SCHWARZ INEQUALITIES ON HADAMARD PRODUCT

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ABSTRACT. In the preceding note, we pointed out the imcomparability of Schwarz inequalities on Hadamard product. Successively, we shall discuss its imcomparability in a general setting. Also we shall show that the equality holds in some inequalities on Hadamard product if and only if operators are diagonal.

1. Introduction.

In the preceding note [7], we pointed out that the following Schwarz inequalities on Hadamard product are imcomparable by calculating 2×2 matrices;

$$(1) \qquad A * B \leq (A^2 * 1)^{1/2}(B^2 * 1)^{1/2} \qquad \text{for } A, B \geq 0$$

and

$$(2) \qquad A * B \leq (A^2 * B^2)^{1/2} \qquad \text{for } A, B \geq 0.$$

The former (1)(resp. (2)) is due to Ando [1](resp. Aujla-Vasudeva [2]).

The Hadamard product of (bounded) linear operators A and B acting on a Hilbert space H is defined by

$$(A * Be_i, e_j) = (Ae_i, e_j)(Be_i, e_j)$$

for a fixed orthonormal basis $\{e_n\}$ of H. J.I.Fujii [4] gave a new look on the Toyama-Marcus-Khan theorem:

$$(3) \qquad A * B = U^*(A \otimes B)U,$$

where U is an isometry of H into $H \otimes H$ defined by $Ue_i = e_i \otimes e_i$ $(i = 1, 2, \cdots)$.

We remark that $E(A) = A * 1$, the conditional expectation onto the diagonal algebra in the sense of Umegaki[8], plays an important role in inequalities on Hadamard product. For instance, it is used in [5] to show that the equality holds in (1) if and only if A and B are diagonal.

On the other hand, (2) is implied by the following

$$(4) \qquad (C * D)(C * D)^* \leq CC^* * DD^* \qquad \text{for operators } C \text{ and } D.$$

The expression (3) is available to prove (4):

$$(C * D)(C * D)^* = U^*(C \otimes D)UU^*(C^* \otimes D^*)U \leq U^*(CC^* \otimes DD^*)U = CC^* * DD^*.$$

In 1995, T.Furuta [6](cf. [9]) established the following useful inequality:

$$(5) \qquad (C * D)(A * B)^{-1}(C * D)^* \leq (CAC^*)^{-1} * (DBD^*)^{-1}$$

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for any operators $A > 0, B > 0, C$ and D. Note that it is a simultaneous extension of (4) and Fiedler type inequality (6):

$$(A * B)^{-1} \leq A^{-1} * B^{-1} \tag{6}$$

for positive invertible operators A and B.

In this note, we show the imcomparabilty between (1) and (2) in a general setting. Precisely we prove that if $E(C)^{1/2} \geq C^{1/2}$ for a positive definite matrices C, then C is diagonal. Moreover, we show that the equalitiy holds in (4) or (6) if and only if operators are diagonal.

2. Imcomparability of Schwarz inequalities.

The following theorem states the imcomparability of inequalities (1) and (2).

Theorem 1. *If a positive semi-definite matrix C satisfies the condition $C^r \leq E(C)^r$ for $1/2 \leq r \leq 1$, then $E(C) = C$.*

Proof. By assumption, it follows that

$$\begin{aligned}
\|E(C)^r - C^r\|_2^2 &= \mathrm{tr}((E(C)^r - C^r)^2) \\
&= \mathrm{tr}(E(C)^{2r}) - \mathrm{tr}(E(C)^r C^r) - \mathrm{tr}(C^r E(C)^r) + \mathrm{tr}(C^r) \\
&= \mathrm{tr}(E(C)^r) + \mathrm{tr}(C^r) - 2\mathrm{tr}(C^{r/2} E(C)^r C^{r/2}) \\
&\leq \mathrm{tr}(E(C)^r) + \mathrm{tr}(C^r) - 2\mathrm{tr}(C^{2r}) \\
&= \mathrm{tr}(E(C)^{2r}) - \mathrm{tr}(C^{2r}) = \mathrm{tr}(E(C)^{2r}) - \mathrm{tr}(E(C^{2r})) \\
&\leq 0,
\end{aligned}$$

because $E(C)^s \leq E(C^s)$ for $s \geq 1$ (cf. [5: Lemma 2]). Therefore we have $E(C)^r = C^r$ i.e., $E(C) = C$. ■

For a positive operator C, the comparability of $C^{1/2}$ and $E(C)^{1/2}$ does occur in the case of $E(C)^{1/2} \geq C^{1/2}$ by Kadison's Schwarz inequality $E(C)^{1/2} \geq E(C^{1/2})$. In this case, a positive semi-definite matrix C is diagonal by Theorem 1. Therefore for positive semi-definite matrices A and B, inequalities (1) and (2) are comparable only if $A^2 * B^2$ is diagonal since $E(A^2 * B^2)^{1/2} = E(A^2)^{1/2} E(B^2)^{1/2}$ holds. The converse is clear.

Remark 1. Generally for an orthogonal projection P, it is well known that $\|Px\| = \|x\|$ implies $Px = x$. Similarly for a matrix C, $\|E(C)\|_2 = \|C\|_2$ implies $E(C) = C$, that is, C is diagonal:

$$\begin{aligned}
\|C - E(C)\|_2^2 &= \mathrm{tr}((C - E(C))^*(C - E(C))) \\
&= \mathrm{tr}(C^*C) + \mathrm{tr}(E(C)^*E(C)) - \mathrm{tr}(C^*E(C)) - \mathrm{tr}(E(C)C^*) \\
&= \mathrm{tr}(C^*C) + \mathrm{tr}(E(C)^*E(C) - \mathrm{tr}(E(C^*E(C))) - \mathrm{tr}(E(E(C)C^*)) \\
&= \mathrm{tr}(C^*C) - \mathrm{tr}(E(C)^*E(C)) = 0.
\end{aligned}$$

3. Diagonal operators.

As a continuation of [5], we show that the equality holds in (4) or (6) if and only if operators are diagonal.

In [5: Theorem 2], we proved the following:

Theorem 2. For a positive operator A, the following are equivalent:

(1) $E(A)^{-1} = E(A^{-1})$.

(2) $A * A^{-1} = 1$.

(3) A is diagonal.

Theorem 2 says that the equality in Kadison's inequality $E(A)^{-1} \le E(A^{-1})$ or Fiedler's inequality $A * A^{-1} \ge 1$ holds for only diagonal operator.

We extend Theorem 2 by using (5) as follows:

Theorem 3. *For positive invertible operators A and B, the equality $(A * B)^{-1} = A^{-1} * B^{-1}$ holds if and only if A and B are diagonal.*

Proof. By (5), it follows that

$$(A * 1)(A * B)^{-1}(A * 1) \le A * B^{-1}.$$

Therefore if $(A * B)^{-1} = A^{-1} * B^{-1}$ is satisfied, then

$$(A * 1)(A^{-1} * B^{-1})(A * 1) \le A * B^{-1}.$$

So, taking the expectation of both sides, we have

$$E(A)^2 E(A^{-1}) E(B^{-1}) \le E(A) E(B^{-1}).$$

Hence it follows that $E(A)E(A^{-1}) \le 1$, and so $E(A^{-1})E(A) = 1$ by Kadison's inequality $E(A)^{-1} \le E(A^{-1})$. This implies that A is diagonal by Theorem 2. Similarly it is proved that B is diagonal.

The converse direction is clear. ∎

For the Schwarz inequality (4) we have

Theorem 4. *For positive invertible operators C and D, the equality $(C * D)^2 = C^2 * D^2$ holds if and only if C and D are diagonal.*

Proof. Putting $c_{ji} = (Ce_i, e_j)$ and $d_{ji} = (De_i, e_j)$, then we have easily by direct calculatios

$$((C * D)^2 e_i, e_i) = \sum_k |c_{ik}|^2 |d_{ik}|^2.$$

On the other hand, we have

$$(C^2 * D^2 e_i, e_i) = (\sum_k |c_{ik}|^2)(\sum_k |d_{ik}|^2).$$

If $(C * D)^2 = C^2 * D^2$ is satisfied, then

$$\sum_k |c_{ik}|^2 |d_{ik}|^2 = (\sum_k |c_{ik}|^2)(\sum_k |d_{ik}|^2).$$

Therefore $|c_{ii}|^2 \sum_{k \ne i} |d_{ik}|^2 = 0$ and $|d_{ii}|^2 \sum_{k \ne i} |c_{ik}|^2 = 0$. As C and D are invertible, we have $c_{ij} = d_{ij} = 0$ $(i \ne j)$.

The sufficiency is evident. ∎

Remark 2. It is essential that C and D are invertible in Theorem 4. For example, put $C = 1 \oplus 0, D = 1 \oplus A$ $(A > 0)$, then $(C * D)^2 = C^2 * D^2 = C$.

References

[1] T.Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard product*, Linear Algebra and Appl. **26** (1979), 203-241

[2] J.S.Aujla and H.L.Vasudeva, *Inequalities involving Hadamard product and operator means*, Math. Japonica **42** (1995), 273-277

[4] J.I.Fujii, *The Marcus-Khan theorem for Hilbert space operators*, Math. Japonica **41** (1995), 531-535

[5] M.Fujii, R.Nakamoto and M.Nakamura, *Conditional expectation and Hadamard product of operators*, Math. Japonica **42** (1995), 239-244

[6] T.Furuta, privately circulated note

[7] K.Kitamura and Y.Seo, *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, Scientiae Mathematicae (to appear).

[8] H.Umegaki, *Conditional expectation in an operator algebra*, Tohoku Math. J. **6** (1954), 177–181

[9] B-Y. Wang and F.Zhang, *Schur complements and matrix inequalities of Hadamard products*, Linear and Multilinear Alg. **43** (1997), 315-326

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