALITY

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ABSTRACT. In this note we present the best bounds for **ReAB** where **A** and **B** are bounded linear operators on a Hilbert space satisfying $\| \mathbf{A} - a \| \leq c$ and $\| \mathbf{B} - b \| \leq d$ for nonzero real numbers a, b, c and d.

Let **H** be a Hilbert space and **L(H)** the Banach algebra consisting of all bounded linear operators on **H**. For two selfadjoint operators **A** and **B** in **L(H)** which satisfy

$$m \leq \mathbf{A} \leq M, \quad n \leq \mathbf{B} \leq N. \tag{1}$$

Strang [1] found the best bounds of the Jordan product **AB+BA.** Fujii,et al.[2] attempted to provide a simple proof of Strang's result. They obtained the following theorem:

Theorem A. Let **A** and **B** be selfadjoint operators in **L(H)** satisfying (1). Then the best lower bound c of **ReAB** is given by

$$\mathbf{ReAB} \geq c = \frac{16MNmn - (M-m)^2(N-n)^2}{8(M+m)(N+n)}. \tag{2}$$

It is regrettable that the above theorem is incorrect when $(M+m)(N+n)$ <0. The following simple example shows (2) is untenable. Let $\mathbf{A}=\frac{1}{2}$ and $\mathbf{B}=-\frac{1}{2}$, then $0 < \mathbf{A} < 1$ and $-1 < \mathbf{B} < 0$. We have a contradictory inequality $-\frac{1}{4} \geq \frac{1}{8}$ by (2). In fact the opposite of (2) holds when $(M+m)(N+n) < 0$. This follows if we write (1) in the form

$$-M \leq -\mathbf{A} \leq -m, \quad n \leq \mathbf{B} \leq N,$$

and use (2) in the case of $(M+m)(N+n) > 0$.

On the other hand Strang's result was generalized in part to one for nonselfadjoint operators in[2] as following:

Theorem B. Let $\mathbf{A}, \mathbf{B} \in \mathbf{L}(\mathbf{H})$. If

$$\| \mathbf{A} - a \| \leq c, \quad \| \mathbf{B} - b \| \leq d \tag{3}$$

for nonzero real numbers a, b, c and d, then the best bound is given by

$$2abRe\mathbf{A}^*\mathbf{B} \geq a^2b^2 - a^2d^2 - b^2c^2.$$

From this theorem the lower bound of $\mathrm{Re}\mathbf{A}^*\mathbf{B}$ (or **ReAB**) is derived for ab >0 and the upper bound for ab <0. In the present paper we shall complete Theorem B and obtain the best lower and upper bound for **ReAB** with the restriction of (3). Strang's theorem becomes a special case of our result. The following is our main theorem in this note:

Theorem. With the hypotheses of Theorem B, we have

$$\alpha \leq \mathbf{ReAB} \leq \beta \tag{4}$$

where α and β are the minimum and maximum of $Re[(a+ce^{i\theta})(b+de^{i\phi})]$ for $\theta, \phi \in [0, 2\pi]$.

From [1] we know that α and β are the least and the greatest of the following $E_j (1 \leq j \leq 5)$ given by

$$E_1 = ab + bc + ad + cd, \quad E_2 = ab + bc - ad - cd,$$
$$E_3 = ab - bc + ad - cd, \quad E_4 = ab - bc - ad + cd,$$
$$E_5 = \frac{1}{2}(ab - \frac{ad^2}{b} - \frac{bc^2}{a}) \ .$$

For the proof of the theorem the following lemma is required:

Lemma. If we set $c = d = 1$ among $E_j (1 \leq j \leq 5)$ given above and α is the least of E_j, then

$$\alpha = E_5 = \frac{1}{2}(ab - \frac{b}{a} - \frac{a}{b})$$

for $ab > 0$,

$$\alpha = E_2 = ab + b - a - 1$$

for $a > 0$ and $b < 0$, and

$$\alpha = E_3 = ab - b + a - 1$$

for $a < 0$ and $b > 0$.

Proof. If $ab > 0$, an easy calculation shows that

$$E_2 - E_5 = \frac{(ab - a + b)^2}{2ab} \geq 0$$

which shows $E_2 \geq E_5$. Similarly, we can verify that E_5 is not greater than E_1, E_3 and E_4. Thus E_5 is the least among $E_j (1 \leq j \leq 5)$ for $ab > 0$.

For the case of $a > 0$ and $b < 0$ we have

$$E_1 - E_2 = 2a + 2 > 0$$

which implies $E_1 > E_2$, and

$$E_5 - E_2 = a + 1 - b - \frac{1}{2}(ab + \frac{b}{a} + \frac{a}{b}) > 0,$$

this leads to $E_5 > E_2$. We can check similarly that $E_3 > E_2$ and $E_4 > E_2$. So E_2 is the least of $E_j (1 \leq j \leq 5)$ in this case. The other inequalities can be shown with the same method. This completes the proof.

The proof of the main theorem:

First we prove the left side of (4). Without loss of generality, we may assume $c = d = 1$. If $ab > 0$ we only need to verify that

$$\mathbf{ReAB} \geq E_5 = \frac{1}{2}(ab - \frac{b}{a} - \frac{a}{b}) = \alpha$$

from the lemma. This is given by Theorem B.

Suppose a>0 and b<0. Set $\mathbf{S}=\mathbf{A}-a$ and $\mathbf{T}= \mathbf{B}-b$, then $\mathbf{S}$ and $\mathbf{T}$ are contractions and

$$\mathbf{AB} = ab + b\mathbf{S} + a\mathbf{T} + \mathbf{ST}.$$

Noticing that $-1 \leq Re\mathbf{P} \leq 1$ for the contraction $\mathbf{P}$, we have

$$\mathbf{ReAB} = ab + bReS + aReT + Re(ST)$$
$$\geq ab + b - a - 1 = E_2 = \alpha$$

by the lemma. The same method is used to the proof in the case of $a < 0$ and $b > 0$. So the left side of (4) is proved.

To prove the right side of (4), we write (3) in the form

$$\| (-\mathbf{A}) - (-a) \| \leq 1, \quad \| \mathbf{B} - b \| \leq 1.$$

By the left side of (4) we have

$$(5) \qquad Re(-\mathbf{A})\mathbf{B} \geq \alpha', \quad i.e. \quad \mathbf{ReAB} \leq -\alpha'$$

where α' is the least of $E'_j (1 \leq j \leq 5)$ given by

$$E'_1 = -ab + b - a + 1, \quad E'_2 = -ab + b + a - 1,$$
$$E'_3 = -ab - b - a - 1, \quad E'_4 = -ab - b + a + 1,$$
$$E'_5 = \frac{1}{2}(-ab + \frac{a}{b} + \frac{b}{a}) .$$

It is obvious that $E'_1 = -E_3, E'_2 = -E_4, E'_3 = -E_1, E'_4 = -E_2$ and $E'_5 = -E_5$. So

$$-\alpha' = -minE'_j = max(-E'_j) = maxE_j = \beta.$$

Hence (5) is just the right side of (4). This concludes the proof.

Corollary. With the same condition of Theorem A we have

$$\tilde{\alpha} \leq \mathbf{ReAB} \leq \tilde{\beta}$$

where α and β are the least and greatest of $\tilde{E}_j (1 \leq j \leq 5)$ given by

$$\tilde{E}_1 = MN, \qquad \tilde{E}_2 = Mn,$$
$$\tilde{E}_3 = mN, \qquad \tilde{E}_4 = mn,$$
$$\tilde{E}_5 = \frac{16MNmn - (M-m)^2(N-n)^2}{8(M+m)(N+n)} .$$

Proof. According to [1] we can translate (1) to (3) in the case of selfadjoint if we take

$$a = \frac{M+m}{2}, b = \frac{M-m}{2}, c = \frac{N+n}{2}, d = \frac{N-n}{2} .$$

Then it follows with a simple computation from the main theorem.

The above corollary is the generalization of Strang's result.

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References

1. W.G.Strang, Eigenvalues of Jordan product, Amer. Math. Mon., **69** (1962) 37-40.
2. J.I. Fujii, et al., Strang's inequality, Math. Japon., **37** (1992)479-486.

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