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SPECTRAL ORDER AND OPERATOR MEANS

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Abstract. Bonsall's proof of the spectral theorem is regarded as an application of the Kubo-Ando theory on operator means. Consequently, the Kato theorem for the lattice operations on the spectral order of positive operators is given a new look in terms of operator means.

1. Introduction. A (bounded linear) operator A acting on a Hilbert space H is positive, say $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. The usual order $A \geq B$ among selfadjoint operators is naturally induced by $A - B \geq 0$. Unfortunately, the selfadjoint operators does not form a complete vector lattice. In 1971, Olson [9] introduced a new order among the selfadjoint operators, by which it becomes a conditionally complete lattice, cf. also [5]: Let E_t (resp. F_t) be the resolution of the identity of A (resp. B), i.e.,

$$A = \int t\, dE_t \quad \text{and} \quad B = \int t\, dF_t. \tag{1}$$

Then the spectral order $A \preceq B$ holds if $E_t \geq F_t$ for all t.

In 1979, Kato [5] realized the join $A \vee B$ of positive operators A and B by

$$\text{s-lim}\ (A^n \nabla B^n)^{\frac{1}{n}} = A \vee B, \tag{2}$$

where ∇ is the arithmetic mean and s-lim the limit in the strong operator topology. Also Ando pointed out that the meet of positive invertible operators A and B is given by the harmonic mean !;

$$\text{s-lim}\ (A^n\ !\ B^n)^{\frac{1}{n}} = A \wedge B, \tag{3}$$

where $A\ !\ B = 2(A^{-1} + B^{-1})^{-1}$ if they are invertible; in general,

$$A\ !\ B = \text{s-lim}_{\epsilon \downarrow 0}(A + \epsilon)\ !\ (B + \epsilon) \tag{4}$$

for $A, B \geq 0$, see Kubo-Ando [7].

On the other hand, Olson characterized the spectral order by their powers for positive operators [9;Theorem 3], cf. also [8] and [10]:

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Theorem O. *For positive operators A and B, $A \preceq B$ if and only if*

$$A^n \leq B^n \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

In addition, several useful properties of the spectral order are given by Uchiyama [11].

In this note, we introduce a harmonic system associated to the harmonic mean and Olson's theorem, and prove that every harmonic system contains a resolution of the identity of each element in it by Bonsall's observation on the spectral theorem. As a consequence, we can give an alternative simple proof to (5). Moreover, as an application of (5), we propose a short proof to Kato's theorem (2) and (3). The latter clarifies some advantage of the parallel sum due to Anderson-Trapp [1].

2. Harmonic systems. A subset Ξ of positive operators is called a harmonic system if Ξ satisfies (i) it contains 1 and is closed in the strong operator topology, (ii) if $A \in \Xi$, then $A^n \in \Xi$ for all $n \in \mathbb{N}$, and (iii) $A,\ B \in \Xi$ implies $A \,!\, B \in \Xi$.

Here we state the following theorem which is nothing but a disguise of Bonsall's constructive spectral theorem in [3]:

Theorem 1. *If a positive operator A with the spectral decomposition (1) belongs to a harmonic system Ξ, then $E_1^{\perp}$ is contained in Ξ.*

As a matter of fact, Bonsall observed that the successive approximation defined by

$$A_{n+1} = 2A_n^2(1 + A_n^2)^{-1}, \quad A_1 = A \tag{6}$$

converges strongly (monotonically) to $E_1^{\perp}$, which is nothing but $A_{n+1} = 1 \,!\, A_n^2$ and proves the theorem.

Furthermore one can construct a sequence converging strongly to $E_t^{\perp}$ by replacing 1 to t^2 in (6):

$$A_{n+1} = 2A_n^2(t^2 + A_n^2)^{-1} = 1 \,!\, (\frac{A_n}{t})^2, \quad A_1 = A \tag{7}$$

Namely we reach Olson's theorem:

Proof of Theorem O. Suppose that A and B have the spectral decompositions (1). We may assume that A and B are invertible. As in the remark above, $E_t^{\perp}$ and $F_t^{\perp}$ are approximated by the sequences $\{A_k\}$ and $\{B_k\}$, defined as in (7), respectively. So we show that $A_k^n \leq B_k^n$ for all $k, n \in \mathbb{N}$ by the induction on k. This is true for $k = 1$ by the assumption. If it is true for some k, then

$$(1 + (\frac{A_k}{t})^{-2})^n \geq (1 + (\frac{B_k}{t})^{-2})^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Taking the inverse on both sides, we have $A_{k+1}^n \leq B_{k+1}^n$ for all $n \in \mathbb{N}$.

In the proof of Theorem O, we approximated $E_t^{\perp}$ by (7). However E_t itself is approximated by

$$A_{n+1}^{(t)} = 1 \,!\, (A_n^{(t)})^2,\ A_1^{(t)} = \frac{1 - A}{1 - t}, \tag{8}$$

where we assume that $\|A\| \leq \|B\| < 1$ for simplicity. Thus Theorem O is also proved by $(A^{(t)})^n \geq (B^{(t)})^n$ for all $n \in \mathbb{N}$, which follows from

$$(1-A)^{-n} = (\sum_{k=0}^{\infty} A^k)^n \leq (1-B)^{-n} = (\sum_{k=0}^{\infty} B^k)^n.$$

Remark. (1) As a simple consequence, the spectral order is reversible in the following sense as well as the usual one: For positive invertible operators A and B, $A \preceq B$ if and only if $B^{-1} \preceq A^{-1}$.

(2) The complexity of a positive invertible operator A is recently introduced in [4] by

$$\kappa(A;x) = \lim_{n\to\infty} \frac{\log(A^n x, x)}{n} \quad \text{for a unit vector } x \in H, \tag{9}$$

by which the complexity order among positive operators is defined as $A \prec_\kappa B$ if $\kappa(A;x) \leq \kappa(B;x)$ for all unit vectors $x \in H$. It is shown that it coincides with the spectral order and consequently Theorem O is obtained [4;Theorem 5.1].

3. Kato's theorem. As an application of Theorem O, we give a simple proof to Kato's realization of the join $A \vee B$ of $A,\ B \geq 0$ under the spectral order.

Theorem 2. (Kato) *For $A,\ B \geq 0$, the join $A \vee B$ is given by*

$$\text{s-lim}(A^n \nabla B^n)^{\frac{1}{n}} = A \vee B. \tag{2}$$

For the sake of convenience, we first cite the Löwner-Heinz inequality;

$$A \geq B \geq 0 \text{ implies } A^\alpha \geq B^\alpha \text{ for } \alpha \in [0,1]. \tag{10}$$

Next we recur the existence of s-lim $(A^n \nabla B^n)^{\frac{1}{n}}$. For $n > m > 0$, the operator concavity of the function $x \to x^{\frac{m}{n}}$ ensures that

$$(\frac{A^n + B^n}{2})^{\frac{m}{n}} \geq \frac{(A^n)^{\frac{m}{n}} + (B^n)^{\frac{m}{n}}}{2} = \frac{A^m + B^m}{2},$$

so that the Löwner-Heinz inequality (10) implies

$$(\frac{A^n + B^n}{2})^{\frac{1}{n}} \geq (\frac{A^m + B^m}{2})^{\frac{1}{m}}. \tag{11}$$

Hence $C_n = (A^n \nabla B^n)^{\frac{1}{n}}$ is a bounded nondecreasing sequence, so that $D =$ s-lim C_n exists.

Proof of Theorem 2. It suffices to show that $A, B \preceq D$ and $D \preceq C$ for $C \succeq A, B$. The former $A \preceq D$ is ensured by

$$A^k = (A^{kn})^{\frac{1}{n}} \leq (A^{kn} + B^{kn})^{\frac{1}{n}} = 2^{\frac{1}{n}}((\frac{A^{kn} + B^{kn}}{2})^{\frac{1}{kn}})^k$$

for all $k,\ n \in \mathbb{N}$ and so $A^k \leq D^k$ for all $k \in \mathbb{N}$ by taking $n \to \infty$.

Next, if $C \succeq A,\ B$, then $C^{kn} \geq A^{kn},\ B^{kn}$ for all $k, n \in \mathbb{N}$. Thus we have

$$C_{kn}^k = (\frac{A^{kn} + B^{kn}}{2})^{\frac{1}{n}} \leq (C^{kn})^{\frac{1}{n}} = C^k$$

by (10) and so $D^k \leq C^k$, i.e., $D \preceq C$. Hence we have $D = A \vee B$.

The above version of Kato's theorem tells us that the join is described by the arithmetic mean, the greatest symmetric (operator) mean; accordingly the harmonic mean will be expected to give the meet since it is the smallest symmetric mean. In the below we give an analogous proof to Ando's theorem [2;Lemma 6.15], in which we need the parallel sum by Anderson-Trapp [1], i.e., $A : B = \frac{1}{2}(A\ !\ B)$.

Theorem 3. *For $A,\ B \geq 0$, the meet $A\ \wedge\ B$ is given by*

$$\text{s-lim } (A^n \ !\ B^n)^{\frac{1}{n}} = A \wedge B. \tag{3}$$

Proof. Put $A_\epsilon = A + \epsilon$ for $\epsilon > 0$. By (11), we have for each $\epsilon > 0$

$$(A_\epsilon^n \ !\ B_\epsilon^n)^{\frac{1}{n}} \leq (A_\epsilon^m \ !\ B_\epsilon^m)^{\frac{1}{m}} \quad \text{if } n > m.$$

Taking $\epsilon \downarrow 0$, both sides above converge decreasingly and hence

$$(A^n \ !\ B^n)^{\frac{1}{n}} \leq (A^m \ !\ B^m)^{\frac{1}{m}} \quad \text{if } n > m.$$

Therefore there exists s-lim $(A^n \ !\ B^n)^{\frac{1}{n}} = E$.

Since $A^{kn} : B^{kn} \leq A^{kn}, B^{kn}$ for $k, n \in \mathbb{N}$, we have

$$((A^{kn} : B^{kn})^{\frac{1}{kn}})^k \leq A^k, B^k,$$

so that $E^k \leq A^k, B^k$. This means that $E \preceq A, B$.

Next, if $0 \leq C \preceq A, B$, then $C^{kn} \leq A^{kn}, B^{kn}$ for $k, n \in \mathbb{N}$. Since $C^{kn} \leq A^{kn} \ !\ B^{kn}$ for $k, n \in \mathbb{N}$, we have $C^k \leq ((A^{kn} \ !\ B^{kn})^{\frac{1}{kn}})^k$ by (10). Taking $n \to \infty$, we have $C^k \leq E^k$ for all $k \in \mathbb{N}$, so that $C \preceq E$. Hence we have $E = A \wedge B$.

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REFERENCES

1. W.N.Anderson and G.E.Trapp, *Shorted operators, II*, SIAM J. Appl. Math., **28** (1975), 60-71.
2. T.Ando, *Majorization, doubly stochastic matrices and comparison of eigenvalues*, Linear Alg. and its Appl., **118** (1989), 163-248.
3. F.F.Bonsall, *A formula for the spectral family of an operator*, J.London Math. Soc., **35** (1960), 321-333.
4. J.-I.Fujii and M.Fujii, *Kolmogorov's complexity for positive operators*, preprint,.
5. M.Fujii and I.Kasahara, *A remark on the spectral order of operators*, Proc. Japan Acad., **47** (1971), 986-988.
6. T.Kato, *Spectral order and a matrix limit theorem*, Linear Multilinear Alg., **8** (1979), 15-19.
7. F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205-224.
8. M.K.Kwong, *Conditions under which the order of two non-negative Hermitian matrices is preserved when taking integral powers*, Linear Alg. and its Appl., **18** (1977), 223-228.
9. M.P.Olson, *The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice*, Proc. Amer. Math. Soc., **28** (1971), 537-544.
10. G.K.Pedersen, *Power ordering of positive operators*, unpublished, privately circulated in 1989.
11. M.Uchiyama, *Commutativity of selfadjoint operators*, Pacific J. Math., **161** (1993), 385-392.

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