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CANCELLATION LAWS FOR BCI-ALGEBRA, ATOMS AND P-SEMISIMPLE BCI-ALGEBRAS

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Abstract. We derive cancellation laws for BCI-algebras and for p-semisimple BCI-algebras, show that the set of all atoms of a BCI-algebra is a p semisimple BCI-algebra and that in a p-semisimple BCI-algebra $\leq$ and $=$ are the same.

1. Introduction. BCI-algebras, first introduced by Iséki in [1], can be defined as follows:
Definition 1 An algebra $\langle X; *, 0\rangle$ of type $(2,0)$ is a BCI-algebra if for all $x, y, z \in X$.

BCI-1 $\quad (x*y)*(x*z) \leq z*y$;
BCI-2 $\quad x*(x*y) \leq y$;
BCI-3 $\quad x \leq x$;
BCI-4 $\quad x \leq y$ and $y \leq x$ imply $x = y$;
BCI-5 $\quad x \leq y$ iff $x*y = 0$.

The following well known properties of BCI-algebras are used below.

(1) $\quad (x*y)*z = (x*z)*y$
(2) $\quad 0*(x*y) = (0*x)*(0*y)$
(3) $\quad x*0 = x$
(4) $\quad x*(x*(x*y)) = x*y$
(5) $\quad x*x = 0$
(6) $\quad x \leq 0 \Rightarrow x = 0$.

2. A Cancellation law for BCI-Algebras.
Theorem 1 If $\langle X; *, 0\rangle$ is a BCI-algebra and $x, y, z \in X$ then:

(i) $x*y \leq x*z \Rightarrow 0*y = 0*z$;
(ii) $y*x \leq z*x \Rightarrow 0*y = 0*z$.

Proof (i) If $x*y \leq x*z$, by BCI-5,

$$(x*y)*(x*z) = 0$$

and so by BCI-1 and BCI-5,

$$0*(z*y) = 0 \qquad -(a)$$

and by (2),

$$(0*z)*(0*y) = 0.$$

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Hence by BCI-5

$$0 * z \leq 0 * y.$$

We now apply the same cancellation procedure to this as we did to $x * y \leq x * z$, this time "cancelling" the 0 to give:

$$0 * y \leq 0 * z$$

$$\therefore \quad 0 * y = 0 * z.$$

(ii) If $y * x \leq z * x$, by BCI-5,

$$(y * x) * (z * x) = 0.$$

BCI-1 and (1) give

$$((y * x) * (z * x)) * (y * z) = 0$$

so

$$0 * (y * z) = 0 \qquad \text{-(b)}$$

giving, as above,

$$0 * y \leq 0 * z.$$

As in (i) this gives $0 * y = 0 * z$.

Corollary If $\langle X; *, 0\rangle$ is a BCI-algebra and $x, y, z \in X$ then

(i) $x * y = x * z \quad \Rightarrow \quad 0 * y = 0 * z$

(ii) $y * x = z * x \quad \Rightarrow \quad 0 * y = 0 * z$.

We have two further properties resulting from the above cancellation laws:

Theorem 2 If $\langle X; *, 0\rangle$ is a BCI-algebra and $x, y, z \in x$ then:

(i) $x \leq x * z \quad \Rightarrow \quad 0 \leq z$

(ii) $x * y \leq x \quad \Rightarrow \quad 0 \leq y$.

Proof (i) If $x \leq x * z$, by (3) $x * 0 \leq x * z$ and so by Theorem 1 (i) $0 * z = 0 * 0$. This gives $0 * z = 0$ ie $0 \leq z$.

(ii) If $x * y \leq x$, by (3), $x * y \leq x * 0$ and so by Theorem 1 (ii) $0 * y = 0 * 0 = 0$, so $0 \leq y$.

3. P-Semisimple Algebras. These were introduced by Lei and Xi in [2] as follows:

Definition 2 A BCI-algebra $\langle X; *, 0\rangle$ is p-semisimple if

$$(\forall x \in X)(0 * x = 0 \quad \Rightarrow \quad x = 0).$$

In these algebras we find that $\leq$ becomes the same as $=$.

Theorem 3 If $\langle X; *, 0\rangle$ is a p-semisimple BCI-algebra and $x, y \in X$ then if $x \leq y$ also $x = y$.

Proof If $x \leq y$, $x * y = 0$ by BCI-5. Also by (5), $x * y = x * x$, so by the corollary to Theorem 1, $0 * y = 0 * x$.

As $(0 * x) * (0 * x) = 0$, we have $(0 * y) * (0 * x) = 0$ and by (2), $0 * (y * x) = 0$.

As BCI-algebras are closed under $*$, $y * x \in X$, so if the algebra is p-semisimple, $y * x = 0$.

By BCI-4, $x = y$.

Our cancellation laws can now be strengthened.

Theorem 4 If $\langle X; *, 0\rangle$ is a p-semisimple BCI-algebra and $x, y, z \in X$ then:

(i) $x * y \leq x * z \quad \Rightarrow \quad y = z$;

(ii) $y * x \leq z * x \quad \Rightarrow \quad y = z$.

Proof (i) If $x * y \leq x * z$, by Theorem 1(i) we get $0 * z = 0 * y$ and so $(0 * z) * (0 * y) = 0$. By (2) this gives $0 * (z * y) = 0$, so if the algebra is p-semisimple we have $z * y = 0$ i.e. $z \leq y$. The result then follows from Theorem 3.

(ii) Similar.

Corollary If $\langle X;*,0\rangle$ is a p-semisimple BCI-algebra and $x,y,z \in X$ then
(i) $x*y = x*z \Rightarrow y = z$;
(ii) $y*x = z*x \Rightarrow y = z$.

4. Atoms. Meng and Xin in [5] introduced the notion of atom and the class of all atoms of a BCI-algebra.

Definition 3 An element of a BCI-algebra $\langle X;*,0\rangle$ is an atom if

$$(\forall z \in X)(z*a = 0 \quad \Rightarrow \quad z = a)$$

Definition 4 $L(X) = \{x \in X \mid a \text{ is an atom of } X\}$

Meng and Xin prove in [5]:

Theorem 5 If $\langle X;*,0\rangle$ is a BCI-algebra then
(i) a is an atom iff $a = 0*(0*a)$;
(ii) $(\forall x \in X)\ \ 0*x \in L(X)$.
((ii) also follows from (4) and (i).)

The following simple representation of $L(X)$ results:

Theorem 6 $L(X) = \{0*x \mid x \in X\}$.

Meng and Xin prove that $L(X)$ is a BCI-algebra. The following result of Lei and Xi [2]:

Theorem 7 If $\langle X;*,0\rangle$ is a BCI-algebra then X is p-semisimple iff $(\forall x \in X)\ \ 0*(0*x) = x$.

and Theorem 5(i) give us:

Theorem 8 If $\langle X;*,0\rangle$ is a BCI-algebra $\langle L(X);*,0\rangle$ is a p-semisimple BCI-algebra.

A final result on $L(X)$ is the following:

Theorem 9 If $\langle X;*,0\rangle$ is a BCI-algebra then $L(L(X)) = L(X)$.

Proof By Theorem 6,

$$\begin{aligned} L(L(X)) &= \{0*x \mid x \in L(X)\} \\ &= \{0*(0*y) \mid y \in X\} \end{aligned}$$

Similarly

$$L(L(L(X))) = \{0*(0*(0*z)) \mid z \in X\},$$

so by (4)

$$L(L(L(X))) = L(X).$$

Hence as $L(L(L(X))) \subseteq L(L(X)) \subseteq L(X)$ we have $L(L(X)) = L(X)$.

5. Powers. In [2] Lei and Xi define a new operation + by:

Definition 5 $x + y = x*(0*y)$

and show that if $\langle X;*,0\rangle$ is a p-semisimple BCI-algebra then $\langle X,+\rangle$ is an abelian group.

In [3] Meng and Wei use the same operation to define powers of elements by:

$$\begin{aligned} x^1 &= x \\ x^{n+1} &= x*(0*x^n), \end{aligned}$$

(though mx instead of x^m might have been in better keeping with +).

The following are new properties of this form of exponentiation:

Theorem 10 If x is an element of a BCI-algebra $\langle X;*,0\rangle$ then:
(i) $(0*x)^n = 0*x^n$;
(ii) $(0*x)^n = (...((0*x)*x)...)*x$
(where there are n xs on the right hand side).

Proof (i) By induction on n.

$n = 1$ - obvious.
Assuming (i) for n,

$$\begin{aligned}(0*x)^{n+1} &= (0*x)*(0*(0*x)^n) \\ &= (0*x)*(0*(0*x^n)) && \text{-(c)} \\ &= 0*(x*(0*x^n)) && \text{by (2)} \\ &= 0*x^{n+1}\end{aligned}$$

(ii) By induction on n.
$n = 1$ - obvious.
Assuming (ii) for n, by (c) above, (1) and (4):

$$\begin{aligned}(0*x)^{n+1} &= (0*(0*(0*x^n)))*x \\ &= (0*x^n)*x \\ &= (0*x)^n*x && \text{by (i)} \\ &= (\dots((0*x)*x)\dots)*x.\end{aligned}$$

as required.

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