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CONTINUOUS DIFFERENTIABILITY AND PARTIAL DERIVATIVES IN (CUN) SPACES

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Abstract. Let X and Y be (CUN) spaces satisfying the condition (p) of S. Nakanishi. The space $\mathcal{L}(X;Y)$ consisting of all continuous linear mappings of X into Y can be treated as a (UCs-N) space under the condition that: each component space (X_m, p_m) of X is locally compact and $X_m \subsetneqq X_{m+1}$ for each $m \in N$ and $Y_n \subsetneqq Y_{n+1}$ for each $n \in N$. The main result of this paper is to show that a mapping of two-variables in (CUB) spaces is continuously differentiable if and only if its partial derivatives are continuous.

1. Introduction. In [5], Prof. S. Nakanishi showed that the analogy of Dieudonné results [1, VIII, 1, 2, 4, 5, 7, 14] in the Banach space also holds in the (CUB) space satisfying the condition (p). In this paper, we study Dieudonné results as a continuation of [5].

In [5], Nakanishi showed that the space $\mathcal{L}(X;Y)$ consisting of all continuous linear mappings of X into Y can be treated as a (CUN) space when X is a locally compact normed space and Y is a (CUN) space satisfying (p). In Section 3, it is shown that, if X and Y are (CUN) spaces satisfying (p), then the space $\mathcal{L}(X;Y)$ can be treated as a (UCs-N) space under the condition that: each component space (X_m, p_m) of X is locally compact and $X_m \subsetneqq X_{m+1}$ for each $m \in N$ and $Y_n \subsetneqq Y_{n+1}$ for each $n \in N$. In Section 4 , we study the continuous differentiability for a mapping in (CUB) spaces. It may be defined as the continuity of the derivative in each component space. We study the continuous differentiability for a mapping of two-variables in (CUB) spaces.

2. Preliminaries. Let us recall the definition of (CUN) spaces and the condition (p) ([5]). Let X be a real or complex vector space, and $X_n (n = 0, 1, \cdots)$ a sequence of vector subspaces of X such that:

(I) $\bigcup_{n=0}^{\infty} X_n = X$.

(II) $X_n \subset X_m$ if and only if $n \leqq m$.

Suppose that in each X_n there is defined a norm p_n in such a way that

(III) if $n \leqq m$, then $p_n(x) \geqq p_m(x)$ for every $x \in X_n$.

For such a collection X_n and $p_n (n = 0, 1, \cdots)$, we define neighborhoods on X as follows: Corresponding to $x \in X$ and $\varepsilon > 0$, a *neighborhood* $V(x, n, \varepsilon)$ *of* x is defined by

$$V(x, n, \varepsilon) = \{y \in X_n : p_n(x - y) < \varepsilon\}$$

for every n with $x \in X_n$ and only for such n. In particular, we denote the neighborhood $V(x, n, 1/2^m)$ by $V(x, n, m)$ etc., and a neighborhood $V(x, n, m)$ in which the third index is m is said to be of *rank* m.

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The space X endowed with these neighborhoods and ranks becomes a ranked vector space. Such a ranked vector space X is called a *ranked countable union space of normed spaces*, or simply a (CUN) *space*. Each normed space (X_n, p_n) is called a *component space* of the (CUN) space X.

(IV) Each component space (X_n, p_n) is complete.

The (CUN) space satisfying the condition (IV) is called a *ranked countable union space of Banach spaces*, or simply a (CUB) *space*.

For a (CUN) space $(X; \{(X_n.p_n)\})$, we consider the following condition (p).

(p) For every n, every bounded set in the normed space (X_n, p_n) is relatively compact as a subset of the normed space (X_m, p_m) for every $m > n$.

Other fundamental terminologies and notations for ranked spaces are referred to [2],[3] and [5].

Lemma 1. *Let* $(X; \{(X_m, p_m)\})$, $(Y; \{(Y_n, q_n)\})$ *be* (CUN) *spaces satisfying* (p). *Let* f *be a continuous mapping into* Y *of an* r*-open set* A *in* X. *Suppose that* $B(\subset A)$ *is an open set in* (X_m, p_m). *Then, if* $x_0 \in B$, *there are an open ball* U *of center* x_0 *in* (X_m, p_m) *with* $U \subset B$ *and an* $n \in N$ *such that:*

(1) *The image* $f(U)$ *is contained in* Y_n.

(2) f *is continuous in* U *as a mapping of* (U, p_m) *into* (Y_n, q_n).

Proof. Since $A \cap X_{m+1}$ is open in (X_{m+1}, p_{m+1}), there is an open ball $V(x_0, m+1, 2\varepsilon') \subset A \cap X_{m+1}$ of center x_0. There is an open ball $V(x_0, m, \varepsilon) \subset B \cap V(x_0, m+1, \varepsilon')$ of center x_0. Put $U = V(x_0, m, \varepsilon)$. By (p), the closure $\bar{U}$ is compact in (X_{m+1}, p_{m+1}). Then, by [5, Proposition 3], there is an n such that $f(\bar{U}) \subset Y_n$ and f is continuous as a mapping of $(\bar{U}, p_{m+1})$ into (Y_n, q_n). Therefore, $f(U) \subset Y_n$ and f is continuous as a mapping of (U, p_m) into (Y_n, q_n).

3. The (UCs-N) space $\mathcal{L}(X;Y)$. Let $(X; \{(X_m, p_m)\})$ and $(Y; \{(Y_n, q_n)\})$ be (CUN) spaces satisfying (p), and each X_m be locally compact. We denote by $\mathcal{L}(X;Y)$ the vector space consisting of all continuous linear mappings of X into Y.

Lemma 2. (cf. [3, Proposition 9]) *Let* T *be a linear mapping of* X *into* Y. *Then,* T *is continuous if and only if the restriction of* T *to* X_m *is a continuous linear mapping of* X_m *into* Y *for every* $m \in N$.

For $T \in \mathcal{L}(X;Y)$, by [5, Proposition 5], for every X_m, there is an $n = n(m)$ such that:

(i) The image of X_m by T is contained in Y_n.

(ii) T is a continuous linear mapping of (X_m, p_m) into (Y_n, q_n).

We put

$$\kappa(m, T) = \min\{n : n \text{ satisfies (i) and (ii) for } m\}.$$

Obviously, this $\kappa(m,T)$ is non-decreasing with respect to m. Further, $\kappa(m,T) \leqq n$ if and only if the restriction of T to X_m is a continuous linear mapping of X_m into Y_n for non-negative integers m and n. For $m, n \in N$, let us put

$$\bar{p}\,^m_{-n}(T) = \sup\{q_n(T(x)) : x \in X_m, p_m(x) \leqq 1\} \quad \text{(which may be finite or infinite).}$$

It is a norm on $\mathcal{L}(X_m; Y_n)$, if $n \geqq \kappa(m,T)$.

Lemma 3. (1) *If* $m \leqq m'$, *then* $\bar{p}\,^m_{-n}(T) \leqq \bar{p}\,^{m'}_{-n}(T)$ *for* $T \in \mathcal{L}(X;Y)$ *and* $n \geqq \kappa(m', T)$.
(2) *For each* $T \in \mathcal{L}(X;Y)$ *and* $m \in N$, *if* $\kappa(m,T) \leqq n \leqq n'$, *then* $\bar{p}\,^m_{-n}(T) \geqq \bar{p}\,^m_{-n'}(T)$.

Let us put

$$\Sigma = \{\lambda = \{n(m)\} : n(0) \leqq n(1) \leqq \cdots, \text{ where } n(m) \in N \text{ for each } m\}.$$

For $\lambda = \{n(m)\} \in \Sigma$ and $\lambda' = \{n'(m)\} \in \Sigma$, define $\lambda \leq \lambda'$ to mean that $n(m) \leqq n'(m)$ for every $m \in N$. Then, Σ, with this ordering $\leq$, is a directed set (see [3]). Corresponding to each $\lambda \in \Sigma$, $\lambda = \{n(m)\}$, define a subset $L_\lambda(X;Y)$ of $\mathcal{L}(X;Y)$ as follows.

$$L_\lambda(X;Y) = \{T \in \mathcal{L}(X;Y) : \kappa(m,T) \leqq n(m) \text{ for every } m \in N\}.$$

We have the following properties as [3, Proposition 10].

Proposition 1. (1) *$L_\lambda(X;Y)$ is a vector subspace of $\mathcal{L}(X;Y)$.*
(2) $\bigcup\{L_\lambda(X;Y) : \lambda \in \Sigma\} = \mathcal{L}(X;Y)$.
(3) *If $\lambda \leq \lambda'$, then $L_\lambda(X;Y) \subset L_{\lambda'}(X;Y)$.*
(4) *In each $L_\lambda(X;Y)$, $\lambda = \{n(m)\}$, $\bar{p}^{\,m}_{-n(m)}$ is a semi-norm on $L_\lambda(X;Y)$ for each m.*
(5) *For any $L_\lambda(X;Y)$, $L_{\lambda'}(X;Y)$, there is a λ'' with $L_\lambda(X;Y) \cap L_{\lambda'}(X;Y) = L_{\lambda''}(X;Y)$.*

Proposition 2. *Suppose that $X_m \subsetneqq X_{m+1}$ for each $m \in N$ and $Y_n \subsetneqq Y_{n+1}$ for each $n \in N$. Then, if $L_\lambda(X;Y) \subset L_{\lambda'}(X;Y)$, we have $\lambda \leq \lambda'$.*

Proof. Put $\lambda = \{n(m)\}$ and $\lambda' = \{n'(m)\}$. Suppose the contrary, then there exists an m' such that $n(m') > n'(m')$. Let $\{t_1, \cdots, t_{k(m)}\}$ be a basis of X_m for every m. Then, every $x \in X$ can be written in the form $x = \xi_1 t_1 + \xi_2 t_2 + \cdots$, where $\xi_j = 0$ if $j > k(m)$ for some m. Let us take a $y_0 \in Y_{n(m')}$ such that $y_0 \notin Y_{n'(m')}$. Consider $T \in \mathcal{L}(X;Y)$ defined by

$$T(x) = \xi_{k(m')} y_0 \qquad (x \in X).$$

We will prove that $T \in L_\lambda(X;Y)$, but $T \notin L_{\lambda'}(X;Y)$. If $m < m'$, then $T(x) = 0$ for every $x \in X_m$. If $m \geqq m'$, then $T(x) = \xi_{k(m')} y_0 \in Y_{n(m')}$ for every $x \in X_m$. Hence, we have $\kappa(m,T) \leqq n(m)$ for every m. Thus, $T \in L_\lambda(X;Y)$. On the other hand, $T(t_{k(m')}) = y_0 \notin Y_{n'(m')}$, so $T \notin L_{\lambda'}(X;Y)$.

Now, we will show that $\mathcal{L}(X;Y)$ can be defined as a (UCs-N) space (cf.[6]). For each $\lambda \in \Sigma$, $\lambda = \{n(m)\}$ and $T \in L_\lambda(X;Y)$, let us put

$$\bar{r}^\lambda_j(T) = \sum_{m=0}^{j} \bar{p}^{\,m}_{-n(m)}(T) \qquad (j \in N).$$

Then, these $\bar{r}^\lambda_j$ are semi-norms on $L_\lambda(X;Y)$ and have the folowing properties.

Lemma 4. (1) *If $j \leqq j'$, then $\bar{r}^\lambda_j(T) \leqq \bar{r}^\lambda_{j'}(T)$ for $T \in L_\lambda(X;Y)$.*
(2) *If $\lambda \leq \lambda'$, then $\bar{r}^\lambda_j(T) \geqq \bar{r}^{\lambda'}_j(T)$ for $T \in L_\lambda(X;Y)$ and $j \in N$.*
(3) *For $T \in L_\lambda(X;Y)$, if $\bar{r}^\lambda_j(T) = 0$ for every $j \in N$, then $T = 0$.*

By Lemma 4(1), for each $\lambda \in \Sigma$, $\lambda = \{n(m)\}$, we can define the space $L_\lambda(X;Y)$ as a (Cs-N) space determined by the countable system of semi-norms $\bar{r}^\lambda_j (j \in N)$. In fact, let us put

$$S(\lambda, j) = \{T \in L_\lambda(X;Y) : \bar{r}^\lambda_j(T) < 1/2^j\} \qquad (j \in N),$$

and we have

$$\mathcal{U}^\lambda(T) = \{T + S(\lambda, j) : j \in N\} \qquad (T \in L_\lambda(X;Y)),$$
$$\mathcal{U}^\lambda_j = \{T + S(\lambda, j) : T \in L_\lambda(X;Y)\} \qquad (j \in N).$$

Then, the space $L_\lambda(X;Y)$ endowed with $\mathcal{U}^\lambda(T)$ $(T \in L_\lambda(X;Y))$ and $\mathcal{U}^\lambda_j$ $(j \in N)$: $(L_\lambda(X;Y), \mathcal{U}^\lambda(T), \mathcal{U}^\lambda_j)$ becomes a (Cs-N) space.

Next, we define the ranked space $(\mathcal{L}(X;Y), \mathcal{U}(T), \mathcal{U}_j)$ as the ranked union space of the (Cs-N) spaces $(L_\lambda(X;Y), \mathcal{U}^\lambda(T), \mathcal{U}_j^\lambda)$ $(\lambda \in \Sigma)$, i.e., as the ranked space $\mathcal{L}(X;Y)$ provided with the family of the preneighborhoods of T: $\mathcal{U}(T)$ $(T \in \mathcal{L}(X;Y))$ and the family of the preneighborhoods of rank j: $\mathcal{U}_j$ $(j \in N)$, which are defined by

$$\mathcal{U}(T) = \bigcup \{\mathcal{U}^\lambda(T) : \lambda \in \Sigma \text{ for which } T \in L_\lambda(X;Y)\},$$
$$\mathcal{U}_j = \bigcup \{\mathcal{U}_j^\lambda : \lambda \in \Sigma\}.$$

Then, by Propositions 2, 1 (5) and [3, Propositions 2 and 4], we have:

Theorem 1. *Let $(X; \{(X_m, p_m)\}), (Y; \{(Y_n, q_n)\})$ be (CUB) spaces satisfying (p), and each X_m be locally compact. Suppose that $X_m \subsetneqq X_{m+1}$ for each $m \in N$ and $Y_n \subsetneqq Y_{n+1}$ for each $n \in N$. Then, the following statements hold.*

(1) *The ranked space $(\mathcal{L}(X;Y), \mathcal{U}(T), \mathcal{U}_j)$ is a (UCs-N) space with component spaces $(L_\lambda(X;Y), \mathcal{U}^\lambda(T), \mathcal{U}_j^\lambda)$ $(\lambda \in \Sigma)$.*
(2) *The ranked space $(\mathcal{L}(X;Y), \mathcal{U}(T), \mathcal{U}_j)$ is a ranked vector space satisfying Hausdorff's axioms (B) and (C) as well as (r-T_1) and having the properties (M_1), (M_2) and (M_3).*

4. Continuous differentiability and partial derivatives. Let us recall the definitions of differentiabilities and derivatives in [5], where we refer to [1] as for those in Banach spaces.

Now, let $(X; \{(X_m, p_m)\}), (Y; \{(Y_n, q_n)\})$ be (CUB) spaces satisfying (p) (both real or both complex), and each X_m be locally compact.

Definition 1. ([5, Definition 2]) Let f be a mapping of an r-open set $A \subset X$ into Y. We will say that f is *differentiable at* $t_0 \in A$ if for every m with $t_0 \in X_m$, there are a neighborhood $B \subset A$ of t_0 in (X_m, p_m) and an n such that $f(B)$ is contained in Y_n, and f is differentiable at t_0 as a mapping of B into (Y_n, q_n). By [5, Lemma 7], the derivative of f at t_0 indicated above is uniquely determined as a linear mapping of (X_m, p_m) into Y. Denote the derivative by u_m. The mapping of X into Y defined by setting $u(t) = u_m(t)$ whenever $t \in X_m$ for every m with $t_0 \in X_m$ is said to be the *derivative of* f *at* t_0. This is well-defined by [5, Lemma 7]. The derivative is continuous linear as a mapping of X into Y, and written $f'(t_0)$ or $\mathrm{D}f(t_0)$.

We define the continuous differentiability for a mapping.

Definition 2. (cf. [5, Definition 5]) Let f be a mapping of an r-open set $A \subset X$ into Y, and differentiable in A. We will say that f is *continuously differerntiable* in A if the derivative $\mathrm{D}f$ is continuous in $A \cap X_m$ as a mapping of $A \cap X_m$ into the (CUB) space $\mathcal{L}(X_m;Y)$ for each m with $A \cap X_m \neq \emptyset$.

We remark that, f is continuously differentiable in A if and only if $\mathrm{D}f$ is continuous in A as a mapping of A into $\mathcal{L}(X_m;Y)$ for each m.

Proposition 3. *Suppose that $X_m \subsetneqq X_{m+1}$ for each $m \in N$ and $Y_n \subsetneqq Y_{n+1}$ for each $n \in N$. Let f be a mapping of an r-open set $A \subset X$ into Y, and differentiable in A. Then, f is continuously differentiable in A if and only if $\mathrm{D}f$ is continuous in A as a mapping of A into the (UCs-N) space $\mathcal{L}(X;Y)$.*

Proof. Supppose that f is continuously differentiable in A. Let $x \in A$ and $r\text{-}\lim x_i = x$ in X, where $x_i \in A$. For each m, we have $r\text{-}\lim \mathrm{D}f(x_i) = \mathrm{D}f(x)$ in $\mathcal{L}(X_m;Y)$. There is an integer $n(m)$ such that $\mathrm{D}f(x)$ and all $\mathrm{D}f(x_i)$ belong to $\mathcal{L}(X_m;Y_{n(m)})$ and $\bar{p}^{\,m}_{-n(m)}(\mathrm{D}f(x) - \mathrm{D}f(x_i)) \to 0$ as $i \to \infty$. Then, we have $\kappa(m, \mathrm{D}f(x)) \leqq n(m)$ and $\kappa(m, \mathrm{D}f(x_i)) \leqq n(m)$ for each i. Without loss of generality, we may assume that $n(0) \leqq n(1) \leqq \cdots$. Then, $\{n(m)\} \in \Sigma$, which is written λ. Hence, for each j, $\bar{r}_j^{\lambda}(\mathrm{D}f(x) - \mathrm{D}f(x_i)) \to 0$ as $i \to \infty$. Consequently, r-lim $\mathrm{D}f(x_i) = \mathrm{D}f(x)$ in the (UCs-N) space $(\mathcal{L}(X;Y), \mathcal{U}(T), \mathcal{U}_j)$. Thus $\mathrm{D}f$ is continuous at $x \in A$. Conversely, suppose that $\mathrm{D}f$ is continuous in A as a mapping of A into the (UCs-N) space $\mathcal{L}(X;Y)$. Let m be a fixed integer, and let $x \in A$ and r-lim $x_i = x$ in X, where $x_i \in A$. Then, r-lim $\mathrm{D}f(x_i) = \mathrm{D}f(x)$ in $\mathcal{L}(X;Y)$. There exists a $\lambda = \{n(m)\} \in \Sigma$ such that $\mathrm{D}f(x)$ and all $\mathrm{D}f(x_i)$ belong to $L_\lambda(X;Y)$ and r-lim $\mathrm{D}f(x_i) = \mathrm{D}f(x)$ in the (Cs-N) space $L_\lambda(X;Y)$, so $\mathrm{D}f(x_i) \to \mathrm{D}f(x)$ in each of $\bar{r}_j^{\lambda}$. Then, $\bar{p}\,^{m}_{-n(m)}(\mathrm{D}f(x) - \mathrm{D}f(x_i)) \to 0$ as $i \to \infty$. Moreover, $\kappa(m, \mathrm{D}f(x)) \leqq n(m)$ and $\kappa(m, \mathrm{D}f(x_i)) \leqq n(m)$ for each i, so $\mathrm{D}f(x)$ and all $\mathrm{D}f(x_i)$ belong to $\mathcal{L}(X_m; Y_{n(m)})$. Thus, $\lim \mathrm{D}f(x_i) = \mathrm{D}f(x)$ in $\mathcal{L}(X_m; Y_{n(m)})$. Hence, r-lim $\mathrm{D}f(x_i) = \mathrm{D}f(x)$ in the (CUB) space $\mathcal{L}(X_m; Y)$. Therefore, f is continuously differentiable in A.

Next, we study the partial derivatives. Let $(X_i; \{(X_{im}, p_{im})\})\,(i = 1, 2)$ and $(Y; \{(Y_n, q_n)\})$ be (CUB) spaces satisfying (p). The product space $X = X_1 \times X_2$ becomes a (CUB) space satisfying (p) (see [5, p.1173]). Let f be a mapping of A into Y. The partial differentiabilities and the partial derivatives of f are similar to those in [1, p.172].

Theorem 2. *Suppose that each $X_{im}(i = 1, 2)$ is locally compact. Let f be a continuous mapping of an r-open set $A \subset X$ into Y. The mapping f is continuously differentiable in A if and only if f is differerntiable at each point with respect to the first and the second variable, and for each m, the mappings $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ are continuous in A as a mapping of A into $\mathcal{L}(X_{1m}; Y)$ and $\mathcal{L}(X_{2m}; Y)$, respectively. Then, at each point $(x_1, x_2) \in A$, the derivative of f is given by*

$$\mathrm{D}f(x_1, x_2) \cdot (t_1, t_2) = \mathrm{D}_1 f(x_1, x_2) \cdot t_1 + \mathrm{D}_2 f(x_1, x_2) \cdot t_2.$$

Proof. The "if "part is proved as follows. Let us take an $(a_1, a_2) \in A$. Let $m \in N$ with $(a_1, a_2) \in X_{1m} \times X_{2m}$. There are open neighborhoods I_i of a_i in (X_{im}, p_{im}) with $I_i \subset A_{a_j}$ and n_i such that:

(1) The image $f_{a_j}(I_i) \subset Y_{n_i}$.
(2) f_{a_j} is continuous in I_i as a mapping of I_i into (Y_{n_i}, q_{n_i}).
(3) f_{a_j} is differentiable at $a_i \in I_i$ as a mapping of I_i into (Y_{n_i}, q_{n_i}),

for $i = 1, 2$ and $j = 2, 1$, respectively (A_{a_j} and f_{a_j} are referred to [1]). By Lemma 1, there are an open ball V of center (a_1, a_2) in $X_{1m} \times X_{2m}$ and an n_3 such that $V \subset A \cap (I_1 \times I_2)$ and f is continuous in V as a mapping of V into Y_{n_3}. Since $\mathrm{D}_i f$ is continuous in A as a mapping of A into $\mathcal{L}(X_{im}; Y)$, by Lemma 1, there are an open ball W of center (a_1, a_2) in $X_{1m} \times X_{2m}$ and an n_4 such that $W \subset A \cap (X_{1m} \times X_{2m})$ and $\mathrm{D}_i f$ is continuous in W as a mapping of W into $\mathcal{L}(X_{im}; Y_{n_4})$, for $i = 1, 2$. Put $U = V \cap W$ and $n_0 = \max\{n_1, n_2, n_3, n_4\}$. Then, f is differentiable at each point in U with respect to the first and the second variable, as a mapping of U into a Banach space Y_{n_0}, and $\mathrm{D}_i f$ is continuous in U as a mapping of U into $\mathcal{L}(X_{im}; Y_{n_0})$, for $i = 1, 2$. By [1, (8.9.1)], f is continuously differentiable in U as a mapping of U into (Y_{n_0}, q_{n_0}) and

$$\mathrm{D}f(a_1, a_2) \cdot (t_1, t_2) = \mathrm{D}_1 f(a_1, a_2) \cdot t_1 + \mathrm{D}_2 f(a_1, a_2) \cdot t_2 \quad \big((t_1, t_2) \in X_{1m} \times X_{2m}\big).$$

For each $(t_1, t_2) \in X_1 \times X_2$, there is an m such that $(a_1, a_2), (t_1, t_2) \in X_{1m} \times X_{2m}$ so that this equality holds. Since $(a_1, a_2) \in A$ is arbitrary, f is differentiable in A and the required equality holds. Moreover, $\mathrm{D}f$ is continuous in U as a mapping of U into $\mathcal{L}(X_m; Y)$ for each m. Since $(a_1, a_2) \in A$ is arbitrary, $\mathrm{D}f$ is continuous in A as a mapping of A into $\mathcal{L}(X_m; Y)$ for each m. Hence, f is continuously differentiable in A. Similarly, the "only if "part is proved by [1, (8.9.1)].

Proposition 4. *Let* $(X;\{(X_m,p_m)\}),(Y;\{(Y_n,q_n)\})$ *be* (CUB) *spaces satisfying* (p). *Let* $I=[\alpha,\beta]\subset\mathbf{R}$ *be a compact interval,* f *a continuous mapping of* $I\times A$ *into* Y*, where* $A\subset X$ *is* r*-open. Then, the mapping* g *which is defined by* $g(z)=\int_\alpha^\beta f(\xi,z)d\xi$ *is continuous in* A.

The proof is similar to that of [1, (8.11.1)].

Proposition 5. (Leibniz's rule) *With the same assumptions as in Proposition 4, suppose in addition that each* X_m *is locally compact, and* f *is continuously differentiable in* $I\times A$ *with respect to the second variable. Then,* g *is continuously differentiable in* A*, and*

$$\mathrm{D}g(z)=\int_\alpha^\beta \mathrm{D}_2 f(\xi,z)d\xi.$$

Proof. Let us take a $z_0\in A$. Let $m\in N$ with $z_0\in X_m$. Then, there are an open neighborhood $U\subset A$ of z_0 in (X_m,p_m) and an n such that for each $\xi\in I$, $\mathrm{D}_2 f$ is continuous at (ξ,z_0) on $I\times U$ in the usual sense as a mapping of $I\times U$ into $\mathcal{L}(X_m;Y_n)$. As in [1, (8.11.2)], g is differentiable at z_0 and we have

$$\mathrm{D}g(z_0)=\int_\alpha^\beta \mathrm{D}_2 f(\xi,z_0)d\xi.$$

Moreover, by Proposition 4, $\mathrm{D}g$ is continuous in A.

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