ON THE ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES OF SEVERAL VARIABLES

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ABSTRACT. In this paper, we consider concerning absolute convergence of the lacunary Fourier series of several variables

$$f(\mathbf{x}) \sim \sum_{\mathbf{m}\in\Lambda} c_{\mathbf{m}} e^{i\mathbf{m}\mathbf{x}} ,$$

where a lattice point set Λ satisfies some gap conditions. We generalize the result of [2;**Theorem**] in the case of several variables.

1. Introduction.

Let $f(x) \in L(\mathbf{T})$ $(\mathbf{T} = [-\pi, \pi])$. Let

$$\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) \tag{1.1}$$

be the Fourier series of f with an infinity of gap (n_k, n_{k+1}), where $\{n_k\}$ $(k \in \mathbf{N})$ is a strictly increasing sequence of natural numbers.

A strictly increasing sequence $\{n_k\}$ $(k \in \mathbf{N})$ of natural numbers is said to satisfy the condition $\mathbf{B}_2$ if $\sup_n R_2(n)$ is finite, where $R_2(n)$ denotes the number of different representations of an integer n in the form

$$n = \varepsilon_1 n_{k_1} + \varepsilon_2 n_{k_2} \qquad (\varepsilon_1, \varepsilon_2 = \pm 1; n_{k_1}, n_{k_2} \in \{n_k\})$$

(cf. [1; Vol.II, p.248]). It is known that any Hadamard gap sequence $\{n_k\}$ satisfies the condition $\mathbf{B}_2$ [1;Vol.II,p.234].

In 1980, J.R.Patadia and V.M.Shah [3] have proved the following theorem concerned with absolute convergence of lacunary Fourier series.

Theorem A. [3; Theorem 1]. *Let $E \subset \mathbf{T}$ be a set of positive measure. Let $\{n_k\}$ $(k \in \mathbf{N})$ be a strictly increasing sequence satisfying the gap condition $\mathbf{B}_2$. If*

$$\sum_{k=1}^{\infty} \left(\frac{\omega^{(2)}(\frac{2\pi}{n_k}, f, E)}{\sqrt{k}} \right)^{\beta} < \infty \qquad (0 < \beta \leq 1),$$

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then for the Fourier series (1.1) *of* $f \in L^2(\mathbf{T})$,

$$\sum_{k=1}^{\infty}\left(|a_{n_k}|^{\beta}+|b_{n_k}|^{\beta}\right)<\infty,$$

where

$$\omega^{(2)}(\delta,f,E)=\sup_{0\le h\le\delta}\left\{\left(\int_E|f(x+h)-f(x-h)|^2dx\right)^{\frac{1}{2}}\right\}.$$

Following Theorem B, we generalized β power stated in the hypothesis and the result in Theorem A to an increasing and concave function.

Theorem B. **[2; Theorem].** *Let* $\varphi(u)$ *defined for* $u \ge 0$ *be an increasing and concave function with* $\varphi(0)=0$ *and let a sequence* $\{n_k\}$ $(k \in \mathbf{N})$ *of natural numbers satisfy the gap condition* $\mathbf{B}_2$. *Let* $E \subset \mathbf{T}$ *be a set of positive measure. If*

$$\sum_{k=1}^{\infty}\varphi\left(\frac{\omega^{(2)}\left(\frac{2\pi}{n_k},f,E\right)}{\sqrt{k}}\right)<\infty,$$

then for the Fourier series (1.1) *of* $f \in L^2(\mathbf{T})$,

$$\sum_{k=1}^{\infty}\left(\varphi(|a_{n_k}|)+\varphi(|b_{n_k}|)\right)<\infty.$$

Furthermore, we extend Theorem B in the case of d-dimension.

Let $f(\mathbf{x}) \in L(\mathbf{T}^d)$ $(\mathbf{T}=[-\pi,\pi])$. It's Fourier series is

$$f(\mathbf{x}) \sim \sum_{\mathbf{m}\in\Lambda} c_{\mathbf{m}}e^{i\mathbf{m}\mathbf{x}} \tag{1.2}$$

where Λ is a lattice point set.

Definition. *A lattice point set* Λ *is said to satisfy the gap condition* $\mathbf{B}_2$, *if for any* $\mathbf{s} \neq \mathbf{0}$,

$$\sharp\left\{(\mathbf{m},\mathbf{m}') : \mathbf{m}'-\mathbf{m}=\mathbf{s}\ ,\ \mathbf{m}\neq\mathbf{m}'\ ,\ \mathbf{m},\mathbf{m}'\in\Lambda\right\}\le M\ ,$$

where $\sharp$ *is the number of pairs* $(\mathbf{m},\mathbf{m}')$ *and* M *is a positive integer.*

Let non-negative number $|\mathbf{m}| = \sqrt{{m_1}^2+\cdots+{m_d}^2}$ where $\mathbf{m} = (m_1,\cdots,m_d)$ and $\Lambda_k = \{\mathbf{m}\in\Lambda\ ;\ |\mathbf{m}|\le k\}$.

Theorem. *Let* $\varphi(u)$ *defined for* $u \ge 0$ *be an increasing and concave function with* $\varphi(0)=0$ *and let* $E \subset \mathbf{T}^d$ *be a set of positive measure. Suppose that a lattice point set* Λ *satisfies the gap condition* $\mathbf{B}_2$ *and that there exists a strictly increasing integer sequence* $\{n_k\}$ *such that* $\sharp\left\{\mathbf{m}\in\Lambda\ ;\ \mathbf{m}\in\Lambda_{n_{k+1}}\setminus\Lambda_{n_k}\right\}\le N$ *where* N *is a positive integer. If*

$$\sum_{k=1}^{\infty}\varphi\left(\frac{\omega^{(2)}\left(\frac{2\pi}{n_k},f,E\right)}{\sqrt{k}}\right)<\infty,$$

then for the Fourier series (1.2) *of* $f \in L^2(\mathbf{T}^d)$,

$$\sum_{\mathbf{m}\in\Lambda} \varphi(|c_{\mathbf{m}}|) < \infty,$$

where

$$\omega^{(2)}(\delta, f, E) = \sup_{|\mathbf{u}|\le\delta} \left\{ \left(\int_E |f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}-\mathbf{u})|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right\} .$$

Remark 1. *There exists the lattice point set* Λ *with the gap condition* $\mathbf{B}_2$ *which does not satisfy the gap condition* $\sharp\left\{\mathbf{m}\in\Lambda \ ;\ \mathbf{m}\in\Lambda_{n_{k+1}}\setminus\Lambda_{n_k}\right\} \le N$ *for any strictly increasing integer sequence* $\{n_k\}$.

Remark 2. *In the case of* 1*-dimension, the gap condition* $\sharp\left\{\mathbf{m}\in\Lambda \ ;\ \mathbf{m}\in\Lambda_{n_{k+1}}\setminus\Lambda_{n_k}\right\} \le N$ *is not necessary.*

2. Proof of Theorem.

To prove Theorem, we need several lemmas.

Lemma 1. [2; Lemma 2]. *When* $m \ge 2$ $(m \in \mathbf{N})$, *the following two conditions are equivalent.*

(I) $$\sum_{k=1}^{\infty} \frac{1}{k^{1-\frac{1}{m}}} \varphi\left(\frac{\omega^{(2)}\left(\frac{2\pi}{n_{[k^{\frac{1}{m}}]}}, f, E\right)}{\sqrt{k^{\frac{1}{m}}}} \right) < \infty,$$

(II) $$\sum_{k=1}^{\infty} \varphi\left(\frac{\omega^{(2)}\left(\frac{2\pi}{n_k}, f, E\right)}{\sqrt{k}} \right) < \infty.$$

Lemma 2. *Let* $E \subset \mathbf{T}^d$ *be a set of positive measure and a lattice point set* Λ *satisfy the gap condition* $\mathbf{B}_2$ $\sharp\{(\mathbf{m},\mathbf{m}') : \mathbf{m}'-\mathbf{m} = \mathbf{s} \ ,\ \mathbf{m}\ne\mathbf{m}' \ ,\ \mathbf{m},\mathbf{m}'\in\Lambda\} \le M$ *for any* $\mathbf{s}\ne\mathbf{0}$ *where* M *is a positive integer. For a constant* $\lambda > 1$, *there exists a natural number* $\nu = \nu(E,\lambda,\Lambda)$ *which has the following property; for any trigonometoric series*

$$P(\mathbf{x}) = \sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\ge\nu}} c_{\mathbf{m}} e^{i\mathbf{m}\mathbf{x}} \quad \textit{such that} \quad \sum_{\mathbf{m}\in\Lambda} |c_{\mathbf{m}}|^2 < \infty \ ,$$

we have

$$\frac{|E|}{\lambda} \sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\ge\nu}} |c_{\mathbf{m}}|^2 \le \int_E |P(\mathbf{x})|^2 d\mathbf{x} \ .$$

Proof of Lemma 2. Suppose that $\Lambda_k = \{\mathbf{m}\in\Lambda; |\mathbf{m}| \le k\}$, $P_k(\mathbf{x}) = \sum_{\mathbf{m}\in\Lambda_k} c_{\mathbf{m}} e^{i\mathbf{m}\mathbf{x}}$, and

$\gamma_{\mathbf{m}} = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \chi_E(\mathbf{x})\overline{e^{i\mathbf{m}\mathbf{x}}}d\mathbf{x}$. Then we have

$$\begin{aligned}
\int_E |P_k(\mathbf{x})|^2 d\mathbf{x} &= \int_E \left(\sum_{\mathbf{m}\in\Lambda_k} c_{\mathbf{m}} e^{i\mathbf{m}\mathbf{x}}\right)\left(\sum_{\mathbf{m}'\in\Lambda_k} \overline{c_{\mathbf{m}'} e^{i\mathbf{m}'\mathbf{x}}}\right) d\mathbf{x} \\
&= \sum_{\mathbf{m}\in\Lambda_k}\sum_{\mathbf{m}'\in\Lambda_k} c_{\mathbf{m}}\overline{c_{\mathbf{m}'}} \int_E \overline{e^{i(\mathbf{m}'-\mathbf{m})\mathbf{x}}} d\mathbf{x} \\
&= |E| \sum_{\mathbf{m}\in\Lambda_k} |c_{\mathbf{m}}|^2 + (2\pi)^d \sum_{\substack{\mathbf{m}\in\Lambda_k \\ \mathbf{m}\neq\mathbf{m}'}}\sum_{\mathbf{m}'\in\Lambda_k} c_{\mathbf{m}}\overline{c_{\mathbf{m}'}}\gamma_{\mathbf{m}'-\mathbf{m}} \quad = \quad I + J.
\end{aligned}$$

As for the term J, we have

$$\begin{aligned}
|J| &\leq (2\pi)^d \sum_{\substack{\mathbf{m}\in\Lambda_k \\ \mathbf{m}\neq\mathbf{m}'}}\sum_{\mathbf{m}'\in\Lambda_k} |c_{\mathbf{m}}c_{\mathbf{m}'}||\gamma_{\mathbf{m}'-\mathbf{m}}| \\
&\leq (2\pi)^d \left(\sum_{\mathbf{m}\in\Lambda_k}\sum_{\mathbf{m}'\in\Lambda_k} |c_{\mathbf{m}}c_{\mathbf{m}'}|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{\mathbf{m}\in\Lambda_k \\ \mathbf{m}\neq\mathbf{m}'}}\sum_{\mathbf{m}'\in\Lambda_k} |\gamma_{\mathbf{m}'-\mathbf{m}}|^2\right)^{\frac{1}{2}} \\
&= (2\pi)^d \left(\sum_{\mathbf{m}\in\Lambda_k} |c_{\mathbf{m}}|^2\right) \left(\sum_{\substack{\mathbf{m}\in\Lambda_k \\ \mathbf{m}\neq\mathbf{m}'}}\sum_{\mathbf{m}'\in\Lambda_k} |\gamma_{\mathbf{m}'-\mathbf{m}}|^2\right)^{\frac{1}{2}} \\
&= (2\pi)^d \left(\sum_{\mathbf{m}\in\Lambda_k} |c_{\mathbf{m}}|^2\right) \left(\sum_{\mathbf{s}}\sum_{\mathbf{m}'-\mathbf{m}=\mathbf{s}} |\gamma_{\mathbf{s}}|^2\right)^{\frac{1}{2}} \\
&= (2\pi)^d \left(\sum_{\mathbf{m}\in\Lambda_k} |c_{\mathbf{m}}|^2\right) \left(\sum_{\mathbf{s}} |\gamma_{\mathbf{s}}|^2 \,\sharp\left\{(\mathbf{m},\mathbf{m}') : \mathbf{m}'-\mathbf{m}=\mathbf{s}\right\}\right)^{\frac{1}{2}} \\
&\leq (2\pi)^d \left(\sum_{\mathbf{m}\in\Lambda_k} |c_{\mathbf{m}}|^2\right) \left(M\sum_{\mathbf{s}} |\gamma_{\mathbf{s}}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Owing to $\sum_{\mathbf{s}} |\gamma_{\mathbf{s}}|^2 < \infty$, there exists a positive number $\rho \geq 0$ such that

$$(2\pi)^d \left(M\sum_{|\mathbf{s}|\geq\rho} |\gamma_{\mathbf{s}}|^2\right)^{\frac{1}{2}} \leq \frac{\lambda-1}{\lambda}|E| .$$

Since there exists a natural number ν such that $\min\{|\mathbf{m}_1-\mathbf{m}_2|\} \geq \rho$ for any $\mathbf{m}_1, \mathbf{m}_2 \in \Lambda_k$ $(|\mathbf{m}_1|, |\mathbf{m}_2| \geq \nu,\ \mathbf{m}_1 \neq \mathbf{m}_2)$, we get

$$|J| \leq (2\pi)^d \left(\sum_{\substack{\mathbf{m}\in\Lambda_k \\ |\mathbf{m}|\geq\nu}} |c_{\mathbf{m}}|^2\right) \left(M\sum_{|\mathbf{s}|\geq\rho} |\gamma_{\mathbf{s}}|^2\right)^{\frac{1}{2}} \leq \frac{\lambda-1}{\lambda}|E| \left(\sum_{\substack{\mathbf{m}\in\Lambda_k \\ |\mathbf{m}|\geq\nu}} |c_{\mathbf{m}}|^2\right) .$$

Therefore,

$$\frac{|E|}{\lambda}\sum_{\substack{\mathbf{m}\in\Lambda_k\\ |\mathbf{m}|\geq\nu}}|c_{\mathbf{m}}|^2\leq\int_E|P_k(\mathbf{x})|^2d\mathbf{x}\ .$$

This inequality follows from $\displaystyle\sum_{\mathbf{m}\in\Lambda}|c_{\mathbf{m}}|^2<\infty$, even if $P_k(\mathbf{x})$ is infinite series $P(\mathbf{x})$.

This completes the proof of Lemma 2.

Lemma 3. *When $d\geq 2$ ($d\in\mathbf{N}$), we have*

$$\int_{\theta=0}^{\frac{\pi}{3}}(\sin\theta)^{d-2}\int_{t=0}^{[\frac{|\mathbf{m}|}{p}]\pi}t^{d-1}|\sin(t\cos\theta)|^2dtd\theta\geq\frac{\pi^2}{12}\left(\frac{\pi}{2}\right)^{d-1}\left(\frac{\sqrt{3}}{2}\right)^{d-2}\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d(d-1)}\ ,$$

where $p\in\mathbf{N}$.

Proof of Lemma 3.

$$\begin{aligned}
&\int_{\theta=0}^{\frac{\pi}{3}}(\sin\theta)^{d-2}\int_{t=0}^{[\frac{|\mathbf{m}|}{p}]\pi}t^{d-1}|\sin(t\cos\theta)|^2dtd\theta\\
=\ &\int_{\theta=0}^{\frac{\pi}{3}}\frac{(\sin\theta)^{d-2}}{(\cos\theta)^d}\int_{v=0}^{[\frac{|\mathbf{m}|}{p}]\pi(\cos\theta)}v^{d-1}|\sin v|^2dvd\theta\\
\geq\ &\int_{\theta=0}^{\frac{\pi}{3}}(\sin\theta)^{d-2}\int_{v=0}^{\frac{1}{2}[\frac{|\mathbf{m}|}{p}]\pi}v^{d-1}|\sin v|^2dvd\theta\\
\geq\ &\frac{\pi}{4}\left(\frac{\pi}{2}\right)^{d-1}\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d}\int_{\theta=0}^{\frac{\pi}{3}}(\sin\theta)^{d-2}d\theta\\
\geq\ &\frac{\pi}{4}\left(\frac{\pi}{2}\right)^{d-1}\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d}\int_{\theta=0}^{\frac{\pi}{3}}\left(\frac{3\sqrt{3}}{2\pi}\theta\right)^{d-2}d\theta\\
=\ &\frac{\pi^2}{12}\left(\frac{\pi}{2}\right)^{d-1}\left(\frac{\sqrt{3}}{2}\right)^{d-2}\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d(d-1)}\ .
\end{aligned}$$

In the previous estimate, we use

$$\begin{aligned}
&\int_{v=0}^{\frac{1}{2}[\frac{|\mathbf{m}|}{p}]\pi}v^{d-1}|\sin v|^2dv=\sum_{j=1}^{[\frac{|\mathbf{m}|}{p}]}\int_{\frac{(j-1)}{2}\pi}^{\frac{j}{2}\pi}v^{d-1}|\sin v|^2dv\\
\geq\ &\sum_{j=1}^{[\frac{|\mathbf{m}|}{p}]}\left(\frac{(j-1)\pi}{2}\right)^{d-1}\int_{\frac{(j-1)}{2}\pi}^{\frac{j}{2}\pi}|\sin v|^2dv=\frac{\pi}{4}\sum_{j=1}^{[\frac{|\mathbf{m}|}{p}]}\left(\frac{(j-1)\pi}{2}\right)^{d-1}\\
\geq\ &\frac{\pi}{4}\left(\frac{\pi}{2}\right)^{d-1}\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d}\ .
\end{aligned}$$

This completes the proof of Lemma 3.

Lemma 4. *If $|\mathbf{m}| \geq 2p$ $(p \in \mathbf{N})$, then*

$$\int_{|\mathbf{u}| \leq \frac{\pi}{p}} |\sin \mathbf{m}\cdot\mathbf{u}|^2 d\mathbf{u} \geq \frac{4\pi^{\frac{3d+1}{2}}}{9\left(4\sqrt{3}p\right)^d d(d-1)\,\Gamma\left(\frac{d-1}{2}\right)} .$$

Proof of Lemma 4. Using Lemma 3,

$$\begin{aligned}
&\int_{|\mathbf{u}| \leq \frac{\pi}{p}} |\sin \mathbf{m}\cdot\mathbf{u}|^2 d\mathbf{u} \\
&= \int_{\theta_{d-1}=0}^{2\pi} \int_{\theta_{d-2}=0}^{\pi} \cdots \int_{\theta_1=0}^{\pi} \int_{r=0}^{\frac{\pi}{p}} |\sin(|\mathbf{m}|\ r\ \cos\theta_1)|^2 \\
&\qquad\qquad r^{d-1}(\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\cdots\sin\theta_{d-2}\,dr d\theta_1\cdots d\theta_{d-1} \\
&\geq \frac{1}{|\mathbf{m}|^d} \int_{\theta_{d-1}=0}^{2\pi} \int_{\theta_{d-2}=0}^{\pi} \sin\theta_{d-2} \int_{\theta_{d-3}=0}^{\pi} (\sin\theta_{d-3})^2 \cdots \int_{\theta_2=0}^{\pi} (\sin\theta_2)^{d-3} \int_{\theta_1=0}^{\frac{\pi}{3}} (\sin\theta_1)^{d-2} \\
&\qquad\qquad \int_{t=0}^{\frac{|\mathbf{m}|}{p}\pi} t^{d-1} |\sin(t\cos\theta_1)|^2 dt d\theta_1\cdots d\theta_{d-1} \\
&\geq \frac{1}{p^d\left([\frac{|\mathbf{m}|}{p}]+1\right)^d} \int_{\theta_{d-1}=0}^{2\pi} \int_{\theta_{d-2}=0}^{\pi} \sin\theta_{d-2} \int_{\theta_{d-3}=0}^{\pi} (\sin\theta_{d-3})^2 \cdots \int_{\theta_2=0}^{\pi} (\sin\theta_2)^{d-3} \\
&\qquad\qquad \int_{\theta_1=0}^{\frac{\pi}{3}} (\sin\theta_1)^{d-2} \int_{t=0}^{[\frac{|\mathbf{m}|}{p}]\pi} t^{d-1} |\sin(t\cos\theta_1)|^2 dt d\theta_1\cdots d\theta_{d-1} \\
&\geq \frac{1}{p^d\left([\frac{|\mathbf{m}|}{p}]+1\right)^d} \frac{\pi^2}{12}\left(\frac{\pi}{2}\right)^{d-1}\left(\frac{\sqrt{3}}{2}\right)^{d-2} \frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{d(d-1)} \\
&\qquad\qquad \int_{\theta_{d-1}=0}^{2\pi} \int_{\theta_{d-2}=0}^{\pi} \sin\theta_{d-2} \int_{\theta_{d-3}=0}^{\pi} (\sin\theta_{d-3})^2 \cdots \int_{\theta_2=0}^{\pi} (\sin\theta_2)^{d-3} d\theta_2\cdots d\theta_{d-1} \\
&\geq \frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{p^d\left([\frac{|\mathbf{m}|}{p}]+1\right)^d} \frac{\pi^2}{12}\left(\frac{\pi}{2}\right)^{d-1}\left(\frac{\sqrt{3}}{2}\right)^{d-2} \frac{1}{d(d-1)} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \\
&\geq \frac{4\pi^{\frac{3d+1}{2}}}{9\left(4\sqrt{3}p\right)^d d(d-1)\,\Gamma\left(\frac{d-1}{2}\right)} .
\end{aligned}$$

In the previous estimate, we use

$$\begin{aligned}
&\int_{\theta_{d-1}=0}^{2\pi} \int_{\theta_{d-2}=0}^{\pi} \sin\theta_{d-2} \int_{\theta_{d-3}=0}^{\pi} (\sin\theta_{d-3})^2 \cdots \int_{\theta_2=0}^{\pi} (\sin\theta_2)^{d-3} d\theta_2\cdots d\theta_{d-1} \\
=\ & (2\pi)\ B\left(1,\frac{1}{2}\right)\ B\left(\frac{3}{2},\frac{1}{2}\right)\cdots B\left(\frac{d-2}{2},\frac{1}{2}\right) \\
=\ & (2\pi)\frac{\Gamma(1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \cdots \frac{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} = (2\pi)\frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^{d-3}}{\Gamma\left(\frac{d-1}{2}\right)} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} ,
\end{aligned}$$

and $$\frac{\left([\frac{|\mathbf{m}|}{p}]-1\right)^d}{\left([\frac{|\mathbf{m}|}{p}]+1\right)^d} \geq \left(\frac{1}{3}\right)^d \quad (|\mathbf{m}| \geq 2p)\ .$$

This completes the proof of Lemma 4.

Lemma 5. *Let $E \subset \mathbf{T}^d$ be a set of positive measure and a lattice point set Λ satisfy the gap condition $\mathbf{B}_2$. When $c_{\mathbf{m}} = 0$ $(|\mathbf{m}| < \nu)$ where ν is as in Lemma 2, we get the following inequality;*

$$\sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\geq q}} |c_{\mathbf{m}}|^2 \leq C\frac{\lambda}{|E|}\omega^{(2)}\left(\frac{2\pi}{q}, f, E\right)^2,$$

where $C = \dfrac{9\left(4\sqrt{3}\right)^d d(d-1)\,\Gamma\left(\frac{d-1}{2}\right)}{16\sqrt{\pi}\,\Gamma\left(\frac{d}{2}+1\right)}$ *and $q \in \mathbf{N}$.*

Proof of Lemma 5. In Lemma 2, putting $P(\mathbf{x}) = f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}-\mathbf{u})$, we get

$$4\frac{|E|}{\lambda}\sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\geq \nu}} |c_{\mathbf{m}}|^2 |\sin \mathbf{m}\cdot\mathbf{u}|^2 \leq \int_E |f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}-\mathbf{u})|^2 d\mathbf{x}\ ,$$

where $c_{\mathbf{m}}$ is the Fourier coefficient of f. Integrating the both sides with respect to $\mathbf{u}$ over $|\mathbf{u}| \leq \frac{\pi}{p}$ $(p \in \mathbf{N})$ and using Lemma 4,

$$\begin{aligned}
&\frac{16\pi^{\frac{3d+1}{2}}}{9\left(4\sqrt{3}p\right)^d d(d-1)\,\Gamma\left(\frac{d-1}{2}\right)}\frac{|E|}{\lambda}\sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\geq 2p}} |c_{\mathbf{m}}|^2 \\
\leq\ & 4\frac{|E|}{\lambda}\sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\geq 2p}} |c_{\mathbf{m}}|^2 \int_{|\mathbf{u}|\leq\frac{\pi}{p}} |\sin \mathbf{m}\cdot\mathbf{u}|^2 d\mathbf{u} \\
\leq\ & \int_{|\mathbf{u}|\leq\frac{\pi}{p}} \left\{\int_E |f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}-\mathbf{u})|^2 d\mathbf{x}\right\} d\mathbf{u} \\
\leq\ & \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\pi}{p}\right)^d \left\{\sup_{|\mathbf{u}|\leq\frac{\pi}{p}} \int_E |f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}-\mathbf{u})|^2 d\mathbf{x}\right\} \\
\leq\ & \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\pi}{p}\right)^d \omega^{(2)}\left(\frac{\pi}{p}, f, E\right)^2 .
\end{aligned}$$

Therefore,

$$\sum_{\substack{\mathbf{m}\in\Lambda \\ |\mathbf{m}|\geq 2p}} |c_{\mathbf{m}}|^2 \leq \frac{9\left(4\sqrt{3}\right)^d d(d-1)\,\Gamma\left(\frac{d-1}{2}\right)}{16\sqrt{\pi}\,\Gamma\left(\frac{d}{2}+1\right)}\frac{\lambda}{|E|}\omega^{(2)}\left(\frac{\pi}{p}, f, E\right)^2 .$$

This completes the proof of Lemma 5.

Lemma 6. (Jensen's inequality concerning concave function) *Let $\varphi(u)$ defined for $u \geq 0$ be an increasing and concave function with $\varphi(0) = 0$. For any infinite sequence* *of non-negative numbers $x_1, x_2, \cdots, x_k, \cdots$ and any infinite sequence of positive numbers $J_1, J_2, \cdots, J_k, \cdots$, we get the following inequality;*

$$\frac{\sum_{k=1}^{\infty} J_k \varphi(x_k)}{\sum_{k=1}^{\infty} J_k} \leq \varphi\left(\frac{\sum_{k=1}^{\infty} J_k x_k}{\sum_{k=1}^{\infty} J_k}\right),$$

where each series in the above inequality converges.

By using lemmas, we shall prove Theorem.

Proof of Theorem.

$$\begin{aligned}
\sum_{\mathbf{m}\in\Lambda} \varphi & (|c_{\mathbf{m}}|) = \sum_{j=1}^{\infty} \sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} \varphi(|c_{\mathbf{m}}|) = \sum_{j=1}^{\infty}\sum_{k=1}^{j^3} \frac{1}{j^3} \sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} \varphi(|c_{\mathbf{m}}|) \\
&\leq \sum_{k=1}^{\infty} \sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3} \sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} \varphi(|c_{\mathbf{m}}|) \\
&\leq \sum_{k=1}^{\infty} \sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3} \sharp\left\{\mathbf{m}\in\Lambda; \mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}\right\} \varphi\left(\frac{\sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|}{\sharp\left\{\mathbf{m}\in\Lambda; \mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}\right\}}\right) \\
&\leq N \sum_{k=1}^{\infty} \sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3} \varphi\left(\sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|\right) \\
&\leq N \sum_{k=1}^{\infty} \left(\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3}\right) \varphi\left(\frac{\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3} \sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|}{\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3}}\right) \\
&\leq 6N \sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}} \varphi\left(2k^{\frac{2}{3}} \sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^3} \sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|\right) \\
&\leq 6N \sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}} \varphi\left(2k^{\frac{2}{3}} \left(\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \frac{1}{j^6}\right)^{\frac{1}{2}} \left(\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \left(\sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|\right)^2\right)^{\frac{1}{2}}\right) \\
&\leq 6N \sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}} \varphi\left(2^4 \sqrt{\frac{3}{5}} \frac{1}{\sqrt{k^{\frac{1}{3}}}} \left(\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \left(\sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|\right)^2\right)^{\frac{1}{2}}\right) \\
&\leq 6N \sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}} \varphi\left(2^4 \sqrt{\frac{3}{5}} \frac{1}{\sqrt{k^{\frac{1}{3}}}} \left(\sum_{j=[k^{\frac{1}{3}}]}^{\infty} \left(\sum_{\mathbf{m}\in\Lambda_{n_{j+1}}\setminus\Lambda_{n_j}} |c_{\mathbf{m}}|^2\right) N\right)^{\frac{1}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \quad 6N\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{3}}}\varphi\left(2^4\sqrt{\frac{3N}{5}}\frac{1}{\sqrt{k^{\frac{1}{3}}}}\left(\sum_{|\mathbf{m}|\geq n_{[k^{\frac{1}{3}}]}}|c_{\mathbf{m}}|^2\right)^{\frac{1}{2}}\right)\\
&\leq \quad 6N\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{3}}}\varphi\left(C'\frac{1}{\sqrt{k^{\frac{1}{3}}}}\ \omega^{(2)}\left(\frac{2\pi}{n_{[k^{\frac{1}{3}}]}},f,E\right)\right)\\
&\leq \quad C''\sum_{k=1}^{\infty}\frac{1}{k^{\frac{2}{3}}}\varphi\left(\frac{1}{\sqrt{k^{\frac{1}{3}}}}\ \omega^{(2)}\left(\frac{2\pi}{n_{[k^{\frac{1}{3}}]}},f,E\right)\right)<\infty\ ,
\end{aligned}
$$

where $\quad C'=2^4\sqrt{\dfrac{3NC\lambda}{5|E|}}$ and $C''=6N\max(1,C')$.

Therefore, owing to Lemma 1 with $m=3$, the proof of Theorem completes.

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