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EFFICIENT SOLUTIONS FOR MULTICRITERIA LOCATION PROBLEMS UNDER THE BLOCK NORM II: APPLICATION TO THE DEVELOPMENT OF NEW PRODUCTS

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ABSTRACT. In this article, the development of new products is considered an application of a location problem. First, principal components analysis is used for a multivariate data. Then, a multicriteria location problem under the block norm is considered for scores of principal components.

1. Introduction. On a plane, demand points $\boldsymbol{y}_i,\ i = 1, 2, \cdots, n$ and the block norm are given. The block norm is the approximation to the road distance. Let Y be a set of all demand points. If a new facility $\boldsymbol{x} \in \boldsymbol{R}^2$ should be located as near as possible for all demand points, then the problem is formulated as

$$(1) \qquad \min_{\boldsymbol{x}\in\boldsymbol{R}^2} (\|\boldsymbol{x} - \boldsymbol{y}_1\|, \|\boldsymbol{x} - \boldsymbol{y}_2\|, \cdots, \|\boldsymbol{x} - \boldsymbol{y}_n\|).$$

The above location problem (1) is known as a multicriteria problem(MCP). MCP is a problem to find an efficient or quasiefficient point. A point $\boldsymbol{x} \in \boldsymbol{R}^2$ is efficient if there is no $\boldsymbol{y} \in \boldsymbol{R}^2$ such that $\|\boldsymbol{y} - \boldsymbol{y}_i\| \leq \|\boldsymbol{x} - \boldsymbol{y}_i\|$ for all i and $\|\boldsymbol{y} - \boldsymbol{y}_j\| < \|\boldsymbol{x} - \boldsymbol{y}_j\|$ for some j. An efficient point $\boldsymbol{x}$ is alternately efficient if there exists $\boldsymbol{y} \neq \boldsymbol{x}$ such that $\|\boldsymbol{y} - \boldsymbol{y}_i\| = \|\boldsymbol{x} - \boldsymbol{y}_i\|$ for all i. An efficient point $\boldsymbol{x}$ is strictly efficient if $\boldsymbol{x}$ is not alternately efficient. A point $\boldsymbol{x} \in \boldsymbol{R}^2$ is quasiefficient if there is no $\boldsymbol{y} \in \boldsymbol{R}^2$ such that $\|\boldsymbol{y} - \boldsymbol{y}_i\| < \|\boldsymbol{x} - \boldsymbol{y}_i\|$ for all i. Let $E(Y), AE(Y), SE(Y)$ and $QE(Y)$ be sets of all efficient, alternately efficient, strictly efficient and quasiefficient points, respectively. By these definitions, $Y \subset SE(Y) \subset E(Y) \subset QE(Y)$. The Stairs algorithm and the Wrapping algorithm to find all efficient and quasiefficient points, both of which are optimal in the sense of the order for the computational time, are given in Kon[2].

On the other hand, various distances or norms are used in multicriteria problems[1,2,4,6,8]. For example, ℓ_1 distance (the rectilinear distance) in [1,8], ℓ_p distances in [8], the one-infinity norm in [6] and the block norm in [2,4]. However most of them are applied to a facility location.

In this article, the development of new products for the market of ready-made clothes is considered and a multicriteria location problem under the block norm is used in it. The original data is a multivariate data. The data consists of candidates for new products and consumers with scores of some variates, which are measured in a common scale. Scores of products represent their characters, and those of consumers represent their tastes. Principal components analysis from the covariance matrix is applied for this data, and two principal components are considered. In $\boldsymbol{R}^2$ which consists of these principal components, MCP under the block norm is considered. Demand points are scores of principal components for consumers. The block norm is determined by using regression coefficients of principal components on variates. In this sense, the block norm makes up for the lost information by principal components analysis. We consider efficient points. The efficient set $E(Y)$ is given by the Stairs Algorithm in [2]. If a product $\boldsymbol{x} \notin E(Y)$, then it means that there is another product $\boldsymbol{y} \in E(Y)$ better than $\boldsymbol{x}$. Perhaps consumers might buy products near their tastes. So, an efficient product $\boldsymbol{y}$ is more salable than a product $\boldsymbol{x}$ which is not efficient. Therefore efficient set can be regarded as a set of salable products. Candidates for new products which are not efficient are modified to be efficient and minimize the cost to modify. An iterative algorithm to determine them is proposed.

In section 2, some properties of the block norm are given. In section 3, the Stairs Algorithm in [2] is given. Finally, the development of new products for the market of ready-made clothes is considered an application of MCP under the block norm in section 4.

2. The block norm. In this section, some notations are prepared and the definition of the block norm and its fundamental properties are given.

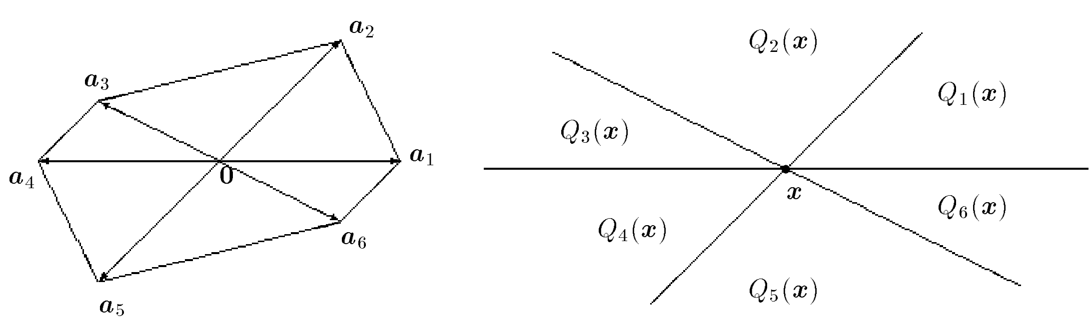
Let $B \subset \boldsymbol{R}^2$ be a bounded polyhedral set such that it is symmetric around the origin and its interior contains the origin. Let

$$\boldsymbol{a}_j = (a_j^1, a_j^2), \ j = 1, 2, \cdots, 2m$$

be extreme points of B. Especially, we set $\boldsymbol{a}_{2m+j} = \boldsymbol{a}_j, \ j = 1, 2$. For each $\boldsymbol{a}_j$, let α_j be an angle such that

$$\boldsymbol{a}_j = \|\boldsymbol{a}_j\|_2 (\cos \alpha_j, \sin \alpha_j)$$

where $\| \cdot \|_2$ is the Euclidean norm. We assume that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m < \pi$, $\alpha_{m+j} = \pi + \alpha_j$, $j = 1, 2, \cdots, m$ and $\alpha_{2m+j} = 2\pi + \alpha_j$, $j = 1, 2$. We set $Q_j(\boldsymbol{x}) = \boldsymbol{x} + \mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\}$, $j = 1, 2, \cdots, 2m$, where $\mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\} = \{\lambda \boldsymbol{a}_j + \mu \boldsymbol{a}_{j+1} : \lambda, \mu \geq 0\}$, and $Q_{2m+j}(\boldsymbol{x}) = Q_j(\boldsymbol{x})$, $j = 1, 2, \cdots, 2m-2$(Figure 1). For $\boldsymbol{x} \in \boldsymbol{R}^2$, we say a cone $Q(\boldsymbol{x})$ with a vertex at $\boldsymbol{x}$ is type r if $Q(\boldsymbol{x}) = \bigcup_{j=j'}^{j'+r-1} Q_j(\boldsymbol{x})$ for some j' ($r = 1, 2, \cdots, 2m$). $Q(\boldsymbol{x})$ is a half plane if it is type m, and is $\boldsymbol{R}^2$ if it is type $2m$. We set $Q^-(\boldsymbol{x}) = 2\boldsymbol{x} - Q(\boldsymbol{x})$ and $Q^0(\boldsymbol{x}) = intQ(\boldsymbol{x})$ where $intQ(\boldsymbol{x})$ is the interior of $Q(\boldsymbol{x})$. Then we set $R(\boldsymbol{x}) = \{Q(\boldsymbol{x}) \subset \boldsymbol{R}^2 : Q(\boldsymbol{x}) \text{ is type } r,$ $r = 1, 2, \cdots, 2m.\}$.



Unit ball($m = 3$)

Figure 1.

The block norm of $\boldsymbol{x} \in \boldsymbol{R}^2$, $\|\boldsymbol{x}\|$, is defined as

$$\|\boldsymbol{x}\| = \inf\{\mu > 0 : \boldsymbol{x} \in \mu B\}. \tag{2}$$

From the above definition, B is an unit ball and $\| \cdot \| : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ is a convex function. $\|\boldsymbol{x}\|$ can be represented as follows[5,7].

$$\|\boldsymbol{x}\| = \min \left\{ \sum_{j=1}^{m} |\gamma_j| : \boldsymbol{x} = \sum_{j=1}^{m} \gamma_j \boldsymbol{a}_j \right\}. \tag{3}$$

By the above representation, $\|\boldsymbol{x}\|$ can be regarded as the shortest path from the origin to $\boldsymbol{x}$ using orientations $\boldsymbol{a}_j, j = 1, 2, \cdots, 2m$. From (3), the following lemma is given[2].

Lemma 1. *For* $\boldsymbol{x} = (x^1, x^2) \in Q_j(\mathbf{0})$,

$$\|\boldsymbol{x}\| = \frac{x^1(a^2_{j+1} - a^2_j) + x^2(a^1_j - a^1_{j+1})}{a^1_j a^2_{j+1} - a^1_{j+1} a^2_j}.$$

By Lemma 1, $\|\cdot\|$ is linear on each $Q_j(\mathbf{0})$, and we have the following lemma.

Lemma 2. *For* $\boldsymbol{x} = (x^1, x^2), \boldsymbol{y} = (y^1, y^2)$ *and* $\boldsymbol{z} = (z^1, z^2)$ *such that* $\boldsymbol{x} \in Q_j(\boldsymbol{y})$ *and* $\boldsymbol{z} \in Q_{m+j}(\boldsymbol{y})$,

$$\|\boldsymbol{x} - \boldsymbol{z}\| = \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y} - \boldsymbol{z}\|.$$

Proof. Without loss of generality, we set $\boldsymbol{y} = (0, 0)$. By Lemma 1,

$$\|\boldsymbol{x}\| = \frac{x^1(a^2_{j+1} - a^2_j) + x^2(a^1_j - a^1_{j+1})}{a^1_j a^2_{j+1} - a^1_{j+1} a^2_j}$$

and

$$\|\boldsymbol{z}\| = \frac{z^1(a^2_{m+j+1} - a^2_{m+j}) + z^2(a^1_{m+j} - a^1_{m+j+1})}{a^1_{m+j} a^2_{m+j+1} - a^1_{m+j+1} a^2_{m+j}}.$$

Since $\boldsymbol{a}_{m+j} = -\boldsymbol{a}_j$ and $\boldsymbol{x} - \boldsymbol{z} \in Q_j(\mathbf{0})$,

$$\|\boldsymbol{x}\| + \|\boldsymbol{z}\| = \frac{(x^1 - z^1)(a^2_{j+1} - a^2_j) + (x^2 - z^2)(a^1_j - a^1_{j+1})}{a^1_j a^2_{j+1} - a^1_{j+1} a^2_j} = \|\boldsymbol{x} - \boldsymbol{z}\|.$$

by Lemma 1. □

3. The Stairs Algorithm. In this section, the Stairs Algorithm in [2] is given.

In efficiency, two following propositions are given in [4].

Proposition 1. $\boldsymbol{x} \notin QE(Y)$ *if and only if* $R(\boldsymbol{x})$ *contains at least one cone* $Q(\boldsymbol{x})$ *such that it is type* m *and* $Q(\boldsymbol{x}) \bigcap Y = \emptyset$.

Proposition 2. *For* $\boldsymbol{x} \in QE(Y)$,

(i) $\boldsymbol{x} \in QE(Y) \backslash E(Y)$ *if and only if* $R(\boldsymbol{x})$ *contains one cone* $Q(\boldsymbol{x})$ *such that it is type* $m - 1$ *and* $Q(\boldsymbol{x}) \bigcap Y = \emptyset$ *and* $(Q^-)^0(\boldsymbol{x}) \bigcap Y \neq \emptyset$.

(ii) $\boldsymbol{x} \in AE(Y)$ *if and only if* $R(\boldsymbol{x})$ *contains one cone* $Q(\boldsymbol{x})$ *such that it is type* $m - 1$ *and* $Q(\boldsymbol{x}) \bigcap Y = \emptyset$ *and* $(Q^-)^0(\boldsymbol{x}) \bigcap Y = \emptyset$.

(iii) $\boldsymbol{x} \in SE(Y)$ *if and only if* $R(\boldsymbol{x})$ *does not contain a cone* $Q(\boldsymbol{x})$ *such that it is type* $m - 1$ *and* $Q(\boldsymbol{x}) \bigcap Y = \emptyset$.

Without loss of generality, we assume that $\alpha_j \neq \pi/2$ for all j. If $\alpha_j = \pi/2$ for some j, then rotate a plane counterclockwise for sufficiently small $\varepsilon > 0$ and reset $\alpha_j = \alpha_j + \varepsilon$ for all j. For each $\boldsymbol{y}_i$ and $\boldsymbol{a}_j$, we set $L_{ij} = \{\boldsymbol{y}_i + \gamma \boldsymbol{a}_j : \gamma \in \boldsymbol{R}\}$. For each α_j, we set $\beta_j = \alpha_j + \pi/2$ and $\boldsymbol{b}_j = (\cos\beta_j, \sin\beta_j)$. For each α_j, we assume that $n(j)$ lines are different among L_{ij}'s, and we sort those lines according to y-intercepts in ascending order. For the simplicity of the notation, we write the above sorted lines as

$$\ell^j_1, \ell^j_2, \cdots, \ell^j_{n(j)}$$

for each α_j. A line ℓ^j_k is the kth line among $\boldsymbol{a}_j$-oriented lines which are sorted. For each L_{ij}, if $L_{ij} = \ell^j_k$, then we set $s(i, j) = k$. For each ℓ^j_k, if $\boldsymbol{y}_i$ is a demand point with maximum(minimum) x-coordinate among demand points on ℓ_k^j, then we set $(k,j)_{\max} = i((k,j)_{\min} = i)$. In the following algorithm, we set

$$p(c) = \begin{cases} 1 & \text{if } 0 < \alpha_c < \pi/2 \text{ or } 3\pi/2 < \alpha_c < 5\pi/2, \\ n(c) & \text{if } \pi/2 < \alpha_c < 3\pi/2 \text{ or } 5\pi/2 < \alpha_c < 3\pi \end{cases}$$

for $c = 2, 3, \cdots, 2m+1$,

$$q(c) = \begin{cases} +1 & \text{if } 0 \le \alpha_c < \pi/2 \text{ or } 3\pi/2 < \alpha_c < 2\pi, \\ -1 & \text{if } \pi/2 < \alpha_c < 3\pi/2 \end{cases}$$

for $c = 1, 2, \cdots, 2m$, and

$$\diamond(c) = \begin{cases} \max & \text{if } 0 \le \alpha_c < \pi/2 \text{ or } 3\pi/2 < \alpha_c < 2\pi, \\ \min & \text{if } \pi/2 < \alpha_c < 3\pi/2 \end{cases}$$

for $c = 1, 2, \cdots, 2m$.

The Stairs Algorithm([2])

Step 0.: Determine ℓ_k^j's, $s(i,j)$'s, $(k,j)_{\max}$'s, and $(k,j)_{\min}$'s.
Step 1.: If $0 \le \alpha_1 < \pi/2$, then set $\boldsymbol{z}^* = \boldsymbol{y}_{(1,1)_{\min}}, r = (1,1)_{\max}$ and $d = 1$, otherwise set $\boldsymbol{z}^* = \boldsymbol{y}_{(n(1),1)_{\max}}, r = (n(1),1)_{\min}$ and $d = n(1)$. Store the route from $\boldsymbol{z}^*$ to $\boldsymbol{y}_r$. Set $\boldsymbol{z} = \boldsymbol{y}_r$ and $c = 1$.
Step 2.: If $s(r, c+1) = p(c+1)$, then set $d = p(c+1), c = c+1$ and go to Step 2.
Step 3.: If $\langle \boldsymbol{y}_{(d+q(c),c)_{\diamond(c)}} - \boldsymbol{z}, \boldsymbol{b}_{c+1} \rangle \le 0$, then set $\boldsymbol{z}' = \boldsymbol{y}_{(d+q(c),c)_{\diamond(c)}}$ and go to Step 4, otherwise set $d = d + q(c)$ and go to Step 3.
Step 4.: Let $\boldsymbol{z}''$ be an intersection point of $\ell_{s(r,c+1)}^{c+1}$ and $\ell_{d+q(c)}^c$. Store the routes from $\boldsymbol{z}$ to $\boldsymbol{z}''$ and from $\boldsymbol{z}''$ to $\boldsymbol{z}'$, i.e. $\boldsymbol{z} \longrightarrow \boldsymbol{z}'' \longrightarrow \boldsymbol{z}'$. If $\boldsymbol{z}' = \boldsymbol{z}^*$, then stop, otherwise set

$$\boldsymbol{z} = \boldsymbol{z}'(= \boldsymbol{y}_{(d+q(c),c)_{\diamond(c)}}), r = (d+q(c),c)_{\diamond(c)}, d = d + q(c)$$

and go to Step 2.

The stored route lines determine the boundary of $E(Y)$. If there exists a region S such that its boundary is a part of the boundary of $E(Y)$ and, for its interior point $\boldsymbol{x}$, $Y \subset Q_{j'}(\boldsymbol{x}) \bigcup Q_{m+j'}(\boldsymbol{x})$ for some j', then $S = Q_{j'}(\boldsymbol{x}_1) \bigcap Q_{m+j'}(\boldsymbol{x}_2)$ for opposite vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$ which are strictly efficient points. Any point in $S \backslash \{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ is an alternately efficient point. Other points are strictly efficient points. The Stairs Algorithm requires $O(n \log n)$ computational time and it is optimal in the sense of the order for the computational time[2].

4. Application to the development of new products. In this section, the development of new products for the market of ready-made clothes is considered an application of MCP under the block norm.

Consider the market of ready-made clothes. It is assumed that manufactured products are characterized by some variates. Some enterprise wants to develop new products and there are candidates for new products A,B,C,D(Table 1). Are there any products better than these new products? Then six consumers are gathered information by questionnaires for taste of products. In this case, characters of products and tastes of consumersare measured in a common scale for common variates. Table 1 shows the result.

Table 1. Characters of products and the result of questionnaires for taste.

Individuals\Variates	x_1 Simple	x_2 Young	x_3 Feminine	x_4 Lighthearted	x_5 Manly	x_6 Gay
Product A	7	8	4	4	8	6
Product B	1	1	7	2	5	9
Product C	2	7	9	3	8	4
Product D	3	5	5	4	8	5
Consumer 1	9	9	6	9	9	2
Consumer 2	1	7	1	5	8	7
Consumer 3	8	5	5	7	6	3
Consumer 4	2	3	5	1	7	8
Consumer 5	1	4	8	4	3	5
Consumer 6	3	8	9	6	3	1

For this data, Table 2 shows the result of principal components analysis from the covariance matrix.

Table 2. Scores of principal components.

Individuals	(z_1:First, z_2:Second)
Product A	$(7.99, -5.35) = \boldsymbol{y}_A$
Product B	$(-1.53, -1.68) = \boldsymbol{y}_B$
Product C	$(4.98, 0.02) = \boldsymbol{y}_C$
Product D	$(4.73, -3.32) = \boldsymbol{y}_D$
Consumer 1	$(13.90, -2.89) = \boldsymbol{y}_1$
Consumer 2	$(4.11, -6.56) = \boldsymbol{y}_2$
Consumer 3	$(9.65, -2.28) = \boldsymbol{y}_3$
Consumer 4	$(0.30, -4.01) = \boldsymbol{y}_4$
Consumer 5	$(2.29, 1.85) = \boldsymbol{y}_5$
Consumer 6	$(8.03, 3.88) = \boldsymbol{y}_6$

Points $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_6$ are demand points in MCP, i.e. $Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_6\}$. A demand point $\boldsymbol{y}_i$ represents a taste of consumer i and $\boldsymbol{y}_A, \boldsymbol{y}_B, \boldsymbol{y}_C, \boldsymbol{y}_D$ represent, respectively, products A,B,C,D (Figure 3). Cumulative rate of contribution up to second component is 0.80. Table 3 shows correlation coefficient for each x_j and z_k.

Table 3. Correlation coefficients.

	z_1	z_2
x_1	0.86	-0.21
x_2	0.83	-0.03
x_3	-0.04	0.91
x_4	0.90	0.08
x_5	0.32	-0.77
x_6	-0.82	-0.53

Form this result, it can be regarded that z_1 means design and z_2 means appeal to the male and female.

Next, an unit ball B which defines the block norm is determined. Table 4 shows regression coefficient of z_k on x_j for each x_j and z_k.

Table 4. Regression coefficients.

	(z_1 , z_2)
x_1	$(1.28, -0.22) = \boldsymbol{a}_1$
x_2	$(1.50, -0.03) = \boldsymbol{a}_2$
x_3	$(-0.07, 1.18) = \boldsymbol{a}_3$
x_4	$(1.75, 0.11) = \boldsymbol{a}_4$
x_5	$(0.68, -1.14) = \boldsymbol{a}_5$
x_6	$(-1.45, -0.66) = \boldsymbol{a}_6$

Then we set $B = \mathcal{K}\{\pm \boldsymbol{a}_1, \pm \boldsymbol{a}_2, \cdots, \pm \boldsymbol{a}_6\}$, where $\mathcal{K}S$ is a convex hull spanned by $S \subset \boldsymbol{R}^2$ (Figure 2). Let $\boldsymbol{x} \in \boldsymbol{R}^2$ be a point which represents a product. Distance between $\boldsymbol{x}$ and $\boldsymbol{y}_i$, $\|\boldsymbol{x} - \boldsymbol{y}_i\|$, is a degree of difference between products $\boldsymbol{x}$ and taste $\boldsymbol{y}_i$ measured in only vectors $\boldsymbol{a}_j$ such that they influence principal components much. The block norm defined in this way can be regarded to make up for the lost information by principal components analysis.

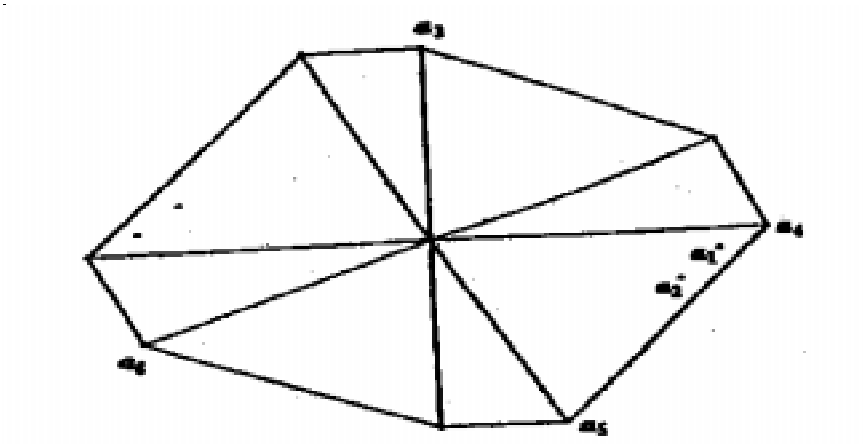


Figure 2. Unit ball B.

Figure 3 shows the result of MCP.

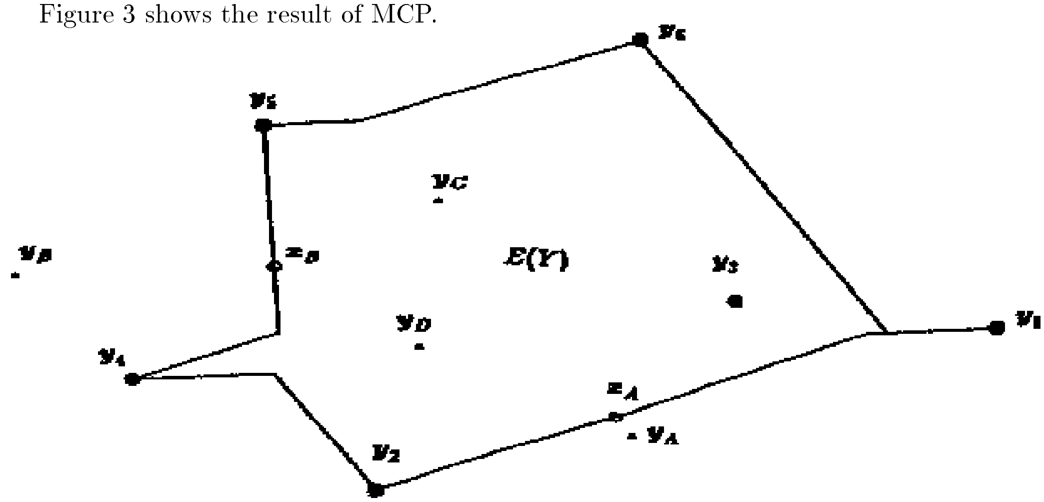


Figure 3. $E(Y)$.

From the above result, products C and D are efficient, but products A and B are not efficient. Hence products A and B should be modified. Here, it is assumed that the cost to modify a product $\boldsymbol{y} \in \boldsymbol{R}^2$ to $\boldsymbol{x} \in \boldsymbol{R}^2$ is $\overline{c} = \overline{c}(\|\boldsymbol{x}-\boldsymbol{y}\|)$, where $\overline{c}$ is strictly increasing in $\|\boldsymbol{x}-\boldsymbol{y}\|$. If a product $\boldsymbol{y}_A$ is modified, then modified product $\boldsymbol{x}_A \in E(Y)$ is determined to minimize the cost to modify. It is similar for a product $\boldsymbol{y}_B$. By the translation, without loss of generality, we set $\boldsymbol{y}_A = \boldsymbol{0}$. Generally, for Y and the block norm in previous sections, the problem is as follows;

$$(4)\qquad \begin{array}{ll} \min & \overline{c}(\|\boldsymbol{x}\|) \\ \text{s.t.} & \boldsymbol{x} \in E(Y). \end{array}$$

Since $\overline{c}$ is strictly increasing in $\|\boldsymbol{x}\|$, the problem (4) is equivalent to the problem

$$(5)\qquad \begin{array}{ll} \min & \|\boldsymbol{x}\| \\ \text{s.t.} & \boldsymbol{x} \in E(Y). \end{array}$$

Since $E(Y)$ is bounded closed set, there exists an optimal solutin for the problem (5).

Theorem 1. Let $\boldsymbol{x}^*$ be an optimal solution of the problem (5). If $\boldsymbol{0} \notin E(Y)$, then $\boldsymbol{x}^* \notin intE(Y)$.

Proof. Assume that $\boldsymbol{x}^* \in intE(Y)$. Then, for sufficiently small $\varepsilon > 0$,

$$(6)\qquad \{\boldsymbol{y} \in \boldsymbol{R}^2 : \|\boldsymbol{y} - \boldsymbol{x}^*\| \leq \varepsilon\} \subset E(Y).$$

For some j, $\boldsymbol{0} \in Q_j(\boldsymbol{x}^*)$. Consider

$$(7)\qquad \overline{\boldsymbol{x}} \in Q_j(\boldsymbol{x}^*) \bigcap Q_j^-(\boldsymbol{0}) \bigcap \{\boldsymbol{y} \in \boldsymbol{R}^2 : \|\boldsymbol{y} - \boldsymbol{x}^*\| = \varepsilon\}.$$

Then $\overline{\boldsymbol{x}} \in E(Y)$ by (6), $\boldsymbol{0} \in Q_j(\overline{\boldsymbol{x}})$ and $\boldsymbol{x}^* \in Q_j^-(\overline{\boldsymbol{x}})$ by (7). By Lemma 2,

$$\|\boldsymbol{x}^*\| = \|\boldsymbol{x}^* - \overline{\boldsymbol{x}}\| + \|-\overline{\boldsymbol{x}}\| = \varepsilon + \|\overline{\boldsymbol{x}}\| > \|\overline{\boldsymbol{x}}\|.$$

This contradicts the optimality of $\boldsymbol{x}^*$. □

We can check $\boldsymbol{0} \in E(Y)$ or not by Proposition 2. By the Stairs Algorithm, $E(Y)$ is given by a sequence of points which represents the polygonal boundary of $E(Y)$. Hence, if $\boldsymbol{0} \notin E(Y)$, then the problem (5) is reduced to the following problem by Theorem 1.

For $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$ such that $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$ and $\boldsymbol{0} \notin [\boldsymbol{x}_1, \boldsymbol{x}_2] = \{(1-\lambda)\boldsymbol{x}_1 + \lambda \boldsymbol{x}_2 : 0 \leq \lambda \leq 1\}$,

$$(8)\qquad \begin{array}{ll} \min & \|\boldsymbol{x}\| \\ \text{s.t.} & \boldsymbol{x} \in [\boldsymbol{x}_1, \boldsymbol{x}_2]. \end{array}$$

Next, an algorithm to solve the problem (8) is considered. For each $\boldsymbol{a}_j$, let ℓ_j be $\boldsymbol{a}_j$-oriented line passing through the origin. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ and intersection points of ℓ_j's and $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ be $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_{q'}$ from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$. We set $f(\lambda) = \|(1-\lambda)\boldsymbol{x}_1 + \lambda \boldsymbol{x}_2\|, 0 \leq \lambda \leq 1$ and $\boldsymbol{p}_i = (1-\lambda_i)\boldsymbol{x}_1 + \lambda_i \boldsymbol{x}_2, i = 1, 2, \cdots, q'-1$. The right differential coefficient of f at λ_i is denoted by $\partial_+ f(\lambda_i)$. Note that $\|\boldsymbol{x}\|$ is linear on each $[\boldsymbol{p}_i, \boldsymbol{p}_{i+1}]$ and piecewise linear and convex on $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ by Lemma 1. In this case, the problem (8) can be solved by the following algorithm.

The Algorithm

Step 1.: Set $r = 1$.

Step 2.: If $\partial_+ f(\lambda_r) > 0$, then stop. $\boldsymbol{p}_r$ is optimal. If $\partial_+ f(\lambda_r) = 0$, then stop. Any $\boldsymbol{x} \in [\boldsymbol{p}_r, \boldsymbol{p}_{r+1}]$ is optimal. If $r = q'-1$, then stop. $\boldsymbol{p}_{q'} = \boldsymbol{x}_2$ is optimal. Otherwise, set $r = r+1$ and go to Step 2.

From the above algorithm, we have $\boldsymbol{x}_A = (7.72, -4.90)$ and $\boldsymbol{x}_B = (2.49, -1.44)$ as modified products (Figure 3).

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