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ON COMPLETELY REGULAR ORDERED SEMIGROUPS

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Abstract. We wrote this paper in an attempt to show the way we pass from the results on ordered semigroups based on ideals to the results on semigroups -without order- based on ideals, and conversely. We tried to use sets instead of elements in the proof of our results as an example to show that in the theory of semigroups -without order- based on ideals, elements do not play any role but the sets. Besides, the results of semigroups -without order- based on ideals can be also obtained either by an easy modification of the results on ordered semigroups (by setting A instead of $(A]$) or as an application of the results on ordered semigroups in the way indicated in this paper.

Completely regular *poe*-semigroups (: ordered semigroups having a greatest element) have been considered by N. Kehayopulu in [3]. The study of *poe*-semigroups gives information about ideal elements, as well. Thus, for example, a *poe*-semigroup S is completely regular if and only if every bi-ideal (resp. bi-ideal element) of S is semiprime. Completely regular *po*-semigroups (: ordered semigroups) have been considered by Sang Keun Lee and Young In Kwon in [14]. As a continuation of the paper by N. Kehayopulu in [3], in an attempt to show the similarity between the theory of semigroups based on ideals and the theory of ordered semigroups based on ideals, we examine here the results given by O. Steinfeld in [16], for ordered semigroups. We also refer the reader to the introduction of [8]. Sang Keun Lee and Young In Kwon based their results on quasi-ideals defined as non-empty subsets Q of S satisfying 1) $QS \cap SQ \subseteq Q$. 2) $a \in Q, \quad S \ni b \leq a \ \Rightarrow \ b \in Q$. This is the definition given by N. Kehayopulu in [3] while later Kehayopulu and Tsingelis changed that definition as follows: A non-empty subset Q of S is called a quasi-ideal of S if and only if 1) $(QS] \cap (SQ] \subseteq Q$. 2) $a \in Q, \quad S \ni b \leq a \ \Rightarrow \ b \in Q$. (Cf., for example, [17,9]). (For $H \subseteq S$, $(H] := \{t \in S \mid t \leq h$ for some $h \in H\}$ [1]). The reason they changed the first definition was that the quasi-ideals should be intersections of right and left ideals -while the first definition failed having this property. Jing Fengjie, independently, in January 1994, gave the same definition of quasi-ideals as a more general than the first one. We tried to use sets instead of elements in the proofs of our results to show that in the corresponding results on semigroups -without order- elements do not play any role, but the sets -which emphasizes the pointless character of such semigroups. The results of O. Steinfeld in [16], can be also obtained either by an easy modification of the results of this paper by setting A instead of $(A]$ or as an application of the results of this paper in the way indicated in this paper. For the way we work to apply the results on ordered semigroups based on ideals to semigroups -without order- we also refer to [12] (cf. also [1,4,5]). A semigroup $(S, .)$ is called completely regular if for every $a \in S$ there exists $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup $(S, .)$ is completely regular if and only if it is regular, left regular and right

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regular (cf. e.g. [15]). Moreover, a semigroup $(S, .)$ is completely regular if and only if it is left regular and right regular (cf. [15; IV.1.2. Proposition]). An ordered semigroup $(S, ., \leq)$ is called completely regular if it is regular, left regular and right regular [3]. O. Steinfeld uses the term "completely semiprime" instead of "semiprime" given by A. H. Clifford. Our terminology is always the same with A. H. Clifford. A semigroup $(S, .)$ is completely regular if and only if the quasi-ideals of S are semiprime [16]. This is not the case for ordered semigroups: The example of the paper shows that there exist ordered semigroups, which are not completely regular, in which the quasi-ideals are semiprime. An ordered semigroup S is completely regular if and only if the bi-ideals of S are semiprime (cf. also [3; Theorems 2,3]). Although for ordered semigroups, the characterization is by means of bi-ideals (instead of quasi-ideals) the Theorem 2 in [16] can be also obtained as application of the Proposition 2 of this paper. On the other hand, an ordered semigroup S is left regular and right regular if and only if the quasi-ideals of S are semiprime. The completely regular ordered semigroups -as intra-regular ordered semigroups- are decomposable into their simple subsemigroups [6].

An ordered semigroup S is called regular (resp. intra-regular) if $A \subseteq (ASA]$ (resp. $A \subseteq (SA^2S])$ for every $A \subseteq S$ [4,6]. S is called left (resp. right) regular if $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S])$ for every $A \subseteq S$ [2]. S is called completely regular if it is regular, left regular and right regular (cf. [3]). The ordered semigroups which are left (resp. right) regular (and so the completely regular ordered semigroups, as well), are intra-regular. Indeed: If $A \subseteq S$ and $A \subseteq (SA^2]$, then

$A \subseteq (S(SA^2](SA^2]] = (S(SA^2)(SA^2)] \subseteq (SA^2S]$.

The definition of the regular, left-right regular subsemigroups of S are the same as in [3]. A non-empty subset Q of S is called a quasi-ideal of S if 1) $(QS] \cap (SQ] \subseteq Q$. 2) $a \in Q,\ S \ni b \leq a \ \Rightarrow\ b \in Q$ [17,9]. A non-empty subset B of S is called a bi-ideal of S if 1) $BSB \subseteq B$. 2) $a \in B,\ S \ni b \leq a \ \Rightarrow\ b \in B$ [3]. A subset T of an ordered semigroup S is called semiprime if for every $A \subseteq S$ such that $A^2 \subseteq T$, we have $A \subseteq T$. Equivalently, if $a \in S$, $a^2 \in T$ implies $a \in T$ [2]. A non-empty subset A of an ordered semigroup S is called a left (resp. right) ideal of S if 1) $SA \subseteq A$ (resp. $AS \subseteq A$). 2) $a \in A, S \ni b \leq a \Rightarrow b \in A$. A is called an ideal of S if it is both a left and a right ideal of S [1]. An ordered semigroup S is regular (resp. intra-regular) if and only if every ideal of S is a regular (resp. intra-regular) subsemigroup of S [12]. Equivalently, if, for every $A \subseteq S$, the ideal $r(\ell(A))$ of S generated by A is a regular (resp. intra-regular) subsemigroup of S. Equivalently, if, for every $a \in S$, the ideal $r(\ell(a))$ of S generated by a is a regular (resp. intra-regular) subsemigroup of S. An ordered semigroup S is left (resp. right) regular if and only if every left (resp. right) ideal of S is a left (resp. right) regular subsemigroup of S. Equivalently if, for every $a \in S$, the left ideal $\ell(a)$ (resp. the right ideal $r(a)$) of S generated by a is a left (resp. right) regular subsemigroup of S. Indeed: Let S be left regular, L a left ideal of S. Then L is a subsemigroup of S. Let $A \subseteq L$. Then

$$A \subseteq (SA^2] \subseteq (S(SA^2]A] = (S(SA^2)A] \subseteq (SA^2A] \subseteq ((SL)A^2] \subseteq (LA^2].$$

Proposition 1. *An ordered semigroup S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S.*

Proof. $\Longrightarrow$. Let Q be a quasi-ideal of S. Since $\emptyset \neq Q \subseteq S$ and

$Q^2 \subseteq QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, Q is a subsemigroup of S.

Let $A \subseteq Q$ $\quad(\Longrightarrow \quad A \subseteq (AQA],\ \ A \subseteq (QA^2],\ \ A \subseteq (A^2Q] \quad ?\,)$

We have

$A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] \subseteq (A^2SA^2] = (A(ASA)A]$,

$ASA \subseteq QSQ \subseteq QS \cap SQ \subseteq Q$.

$$\begin{aligned} A &\subseteq (ASA] \subseteq (AS(SA^2)] \subseteq (ASA^2] \subseteq (AS(SA^2]A] \\ &= (AS(SA^2)A] \subseteq ((ASA)A^2] \subseteq (QA^2]. \\ A &\subseteq (ASA] \subseteq ((A^2S]SA] \subseteq (A^2SA] \subseteq (A(A^2S]SA] \\ &= (A(A^2S)SA] \subseteq (A^2(ASA)] \subseteq (A^2Q]. \end{aligned}$$

$\Longleftarrow$. Since S is a quasi-ideal of S. □

The ordered semigroups in which the quasi-ideals are semiprime are not completely regular, in general. We show it by the following

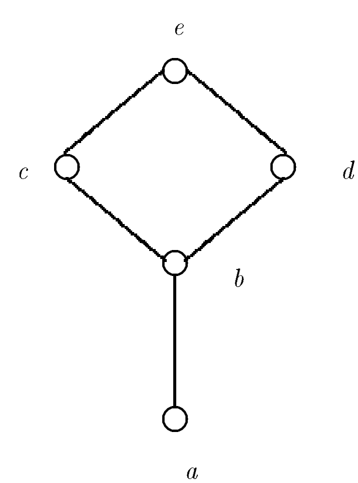
Example 1. We consider the ordered semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication and the order below:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\begin{aligned} \leq \ : \ = \ &\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c), \\ &(c,e),(d,d),(d,e),(e,e)\}. \end{aligned}$$

We give the covering relation "$\prec$" and the figure of S.

$$\prec \ = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}.$$



For an easy way to check that this is an ordered semigroup we refer to [7,11].
The quasi-ideals of S are the sets:

$\{a,b\}$, $\{a,b,c\}$, $\{a,b,d\}$ and S.

Every quasi-ideal of S is semiprime, but S is not completely regular (in fact, S is not regular).

Proposition 2. *An ordered semigroup S is left regular and right regular if and only if the quasi-ideals of S are semiprime.*

Proof. $\Longrightarrow$. Let Q be a quasi-ideal of S, $A \subseteq S$, $A^2 \subseteq Q$. Then
$A \subseteq (A^2S] \subseteq (QS]$, $A \subseteq (SA^2] \subseteq (SQ]$, $A \subseteq (QS] \cap (SQ] \subseteq Q$.
$\Longleftarrow$. Let $\emptyset \neq A \subseteq S$ $\quad(\Longrightarrow A \subseteq (SA^2],\ A \subseteq (A^2S]\ ?)$
We consider the quasi-ideal $q(A^2)$ of S generated by A^2. Since $A^2 \subseteq q(A^2)$, and $q(A^2)$ is semiprime, we have
$A \subseteq q(A^2) = r(A^2) \cap \ell(A^2)$ (cf. [10; Remark 1]).
Then $A \subseteq r(A^2) = (A^2 \cup A^2S]$ and $A \subseteq \ell(A^2) = (A^2 \cup SA^2]$ (cf. [10; Remark 1], also [13; p. 318]).
From $A \subseteq (A^2 \cup A^2S]$, we have $A^2 \subseteq (A^2 \cup A^2S](A] \subseteq (A^3 \cup A^2SA] \subseteq (A^2S]$,
then $A \subseteq (A^2 \cup A^2S] \subseteq ((A^2S] \cup A^2S] = ((A^2S]] = (A^2S]$.
From $A \subseteq (A^2 \cup SA^2]$, we have $A^2 \subseteq (A](A^2 \cup SA^2] \subseteq (A^3 \cup ASA^2] \subseteq (SA^2]$,
then $A \subseteq ((SA^2] \cup SA^2] = ((SA^2]] = (SA^2]$. $\square$

Remark. An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Indeed: $\Longrightarrow$. Let $A \subseteq S$. Then

$$A \subseteq (ASA] \subseteq ((A^2S]S(SA^2]] = ((A^2S)S(SA^2)] \subseteq (A^2SA^2].$$

$\Longleftarrow$. Let $A \subseteq S$. We have

$$A \subseteq (A^2SA^2] \subseteq (ASA],\ (A^2S],\ (SA^2]. \quad \square$$

Proposition 3. *An ordered semigroup S is completely regular if and only if the bi-ideals of S are semiprime.*

Proof. $\Longrightarrow$. Since S is regular, the bi-ideals and the quasi-ideals of S coincide (cf., for example, [8; Lemma 2]). The implication follows by Proposition 2.

$\Longleftarrow$. Let $\emptyset \neq A \subseteq S$. The set $(A^2SA^2]$ is a bi-ideal of S. Indeed:

$\emptyset \neq (A^2SA^2] \subseteq S$.

$$(A^2SA^2]S(A^2SA^2] = (A^2SA^2](S](A^2SA^2] \subseteq (A^2SA^2SA^2SA^2] \subseteq (A^2SA^2].$$

If $x \in (A^2SA^2]$ and $S \ni y \leq x$, then $y \in (A^2SA^2]$.
Since $A^8 \subseteq (A^2SA^2]$, by hypothesis, we have A^4, $A^2 \subseteq (A^2SA^2]$, and $A \subseteq (A^2SA^2]$. $\square$

Applications. We apply the Propositions 1,3 to semigroups -without order. *A semigroup $(S,.)$ is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S* [16]. In fact: $\Longrightarrow$. Let $(S,.)$ be a completely regular semigroup, Q a quasi-ideal of S. Let "$\leq$" be the relation on S defined by $\leq\ := \{(x,y) \mid x = y\}$. Then $(S,.,\leq)$ is an ordered semigroup and $(S,.,\leq)$ is completely regular. Q is a quasi-ideal of $(S,.,\leq)$. Indeed:
$\emptyset \neq Q \subseteq S$. $(QS] \cap (SQ] \subseteq Q$: Let $t \in (QS]$, $t \in (SQ]$. Then $t \leq q_1s_1$ for some $q_1 \in Q$, $s_1 \in S$, $t \leq s_2q_2$ for some $s_2 \in S$, $q_2 \in Q$. Since $(t, q_1s_1) \in\leq$ and $(t, s_2q_2) \in\leq$, we have $t = q_1s_1 = s_2q_2 \in QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$. If $q \in Q$, $S \ni t \leq q$, then

$t = q \in Q$. By Proposition 1, Q is a completely regular subsemigroup of $(S, ., \leq)$. Then Q is a completely regular subsemigroup of $(S, .)$. $\Longleftarrow$. Since S is a quasi-ideal of S.
A semigroup $(S, .)$ is completely regular if and only if every quasi-ideal of S is semiprime [16]. In fact:
$\Longrightarrow$. Let Q be a quasi-ideal of $(S, .)$, $a \in S$, $a^2 \in Q$ $(\Longrightarrow a \in Q\ ?\ )$
$(S, ., \leq)$ is completely regular, Q a quasi-ideal of $(S, ., \leq)$. Then Q is a bi-ideal of $(S, ., \leq)$. Indeed: $QSQ \subseteq (QS] \cap (SQ] \subseteq Q$. By Proposition 3, Q is semiprime. Since $a^2 \in Q$, we have $a \in Q$.
$\Longleftarrow$. Suppose each quasi-ideal of $(S, .)$ is semiprime ($\Longrightarrow (S, .)$ is completely regular ?)
$(S, ., \leq)$ is an ordered semigroup. We prove that every bi-ideal of $(S, ., \leq)$ is semiprime. Then, by Proposition 3, $(S, ., \leq)$ is completely regular. Then $(S, .)$ is completely regular.
Let B be a bi-ideal of $(S, ., \leq)$, $a \in S$, $a^2 \in B$ $(\Longrightarrow\ a \in B\ ?\ )$
Since $a^2 \in B$, we have $a^2 S a^2 \subseteq BSB \subseteq B$. We prove that $a \in a^2 S a^2$. Then we have $a \in B$.
$a \in a^2 S a^2$. In fact: $a^2 S \cap S a^2$ is a quasi-ideal of $(S, .)$. Indeed:
$\emptyset \neq a^2 S \cap S a^2 \subseteq S$ $(a^3 \in a^2 S \cap S a^2)$.
$(a^2 S \cap S a^2) S \cap S (a^2 S \cap S a^2) \subseteq a^2 S^2 \cap S^2 a^2 \subseteq a^2 S \cap S a^2$.
By hypothesis, $a^2 S \cap S a^2$ is semiprime. Since $a^4 \in a^2 S \cap S a^2$, we have $a^2 \in a^2 S \cap S a^2$, $a \in a^2 S \cap S a^2$. Then $a = a^2 x = y a^2$ for some $x, y \in S$. Then

$$\begin{aligned} a &= a(ax) = (a^2 x)(y a^2) x = a^2 (xy)(a^2 x) = a^2 (xy)(y a^2) \\ &= a^2 (x y^2) a^2 \in a^2 S a^2. \quad \square \end{aligned}$$

We finally give an example of completely regular semigroups.

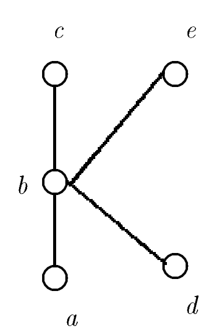
Example 2. The ordered semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication and the order below is a completely regular ordered semigroup.

$\cdot$	a	b	c	d	e
a	d	b	b	d	e
b	b	b	b	b	e
c	b	b	c	b	e
d	d	b	b	d	e
e	b	b	e	b	e

$$\begin{aligned} \leq: \ = \ &\{(a,a),(a,b),(a,c),(a,e),(b,b),(b,c),(b,e),(c,c),(d,b),(d,c), \\ &(d,d),(d,e),(e,e)\}. \end{aligned}$$

We give the covering relation "$\prec$" and the figure of S.

$$\prec\ = \{(a,b),(b,c),(b,e),(d,b)\}.$$



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