ON A. GROTHENDIECK'S PROBLEMS CONCERNING (F) AND (DF)–SPACES

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For non-experts only

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Abstract. This is an expository article devoted to the answers to the ten open problems from the fundamental paper [16] of A. Grothendieck, some of which were given in the recent years.

A significant contribution to the development of the theory of locally convex spaces (*lcs*), as a part of functional analysis, was given in the fifties by a group of French mathematicians (L. Schwartz, N. Bourbaki, J. Dieudonné, A. Grothendieck). The dominant part in their work, concerning this part of functional analysis, was devoted to the so called duality theory. It was especially applied to the investigation of metrizable, i.e. Fréchet *lcs*, which are "nearest" to the normed and Banach spaces [14]. The article [16] of A. Grothendieck "on (F) and (DF) spaces" takes a special place in the theory of *lcs*. Apart from solving nearly all the problems of L. Schwartz and J. Dieudonné, posed in [14], a significant class of spaces—"(DF)-spaces"—was introduced in it. This class contains a lot of important functional spaces—we mention just a few, following the paper [16] itself:

a) The spaces $(\mathcal{E}^{(m)})$, $(\mathcal{E}')$, $({\mathcal{D}'_{L_p}}^{(m)})$, $(\mathcal{D}'_{L_p})$, introduced by L. Schwartz in [28] for application in the theory of distributions; also the space (C) of continuous functions with compact support.

b) The spaces of locally-analytic functions $\mathcal{H}(K)$.

c) A lot of examples of functional spaces defined by the condition that their elements belong to at least one normed space from a certain sequence.

d) Miscellaneous examples of the spaces of linear mappings $L(E, H)$, where E is an (F)-space, and H is a (DF)-space, with the topology of bounded convergence.

Investigating the features of (F)-spaces and the new class of (DF)-spaces, a lot of new problems arose, some of which remained unsolved. These unsolved questions (there are 10 of them) are given at the end of the paper [16]. These questions, together with a lot of others which were posed later in connection with (F)- and (DF)-spaces and their generalizations, played in the past 40 years one of the main themes to be considered by many mathematicians in this field. The present article is of expository character and its goal is to show the actuality of Grothendieck's work. Namely, some of the mentioned ten problems were solved only in the recent years.

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We begin with a brief survey of the basic properties of a metrizable *lcs*, its strong dual and bidual. So the need for introducing the class of (DF)-spaces will be clear.

If (E,t) is a metrizable *lcs*, then it has a fundamental sequence of bounded subsets if and only if it is normed. If (B_n) is a sequence of bounded sets in a metrizable *lcs*, then there exist scalars $\rho_n > 0$ so that $\bigcup_{n=1}^{\infty} \rho_n B_n$ is a bounded subset in the space E. The strong dual of each metrizable *lcs* has a fundamental sequence of bounded subsets and it is metrizable itself if and only if the given space is normed. If (E,t) is a metrizable *lcs*, then the so called natural topology in its bidual (the topological dual of its strong dual space) is equal to the strong topology. For each metrizable *lcs* the following two equivalent conditions of its strong dual are the main motivation for introducing the class of (DF)-spaces:

(1) If a union $A = \bigcup_{n=1}^{\infty} A_n$ of countably many $\beta(E',E)$-equicontinuous subsets of the space E'' is $\sigma(E'',E')$-bounded, then A is itself a $\beta(E',E)$-equicontinuous subset of E''.

(2) If (V_n) is a sequence of closed absolutely convex $\beta(E',E)$-neighbourhoods of origin, such that the intersection $V = \bigcap_{n=1}^{\infty} V_n$ is an absorbent subset in the topological dual E', then V is also a $\beta(E',E)$-neighbourhood of origin.

1. One of the direct corollaries of the property (1) (or the equivalent property (2)) is that each separable bounded subset of the completion $\hat{E}$ of the metrizable *lcs* E, is contained in the closure of some bounded set from E. The problem 1 in the Grothendieck's paper now reads as follows:

> *Is each bounded subset of the completion $\hat{E}$ of the metrizable lcs E contained in the closure of some bounded subset of the space E (i.e. can the separability condition be dropped in the previous proposition)?*

The first negative answer to this question was given by J. Dieudonné [13], using the continuum-hypothesis. J. Amemiya did that without continuum-hypothesis in the paper [3]. See also G. Köthe [21], p. 404.

2. When metrizable *lcs*'s are concerned, we always have $\hat{E} \subset E'' = (E', \beta(E',E))'$. In the general case, $\hat{E}$ and E'' are uncomparable subspaces of the algebraic dual E'^*. Therefore it is natural to consider spaces (E,t) for which $(E', \beta(E',E))$ is a barrelled space, i.e. for which each $\sigma(E'',E')$-bounded subset of the bidual E'' is contained in the closure of some t-bounded subset (the closure is taken with respect to the topology $\sigma(E'',E')$). Such spaces are called distinguished. Grothendieck showed the following:

(3) For each metrizable *lcs* E, the space $H = (E', \beta(E',E))$ is barrelled (i.e. E is distinguished) if and only if H is bornological and if and only if H is quasibarrelled.

In connection with this characterization of the strong dual of an (F)-space, the problem 2 was formulated:

> *Is for each (F)-space E the strong topology of its dual equal to the Mackey topology, i.e. is $\beta(E',E) = \tau(E',E'')$?*

Of course, in the case when the (F)-space E is semi-reflexive, the answer to this question is positive. But, the negative answer in the general case was given by Y. Komura in the paper [20]; see also G. Köthe [21], pp. 388 and 404.

3. The condition (1) inspired Grothendieck to define the class of (DF)-spaces. Thus, an *lcs* (E,t) is a space of type (DF) if it has a fundamental sequence of bounded subsets and if each strongly bounded union of countably many t-equicontinuous subsets of its dual is t-equicontinuous. Of course, each normed space is of that kind. So is the strong dual of a metrizable *lcs*. It can be easily proved that each bidual (with strong topology) of a (DF)-space is also a (DF)-space; S. Dierolf [10] proved that the same is true when the bidual is endowed with the natural topology.

One of the important properties of strong duals of metrizable spaces which were proved by Grothendieck is the above mentioned property (3). The question 3 read as follows:

> *Is the mentioned property (3) valid for an arbitrary (DF)-space H? Particularly, is every quasibarrelled space H bornological?*

The negative answer to this question was given in the same year when Grothendieck's paper was published. Examples of barrelled (even complete) spaces which are not bornological were independently constructed by L. Nachbin [25] and T. Shirota [29]. Later Y. Komura [19] constructed an example of a barrelled (DF)-space which is not bornological. Unfortunately Komura's proof contained a gap. Examples of quasibarrelled (and even barrelled) (DF)-spaces which are not bornological were given by M. Valdivia [33], [35].

4. In the theorem 5 of his paper, Grothendieck proved that each (DF)-space whose bounded subsets are metrizable is quasibarrelled. A criterion for an (F)-space to be distinguished can be easily derived from there—it is sufficient that the bounded subsets of its strong dual are metrizable. It was natural to pose the question of inverse statements, i.e.:

> *Are bounded subsets of the strong dual E' metrizable for each distinguished space E of the type (F)? More generally, if H is a quasibarrelled (DF)-space, are all of its bounded subsets metrizable?*

The negative answer to the posed question was given by I. Amemiya in the above mentioned paper [3]; see also G. Köthe [21], p. 405.

We mention that K.-D. Bierstedt and J. Bonet in [4] showed that for a metrizable *lcs* E the property that bounded sets in its strong dual are metrizable is equivalent to the important property that E satisfies Heinrich's density condition.

5. If the answer to the question no. 4 had been positive, the next question would not have been posed—it would have the positive answer.

> *Does every quasibarrelled space H of the type (DF) have the distinguished strong dual, i.e. the barrelled strong bidual? Is it true at least in the case when H is the strong dual of an (F)-space, i.e. is the strong bidual of a distinguished (F)-space distinguished?*

The negative answer to this question could be suspected already on the base of I. Amemiya's example [3] of a bornological (DF)-space whose bidual is not bornological—see G. Köthe [21], p. 436. However, the full (negative) answer to this question was given only by J. Bonet, S. Dierolf, C. Fernandez [7]—using some new techniques in the theory of *lcs*'s. Namely, they consider (F)-spaces of Moscatelli type of the form

$$E = \{\, (y_k)_{k\in\mathbf{N}} \in Y^{\mathbf{N}} \mid \big(f(y_k)\big)_{k\in\mathbf{N}} \in c_0(X)\,\},$$

where Y, X are Banach spaces, and $f\colon Y \to X$ a continuous linear map. Such spaces are always distinguished. However, for its bidual E'' it is proved that it is distinguished if and only if the mapping f is open onto its range.

It should be mentioned that S. Dierolf in [11] related the given result with some results of R. Meise and D. Vogt [23] concerning short exact sequences of *lcs*.

6. The question 6 is of particular interest and of slightly more general character. Namely, it is known (see e.g. [16], propositions 4 and 5) that if E is an *lcs*, and F its quasibarrelled vector subspace whose strong dual is bornological (or if F is a (DF)-space), then in the space E'/F° the factor-topology of the strong topology of the space E' coincides with the strong topology of the dual of the space F. It is also known that in the general case these two topologies are different (see e.g. G. Köthe [21], p. 434). Therefore the following question is of interest:

> *Do both strong topologies in the space E'/F° (E—an arbitrary lcs, F—its closed subspace) have the same bounded subsets?*

This question was answered in negative by S. Dierolf [10] in 1981. Her example also showed that in the Proposition 4 of [16], the quasibarrelledness-hypothesis was not superfluous.

7. For the seventh question A. Grothendieck said that he considered it the most important.

> *Let E and F be (F)-spaces and $B(E,F)$ the space of all continuous bilinear forms on $E \times F$, equipped with the topology of uniform convergence on products $A \times B$ of bounded sets $A \subset E$ and $B \subset F$. Is $B(E,F)$ a space of the type (DF)? Is $L(E,H)$, equipped with topology of bounded convergence, a space of the type (DF) if E is of type (F), and H of type (DF)? The same question if E and H are normed spaces.*

The answer to this question is negative, too. This important problem, which is connected with some questions concerning tensor products (see the other fundamental article of Grothendieck [17]), was solved in the recent years by J. Taskinen [30], [31]; later J. Bonet and A. Galbis [8] gave more elementary proofs for the mentioned negative answer, modifying Taskinen's counterexample. S. Dierolf gave a direct proof that $L_b(E, l_2)$ is not a (DF)-space if E is the Fréchet space of Taskinen.

The mentioned important question inspired a lot of mathematicians to obtain the positive answer by certain modifications. We mention here the one given by J. Bonet, J.C. Diaz and J. Taskinen [5].

8. The question number 8 consists of several problems concerning biduals of strict inductive limits of sequences of *lcs*'s.

> *Does the bidual of the strict inductive limit E of a sequence of lcs's E_n coincide with the inductive limit of the biduals E_n''? Is it true when E_n are (F)-spaces or distinguished (F)-spaces? Is E'' complete? (The last is true when E_n are distinguished (F)-spaces.)*

M. Valdivia [34] gave a negative answer to the first question in 1979. The negative answers to all of the quoted questions were relatively recently given by J. Bonet and S. Dierolf in [6]. They, namely, presented a method of construction of strict (LF)-spaces $E = \operatorname{ind} E_n$ such that the strong bidual E_b'' of E and the (stronger) inductive limit topology of $E_{ind}'' = \operatorname{ind} E_{n,b}''$ do not even have the same bounded subsets. Also, they provided an example when even in the case of distinguished spaces E_n, E_b'' need not be an (LF)-space.

9. An *lcs* is said to be totally reflexive if all of its quotients with respect to arbitrary closed subspaces are reflexive. The question 9 posed by Grothendieck was:

> *Is the product of two totally reflexive (F)-spaces always totally reflexive?*

This is one of the two questions having the positive answer. It was recently given by M. Valdivia in [36]. Namely, he used his characterization of a totally reflexive Fréchet space as a space isomorphic to a closed subspace of a product of countably many reflexive Banach spaces.

10. Finally, the question number 10 was:

Is every metrizable (M)-space separable?

An *lcs* is called Montel ((M)-space) if it is barrelled and each bounded subset in it is relatively compact. The positive answer to this question was announced by the author himself, and it was published in the paper [12] of J. Dieudonné; see also G. Köthe [21], p. 370.

11. There are a lot of other important results, given in the past 40 years in connection with (F) and (DF) spaces. We refer the interested reader to [9] and [23]. Especially, in the last ten years there were some significant contributions on (F) spaces. However, we have restricted ourselves in this article to the solutions of 10 Grothendieck's problems, and we shall mention in the end only some results concerning (DF) spaces.

1° Already in the paper of Grothendieck it was stated that an arbitrary (closed) subspace of a (DF)-space need not be of the same kind. However, M. Valdivia [32] proved that each subspace of finite codimension of a (DF)-space is always of the type (DF).

2° Another characterization of (DF)-spaces was obtained by B. Mirković [24]. He showed that the condition for a space to have a fundamental sequence of bounded subsets could be weakened in the following way.

Let $\mathcal{M}$ be a family of bounded sets of an *lcs* (E,t) satisfying the following requests: (a) $E = \bigcup\{ M \mid M \in \mathcal{M}\}$, (b) $\lambda M \in \mathcal{M}$ for every $M \in \mathcal{M}$ and every $\lambda \geq 0$, (c) for every M_1, $M_2 \in \mathcal{M}$, there exists $M \in \mathcal{M}$ such that $M_1 \cup M_2 \subset M$. The space (E,t) is called a space of the type $\mathcal{M}$-(DF) if it has the following properties: 1° the family $\mathcal{M}$ has a fundamental sequence of absolutely convex sets; 2° if $(U_n)_{n\in\mathbf{N}}$ is a sequence of closed absolutely convex t-neighbourhoods of zero and $U = \bigcap_{n\in\mathbf{N}} U_n$ abosrbs each set $M \in \mathcal{M}$, then U is a t-neighbourhood of zero. The theorem 5 [24] is:

Every space of the type $\mathcal{M}$-(DF) is a space of the type (DF).

In other words, together with the condition 2°, the (weaker) condition 1° implies that the family $\mathcal{B}$ of all bounded sets in (E,t) has a fundamental sequence.

Proof. Let $t_{\mathcal{M}}$ denote the locally convex topology on the dual space E' of uniform convergence on the sets $M \in \mathcal{M}$. Since the family $\mathcal{M}$ has a fundamental sequence, topology $t_{\mathcal{M}}$ is metrizable, hence the space $(E', t_{\mathcal{M}})$ is bornological. Clearly, the strong topology $\beta(E', E)$ is finer than $t_{\mathcal{M}}$. To prove the converse, let i denote the identical mapping from $(E', t_{\mathcal{M}})$ onto $(E', \beta(E', E))$. According to [21; 28.3], i is continuous if it maps each 0-sequence (f_n) from $(E', t_{\mathcal{M}})$ into a bounded set in $(E', \beta(E', E))$. But such a set is a countable union of equicontinuous subsets $\{f_n\}$ of E', which is bounded in $(E', t_{\mathcal{M}})$; since E is an $\mathcal{M}$-(DF) space, it is equicontinuous, hence bounded in $(E', \beta(E', E))$. So the space $(E', \beta(E', E))$, being equal to $(E', t_{\mathcal{M}})$, is metrizable. It means that the family $\mathcal{B}$ of all bounded subsets in E has a fundamental sequence. The other condition is satisfied, and so (E,t) is a space of the type (DF). $\square$

3° The spaces satisfying only the condition (2) from the definition of (DF)-spaces (and not having a fundamental sequence of bounded subsets in general) were studied by many authors. They were firstly introduced by T. Husain in [18] and called countably-quasibarrelled spaces.

4° Spaces of the type (DF) can be introduced even in the category of (non-locally convex) linear topological spaces, whereas, of course, the usage of duality theory is avoided. For details we refer to the work of B. Ernst [15], J. P. Ligaud [22] and N. Adasch, B. Ernst and D. Keim [2].

5° One of the most important properties of (DF)-spaces proved by A. Grothendieck is the following:

Let H be a space of the type (DF). In order to an absolutely convex set U in H to be a zero-neighbourhood, it is sufficient that its intersection with each absolutely convex bounded set $A \subset H$ be a zero-neighbourhood (in the topology which H induces in A).

Stated in another way:

Let H be a (DF) and G an arbitrary *lcs*. A linear mapping from H to G is continuous if its restrictions to all bounded subsets of H are continuous.

The given property served as an inspiration for introducing special classes of *lcs*'s, respectively linear topological spaces, which do not have to be (DF), but possess the mentioned property—see K. Noureddine [26], [27] ("*b*-spaces"), resp. N. Adasch, B. Ernst [1] ("locally topological spaces").

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References

1. N. Adasch, B. Ernst, *Lokaltopologische Vektorräume*, Collect. Math. **25** (1974), 255–274.
2. N. Adasch, B. Ernst, D. Keim, *Topological vector spaces. The theory without convexity conditions*, Lecture Notes in Mathematics 639, Springer, Berlin-Heidelberg-New York, 1978.
3. I. Amemiya, *Some examples of (F)- and (DF)-spaces*, Proc. Japan Acad. **33** (1957), 169–171.
4. K.-D. Bierstedt, J. Bonet, *Stefan Heinrich's density condition for Fréchet spaces and the characterization of distinguished Köthe echelon spaces*, Math. Nachr. **135** (1988), 149–180.
5. J. Bonet, J. C. Diaz, J. Taskinen, *Tensor stable Fréchet and (DF)-spaces*, Collect. Math. **42** (1991), 199–236.
6. J. Bonet, S. Dierolf, *A note on biduals of strict (LF)-spaces*, Results Math. **13** (1988), 23–32.
7. J. Bonet, S. Dierolf, C. Fernández, *The bidual of a distinguished Fréchet space need not be distinguished*, Arch. Math. (Basel) **57** (1991), 475–478.
8. J. Bonet, A. Galbis, *A note on Taskinen's counterexamples on the problem of topologies of Grothendieck*, Proc. Edinburgh Math. Soc. **32** (1989), 281–283.
9. P. P. Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Math. Studies 131, North-Holland, Amsterdam, 1987.
10. S. Dierolf, *On two questions of A. Grothendieck*, Bul. Soc. Roy. Sci. Liège **50** (1981), 282–286.
11. ———, *On the three-space-problem and the lifting of bounded sets*, Collect. Math. **44** (1993), 81–89.
12. J. Dieudonné, *Sur les espaces de Montel métrizables*, C. R. Acad. Sci. Paris **238** (1954), 194–195.
13. ———, *Bounded sets in (F)-spaces*, Proc. Amer. Math. Soc. **6** (1955), 729–731.
14. J. Dieudonné, L. Schwartz, *La dualité dans les espaces (F) et (LF)*, Ann. Inst. Fourier (Grenoble) **1** (1950), 61–101.
15. B. Ernst, *Ultra-(DF)-Räume*, J. reine angew. Math. **258** (1973), 87–102.
16. A. Grothendieck, *Sur les espaces (F) et (DF)*, Summa Bras. Math. **3** (1954), 57–123.
17. ———, *Produit tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).
18. T. Husain, *Two new classes of locally convex spaces*, Math. Annalen **166** (1966), 289–299.
19. Y. Komura, *On linear topological spaces*, Kumamoto J. Science, Series A **5** (1962), 148–157.
20. ———, *Some examples on linear topological spaces*, Math. Annalen **153** (1964), 150–162.
21. G. Köthe, *Topological Vector Spaces I*, Springer, Berlin-Heidelberg-New York, 1983.
22. J. P. Ligaud, *Espaces DF non nécessairement localment-convexes*, C. R. Acad. Sci. Paris **275** (1972), 283–285.
23. R. Meise, D. Vogt, *Einführung in Funktionalanalysis*, Vieweg, Wiesbaden, 1992.
24. B. Mirković, *On locally convex spaces of the type (DF) defined by an arbitrary family of bounded sets*, Mat. Vesnik **11 (26)** (1974), 127–130.
25. L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. USA **40** (1954), 471–474.
26. K. Noureddine, *Nouvelles classes d'espaces localment convexes*, Publ. dép. mat. Lyon **10** (1973), 105–122.

27. ———, *Note sur les espaces D_b*, Math. Annalen **219** (1976), 97–103.
28. L. Schwartz, *Théorie des distributions, t.1,2*, Act. sci et ind., 1091, 1122, Paris, 1950–1951.
29. T. Shirota, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad. **30** (1954), 294–298.
30. J. Taskinen, *Counterexamples to "Problème des topologies" of Grothendieck*, Ann. Acad. Sci. Fenn., Ser. A.I. Dissertationes **63** (1986).
31. ———, *The projective tensor product of Fréchet-Montel spaces*, Studia Math. **91** (1988), 17–30.
32. M. Valdivia, *On DF spaces*, Math. Annalen **191** (1971), 38–43.
33. ———, *A class of quasibarrelled (DF)-spaces which are not bornological*, Math. Z. **136** (1974), 249–251.
34. ———, *Solution to a problem of Grothendieck*, J. reine angew Math. **305** (1979), 116–121.
35. ———, *A characterization of Köthe Schwartz spaces*, in North-Holland Math. Studies 35, North-Holland, Amsterdam, 1979, pp. 409–419.
36. ———, *A characterization of totally reflexive Fréchet spaces*, Math. Z. **200** (1989), 327–346.

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