NOTES ON TOPOLOGICAL BCK-ALGEBRAS

Dong Soo Lee, Dong Neun Ryu

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Abstract. In this paper we will define a topological BCK-algebra and find some properties of this structure. Especially we will give a filter base generating a BCK-algebra topolgy which is a fundamental neighborhood system of zero for that topology and find some properties of that BCK-algebra topology.

1. Introduction

Since K. Iseki and Imai gave an algebraic formulation for the BCK-propositional calculus system, numerous mathematical papers have been written investigating the algebraic properties of the BCK-algebras. Especially R.Alo and E.Deeba attempted to study the topological aspects of the BCK-structure [1]. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have been made to study the topological structures making the star operation of BCK-algebra continuous.

Theories of topological groups, topological rings and topological modules are well known and still investigated by many mathematicians. Even topological universal algebraic structures have been studied by some authors.

In this paper we initiate the study of topological BCK-algebras. We need some preliminary materials that are necessary for the development of the paper. Section 2 contains some basic knowledges of the BCK-algebras which are needed for studying this topic. And we will define a topological BCK-algebra and study some general facts for topological BCK-algebras.

In section 3, we will find a filter base generating a BCK-topology. In fact such filter base is a fundamental neigborhood system of zero with respect to the topology generated by that filter base. We know that the ideal topology which is introduced by Alo and Deeba satisfies the properties of such filter base introduced by us. We will study some properties of BCK-topolgy whcih is generated by such filter base introduced in this paper.

2. Topological BCK-algebra

A non-empty set X together with a binary operation $\star$ and a zero element 0 is said to be a BCK-algebra if the following axioms are satisfied for all $x, y, z \in X$

(1) $((x \star y) \star (x \star z)) \star (z \star y) = 0$
(2) $(x \star (x \star y)) \star y = 0$
(3) $x \star x = 0$
(4) $0 \star x = 0$

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(5) $x \star y = 0$ and $y \star x = 0$ imply that $x = y$

An order relation can be defined for all x and y in X to be $x \leq y$ if and only if $x \star y = 0$. It is clear that this order relation on X is a partial ordering. Under the definion of BCK-algebras we can get the following properties very easily.

Remark. *In any BCK-algebra X, the following relations hold for all $x, y, z \in X$*

(1) $x \leq y$ *implies* $x \star z \leq y \star z$ *and* $z \star y \leq z \star x$
(2) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$
(3) $((x \star y) \star (z \star y)) \star (x \star z) = 0$
(4) $(x \star y) \star z = (x \star z) \star y$
(5) $x \star y \leq x$

A non-empty subset I of a BCK-algebra X is said to be an ideal of X if (1) $0 \in I$ (2) $x \in I$ and $y \star x \in I$ imply that $y \in I$. We can define a congruence relation on a BCK-algebra X with its ideal. For any x and y in X, $x \sim y$ is defined by $x \star y \in A$ and $y \star x \in A$. We denote the equivalent class containig x as C_x. Then we can prove that the set of all equivalent classes $\{C_x\}$ is a BCK-algebra. We will define the topological BCK-algebra and describe some properties of topological BCK-algebras.

Definition 2.1. *A topology , on a BCK-algebra X is BCK-algebra topology and X furnished with , is a topological BCK-algebra if $(x, y) \to x \star y$ is continuous from $X \times X$ with its cartesian product topology to X with the topology , . In this case abbreviately we call X a TBCK-algebra.*

In fact some properties of TBCK-algebras are parallel to those of topological groups.

Proposition 2.2. *If $\{0\}$ is an open set of a TBCK-algebra X, X is discrete.*

Proof. For every $x \in X$, there exist some open neighborhoods V and U of x such that $U \star V = \{0\}$ since $x \star x = 0$ and $\{0\}$ is open. Let $W = U \cap V$. Then $W \star W = \{0\}$. This implies that $W = \{x\}$.

Proposition 2.3. *$\{0\}$ is closed in a TBCK-algebra X if and only if X is Hausdorff.*

Proof. Assume that x and y are different in X. Then $x \star y \neq 0$ or $y \star x \neq 0$. We can assume $x \star y \neq 0$. Then there exist some neighborhoods U and V of x and y respectively, such that

$$U \star V \subset X \setminus \{0\}$$

Thus $U \cap V = \emptyset$

Proposition 2.4. *Let A be an ideal of a TBCK-algebra X. If 0 is an interior point of A, then A is open.*

Proof. For every $x \in A$ there exists a neighborhood V and W of x such that

$$V \star W \subset A$$

since A is a neighborhood of 0. Thus $V \subset A$ for $x \in A$

Proposition 2.5. *If A is an open ideal of a TBCK-algebra X. Then A is also closed.*

Proof. Let $x \notin A$. Then there exists an open neighborhood V of x such that $V \star V \subset A$ since $x \star x = 0$. If some y is contained in $V \cap A$, then $V \subset A$ by properties of ideals. This is contradiction. Thus $V \subset A^c$.

The following example shows some different properties of TBCK-algebras from those of topological groups.

Example. *Let $X = \{0, a, b, c\}$ and $\star$ be defined as following : $c \star a = c$, $c \star b = c$, $a \star c = a$, $b \star c = b$, $a \star b = 0$, $b \star a = b$, $x \star 0 = x$, $x \star x = 0$ and $0 \star x = 0$ for every $x \in X$. We know that X is a BCK-algebra with the operation $\star$ containing ideal $\{0, c\}$. Let*

$$, \ = \{\emptyset, \{0, a\}, \{b\}, \{c\}, \{0, a, b\}, \{b, c\}, \{0, a, c\}, X\}$$

Then , is a BCK-topology. Under this topology, c is an interior point of an ideal $\{0, c\}$. But $\{0, c\}$ is not open.

3. Some Neighborhood System and related results

In this section we will give a filter base on X generating a BCK topology on X. For arbitrary $a \in X$ and any subset $V \subset X$, $V(a)$ will be defined as $V(a) = \{x \in X \mid x \star a \in V$ and $a \star x \in V\}$. With this, we can get the following theorem.

Theorem 3.1. *Let X be a BCK-algebra. If Ω is a filter base on X satisfying*

(1) *For every $v \in V \in \Omega$ there exists $U \in \Omega$ such that $U(v) \subset V$*
(2) *For every $v \in V \in \Omega$ if $x \star v = 0$ then $x \in V$*
(3) *For every $V \in \Omega$ there exists $W \in \Omega$ such that $W(w) \subset V$ for every $w \in W$ (We will denote it as $W(W) \subset V$)*

then there is a BCK topology on X for which Ω is a fundamental system of neighborhoods of zero.(that is the filter base generating the filter of all neighborhoods of zero)

Proof. Let , $= \{O \subset X \mid$ For every $a \in O$, there exists $V \in \Omega$ such that $V(a) \subset O\}$. At first we can prove that , is a topology on X. Clearly X and $\emptyset$ beong to , . Let $\{O_\alpha\}$ be a family of open sets. Then for every $a \in \bigcup O_\alpha$, $a \in O_\alpha$ for some α. Thus there exists V such that $V(a) \subset O_\alpha$. Assume that O_1 and O_2 belong to , . Let $a \in O_1 \bigcap O_2$. Then there exist V_1 and V_2 such that $V_1(a) \subset O_1$ and $V_2(a) \subset O_2$ respectively. Since Ω is a filter base, there exists $V \in \Omega$ such that $V \subset V_1 \cap V_2$. Then we have

$$V(a) \subset (V_1 \cap V_2)(a) \subset V_1(a) \cap V_2(a) \subset O_1 \cap O_2$$

and so $O_1 \cap O_2 \in$, .

Next we will prove that $V(a)$ is open. Let $x \in V(a)$. We know that $x \star a \in V$ and $a \star x \in V$. Then by (1) of properties of Ω there exits U_1 and U_2 respectively such that $U_1(x \star a) \subset V$ and $U_2(a \star x) \subset V$. Choose $W \in \Omega$ such that $W \subset U_1 \cap U_2$. If $y \in W(x)$ then $x \star y \in W$ and $y \star x \in W$. Since

$$(x \star a) \star (y \star a) \leq x \star y$$

and

$$(y \star a) \star (x \star a) \leq y \star x$$

we know that $(x \star a) \star (y \star a)$ and $(y \star a) \star (x \star a)$ are contained in W by the property (2) of this filter Ω. Thus

$$(y \star a) \in W(x \star a) \subset U_1(x \star a) \subset V$$

Hence $y \star a \in V$. Similarly we know that $a \star y \in V$. Thus $y \in V(a)$ that is $W(x) \subset V(a)$ and $V(a)$ is open.

Every $V \in \Omega$ contains zero because $0 \star x = 0$ for every $x \in X$ ((2) of properties of Ω) and V is open by (1) of properties of Ω. Conversely if V is a neighborhood of zero, then there exists an open set O such that $0 \in O \subset V$. Thus there exists some $U \in \Omega$ such that $0 \in U = U(0) \subset O \subset V$. Thus Ω is the filter base of neighborhoods of 0 with respect to the topology , . Finally we will prove that $\star$ is continuous. Let x and y be any elements of X. Since every open set containing $x \star y$ contains $V(x \star y)$ for some $V \in \Omega$. It is sufficient to show that $(W(x)) * (W(y)) \subset V(x \star y)$ for some $W \in \Omega$. In fact we can choose $W \in \Omega$ such that $W(W) \subset V$ by(3). Let $u \in W(x)$ and $v \in W(y)$. Then

$$\begin{aligned}
&((x \star y) \star (u \star v)) \star (x \star u) \\
&= ((x \star y) \star (x \star u)) \star (u \star v) \\
&= ((x \star (x \star u)) \star y) \star (u \star v) \\
&\leq (u \star y) \star (u \star v) \\
&\leq v \star y
\end{aligned}$$

Since $v \star y$ is contained in W, $((x \star y) \star (u \star v)) \star (x \star u)$ is also contained in W by (2). And $(x \star u) \star ((x \star y) \star (u \star v))$ is contained in W because it is less than $x \star u$. Thus $(x \star y) \star (u \star v) \in W(x \star u) \subset V$. By similar method we know that $(u \star v) \star (x \star y) \in V$. Thus $u \star v$ is contained in $V(x \star y)$ which shows that $W(x) \star W(y) \subset V(x \star y)$.

We call above BCK-topology , the BCK-topology generated by the filter base Ω. We can find several examples of filter bases satisfying the condtions of theorem 3.1.

Example. *Let Ω be a filter base of ideals of a BCK-algebra X. Then Ω is a filter base satisfying conditions of theorem 3.1. Since for every $x \in I$, $I(x) \subset I$ where I is an ideal of X, we know that codition (1) and (3) are satisfied. If $y \leq x$ and $x \in I$ then $y \star x = 0 \in I$ implies $y \in I$.Thus (2) is satisfied.*

The following remark shwows that condition (3) of hypotheses of theorem 3.1 could be rejected with some BCK-algebras having special properties.

Remark. *Let a BCK-algeba X have a filter base Ω satisfying conditions (1) and (2). If X is T_1 space with respect to the topology generated by Ω in sense in theorem 3.1 and $x \star y \neq 0$ implies $y \star x = 0$, then condition (3) is satisfied.*

Proof. Pick some $v \in V \in \Omega$. We can find some $W_1 \in \Omega$ such that $W_1(v) \subset V$ and $W_2 \in \Omega$ such that $v \notin W_2$. Let $W = W_1 \cap W_2$. Assume that for any $w \in W$, $x \star w \in W$ and $w \star x \in W$. If $y \star v = 0$ then $y \in V$. If $y \star v \neq 0$ then $y \star v \in W$ since $y \star v \leq y \star w$. Thus $y \in V$.

Here after X means a TBCK-algebra whose topology is the topology , generated by a filter base Ω satisfying the conditions of theorem 3.1. For any subset A of X, $V(A) = \bigcup_{a \in A} V(a)$ is cleary an open neighborhood of A. We can get the following theorem.

Theorem 3.2. *For any subset A of X, $\overline{A} = \bigcap\{V(A) \mid V \in \Omega\}$ where $\overline{A}$ is the closure of A.*

Proof. Let $b \in \overline{A}$ and $V \in \Omega$. Since $V(b)$ is a neighborhood of b, $V(b) \bigcap A \neq \emptyset$. Thus there exists $a \in A$ such that $a \star b \in V$ and $b \star a \in V$. Hence $b \in V(a)$ and $b \in \bigcap\{V(A) \mid V \in \Omega\}$. Conversely if $b \in \bigcap\{V(A) \mid V \in \Omega\}$ then for any $W \in \Omega$, $b \in W(A)$. Thus $W(b) \bigcap A \neq \emptyset$

From this theorem we can get the following corollary that is the similar result to that of topological group theory.

Corollary. *There exist a fundamental neighborhood system whose elements are closed.*

Proof. For every $U \in \Omega$ there exist a closed neighborhood contained in U since $V \subset \overline{V} \subset V(V) \subset U$.

The following theorem shows that every neighborhood of a compact set contains a neighborhood $W(A)$ for some $W \in \Omega$.

Theorem 3.3. *Let A be a compact subset of X and U be a neighborhood of A. Then there exists $W \in \Omega$ such that $A \subset W(A) \subset U$.*

Proof. Since U is a neighborhood of A, for every $a \in A$ there exists $V_a \in \Omega$ such that $V_a(a) \subset U$. By property (3) we can choose $W_a \in \Omega$ such that $W_a(W_a) \subset V_a$. Since $A \subset \bigcup_{a \in A} W_a(a)$, there exist $a_1, a_2, \ldots, a_n$ such that

$$A \subset W_{a_1}(a_1) \cup W_{a_2}(a_2) \cup \cdots \cup W_{a_n}(a_n)$$

. Let $W = \bigcap W_{a_i}$. It is sufficient to show that $W(a) \subset U$ for every $a \in A$. Since $a \in W_{a_i}(a_i)$ for some a_i, $a \star a_i \in W_{a_i}$ and $a_i \star a \in W_{a_i}$. If $a \star y \in W$ and $y \star a \in W$, then

$$(a \star a_i) \star (y \star a_i) \leq a \star y \in W$$

Thus

$$(a \star a_i) \star (y \star a_i) \in W$$

and similarly

$$(y \star a_i) \star (a \star a_i) \in W$$

Hence

$$y \star a_i \in W_{a_i}(a \star a_i) \subset W_{a_i}(W_{a_i}) \subset V_{a_i}$$

and

$$a_i \star y \in V_{a_i}$$

by similar method. It shows that $y \in V_{a_i}(a_i) \subset U$ and $W(a) \subset U$. Thus $W(A) \subset U$.

Theorem 3.4. *Let K be compact subset of X and F be closed subset of X. If $K \bigcap F = \emptyset$ then there exists $V \in \Omega$ such that $V(K) \bigcap V(F) = \emptyset$*

Proof. Since $X \setminus F$ is a neighborhood of K, there exists $W \in \Omega$ such that $W(K) \subset X \setminus F$ by theorem 3.3. Let V be contained in Ω such that $V(V) \subset W$. Then

$$V(K) \cap V(F) = \emptyset$$

Suppose that y is contained in $V(K) \cap V(F)$. $y \in V(k)$ for some $k \in K$ and $y \in V(f)$ for some $f \in F$. Since

$$(k \star y) \star (k \star f) \leq f \star y \in V$$

and

$$(k \star f) \star (k \star y) \leq y \star f \in V$$

we know that $k \star f \in W$. Similarly $f \star k \in W$. Thus $f \in W(k)$. But it is contradiction on the fact that

$$W(K) \subset X \setminus F$$

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Department of Mathematics, University of Ulsan, Ulsan, Korea 680-749

E-mail: dslee@uou.ulsan.ac.kr