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ON COMPLETE AND STRONGLY STONIAN MV-ALGEBRAS

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Abstract. We solve positively a conjecture of L. P. Belluce by using the notion of singular element of an MV-algebra. This concept implies a decomposition theorem for complete MV-algebras, formally analogous to that one for lattice-ordered complete groups. We also prove that strongly stonian MV-algebras correspond, via the well known functor , , to lattice-ordered Abelian groups with strong unit which are strongly projectable.

1. Introduction. After Mundici [1], the theory of MV-algebras of Chang [2] turned out to be intimely connected with lattice-ordered Abelian groups with strong unit, AF C^*-algebras [1], fuzzy set theory [3], [4], just to cite some notewhorty applications.

We recall that an MV-algebra $A = (A, \oplus, \odot, -0, 1)$ is a system such that $(A, \oplus, 0)$ is an Abelian monoid, $x \oplus 1 = 1$, $\bar{\bar{x}} = x$, $\bar{0} = 1$, $x \odot y = \overline{(\bar{x} + \bar{y})}$, $\overline{(\bar{x} \oplus y)} \oplus y = \overline{(\bar{y} \oplus x)} \oplus x$ for all x, $y \in A$.

A is said complete iff the underlying bounded distributive lattice $(A, \vee, \wedge, \leq)$, defined via the stipulations $x \vee y = (x \odot \bar{y}) \oplus y$, $x \wedge y = (x \oplus \bar{y}) \odot y$ and $x \leq y$ iff $x \wedge y = x$ for all x, $y \in A$, is complete. For brevity, we write $x \odot y = xy$ for all x, $y \in A$ from now on.

For all the unexplained notions on MV-algebras and lattice-ordered groups, we refer to [2], [1] and [7], [8], respectively.

The following is due to Lacava [5] (a full functorial description is given in [1]):

Proposition 1. *Let G be an Abelian lattice-ordered group with strong unit u and $A = [0, u]$ be the unit interval of G. For each x, $y \in A$ define $x \oplus y = (x+y) \wedge u$, $x \odot y = 0 \vee (x+y-u)$, $\bar{x} = u - x$, $u = 1$, (here $+$, $\vee$, $\wedge$ are operations in G). Then A is an MV-algebra denoted by $A =$, (G, u) and the lattice operations on A agree with those of G. Conversely, let A be an MV-algebra. Then there exists (up to isomorphisms) an Abelian lattice-ordered group G with strong unit u such that $A =$, (G, u).*

From now on we tacitly use this proposition. The functor , is a categorical equivalence [1], thus we rely on its properties in order to solve an open question of L. P. Belluce [4], already solved in other way by U. Höhle [9], using properly MV-machinery. After the concept of singular MV-algebra, we show that a complete MV-algebra is direct product of a divisible MV-algebra and a singular MV-algebra, obtained from the analogous result of lattice-ordered groups using , . We also prove that strongly stonian MV-algebras and Abelian lattice-ordered groups with strong unit which are strongly projectable, correspond via , .

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2. Definition and results. All the definitions used here draw inspiration from the analogous ones for lattice-ordered groups.

A positive element s of an MV-algebra A is said singular iff for each $x \in A$, $x \leq s$, then $x \wedge s\bar{x} = 0$ ([8], Def. 6.9).

Theorem 2. *Let $s \in A - \{0\}$ and $A = \Gamma(G, u)$. Then s is singular in A iff s is singular in G.*

Proof. For $x \leq s$, we have $s\bar{x} = 0 \vee (s + u - x - u)$. Then the thesis is immediate since $x \wedge s\bar{x} = 0$ iff $x \wedge (s - x) = 0$ for all $x \leq s$. □

We call singular an MV-algebra iff each element $a \in A - \{0\}$ majorizes a singular element cfr. ([7], 11.2.4) or ([8], Def. 54.2).

Lemma 3. *Let $A = \Gamma(G, u)$. Then A is singular iff G is singular.*

Proof. Let A be singular and $g > 0$ be an element of G. If $a = g \wedge u$, then $a > 0$ since u is also a weak unit for G ([8], Prop. 54.19). Since $a \in A$, there exists a singular element s in A such that $0 < s \leq a \leq g$. s is also singular in G by Theorem 2 and hence G is singular. The converse is trivial by Theorem 2. □

Following ([4], Prop. 15), we say that an MV-algebra A is strongly atomless iff for each $a \in A - \{0\}$, there is $x \in A$, $0 < x < a$, with $x \wedge a\bar{x} > 0$.

Following [6], an MV-algebra A is divisible iff for each $a \in A - \{0\}$ and integer $n > 0$, there exist a unique least element $b \in A$ such that

$$\underbrace{b \oplus b \oplus \ldots \oplus b}_{n-\text{times}} = a \quad \text{and} \quad a \cdot (\underbrace{\bar{b} \odot \bar{b} \ldots \bar{b}}_{(n-1)-\text{times}}) = b.$$

It is known that A is injective iff A is complete and divisible [6]. L. P. Belluce ([4], Cor. 2) proves that A injective implies A complete and strongly atomless, conjecturing that the converse of this implication is also true. We confirm this conjecture proving the following:

Theorem 4. *Let A be a complete MV-algebra. Then A is strongly atomless iff A is divisible.*

Proof. Let G be such that $A = \Gamma(G, u)$. A is strongly atomless iff G has not singular elements by Theorem 2. Further, A is divisible iff G is divisible ([5], Prop. 1.2) and A is complete iff G is complete ([10], Thm. 3.1). Thus the thesis follows from ([7], 11.2.13) or ([8], Thm. 54.13). □

3. Strongly stonian MV-algebras. In this Section we show that strongly stonian MV-algebras and Abelian lattice-ordered groups with strong unit which are strongly projectable are correspondent via the functor Γ. In order to get this result, we first recall some related definitions.

An MV-algebra A is strongly stonian iff for each each subset $\emptyset \neq X \subseteq A$, $X^{\perp_A} = \text{id}(e)$ for some $e \in B(A)$, where $B(A) = \{b \in A \ : \ b \oplus b = b\}$ is the Boolean subalgebra of A, $X^{\perp_A} = \{x \in A \ : \ x \wedge a = 0 \ \forall y \in X\}$ is the polar of X in A and $\text{id}(e) = \{x \in A \ : \ x \leq e\}$ is the ideal of A generated by e.

If $\text{id}(X)$ denotes the ideal generated by X in A and since $(\text{id}(X))^{\perp_A} = X^{\perp_A}$ [3], the present definition is clearly equivalent to that one given in [4]. For sake of completeness, we remember that a lattice-ordered group G is strongly projectable iff any polar P in G is a cardinal summand, i.e. $G = P \times P^{\perp_G}$ ([8], Def. 18.1), "$\times$" being the internal direct product ([7], 3.5.5).

Let $A = \,, (G,u)$ and $\emptyset \neq X$ be a subset of $[0,u]$. If $X^{\perp_G} = \{g \in G \,:\, |g| \wedge x = 0 \;\forall x \in X\}$ is the polar of X in G, evidently $X^{\perp_A} = X^{\perp_G} \cap [0,u]$. Now we prove that

Theorem 5. *Let $A = \,, (G,u)$. Then G is strongly projectable iff A is strongly stonian.*

Proof. Let G be strongly projectable and $\emptyset \neq X$ be a subset of $[0,u]$. By ([8], Thm. 18.5), the set $X^{\perp_A} = X^{\perp_G} \cap [0,u]$ has a supremum e in G, which is also the supremum of $X^{\perp_A}$ in A. Indeed, $e \in X^{\perp_A}$ as it is easily seen and clearly $e \in B(A)$.

Conversely, let A be strongly stonian and let $g > 0$, $X \subseteq [0,g]$. By ([8], Thm. 18.5), we must show that the set $Y = X^{\perp_G} \wedge [0,u]$ has supremum in G. By ([1], Prop. 3.1), let $g_1, g_2, \ldots, g_n \in A$ be such that $g = g_1 + g_2 + \ldots + g_n$. Let $y \in Y$ and $y_1, y_2, \ldots, y_m \in A$ such that $y = y_1 + y_2 + \ldots + y_m$. Without loss of generality, we can assume $m = n$.

Since $y \leq g$, then $y_i \leq g_i$ for any $i = 1, 2, \ldots, n$ by ([1], Prop. 3.1). Let $x \in X$ and similarly as before, let $x_1, x_2, \ldots, x_n \in A$ be such that $x = x_1 + x_2 + \ldots, x_n$, $x_i \leq g_i$, for any $i = 1, 2, \ldots, n$. Define the sets $X_i = \{x_i \in [0, g_i] \,:\, x \in X\} \subseteq [0,u]$. Since $y \wedge x = 0$ for all $x \in X$, we have $y_i \wedge x_i \leq y \wedge x = 0$ for all $x_i \in X_i$ and $i = 1, 2, \ldots, n$. This means $y_i \in X^{\perp_{A_i}} = \mathrm{id}(e_i)$ for some $e_i \in B(A)$. By setting $e = e_1 + e_2 + \ldots + e_n$, we have $y \leq e$ for all $y \in Y$. Clearly $e \leq g$ since, for any $i = 1, 2, \ldots, n$, g_i is an upper bound for $X^{\perp_{A_i}}$. By ([1], Prop. 3.1), $e \wedge x = (e_1 \wedge x_1) + \ldots + (e_n \wedge x_n) = 0$ for all $x \in X$, i.e. $x \in X^{\perp_G}$ and $e \in A$. Clearly e is the supremum of Y in G. □

Since a complete lattice-ordered group is strongly projectable ([8], Prop. 5.4) or ([7], 11.2.4), then any complete MV-algebra is strongly stonian by the above theorem (cfr. [4], Prop. 18).

4. A decomposition theorem. In this Section we prove a decomposition theorem for complete MV-algebras using the following well known lemma (cfr., e.g., ([11], Lemmas 3.1 and 3.2] involving the functor , .

Lemma 6. *Let G be an Abelian lattice-ordered group with strong unit u and $G = G_1 \times G_2$. If u_i is the component of u in G_i, then u_i is a strong unit of G_i, $i = 1, 2$. Further, $A = A_1 \times A_2$, where $A = \,, (G,u)$, $A_i = \,, (G_i, u_i)$ for $i = 1, 2$.*

Now we are in position to prove the following:

Theorem 7. *Let A be a complete MV-algebra. Then $A = A_1 \times A_2$, where A_1 (resp. A_2) is a complete divisible (resp. singular) MV-algebra.*

Proof. Let $A = \,, (G,u)$. G is complete ([10], Thm. 3.1), then $G = G_1 \times G_2$, where G_1 (resp. G_2) is complete and divisible (resp. singular) by ([8], Thm. 54.14) or ([7], 11.2.15). Further, if $u = u_1 + u_2$, then u_i is a strong unit for G_i, $i = 1, 2$, by Lemma 6 which implies also that $A = A_1 \times A_2$, where $A_1 = \,, (G_1, u_1)$ (resp. $A_2 = \,, (G_2, u_2)$) is complete and divisible (resp. singular) by ([5], Prop. 12) (resp. Lemma 3). □

An exact description of the MV-algebras A_1 and A_2 can be obtained recalling that $G_1 = S^{\perp_G}$ and $G_2 = S^{\perp_G \perp_G}$, where S is the set of singular elements of G ([8], Thm. 54.14). The following lemma is useful for this description:

Lemma 8. *Let $A = \,, (G,u)$ and $s' \in S$. Then s' is singular in A iff $s' = s \wedge u$ with $s \in S$.*

Proof. If $s' = s \wedge u$ with $s \in S$, then $s' > 0$ is a minorant of s, hence $s' \in S$ by ([7], 11.2.9) or ([8], Cor. 54.8). Then s' is singular in A by Theorem 2. The converse is trivial. □

In virtue of this Lemma, the set $T = \{s \wedge u \ : \ s \in S\} \subseteq [0,u]$ coincides with the singular elements of A. Clearly $T^{\perp_G} = S^{\perp_G}$ and then $T^{\perp_A} = S^{\perp_G}$ which implies $A = (S^{\perp_G} \times S^{\perp_G \perp_G}) \cap [0,u] = (S^{\perp_G} \cap [0,u]) \times (S^{\perp_G \perp_G} \cap [0,u]) = T^{\perp_A} \times T^{\perp_A \perp_A}$, i.e. $A_1 = T^{\perp_A}$, $A_2 = T^{\perp_A \perp_A}$. Since A is strongly stonian, we have $A_1 = \mathrm{id}(u_1)$, $A_2 = \mathrm{id}(\bar{u}_1)$ since $u_2 = u - u_1$ and $u_1 \in B(A)$.

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