N–FOLD COMMUTATIVE BCK–ALGEBRAS

Hua Xie and Yisheng Huang

Received March 8, 1995

Abstract. In this paper we introduce the concept of n-fold commutative BCK-algebras which is a kind of generalization of one of commutative BCK-algebras, and investigate the relations among a number of special types of BCK-algebras, and prove the existence of n-fold commutative BCK-algebras. Finally we show that if $n \geq 2$, the n-fold commutative BCK-algebraic class does not form a variety.

As it is well known, the commutative, implicative and positive implicative BCK-algebraic classes are three important sub-classes in BCK-algebras. The generalizing work of these concepts have been always followed with interest. H. Yutani [12] introduced quasi-commutative BCK-algebras and K. Iséki [5] proved its existence. J. Meng and X. L. Xin in [7], [8] and [9] introduced and discussed commutative, implicative and positive implicative BCI-algebras, respectively. Lanqing Chen [1] introduced n-fold positive implicative BCK-algebras. Y. Huang and Z. Chen [3] introduced n-fold implicative BCK-algebras recently. In this paper we shall introduce the concept of n-fold commutative BCK-algebras which is another generalization of commutative BCK-algebraic concept and we shall obtain a few interesting properties similar to the properties of commutative BCK-algebras.

For the concerned fundamental concepts and properties please refer to [6]. We appoint $x * y^n = (\dots((x * y) * y) * \dots) * y$, (y occurs n times). For convenience to the following discussion, let us recall some concepts (see [1], [3] and [4]).

Let X be a BCK-algebra. Then X is called n-fold positive implicative if there exists a fixed natural number n such that $x * y^{n+1} = x * y^n$ for all x, y in X. X is called n-fold implicative if there exists a fixed natural number n such that $x * (y * x^n) = x$ for all x, y in X. X is called normal if for any a in X, the set $R_a = \{x \in X \mid x * a = x\}$ is an ideal of X.

Definition 1. A BCK-algebra X is called n-fold commutative if there exists a fixed natural number n such that for any x, y in X, $x * y = x * (y * (y * x^n))$.

In the definition, if $n = 1$, by $x * y = x * (y * (y * x))$, we have

$$x * (x * y) = x * (x * (y * (y * x))) \leq y * (y * x)$$

and hence by the symmetry, $x * (x * y) = y * (y * x)$ for any x, y in X, that is, X is a commutative BCK-algebra.

It is obvious.

(1991) Mathematics Subject Classitication. 06F35, 08A05 .

Proposition 1. Let X be a BCK-algebra and m, n two natural numbers with $m \geq n$. If X is n-fold commutative (n-fold positive implicative, n-fold implicative) then X is m-fold commutative (m-fold positive implicative, m-fold implicative).

It is the same as n-fold positive implicative and n-fold implicative BCK-algebras, the n-fold commutative proper BCI-algebras do not exist.

Theorem 1. Let X be a BCI-algebra and n a natural number. If for all x, y in X, there is $x * y = x * (y * (y * x^n))$ then X must be a BCK-algebra.

Proof. The fact that X is a BCK-algebra is got by

$$0 * x = 0 * (x * (x * 0^n)) = 0 * (x * x) = 0 * 0 = 0$$

for all x in X. □

The following equivalent conditions in n-fold commutative BCK-algebras are the generalization of corresponding equivalent conditions in commutative BCK-algebras (in detail, see [10], Theorem I.5.6).

Theorem 2. Let X be a BCK-algebra. Then the following are equivalent: for any x, y, z in X,

(1) X is n-fold commutative;
(2) $x * (x * y) \leq y * (y * x^n)$;
(3) $x \leq z$ and $z * y \leq z * x^n$ imply $x \leq y$;
(4) $x, y \leq z$ and $z * y \leq z * x^n$ imply $x \leq y$;
(5) $x \leq y$ implies $x \leq y * (y * x^n)$.

Proof. (1) $\Longleftrightarrow$ (2) Suppose that X is n-fold commutative then

$$x * (x * y) = x * (x * (y * (y * x^n))) \leq y * (y * x^n),$$

that is, (2) holds. Conversely, the inequality $x * y \leq x * (y * (y * x^n))$ holding is natural. Next by (2), $x * (y * (y * x^n)) \leq x * (x * (x * y)) = x * y$. Hence $x * y = x * (y * (y * x^n))$ and X is n-fold commutative.

(2) $\Longrightarrow$ (3) If $x \leq z$ and $z * y \leq z * x^n$ then $x * z = 0$ and $(z * y) * (z * x^n) = 0$. So by (2), $x * y = (x * (x * z)) * y \leq (z * (z * x^n)) * y = (z * y) * (z * x^n) = 0$. Hence $x \leq y$.

(3) $\Longrightarrow$ (4) It is trivial.

(4) $\Longrightarrow$ (5) Let $x \leq y$ and put $u = y * (y * x^n)$ then $x, u \leq y$ and

$$y * u = y * (y * (y * x^n)) \leq y * x^n.$$

Thus by (4), $x \leq u$, i.e., $x \leq y * (y * x^n)$.

(5) $\Longrightarrow$ (2) By $x * (x * y) \leq x$, we have $y * x^n \leq y * (x * (x * y))^n$, consequently,

$$y * (y * (x * (x * y))^n) \leq y * (y * x^n).$$

Now by $x * (x * y) \leq y$ and (5),

$$x * (x * y) \leq y * (y * (x * (x * y))^n).$$

Hence $x * (x * y) \leq y * (y * x^n)$ and (2) holds. □

Remark 1. It is more convenient for using the condition (5) to check whether a BCK-algebra is n-fold commutative or not.

Example 1. According to Remark 1 we can easily verify that the BCK-algebra B_{5-4-12} (see [10], appendix) is 3-fold commutative but not 2-fold commutative where the $*$-operation of B_{5-4-12} is defined by **Table 1**.

Example 2. Let $X = \{0, 1, 2, \dots, a\}, a \geq 4$ with the $*$-operation in **Table 2**. Then X is a BCK-algebra (see [2], Example 1). It is easy to verify that X is 2-fold commutative but not commutative since $3 * (3 * 2) = 1 \neq 2 = 2 * (2 * 3)$. However X is neither 2-fold positive implicative nor 2-fold implicative since $a * 1^3 = 0 \neq 1 = a * 1^2$ and $1 * (a * 1^2) = 0 \neq 1$. Noting that $x * y^3 = 0$ for any nonzero y in X, we see an interesting fact: for this algebra, it is 3-fold commutative, 3-fold positive implicative and 3-fold implicative.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	0
4	4	2	1	2	0

Table 1

$$x * y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{if } y = 0 \\ x - y, & \text{if } x = a \\ 1, & \text{if } 0 < y < x < a \end{cases}$$

Table 2

Example 3. The set $X = \{0, 1, 2, \dots\}$ with the $*$-operation $x * y = \max\{0, x - y\}$ forms a commutative BCK-algebra and thus by Proposition 1, X is n-fold commutative for all natural number n but it is neither n-fold positive implicative nor n-fold implicative, in fact, put $x = n + 1$ and $y = 1$ then $x * y^{n+1} = 0 \neq 1 = x * y^n$ and $y * (x * y^n) = 0 \neq y$.

Let us discuss the relations among n-fold commutative, n-fold positive implicative and n-fold implicative BCK-algebras.

We know that any n-fold implicative BCK-algebra is n-fold positive implicative but the inverse is false (see [3]).

Theorem 3. Let X be a BCK-algebra. If X is n-fold implicative then X is n-fold commutative but the inverse is false.

Proof. Suppose that $x \leq y$. Since X is n-fold implicative, we have

$$x = x * (y * x^n) \leq y * (y * x^n)$$

and so X is n-fold commutative by Theorem 2. But from Example 2 or Example 3, we see that the inverse is not true. □

Theorem 4. Let X be a BCK-algebra. Then X is n-fold implicative if and only if X is both n-fold positive implicative and n-fold commutative.

Proof. It suffices to prove the part "if". Let $x, y \in X$ and $u = y * x^n$. Since X is n-fold positive implicative, $u * x^n = u$. So by X n-fold commutative and Theorem 2, we have

$$x * (x * (y * x^n)) = x * (x * u) \leq u * (u * x^n) = u * u = 0.$$

Likewise $x * (y * x^n) \leq x$. Hence $x * (y * x^n) = x$, as required. □

Let us consider the normality of n-fold commutative BCK-algebras. First of all, we give a lemma.

Lemma 1. Let X be an n-fold commutative BCK-algebra and $x, a \in X$. Then $x*a = x$ if and only if $a * x = a$.

Proof. If $x * a = x$ then $x * a^n = x$ and so since X is n-fold commutative, we have

$$a * (a * x) \leq x * (x * a^n) = x * x = 0,$$

in addition, $a * x \leq a$. Hence $a * x = a$. Exchanging x and a above, we get the proof of sufficiency. □

Theorem 5. Every n-fold commutative BCK-algebra X is normal.

Proof. Let $a \in X$. By Lemma 1, the set $R_a = \{x \in X \mid a * x = a\}$. Clearly $0 \in R_a$. If $x, y * x \in R_a$, $a * x = a$ and $a * (y * x) = a$. So $a = a * (y * x) = (a * x) * (y * x) \leq a * y$. Also $a * y \leq a$. Hence $a * y = a$, i.e., $y \in R_a$. We have proved that R_a is an ideal of X and X is normal. □

For the finite BCK-algebras, we have the following equivalent conditions:

Theorem 6. Let X be a nonzero finite BCK-algebra. Then the following are equivalent:

(1) There exists a natural number n such that X is n-fold implicative;
(2) There exists a natural number m such that X is m-fold commutative;
(3) X is normal;
(4) X is a sum of a simple ideal family of X, i.e., X itself regarded as an ideal can be generated by a simple ideal family of X.

Proof. In the proof we appoint that $\langle A \rangle$ and $\langle x \rangle$ denote the ideals generated by the sets A and $\{x\}$, respectively.

(1) $\Longrightarrow$ (2) According to Theorem 3, X is m-fold commutative as long as we put $m = n$.

(2) $\Longrightarrow$ (3) It is a direct consequence of Theorem 5.

(3) $\Longrightarrow$ (4) By X nonzero finite, it contains at least a simple ideal. Let A be the ideal generated by all simple ideals of X. We shall prove $X = A$, otherwise there is $A \subset X$ (proper contained). Since X is finite, $X - A$ has at least a minimal element b with respect to BCI-ordering. Suppose that a is a maximal solution of the inequality $x < b$, i.e., $a \leq b$ with $a \neq b$ and for any solution c, $a \leq c$ implies $a = c$ then clearly $a \in A$ and $a \neq 0$. From this we can see that $b * a = b$, i.e., $b \in R_a$ but $a \notin R_a$ since $a * a = 0 \neq a$. Note that $a \leq b \in R_a$, we see that R_a is not an ideal of X, a contradiction with X normal.

(4) $\Longrightarrow$ (1) As $x \geq x * y \geq x * y^2 \geq \ldots$ for any x, y in X, the fact that X is finite implies that there exists a natural number n such that X is n-fold positive implicative. Next, in fact, the condition (4) is clearly equivalent to that X is generated by the atoms of X where a is called an atom if $x \in \langle a \rangle$ and $x \neq 0$ imply $\langle x \rangle = \langle a \rangle$. Now, for any nonzero element x in X, we denote $A = \{a \leq x \mid a \text{ is an atom of } X\}$ and $B = \{a \not\leq x \mid a \text{ is an atom of } X\} \cup \{0\}$ then we can easily verify that $\langle x \rangle = \langle A \rangle$, $\langle x \rangle \cap \langle B \rangle = \{0\}$ and $X = \langle B \cup \{x\} \rangle$. As X is n-fold positive implicative, we have $(y * x^n) * x = y * x^n$ for all y in X. Thus $y * x^n \in \langle B \rangle$ by $X = \langle B \cup \{x\} \rangle$. Then $x * (x * (y * x^n)) \leq x$, $y * x^n$ imply $x * (x * (y * x^n)) \in \langle x \rangle \cap \langle B \rangle = \{0\}$, that is, $x * (x * (y * x^n)) = 0$. Also $x * (y * x^n) \leq x$. We have $x * (y * x^n) = x$ and (1) holds. □

According to (5) of Theorem 2 and (4) of Theorem 6 and the appendix in [10] entitled Complete Classification Tables of BCK-algebras with Order $n \leq 5$ and Its Ideals, we can easily verify the n-fold commutatity of a BCK-algebra with order $n \leq 5$. The calculating results are listed in the following:

There are altogether ten 2-fold commutative but not commutative BCK-algebras and they are B_{4-3-1} and $B_{5-4-k}, k = 1, 2, 3, 4, 5, 6, 7, 8, 9$.

There is only one 3-fold commutative but not 2-fold commutative BCK-algebra and it is B_{5-4-12}.

There are no 4-fold commutative BCK-algebras with order $n \leq 5$.

Let us consider the existence of n-fold commutative BCK-algebras.

Theorem 7. There exists an $(n+1)$-fold commutative but not n-fold commutative BCK-algebra for any natural number n.

Proof. The set $X = \{0, 1, 2, \ldots, n\}$ with the $*$-operation defined by $u * v = \max\{0, u - v\}$ for all u, v in X forms a commutative BCK-algebra with order $n+1$. We make an extension from X to $\overline{X} = X \cup \{a, b\}$ where the $*$-operation on $\overline{X}$ is defined by **Table 3** (in which u, v are in X) and its operation table is listed in **Table 4**.

We shall prove that $\overline{X}$ is an $(n+1)$-fold commutative BCK-algebra but not n-fold commutative. By Theorem I.1.6 in [10] and (5) of Theorem 2, it suffices to verify that the following conditions hold:

(1) $((x * y) * (x * z)) * (z * y) = 0$;
(2) $x * (0 * y) = x$;
(3) $x * y = 0$ and $y * x = 0$ imply $x = y$;
(4) $x \leq y$ implies $x \leq y * (y * x^{n+1})$ and there exist x, y in $\overline{X}$ such that $x \leq y$ but the inequality $x \leq y * (y * x^n)$ does not hold.

$$
\begin{aligned}
&u * v = \max\{0, u - v\},\\
&u * a = \begin{cases} 0, & \text{if } u = 0 \\ u - 1, & \text{if } u \neq 0 \end{cases},\\
&a * u = \begin{cases} a, & \text{if } u = 0 \\ 1, & \text{if } u \neq 0 \end{cases},\\
&u * b = 0,\\
&b * u = \begin{cases} b, & \text{if } u = 0 \\ n - u + 1, & \text{if } u \neq 0 \end{cases},\\
&a * a = a * b = b * b = 0,\\
&b * a = n.
\end{aligned}
$$

Table 3

$*$	0	1	2	3	$\ldots$	$n-1$	n	a	b
0	0	0	0	0	$\ldots$	0	0	0	0
1	1	0	0	0	$\ldots$	0	0	0	0
2	2	1	0	0	$\ldots$	0	0	1	0
3	3	2	1	0	$\ldots$	0	0	2	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$n-1$	$n-2$	$n-3$	$n-4$	$\ldots$	0	0	$n-2$	0
n	n	$n-1$	$n-2$	$n-3$	$\ldots$	1	0	$n-1$	0
a	a	1	1	1	$\ldots$	1	1	0	0
b	b	n	$n-1$	$n-2$	$\ldots$	2	1	n	0

Table 4

It is easy to see from **Table 4** that (2) and (3) hold. In order to prove (1) we replace the left side of equality (1) by the symbol (xyz), that is, $(xyz) = ((x * y) * (x * z)) * (z * y)$, and suppose that the elements u, v, w are in X and x, y are in $\overline{X}$. Then the following are obvious: $(uvw) = (0xy) = (xby) = (xxy) = (xyx) = (yxx) = 0$. The remainder to verify is the following nine kind of **cases:**

$$(auv) = (uav) = (uva) = (buv) = (uvb) = (aub) = (uab) = (bau) = (bua) = 0.$$

Case 1: $(auv) = 0$. If $u = 0$ then it suffices to verify the case that $v \neq 0$, in fact,

$$(auv) = (a * 1) * v = 1 * v = 0.$$

If $u \neq 0$ then when $v = 0$, $(auv) = 1 * a = 1 - 1 = 0$ and when $v \neq 0$,

$$(auv) = (1 * 1) * (v * u) = 0.$$

Case 2: $(uav) = 0$. It suffices to verify the case that $u \neq 0$. If $v = 0$,

$$(uav) = ((u - 1) * u) * 0 = 0.$$

If $v \neq 0$ then when $u \leq v$, $(uav) = ((u-1) * 0) * (v-1) = (u-1) * (v-1) = 0$ and when $u > v$,

$$\begin{aligned}(uav) &= ((u-1) * (u-v)) * (v-1) = ((u-1) * (v-1)) * (u-v) \\ &= ((u-1) - (v-1)) * (u-v) = (u-v) * (u-v) = 0.\end{aligned}$$

Case 3: $(uva) = 0$. We need only to check the case that $u \neq 0$. If $v = 0$ then

$$(uva) = (u * (u-1)) * a = 1 * a = 1 - 1 = 0.$$

If $v \neq 0$ then $(uva) = ((u * v) * (u-1)) * 1 = 0 * 1 = 0$.

Case 4: $(buv) = 0$. If $u = 0$ then it suffices to verify that $v \neq 0$, in fact,

$$(buv) = (b * (n - v + 1)) * v = (n - (n - v + 1) + 1) * v = v * v = 0.$$

If $u \neq 0$ then when $v = 0$, $(buv) = ((n - u + 1) * b) * 0 = 0$ and when $v \neq 0$, if $u \leq v$ then

$$(buv) = ((n-u+1) * (n-v+1)) * (v * u) = ((n-u+1) - (n-v+1)) * (v-u) = 0$$

and if $u > v$ then $(buv) = ((n-u+1) * (n-v+1)) * 0 = 0$.

Case 5: $(uvb) = 0$. If $v = 0$, $(uvb) = (u * 0) * b = 0$. If $v \neq 0$,

$$(uvb) = ((u * v) * 0) * (n - v + 1) = 0.$$

Case 6: $(aub) = 0$. If $u = 0$, $(aub) = (a * 0) * b = 0$. If $u \neq 0$,

$$(aub) = (1 * 0) * (n - u + 1) = 0.$$

Case 7: $(uab) = 0$. It suffices to verify the case that $u \neq 0$, in fact,

$$(uab) = ((u - 1) * 0) * n = 0.$$

Case 8: $(bau) = 0$. If $u = 0$, $(bau) = (n * b) * 0 = 0$. If $u \neq 0$,

$$(bau) = (n * (n - u + 1)) * (u - 1) = (n - (n - u + 1)) * (u - 1) = (u - 1) * (u - 1) = 0.$$

Case 9: $(bua) = 0$. If $u = 0$, $(bua) = (b * n) * a = (n - n + 1) * a = 1 * a = 1 - 1 = 0$. If $u \neq 0$, $(bua) = ((n - u + 1) * n) * 1 = 0 * 1 = 0$.

So far we have proved that $\overline{X}$ is a BCK-algebra.

Let us prove (4). It is easy to check that for any x, y in $\bar{X}$, if $x \leq y$ and $x \neq a$ then $x = y*(y*x)$ and hence $x \leq y*(y*x^{n+1})$. On the other hand, for the case $x = a$, the solutions satisfying the inequality $x \leq y$ are $y = a$ and $y = b$, and it is trivial that $x \leq y * (y * x^{n+1})$ if $x = a$ and $y = a$. If $x = a$ and $y = b$ then we have $b * a^k = n - k + 1$, $k = 1, 2, \ldots, n, n + 1$, and hence

$$b * (b * a^k) = \begin{cases} k, & \text{if } 1 \leq k \leq n, \\ b, & \text{if } k = n + 1. \end{cases}$$

From this we see that $a \leq b * (b * a^{n+1})$ but not $a \leq b * (b * a^n)$. We have already proved that X is an $(n + 1)$-fold commutative BCK-algebra but not n-fold commutative. □

Finally let us consider the problem whether the n-fold commutative BCK-algebraic class forms a variety or not.

A. Worński in [11] proved that the algebra $(X; *, 0)$ of type $(2, 0)$ is a BCK-algebra but it can not be defined by a set of identities in which $X = N \cup A \cup B$, $N = \{0, 1, 2, 3, \ldots\}$, $A = \{a_n \mid n \in N\}$ and $B = \{b_n \mid n \in N\}$, and the $*$-operation is defined as follows: for all m, n in N,

$$\begin{aligned} &m * n = \max\{0, m - n\}, \\ &n * a_m = n * b_m = 0, \\ &a_m * n = a_{m+n}, \\ &b_m * n = b_{m+n}, \\ &a_n * a_m = b_n * b_m = m * n, \\ &a_n * b_m = b_n * a_m = (m + 1) * n. \end{aligned}$$

Since the commutative BCK-algebras can be defined by a set of identities, X is not commutative. We shall prove that X is n-fold commutative for all $n \geq 2$. It suffices to prove that X is 2-fold commutative by Proposition 1. Observing the feature of $*$-operation, by (5) of Theorem 2, we need only to verify that the following four inequalities hold: for all m, n in N,

(1) $m \leq n * ((n * m) * m)$ where $m \leq n$;
(2) $m \leq a_n * ((a_n * m) * m)$;
(3) $a_m \leq a_n * ((a_n * a_m) * a_m)$ where $n \leq m$;
(4) $a_m \leq b_n * ((b_n * a_m) * a_m)$ where $n + 1 \leq m$.

The verification is as follows:

(1) It is trivial.

(2) $m \leq 2m = (n + 2m) * n = a_n * a_{n+2m} = a_n * (a_{n+m} * m) = a_n * ((a_n * m) * m)$.

(3) First it is obvious that $(a_n * a_m) * a_m = 0$. Next since $n \leq m$, $a_m * a_n = n * m = 0$, i.e., $a_m \leq a_n$. Hence $a_m \leq a_n = a_n * 0 = a_n * ((a_n * a_m) * a_m)$.

(4) It is also obvious that $(b_n * a_m) * a_m = 0$. Next if $n+1 \leq m$, $a_m * b_n = (n+1) * m = 0$, i.e., $a_m \leq b_n$. Therefore $a_m \leq b_n = b_n * ((b_n * a_m) * a_m)$.

Summarizing the above facts, we obtain the following theorem.

Theorem 8. The n-fold commutative BCK-algebeaic class with $n \geq 2$ dose not form a variety.

References

1. L. Chen, *n-fold positive implicative BCK-algebras*, *Selected Papers on Fourth National BCK- and BCI-algebraic Academic Conference*, Fuzhou, 1989.
2. A. Grzaślewicz, *On some problem on BCK-algebras*, Math. Japonica **25** (1980), 497–500.
3. Y. Huang and Z. Chen, *n-fold implicative BCK-algebras*, Math. Japonica **46** (1997), 63–67.
4. Y. Huang and Z. Chen, *Normal BCK-algebras*, Math. Japonica **45** (1997), 541–546.
5. K. Iséki, *On the existence of quasi-commutative BCI-algebras*, Math. Seminar Notes **8** (1980), 181–186.
6. K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1–26.
7. J. Meng and X. L. Xin, *Commutative BCI-algebras*, Math. Japonica **37** (1992), 569–572.
8. J. Meng and X. L. Xin, *Implicative BCI-algebras*, *Selected Papers on BCK− and BCI-algebras*, vol. 1, 1992, pp. 50–53.
9. J. Meng and X. L. Xin, *Positive implicative BCI-algebras*, *Selected Papers on BCK- and BCI-algebras*, vol. 1, 1992, pp. 45–49.
10. J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Korea, 1994.
11. A. Worński, *BCK-algebras do not form a variety*, Math. Japonica **28** (1983), 211–213.
12. H. Yutani, *Quasi-commutative BCK-algebras and congruence relations*, Math. Seminar Notes **5** (1977), 469–480.

Department of Mathematics, Longyan Normal College, Longyan, Fujian 364000, P. R. China