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OPERATOR INEQUALITIES ON HADAMARD PRODUCT ASSOCIATED WITH KADISON'S SCHWARZ INEQUALITIES

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Abstract. We shall give a complementary inequality to the Fiedler inequality on the Hadamard product, which implies the Kantorovich inequality. Moreover we discuss inequlities due to Ando and Aujla-Vasudeva.

1. Introduction. For a fixed orthonormal basis $\{e_n\}$ of a Hilbert space H, the Hadamard product $A * B$ of (bounded) operators A and B acting on H is defined by

$$((A*B)e_i, e_j) = (Ae_i, e_j)(Be_i, e_j).$$

For convenience, we cite Kadison's Schwarz inequalities: If A is a positive operator on H, then

$$(Ax,x)^2 \le (A^2x,x) \qquad \text{for every unit vector } x \in H \tag{1}$$

and if A is positive and invertible, then

$$(Ax,x)^{-1} \le (A^{-1}x,x) \qquad \text{for every unit vector } x \in H. \tag{2}$$

Related to them, several operator inequalities on the Hadamard product are given, see [1], [3], [7]: For instance,

$$A^{\frac{1}{2}} * B^{\frac{1}{2}} \le (\frac{A+B}{2}) * 1 \qquad \text{for } A, B \ge 0 \tag{3}$$

and

$$A * A^{-1} \ge 1 \qquad \text{for } A > 0. \tag{4}$$

Actually, we have to remark that (3) and (4) imply (1) and (2) respectively. We note that (3) is followed from the arithmetic-geometric mean inequality and that (4) is well-known as the Fiedler inequality, also see [1].

Now, the Kantorovich inequality is complementary to (2), that is, if A is a positive operator such that $0 < m \le A \le M$, then

$$(Ax,x)(A^{-1}x,x) \le \frac{(M+m)^2}{4mM} \tag{5}$$

for every unit vector $x \in H$.

In this note, we give a complementary inequality to the Fiedler inequality on the Hadamard product: If A is a positive operator such that $0 < m \le A \le M$, then

$$(A^2 * 1)^{\frac{1}{2}}(A^{-2} * 1)^{\frac{1}{2}} \le \frac{M^2+m^2}{2Mm}. \tag{6}$$

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Since it implies the Kantorovich inequality (5), we call it Kantorovich inequality on the Hadamard product. Moreover we discuss inequalities due to Ando [1] and Aujla-Vasudeva [2].

2. Kantorovich inequality as complementary Fiedler inequality

Following [8], [3], the Hadamard product is expressed as the following deformation of the tensor product, which is one of the most powerful tools for the study of the Hadamard product of operators.

Theorem A. *Let* $\{e_n\}$ *be a fixed orthonormal basis on a Hilbert space* H *and* U *an isometry of* H *into* $H \otimes H$ *such that* $Ue_n = e_n \otimes e_n$. *Then the Hadamard product* $A * B$ *of operators* A *and* B *on* H *for* $\{e_n\}$ *is expressed as*

$$A * B = U^*(A \otimes B)U.$$

Kijima [5] obtained the following theorem on the tensor product in connection with the Kantorovich inequality:

Theorem B (Kijima). *Let* A *and* B *be positive operators such that* $0 < m_1 \le A \le M_1$ *and* $0 < m_2 \le B \le M_2$. *Then*

$$\frac{M_2}{m_1}(A \otimes B^{-1}) + \frac{M_1}{m_2}(A^{-1} \otimes B) \le 1 + \frac{M_1 M_2}{m_1 m_2}.$$

We shall cite here a point of his proof for convenience. Since $\frac{m_1}{M_2} \le A \otimes B^{-1} \le \frac{M_1}{m_2}$ and $\frac{m_2}{M_1} \le A^{-1} \otimes B \le \frac{M_2}{m_1}$, we have

$$(\frac{M_2}{m_1} - A^{-1} \otimes B)(\frac{M_1}{m_2} - A \otimes B^{-1}) \ge 0,$$

which is eqivalent to the desired inequality.

For positive invertible operators A and B, the geometric (resp. arithmetic) operator mean is defined as $A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}$ (resp. $A \nabla B = \frac{1}{2}(A + B)$), and the arithmetic-geometric mean inequality yields $A \sharp B \le A \nabla B$.

Theorem 1. *If* A *is a positive operator on* H *such that* $0 < m \le A \le M$, *then*

$$A * A^{-1} \le (A^2 * 1)^{\frac{1}{2}} (A^{-2} * 1)^{\frac{1}{2}} \le \frac{M^2 + m^2}{2Mm}.$$

Proof. Applying Theorem B for $B = 1$, we have

$$\frac{1}{m^2}(A^2 \otimes 1) + M^2(A^{-2} \otimes 1) \le 1 + \frac{M^2}{m^2},$$

so that Theorem A implies

$$\frac{1}{m^2}(A^2 * 1) + M^2(A^{-2} * 1) \le 1 + \frac{M^2}{m^2}.$$

Since $A^2 * 1$ and $A^{-2} * 1$ commute, the arithmetic-geometric mean inequality ensures

$$\frac{M}{m}(A^2 * 1)^{\frac{1}{2}} (A^{-2} * 1)^{\frac{1}{2}} = \frac{1}{m^2}(A^2 * 1) \sharp M^2 (A^{-2} * 1) \le \frac{M^2 + m^2}{2m^2},$$

which is the desired inequality. On the other hand, it follows from [2, Theorem 4.1] that

$$A * A^{-1} = (A^2 \sharp 1) * (1 \sharp A^{-2}) \leq (A^2 * 1) \sharp (1 * A^{-2}) = (A^2 * 1)^{\frac{1}{2}} (A^{-2} * 1)^{\frac{1}{2}}.$$

Corollary 2. *If A is a positive operator on H such that $0 < m \leq A \leq M$, then*

$$A * A^{-1} * 1 \leq \frac{(M+m)^2}{4Mm}.$$

Proof. It follows from [4, Theorem 5] that

$$(A^2 * A^{-2}) * 1 = (A^2 * 1) * (A^{-2} * 1) = (A^2 * 1)(A^{-2} * 1) \leq (\frac{M^2 + m^2}{2Mm})^2$$

by Theorem 1, which is equivalent to the desired inequality.

Remark. It follows that Corollary 2 implies the Kantorovich inequality. In fact, for a given unit vector x, we take a complete orthonormal basis $\{e_i\}$ with $e_1 = x$, then it implies

$$\begin{aligned}(Ax, x)(A^{-1}x, x) &= ((A \otimes A^{-1})Ux, Ux) = (A * A^{-1}x, x)(x, x) \\ &= ((A * A^{-1}) \otimes 1)Ux, Ux) = ((A * A^{-1} * 1)x, x) \leq \frac{(m+M)^2}{4Mm}.\end{aligned}$$

Thus we may call it Kantorovich inequality on the Hadamard product.

Also, J.I.Fujii [3] pointed out under the hypothesis of Theorem 1

$$((A * A^{-1})e_i, e_i) \leq \frac{(M+m)^2}{4Mm}$$

whereas

$$A * A^{-1} \leq \frac{(M+m)^2}{4Mm}$$

does not hold in general: If $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $m = \frac{3-\sqrt{5}}{2} \leq A \leq \frac{3+\sqrt{5}}{2} = M$ and so $\frac{(M+m)^2}{4Mm} = \frac{9}{4}$. On the other hand, we have $\|A * A^{-1}\| = 3 > \frac{9}{4}$. This example might clarify the meaning of Corollary 2 and consequently Theorem 1.

3. Schwarz inequlities on the Hadamard product.

As extensions of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A * B \leq (A^2 * 1)^{\frac{1}{2}} (B^2 * 1)^{\frac{1}{2}} \qquad \text{for} \qquad A, B \geq 0,$$

and Aujla and Vasudeva [2] gave an alternative estimation that

$$A * B \leq (A^2 * B^2)^{\frac{1}{2}} \qquad \text{for} \qquad A, B \geq 0.$$

In this section, we shall show that $(A^2 * 1)^{\frac{1}{2}} (B^2 * 1)^{\frac{1}{2}}$ and $(A^2 * B^2)^{\frac{1}{2}}$ are incomparable for 2-square positive definite matrices A and B, also see [6]. We consider the case of $A = B$: Put $A = \begin{pmatrix} a & c \\ c & \frac{k}{a} \end{pmatrix}$ where $a > 0$, $k \geq 1$ and $c \in \mathbb{R}$. We may assume that $c = 1$. Then we have

$$(A * A)^{\frac{1}{2}} = \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} \begin{pmatrix} a^2 + \sqrt{\alpha\beta} & 1 \\ 1 & \frac{k^2}{a^2} + \sqrt{\alpha\beta} \end{pmatrix}$$

where $\alpha + \beta = a^2 + \frac{k^2}{a^2}$ and $\alpha\beta = k^2 - 1$. Also, since $\sqrt{\alpha\beta} = \sqrt{k^2-1}$ and $\sqrt{\alpha} + \sqrt{\beta} = \frac{1}{a}\sqrt{a^4 + k^2 + 2a^2\sqrt{k^2-1}}$, it follows that

$$(A * A)^{\frac{1}{2}} - A * 1 = \frac{1}{\sqrt{\alpha} + \sqrt{\beta}} \begin{pmatrix} P(1) & 1 \\ 1 & P(2) \end{pmatrix}$$

where

$$\begin{aligned} P(1) &= (a^2 + \sqrt{k^2-1}) - \sqrt{a^4 + k^2 + 2a^2\sqrt{k^2-1}} \\ P(2) &= \frac{1}{a^2}((k^2 + a^2\sqrt{k^2-1}) - k\sqrt{a^4 + k^2 + 2a^2\sqrt{k^2-1}}. \end{aligned}$$

Since $P(1) < 0$ and $P(2) < 0$, it suffices to show that $P(1)P(2) - 1 < 0$. Then it follows that

$$\begin{aligned} &(P(1)P(2) - 1)a^2 \\ &= (k + \sqrt{k^2-1})\sqrt{a^4 + k^2 + 2a^2\sqrt{k^2-1}}(\sqrt{a^4 + k^2 + 2a^2\sqrt{k^2-1}} - (a^2 + k)) < 0 \end{aligned}$$

since $a^4 + k^2 + 2a^2\sqrt{k^2-1} - (a^2+k)^2 = 2a^2(\sqrt{k^2-1} - k) < 0$.

Therefore we have

$$(A * A)^{1/2} - A * 1 \not\leq 0.$$

In the remainder, we shall discuss Kadison's Schwarz inequlity. We note that (1) is proved along with our idea: Since

$$(A \otimes 1) \sharp (1 \otimes B) \leq (A \otimes 1) \nabla (1 \otimes B)$$

by the arithmetic- geometric mean inequality, Theorem A implies (3) easily. Moreover (1) is obtained by putting $A = B$ in (3).

Noting that (1) is equivalent to the inequality

$$(A^{\frac{1}{2}}x, x) \leq (Ax, x)^{\frac{1}{2}} \qquad \text{for every unit vector } x \in H,$$

we give a proof to the Hölder-McCarthy inequality: For $t \in [0, 1]$

$$(A^t x, x) \leq (Ax, x)^t \qquad \text{for every unit vector } x \in H. \tag{7}$$

The following proof is due to Prof. R. Nakamoto. It is along with Pedersen's technique used in [9]. Let $I = \{t \in [0,1] : (A^t x, x) \leq (Ax, x)^t \quad \text{for every unit vector } x \in H\}$. It follows that I is closed and contains 0 and 1. We then prove that I is convex. If $s, t \in I$, then

$$\begin{aligned} (A^{\frac{s+t}{2}}x, x) &= (A^{\frac{s}{2}}x, A^{\frac{t}{2}}x) \leq \|A^{\frac{s}{2}}x\|\|A^{\frac{t}{2}}x\| \\ &= (A^s x, x)^{\frac{1}{2}}(A^t x, x)^{\frac{1}{2}} \leq (Ax, x)^{\frac{s}{2}}(Ax, x)^{\frac{t}{2}} = (Ax, x)^{\frac{s+t}{2}} \end{aligned}$$

Thus I contains $\frac{s+t}{2}$, so that I is convex.

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