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OPERATOR-COMPACT AND OPERATOR-CONNECTED SPACES

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Abstract. In this paper we discuss some properties of operator-compact spaces and introduce the concept of operator connected spaces.

In [1], Kasahara introduced the concept of an operator associated to a topology and gave some definitions which are equivalent to the usual ones when the operator involved is the identity operator. We will be using his definitions in somewhat modified form and his results to prove properties similar to the usual ones in General Topology. Introducing a concept of stable operators with respect to a subset, we can correct a small mistake in Theorem 6 of [1]. Throughout this paper we consider non-empty topological spaces on which no separation axioms are assumed unless explicitely stated and the simbol ■ is used to indicate the end or amission a proof.

Definition 1. *Let (X,τ) be a topological space, B be a subset of X and α be an operator from τ to $P(X)$, i.e $\alpha:\tau\to P(X)$. We say that α is an operator an τ if:*
($\mathcal{O}$) $U\subset\alpha(U)$ for every $U\in\tau$.
We say that the operator α on τ is stable with respect to B *if:*
($\mathcal{S}$) α induces an operator $\alpha_B:\tau_B\to P(B)$ such that $\alpha_B(U\cap B)=\alpha(U)\cap B$ for every $U\in\tau$ where τ_B is the relative topology on B.

For a subset A of a topological space (X,τ), the closure of A respect to τ and the interior of A respect to τ and the interior of A respect to τ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. For a subspace (B,τ_B) of (X,τ) and its subset E of (B,τ_B), the closure of E respect τ_B id denoted by $\mathrm{Cl}_B(E)$.

For a dense subset B of (X,τ), we have a stable operator with respect to the set B as follows:

Proposition 1. *Let $\alpha:\tau\to P(X)$ be the closure operator defined by $\alpha(U)=Cl(U)$ for every $U\in\tau$. If a subset B is dense in (X,τ), then α satisfies $(\mathcal{O}_1)$ and it is stable with respect to B.*

Proof. Let $U\in\tau$ and let $\alpha_B(U\cap B)=\mathrm{Cl}_B(U\cap B)$, Then, it is known that if B is dense, $\mathrm{Cl}(U)=\mathrm{Cl}(U\cap B)$ holds for $U\in\tau$. Therefore, α satisfies $(\mathcal{O}_1)$ and it is stable with respect to the dense set B because $\alpha_B(U\cap B)=\mathrm{Cl}(U\cap B)\cap B=\mathrm{Cl}(U)\cap B=\alpha(U)\cap B$ hold. ■

Proposition 2. *Let $\alpha:\tau\to P(X)$ be an operator satisfying the following two properties:*
$(\mathcal{O}_1)$ *$U\subset\alpha(U)$ for every $U\in\tau$,*
$(\mathcal{O}_2)$ *$\alpha(\emptyset)=\emptyset$.*

If α is stable with respect to all of proper closed set of (X,τ), then α is the identity operator.

Proof. Let $U\in\tau$ and $U\neq\emptyset$. Then $X\backslash U$, say F, is a proper closed set. Now we have that $\alpha(U)\cap F=\alpha_F(U\cap F)=\alpha_F(\emptyset)=\alpha_F(\emptyset\cap F)=\alpha(\emptyset)\cap F=\emptyset\cap F=\emptyset$ by $(\mathcal{S})$ and $(\mathcal{O}_2)$. Then we have $\alpha(U)\cap(X\backslash U)=\emptyset$ and hence $\alpha(U)\subset U$. By using $(\mathcal{O}_1)$ it is proved that $U=\alpha(U)$ for any non-empty $U\in\tau$. Since $\alpha(\emptyset)=\emptyset$ by $(\mathcal{O}_2)$, we proved that $\alpha(U)=U$ for any $U\in\tau$. ■

The example 1 shows that there exists a stable and non-identity operation which does not satisfy the property $(\mathcal{O}_2)$.

Example 1. *Let $\alpha:\tau\to P(X)$ and $\alpha_B:\tau_B\to P(B)$ be operators defined by $\alpha(U)=X$ and $\alpha_B(U\cap B)=B$ for every $U\in\tau$ where B is a subset of X. Then, α satisfiesis $(\mathcal{O}_1)$ and it is stable with respect to B. However $\alpha(\emptyset)\neq\emptyset$.*

Example 2. *Let $X=R$ (the set of all real numbers) with usual topology τ. Let $B=[0,1]$ be a closed interval of X. Then the closure operator α is not the identity and it satisfies $(\mathcal{O}_1)$ and $(\mathcal{O}_2)$. However $Cl_B(U\cap B)\neq Cl(U)\cap B$ where $U=(-1,0)\cup\left(\frac{1}{3},\frac{1}{2}\right)$.*

Definition 2. *([2])Let (X,τ) be a topological space and α be an operator on τ. A subset A of X is said to be α-open if for each $x\in A$ there exists an τ-open neighborhood U of x such that $\alpha(U)\subset A$. A subset B of X is α-closed if its complement $X\backslash B$ is α-open.*

Note that the family τ_α of all α-open sets is a subset of τ.

Definition 3. *([1])Let (X,τ) be topological space and α be an operator on τ. We say that (X,τ) is α-regular if for every $x\in X$ and every τ-open neighborhood U of x there exists a τ-open neighborhood V of x such that $\alpha(V)\subset U$.*

Definition 4. *([1]) Let (X,τ) be topological space and α be an operator on τ. We say that α is a regular operator if for every $x\in X$ and every pair U,V of τ-open neigborhood of x there exists a τ-open neighborhood W of x such that $\alpha(W)\subseteq\alpha(U)\cap\alpha(V)$.*

In general the family τ_α is not a topology, but if α is a regular operator, then τ_α is a toplogy on X.

Definition 5. *([2]) Let (X,τ) be a topological space and α be an operator on τ. We say that (X,τ) is an α-T_2 space if for every pair x,y of distinct points of X there exists τ-open sets U,V such that $x\in U$, $y\in V$ and $\alpha(U)\cap\alpha(V)=\phi$.*

Clearly if the space (X,τ) is α-T_2 it is also T_2.

Definition 6. *([2]) Let (X,τ) and (Y,φ) be two topological space and α,β be operators on τ and φ respectively. We say that a function $f:(X,\tau)\to(Y,\varphi)$ is (α,β)-continuous if for each point $x\in X$ and every φ-open neighborhood V of $f(x)$, there exists a τ-open neigborhood U of x such that $f(\alpha(U))\subseteq\beta(V)$.*

Theorem 1. *Let* (X,τ) *and* (Y,φ) *be two topological spaces and* α *be an operator on* τ. *If* $f:(X,\tau)\to(Y,\varphi)$ *is* (α,id)*-continuous map, then* f *is a continuous map in the usual sence.*

Proof. Let $x\in X$ and V an φ open neighborhood of $f(x)$. Since f is an (α,id)-continuous, there exists a τ-open neighborhood U of x such that $f(\alpha(U))\subseteq V$. But since $U\subseteq\alpha(U)$, we have that $f(U)\subseteq f(\alpha(U))\subseteq V$, which implies that f is continuous. ■

Theorem 2. *Let* (X,τ), (Y,φ) *and* (Z,ψ) *be topological spaces and* α, β, γ *be operators on* τ, φ *and* ψ *respectively. If* $f:(X,\tau)\to(Y,\psi)$ *is* (α,β)*-continuous and* $g:(Y,\varphi)\to(Z,\psi)$ *is* (β,γ)*-continuous map, then* $g\circ f$ *is* (α,γ)*-continuous.*

Definition 7. *([1]) Let* (X,τ) *be a topological space and* α *be an operator on* τ. *A subset* K *of* X *is said to be* α*-compact if for every* τ*-open cover , of* K, *there exists a finite collection* $\{C_1,\dots,C_n\}$ *of , such that* $K\subseteq\bigcup_{i=1}^{n}\alpha(C_i)$.

It is easy to see that every compact subset K of X is α-compact for each operator α on τ. Howerver if (X,τ) is α-compact and (X,τ) is α-regular, then (X,τ) is compact.

Theorem 3. *Let* (X,τ) *be a topological space,* A *be a subset of* X, K *be a subset of* A *and* α *be an operator on* τ. *If* A *is* α*-compact and* K *is* α*-closed, then* K *is* α*-compact.*

Proof. Let , be a τ-open covering of K. Since K is α-closed, then $X\backslash K$ is α-open and therefore for every $x\in A\backslash K$ there exists a τ-neighborhood V_x such that $\alpha(V_x)\subseteq X\backslash K$. In this way we obtain that $\Phi=\ ,\ \cup\{V_x : x\in A\backslash K \text{ and } \alpha(V_x)\subseteq X\backslash K\}$ is a τ-open cover of A. Since A is α-compact there exists a finite collection $\{U_1,U_2,\dots,U_n\}$ of Φ such that $A\subseteq\bigcup_{i=1}^{n}\alpha(U_i)$. Since $\alpha(V_x)\subseteq X\backslash K$ for $x\in A\backslash K$, there exists a subcollection $\{U_{i1},U_{i2},\dots,U_{ij}\}\subseteq\{U_1,\dots,U_n\}$ where $j\le n$ such that $\{U_{i1},U_{i2},\dots,U_{ij}\}\subset\ ,$ and therefore $K\subseteq\bigcup_{k=1}^{j}\subseteq\alpha(U_{ik})$. So K is α-compact. ■

Corollary 1. *Let* (X,τ) *be a topological space,* K *be a subset of* X *and* α *be an operator on* τ. *If* (X,τ) *is* α*-compact and* K *is* α*-closed, then* K *is* α*-compact.*

Proof. The proof follows if we take $X=A$ in the theorem 3. ■

In the above corollary the hypothesis that K is α-closed is necessary as we show in the following example.

Example 3. *Let* N *be the set of all natural numbers with the discrete topology* τ, *and let* i_0 *be a fixed odd number. We define* $\alpha:\tau\to P(N)$ *as follows:*

$$\alpha(\{n\})=\begin{cases}\{2i : i\in N\} & \textit{if } n \textit{ is an even number}\\ \{2i+1 : i\in N\} & \textit{if } n=i_0\\ \{n\} & \textit{if } n \textit{ is an odd number} \neq i_o\end{cases}$$

and $\alpha(A)=N$ *for the rest.*

Notice that N is α-compact, $N\backslash\{i_0\}$ is a closed set but is not α-closed and $N\backslash\{i_0\}$ is not α-compact. ■

Theorem 4. *Let* (X,τ) *be a topological space and* α *be regular operator on* τ. *If* X *is* α-T_2 *and* $K\subseteq X$ *is* α*-compact, then* K *is* α*-closed.*

Proof. We need to prove that $X\backslash K$ is α-open. So let $x_0 \in X\backslash K$. For each $y \in K$, there exists τ-open neighborhoods U_y and V_y such that $x_0 \in U_y$, $y \in V_y$ and $\alpha(U_y) \cap \alpha(V_y) = \emptyset$. In this way we construct an open cover , $= \{V_y : y \in K\}$ of K. Since K is α-compact, there exists a finite collection $\{V_{y_1}, \dots , V_{y_n}\}$ of , such that

$$K \subseteq \bigcup_{i=1}^{n} \alpha(V_{y_i}).$$

Let $U = \bigcap_{i=1}^{n} U_{y_i}$. We can see that U is a τ-open neigborhood of x_0, but it does not have to happen that $\alpha(U) \subseteq X\backslash K$. Here we need the regularity of α to achieve our purpose.

Since $U_{y_1}, \dots , U_{y_n}$ are τ-open neighborhoods of x_0, then using the regularity of α there exists a τ-open neighborhood W of x_0, such that $W \subseteq \alpha(W) \subseteq X\backslash K$. This implies that $X\backslash K$ is α-open, and hence K is α-closed. ■

Theorem 5. *Let (X, τ) be a topological space, K be a subset of X and α be operator on τ which is stable with respect to K. If (X, τ) is α-compact and K is α-closed, then K is α_K-compact.*

Proof. Let $\Phi = \{U_\beta\}_{\beta \in I}$ an open cover of K by τ_K-open sets. Lets $\Phi^* \subseteq \tau$ be the sets of all τ-open sets such that for each $V \in \Phi^*$, $V \cap K \in \Phi$. Since $X\backslash K$ is τ-open, we can take a τ-open cover of $X\backslash K$ say $\Psi = \{W_x \in \tau : \alpha(W_x) \subseteq X\backslash K, x \in X\backslash K\}$. Then the collection $\Phi^* \cup \Psi$ is a τ-open cover of X. Since X is α-compact, we have two finite subcollections $\{V_1, \dots , V_n\} \subseteq \Phi^*$ and $\{W_1, \dots , W_m\} \subseteq \Psi$ such that

$$X = \left\{ \bigcup_{i=1}^{n} \alpha(V_i) \right\} \bigcup \left\{ \bigcup_{j=1}^{m} \alpha(W_j) \right\}.$$

Then

$$K = \left\{ \bigcup_{i=1}^{n} \alpha(V_i) \cap K \right\} \bigcup \left\{ \bigcup_{j=1}^{m} \alpha(W_j) \cap K \right\} = \left\{ \bigcup_{i=1}^{n} \alpha_K(V_i \cap K) \right\} = \bigcup_{i=1}^{n} \alpha_K(U_i),$$

since $\alpha(W_j) \cap K = \phi$ for $j = 1, 2, \dots , m$ and α is stable with respect to K. Therefore K is α_K-compact. ■

At this point we would like to point out a small mistake in Theorem 6 of [1]. In fact, in [1:Th. 6] $\alpha_K : \tau_K \to P(K)$ satisfying $\alpha_K(G \cap K) = \alpha(G) \cap K$ with $G \in \tau$, is not necessarily well defined in some cases as follows: Let α be an operator on τ defined as $\alpha(G) = Cl(G)$ for every $G \in \tau$ and $X = R$ with the usual topology $\tau, \alpha(G) = Cl(G)$ for every $G \in \tau$ and $K = [0, 1]$. If G= $(\frac{1}{4}, \frac{1}{2}) \cup (1, 2)$ and $G' = (-1, 0) \cup (\frac{1}{4}, \frac{1}{2})$, we have on one hand

$$\alpha(G) \cap K = \left(\left[\frac{1}{4}, \frac{1}{2} \right] \cup [1, 2] \right) \cap K = \left[\frac{1}{4}, \frac{1}{2} \right] \cup \{1\}$$

and on the other hand

$$\alpha(G') \cap K = \left[\frac{1}{4}, \frac{1}{2} \right] \cup \{0\},$$

which indicates that α_K is not well defined. Then Theorem 6 of [1] can be stated as follows.

Theorem 6. *Let (X, τ) be a topological space and K be a subset of X. Let α be an operator on τ and stable with respect to K. Then K is α-compact if and only if K is α_K compact.*

Proof. It is the same as in [1]. ■

Theorem 7. *Let K be a subset of X and let $\alpha : \tau \to P(X)$ and $\alpha_K : \tau_K \to P(K)$ be operators satisfying the following properties:*

$(\mathcal{S}')$ $\alpha_K(V \cap K) \subset \alpha(V) \cap K$ for any open set V of X such that $V \cap K \neq \emptyset$. If K is α_K-compact in (K, τ_K), then K is α-compact.

Proof. Let $\mathcal{C}$ be an open cover of K. Then $\{G \cap K | G \in \mathcal{C}\} \subset \tau_K$ is a cover of K and so there exists a finite subfamily $\{G_1, G_2, \dots, G_n\}$ of $\mathcal{C}$ such that $K = \cup_{i=1}^{n} \alpha_K(G_i \cap K) \subset \cup_{i=1}^{n} \alpha(G_i) \cap K \subset \cup_{i=1}^{n} \alpha(G_i)$. Therefore, K is α-compact. ■

In the following examples, the sets K and operations α and α_K satisfy the condition $(\mathcal{S}')$ in Theorem 20.

Example 4. *Let K be an open set in (X, τ). Then, any operation $\alpha : \tau \to P(X)$ satisfying $(\mathcal{O}_1)$ induces an operation $\alpha_K : \tau_K \to P(K)$ as follows: $\alpha_K(V \cap K) = \alpha(V \cap K) \cap K$ for each $V \in \tau$. Moreover, if α is monotone, (i.e., $\alpha(A) \subset \alpha(B)$ for sets A and B such that $A \subset B$), $\alpha_K(V \cap K) \subset \alpha(V) \cap K$ for each $V \in \tau$. Therefore, the condition $(\mathcal{S}')$ is satisfied.*

Example 5. *Let K be a preopen set of (X, τ), that is, $K \subset Int(Cl(K))$ holds by definition. Let $\alpha : \tau \to P(X)$ be the Interior-Closure operation, i.e, $\alpha(V) = Int(Cl(V))$ and $\alpha_K(V \cap K) = Int_K(Cl_K(V \cap K))$ for every set $V \in \tau$. Then, it is known that $Int(Cl(V \cap K)) \cap K = Int_K(Cl_K(V \cap K))$ holds if K is preopen in (X, τ). Therefore, the condition $(\mathcal{S}')$ is satisfied, that is, $\alpha_K(V \cap K) = Int(Cl(V \cap K) \cap K) \subset Int(Cl(V)) \cap K = \alpha(V) \cap K$ for each $V \in \tau$.*

Theorem 8. *The finite union of α-compact subsets of X is α-compact.*

In the end of this paper, we conclude operator-connected spaces.

Definition 8. *Let (X, τ) be a topological space and α be an operator on τ. X is said to be α-connected if there is not function $f : (X, \tau) \to \{0, 1\}$ which is (α, id)-continuous and onto.*

Theorem 9. *Let (X, τ) be a topological space and α be an operator on τ. If X is connected then it is α-connected.*

Proof. The proof is straightforward from Theorem 1 and Definition 8. ■

Theorem 10. *Let (X, τ) and (Y, τ') be topological spaces and α, β be operator on τ and τ' respectively. If $f : (X, \tau) \to (Y, \tau')$ is an onto (α, β)-continuous function and (X, τ) is α-connected, then (Y, τ') is β-connected.*

Proof. If there exists a function $g : (Y, \tau') \to \{0, 1\}$ wich is (β, id)-continouos, then by Theorem 12, $g \circ f : (X, \tau) \to \{0, 1\}$ is (α, id)-continuous, which implies that X is not α-connected. ■

Theorem 11. *Let (X, τ) be a topological space and α be operator on τ such that the composite operator $\alpha \circ \alpha$ is well defined and $\alpha \circ \alpha = \alpha$. If (X, τ) is α-connected, then there are not τ-open sets U, V such that $\alpha(U)$ and $\alpha(V)$ are open and form a partition of X. If in addition α is additive, then the converse is also true.*

Proof. Let U and V be open sets that $\alpha(U)$ and $\alpha(V)$ form an open partition of X.

Define $f : X \to \{0,1\}$ as follows

$$f(x) = \begin{cases} 0 & if \quad x \in \alpha(U) \\ 1 & if \quad x \in \alpha(V), \end{cases}$$

f is an onto map and (α, id)-continuous. Then X is not α-connected.

Now suppose that there exists $f : X \to \{0,1\}$ such that f is onto and (α, id)-continuous. Let $x \in f^{-1}(0)$, then there exists an open set V_x such that $f(\alpha(V_x)) = \{0\}$.

So, the set $U = \bigcup\limits_{x \in f^{-1}(0)} \alpha(V_x) = f^{-1}(0)$.

We claim that $\alpha(U) = U$. In fact,

$$\begin{aligned} \alpha(U) &= \alpha\left(\bigcup_{x \in f^{-1}(0)} \alpha(V_x)\right) = \alpha\left(\alpha\left(\bigcup_{x \in f^{-1}(0)} V_x\right)\right) \\ &= \alpha\left(\bigcup_{x \in f^{-1}(0)} V_x\right) = \bigcup_{x \in f^{-1}(0)} \alpha(V_x) = U. \end{aligned}$$

Also since f is continuous, we have that $\alpha(U)$ is open. Now if we proceed in similar way with $f^{-1}(1)$, we get an open set V such that $\alpha(V) = V$ and they form an open partition of X.

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