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A NOTE ON STRONGLY REGULAR ORDERED SEMIGROUPS

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Abstract. We examine the results concerning the completely regular semigroups given by O. Steinfeld in [7], for ordered semigroups. The concept of completely regular semigroups is generalized in ordered semigroups in two ways. The first one is the completely regular ordered semigroups studied previously by the same author. The second is the strongly regular ordered semigroups introduced in this paper. We have seen in [5] how we apply our results on completely regular ordered semigroups to get the corresponding results on semigroups -without order- given by O. Steinfeld in [7]. Here we show how we apply the results on strongly regular ordered semigroups to get the same results of O. Steinfeld.

An ordered semigroup is called completely regular if it is regular, left regular and right regular (cf. [1,5]). We introduce here the concept of strongly regular ordered semigroups which corresponds to the property "$a = axa$, $ax = xa$" of semigroups -without order. A semigroup S is regular, left regular and right regular if and only if for every $a \in S$ there exists $x \in S$ such that $a = axa$ and $ax = xa$. Such semigroups are called completely regular. We can get this characterization of completely regular semigroups as an easy consequence of the Proposition IV.1.2 in [6]. A very easy independent proof, due to M. Tsingelis, is the following: $\Longrightarrow$. Let $a \in S$. Since S is left and right regular, there exist $t, z \in S$ such that $a = ta^2 = a^2 z$. Then we have

$ata = at(a^2z) = a(ta^2)z = a^2z = a, \quad aza = (ta^2)za = t(a^2z)a = ta^2 = a.$

For the element $x := aztazta$ of S, we have

$axa = a(aztazta)a = (a^2z)taz(ta^2) = (ata)za = aza = a.$

$ax = a(aztazta) = (a^2z)tazta = (ata)zta = azta,$

$xa = (aztazta)a = aztaz(ta^2) = azt(aza) = azta.$

$\Longleftarrow$. It is obvious. Moreover, a semigroup S is completely regular if and only if it is left regular and right regular. The property "$a = axa, \ \ ax = xa$" is, naturally, extended to ordered semigroups by the property "$a \le axa, \ \ ax = xa$".

The left (resp. right) regular ordered semigroups are intra-regular [5]. While the property "$a = axa, \ \ ax = xa$" characterizes the completely regular semigroups (without order), the ordered semigroups having the corresponding condition and the completely regular ordered semigroups are two different concepts. This leads us to introduce the concept of strongly regular ordered semigroups. O. Steinfeld proved in [7] that a semigroup S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S. The same result for ordered semigroups also holds: An ordered semigroup S is completely regular if and only if every quasi-ideal of S is a completely regular subsemigroup of S [5]. The following question is natural: Can the strongly regular ordered semigroups be characterized at the same way ? We give an example of strongly regular ordered semigroups before

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answering this question. The completely regular ordered semigroups having a greatest element "e" (: *poe*-semigroups) have been considered by the same author in [1]. The study of completely regular *poe*-semigroups gives information about ideal elements, as well. This paper is a continuation of [5]. For the necessary definitions we refer to [5]. O. Steinfeld uses the term "completely semiprime" instead of "semiprime" given by A. H. Clifford. Our terminology is always the same with the terminology given by A.H. Clifford.

Definition. An ordered semigroup $(S, ., \leq)$ is called *strongly regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$ and $ax = xa$.

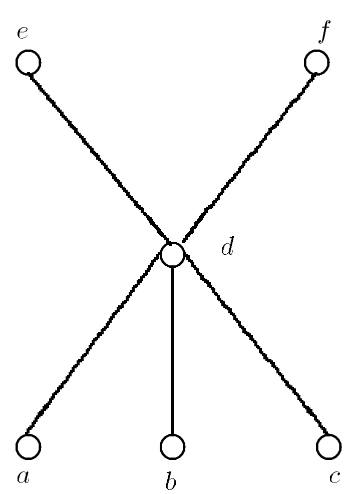
Example. We consider the ordered semigroup $S = \{a, b, c, d, e, f\}$ defined by the following multiplication, order.

.	a	b	c	d	e	f
a	b	c	d	d	d	d
b	c	d	d	d	d	d
c	d	d	d	d	d	d
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f

$$\begin{aligned} \leq \ := \ & \{(a,a),(a,d),(a,e),(a,f),(b,b),(b,d),(b,e),(b,f),(c,c),(c,d),\\ & (c,e),(c,f),(d,d),(d,e),(d,f),(e,e),(f,f)\}. \end{aligned}$$

We give the covering relation " $\prec$ " and the figure of S.

$\prec = \{(a,d),(b,d),(c,d),(d,e),(d,f)\}$.



This is strongly regular. For an easy way to check that this is an ordered semigroup we refer to [2,4].

Proposition. *An ordered semigroup $(S,.,\leq)$ is strongly regular if and only if every quasi-ideal of S is a strongly regular subsemigroup of S.*

Proof. $\Longrightarrow$. Let Q be a quasi-ideal of S. Since $\emptyset \neq Q \subseteq S$ and
$Q^2 \subseteq QS \cap SQ \subseteq (QS] \cap (SQ] \subseteq Q$, Q is a subsemigroup of S.
Let $a \in Q$ $(\Longrightarrow \exists\ x \in Q\ :\ a \leq axa$ and $ax = xa$?)
Since $a \in S$, S strongly regular, there exists $y \in S$ such that $a \leq aya$ and $ay = ya$. We put $x := yay$. Then we have
$x = (ya)y = (ay)y = ay^2 \in QS \subseteq (QS]$,
$x = y(ay) = y(ya) = y^2a \in SQ \subseteq (SQ]$.
Then $x \in (QS] \cap (SQ] \subseteq Q$. Moreover,
$a \leq aya \leq ay(aya) = a(yay)a = axa$, and
$ax = a(yay) = (ya)(ay) = (yay)a = xa$.
$\Longleftarrow$. Since S is a quasi-ideal of S.

Remark 1. An ordered semigroup S is strongly regular if and only if, for every $x \in S$, the quasi-ideal $q(x)$ of S generated by x is a strongly regular subsemigroup of S. Indeed: Suppose $q(x)$ is a strongly regular subsemigroup of S for every $x \in S$, and let $a \in S$. Since $q(a)$ is strongly regular, there exists $x \in q(a) \subseteq S$ such that $a \leq axa$ and $ax = xa$. Hence S is strongly regular.

Remark 2. As an application of the Proposition of this note, we get the Theorem 1 in [7] in the following way. Besides, the Theorem 1 in [7] can be also obtained as an easy modification of the proof of the same Proposition.

Let $(S,.)$ be a completely regular semigroup, Q a quasi-ideal of $(S,.)$. Let " $\leq$ " be the order relation on S defined by $\leq := \{(x,y) \mid x = y\}$. Then $(S,.,\leq)$ is a strongly regular ordered semigroup, and Q is a quasi-ideal of $(S,.,\leq)$. By the Proposition, Q is a strongly regular subsemigroup of $(S,.,\leq)$. Then Q is a completely regular subsemigroup of $(S,.)$. Indeed: Let $a \in Q$. Then there exists $x \in Q$ such that $a \leq axa$ and $ax = xa$. Then $a = axa$ and $ax = xa$.

As a conclusion: The concept of completely regularity defined in semigroups -without order- is generalized in ordered semigroups in two ways. In ordered semigroups we have the concept which corresponds to the concept "the semigroup is regular, left regular and right regular" of semigroups, and the concept which corresponds to the property "$a = axa$, $ax = xa$" of semigroups. Thus we have the concept of completely regular ordered semigroups and the concept of strongly regular ordered semigroups. The strongly regular ordered semigroups are completely regular.

Problem 1. Find an example of a completely regular ordered semigroup which is not strongly regular. One could search for such an example of an ordered semigroup of finite order (using Computer).

Remark 3. In strongly regular ordered semigroups the bi-ideals and the quasi-ideals coincide. In fact, the strongly regular ordered semigroups are regular, and in regular ordered semigroups the bi-ideals and the quasi-ideals coincide (cf. [3; the Corollary]). Concerning the analogous of the Theorem 2 in [7]: Let S be a strongly regular ordered semigroup. Then since S is completely regular and regular, by the Proposition 3 in [5], the quasi-ideals of S are semiprime. Conversely, if S is an ordered semigroup and if the quasi-ideals of S are semiprime, then S is not strongly regular, in general This is a consequence of the Example 1 in [5]. The following Problem is natural:

Problem 2. Using Computer, find an example of a finite ordered semigroup S which is not strongly regular and the bi-ideals of S are semiprime. Then, this is also an example of an ordered semigroup which is completely regular and not strongly regular (cf. [5; Proposition 3]).

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