Scientiae Mathematicae **Vol. 1** No. 2(1998), 189–193

ON MANN AND ISHIKAWA ITERATION PROCESSES OF STRONGLY PSEUDOCONTRACTIVE MAPPINGS

Liwei Liu

Received March 6, 1995; revised June 29, 1995

Abstract. Let E be a real Banach space with a uniformly convex dual E^*, let the modulus of convexity of E^* satisfy $\delta_{E^*}(\epsilon) \ge C\epsilon^q$ for some $q \ge 2$ and $C > 0$, and let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be a continuous strongly pseudocontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences with $0 \le \alpha_n \le \beta_n < 1, \sum_n \alpha_n \beta_n^{p-1} < \infty$, and $1/p + 1/q = 1$. Then the sequence $\{x_n\}$ generated by $x_1 \in K, x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, y_n = (1-\beta_n)x_n + \beta_n T x_n, n \ge 1$, converges strongly to the unique fixed point of T. Furtheremore, error estimates of Mann iteration scheme are given.

1. Introduction. Let E be a Banach space, and $K \subset E$. A mapping $T : K \to K$ is called a strong pseudocontraction if there exists $t > 1$ such that the inequality

$$\|x - y\| \le \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all x, y in K and $r > 0$. If $t = 1$, then T is called pseudocontractive. A mapping U with domain $D(U)$ and range $R(U)$ in E is called accretive if the inequality

$$\|x - y| \le \|x - y + s(Ux - Uy)\| \tag{1}$$

holds for every $x, y \in D(U)$ and $s > 0$. U is said to be strongly accretive [2] if $U - kI$ is accretive for some $k > 0$. Strongly accretive mappings are sometimes also called strictly accretive. A mapping T is pseudocontractive if and only if $(I - T)$ is accretive [1, Prop. 1].

In [3], Chidume proved that if E is a real Banach space with a uniformly convex dual E^*, and $T : K \to K$ is a continuous strongly pseudocontractive map, then Mann's iteration sequence ([6]) converges strongly to a fixed point of T. But, he has not solved the question of whether the Ishikawa iteration method [8] converges for this class of nonlinear maps (see Remark 5 of [3]).

The main purpose of this paper is to answer this question. We prove the following result: If E is a real Banach space with a uniformly convex dual E^*, the modulus of convexity of E^* satisfies $\delta_{E^*}(\epsilon) \ge C\epsilon^q$ for some $q \ge 2$ and $C > 0, K$ is a nonempty closed convex and bounded subset of E and $T : K \to K$ is a continuous strongly pseudocontractive map, then the Ishikawa iteration method converges strongly to the unique fixed point of T. Furthermore, error estimates of Mann iteration scheme are given.

2. Preliminaries. Let E be a reflexive Banach space. Then we denote by J_φ the duality map from E to 2^{E^*} corresponding to the gauge function $\varphi(t) = t^{p-1} (p > 1)$ defined by

$$J_\varphi x = \{f \in E^* : \|f\|\|x\| = \langle x, f\rangle, \|f\| = \|x\|^{p-1}\}.$$

When $p=2, J_\varphi$ is the normalized duality map (see [7, p.126]). The duality map J_φ can be equivalently defined as the subdifferential of the convex function $\Phi(x)=\dfrac{\|x\|^p}{p}$, i.e.,

$$f \in J_\varphi x \Longleftrightarrow f \in \partial\Phi(x)=\{g \in E^* : \frac{\|z\|^p}{p}-\frac{\|x\|^p}{p} \geq \langle z-x, g\rangle, z \in E\}. \tag{2}$$

It is known that $J_\Phi(\lambda x)=\lambda^{p-1}J_\Phi x$ for all $\lambda \geq 0$ and if E^* is strictly convex, then J_Φ is singlevalued. We shall denote the singlevalued duality map J_φ by j.

We state the following lemma (see [5, p.313]).

Lemma. *Let E be a real Banach space with a uniformly convex dual E^*. Assume that the modulus of convexity of E^* satisfies $\delta_{E^*}(\epsilon) \geq C\epsilon^q$ for some $q \geq 2$ and $C>0$. Then duality mapping $j : E \to E^*$ with gauge function $\varphi(t)=t^{p-1}(1/p+1/q=1)$ is Hölder continuous with exponent $p-1$, i.e, there exists $r>0$ such that*

$$\|jx-jy\| \leq r\|x-y\|^{p-1}, x, y \in E. \tag{3}$$

Proof. Let $x, y \in E$. Without loss of generality, we assume that $\|x\|=1$ and $0<\|y\| \leq 1$.

(a) If $\|x\|=\|y\|=1$, then by convexity of E^* and condition, we have

$$(1/2)\|jx+jy\| \leq 1-\delta_{E^*}(\|jx-jy\|) \leq 1-C\|jx-jy\|^q.$$

On the other hand, since $\langle x-y, jx\rangle \geq \dfrac{\|x\|^p}{p}-\dfrac{\|y\|^p}{p}=0$,

$$\langle x, jx+jy\rangle=\langle x, jx\rangle+\langle y, jy\rangle+\langle x-y, jy-jx\rangle+\langle x-y, jx\rangle \geq 2-\|x-y\|\|jx-jy\|.$$

Hence,

$$\|jx-jy\| \leq (1/2C)^{p-1}\|x-y\|^{p-1}.$$

(b) If $\|y\|<\|x\|=1$, then by (a), we have

$$\|j(\alpha x)-jy\| \leq (1/2C)^{p-1}\|\alpha x-y\|^{p-1}, \tag{4}$$

where $\alpha=\|y\|$. Because $1-\alpha=\|x\|-\|y\| \leq \|x-y\|$, and $1-\alpha^{p-1} \leq 1-\alpha \leq (1-\alpha)^{p-1}(1<p \leq 2)$,

$$\begin{aligned}\alpha^{p-1}\|jx-jy\| &= \|\alpha^{p-1}(jx-jy)-(1-\alpha^{p-1})(jy)+(1-\alpha^{p-1})(jy)\| \\ &\leq \|\alpha^{p-1}jx-jy\|+\alpha^{p-1}(1-\alpha^{p-1}) \leq \|j(\alpha x)-jy|+\alpha^{p-1}\|x-y\|^{p-1},\end{aligned} \tag{5}$$

$$\begin{aligned}\|\alpha x-y\|^{p-1} &= \|\alpha(x-y)+(1-\alpha)(-y)\|^{p-1} \\ &\leq (\|\alpha(x-y)\|+(1-\alpha)\|y\|)^{p-1} \leq 2^{p-1}\alpha^{p-1}\|x-y\|^{p-1}.\end{aligned} \tag{6}$$

Thus, using (4), (5) and (6), we get

$$\|jx-jy\| \leq (1+1/C^{p-1})\|x-y\|^{p-1}.$$

The result follows by making $r=1+1/C^{p-1}$.

Remark 1. (a) The modulus of convexity of $L_p(1<p<\infty)$ satisfies $\delta_{L_p}(\epsilon) \geq C\epsilon^{\max(2,p)}$ for some $C>0$ (see, e.g., [9, 10])

(b) We always have sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n \geq 1}$ satisfying:

(i) $0 \leq \alpha_n \leq \beta_n < 1, n \geq 1$,
(ii) $\sum_n \alpha_n=\infty$,
(iii) $\sum_n \alpha_n\beta_n^{p-1}<\infty, q \geq 2, 1/p+1/q=1$.

For example, let $\alpha_n = \dfrac{1}{n}$ and $\beta_n = \dfrac{1}{n^{1/p}}$.

Main result.

Theorem 1. *Let E be a real Banach space with a uniformly convex dual E^*, let the modulus of convexity of E^* satisfy $\delta_{E^*}(\epsilon) \geq C\epsilon^q$ for some $q \geq 2$ and $C > 0$, and let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be a continuous strongly pseudocontractive mapping of K into itself. Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be real sequences satisfying*:

(i) $0 \leq \alpha_n \leq \beta_n < 1, n \geq 1,]$
(ii) $\sum_n \alpha_n = \infty$ and $\lim\limits_{n\to\infty} \beta_n = 0,$
(iii) $\sum_n \alpha_n \beta_n^{p-1} < \infty, 1/p + 1/q = 1.$

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ in K by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \tag{7}$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, n \geq 1. \tag{8}$$

Then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. The existence of a fixed point follows from Deimling [11]. Let x^* denote a fixed point of T. Since T is strongly pseudocontractive, $(I-T)$ is strongly accretive, i.e., $(1-k)I - T$ is accretive, where $k = (t-1)/t$. It follows that inequality

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq k\|x-y\|^p \tag{9}$$

holds for every $x, y \in K$. Indeed, for any $s > 0$, by (1),

$$\begin{aligned} 0 &\leq \|x - y + s[((1-k)I-T)x - ((1-k)I-T)y]\|^p - \|x-y\|^p \\ &\leq ps\langle ((1-k)I-T)x - ((1-k)I-T)y, j[x-y+s[((1-k)I-T)x - ((1-k)I-T)y]\rangle \end{aligned}$$

i.e.,

$$0 \leq \langle ((1-k)I-T)x - ((1-k)I-T)y, j[x-y+s[(1-k)I-T)x - ((1-k)I-T)y]\rangle.$$

Let $s \to 0$. By continuity of j, we obtain (9). So using (3), (8) and (9),

$$\begin{aligned} \langle Ty_n - Tx^*, j(x_n - x^*)\rangle &= \langle Ty_n - Tx^*, j(y_n - x^*)\rangle \\ &\quad + \langle Ty_n - Tx^*, j(x_n - x^*) - j(y_n - x^*)\rangle \\ &\leq (1-k)\|y_n - x^*\|^p + r\|Ty_n - Tx^*\|\|x_n - x^* - (y_n - x^*)\|^{p-1} \\ &= (1-k)\|y_n - x^*\|^p + r\beta_n^{p-1}\|x_n - Tx_n\|^{p-1}\|Ty_n - Tx^*\|. \end{aligned} \tag{10}$$

By (2), (3) and (9),

$$\begin{aligned} \|y_n - x^*\|^p &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - Tx^*)\|^p \\ &\leq (1-\beta_n)^p\|x_n - x^*\|^p + p\beta_n\langle Tx_n - Tx^*, j(y_n - x^*)\rangle \\ &= (1-\beta_n)^p\|x_n - x^*\|^p + p\beta_n\langle Tx_n - Tx^*, j(1-\beta_n)(x_n - x^*)\rangle \\ &\quad + p\beta\langle Tx_n - Tx^*, j(y_n - x^*) - j(1-\beta_n)(x_n - x^*)\rangle \\ &\leq [(1-\beta_n)^p + (1-k)\beta_n(1-\beta_n)^{p-1}]\|x_n - x^*\|^p + pr\beta_n^p\|Tx_n - Tx^*\|^p. \end{aligned} \tag{11}$$

By (7) and (8)

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^p \le (1-\alpha_n)^p \|x_n - x^*\|^p + p\alpha_n \langle Ty_n - Tx^*, j(x_{n+1} - x^*)\rangle \\
&= (1-\alpha_n)^p \|x_n - x^*\|^p + p\alpha_n \langle Ty_n - Tx^*, j(x_{n+1} - x^*) - j(1-\alpha_n)(x_n - x^*)\rangle \\
&\quad p\alpha_n \langle Ty_n - Tx^*, j(1-\alpha_n)(x_n - x^*)\rangle \\
&\le (1-\alpha_n)^p \|x_n - x^*\|^p + pr\alpha_n^p \|Ty_n - Tx^*\|^p \\
&\quad + p\alpha_n (1-\alpha_n)^{p-1} \langle Ty_n - Tx^*, j(x_n - x^*)\rangle \\
&\le (1-\alpha_n)^p \|x_n - x^*\|^p + pr\alpha_n^p \|Ty_n - Tx^*\|^p \\
&\quad + p(1-k)\alpha_n (1-\alpha_n)^{p-1} \|y_n - x^*\|^p \\
&\quad + pr\alpha_n (1-\alpha_n)^{p-1} \beta_n^{p-1} \|Ty_n - Tx^*\| \|x_n - Tx_n\|^{p-1} \\
&\le \{(1-\alpha_n)^p + p\alpha_n (1-\alpha_n)^{p-1} [(1-k)(1-\beta_n)^p \\
&\quad + p(1-k)^2 \beta_n (1-\beta_n)^{p-1}]\} \|x_n - x^*\|^p \\
&\quad + pr\alpha_n^p \|Ty_n - Tx^*\|^p + p^2 r(1-k)\alpha_n (1-\alpha_n)^{p-1} \beta_n^p \|Tx_n - Tx^*\|^p \\
&\quad + pr\alpha_n (1-\alpha_n)^{p-1} \beta_n^{p-1} \|Ty_n - Tx^*\| \|x_n - Tx_n\|^{p-1}.
\end{aligned}
$$

For sufficiently large n, (i) and (ii) implies $(1-k)(1-\beta_n)^p + p(1-k)^2\beta_n(1-\beta_n)^{p-1} \le 1 - k + k^2$, $\frac{1}{2}pk(1-k) < pk(1-k) - (p-1)\alpha_n$ and $\alpha_n^p \le \alpha_n \beta_n^{p-1}$. Hence

$$
\begin{aligned}
&(1-\alpha_n)^p + p\alpha_n (1-\alpha_n)^{p-1} [(1-k)(1-\beta_n)^p + p(1-k)^2 \beta_n (1-\beta_n)^{p-1}] \\
&\le (1-\alpha_n)^{p-1} [1 - \alpha_n + p\alpha_n (1 - k + k^2)] \\
&\le [1 - (p-1)\alpha_n][1 - \alpha_n + p\alpha_n (1 - k + k^2)] \\
&= 1 - pk(1-k)\alpha_n + (p-1)\alpha_n^2 - p(p-1)\alpha_n^2 (1 - k + k^2) \\
&\le 1 - pk(1-k)\alpha_n + (p-1)\alpha_n^2 = 1 - \alpha_n \{pk(1-k) - (p-1)\alpha_n\}.
\end{aligned}
$$

Thus,

$$\|x_{n+1} - x^*\|^p \le [1 - \frac{pk(1-k)\alpha_n}{2}] \|x_n - x^*\|^p + M\alpha_n \beta_n^{n-1}$$

for some constant $M > 0$, since K is bounded. The result of the argument now follows as in [3, 4] to give that $\{x_n\}$ converges strongly to the unique fixed point of T.

Theorem 2. *Let E, K and T be as in Theorem 1. Then there exists a sequence $\{C_n\}$ such that the sequence $\{x_n\}$ generated by $x_1 \in K$,*

$$x_{n+1} = (1 - C_n)x_n + C_n T x_n, n \ge 1$$

converges strongly to the unique fixed point x^* of T, and the estimate $\|x_n - x^*\| \le O(1/n^{1q})$ holds.

Proof. Let $d = \sup_{x \in K} \|x - x^*\|, h = (pr)^{q/p}(\text{diam } TK)^q / d^q$, $C_n = k^{q/p}(h + nk^q)$ and $d_n = 1/[h + (n-1)k^q]^{1/q}$, where x^* and k are as in proof Theorem 1. Then $0 < C_n < 1$, $\lim_{n \to \infty} C_n = 0$ and $\sum_n C_n^p < \infty (1/p + 1/q = 1)$. In (11), replacing β_n and y_n by C_n and x_{n+1}, respectively, we obtain

$$\|x_{n+1} - x^*\|^p \le [(1 - C_n)^p + p(1-k)C_n(1 - C_n)^{p-1}] \|x_n - x^*\|^p + prC_n^p \|Tx_n - Tx^*\|^p.$$

Since $(1+\alpha)^p \geq 1+p\alpha$ when $p>1$ and $\alpha>0$, then

$$\begin{aligned}&(1-C_n)^p + p(1-k)C_n(1-C_n)^{p-1} = (1-C_n)^p[1+p(1-k)C_n(1-C_n)^{-1}]\\&\leq (1-C_n)^p[1+(1-k)C_n(1-C_n)^{-1}]^p = (1-kC_n)^p.\end{aligned}$$

So,

$$\|x_{n+1}-x^*\|^p \leq (1-kC_n)^p\|x_n-x^*\|^p + prC_n^p\|Tx_n-Tx^*\|^p.$$

Next, using the induction, we show that $\|x_n-x^*\| \leq dndh^{1q}$ for each n. Indeed, $\|x_1-x^*\| \leq d = d_1h$. Now assume $\|x_n-x^*\| \leq d_ndh$. Then

$$\begin{aligned}\|x_{n+1}-x^*\|^p &\leq [(1-kC_n)^pd_n^p + C_n^p]d^ph^{p/q}\\&= \left\{\left(\frac{h+(n-1)k^q}{h+nk^q}\right)^p \frac{1}{[h+(n-1)k^q]^{p/q}} + \frac{k^q}{(h+nk^q)^p}\right\}d^ph^{p/q}\\&= \frac{d^ph^{p/q}}{(h+nk^q)^{p/q}} = d_{n+1}^pd^ph^{p/q}.\end{aligned}$$

So, $\|x_{n+1}-x^*\| \leq d_{n+1}dh^{1/q}$. Since $d_n = O(n^{-1/q})$, the error estimate of the theorem is also established. The proof is complete.

Remark 2. If $E = Lp$ and $1<p<\infty$, then by Remark 1(a),

$$\begin{aligned}&\|x_n-x^*\| \leq O(n^{-(p-1)/p}) \leq O(n^{-(p-1)/2}), \quad \text{if} \quad 1<p<2,\\&\|x_n-x^*\| \leq O(n^{-1/2}), \quad \text{if} \quad p\geq 2.\end{aligned}$$

References

1. F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.
2. W. A. Kirk and R. Schoneberg, *Mapping theorem local expansion in metric and Banach spaces*, J. Math. Anal. Appl. **72** (1979), 114–121.
3. C. E. Chidume, *Approximation of fixed points of strongly pseudocontractive mapping*, Proc. Amer. Math. Soc. **120** (1994), 545–551.
4. C. E. Chidume, *Iterative approximation of fixed points of Lipschitzian strictly pseudocontractive mappings*, Proc. Amer. Math. Soc. **99** (1987), 283–288.
5. S. Reich, *The range of sums of accretive and monotone operators*, J. Math. Anal. Appl. **68** (1979), 310–317.
6. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 508–510.
7. D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, editura Academiei, Bucurseti, Romania, 1978.
8. S. Ishikawa, *Fixed points by a new iteration methods*, Proc. Amer. Math. Soc. **149** (1974), 147–150.
9. O. Hanner, *On the uniform convexity of L^p and ℓ^p*, Ark. Mat. **3** (1956), 239–244.
10. P-K. Lin, *Strongly unique best approximation in uniformly convex Banach spaces*, J. Approx. theory **56** (1989), 101–107.
11. K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 283–288.

Department of Mathematics, Nanchang University, Nanchang, Jiangxi; 330047, The People's Republic of China