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EQUIVALENCE RELATION BETWEEN GENERALIZED FURUTA INEQUALITY AND RELATED OPERATOR FUNCTIONS

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1. INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by : $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is strictly positive (denoted by : $T > 0$) if T is positive and invertible. As an extension of the famous Löwner-Heinz inequality, we have established the following;

Theorem F (Furuta inequality) [4].

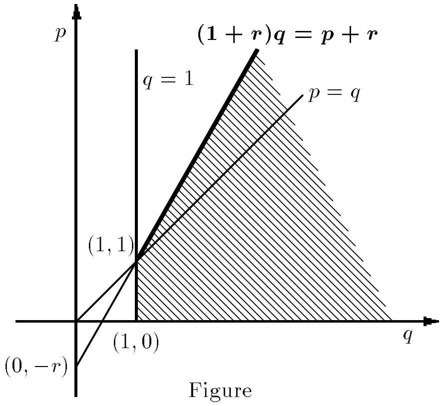
If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

Alternative proofs of Theorem F are given in [2] and [10] and also an elementary one page proof in [5]. It is shown in [11] that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F.

2. RESULT

We shall show the following equivalence relation between generalized Furuta inequality and related operator functions.

Theorem 1. *The following* (i), (ii), (iii) *and* (iv) *hold and follow from each other.*

(i) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad \text{holds for } s \geq 1 \text{ and } r \geq t.$$

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \quad \text{holds for } s \geq 1 \text{ and } r \geq t.$$

(iii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$.

(iv) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq t$,*

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$$G_{p,q,t}(A,B,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r\geq t$ *and* $s\geq 1$ *such that* $(p-t)s\geq q-t$.

(i) and (iii) in Theorem 1 have been obtained in [6] and an excellent mean theoretic proof is shown in [3]. (i) interpolates the inequality equivalent to the main result of the log majorization [1] and Theorem F itself. Recently (ii) is shown in [8] and (iv) is also shown in [7] as an extension of (iii) and [9]. We need the following result to give a proof of Theorem 1.

Lemma A [6]. *Let* $A>0$ *and* B *be an invertible operator. For any real number* λ

$$(BAB^*)^\lambda=BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*.$$

Proof of Theorem 1. We may assume that A and B are both invertible.

(iv) $\Longrightarrow$ (iii). We have only to put $q=1$ in (iv).

(iii) $\Longrightarrow$ (i). $A\geq B\geq 0$ and monotonicity of $F_{p,t}(A,B,r,s)$ ensure

$$A^{1-t}\geq A^{\frac{-t}{2}}BA^{\frac{-t}{2}}=F_{p,t}(A,B,t,1)\geq F_{p,t}(A,B,r,s)$$

so that we have (i).

(i) $\Longrightarrow$ (ii). Put $A_1=A^q$ and $B_1=B^q$ for $q\in[0,1]$. Then $A_1\geq B_1\geq 0$ holds by Löwner-Heinz inequality. Put $p_1=\frac{p}{q}\geq 1$, $t_1=\frac{t}{q}$ and $r_1=\frac{r}{q}$. Then we have only to apply (i) on $A_1\geq B_1$.

(ii) $\Longrightarrow$ (iv). Put $q=t$ in (ii). Then if $A\geq B\geq 0$, then for each $t\in[0,1]$ and $p\geq t$

$$A^r\geq\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r}{(p-t)s+r}}\qquad\text{for } s\geq 1 \text{ and } r\geq t. \tag{2.1}$$

(a) *Decreasing of* $G_{p,q,t}(A,B,r,s)$ *for* s. Put $D=A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}$. Applying Lemma A to (2.1) and Löwner-Heinz inequality, we obtain for each $t\in[0,1]$, $p\geq t$, $s\geq 1$ and $r\geq t$

$$(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}}\geq D^w\qquad\text{for } s\geq w\geq 0. \tag{2.2}$$

Then we have

$$\begin{aligned}
f(s)&=\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}\\
&=(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}}\\
&=\{(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{(p-t)(s+w)+r}{(p-t)s+r}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}}\\
&=\{A^{\frac{r}{2}}D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}}D^{\frac{s}{2}}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}}\qquad\text{by Lemma A}\\
&\geq(A^{\frac{r}{2}}D^{s+w}A^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}}\\
&=f(s+w)
\end{aligned}$$

and the last inequality holds by (2.2) and Löwner-Heinz inequality since $\frac{q-t+r}{(p-t)(s+w)+r}\in[0,1]$ holds, so the proof of (a) is complete since $G_{p,q,t}(A,B,r,s)=A^{\frac{-r}{2}}f(s)A^{\frac{-r}{2}}$.

(b) *Decreasing of* $G_{p,q,t}(A,B,r,s)$ *for* r. Applying Löwner-Heinz inequality to (2.1), if $A\geq B\geq 0$, then for each $t\in[0,1]$, $p\geq t$, $s\geq 1$ and $r\geq t$

$$A^u\geq(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}\qquad\text{for } r\geq u\geq 0. \tag{2.3}$$

Then we have

$$\begin{aligned}
G_{p,q,t}(A,B,r,s)&=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}\\
&=D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}D^{\frac{s}{2}}\qquad\text{by Lemma A}\\
&=D^{\frac{s}{2}}\{(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}}\\
&=D^{\frac{s}{2}}\{D^{\frac{s}{2}}A^{\frac{r}{2}}(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}A^{\frac{r}{2}}D^{\frac{s}{2}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}}\qquad\text{by Lemma A}\\
&\geq D^{\frac{s}{2}}(D^{\frac{s}{2}}A^{r+u}D^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}}\\
&=G_{p,q,t}(A,B,r+u,s)
\end{aligned}$$

and the last inequality holds by (2.3) and Löwner-Heinz inequality since $\frac{q-t-(p-t)s}{(p-t)s+r+u}\in[-1,0]$. Consequently we obtain (iv) by (a) and (b), so the proof of Theorem 1 is complete.

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