OPERATOR FUNCTIONS INVOLVING ORDER PRESERVING INEQUALITIES

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Dedicated to the memory of the late Professor Katsutoshi Takahashi with deep sorrow

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Abstract. We shall obtain a new characterization of chaotic order in terms of parametric operator functions via Furuta inequality, which gives a precise estimation of the already known characterization [4][8] of chaotic order and a parallel result to [11]. Moreover we shall show some applications of parametric operator functions in [11] and [12] via Furuta inequality, which are extensions of recent Kamei's results [15] and [16].

§1 Introduction

In what follows, a capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible. The α-mean is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$$

for any $\alpha \in [0,1]$ for positive operators A and B by [17]. Also $A \natural_s B$ is defined by

$$A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$$

for any real number s and for positive invertible operators A and B.

We write $A \gg B$ if $log A \geq log B$ for invertible positive operator A and B, which is called the chaotic order [4]. It is well known that $A \gg B$ iff $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geq 0$ in [1] and also $A \gg B$ iff $A^t \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}}$ for all $p, t \geq 0$ in [4][8].

Recently E.Kamei [15] has obtained the following excellent result.

Theorem A [15]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq \delta_2 \geq \delta_1 \geq 1$,*

$$(A^t \natural_{\frac{\delta_2 - t}{p-t}} B^p)^{\frac{1}{\delta_2}} \geq (A^t \natural_{\frac{\delta_1 - t}{p-t}} B^p)^{\frac{1}{\delta_1}},$$

that is, for each $t \leq 0$, $f(\delta) = (A^t \natural_{\frac{\delta - t}{p-t}} B^p)^{\frac{1}{\delta}}$ is increasing for δ such that $p \geq \delta \geq 1$.

Very recently we obtained the following which is an extension of Theorem A.

Theorem B [11]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq 1$,*

$$H_{p,t}(A, B, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is increasing for s such that $1 \geq s \geq \frac{1-t}{p-t}$ and decreasing for r such that $0 \geq r \geq t$.

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Corollary C [11]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \leq 0$ and $p \geq 1$,*

$$A \geq B \geq (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \geq A^{t-r} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \geq A^t \natural_{\frac{1-t}{p-t}} B^p \tag{1.1}$$

holds for $0 \geq r \geq t$ and $1 \geq s \geq \frac{1-t}{p-t}$.

Very recently among others, E.Kamei [16] shows the following interesting result.

Theorem D [16]. *If $A \geq B > 0$, then*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geq A^u \natural_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

for $\beta \geq p \geq 1 \geq t \geq 0$, $p \neq t$, $u \leq 0$ and $\beta \geq \delta \geq 0$.

On the other hand, we obtained the following result via Furuta inequality.

Theorem E [12]. *If $A \geq B \geq 0$ with $A > 0$, then for each $q \geq 0$, $t \in [0,1]$ and $p \geq t$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

We shall show an extension of Theorem D as an application of Theorem E.

§2 Parametric characterization of chaotic order

We shall show the following characterization of chaotic order in terms of a parametric operator function.

Theorem 2.1. *Let A and B be positive invertible operators. Then the following* (i), (ii) *and* (iii) *are mutually equivalent;*

(i) $A \gg B$ (i.e., $logA \geq logB$).

(ii) *For each $t \leq 0$ and $p \geq 0$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is increasing for s such that $1 \geq s \geq \frac{-t}{p-t}$ and decreasing for r such that $0 \geq r \geq t$.

(iii) *For each $0 \geq r \geq t$ and $p \geq 0$,*

$$I \geq A^{t-r} \natural_{\frac{r-t}{p+r-t}} B^p \geq A^{t-r} \natural_{\frac{r-t}{(p-t)s+r}} (A^t \natural_s B^p) \geq A^t \natural_{\frac{-t}{p-t}} B^p \tag{2.1}$$

holds for $1 \geq s \geq \frac{-t}{p-t}$.

Remark 1. We recall that $A \gg B$ iff $A^{-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ for all $p \geq 0$ and $t \leq 0$, that is, $I \geq A^t \natural_{\frac{-t}{p-t}} B^p$ for all $p \geq 0$ and $t \leq 0$ in [4][8], so that the equivalence relation between (i) and (iii) of Theorem 2.1 shows a precise estimation of this characterization of chaotic order. Theorem 2.1 can be considered as a parallel result to Theorem B and Corollary C.

The following Remark 2 is an immediate consequence of Theorem 2.1 and Corollary C.

Remark 2. *Let A and B be positive invertible operators. Then the following parallel results* (i)*and* (ii) *hold;*

(i) $A \geq B \iff$ *for each $t \leq 0$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \geq A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{r}{2}} \tag{2.2}$$

holds for $0 \geq r \geq t$ and $1 \geq s \geq \frac{1-t}{p-t}$.

(ii) $A \gg B$ (i.e., $logA \geq logB$)$\iff$ *for each $t \leq 0$ and $p \geq 0$,*

$$A^{r-t} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \geq A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}} A^{\frac{r}{2}} \tag{2.3}$$

holds for $0 \geq r \geq t$ *and* $1 \geq s \geq \frac{-t}{p-t}$.

§3 Proofs of the results in §2

We cite the following results to give a proof of Theorem 2.1.

Theorem F (Furuta inequality) [6] **.**

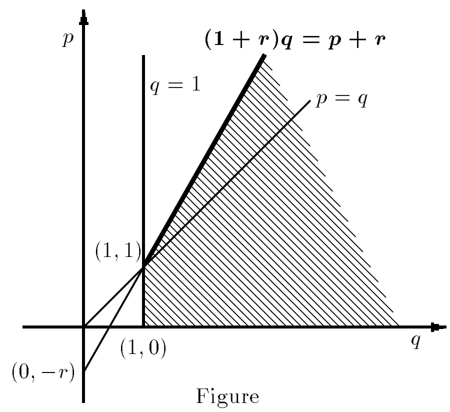
If $A \geq B \geq 0$, *then for each* $r \geq 0$

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

Theorem F ensures the famous Löwner-Heinz inequality when we put $r = 0$ in (i) or (ii) of Theorem F; $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. Alternative proofs of Theorem F are given in [3][14] and one page elementary proof is in [7]. It is shown in [18] that the domain drawn for p q and r in Fugure is the best possible one for (i) and (ii) of Theorem F.

Lemma 1 [9]. *Let A be invertible operator and let B be positive invertible operator. For any real number λ,*

$$(ABA^*)^\lambda = AB^{\frac{1}{2}}(B^{\frac{1}{2}}A^*AB^{\frac{1}{2}})^{\lambda-1}B^{\frac{1}{2}}A^*.$$

Lemma 2 [4][8].

(i) *If* $A \gg B$, *then for a fixed* $q \geq 0$ *and* $t \leq 0$

$$F_q(p) = (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}}$$

is decreasing for $p \geq q$.

(ii) $A \gg B$ *iff* $A^t \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{t}{p+t}}$ *for all* $p \geq 0$ *and* $t \geq 0$.

Lemma 3. *If* $C \geq D > 0$, *then the following inequality holds for* $p \geq q \geq 1$ *and* $0 \geq r \geq t$

$$C^{1-t+r} \geq (C^{\frac{r-t}{2}} D^q C^{\frac{r-t}{2}})^{\frac{1-t+r}{q-t+r}} \geq \{C^{\frac{r}{2}}(C^{\frac{-t}{2}} D^p C^{\frac{-t}{2}})^{\frac{q-t}{p-t}} C^{\frac{r}{2}}\}^{\frac{1-t+r}{q-t+r}}.$$

Proof of Lemma 3. As $log t$ is operator monotone function, that is, $C \geq D > 0$ ensures $C \gg D$, so that by (i) of Lemma 2 we have the following (3.1) for $p \geq q \geq 1$ and $t \leq 0$

(3.1) $$C^{\frac{-t}{2}} D^q C^{\frac{-t}{2}} \geq (C^{\frac{-t}{2}} D^p C^{\frac{-t}{2}})^{\frac{q-t}{p-t}}.$$

Multiplying $C^{\frac{r}{2}}$ on the both sides of (3.1), we have

(3.2) $$C^{\frac{r-t}{2}} D^q C^{\frac{r-t}{2}} \geq C^{\frac{r}{2}}(C^{\frac{-t}{2}} D^p C^{\frac{-t}{2}})^{\frac{q-t}{p-t}} C^{\frac{r}{2}} \quad \text{for } 0 \geq r \geq t.$$

Then we have

(3.3) $$C^{1-t+r} \geq (C^{\frac{r-t}{2}} D^q C^{\frac{r-t}{2}})^{\frac{1-t+r}{q-t+r}}$$
$$\geq \{C^{\frac{r}{2}}(C^{\frac{-t}{2}} D^p C^{\frac{-t}{2}})^{\frac{q-t}{p-t}} C^{\frac{r}{2}}\}^{\frac{1-t+r}{q-t+r}} \quad \text{for } 0 \geq r \geq t,$$

where the first inequality follows by (ii) of Theorem F and the second one follows by applying Löwner-Heinz inequality to (3.2) since $\frac{1-t+r}{q-t+r} \in [0,1]$.

Proof of Theorem 2.1. (i) $\Longrightarrow$ (ii).

(a) *Proof of the result that $F_{p,t}(A,B,r,s)$ is increasing for s.*

$A \gg B$ ensures the following (3.4) for $p \geq q \geq 0$ and $t \leq 0$

$$A^{\frac{-t}{2}}B^qA^{\frac{-t}{2}} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}} \quad \text{by (i) of Lemma 2.} \tag{3.4}$$

Multiplying $A^{\frac{r}{2}}$ on the both sides of (3.4), we have

$$A^{\frac{r-t}{2}}B^qA^{\frac{r-t}{2}} \geq A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}} \quad \text{for } 0 \geq r \geq t. \tag{3.5}$$

Then we have

$$\begin{aligned} A^{r-t} &\geq (A^{\frac{r-t}{2}}B^qA^{\frac{r-t}{2}})^{\frac{r-t}{q-t+r}} \\ &\geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{r-t}{q-t+r}} \quad \text{for } 0 \geq r \geq t, \end{aligned} \tag{3.6}$$

where the first inequality follows by (ii) of Lemma 2 and the second one follows by applying Löwner-Heinz inequality to (3.5) since $\frac{r-t}{q-t+r} \in [0,1]$. In (3.6) put $C = A^{r-t}$ and $D = \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{r-t}{q-t+r}}$. Then $C \geq D > 0$ by (3.6), so that by Lemma 3 for $p_1 \geq q_1 \geq 1$ and $0 \geq r_1 \geq t_1$, we have

$$C^{1-t_1+r_1} \geq (C^{\frac{r_1-t_1}{2}}D^{q_1}C^{\frac{r_1-t_1}{2}})^{\frac{1-t_1+r_1}{q_1-t_1+r_1}} \geq \{C^{\frac{r_1}{2}}(C^{\frac{-t_1}{2}}D^{p_1}C^{\frac{-t_1}{2}})^{\frac{q_1-t_1}{p_1-t_1}}C^{\frac{r_1}{2}}\}^{\frac{1-t_1+r_1}{q_1-t_1+r_1}}. \tag{3.7}$$

In (3.7), put

$$p_1 = \frac{q-t+r}{r-t}, \qquad q_1 = \frac{q'-t+r}{r-t}$$

for $p \geq q \geq q' \geq 0$. Then $p_1 \geq q_1 \geq 1$. Also put $r_1 = t_1 = \dfrac{r}{r-t} \leq 0$. Then

$$C^{\frac{r_1}{2}} = C^{\frac{t_1}{2}} = A^{\frac{r}{2}}, \qquad \frac{q_1-t_1}{p_1-t_1} = \frac{q'-t}{q-t}$$

and

$$D^{p_1} = A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}.$$

Therefore (3.7) implies

$$C \geq D \geq \{A^{\frac{r}{2}}[A^{\frac{-r}{2}}A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}A^{\frac{-r}{2}}]^{\frac{q'-t}{q-t}}A^{\frac{r}{2}}\}^{\frac{r-t}{q'-t+r}},$$

that is,

$$A^{r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{r-t}{q-t+r}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{q'-t}{p-t}}A^{\frac{r}{2}}\}^{\frac{r-t}{q'-t+r}}, \tag{3.8}$$

for $p \geq q \geq q' \geq 0$ and $0 \geq r \geq t$. Replacing $s = \frac{q-t}{p-t}$ and and $s' = \frac{q'-t}{p-t}$ in (3.8), then $1 \geq s \geq s' \geq \frac{-t}{p-t}$ since $p \geq q \geq q' \geq 0$, so the proof of (a) is complete by (3.8).

(b) *Proof of the result that $F_{p,t}(A,B,r,s)$ is decreasing for r.*

We recall the following (3.9) by applying Löwner-Heinz theorem to (3.8)

$$A^u \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{u}{(p-t)s+r}} \quad \text{for } r-t \geq u \geq 0 \tag{3.9}$$

Put $D = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{s}{2}}$. Then

$$\begin{aligned} F_{p,t}(A,B,r,s) &= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}A^{\frac{-r}{2}} \\ &= D(DA^rD)^{\frac{-t-(p-t)s}{(p-t)s+r}}D \quad \text{by Lemma 1} \\ &= D\{(DA^rD)^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{-t-(p-t)s}{(p-t)s+r+u}}D \\ &= D\{DA^{\frac{r}{2}}(A^{\frac{r}{2}}D^2A^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}A^{\frac{r}{2}}D\}^{\frac{-t-(p-t)s}{(p-t)s+r+u}}D \quad \text{by Lemma 1} \end{aligned}$$

$$\geq D(DA^{\frac{r}{2}}A^{u}A^{\frac{r}{2}}D)^{\frac{-t-(p-t)s}{(p-t)s+r+u}}D$$
$$= D(DA^{r+u}D)^{\frac{-t-(p-t)s}{(p-t)s+r+u}}D$$
$$= F_{p,t}(A,B,r+u,s),$$

where the last inequality follows by (3.9) and Löwner-Heinz theorem since $\frac{-t-(p-t)s}{(p-t)s+r+u} \in [-1,0]$ and finally taking inverses on the both sides. So the proof of (b) is complete.

Whence the proof of (i) $\Longrightarrow$ (ii) is complete.

(ii) $\Longrightarrow$ (iii). Assume (ii). Then for each $t \leq 0$ and $p \geq 0$, $F_{p,t}(A,B,r,s)$ is increasing for s such that $1 \geq s \geq \frac{-t}{p-t}$ and decreasing for r such that $0 \geq r \geq t$, so that for each $t \leq 0$ and $p \geq 0$,

$$F_{p,t}(A,B,t,1) \geq F_{p,t}(A,B,r,1) \geq F_{p,t}(A,B,r,s) \geq F_{p,t}(A,B,r,\tfrac{-t}{p-t})$$

holds for $0 \geq r \geq t$ and $1 \geq s \geq \frac{-t}{p-t}$, that is,

$$A^{-t} \geq A^{\frac{-r}{2}}(A^{\frac{r-t}{2}}B^{p}A^{\frac{r-t}{2}})^{\frac{r-t}{p+r-t}}A^{\frac{-r}{2}}$$
$$\geq A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}A^{\frac{-r}{2}}$$
$$\geq (A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{\frac{-t}{p-t}}.$$

Multiplying $A^{\frac{t}{2}}$ on both sides of the inequalities stated above, we have (iii).

(iii) $\Longrightarrow$ (i). Assume (iii). Then $I \geq A^{t}\natural_{\frac{-t}{p-t}}B^{p}$ holds for each $t \leq 0$ and $p \geq 0$, that is,

$$A^{-t} \geq (A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$$

holds for each $t \leq 0$ and $p \geq 0$ and this is equivalent to $A \gg B$ by (ii) of Lemma 2. Whence the proof of Theorem 2.1 is complete.

§4 $H_{p,t}(A,B,r,s)$ in Theorem A and $F_{p,t}(A,B,r,s)$ in Theorem 2.1

We cite the following Theorem G which interpolates Theorem F and the inequality equivalent to the main result of log majorization by Ando-Hiai [2].

Theorem G [5][9]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$G_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for both r and s such that $r \geq t$ and $s \geq 1$.

We remark that Theorem 2.1 is parallel result to Theorem B and Theorem G and the operator function $H_{p,t}(A,B,r,s)$ in Theorem B is the same form as $G_{p,t}(A,B,r,s)$ in Theorem G, that is, $F_{p,t}(A,B,r,s)$ in Theorem 2.1 is a slightly variation of $H_{p,t}(A,B,r,s)$ in Theorem B and we can see a nice contrast of the ranges of the parameters t, r and s in Theorem B, Theorem G and Theorem 2.1 , that is, one of Theorem B is

(4.1) $$t \leq 0,\ p \geq 1,\ 1 \geq s \geq \tfrac{1-t}{p-t} \text{ and } 0 \geq r \geq t$$

one of Theorem G is

(4.2) $$t \in [0,1],\ p \geq 1,\ s \geq 1 \text{ and } r \geq t.$$

one of Theorem 2.1 is

(4.3) $$t \leq 0,\ p \geq 0,\ 1 \geq s \geq \tfrac{-t}{p-t} \text{ and } 0 \geq r \geq t.$$

The form of the operator function $H_{p,t}(A,B,r,s)$ in Theorem B is very important in order to research several problems associated with operator functions and several examples of its importance can be found in [5][9][10][12] and [13].

§5 An application of another parametric operator function.

In this chapter, we shall show an extension of Theorem D by using Theorem E.

Theorem 5.1. *If $A \geq B \geq 0$ with $A > 0$, then for each $q \geq 0$, $t \in [0,1]$ and $p \geq t$,*

$$\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

holds for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

Theorem 5.2. *If $A \geq B \geq 0$ with $A > 0$, then for each $q \geq 0$, $t \in [0,1]$ and $p \geq max\{q,t\}$,*

(i)
$$B^q \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

and

(ii)
$$B^q \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{q-t+r}{(p-t)s+t}}A^{\frac{t-r}{2}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r-t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

hold for $r \geq t$ and $s \geq 1$.

Theorem 5.3. *If $A \geq B \geq 0$ with $A > 0$, then for each $q, t \in [0,1]$ and $p \geq max\{q,t\}$,*

(i)
$$A^q \geq B^q \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

and

(ii)
$$A^q \geq B^q \geq A^{\frac{t-r}{2}}(A^{\frac{r-t}{2}}B^pA^{\frac{r-t}{2}})^{\frac{q-t+r}{(p-t)s+t}}A^{\frac{t-r}{2}}$$
$$\geq A^{\frac{t-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{t-r}{2}}$$

hold for $r \geq t$ and $s \geq 1$.

When we replace $\beta = (p-t)s+t$, $u = t-r \leq 0$ and $\delta = q$ in Theorem 5.1, Theorem 5.2 and Theorem 5.3 become the following Theorem 5.1', Theorem 5.2' and Theorem 5.3' respectively.

Theorem 5.1'. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geq A^u \sharp_{\frac{\delta-u}{\beta-u}}(A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

for $\beta \geq p \geq t$, $p \neq t$, $u \leq 0$ and $\beta \geq \delta \geq 0$.

Theorem 5.2'. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$*

(i)
$$B^\delta \geq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geq A^u \sharp_{\frac{\delta-u}{\beta-u}}(A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

and

(ii)
$$B^\delta \geq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \geq A^u \sharp_{\frac{\delta-u}{\beta-u}}(A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

hold for $\beta \geq p \geq max\{\delta,t\}$, $p \neq t$, $u \leq 0$.

Theorem 5.3'. *If $A \geq B \geq 0$ with $A > 0$, then for each $\delta, t \in [0,1]$*

(i)
$$A^\delta \geq B^\delta \geq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geq A^u \sharp_{\frac{\delta-u}{\beta-u}}(A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

and

(ii)
$$A^\delta \geq B^\delta \geq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \geq A^u \sharp_{\frac{\delta-u}{\beta-u}}(A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

hold for $\beta \geq p \geq max\{\delta,t\}$, $p \neq t$, $u \leq 0$.

We remark that Theorem 5.1' easily implies Theorem D.

Proof of Theorem 5.1. Theorem E ensures $G_{p,q,t}(A,B,t,s) \geq G_{p,q,t}(A,B,r,s)$ for each $q \geq 0$, $t \in [0,1]$ and $p \geq t$, that is,

$$A^{\frac{t}{2}} G_{p,q,t}(A,B,t,s) A^{\frac{t}{2}} \geq A^{\frac{t-r}{2}} \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}} \tag{5.1}$$

holds for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$. Then we have Theorem 5.1 by refining (5.1).

Proof of Theorem 5.2.

(i) Theorem E ensures $G_{p,q,t}(A,B,t,1) \geq G_{p,q,t}(A,B,t,s) \geq G_{p,q,t}(A,B,r,s)$ under conditions in Theorem 5.2, that is, (i) holds.

(ii) Theorem E ensures $G_{p,q,t}(A,B,t,1) \geq G_{p,q,t}(A,B,r,1) \geq G_{p,q,t}(A,B,r,s)$ under conditions in Theorem 5.2, that is, (ii) holds.

Proof of Theorem 5.3. We have onlt to apply Löwner-Heinz theorem to Theorem 5.2.

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