

THE SPECTRAL SPACE OF MV-ALGEBRAS IS A STONE SPACE

JIE MENG AND YOUNG BAE JUN[†]

Received July 20, 1995

ABSTRACT. Let X be an MV-algebra and let $\text{Spec}(X)$ be the set of all prime ideals of X . For any ideal I of X , denote $S(I) = \{P \in \text{Spec}(X) : I \not\subseteq P\}$ and $T(X) = \{S(I) : I \in \mathcal{I}(X)\}$ where $\mathcal{I}(X)$ is the set of all ideals of X . On base of the notes [2] and [10] we prove that the spectral space $(\text{Spec}(X), T(X))$ is a Stone space.

In order to study many value logics by means of algebraic method, C. C. Chang ([6, 7]) introduced MV-algebras. Recently, the algebraic theory of MV-algebras is intensively studied (see [4], [5] and [9]). M. Palasinski ([17]) and C. S. Hoo and P. V. R. Murty ([10]) introduced topologies on the set of all prime ideals of a commutative BCK-algebra by different way, respectively. M. Aslam, E. Y. Deeba and A. B. Thaheem ([2]) developed this theory and applied it to investigate some properties of commutative BCK-algebras. We know from [16] that MV-algebras are categorically equivalent to bounded commutative BCK-algebras.

In this note we will discuss relationship between prime ideal space of an MV-algebra and Stone space.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies

- (1) $((x * y) * (x * z)) * (z * y) = 0$,
- (2) $(x * (x * y)) * y = 0$,
- (3) $x * x = 0$,
- (4) $0 * x = 0$,
- (5) $x * y = y * x = 0$ implies $x = y$,
- (6) $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra satisfying the condition $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, is said to be *commutative*. If there is an element $1 \in X$ such that $x * 1 = 0$ for all $x \in X$, then X is called *bounded*. For short we call a bounded commutative BCK-algebra an *MV-algebra*. Throughout this paper X will always denote an MV-algebra without mentioned otherwise. We write $Nx = 1 * x$ and $x \vee y = N(Nx \wedge Ny)$. T. Traczyk ([19]) proved that $(X; \vee, \wedge, 0, 1)$ is a bounded distributive lattice. A nonempty subset I of X is an *ideal* ([11]) if it satisfies

- (i) $0 \in I$,
- (ii) $x * y, y \in I$ implies $x \in I$.

Any ideal I satisfies

- (iii) $x \leq y$ and $y \in I$ imply $x \in I$,
- (iv) $x, y \in I$ implies $x \vee y \in I$.

This shows that an ideal must be a lattice ideal, but the inverse need not hold.

1991 Mathematics Subject Classification. 54A10, 06F35, 06D05.

Keywords and phrases. MV-algebra, ideal, spectrum, lattice-filter, stone space.

[†] Supported by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. BSRI-95-1406.

An ideal I is said to be *prime* if

(v) $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

The set of all ideals of X is denoted by $\mathcal{I}(X)$; the set of all prime ideals of X is denoted by $\text{Spec}(X)$, called the *spectrum* of X (see [2]). For a subset A of X , we call the least ideal containing A the *generated ideal* by A , written (A) . K. Iséki ([11]) proved that $x \in (A)$ if and only if there are $a_1, a_2, \dots, a_n \in A$ such that $(\dots(x * a_1) * \dots) * a_n = 0$. We write (a) for $(\{a\})$. Define $S(A) = \{P \in \text{Spec}(X) : A \not\subseteq P\}$. For short denote $S(a)$ instead of $S(\{a\})$.

Lemma 1 ([2]). *The family $T(X) = \{S(I) : I \in \mathcal{I}(X)\}$ forms a topology on $\text{Spec}(X)$.*

Because $S(A) = S((A))$ and, in particular, $S(a) = S((a))$, we know that $T_0(X) = \{S(a) : a \in X\} \subseteq T(X)$.

Lemma 2 ([10]). *$T_0(X)$ is a base of $T(X)$.*

The topological space $(\text{Spec}(X), T(X))$ is called the *spectral space* of X ([2]).

A *ring of sets* is a nonempty set R of subsets of a set S such that if $A, B \in R$ then $A \cup B \in R$ and $A \cap B \in R$ (see [3; II.2, Definition 2]).

In what follows we will prove that $T_0(X)$ is a ring of sets.

Proposition 1. *For any x, y in X , $S(x) \cup S(y) = S(x \vee y)$ and $S(x) \cap S(y) = S(x \wedge y)$, i.e., $T_0(X)$ is a ring of sets.*

Proof. By (iii) and (iv) we have

$$\begin{aligned} P \in S(x) \cup S(y) &\Leftrightarrow P \in S(x) \text{ or } P \in S(y) \\ &\Leftrightarrow x \notin P \text{ or } y \notin P \\ &\Leftrightarrow x \vee y \notin P \\ &\Leftrightarrow P \in S(x \vee y), \end{aligned}$$

so $S(x) \cup S(y) = S(x \vee y)$. Taking use of properties of prime ideals it follows that

$$\begin{aligned} P \in S(x) \cap S(y) &\Leftrightarrow P \in S(x) \text{ and } P \in S(y) \\ &\Leftrightarrow x \notin P \text{ and } y \notin P \\ &\Leftrightarrow x \wedge y \notin P \\ &\Leftrightarrow P \in S(x \wedge y), \end{aligned}$$

thus $S(x) \cap S(y) = S(x \wedge y)$. This completes the proof.

Proposition 2. *If A is a compact open subset of $(\text{Spec}(X), T(X))$, then there is $x \in X$ such that $A = S(x)$.*

Proof. Since A is open and $T_0(X)$ is a base of $T(X)$, there is $\{x_\alpha : \alpha \in B\} \subseteq X$ such that $A = \cup\{S(x_\alpha) : \alpha \in B\}$. Using compactness of A and Proposition 1, we have a finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of B satisfying

$$A = S(x_{\alpha_1}) \cup S(x_{\alpha_2}) \cup \dots \cup S(x_{\alpha_n}) = S(x_{\alpha_1} \vee x_{\alpha_2} \vee \dots \vee x_{\alpha_n}),$$

proving the proposition.

We will need the notion of lattice-filters and some results.

Definition 1 ([1]). Let $(X; *, 0)$ be a commutative BCK-algebra. A nonempty subset D of X is called a *lattice-filter* of X if it satisfies:

- (D₁) $x \leq y, x \in D$ imply $y \in D$,
- (D₂) $x, y \in D$ implies $x \wedge y \in D$.

For a nonempty subset A of X , $\langle A \rangle$ denote the least lattice-filter containing A , called the *lattice-filter generated* by A .

Lemma 3 ([1]). *Let X be a commutative BCK-algebra and A a nonempty subset of X . Then*

$$\langle A \rangle = \{x \in X : a_1 \wedge a_2 \wedge \dots \wedge a_n \leq x \text{ for some } a_1, a_2, \dots, a_n \in A\}.$$

C. S. Hoo and P. V. R. Murty gave the prime ideal theorem in BCK- algebras as follows.

Lemma 4 ([10]). *Suppose A is a nonempty subset of X and \wedge -closed. Let I be an ideal of X and $I \cap A = \emptyset$. Then there is a prime ideal P of X such that $I \subseteq P$ and $A \cap P = \emptyset$.*

The following result is important for compactness of spectral space.

Proposition 3. *Let A and B be nonempty subsets of X . If $\cap\{S(a) : a \in A\} \subseteq \cup\{S(b) : b \in B\}$, then there are nonempty finite subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $\cap\{S(a) : a \in A_0\} \subseteq \cup\{S(b) : b \in B_0\}$.*

Proof. At first we prove $\langle A \rangle \cap (B] \neq \emptyset$. If not, then by Lemma 4 there exists $P \in \text{Spec}(X)$ such that (i) $\langle A \rangle \cap P = \emptyset$ and (ii) $(B] \subseteq P$. The statement (i) shows that for all $a \in A$, we have $a \notin P$, i.e., $P \in S(a)$; hence $P \in \cap\{S(a) : a \in A\}$. On the other hand the statement (ii) shows that for all $b \in B$, we get $b \in P$, i.e., $P \notin S(b)$, and so $P \notin \cup\{S(b) : b \in B\}$. This contradicts to $\cap\{S(a) : a \in A\} \subseteq \cup\{S(b) : b \in B\}$. Hence $\langle A \rangle \cap (B] \neq \emptyset$. Choose $a \in \langle A \rangle \cap (B]$. By Lemma 3, $a \in \langle A \rangle$ implies that there are $a_1, a_2, \dots, a_n \in A$ such that $a_1 \wedge a_2 \wedge \dots \wedge a_n \leq a$, and hence $S(a_1 \wedge a_2 \wedge \dots \wedge a_n) \subseteq S(a)$. Taking use of Proposition 1 and induction, we know $S(a_1 \wedge a_2 \wedge \dots \wedge a_n) = S(a_1) \cap S(a_2) \cap \dots \cap S(a_n)$, therefore

$$(7) \quad S(a_1) \cap S(a_2) \cap \dots \cap S(a_n) \subseteq S(a).$$

It follows from $a \in (B]$ that there are $b_1, b_2, \dots, b_m \in B$ such that $(\dots(a * b_1) * \dots) * b_m = 0$. We assert that

$$(8) \quad S(a) \subseteq S(b_1) \cup \dots \cup S(b_m).$$

In fact, if $P \in S(a)$ then $a \notin P \in \text{Spec}(X)$. Suppose $P \notin S(b_1) \cup \dots \cup S(b_m)$, that is, $P \notin S(b_i)$ ($i = 1, \dots, m$). Then $b_i \in P$ ($i = 1, \dots, m$), and so $a \in P$. This is a contradiction. This proves that (8) holds. By (7) and (8) we have

$$S(a_1) \cap \dots \cap S(a_n) \subseteq S(b_1) \cup \dots \cup S(b_m),$$

proving the proposition.

This proposition is a base of compactness of spectral space. It has some interesting corollaries.

Corollary 4 ([10]). *Every $S(a)$ is a compact open subset of $\text{Spec}(X)$.*

Proof. It follows from taking $A = \{a\}$ in Proposition 3.

Combining Proposition 2 we have

Corollary 5. *An open subset O of $\text{Spec}(X)$ is compact if and only if there is $a \in X$ such that $O = S(a)$.*

Corollary 6. *The spectral space $\text{Spec}(X)$ is compact.*

Proof. It is easy to see that $\text{Spec}(X) = S(1)$, so we take $a = 1$ in Corollary 4 to obtain $\text{Spec}(X)$ being compact.

Corollary 7. *The family of compact open subsets of $\text{Spec}(X)$ is a ring of sets and is a base of X .*

Proof. This is a consequence of Lemma 2 and Proposition 1.

In order to mention our main result we need the following.

Definition 2 ([3]). A *Stone space* is a topological space X satisfying:

- (S₁) X is a T_0 -space,
- (S₂) The family of compact open subsets of X is a ring of sets and a base for X ,
- (S₃) If $\{X_s : s \in S\}$ and $\{Y_t : t \in T\}$ are nonempty families of nonempty compact open sets and $\cap\{X_s : s \in S\} \subseteq \cup\{Y_t : t \in T\}$ then there exist finite nonempty subsets $S_0 \subseteq S$ and $T_0 \subseteq T$ such that $\cap\{X_s : s \in S_0\} \subseteq \cup\{Y_t : t \in T_0\}$.

Combining Lemma 2, Corollary 6 and Proposition 3, we obtain the main result.

Theorem. *The spectral space $(\text{Spec}(X), T(X))$ is a Stone space.*

REFERENCES

1. B. Ahmad, *Dual ideals in BCK-algebras I*, Math. Seminar Notes **10** (1982), 243 - 250.
2. M. Aslam, E. Y. Deeba and A. B. Thaheem, *On Spectral properties of BCK-algebras*, Math. Japon. **38** (1993), 1121 - 1128.
3. R. Balbes and P. Dwinger, *Distributive lattice*, University of Missouri Press (1974).
4. L. P. Belluce, *Semisimple algebras of infinite valued logic and bold fuzzy set theory*, Can. J. Math. **XXXVIII** (1986), 1356 - 1379.
5. L. P. Belluce, *Semisimple and complete MV-algebras*, Algebra Universalis **29** (1992), 1 - 9.
6. C. C. Chang, *Algebraic analysis on many value logic*, Trans. Amer. Math. Soc. **88** (1958), 467 - 490.
7. C. C. Chang, *A new proof of the completeness of the Lukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1959), 74 - 80.
8. A. Filipoiu, *About Baer extensions of MV-algebras*, Math. Japon. **40** (1994), 236 - 241.
9. C. S. Hoo, *MV-algebras, ideals and semisimplicity*, Math. Japon. **34** (1989), 563 - 583.
10. C. S. Hoo and P. V. R. Murty, *The ideals of a bounded commutative BCK-algebras*, Math. Japon. **32** (1987), 723 - 733.
11. K. Iséki, *On ideals in BCK-algebras*, Math. Seminar Notes **3** (1975), 1 - 12.
12. K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japon. **21** (1976), 351 - 366.
13. J. Meng, *On ideals of BCK-algebras*, Math. Japon. **40** (1994), 143 - 154.
14. J. Meng, *Ideals in BCK-algebras (IV)*, Pure and Appl. Math. (in China) **10** (19), 13 - 18.
15. J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co. Seoul, Korea (1994).
16. D. Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon. **31** (1986), 889 - 894.
17. M. Palasinski, *On a space of prime ideals of a BCK-algebra*, Math. Seminar Notes **10** (1982), 463 - 465.
18. M. H. Stone, *Topological representations of distributive lattices and Brouwerian logics*, Casopis Pest. Math. **67** (1937), 1- 25.
19. T. Traczyk, *On the variety of bounded commutative BCK-algebras*, Math. Japon. **24** (1979), 283 - 292.
20. H. Yutani, *On a system of axioms of a commutative BCK-algebras*, Math. Seminar Notes **5** (1977), 427 - 432.

DEPARTMENT OF MATHEMATICS EDUCATION, GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701,
KOREA

E-mail: ybjun@nongae.gsnu.ac.kr