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DETERMINANT FOR POSITIVE OPERATORS

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Abstract. Since the normalized determinant of a positive-definite matrix is the geometric mean of the eigenvalues, we introduce the determinant of a positive operator as a (continuous) geometric mean of the spectrum. Some inequalities as the geometric mean hold; for example, the determinant is not greater than the trace which is the arithmetic mean. In addition we discuss elementary properties.

1. Introduction. There are some attempts to extend the notion of the determinant for matrices. In '50s, Fuglede and Kadison [4,5] defined the determinant on invertible operators A in a II_1-factor M with the canonical (normalized) trace Tr as

$$\Delta(A) = \exp \mathrm{Tr}(\log |A|)$$

and discussed the properties of this determinant. Afterwards, Arveson [2] developed it in general von Neumann algebras and investigated some additional properties.

Here, note that the determinant of a matrix is the product of all the eigenvalues, which contrasts with the fact that the trace of it is the sum of them. The normalization of the trace in $\Delta(A)$ yields another view for the determinants. For a positive-definite $n \times n$ matrix A with the spectrum $\sigma(A) = \{t_1, ..., t_n\}$, the determinant in their sense is just the geometric mean

$$\prod_{i=1}^{n} t_i^{1/n}$$

while the normalized trace is the arithmetic mean $\sum_{i=1}^{n} t_i/n$. (Hardy, Littlewood and Pólya [6 ;6.21] also discussed these means in this way.) So their determinant for positive operators is considered as the 'continuous' geometric mean, which reminds us of the product integral introduced by G.Birkhoff [3].

Throughout this paper, let A be a positive invertible operators on a Hilbert space H and x a unit vector in H. In this paper, we consider the *determinant* $\Delta_x(A)$ *for* A *at* x defined as

$$\Delta_x(A) = \exp\langle \log(A)x, x\rangle.$$

As in the above, we discuss it as a continuous (weighted) geometric mean (with the weight x) and observe some inequalities around the determinant from this point of view. Note that this definition is easily extended to that at a state on a suitable operator algebra via the GNS representation.

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2. Continuous means. By the definition, we immediately have

$$\Delta_x(tA) = t\Delta_x(A) \text{ and } \Delta_x(t) = t$$

for all positive numbers t. We also have the (norm) continuity for the maps $x \mapsto \Delta_x(A)$ and $A \mapsto \Delta_x(A)$. Moreover the latter map is monotone by the operator monotonicity of the logarithm:

Theorem 1. *The map $A \mapsto \Delta_x(A)$ is monotone: $A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$.*

It is easy to see

$$(*) \qquad \Delta_x\left(\sum_{i=1}^n t_i E_i\right) = \prod_{i=1}^n t_i^{\langle E_i x, x\rangle}$$

for the projections E_i with $\sum_i E_i = 1$. Then (*) prompts us to consider another 'product' integral for a positive operator A after G.Birkhoff [3]. By the simple functions $A_n = \sum_{i=1}^n t_i^{(n)} E_i^{(n)}$ of A converging uniformly to $A = \int_m^M t dE_t$, we define

$$^{\Pi}\!\!\int_m^M t d\langle E_t x, x\rangle = \lim_{n\to\infty} \prod_{i=1}^n t_i^{(n)\ \langle E_i^{(n)} x, x\rangle}.$$

This definition makes sense by the above properties and it also shows

$$^{\Pi}\!\!\int_m^M t d\langle E_t x, x\rangle = \Delta_x(A).$$

Thus we may say that the determinant for positive operators is a continuous weighted geometric mean with the weight x. Similarly we may consider $\langle A^{-1}x, x\rangle^{-1}$ as a continuous harmonic mean. Thereby a continuous version of the harmonic-geometric-arithmetic mean inequality is the following basic one:

Theorem 2. *The determinant $\Delta_x(A)$ is not greater (resp. smaller) than $\langle Ax, x\rangle$ (resp. $\langle A^{-1}x, x\rangle^{-1}$).*

Immediately we have the following inequalities:

Corollary 3. $\|A^{-1}\|^{-1} \le \Delta_x(A) \le r(A) = \|A\|$.

Moreover it is well-known that the *power (arithmetic) means*

$$M_r(x_1, ..., x_n) = \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r}$$

make a path of means from the harmonic one at $r = -1$ to the arithmetic one at $r = 1$ via the geometric one at $r = 0$ (precisely the limit as $r \to 0$). Since $\langle A^r x, x\rangle^{1/r}$ is considered as a continuous power mean from the above viewpoint, we have an extension of Theorem 2:

Theorem 4. *For positive invertible operator A, $\langle A^t x, x\rangle^{1/t}$ converges monotone decreasingly (resp. increasingly) to $\Delta_x(A)$ as $t \downarrow 0$ (resp. $t \uparrow 0$).*

Proof. The monotonicity follows from the Jensen inequality (e.g., [1; Theorem IV.1]);

$$0 \le \frac{s}{t} \le 1 \text{ implies } \langle A^t x, x\rangle^{s/t} \le \langle A^{t(s/tx,x\rangle}) = \langle A^s x, x\rangle.$$

As for the convergence, the l'Hospital theorem shows

$$\lim_{t\downarrow 0}\frac{\log\langle A^t x,x\rangle}{t}=\lim_{t\downarrow 0}\frac{d\langle A^t x,x\rangle/dt}{\langle A^t x,x\rangle}=\lim_{t\downarrow 0}\frac{\langle A^t\log Ax,x\rangle}{\langle A^t x,x\rangle}=\langle\log Ax,x\rangle,$$

so that we have the required convergence. Similarly we have the case $t\uparrow 0$.

In the Kubo-Ando theory of operator means [7], the notion of duality was introduced. In terms of the power mean, M_r is the *dual* of M_{-r};

$$M_r(x_1^{-1},...,x_n^{-1})=M_{-r}(x_1,...,x_n)^{-1}.$$

In particular, the geometric mean is selfdual, which reflects the following property:

Corollary 5. $\Delta_x(A^{-1})=\Delta_x(A)^{-1}$.

3. Inequalities. In this section we observe various inequalities around the determinant. First we pose the well-known Ky Fan inequality, which follows from the operator concavity of the logarithm:

Theorem 6. *If* $A,B\geq 0$, $\alpha>0,\beta>0$ *and* $\alpha+\beta=1$, *then*

$$\Delta_x(\alpha A+\beta B)\geq\Delta_x(A)^{\alpha}\Delta_x(B)^{\beta}.$$

Next we discuss Arveson's inequality in [2; Corollary 4.3.3 (i)], which is an extension of the Ky Fan inequality. To see this, he shows his formula [2; Proposition 4.3.2]. Note that $\Delta_x(AB)=\Delta_x(A)\Delta_x(B)$ for commuting A and B.

Theorem 7 (Arveson). *The determinant* $\Delta_x(A)$ *is the infimum of the set*

$$\{\langle ABx,x\rangle\mid\Delta_x(B)\geq 1,\ B\in\{A\}'\}.$$

Proof. Since a positive operator B commutes with A and $\Delta_x(B)\geq 1$, we have

$$\Delta_x(AB)=\Delta_x(A)\Delta_x(B)\geq\Delta_x(A).$$

On the other hand, consider $B=\Delta_x(A)A^{-1}$. Then $\Delta_x(B)=1$ and

$$\langle ABx,x\rangle=\Delta_x(A)\langle AA^{-1}x,x\rangle=\Delta_x(A),$$

which shows the above formula.

Now we have Arveson's inequality for commuting operators similarly, but it is an open problem for noncommuting operators:

Corollary 8. *If* A *and* B *commutes, then* $\Delta_x(A+B)\geq\Delta_x(A)+\Delta_x(B)$.

Finally we estimate the difference between $\Delta_x(A)$ and $\langle Ax,x\rangle$ using concavity of the logarithm based on the following result:

Lemma 9. *Let a function $f(t)$ be monotone increasing, concave and differentiable on $[m, M]$. Then*

$$h(t) \equiv t - f^{-1}(at+b) \leq \frac{f(\mu)(M-m) + f(m)M - f(M)m}{f(M) - f(m)} - \mu$$

where $f'(\mu) = a \equiv (f(M) - f(m))/(M-m)$ and $b = (Mf(m) - mf(M))/(M-m)$.

Proof. We can take $t_0 \in [m, M]$ with $f(\mu) = at_0 + b$. Since f^{-1} is convex, then h is concave and the maximum is $h(t_0)$ by

$$h'(t_0) = 1 - \frac{a}{f'\left(f^{-1}(at_0+b)\right)} = 1 - \frac{a}{f'(\mu)} = 0.$$

Then $h(t_0)$ is equal to the right hand in the required inequality.

For $f(t) = \log t$, note that the constant a in the above lemma is equal to the inverse of the logarithmic mean $L(M, m)$. So we have the following estimation:

Theorem 10. *If A is a positive operator such that $0 < m \leq A \leq M$, then*

$$\langle Ax, x \rangle - \Delta_x(A) \leq L(M,m)\left(\log L(M,m) + \frac{M\log m - m \log M}{M-m} - 1\right).$$

Proof. Putting $t = \langle Ax, x\rangle$ and $f(t) = \log t$, we have

$$\langle Ax, x \rangle - \Delta_x(A) \leq t - e^{at+b} = t - f^{-1}(at+b)$$

for A and b as in Lemma 9. Since $f'(\mu) = \frac{1}{\mu}$, then $\mu = 1/a = L(M,m)$. Thus we have the required inequality by Lemma 9.

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References

[1] T.Ando, *Topics on operator inequalities*, Hokkaido Univ. Lecture Note, 1978.
[2] W.B.Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642.
[3] G.Birkhoff, *On product integration*, J.Math. and Phys. **16** (1937), 104–132.
[4] B.Fuglede and R.V.Kadison, *On determinants and a property of the trace in finite factors*, Proc. Nat. Acad. Sci. U.S.A. **36** (1951), 425–431.
[5] B.Fuglede and R.V.Kadison, *Determinant theory in finite factors*, Ann. of Math. **55** (1952), 520–530.
[6] G.Hardy, J.E.Littlewood and G.Pólya, *Inequalities (2nd. Ed.)*, Cambridge Univ. Press, 1951.
[7] F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.

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